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On vector measures which have everywhere infinite variation or noncompact range

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Abstract

Let X be an infinite-dimensional Banach space, let Σ be a σ-algebra of subsets of a set S, and denote by $ca(\Sigma, X)$ the Banach space of X-valued measures on Σ equipped with the uniform norm. We say that a nonzero $\mu \in ca(\Sigma, X)$ is everywhere of infinite variation [has everywhere noncompact range] if $|\mu|(A) = \infty$ or 0 [$\{\mu(E) : E \in \Sigma,\ E \subset A\}$ is not relatively compact or equals $\{0\}$] for every $A \in \Sigma$. Let λ be a nonatomic probability measure on Σ, and denote by $ca(\Sigma, \lambda, X)$ the closed subspace of $ca(\Sigma, X)$ consisting of λ-continuous measures. Analogously as above, we define measures in $ca(\Sigma, \lambda, X)$ that are λ-everywhere of infinite variation or have λ-everywhere noncompact range. Using the Dvoretzky–Rogers theorem, we give two constructions of an absolutely convergent series of λ-simple measures in $ca(\Sigma, \lambda, X)$ such that the sum of each of its subseries is λ-everywhere of infinite variation. In particular, the normed space $\mathcal{P}(\lambda, X)$ of Pettis λ-integrable functions with values in X lacks property (K), and so is incomplete. These results refine and improve some earlier results of E. Thomas, and L. Janicka and N. J. Kalton. One of the constructions also yields the existence of an infinite-dimensional closed subspace in $ca(\Sigma, \lambda, X)$ all of whose nonzero members are λ-everywhere of infinite variation. Moreover, modifying some ideas of R. Anantharaman and K. M. Garg, we prove that the measures that are λ-everywhere of infinite variation form a dense G_δ-set in $ca(\Sigma, \lambda, X)$. From this we derive an analogous result on measures that are everywhere of infinite variation and the closed subspace of $ca(\Sigma, X)$ consisting of nonatomic measures. Similar results concerning measures that have [λ-] everywhere noncompact range are also established. In this case, the existence of X-valued measures with noncompact range must, however, be postulated. We also prove that the measures of σ-finite variation form an $F_{\sigma\delta}$-, but not F_σ-, subset of $ca(\Sigma, \lambda, X)$, and the same is true for $\mathcal{P}(\lambda, X)$ provided that X is separable. Finally, we consider the special case when X is a Banach lattice and, for X nonisomorphic to an AL-space, we note analogues of some of the results above for positive X-valued measures on Σ.

1. Introduction

In general, our terminology and notation, and the basic facts concerning Banach spaces and vector measures that we use here are standard, as in [DS], [LT] and [DU]. Throughout this paper,

- X denotes an infinite-dimensional (real or complex) Banach space,
- Σ denotes a σ-algebra of subsets of a set S,
- λ denotes a probability measure on Σ assumed, with the exception of Sections 3 and 7, to be nonatomic (= atomless),
- $([0,1], \mathcal{B}, m)$ denotes the standard Borel–Lebesgue measure space, with $\mathcal{B} =$ Borel σ-algebra on the interval $[0,1]$ and $m =$ Lebesgue measure.

We denote by $ca(\Sigma, X)$ the Banach space of all (countably additive) measures $\mu : \Sigma \to X$ equipped with the uniform norm

$$\|\mu\| := \sup_{A \in \Sigma} \|\mu(A)\|,$$

and by $cca(\Sigma, X)$ its closed subspace consisting of measures with relatively (norm) compact range. Further, we let $ca(\Sigma, \lambda, X)$ stand for the closed subspace of $ca(\Sigma, X)$ consisting of λ-continuous measures, and $cca(\Sigma, \lambda, X)$ for the closed subspace $cca(\Sigma, X) \cap ca(\Sigma, \lambda, X)$ of $ca(\Sigma, \lambda, X)$.

The variation (measure) of a scalar or vector measure μ is denoted by $|\mu|$. We use $bvca(\Sigma, X)$ $[bvca(\Sigma, \lambda, X)]$ as the notation for the Banach space of all measures in $ca(\Sigma, X)$ $[ca(\Sigma, \lambda, X)]$ which are of bounded variation, equipped with the variation norm $\mu \to |\mu|(S)$. Let us mention at this point that the uniform norm in $ca(\Sigma, X)$ is equivalent to another frequently used norm in this space, namely $\mu \to \sup\{|x^*\mu|(S) : x^* \in X^*,\ \|x^*\| \le 1\}$ (cf. [DU, Prop I.1.11]).

We now introduce the basic concepts used in the sequel. We say that a nonzero measure $\mu : \Sigma \to X$ is *everywhere of infinite variation* if $|\mu|(A) = \infty$ or 0 for every $A \in \Sigma$. (In [AG, p. 22] such a measure is called nowhere of finite variation.) If $\mu \ll \lambda$ and $|\mu|(A) = \infty$ whenever $\lambda(A) > 0$, then we say that μ is λ-*everywhere of infinite variation*. Clearly, if μ is λ-everywhere of infinite variation, then it is everywhere of infinite variation; the converse holds provided that $\lambda(A) = 0$ whenever $|\mu|(A) = 0$. Also note that a nonzero measure $\mu \ll \lambda$ is everywhere of infinite variation if and only if there is $B \in \Sigma$ such that μ is concentrated on B and λ-everywhere of infinite variation on B. (That is, $|\mu|(A) = \infty$ whenever $A \subset B$ and $\lambda(A) > 0$.)

Likewise, we say that a nonzero measure $\mu : \Sigma \to X$ has *everywhere noncompact range* if, for every $A \in \Sigma$, the set $\{\mu(E) : E \in \Sigma,\ E \subset A\}$ is not relatively compact or equals $\{0\}$. Measures that have λ-*everywhere noncompact range* are defined analogously.

Obviously, a measure that is everywhere of infinite variation or has everywhere noncompact range is necessarily nonatomic.

The present paper deals with the existence and constructions of such "pathological" measures, and with the "dense and G_δ" type properties of various sets of such measures. In order to place our results in a proper perspective, we first make quite a detailed survey of the earlier results concerning measures with everywhere infinite variation. (We do not know if measures with everywhere noncompact range have been considered in the literature.) We call the reader's attention to the fact that most of the results we mention are stated here in a more general form than originally proved, and sometimes provided with alternative arguments or additional information.

Measures of infinite variation with values in Banach spaces are intimately related to unconditionally but not absolutely convergent series and thus, in general, the very existence of such measures depends in an essential way on the Dvoretzky–Rogers theorem (see [LT, Thm. 1.c.2]). Indeed, a vector measure μ is of infinite variation if and only if there is a disjoint sequence (A_n) of sets in its domain such that $\sum_{n=1}^{\infty} \|\mu(A_n)\| = \infty$. On the other hand, given an unconditionally but not absolutely convergent series $\sum_{n=1}^{\infty} x_n$, the measure μ on $2^{\mathbb{N}}$ defined by $\mu(A) = \sum_{n\in A} x_n$ is of infinite variation, and it is easy to produce such purely atomic measures of infinite variation on every infinite σ-algebra of sets. Moreover, on every σ-algebra that admits a nonatomic finite positive measure, it is easy to construct a nonatomic vector measure of infinite variation. For instance, such is the measure $\mu : \mathcal{B} \to X$ defined by $\mu(A) = \sum_{n=1}^{\infty} 2^n m(A \cap (2^{-n-1}, 2^{-n}]) \cdot x_n$, where $\sum_{n=1}^{\infty} x_n$ is an unconditionally but not absolutely convergent series in X. The measure $\mu : \mathcal{B} \to L_p(m)$ $(1 < p < \infty)$ that assigns to every $B \in \mathcal{B}$ its characteristic function χ_B is even worse in this respect—it is m-everywhere of infinite variation (cf. [DU, Ex. I.1.16]). (Furthermore, it has m-everywhere noncompact range; cf. [DU, Ex. IX.1.1].) Now, it is natural to ask whether similar examples can be found in every infinite-dimensional Banach space X. As will be explained in a moment (after (**JK**) below), this question is equivalent to the problem of the existence of X-valued measures with non-σ-finite variation. It was first solved by Thomas [T1, p. 90] who published in 1974 the following result.

(**T**) *If $\sum_{n=1}^{\infty} x_n$ is an unconditionally but not absolutely convergent series in X, then there is a sequence (f_n) in $L_1(m)$ with $\|f_n\|_1 \le 1$ such that the measure $\mu \in cca(\mathcal{B}, m, X)$ defined by*

$$\mu(A) = \sum_{n=1}^{\infty} \int_A f_n \, dm \cdot x_n$$

has non-σ-finite variation.

Very soon, in 1977, an alternative (and technically much simpler) solution to the question above was published by Janicka and Kalton [JK]. In order to present their approach in a clear and concise way, let us first introduce two additional subspaces of $ca(\Sigma, \lambda, X)$:

- $ca_\sigma(\Sigma, \lambda, X)$, consisting of all measures which are of σ-finite variation,
- $ca_s(\Sigma, \lambda, X)$, consisting of all λ-simple measures.

Here, by a λ-*simple measure* we mean an arbitrary measure μ of the form

$$\mu(A) = \sum_{i=1}^{n} \lambda(A \cap A_i) x_i,$$

where $\{A_1, \ldots, A_n\}$ is a Σ-partition of S and $\{x_1, \ldots, x_n\} \subset X$. Clearly, λ-simple measures coincide with indefinite λ-integrals of Σ-simple functions $f : S \to X$. Moreover, $ca_s(\Sigma, \lambda, X)$ is dense in $cca(\Sigma, \lambda, X)$ [DU, Thm. VIII.1.5].

Also, recall that a complete metrizable topological vector space is called an *F-space.*

The main result of Janicka and Kalton ([JK, Thm. 1 and its proof) may now be stated as follows. (The restriction in [JK] to the measure space $([0,1], \mathcal{B}, m)$ is inessential; the definition of the F-space topology occurring below is recalled before Theorem 5.2.)

(**JK**) *The subspace* $ca_\sigma(\Sigma, \lambda, X)$ *admits an F-space topology* τ *which is stronger than the norm topology induced from* $ca(\Sigma, \lambda, X)$, *and is strictly stronger than the norm topology on* $ca_s(\Sigma, \lambda, X)$.

In consequence, by the closed graph theorem, $cca(\Sigma, \lambda, X) \setminus ca_\sigma(\Sigma, \lambda, X) \neq \emptyset$.

Now, as observed in [JK], by an easy exhaustion argument it can be seen that for every $\mu \in ca(\Sigma, \lambda, X)$ there is a λ-maximal set S_0 in Σ on which $|\mu|$ is σ-finite. Hence, if μ has non-σ-finite variation (on S), then $\lambda(S \setminus S_0) > 0$, and $|\mu|(A) = \infty$ for all $A \subset S \setminus S_0$ with $\lambda(A) > 0$. We may thus say that μ is λ-everywhere of infinite variation on $S \setminus S_0$. (A similar observation holds for measures in $ca(\Sigma, X)$ that are of non-σ-finite variation.)

Using this observation, (**JK**) and the isomorphism of standard measure spaces, Janicka and Kalton [JK, Thm. 2] readily show that there exist measures in $cca(\mathcal{B}, m, X)$ which are m-everywhere of infinite variation.

In fact, a more elementary argument (cf. the proof of Proposition 6.3) can be employed to show that, in the general case, there exist measures $\mu \in cca(\Sigma, \lambda, X)$ which are λ-everywhere of infinite variation. To see this, first note that from (**JK**) and the observation above it follows that for every $A \in \Sigma$ with $\lambda(A) > 0$ we can find $C \in \Sigma$ with $C \subset A$ such that: $(*)$ $\lambda(C) > 0$ and there exists a measure in $cca(\Sigma, \lambda, X)$ which is concentrated on C and is λ-everywhere of infinite variation on C. Next, take a maximal disjoint family consisting of sets $C \in \Sigma$ satisfying condition $(*)$. It is countable, say $\{C_n : n \in \mathbb{N}\}$, and $\lambda(S \setminus \bigcup_{n=1}^{\infty} C_n) = 0$. By $(*)$, for every n there exists $\mu_n \in cca(\Sigma, \lambda, X)$ which is concentrated on C_n and is λ-everywhere of infinite variation on C_n; it can be assumed that $\|\mu_n\| \le 2^{-n}$. Finally, the measure $\mu := \sum_{n=1}^{\infty} \mu_n$ is clearly in $cca(\Sigma, \lambda, X)$ and is λ-everywhere of infinite variation.

Constructions of X-valued measures which are m-everywhere of infinite variation as sums of series of m-simple measures were also given by A. Szankowski (unpublished) in 1976, and Rodríguez-Piazza [R-P, Proof of Thm. 3.4] in 1991. They were based on the Dvoretzky theorem and some results on 1-absolutely summing operators, respectively. Szankowski's construction is sketched in the remark after the proof of Theorem 2.3 below.

The following result was obtained by Anantharaman and Garg [AG, Cor. 2.5] in 1983. (In view of what was said above, their restriction to $([0,1], \mathcal{B}, m)$ is inessential, as easily seen by inspecting [AG].)

(**AG**) *The measures that are λ-everywhere of infinite variation form a dense G_δ-subset of $cca(\Sigma, \lambda, X)$.*

Let us stress, however, that their proof is not a Baire category proof of the existence of such measures; on the contrary, it makes use of the fact that such measures do exist.

Before presenting our main results, let us make the following digression, which is only loosely relevant to the main body of the paper.

The existence of vector measures with non-σ-finite variation is closely connected with the problem of completeness of spaces of Pettis integrable functions. Let $\mathcal{P}(\lambda, X)$ denote the normed space of all Pettis λ-integrable functions $f : S \to X$, with the norm $\|f\| = \sup_{A\in\Sigma} \|\int_A f\, d\lambda\|$, where, of course, functions f, g with $\|f - g\| = 0$ are identified. Obviously, the map $f \to \int_{(\cdot)} f\, d\lambda$ is a linear isometric embedding of $\mathcal{P}(\lambda, X)$ into $ca(\Sigma, \lambda, X)$, and so $\mathcal{P}(\lambda, X)$ may be considered a subspace of $ca(\Sigma, \lambda, X)$.

Now, $ca_s(\Sigma, \lambda, X) \subset \mathcal{P}(\lambda, X) \subset ca_\sigma(\Sigma, \lambda, X)$ (see [R, Lemma 2], [Mu, Cor. 1] or [VTC, Prop. II.3.5 (a)] for the second inclusion), and taking closures in $ca(\Sigma, \lambda, X)$ gives $cca(\Sigma, \lambda, X) = \overline{ca_s(\Sigma, \lambda, X)} \subset \overline{\mathcal{P}(\lambda, X)}$. However, since $cca(\Sigma, \lambda, X) \setminus ca_\sigma(\Sigma, \lambda, X) \neq \emptyset$ by (**JK**), it follows that $\mathcal{P}(\lambda, X) \neq \overline{\mathcal{P}(\lambda, X)}$ so that the space $\mathcal{P}(\lambda, X)$ is not complete. We have repeated here the argument used in [JK, Thm. 3] in the case $\lambda = m$; the result itself was independently obtained also by Thomas in [T2, p. 131] as a consequence of (**T**). Apparently unaware of [JK], Heiliö [He, 4.4] introduced a topology in $\mathcal{P}(\lambda, X)$ which is, for separable X, the restriction of the Janicka–Kalton topology, and applied it to establish the incompleteness of $\mathcal{P}(\lambda, X)$ in this case. Let us also mention that already Pettis himself, using an ingenious argument, proved that the space $\mathcal{P}(m, L_2(m))$ is not complete [Pe, 9.4]. Moreover, Rybakov [R, Ex. on p. 62] showed that $\mathcal{P}(m, c_0)$ is not complete. Their constructions can be viewed as special cases of that used in the proof of our Theorem 2.3.

On the other hand, quite surprisingly, it was recently shown in [DFP, Thm. 2] that $\mathcal{P}(\lambda, X)$ is always barrelled. Now, many intermediate properties between completeness and barrelledness are known in the literature, and one may wonder if $\mathcal{P}(\lambda, X)$ enjoys some of them. To be more specific consider, for a normed space Z, the following conditions: (1) Z is complete, (2) Z has property (K), (3) Z is a Baire space, (4) Z is ultrabarrelled, and (5) Z is barrelled. We refer the reader to [PB] for the definitions and more information. Let us recall, however, that the space Z has *property* (K) if every sequence (z_n) in Z with $z_n \to 0$ has a subsequence (z_{n_k}) such that the series $\sum_{k=1}^{\infty} z_{n_k}$ converges in Z. As for the notion of ultrabarrelledness, it is enough for our purposes to know that Z is ultrabarrelled if and only if every closed-graph linear map from Z to any F-space is continuous (see [W, Prop. 7 on p. 15]).

It is known that (1)⇒(2)⇒(3)⇒(4)⇒(5), and that none of these implications is reversible, in general. (The second implication is a particular case of a more general result [BKL, Thm. 2].) Returning to the space $\mathcal{P}(\lambda, X)$, in [DFP] it was also shown that $\mathcal{P}(m, L_2(m))$ does not have property (K). The problem of extending this last result was, in fact, the starting point of our research.

Now, the question raised above concerning the properties of $\mathcal{P}(\lambda, X)$ stronger than barrelledness turns out to be quite easy, and to answer it is just enough to apply (**JK**) and the closed graph theorem. The following holds true:

($\star$) *No subspace of $ca(\Sigma, \lambda, X)$ which contains $ca_s(\Sigma, \lambda, X)$ and is contained in $ca_\sigma(\Sigma, \lambda, X)$ is ultrabarrelled. In particular, the space $\mathcal{P}(\lambda, X)$ is not ultrabarrelled, and so is neither Baire nor has property (K).*

Indeed, suppose a subspace $\mathcal{M}$ such that $ca_s(\Sigma, \lambda, X) \subset \mathcal{M} \subset ca_\sigma(\Sigma, \lambda, X)$ is ultrabarrelled in the norm topology. Then, by the closed graph theorem, the identity map from $\mathcal{M}$ with its norm topology into the space $ca_\sigma(\Sigma, \lambda, X)$ with the F-space topology τ provided by (**JK**) is continuous. It follows that both the topologies coincide on $\mathcal{M}$, hence a fortiori on $ca_s(\Sigma, \lambda, X)$, in contradiction with (**JK**).

We now give a more detailed description of the contents of the present paper.

In Section 2, Theorem 2.3, we combine "small" simple measures of "large" variation, used in [JK] and [R-P], with an idea of Szankowski's proof mentioned above, to construct explicitly an absolutely convergent series of λ-simple measures such that the sum of each of its subseries is λ-everywhere of infinite variation. Thus this result yields two things at the same time: the existence of measures in $cca(\Sigma, \lambda, X)$ with λ-everywhere infinite variation, and the lack of property (K) in $\mathcal{P}(\lambda, X)$ (in particular, the incompleteness of $\mathcal{P}(\lambda, X)$). Moreover, as one of the consequences of our construction, we obtain the existence of an infinite-dimensional closed subspace in $cca(\Sigma, \lambda, X)$ all of whose nonzero members are λ-everywhere of infinite variation (Corollary 2.5). Our Theorem 2.7 is more in the spirit of Thomas: it shows how one can use an arbitrary unconditionally but not absolutely convergent series $\sum_{n=1}^{\infty} x_n$ to produce a series of λ-simple measures that has the property mentioned above. Its consequence, Corollary 2.8, shows that in Thomas' result (**T**) a desired measure μ can be obtained already by means of a very simple sequence (f_n), namely, $f_n = c_n\chi_{S_n}$.

Section 3 is auxiliary and is mostly concerned with semicontinuity type properties of some maps related to the map $\mu \to |\mu|$ on $ca(\Sigma, \lambda, X)$, needed in the next two sections.

In Section 4 we give two Baire category proofs of the existence of measures that are λ-everywhere of infinite variation. In fact, we show in Theorem 4.1 that such measures form dense G_δ-sets in both spaces $ca(\Sigma, \lambda, X)$ and $cca(\Sigma, \lambda, X)$. The latter case is, of course, the result of [AG] stated above in its generalized form (**AG**). A similar result for the set of those measures in $ca(\Sigma, \lambda, X)$ (or $cca(\Sigma, \lambda, X)$) which are everywhere of infinite variation is given in Theorem 4.5.

In Section 5 we show that, in contrast to the results just mentioned, the measures in $ca(\Sigma, \lambda, X)$ or $cca(\Sigma, \lambda, X)$ that are of non-σ-finite variation form a set of exact Borel class $G_{\delta\sigma}$ (Theorems 5.1 and 5.2). In the same vein, $\mathcal{P}(\lambda, X)$ is a set of exact Borel class $F_{\sigma\delta}$ provided that X is separable (Theorem 5.6). The latter is related, via [BKL, Cor.], to the result $(\star)$ above.

Section 6 is devoted to vector measures with noncompact range. We show in Theorem 6.4 that, whenever such measures exist, the measures that have λ-everywhere noncompact range form a dense G_δ-subset of $ca(\Sigma, \lambda, X)$. This combined with Theorem 4.1 yields that the measures with λ-everywhere infinite variation and λ-everywhere noncompact range form a dense G_δ-subset of $ca(\Sigma, \lambda, X)$ (Corollary 6.5). An analogue of Theorem 6.4 for λ-continuous measures with everywhere noncompact range is provided by Theorem 6.8 (a).

In Section 7, we first consider sets of a certain form in, roughly speaking, c_0-sums of normed spaces (Theorem 7.1) or in Banach spaces with Schauder decompositions (Corollary 7.3), and prove that they are G_δ-subsets of those spaces. Then, using the latter result, we arrive quickly at Theorem 7.4 in which we give precise analogues of Theorems 4.5 and 6.8 (b) for the subspaces of $ca(\Sigma, X)$ and $cca(\Sigma, X)$ consisting of nonatomic measures. It should be noted that Theorems 4.5 and 7.4 (a) solve a problem posed at the end of [AG].

Finally, in Section 8, we consider the special case when X is a Banach lattice and, for X nonisomorphic to an AL-space, we obtain in Theorem 8.2 an analogue of Theorem 2.3 for positive measures from Σ to X.

In Sections 4 and 6 we shall make use of some standard properties of the submeasure majorant $\overline{\mu}$ of a measure $\mu : \Sigma \to X$, which is defined by

$$\overline{\mu}(A) = \sup\{\|\mu(B)\| : B \in \Sigma \text{ and } B \subset A\} \quad \text{for all } A \in \Sigma.$$

Namely, $\overline{\mu}$ is increasing, σ-subadditive, and order continuous (i.e., for every decreasing sequence (A_n) in Σ with empty intersection we have $\overline{\mu}(A_n) \to 0$); see [D1, 2.1, 5.2, 5.3] for details. A set $A \in \Sigma$ is called *μ-null* if $\overline{\mu}(A) = 0$, i.e., $\mu(B) = 0$ for all $B \in \Sigma$ with $B \subset A$. Also, note that $\overline{\mu}(S) = \|\mu\|$.

Convention. All subsets of S occurring in our considerations are assumed (often tacitly) to belong to Σ. Thus, for instance, if $A \in \Sigma$ and we take a set $B \subset A$, then it is to be understood that, in addition, $B \in \Sigma$.

2. Vector measures with λ-everywhere infinite variation represented by series of simple measures

The proofs of the main results of this section, Theorems 2.3 and 2.7, provide us with (essentially) explicit constructions of vector measures with λ-everywhere infinite variation. In either case, such measures are obtained as sums of series of λ-simple measures of the type used in [JK], and described in the following lemma.

2.1. LEMMA. *Let $z_1, \ldots, z_n$ be nonzero elements in a normed space Z. Denote*

$$c = \sup\Big\{\Big\|\sum_{i=1}^{n} t_i z_i\Big\| : 0 \le t_i \le 1\Big\} \quad \text{and} \quad r = \sum_{i=1}^{n} \|z_i\|,$$

and let $\{S_1, \ldots, S_n\}$ be a Σ-partition of S with $\lambda(S_i) = \|z_i\|/r$ $(i = 1, \ldots, n)$. Then the λ-simple measure $\mu : \Sigma \to \operatorname{lin}\{z_1, \ldots, z_n\} \subset Z$ defined by

$$\mu(A) = \sum_{i=1}^{n} \frac{\lambda(A \cap S_i)}{\lambda(S_i)} z_i = r \sum_{i=1}^{n} \lambda(A \cap S_i) \frac{z_i}{\|z_i\|}$$

has the following properties: $\|\mu\| = c$ *and* $|\mu| = r\lambda$.

P r o o f. The first property follows from the nonatomicity of λ. On the other hand, for every $A \in \Sigma$ we have

$$|\mu|(A) = \sum_{i=1}^{n} |\mu|(A \cap S_i) = r \sum_{i=1}^{n} \lambda(A \cap S_i) = r\lambda(A). \ \blacksquare$$

The λ-simple measures used in our constructions will be required to have small norms and uniformly large (when compared with λ) variations, and to be located, roughly speaking, in mutually "orthogonal" subspaces of X. As will be seen from Theorem 2.7 below, such measures can be obtained from any unconditionally but not absolutely convergent series. We could adopt this approach in the proof of Theorem 2.3 as well; however, for the sake of clarity, we prefer to rely on the following

2.2. PROPOSITION. *For every $r > 0$ there is $m \in \mathbb{N}$ such that for every normed space Z of dimension $\ge m$ there exists a λ-simple measure $\mu : \Sigma \to Z$ with $\|\mu\| \le 1$ and $|\mu| = r\lambda$.*

P r o o f. By the Dvoretzky–Rogers theorem (or rather its proof, see [LT, p. 17]), for every $r > 0$ there is $m \in \mathbb{N}$ such that if Z is a normed space with $\dim Z \ge m$, then one can find nonzero elements $z_1, \ldots, z_n$ in Z for which

$$\Big\|\sum_{i=1}^{n} t_i z_i\Big\| \le 1 \quad \text{whenever } |t_i| \le 1, \quad \text{and} \quad \sum_{i=1}^{n} \|z_i\| = r.$$

Thus, the assertion follows from Lemma 2.1. $\blacksquare$

2.3. Theorem. *There exists a sequence of* λ*-simple measures* $\mu_n : \Sigma \to X$ *with the following properties.*

(a) *The sequence* (μ_n) *is a basic sequence in each of the Banach spaces* $cca(\Sigma,\lambda,X)$ *and* $bvca(\Sigma,\lambda,X)$.

(b) *The series* $\sum_{n=1}^{\infty}\mu_n$ *is absolutely* (*hence subseries*) *convergent in* $cca(\Sigma,\lambda,X)$.

(c) *For every infinite subset* M *of* $\mathbb{N}$, *the measure* $\mu_M := \sum_{n\in M}\mu_n$ *is* λ*-everywhere of infinite variation.*

(d) *Every sequence* (M_k) *of disjoint infinite subsets of* $\mathbb{N}$ *has a subsequence* (N_k) *such that the measures* μ_{N_k} (*as defined in* (c)) *form a basic sequence in* $cca(\Sigma,\lambda,X)$, *and every nonzero measure in the closed linear span of* (μ_{N_k}) *in* $cca(\Sigma,\lambda,X)$ *is* λ*-everywhere of infinite variation.*

Proof. We may assume that X has a Schauder basis (x_n). Denote by P_n the associated projections from X onto $\mathrm{lin}\{x_1,\dots,x_n\}$ $(n\in\mathbb{N})$, and let $b=\sup_n\|P_n\|$ be the basis constant of (x_n).

Applying Proposition 2.2, we find a sequence (X_n) of finite-dimensional subspaces in X of the form
$$X_n=\mathrm{lin}\,\{x_k : k_{n-1}<k\le k_n\},$$
where $0=k_0<k_1<\dots$, and, for every n, a λ-simple measure $\mu_n:\Sigma\to X$ such that $\mu_n(\Sigma)\subset X_n$, $\|\mu_n\|\le 2^{-n}$ and $|\mu_n|=2^n\lambda$. Thus condition (b) is obviously satisfied by (μ_n).

To verify (a), let $m<n$ and let $a_1,\dots,a_n$ be arbitrary scalars. Then, for every $A\in\Sigma$,
$$\Big\|\sum_{i=1}^{m}a_i\mu_i(A)\Big\|=\Big\|P_{k_m}\Big(\sum_{i=1}^{n}a_i\mu_i(A)\Big)\Big\|\le b\Big\|\sum_{i=1}^{n}a_i\mu_i(A)\Big\|.$$
From this we easily get
$$\Big\|\sum_{i=1}^{m}a_i\mu_i\Big\|\le b\Big\|\sum_{i=1}^{n}a_i\mu_i\Big\|\quad\text{and}\quad\Big|\sum_{i=1}^{m}a_i\mu_i\Big|\le b\Big|\sum_{i=1}^{n}a_i\mu_i\Big|.$$
Therefore, (a) follows from [LT, Prop. 1.a.3].

Now, let μ_M be as defined in (c). Then, for every $A\in\Sigma$ and $n\in M$, we have
$$2b\|\mu_M(A)\|\ge\|Q_n\mu_M(A)\|=\|\mu_n(A)\|,$$
where $Q_n=P_{k_n}-P_{k_{n-1}}$ is the natural projection from X onto X_n. Hence
$$2b|\mu_M|(A)\ge|\mu_n|(A)=2^n\lambda(A),$$
and thus also the assertion of (c) holds.

Finally, let (M_k) be as required in (d), and for every k put $\tau_k=\mu_{M_k}$. We first verify the following strengthening of (c):

If (c_k) is a nonzero sequence of scalars such that $\tau(A)=\sum_{k=1}^{\infty}c_k\tau_k(A)$ is defined for all $A\in\Sigma$, then the measure τ is λ-everywhere of infinite variation.

For, let some $c_j \neq 0$. Then, as in (c) above, for every $n \in M_j$ and $A \in \Sigma$ we have $Q_n\tau(A) = c_j\mu_n(A)$. Hence, $2b|\tau|(A) \geq 2^n|c_j|\lambda(A)$, so that $|\tau|(A) = \infty$ if $\lambda(A) > 0$.

To finish, it is enough to observe that the normalized sequence $(\tau_k/\|\tau_k\|)$ is coordinatewise—with respect to the basic sequence (μ_n)—convergent to zero, and apply a well-known extraction principle for basic sequences [LT, Prop. 1.a.12]. ∎

Remark. We describe briefly Szankowski's construction mentioned in the Introduction. Let Z_n denote the 2^{4n}-dimensional Hilbert space, and define a measure $\nu_n : \mathcal{B} \to Z_n$ by $\nu_n(A) = 2^n(m(A \cap I_1), \ldots, m(A \cap I_{2^{4n}}))$, where the I_i's, $i = 1, \ldots, 2^{4n}$, form a partition of $[0, 1]$ into intervals of length 2^{-4n}. Then $\|\nu_n\| = 2^{-n}$ and $|\nu_n| = 2^n m$. Apply the Dvoretzky theorem [Pi, Thm. 4.1] to select integers k_n increasing so fast that for every n there exists an embedding $J_n : Z_n \to X_n$ with $\|z\| \leq \|J_n(z)\| \leq 2\|z\|$ for all $z \in Z_n$, where the X_n's are defined as in our proof above. (Actually, the mutually "orthogonal" subspaces X_n were originally obtained in a more direct way, without using a Schauder basis.) Finally, set $\mu_n = J_n \circ \nu_n$. Then, as in the proof above, it is easily seen that the measure $\mu := \sum_{n=1}^{\infty} \mu_n$ is m-everywhere of infinite variation.

It is evident that Szankowski's construction carries over to the general case and yields a sequence (μ_n) as required in Theorem 2.3.

2.4. Corollary. *The space* $\mathcal{P}(\lambda, X)$ *does not have property* (K).

2.5. Corollary. *The space* $cca(\Sigma, \lambda, X)$ *contains a closed infinite-dimensional subspace* $\mathcal{M}$ *such that every nonzero* $\nu \in \mathcal{M}$ *is* λ*-everywhere of infinite variation.*

Remark. For a trivial reason, in general one cannot expect a subspace $\mathcal{M}$ of $cca(\Sigma, \lambda, X)$ as in Corollary 2.5 to be nonseparable. Indeed, if both $L_1(\lambda)$ and X are separable, so is $cca(\Sigma, \lambda, X)$. (This follows immediately from the density of $ca_s(\Sigma, \lambda, X)$ in $cca(\Sigma, \lambda, X)$, but can also be easily seen by embedding X isometrically into a Banach space Y with a Schauder basis, e.g., $Y = C[0, 1]$, and first proving that $cca(\Sigma, \lambda, Y)$ is separable. Cf. the proof of Theorem 2.6 below.)

Our next result shows that the construction employed in Theorem 2.3 is quite natural.

2.6. Theorem. *Let X have a Schauder basis (x_n), with the associated coefficient functionals (x_n^*), and let $\mu \in cca(\Sigma, \lambda, X)$ be λ-everywhere of infinite variation. Then there exists a sequence of indices $0 = n_0 < n_1 < \ldots$ such that the sequence (μ_k) of finite-dimensional measures defined by*

$$\mu_k = \sum_{i=n_{k-1}+1}^{n_k} x_i^*\mu \cdot x_i$$

satisfies conditions (a) *through* (d) *of Theorem* 2.3, *and* $\mu = \sum_{k=1}^{\infty} \mu_k$ *holds in* $cca(\Sigma, \lambda, X)$.

Proof. Let $P_n : X \to \text{lin}\{x_1, \ldots, x_n\}$ ($n \in \mathbb{N}$) be the projections associated with the basis (x_n). For convenience, we will assume that the basis (x_n) is monotone, that is, $\|P_n\| = 1$ for all n. (This can be achieved by passing to the equivalent norm in X given by the formula $\|x\|' = \sup_n \|P_n(x)\|$.)

Consider the one-dimensional measures $\gamma_n = x_n^*\mu \cdot x_n$ ($n \in \mathbb{N}$). Then μ has the following expansion:

$$\mu(A) = \sum_{n=1}^{\infty} \gamma_n(A) = \lim_{n\to\infty} P_n(\mu(A)) \quad \text{for all } A \in \Sigma.$$

Moreover, since the set $\mu(\Sigma)$ is relatively compact and the projections P_n are equicontinuous, the above expansion converges uniformly for $A \in \Sigma$. We also have

$$\Big\|\sum_{i=1}^{n} a_i\gamma_i(A)\Big\| \le \Big\|\sum_{i=1}^{n+1} a_i\gamma_i(A)\Big\|$$

for all $n \in \mathbb{N}$, scalars a_i, and $A \in \Sigma$. It follows that

$$(*) \qquad \Big\|\sum_{i=1}^{n} a_i\gamma_i\Big\| \le \Big\|\sum_{i=1}^{n+1} a_i\gamma_i\Big\| \quad \text{and} \quad \Big|\sum_{i=1}^{n} a_i\gamma_i\Big| \le \Big|\sum_{i=1}^{n+1} a_i\gamma_i\Big|.$$

In consequence, the $\gamma_n \neq 0$ form a monotone basic sequence in $ca(\Sigma, \lambda, X)$. (In fact, our argument shows that the subspaces $L_1(\lambda)\cdot x_n$ form a monotone Schauder decomposition of $cca(\Sigma, \lambda, X)$.)

Define a sequence $(\varphi_n)_{n=0}^{\infty}$ of finite-dimensional (hence of bounded variation) measures in $cca(\Sigma, \lambda, X)$ by

$$\varphi_0 = 0 \quad \text{and} \quad \varphi_n = \gamma_1 + \ldots + \gamma_n \ (= P_n\circ\mu) \quad \text{for } n = 1, 2, \ldots$$

From $(*)$ it follows that for every $A \in \Sigma$ the sequence $(|\varphi_n|(A))$ is increasing. On the other hand, since $\varphi_n \to \mu$ pointwise on Σ, we have $|\mu|(A) \le \lim_n |\varphi_n|(A)$ for every $A \in \Sigma$. Therefore, $|\varphi_n|(A) \uparrow \infty$ if $\lambda(A) > 0$.

Now, let $f_n = d|\varphi_n|/d\lambda$ for $n = 1, 2, \ldots$ Then, by the above, $f_n \le f_{n+1}$ a.e. for every n, and $\lim_n f_n(s) = \infty$ for a.a. $s \in S$. Applying the convergence of (φ_n) in $cca(\Sigma, \lambda, X)$, the fact that each f_n is finite a.e., and a version of Egorov's theorem, we can easily find a sequence $(E_k)_{k=1}^{\infty}$ in Σ with $\lambda(S\setminus E_k) \to 0$, a sequence $(r_k)_{k=0}^{\infty}$ in $\mathbb{R}_+$ with $r_0 = 0$, and an increasing sequence $(n_k)_{k=0}^{\infty}$ in $\mathbb{N} \cup \{0\}$ with $n_0 = 0$ such that

$$\sum_{k=1}^{\infty} \|\mu_k\| < \infty \quad \text{and} \quad k + r_{k-1} \le f_{n_k} \le r_k \quad \text{on } E_k \quad (k = 1, 2, \ldots),$$

where

$$\mu_k = \varphi_{n_k} - \varphi_{n_{k-1}} \quad \text{for } k = 1, 2, \ldots$$

Then, for every $k \ge 1$ and $A \subset E_k$,

$$(k + r_{k-1})\lambda(A) \le |\varphi_{n_k}|(A) \le r_k\lambda(A),$$

whence

$$(**) \qquad |\mu_k|(A) \ge |\varphi_{n_k}|(A) - |\varphi_{n_{k-1}}|(A) \ge k\lambda(A).$$

We have thus obtained a sequence of measures (μ_k) of the required form which clearly satisfies assertions (a) and (b) of Theorem 2.3, and $\mu = \sum_{k=1}^{\infty} \mu_k$ in $cca(\Sigma, \lambda, X)$. To finish, use $(**)$ and the fact that $\lambda(S \setminus E_k) \to 0$, and slightly modify the proofs of parts (c) and (d) of Theorem 2.3. (Cf. the final part of the second proof of Theorem 4.1 below.) ■

Remarks. (a) If desired, the measures μ_k constructed above can be replaced by λ-simple measures μ'_k, each with the same range space as μ_k, and having $|\mu_k - \mu'_k|(S)$ so small that the new sequence (μ'_k) will still satisfy conditions (a) through (d) of Theorem 2.3. (For this new sequence, condition $(**)$ may take, e.g., the following form: $|\mu'_k|(A) \ge (k/2)\lambda(A)$ whenever $A \subset E'_k$, for some $E'_k \subset E_k$ with $\lambda(S \setminus E'_k) \to 0$.)

(b) The essential part of the construction carried out in the proof of Theorem 2.6 is generalized in the following

LEMMA. *Let $\mu = \sum_{n=1}^{\infty} \gamma_n$ in $ca(\Sigma, \lambda, X)$, where μ is λ-everywhere of infinite variation, while each γ_n is of σ-finite variation. Then there exist integers $0 = n_0 < n_1 < n_2 < \ldots$ and $E_k \in \Sigma$ such that for every $k \in \mathbb{N}$*

$$\text{(i)} \qquad \lambda(S \setminus E_k) < 2^{-k},$$

$$\text{(ii)} \qquad \Big\| \sum_{n=n_k+1}^{m} \gamma_n \Big\| < 2^{-k} \quad \textit{whenever } m > n_k,$$

$$\text{(iii)} \qquad \Big| \sum_{n=n_{k-1}+1}^{n_k} \gamma_n \Big|(A) \ge k\lambda(A) \quad \textit{whenever } A \subset E_k.$$

Proof. We proceed by induction. Set $\varphi_n = \gamma_1 + \ldots + \gamma_n$ and note that $\varphi_n \to \mu$ in $ca(\Sigma, \lambda, X)$ by assumption. To define n_{k+1}, choose $E \in \Sigma$ and $r > 0$ with $\lambda(S \setminus E) < 2^{-(k+2)}$ and

$$|\varphi_{n_k}|(A) \le r\lambda(A) \quad \text{whenever } A \subset S \setminus E.$$

(This is possible by the Radon–Nikodym theorem.) By assumption and Lemma 4.4 below, we can find $n_{k+1} > n_k$ and $F \in \Sigma$ so that $\lambda(S \setminus F) < 2^{-(k+2)}$,

$$|\varphi_{n_{k+1}}|(A) \ge (r + k + 1)\lambda(A) \quad \text{whenever } A \subset F,$$

and condition (ii) holds with k replaced by $k + 1$. Finally, put $E_{k+1} = E \cap F$. Then also conditions (i) and (iii) hold with k replaced by $k + 1$. ■

In the proof below we make use of the following well-known property of the Banach space ℓ_∞ (see, e.g., [LR, p. 326]):

If $a > 1$ and V is a finite-dimensional subspace of ℓ_∞, then there exists a finite-rank projection P in ℓ_∞ such that $V \subset P(\ell_\infty)$ and $\|P\| \le a$.

For the reader's convenience, we include a self-contained proof of this assertion. Denote by R_n the natural projections from ℓ_∞ onto $\ell_\infty^n := \mathrm{lin}\{e_i : i \le n\}$ $(n \in \mathbb{N})$. It is enough to find n and a linear map $S : \ell_\infty^n \to \ell_\infty$ such that $\|S\| \le a$, $R_n S(z) = z$ for all $z \in \ell_\infty^n$, and $SR_n(v) = v$ for all $v \in V$. Then $P = SR_n$ has the desired properties.

Denote by ξ_i the coordinate functionals on ℓ_∞ $(i \in \mathbb{N})$. Since $\sup_{i\le n} |\xi_i(x)| \uparrow \|x\|$ for every $x \in \ell_\infty$ and $\dim V < \infty$, by Dini's theorem there is n for which the map $R := R_n|V$ satisfies $a^{-1}\|v\| \le \|R(v)\| \le \|v\|$ for all $v \in V$. Its inverse $R^{-1} : W = R(V) \to \ell_\infty$ is of the form $R^{-1}(w) = (\eta_i(w))$, where $\eta_i \in W^*$ for $i \in \mathbb{N}$, and $\eta_i = \xi_i|W$ for $i \le n$. Clearly, $\|\eta_i\| \le a$ for $i \in \mathbb{N}$. Finally, put $S(z) = (\zeta_i(z))$, where each $\zeta_i \in (\ell_\infty^n)^*$ is a norm preserving extension of η_i, and $\zeta_i = \xi_i|\ell_\infty^n$ for $i \le n$.

2.7. Theorem. *Let $\sum_{i=1}^\infty x_i$ be an unconditionally but not absolutely convergent series in X with nonzero terms. Then there exist sequences (m_k) and (n_k) in $\mathbb{N}$ with $m_1 \le n_1 < m_2 \le n_2 < \ldots$ such that if, for every k, we take any Σ-partition $\{S_i : m_k \le i \le n_k\}$ of S with $\lambda(S_i) = \|x_i\|/d_k$, where $d_k = \sum_{i=m_k}^{n_k} \|x_i\|$, and define a λ-simple measure $\mu_k : \Sigma \to X$ by*

$$\mu_k(A) = \sum_{i=m_k}^{n_k} \frac{\lambda(A \cap S_i)}{\lambda(S_i)} x_i = d_k \sum_{i=m_k}^{n_k} \lambda(A \cap S_i) \frac{x_i}{\|x_i\|},$$

then the series $\sum_{k=1}^\infty \mu_k$ is absolutely convergent in $cca(\Sigma, \lambda, X)$, and the sum of each of its subseries is λ-everywhere of infinite variation.

Proof. We may assume that the space X is separable and, therefore, that it is (via a linear isometry) a subspace of ℓ_∞ (see [M-N, Thm. 2.1.14). By the property of ℓ_∞ recalled above, there exists a sequence (P_n) of finite-rank projections in ℓ_∞ such that $\{x_1, \ldots, x_n\} \subset P_n(\ell_\infty)$ and $\|P_n\| \le 2$ for every $n \in \mathbb{N}$.

Fix a sequence $(r_k)_{k=0}^\infty$ in $\mathbb{R}_+$ such that

$$r_0 = 0 \quad \text{and} \quad r_k - \sum_{j=0}^{k-1} r_j \ge k + 1 \quad \text{for all } k \ge 1.$$

By an easy induction, we obtain sequences (m_k) and (n_k) in $\mathbb{N}$ with $m_1 \le n_1 < m_2 \le n_2 < \ldots$ such that for every k,

$$\sum_{i=m_{k+1}}^\infty \|P_{n_k}(x_i)\| \le 1,$$

$$r_k \le \sum_{i=m_k}^{n_k} \|x_i\| \le r_k + 2^{-k}, \quad \text{and} \quad \Big\|\sum_{i=m_k}^{n_k} t_i x_i\Big\| \le 2^{-k} \quad \text{whenever } |t_i| \le 1.$$

Now, define the measures μ_k as explained in the theorem. By Lemma 2.1, $\|\mu_k\| \le 2^{-k}$ and $|\mu_k| = d_k\lambda$, so that

$$r_k\lambda \le |\mu_k| \le (r_k + 2^{-k})\lambda.$$

Define measures λ_i and τ_k on Σ by

$$\lambda_i(A) = \frac{\lambda(A \cap S_i)}{\lambda(S_i)} \quad \text{and} \quad \tau_k(A) = \sum_{j=k+1}^{\infty} \sum_{i=m_j}^{n_j} \lambda_i(A) P_{n_k}(x_i).$$

We then have

$$|\tau_k|(A) \le \sum_{j=k+1}^{\infty} \sum_{i=m_j}^{n_j} \lambda_i(A) \|P_{n_k}(x_i)\| \le 1.$$

Finally, define $\mu : \Sigma \to X$ by $\mu(A) = \sum_{k=1}^{\infty} \mu_k(A)$. It follows that for every $k \in \mathbb{N}$ and $A \in \Sigma$,

$$\begin{aligned}
2|\mu|(A) \ge |P_{n_k} \circ \mu|(A) &= \Big|\sum_{j=1}^{k} \mu_j + \tau_k\Big|(A) \ge |\mu_k|(A) - \sum_{j=1}^{k-1} |\mu_j|(A) - |\tau_k|(A) \\
&\ge r_k \lambda(A) - \sum_{j=1}^{k-1} (r_j + 2^{-j}) \lambda(A) - 1 \\
&\ge \Big(r_k - \sum_{j=1}^{k-1} r_j - 1\Big) \lambda(A) - 1 \ge k\lambda(A) - 1.
\end{aligned}$$

Hence $|\mu|(A) = \infty$ if $\lambda(A) > 0$.

In the same manner it can be shown that if M is an infinite subset of $\mathbb{N}$, then $\mu_M = \sum_{k \in M} \mu_k$ is λ-everywhere of infinite variation. ∎

2.8. COROLLARY. *Let $\sum_{i=1}^{\infty} x_i$ be an unconditionally but not absolutely convergent series in X. Then there exist a sequence (c_i) of reals and a sequence (S_i) in Σ with $c_i \lambda(S_i) = 1$ or 0 for every i, and such that the measure $\mu \in cca(\Sigma, \lambda, X)$ defined by*

$$\mu(A) = \sum_{i=1}^{\infty} c_i \lambda(A \cap S_i) x_i$$

is λ-everywhere of infinite variation.

The following lemma will be needed in the proof of Theorem 4.1.

2.9. LEMMA. *If $\mu, \nu \in ca(\Sigma, \lambda, X)$ and μ is λ-everywhere of infinite variation, then so is $t\mu + \nu$ for all but countably many scalars t.*

Proof. Define

$$E_t = \Big\{ s \in S : \frac{d|t\mu + \nu|}{d\lambda}(s) < \infty \Big\}.$$

The measure $(t - t')\mu = (t\mu + \nu) - (t'\mu + \nu)$ is of σ-finite variation on $E_t \cap E_{t'}$. Therefore, $\lambda(E_t \cap E_{t'}) = 0$ whenever $t \ne t'$. Hence $\lambda(E_t) = 0$ for all but countably many t, and we are done. ∎

We conclude this section by pointing out that, in general, if $\mathcal{A}$ is an algebra generating Σ and a measure $\mu : \Sigma \to X$ is such that $|\mu|(A) = \infty$ for every $A \in \mathcal{A}$ with $\lambda(A) > 0$, then μ need not be λ-everywhere of infinite variation on Σ. This is a consequence of the following

2.10. Proposition. *If (A_i) is a sequence of sets in Σ of strictly positive λ measure, then there exists $\mu \in cca(\Sigma, \lambda, X)$ of σ-finite variation such that $|\mu|(A_i) = \infty$ for all i.*

Proof. As is well-known, there exists a Σ-partition $\{S_j : j \in \mathbb{N}\}$ of S with $\lambda(A_i \cap S_j) > 0$ for all i, j. Now, using Proposition 2.2, we can find a sequence (μ_j) of λ-simple (hence of finite variation) measures from Σ to X such that

$$\|\mu_j\| \le 2^{-j} \quad \text{and} \quad |\mu_j|(A_i \cap S_j) \ge 1, \quad j = 1, 2, \ldots, \quad i = 1, \ldots, j,$$

and μ_j is concentrated on $\bigcup_{i=1}^{j} A_i \cap S_j$. Define $\mu = \sum_{j=1}^{\infty} \mu_j$. Then $|\mu| = \sum_{j=1}^{\infty} |\mu_j|$, and it follows easily that $|\mu|(S_j) < \infty$ and $|\mu|(A_i) = \infty$. ∎

Remark. In order to verify the assertion used at the beginning of the above proof, it is enough to see how to find an $S_1 \in \Sigma$ with $\lambda(A_i \cap S_1) > 0$ and $\lambda(A_i \setminus S_1) > 0$ for all i. A Baire category proof of the existence of such S_1 appeared in [K, Thm. 1]; a direct construction is, however, completely elementary: Choose $B_i \subset A_i$ so that $\sum_{i>k} \lambda(B_i) < \lambda(B_k)$ for every k, and next choose $D_k \subset C_k := B_k \setminus \bigcup_{i>k} B_i$ so that $0 < \lambda(D_k) < \lambda(C_k)$. Then $S_1 := \bigcup_k D_k$ is as required; in particular, $\lambda(A_k \setminus S_1) > 0$ because $A_k \setminus S_1 \supset C_k \setminus D_k$.

3. Semicontinuity of some maps related to the variation map

In this section, λ stands for an arbitrary finite positive measure on Σ. (Besides, the standing assumption that the Banach space X is infinite-dimensional will not be used here.) We denote by $L_0(\lambda, \overline{\mathbb{R}}_+)$ the lattice of all (λ-equivalence classes of) Σ-measurable functions from S to $\overline{\mathbb{R}}_+$. Recall that this lattice has the countable sup property, that is, every set $F \subset L_0(\lambda, \overline{\mathbb{R}}_+)$ has a countable subset F_0 such that $\sup F = \sup F_0$ (cf. [DS, Thm. IV.11.6). Also note that, for every $\mu \in ca(\Sigma, \lambda, X)$, the variation measure $|\mu|$ has a (unique) Radon–Nikodym derivative with respect to λ,

$$\frac{d|\mu|}{d\lambda} =: f_\mu \in L_0(\lambda, \overline{\mathbb{R}}_+).$$

Thus $|\mu|(A) = \int_A f_\mu \, d\lambda$ for all $A \in \Sigma$.

Proposition 3.1 (a), (b) below is a rather well-known result; it is included here, and provided with a proof, mainly for the reader's convenience. (See [Mu, Prop. 1]; and [L, Thm. 4.2], [AG, Proof of Thm. 2.4] and [R-P, Prop. II.1.4d)] for some versions of (a) and (b), respectively.) We refer the reader to [DS, Chap. IV.10] for the integration of measurable scalar functions with respect to vector measures. (See also [DU, pp. 5–6] for the more elementary integration of bounded measurable functions, which is sufficient for some of our purposes.) We also note that part (b) of Proposition 3.1 is used in the sequel only in the proofs of Lemma 4.3 and Proposition 4.6, where it can be replaced by the simpler assertion (C) of the proof below.

3.1. PROPOSITION. (a) *Let* $\mu \in ca(\Sigma, \lambda, X)$. *If*

$$f_{x^*} := \frac{d(x^*\mu)}{d\lambda} \quad \textit{for } x^* \in X^*,$$

then $f_\mu = \sup\{|f_{x^*}| : \|x^*\| \le 1\}$ *in the lattice* $L_0(\lambda, \overline{\mathbb{R}}_+)$.

(b) *Let* $\mu \in ca(\Sigma, X)$, *and let* f *be a* μ*-integrable scalar function. If* $\mu_f := \int_{(\cdot)} f\, d\mu$, *then*

$$|\mu_f|(A) = \int_A |f|\, d|\mu| \quad \textit{for all } A \in \Sigma.$$

(c) *If* μ, f *and* μ_f *are as in* (b) *and* $|f| \ge c\chi_S$, *where* $c \ge 0$, *then*

$$c\mu(\Sigma) \subset 4\,\overline{\text{abs conv}}\,\mu_f(\Sigma).$$

Proof. (a): Denote $g = \sup\{|f_{x^*}| : \|x^*\| \le 1\}$. If $\|x^*\| \le 1$, then $|x^*\mu| \le |\mu|$. It follows that $g \le f_\mu$. On the other hand, if $A \in \Sigma$ and $x^* \in X^*$ is chosen so that $\|\mu(A)\| = |x^*\mu(A)|$, then $\|\mu(A)\| \le |x^*\mu|(A) = \int_A |f_{x^*}|\, d\lambda \le \int_A g\, d\lambda$. From this it follows that $|\mu|(A) \le \int_A g\, d\lambda$ for every $A \in \Sigma$, hence $f_\mu \le g$.

(b): Choose as λ any "control" measure for μ (see [DU, I.2.6]). Then, using the notation of part (a), we have $|x^*\mu_f| = \int_{(\cdot)} |f|\, d|x^*\mu| = \int_{(\cdot)} |f||f_{x^*}|\, d\lambda$, so that $d|x^*\mu_f|/d\lambda = |f||f_{x^*}|$. Hence, by (a), $|\mu_f| = \int_{(\cdot)} |f| f_\mu\, d\lambda = \int_{(\cdot)} |f|\, d|\mu|$.

A direct proof of (b) goes as follows (cf. [L, Proof of Thm. 4.2]): For all $A \in \Sigma$ and $x^* \in X^*$ with $\|x^*\| \le 1$, we have $|x^*\mu_f(A)| \le \int_A |f|\, d|x^*\mu| \le \int_A |f|\, d|\mu|$. Hence $\|\mu_f(A)\| \le \int_A |f|\, d|\mu|$. In consequence, $|\mu_f|(A) \le \int_A |f|\, d|\mu|$.

To prove the converse inequality, first observe that

(C) if $|f| \ge c\chi_B$, where $c \ge 0$ and $B \in \Sigma$, then $|\mu_f|(B) \ge c|\mu|(B)$.

In fact, fix $C \subset B$ and choose $x^* \in X^*$ with $\|x^*\| = 1$ and $|x^*\mu(C)| = \|\mu(C)\|$. Then

$$c\|\mu(C)\| \le \int_C |f|\, d|x^*\mu| = |x^*\mu_f|(C) \le |\mu_f|(C),$$

from which (C) follows. By (C), for every positive Σ-simple function g with $g \le |f|$ and $A \in \Sigma$ we have

$$|\mu_f|(A) \ge \int_A g\, d|\mu|,$$

and taking the supremum over all such functions g yields $|\mu_f|(A) \ge \int_A |f|\, d|\mu|$.

(c): We may assume that $\mu \ne 0$ and $c = 1$. Moreover, it is enough to show that $\mu(S) \in 4\,\overline{\text{abs conv}}\,\mu_f(\Sigma)$. Fix $\varepsilon > 0$ and choose $S_0 \in \Sigma$ so that $f|S_0$ is bounded and $\overline{\mu}(S \setminus S_0) < \varepsilon/2$. There exist scalars t_i and a Σ-partition $\{S_1, \ldots, S_n\}$ of S_0 with $|t_i| \geq 1$ and

$$\Big| f(s) - \sum_{i=1}^{n} t_i \chi_{S_i}(s) \Big| < \frac{\varepsilon}{8\overline{\mu}(S)} \quad \text{for } s \in S_0.$$

Hence

$$\Big| \sum_{i=1}^{n} \frac{1}{t_i} \chi_{S_i}(s) f(s) - \chi_{S_0}(s) \Big| < \frac{\varepsilon}{8\overline{\mu}(S)} \quad \text{for } s \in S_0.$$

It follows that

$$\Big\| \sum_{i=1}^{n} \frac{1}{t_i} \mu_f(S_i) - \mu(S_0) \Big\| < \frac{\varepsilon}{2},$$

and so

$$\Big\| \sum_{i=1}^{n} \frac{1}{t_i} \mu_f(S_i) - \mu(S) \Big\| < \varepsilon.$$

The assertion follows (cf. [DU, Lemma IX.1.3 (c) and its proof]). ■

In our next proposition, we consider the space $L_\infty(\lambda)$ equipped with its weak* topology $\sigma(L_\infty(\lambda), L_1(\lambda))$. For $f \in L_\infty(\lambda)$ and $\mu \in ca(\Sigma, \lambda, X)$, we define

$$v(f, \mu) = |\mu_f|(S), \quad \text{where } \mu_f = \int_{(\cdot)} f \, d\mu.$$

The first part of the proposition is an extension of an analogous result established in the course of the proof of Thm. 2.4 in [AG] for the space $cca(\Sigma, \lambda, X)$. That the second part follows from the first one is, in fact, a particular case of a general result on lower semicontinuous functions on products of topological spaces.

3.2. Proposition. *Let K be a weak*-compact subset of $L_\infty(\lambda)$. Then the maps*

$$v : K \times ca(\Sigma, \lambda, X) \to \overline{\mathbb{R}}_+, \quad (f, \mu) \to v(f, \mu),$$

and

$$v_K : ca(\Sigma, \lambda, X) \to \overline{\mathbb{R}}_+, \quad \mu \to \inf_{f \in K} v(f, \mu),$$

are lower semicontinuous.

Proof. In view of the definition of the variation, to prove the first assertion it is enough to verify that for every $A \in \Sigma$ the map

$$\psi : K \times ca(\Sigma, \lambda, X) \to \mathbb{R}_+, \quad (f, \mu) \to \Big\| \int_A f \, d\mu \Big\|,$$

is lower semicontinuous.

Assume, as we may, that K is contained in the unit ball of $L_\infty(\lambda)$, and fix $(f_0, \mu_0) \in K \times ca(\Sigma, \lambda, X)$ and a real number c with $d := \psi(f_0, \mu_0) > c$. Choose

$x_0^* \in X^*$ with $\|x_0^*\| = 1$ so that

$$\Big|\int_A f_0\, d(x_0^*\mu_0)\Big| = \Big|\Big\langle x_0^*, \int_A f_0\, d\mu_0\Big\rangle\Big| = d,$$

and define a weak*-neighborhood of f_0 in K by

$$U = \Big\{f \in K : \Big|\int_A (f - f_0)g\, d\lambda\Big| < \varepsilon\Big\},$$

where $g = d(x_0^*\mu_0)/d\lambda$ and $\varepsilon = (d - c)/2$.

Then, if $f \in U$, $\mu \in ca(\Sigma, \lambda, X)$ and $\|\mu - \mu_0\| < \varepsilon/4$, we have

$$\begin{aligned}\psi(f, \mu) &\ge \Big|\Big\langle x_0^*, \int_A f\, d\mu\Big\rangle\Big| \\ &\ge \Big|\Big\langle x_0^*, \int_A f_0\, d\mu_0\Big\rangle\Big| - \Big|\Big\langle x_0^*, \int_A f\, d(\mu - \mu_0)\Big\rangle\Big| - \Big|\int_A (f - f_0)g\, d\lambda\Big| \\ &\ge d - 4\|\mu - \mu_0\| - \varepsilon > d - 2\varepsilon = c,\end{aligned}$$

which proves the lower semicontinuity of ψ.

Now, we verify that v_K is lower semicontinuous. Fix $\mu \in ca(\Sigma, \lambda, X)$ and $c \in \mathbb{R}$ with $v_K(\mu) > c$. Then $v(f, \mu) > c$ for every $f \in K$. Since v is lower semicontinuous, there exist an open weak*-neighborhood U_f of f in K and an open neighborhood $\mathcal{V}_f$ of μ in $ca(\Sigma, \lambda, X)$ such that $v(g, \nu) > c$ for all $(g, \nu) \in U_f \times \mathcal{V}_f$. Since K is weak*-compact, there exists a finite subset $\{f_1, \ldots, f_j\}$ of K such that $K = \bigcup_{i=1}^j U_{f_i}$. Then $\mathcal{V} := \bigcap_{i=1}^j \mathcal{V}_{f_i}$ is a neighborhood of μ in $ca(\Sigma, \lambda, X)$, and for every $\nu \in \mathcal{V}$ we have $v(g, \nu) > c$ whenever $g \in K$. Since the function $g \to v(g, \nu)$, being lower continuous, attains its minimum on K, we obtain $v_K(\nu) > c$. This proves that v_K is lower semicontinuous. ■

Given $h \in L_0(\lambda, \overline{\mathbb{R}}_+)$ and a number $\delta > 0$, we define

$$L(h, \delta) = \{f \in L_0(\lambda, \overline{\mathbb{R}}_+) : \lambda(\{s \in S : f(s) \le h(s)\}) < \delta\}.$$

Note that if $\mu \in ca(\Sigma, \lambda, X)$, then

$$E(\mu, h) := \{s \in S : f_\mu(s) \le h(s)\}$$

can be characterized as a λ-maximal C set in Σ such that $|\mu|(A) \le \int_A h\, d\lambda$ for all $A \subset C$.

3.3. Proposition. (a) *The map*

$$w : ca(\Sigma, \lambda, X) \to L_0(\lambda, \overline{\mathbb{R}}_+), \qquad \mu \to f_\mu,$$

is lower semicontinuous in the sense that for every $h \in L_0(\lambda, \overline{\mathbb{R}}_+)$ *and* $\delta > 0$ *the set*

$$\mathcal{W}(h, \delta) := w^{-1}(L(h, \delta)) = \{\mu \in ca(\Sigma, \lambda, X) : \lambda(E(\mu, h)) < \delta\}$$

is open in $ca(\Sigma, \lambda, X)$. *Equivalently*,

(b) *For every* $h \in L_0(\lambda, \overline{\mathbb{R}}_+)$, *the map*

$$w_h : ca(\Sigma, \lambda, X) \to \mathbb{R}_+, \qquad \mu \to \lambda(E(\mu, h)),$$

is upper semicontinuous.

Proof. In spite of the equivalence of (a) and (b), we shall give two separate proofs.

(a): Fix $\mu \in \mathcal{W}(h, \delta)$, and let $f = f_\mu$. Thus there is $E \in \Sigma$ with $\lambda(E) < \delta$ such that $f > h$ on $S \setminus E$. As easily seen, it can be assumed that for some $\rho > 0$,

$$f > h + 2\rho \quad \text{ on } S \setminus E.$$

Define $B = \{x^* \in X^* : \|x^*\| \le 1\}$ and let, for every $x^* \in B$, $f_{x^*} = d(x^*\mu)/d\lambda$. By Proposition 3.1 (a), $f = \sup_{x^* \in B} |f_{x^*}|$ in the lattice $L_0(\lambda, \overline{\mathbb{R}}_+)$. Moreover, since $L_0(\lambda, \overline{\mathbb{R}}_+)$ has the countable sup property, there exists a sequence (x_n^*) in B such that $f = \sup_n |f_{x_n^*}|$. Now, let $f_n = \sup_{1 \le i \le n} |f_{x_i^*}|$ for $n = 1, 2, \ldots$, and let $\eta = (\delta - \lambda(E))/2$. Since $f_n \uparrow f$ a.e. and $f > h + 2\rho$ on $S \setminus E$, by a version of Egorov's theorem we can find a set $F \subset S \setminus E$ with $\lambda(F) < \eta$ and $m \in \mathbb{N}$ such that $f_m \ge h + \rho$ on $S \setminus (E \cup F)$.

We finish by verifying that the open ball in $ca(\Sigma, \lambda, X)$ with center μ and radius $\varepsilon := \rho\eta/(4m)$ is contained in $\mathcal{W}(h, \delta)$. Fix $\nu \in ca(\Sigma, \lambda, X)$ with $\|\mu - \nu\| < \varepsilon$, and set $g = f_\nu$ and $g_{x^*} = d(x^*\nu)/d\lambda$ for $x^* \in B$. Then, for $1 \le i \le m$, we have $|x_i^*\mu - x_i^*\nu|(S) < \rho\eta/m$; hence $\lambda(G_i) < \eta/m$, where $G_i = \{s \in S : |f_{x_i^*}(s) - g_{x_i^*}(s)| > \rho\}$. Therefore, if $G := G_1 \cup \ldots \cup G_m$, then $\lambda(G) < \eta$ and $\sup_{1 \le i \le m} |f_{x_i^*} - g_{x_i^*}| \le \rho$ on $S \setminus G$. Now, if $H := E \cup F \cup G$, then $\lambda(H) < \delta$, and on $S \setminus H$ we have

$$\begin{aligned} g = \sup_{x^* \in B} |g_{x^*}| &\ge \sup_{1 \le i \le m} |g_{x_i^*}| \\ &\ge \sup_{1 \le i \le m} |f_{x_i^*}| - \sup_{1 \le i \le m} |f_{x_i^*} - g_{x_i^*}| > (h + \rho) - \rho = h. \end{aligned}$$

(b): We have to show that for every $\delta > 0$ the set

$$\mathcal{C}(h, \delta) := \{\mu \in ca(\Sigma, \lambda, X) : \lambda(E(\mu, h)) \ge \delta\}$$

is closed in $ca(\Sigma, \lambda, X)$. Let μ_0 be in the closure of $\mathcal{C}(h, \delta)$. Choose a sequence (μ_n) in $\mathcal{C}(h, \delta)$ so that $\|\mu_0 - \mu_n\| \le 2^{-n}$, and set $E_n = E(\mu_n, h)$. Thus, for every n,

$$\lambda(E_n) \ge \delta \quad \text{and} \quad \|\mu_n(A)\| \le \gamma(A) := \int_A h \, d\lambda \quad \text{whenever } A \subset E_n.$$

Define

$$D_k = \bigcup_{n=k}^{\infty} E_n \quad \text{and} \quad D = \bigcap_{k=1}^{\infty} D_k;$$

clearly, $\lambda(D) \ge \delta$. Therefore, it is enough to show that $D \subset E(\mu_0, h)$, to conclude that $\mu_0 \in \mathcal{C}(h, \delta)$. Fix $A \subset D$ and $k \in \mathbb{N}$. Then A can be written as the union of a disjoint sequence $(A_n)_{n=k}^{\infty}$ such that $A_n \subset E_n$ for $n \ge k$.

Therefore,

$$\|\mu_0(A)\| \le \sum_{n=k}^{\infty} \|\mu_0(A_n)\| \le \sum_{n=k}^{\infty} \|\mu_0(A_n) - \mu_n(A_n)\| + \sum_{n=k}^{\infty} \|\mu_n(A_n)\|$$
$$\le \sum_{n=k}^{\infty} 2^{-n} + \sum_{n=k}^{\infty} \gamma(A_n) = 2^{-k+1} + \gamma(A).$$

Since k can be arbitrarily large, we obtain $\|\mu_0(A)\| \le \gamma(A)$. Hence $D \subset E(\mu_0, h)$.

4. Sets of λ-continuous measures with (λ-) everywhere infinite variation

4.1. THEOREM. *Let $\mathcal{M}$ be a subspace of $ca(\Sigma, \lambda, X)$ containing $cca(\Sigma, \lambda, X)$. Then the measures in $\mathcal{M}$ that are λ-everywhere of infinite variation form a dense G_δ-set.*

In view of Lemma 2.9 and Theorem 2.3 (c), we only need to establish the "G_δ" assertion. We shall give two proofs of this assertion, differing in the way a required sequence of dense open sets is produced. In fact, those open sets turn out to be dense in view of Lemma 4.2. Therefore, if $\mathcal{M}$ is, additionally, closed (say $\mathcal{M} = cca(\Sigma, \lambda, X)$, in which case Theorem 4.1 coincides with (**AG**); see p. 8), we can appeal to the Baire category theorem instead of Lemma 2.9 and Theorem 2.3 (c).

For $r > 0$, define

$$\mathcal{V}_r = \{\mu \in \mathcal{M} : |\mu| \ge r\lambda\}.$$

4.2. LEMMA. *For every $r > 0$, the set $\mathcal{V}_r$ is dense in $\mathcal{M}$.*

Proof. Let $\mu_0 \in \mathcal{M}$, and define $f = d|\mu_0|/d\lambda$ and $E = \{s \in S : f(s) \ge r\}$. Then

$$|\mu_0|(A) \ge r\lambda(A) \quad \text{whenever } A \subset E.$$

Now, given $\varepsilon > 0$, using Proposition 2.2 we find $\nu \in ca_s(\Sigma, \lambda, X)$ such that

$$\|\nu\| < \varepsilon \quad \text{and} \quad \nu \in \mathcal{V}_{2r}.$$

Define a measure $\mu : \Sigma \to X$ by

$$\mu(A) = \mu_0(A) + \nu(A \cap (S \setminus E)).$$

Then, clearly, $\|\mu - \mu_0\| < \varepsilon$. Moreover, $|\mu|(A) = |\mu_0|(A)$ when $A \subset E$, and $|\mu|(A) \ge |\nu|(A) - |\mu_0|(A) \ge r\lambda(A)$ when $A \subset S \setminus E$. It follows that $\mu \in \mathcal{V}_r$. ■

Our first proof of Theorem 4.1 is a modification of the proof of Theorem 2.4 in [AG].

As in [AG], we will make use of the fact that the set

$$P := \{f \in L^\infty(\lambda) : 0 \le f \le 1\},$$

is compact under the induced weak* topology $\sigma(L_\infty(\lambda), L_1(\lambda))$. Actually, what will be needed below is that for every $k > 0$ the set

$$P_k := \Big\{ f \in P : \int_S f\, d\lambda \ge k^{-1} \Big\}$$

is weak*-closed, and so compact in P.

For all $c > 0$ and $k > 0$, define

$$\mathcal{V}_{c,k} = \{\mu \in \mathcal{M} : v(f,\mu) > c \text{ for all } f \in P_k\}.$$

Thus, using the notation of Proposition 3.2,

$$\mathcal{V}_{c,k} = \{\mu \in \mathcal{M} : v_{P_k}(\mu) > c\}.$$

Denote by $\mathcal{V}_\infty$ the subset of $\mathcal{M}$ considered in Theorem 4.1, i.e.

$$\mathcal{V}_\infty = \{\mu \in \mathcal{M} : \mu \text{ is } \lambda\text{-everywhere of infinite variation}\}.$$

Recall that the subsets $\mathcal{V}_r$ of $ca(\Sigma, \lambda, X)$ have been defined before Lemma 4.2.

4.3. LEMMA. *For all $c > 0$ and $k > 0$, the set $\mathcal{V}_{c,k}$ is open and dense in $\mathcal{M}$, and $\mathcal{V}_\infty \subset \mathcal{V}_{c,k}$.*

Proof. From the second part of Proposition 3.2 it follows immediately that $\mathcal{V}_{c,k}$ is open in $\mathcal{M}$. Now, observe that $\mathcal{V}_r \subset \mathcal{V}_{c,k}$ for $r = 4ck^2$. Indeed, let $\mu \in \mathcal{V}_r$. Take $f \in P_k$, and set $A = \{s \in S : f(s) \ge (2k)^{-1}\}$. Then, as easily seen, $\lambda(A) > (2k)^{-1}$, and, by Proposition 3.1 (b),

$$v(f,\mu) \ge \int_A f\, d|\mu| \ge \frac{1}{2k}|\mu|(A) \ge \frac{1}{2k} r\lambda(A) > c;$$

thus $\mu \in \mathcal{V}_{c,k}$. Hence, by Lemma 4.2, $\mathcal{V}_{c,k}$ is dense in $\mathcal{M}$. To finish, observe that $\mathcal{V}_\infty \subset \mathcal{V}_r$. ∎

First proof of Theorem 4.1. By Lemma 4.3, $\mathcal{V} := \bigcap_{n=1}^\infty \mathcal{V}_{n,n}$ is a G_δ-subset of $\mathcal{M}$, and it contains $\mathcal{V}_\infty$. If $\mu \in \mathcal{V}$, then for every n and $0 \ne f \in P$ we have $v(f,\mu) > n$; hence $v(f,\mu) = \infty$. In particular, if $A \in \Sigma$ and $\lambda(A) > 0$, then $v(\chi_A, \mu) = |\mu|(A) = \infty$. Thus $\mathcal{V} \subset \mathcal{V}_\infty$. ∎

The basic ingredient of our second proof of Theorem 4.1 is Lemma 4.4 below. Given $r > 0$ and $\delta > 0$, set

$$\mathcal{W}_{r,\delta} = \mathcal{W}(r\chi_S, \delta) \cap \mathcal{M};$$

see Proposition 3.3 for the definition of $\mathcal{W}(h,\delta)$. Thus $\mu \in \mathcal{M}$ is in $\mathcal{W}_{r,\delta}$ if and only if there is a set $E \in \Sigma$ with $\lambda(E) < \delta$ such that $|\mu|(A) > r\lambda(A)$ whenever $A \subset S \setminus E$ and $\lambda(A) > 0$.

4.4. LEMMA. *For all $r > 0$ and $\delta > 0$, the set $\mathcal{W}_{r,\delta}$ is open and dense in $\mathcal{M}$, and $\mathcal{V}_\infty \subset \mathcal{V}_{r'} \subset \mathcal{W}_{r,\delta}$ for $0 < r < r'$.*

Proof. The inclusions are obvious, and the other assertions follow directly from Lemma 4.2 and Proposition 3.3. ∎

S e c o n d p r o o f o f T h e o r e m 4.1. For every $n \in \mathbb{N}$, let $\mathcal{W}_n = \mathcal{W}_{n,1/n}$; by Lemma 4.4, this is an open subset of $\mathcal{M}$ containing $\mathcal{V}_\infty$. Hence $\mathcal{W} := \bigcap_{n=1}^\infty \mathcal{W}_n$ is a G_δ-subset of $\mathcal{M}$, and it contains $\mathcal{V}_\infty$. If $\nu \in \mathcal{W}$, then there exists a sequence (E_n) in Σ such that $\lambda(E_n) < n^{-1}$ and $|\nu|(A) \geq n\lambda(A)$ whenever $A \subset S \setminus E_n$. Hence if $A \in \Sigma$ and $\lambda(A) > 0$, then from the inequalities

$$|\nu|(A) \geq |\nu|(A \cap (S \setminus E_n)) \geq n\lambda(A \cap (S \setminus E_n))$$

it follows immediately that $|\nu|(A) = \infty$. ■

4.5. THEOREM. *Let $\mathcal{M}$ be a subspace of $ca(\Sigma, \lambda, X)$ containing $cca(\Sigma, \lambda, X)$. Then the measures in $\mathcal{M}$ that are everywhere of infinite variation form a dense G_δ-set.*

P r o o f. Since the set in question contains all measures in $\mathcal{M}$ that are λ-everywhere of infinite variation, its density follows from Theorem 4.1. Therefore, it is enough to consider it only in the case where $\mathcal{M} = ca(\Sigma, \lambda, X)$, and show that it is a G_δ-subset of this space.

For every $n \in \mathbb{N}$ define $\mathcal{G}_n$ to be the set of all $\mu \in ca(\Sigma, \lambda, X)$ with the following property: There are a set $E_n \in \Sigma$ with $\lambda(E_n) > 0$ and a subset F_n of E_n with $\lambda(F_n) < n^{-1}\lambda(E_n)$ such that

$$\overline{\mu}(S \setminus E_n) < 2^{-n} \quad \text{and} \quad |\mu|(A) > n\lambda(A) \quad \text{if } A \subset E_n \setminus F_n \text{ and } \lambda(A) > 0.$$

From Lemma 4.4 it follows easily that $\mathcal{G}_n$ is open in $ca(\Sigma, \lambda, X)$. Set $\mathcal{G} = \bigcap_{n=1}^\infty \mathcal{G}_n$.

Let $\mu \in ca(\Sigma, \lambda, X)$ be of everywhere infinite variation, and let $E \in \Sigma$ be a λ-minimal set on which μ is concentrated. Then $\overline{\mu}(S \setminus E) = 0$, and $|\mu|(A) = \infty$ for all $A \subset E$ with $\lambda(A) > 0$. Hence $\mu \in \mathcal{G}$.

Now, let $\mu \in \mathcal{G}$ and let E_n and F_n be as above. Clearly, $\mu \neq 0$. Set

$$\widetilde{E}_k = \bigcap_{n=k}^\infty E_n, \qquad E = \bigcup_{k=1}^\infty \widetilde{E}_k.$$

Since $\overline{\mu}(S \setminus E) \leq \overline{\mu}(S \setminus \widetilde{E}_k) \leq \sum_{n=k}^\infty \overline{\mu}(S \setminus E_n) < 2^{-k+1}$ for all k, we have $\overline{\mu}(S \setminus E) = 0$.

Now, let $A \subset E$ and $\lambda(A) > 0$. Then for every k, $\widetilde{E}_k \subset E$ and

$$|\mu|(A) \geq |\mu|(A \cap \widetilde{E}_k \setminus F_k) \geq k\lambda(A \cap \widetilde{E}_k \setminus F_k) \geq k[\lambda(A \cap \widetilde{E}_k) - \lambda(F_k)].$$

Since $\lambda(A \cap \widetilde{E}_k) \to \lambda(A)$ and $\lambda(F_k) \to 0$, it follows that $|\mu|(A) = \infty$. Thus μ is everywhere of infinite variation. ■

R e m a r k. For $\mathcal{M} = cca(\Sigma, \lambda, X)$, Theorem 4.1 can be easily deduced from Theorem 4.5, the Baire category theorem and the following result of Anantharaman and Garg ([AG, Thm. 2.1]; see also **MR** 86d:28014 for a simplification of the proof of that theorem): *The measures in $cca(\Sigma, \lambda, X)$ that are equivalent to λ form a dense G_δ-set.*

It is easy to obtain an analogue of Corollary 2.5 for measures in $cca(\Sigma, \lambda, X)$ of everywhere infinite variation: Take any such a measure μ and a disjoint sequence (E_n) in Σ with $\lambda(E_n) > 0$, and set $\mu_n(A) = \mu(A \cap E_n)$. Then the sequence (μ_n) is basic in $cca(\Sigma, \lambda, X)$ and every nonzero measure in its closed linear span is of everywhere infinite variation. Our next result is a version of Corollary 2.5 which is more suitable for the present setting. We first make some observations.

Given $\mu \in ca(\Sigma, \lambda, X)$, let $L_1(\mu)$ denote the Banach space of all μ-integrable scalar functions f on S, with the norm $\|f\| := \sup_{A\in\Sigma} \| \int_A f\, d\mu\|$ (and obvious identifications). Then, setting $I_\mu(f) = \mu_f$ $(= \int_{(\cdot)} f\, d\mu)$, we obtain a linear isometric embedding of $L_1(\mu)$ in $ca(\Sigma, \lambda, X)$ (or $cca(\Sigma, \lambda, X)$ if μ is in $cca(\Sigma, \lambda, X)$). Note that the range of I_μ coincides with $\overline{\mathrm{lin}}\{\mu_E : E \in \Sigma\}$, where each measure μ_E is defined on Σ by $\mu_E(A) = \mu(A \cap E)$. Moreover, as follows directly from Proposition 3.1 (b) (or, in fact, already from the assertion (C) in its proof), if μ is λ-everywhere of infinite variation, $f \in L_1(\mu)$, and S_f is the support of f, then $|\mu_f|(A) = \infty$ whenever $A \subset S_f$ and $\lambda(A) > 0$.

Now, choose any $\mu \in cca(\Sigma, \lambda, X)$ which is of λ-everywhere infinite variation and set $\mathcal{M} = I_\mu\big(L_1(\mu)\big)$. Then, using the observations above, we conclude that $\mathcal{M}$ has all the properties required in the following

4.6. PROPOSITION. *The space $cca(\Sigma, \lambda, X)$ contains a closed subspace $\mathcal{M}$ which, apart from zero, consists of measures with everywhere infinite variation and is such that, for every $A \in \Sigma$ with $\lambda(A) > 0$, its subspace consisting of measures concentrated on A is infinite-dimensional.*

5. Borel complexity of some spaces of vector measures

In view of Theorem 4.5, the second part of the following result is somewhat unexpected.

5.1. THEOREM. *The subspace $ca_\sigma(\Sigma, \lambda, X)$ is an $F_{\sigma\delta}$-, but not F_σ-, subset of $ca(\Sigma, \lambda, X)$.*

P r o o f. The second assertion follows immediately from Theorem 5.2 below. In order to establish the first one, for μ in $ca(\Sigma, \lambda, X)$ set $f_\mu = d|\mu|/d\lambda$. Then $\mu \in ca_\sigma(\Sigma, \lambda, X)$ if and only if f_μ is finite λ-a.e., hence if and only if for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $\lambda(\{s \in S : f_\mu(s) \le n\}) \ge 1 - 1/k$. Using the notation introduced before Lemma 4.4 with $\mathcal{M} = ca(\Sigma, \lambda, X)$, the latter means that $\mu \in ca(\Sigma, \lambda, X) \setminus \mathcal{W}_{n,1-1/k}$. Thus

$$ca_\sigma(\Sigma, \lambda, X) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} (ca(\Sigma, \lambda, X) \setminus \mathcal{W}_{n,1-1/k}),$$

and we obtain the desired conclusion by appealing to Lemma 4.4. ∎

Denote by τ the F-space topology on $ca_\sigma(\Sigma, \lambda, X)$ introduced by Janicka and Kalton [JK]. A base of neighborhoods of zero for τ consists of the sets $\mathcal{U}_\varepsilon$, $\varepsilon > 0$, where $\mu \in ca_\sigma(\Sigma, \lambda, X)$ is in $\mathcal{U}_\varepsilon$ if and only if $\|\mu\| \le \varepsilon$ and there exists $E \in \Sigma$ with $\lambda(E) \le \varepsilon$ and $|\mu|(S \setminus E) \le \varepsilon$.

5.2. THEOREM. *Let $\mathcal{M}$ be a τ-closed subspace of $ca_\sigma(\Sigma, \lambda, X)$ containing $ca_s(\Sigma, \lambda, X)$. Then $\mathcal{M}$ is not an F_σ-subset of $ca(\Sigma, \lambda, X)$.*

Proof. Suppose, to get a contradiction, that $\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$, where the $\mathcal{F}_n$'s are closed in $ca(\Sigma, \lambda, X)$. By the Baire category theorem applied to τ restricted to $\mathcal{M}$, we can find $n \in \mathbb{N}$, $\varepsilon > 0$ and $\varphi \in \mathcal{M}$ with $\mathcal{U}_\varepsilon \cap \mathcal{M} \subset \mathcal{F}_n + \varphi$. It follows that the closure of $\mathcal{U}_\varepsilon \cap \mathcal{M}$ in $ca(\Sigma, \lambda, X)$ is contained in $\mathcal{M}$. However, this is false for every $\varepsilon > 0$. Indeed, fix a set $E \in \Sigma$ with $0 < \lambda(E) \le \varepsilon$. By Theorem 2.3 (or 2.7), there exists a sequence (ν_n) of measures in $ca_s(\Sigma, \lambda, X)$ of norm $\le \varepsilon$ which is norm convergent to a measure ν with λ-everywhere infinite variation. Now define

$$\mu_n(A) = \nu_n(A \cap E) \quad \text{and} \quad \mu(A) = \nu(A \cap E) \qquad \text{for all } E \in \Sigma.$$

Then (μ_n) is a sequence in $\mathcal{U}_\varepsilon \cap \mathcal{M}$ which converges in norm to the measure $\mu \notin \mathcal{M}$. ∎

Remark. The subspace $\mathcal{M}$ of Theorem 5.2 is not a G_δ-subset of $ca(\Sigma, \lambda, X)$. Indeed, otherwise it would be closed by a result due to Mazur and Sternbach (see [BP, Prop. VIII.2.1]).

We note that Theorem 5.2 also applies to $\mathcal{M} = ca_\sigma(\Sigma, \lambda, X) \cap cca(\Sigma, \lambda, X)$, and so an analogue of Theorem 5.1 holds for this subspace. We shall prove that the same is true for $\mathcal{P}(\lambda, X)$ provided that X is separable. To this end we shall introduce some more notation and formulate three lemmas.

We denote by $L_0(\lambda, X)$ the F-space of (λ-equivalence classes of) strongly measurable functions from S to X, with the topology of convergence in λ measure (cf. [VTC, II.3]).

The first lemma is due essentially to Heiliö [He, Thm. 4.4.2]. We include a proof of it for the reader's convenience.

5.3. LEMMA. *If X is separable, then $\mathcal{P}(\lambda, X)$ is τ-closed in $ca_\sigma(\Sigma, \lambda, X)$.*

Proof. Let (f_n) be a τ-Cauchy sequence in $\mathcal{P}(\lambda, X)$ and set $\mu_n = \int_{(\cdot)} f_n \, d\lambda$. Denote by μ the τ-limit of (μ_n) in $ca_\sigma(\Sigma, \lambda, X)$. We only need to show that $d\mu/d\lambda$ exists in the Pettis sense. Since

$$\|f_n(\cdot) - f_m(\cdot)\| = \frac{d|\mu_n - \mu_m|}{d\lambda}(\cdot) \qquad \lambda\text{-a.e.}$$

(cf. [DU, Thm. II.2.4 (iv)]), (f_n) is a Cauchy sequence in $L_0(\lambda, X)$. Denote by f its limit in $L_0(\lambda, X)$. We then have $x^* f_n \to x^* f$ in $L_0(\lambda)$ for each $x^* \in X^*$. On the other hand, $(x^* f_n)$ is a Cauchy sequence in $L_1(\lambda)$ for each $x^* \in X^*$. It follows that $x^* f \in L_1(\lambda)$ and

$$\int_E x^* f \, d\lambda = x^* \mu(E) \qquad \text{for all } x^* \in X^* \text{ and } E \in \Sigma,$$

which completes the proof. ∎

The next two lemmas are well known and straightforward; cf. [Ha, § 44] for the second one.

5.4. Lemma. *If X is separable and Z is a closed subspace of X, then* $\mathcal{P}(\lambda, Z) = \mathcal{P}(\lambda, X) \cap ca(\Sigma, \lambda, Z)$.

5.5. Lemma. *Let T and Z be metric spaces and let (Φ_n) be a sequence of continuous maps from T to Z. Then*

$$\{t \in T : (\Phi_n(t)) \text{ is Cauchy in } Z\}$$

is an $F_{\sigma\delta}$-subset of T.

5.6. Theorem. *If X is separable, then $\mathcal{P}(\lambda, X)$ is an $F_{\sigma\delta}$-, but not F_σ-, subset of $ca(\Sigma, \lambda, X)$.*

Proof. The second assertion follows directly from Theorem 5.2 and Lemma 5.3.

We shall establish the first assertion under the additional assumption that X has a Schauder basis, say (x_n). (The general case then follows by Lemma 5.4 and the classical result mentioned in the remark following Corollary 2.5.) Denote by (x_n^*) the associated coefficient functionals and define the maps Φ_n from $ca(\Sigma, \lambda, X)$ to $L_0(\lambda, X)$ by

$$\Phi_n(\mu) = \sum_{i=1}^{n} \frac{d(x_i^* \mu)}{d\lambda} x_i, \qquad n = 1, 2, \ldots$$

In view of [LM, Thm. 2], we have

$$\mathcal{P}(\lambda, X) = \{\mu \in ca(\Sigma, \lambda, X) : (\Phi_n(\mu)) \text{ converges in } L_0(\lambda, X)\}.$$

An appeal to Lemma 5.5 completes the proof.

6. Sets of (λ-continuous) measures with (λ-) everywhere noncompact range

In this section, we use the notation

$$\Sigma(A) = \{E \in \Sigma : E \subset A\} \quad \text{for } A \in \Sigma.$$

We say that a nonzero measure $\mu \in ca(\Sigma, \lambda, X)$ [$\mu \in ca(\Sigma, X)$] has λ-*everywhere* [*everywhere*] *noncompact range* if $\mu(\Sigma(A))$ is not relatively compact for every $A \in \Sigma$ with $\lambda(A) > 0$ [$\overline{\mu}(A) > 0$]. If this holds for all $A \subset B$ with $\lambda(A) > 0$ [$\overline{\mu}(A) > 0$], where $B \in \Sigma$ and $\lambda(B) > 0$ [$\overline{\mu}(B) > 0$], then we say that μ has λ-*everywhere* [*everywhere*] *noncompact range on* B. Clearly, if μ has λ-everywhere noncompact range, then it has everywhere noncompact range; the converse holds provided that $\lambda(A) = 0$ whenever $\overline{\mu}(A) = 0$. Also note that a nonzero measure $\mu \ll \lambda$ has everywhere noncompact range if and only if there is $B \in \Sigma$ such that μ is concentrated on B and has λ-everywhere noncompact range on B.

6.1. PROPOSITION. (a) *For every $\mu \in ca(\Sigma, \lambda, X)$ there exists a unique maximal* (*up to λ-null sets*) *set $C_\mu \in \Sigma$ such that $\mu(\Sigma(C_\mu))$ is relatively compact.*

(b) *For every $\mu \in ca(\Sigma, X)$ there exists a unique maximal* (*up to μ-null sets*) *set $D_\mu \in \Sigma$ such that $\mu(\Sigma(D_\mu))$ is relatively compact.*

Proof. To establish (a), consider a maximal disjoint family $\mathcal{C}$ consisting of sets $C \in \Sigma$ with $\lambda(C) > 0$ such that $\mu(\Sigma(C))$ is relatively compact. Then $\mathcal{C}$ is countable and it is easy to see that its union C_μ has the required properties.

The proof of (b) is analogous. ■

The proof of our next result resembles that of Proposition 3.3 (b).

6.2. PROPOSITION. (a) *The map*

$$\kappa : ca(\Sigma, \lambda, X) \to \mathbb{R}_+, \qquad \mu \to \lambda(C_\mu),$$

is upper semicontinuous.

(b) *The map*

$$\overline{\kappa} : ca(\Sigma, X) \to \mathbb{R}_+, \qquad \mu \to \overline{\mu}(D_\mu),$$

is upper semicontinuous.

Proof. (a): We shall show that for every $r > 0$ the set

$$\mathcal{K}_r := \{\mu \in ca(\Sigma, \lambda, X) : \lambda(C_\mu) \ge r\}$$

is closed in $ca(\Sigma, \lambda, X)$.

Let μ_0 be in the closure of $\mathcal{K}_r$ in $ca(\Sigma, \lambda, X)$. For every $n \in \mathbb{N}$ choose $\mu_n \in \mathcal{K}_r$ with $\|\mu_0 - \mu_n\| \le 2^{-n}$ and denote $C_n = C_{\mu_n}$. Thus $\lambda(C_n) \ge r$ and $\mu_n(\Sigma(C_n))$ is relatively compact. Define

$$B_k = \bigcup_{n=k}^{\infty} C_n \quad \text{and} \quad B = \bigcap_{k=1}^{\infty} B_k;$$

clearly, $\lambda(B) \ge r$. Therefore, it is enough to show that $\mu_0(\Sigma(B))$ is relatively compact. To this end, for every k choose $n_k \ge k$ so that

$$\lambda(B_k \setminus A_k) < 2^{-k}, \quad \text{where} \quad A_k = \bigcup_{n=k}^{n_k} C_n,$$

and set

$$\widetilde{A}_m = \bigcap_{k=m}^{\infty} A_k \quad \text{for } m = 1, 2, \ldots$$

Then

$$\widetilde{A}_m \subset B \quad \text{and} \quad \lambda(B \setminus \widetilde{A}_m) < 2^{-m+1}.$$

Therefore, as easily seen, the proof will be complete once we show that for every m the set $\mu_0(\Sigma(\widetilde{A}_m))$ is relatively compact. Let $A \in \Sigma(\widetilde{A}_m)$. Fix $k \ge m$ and define

$$E_k = C_k \cap A \quad \text{and} \quad E_n = \Big(C_n \setminus \bigcup_{j=k}^{n-1} C_j\Big) \cap A \quad \text{for } k < n \le n_k,$$

and

$$z = \sum_{n=k}^{n_k} \mu_n(E_n).$$

Then z is an element of the relatively compact set $\sum_{n=k}^{n_k} \mu_n(\Sigma(C_n))$, and

$$\|\mu_0(A) - z\| \le \sum_{n=k}^{n_k} \|\mu_0(E_n) - \mu_n(E_n)\| \le \sum_{n=k}^{n_k} 2^{-n} < 2^{-k+1}.$$

It follows that $\mu_0(\Sigma(\widetilde{A}_m))$ is relatively compact.

(b): We modify the proof of (a) by replacing $\mathcal{K}_r$ with

$$\mathcal{L}_r := \{\mu \in ca(\Sigma, X) : \overline{\mu}(D_\mu) \ge r\},$$

C_n with $D_n := D_{\mu_n}$, and λ with $\overline{\mu}_0$. Since $\overline{\mu}_n(D_n) \le \overline{\mu}_n(B_n) \le \overline{\mu}_0(B_n) + 2^{-n}$, we have $\overline{\mu}_0(B) \ge r$. The rest of the argument does not require any changes. ■

If $cca(\Sigma, \lambda, X) \ne ca(\Sigma, \lambda, X)$, then we will say that X has the *Noncompact Range Property* (*NCRP*). As shown in [D3, Thm. 2], this is indeed a property of X itself, i.e., it is independent of a particular choice of the nonatomic probability measure space (S, Σ, λ). Moreover, it is equivalent to the existence of a noncompact bounded linear operator from ℓ_∞ to X.

6.3. Proposition. *If X has the* (*NCRP*), *then there exists $\mu \in ca(\Sigma, \lambda, X)$ that is λ-everywhere of infinite variation and has λ-everywhere noncompact range.*

Proof. As explained above, the assumption guarantees that on every nonzero nonatomic finite positive measure space there exists an absolutely continuous measure with values in X whose range is not relatively compact. Applying this along with Proposition 6.1 (a), we obtain that every $A \in \Sigma$ with $\lambda(A) > 0$ has a subset C with $\lambda(C) > 0$ such that there exists a measure in $ca(\Sigma, \lambda, X)$ which is concentrated on C and has λ-everywhere noncompact range on C. Now, consider a maximal disjoint family $\mathcal{C}$ consisting of such sets C. It is countable, say $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$, and $\lambda(S \setminus \bigcup_{n=1}^{\infty} C_n) = 0$. For every n choose a measure $\mu_n \in ca(\Sigma, \lambda, X)$ which is concentrated on C_n and has λ-everywhere noncompact range on C_n; it can be assumed that $\|\mu_n\| \le 2^{-n}$. Then it is clear that the measure $\mu_0 := \sum_{n=1}^{\infty} \mu_n$ has λ-everywhere noncompact range.

To finish, let $A_0 \in \Sigma$ be a λ-maximal set on which $|\mu_0|$ is σ-finite, and take ν in $cca(\Sigma, \lambda, X)$ with λ-everywhere infinite variation (see Theorem 2.3, 2.7 or 4.1). Then the measure μ defined by the formula $\mu(A) = \mu_0(A) + \nu(A \cap A_0)$ is as desired. ■

6.4. Theorem. *If X has the* (*NCRP*), *then the measures that have λ-everywhere noncompact range form a dense G_δ-subset of $ca(\Sigma, \lambda, X)$.*

P r o o f. Let $\mathcal{K}'$ denote the set the theorem is concerned with. Using the notation $\mathcal{K}_r$ employed in the proof of Proposition 6.2 (a), we have

$$\mathcal{K}' = ca(\Sigma, \lambda, X) \Big\backslash \bigcup_{n=1}^{\infty} \mathcal{K}_{1/n}.$$

Hence, by Proposition 6.2 (a), $\mathcal{K}'$ is a G_δ-set in $ca(\Sigma, \lambda, X)$.

By Proposition 6.3, $\mathcal{K}' \neq \emptyset$. Fix $\mu \in \mathcal{K}'$ and $\nu \in ca(\Sigma, \lambda, X)$. For scalars t, t' with $t \neq t'$ we have $\lambda(C_{t\mu+\nu} \cap C_{t'\mu+\nu}) = 0$. Hence $\lambda(C_{t\mu+\nu}) = 0$, i.e., $t\mu + \nu \in \mathcal{K}'$ for all but countably many t. This implies the density assertion. ∎

As a direct consequence of Theorems 4.1 and 6.4 and the Baire category theorem, we now obtain the following

6.5. Corollary. *If X has the (NCRP), then the measures that are λ-everywhere of infinite variation and have λ-everywhere noncompact range form a dense G_δ-subset of $ca(\Sigma, \lambda, X)$.*

Corollary 6.7 below is an analogue of Corollary 2.5 for the present setting. In contrast to the latter, we were able to prove it only under an additional assumption on X.

6.6. Proposition. *Assume that X has an unconditional Schauder decomposition $X = \sum_{n=1}^{\infty} X_n$ with $0 < \dim X_n < \infty$ for all $n \in \mathbb{N}$. For every $M \subset \mathbb{N}$, let P_M denote the natural projection from X onto its subspace $\sum_{n\in M} X_n$. If a bounded subset E of X is not relatively compact, then there exists a partition $\{M_i : i \in \mathbb{N}\}$ of $\mathbb{N}$ such that $P_{M_i}(E)$ is not relatively compact for every $i \in \mathbb{N}$.*

P r o o f. Set $n_0 = 0$ and $P_0(x) = 0$ for all $x \in X$. By a compactness criterion for sets in Banach spaces with finite-dimensional Schauder decompositions (cf. [DS, Cor. IV.5.5]), we can find a sequence (z_k) in E and a sequence $n_1 < n_2 < \ldots$ in $\mathbb{N}$ such that for some $\varepsilon > 0$,

$$\|(P_{n_k} - P_{n_{k-1}})(z_k)\| > \varepsilon \quad \text{for all } k \in \mathbb{N},$$

where $P_n = P_{\{1,\ldots,n\}}$. Let $\{N_i : i \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinite sets, and define

$$M_i = \bigcup_{k\in N_i} \{n \in \mathbb{N} : n_{k-1} < n \le n_k\}.$$

Since

$$(P_{n_k} - P_{n_{k-1}})P_{M_i} = P_{n_k} - P_{n_{k-1}} \quad \text{for } k \in N_i \text{ and } i \in \mathbb{N},$$

the same compactness criterion as above shows that $P_{M_i}(\{z_k : k \in N_i\})$ is not relatively compact, and we are done. ∎

6.7. Corollary. *If X is as in Proposition 6.6 and has the (NCRP), then there exists a closed infinite-dimensional subspace $\mathcal{M}$ of $ca(\Sigma, \lambda, X)$ which, apart from zero, consists of measures with λ-everywhere infinite variation and λ-everywhere noncompact range.*

P r o o f. Take $\nu \in ca(\Sigma, \lambda, X) \setminus cca(\Sigma, \lambda, X)$, and let (M_i) be a partition of $\mathbb{N}$ provided by Proposition 6.6 applied with $E = \nu(\Sigma)$. Denote $Q_i = P_{M_i}$ and $Y_i = Q_i(X)$. Then $Q_i \circ \nu \notin cca(\Sigma, \lambda, Y_i)$, so that Y_i has the (NCRP). Applying Proposition 6.3, for every $i \in \mathbb{N}$ choose $\mu_i \in ca(\Sigma, \lambda, Y_i)$ which is λ-everywhere of infinite variation and has λ-everywhere noncompact range. Then it is easily seen that (μ_i) is an (unconditional) basic sequence in $ca(\Sigma, \lambda, X)$ (cf. the proof of part (a) of Theorem 2.3). Moreover, its closed linear span $\mathcal{M}$ has the required properties. Indeed, if $0 \neq \mu \in \mathcal{M}$, then μ is of the form $\mu = \sum_{i=1}^{\infty} c_i \mu_i$, where at least one of the coefficients, say c_j, is not zero. Now, let $A \in \Sigma$ and $\lambda(A) > 0$. Then $Q_j(\mu(\Sigma(A))) = c_j \mu_j(\Sigma(A))$, and since $\mu_j(\Sigma(A))$ is not relatively compact, neither is $\mu(\Sigma(A))$. Also, $|\mu|(A) \geq \|Q_j\|^{-1} |c_j| |\mu_j|(A) = \infty$. ∎

R e m a r k. Using a similar argument to that in the preceding proof, the following can be shown: Let Z be a Banach space, and let $L(Z, X)$ denote the usual Banach space of all bounded linear operators from Z to X. If X is as in Proposition 6.6 and $L(Z, X)$ contains a noncompact operator, then it also contains a closed infinite-dimensional subspace which, apart from zero, consists of noncompact operators.

In connection with our next result see also Theorem 7.4 (b) below.

6.8. THEOREM. *Assume that X has the (NCRP). Then*

(a) *The measures that have everywhere noncompact range form a dense G_δ-subset of $ca(\Sigma, \lambda, X)$.*

(b) *The measures that have everywhere noncompact range form a G_δ-subset of $ca(\Sigma, X)$.*

P r o o f. (a): The density of the set in part (a) is assured by Theorem 6.4, and the fact that it is of type G_δ is obviously a consequence of part (b). It can also be verified as follows.

For every n, let $\mathcal{H}_n$ denote the set of all $\mu \in ca(\Sigma, \lambda, X)$ for which there exists $E \in \Sigma$ such that

$$\lambda(E \cap C_\mu) < n^{-1} \lambda(E) \quad \text{and} \quad \overline{\mu}(S \setminus E) < 2^{-n}.$$

By Proposition 6.2 (a) applied to the measure space $(E, \Sigma(E), \lambda|\Sigma(E))$, we easily see that $\mathcal{H}_n$ is open in $ca(\Sigma, \lambda, X)$. Set $\mathcal{H} = \bigcap_{n=1}^{\infty} \mathcal{H}_n$. It is clear that $\mathcal{H}$ contains all measures with everywhere noncompact range.

Conversely, let $\mu \in \mathcal{H}$. Then there exist sets $E_n \in \Sigma$ such that

$$\lambda(E_n \cap C_\mu) < n^{-1} \lambda(E_n) \quad \text{and} \quad \overline{\mu}(S \setminus E_n) < 2^{-n}.$$

Define $\widetilde{E}_k$ and E as in the proof of Theorem 4.5. Then, by the same argument as therein, $\overline{\mu}(S \setminus E) = 0$. Moreover, we have $\lambda(\widetilde{E}_k \cap C_\mu) \leq \lambda(E_k \cap C_\mu) < k^{-1}$ and $\lambda(\widetilde{E}_k \cap C_\mu) \to \lambda(E \cap C_\mu)$, hence $\lambda(E \cap C_\mu) = 0$. Consequently, μ has everywhere noncompact range.

(b): Using the notation $\mathcal{L}_r$ employed in the proof of Proposition 6.2 (b), we can represent the set part (b) is concerned with as

$$ca(\Sigma, X) \setminus \bigcup_{n=1}^{\infty} (\mathcal{L}_{1/n} \cup \{0\}),$$

and so (b) follows from Proposition 6.2 (b). ∎

The following is a direct consequence of Theorems 4.5 and 6.8 (a) and the Baire category theorem.

6.9. COROLLARY. *If X has the (NCRP), then the measures that are everywhere of infinite variation and have everywhere noncompact range form a dense G_δ-subset of $ca(\Sigma, \lambda, X)$.*

The argument used to establish Proposition 4.6 together with Propositions 3.1 (c) and 6.3 yields

6.10. PROPOSITION. *If X has the (NCRP), then $ca(\Sigma, \lambda, X)$ contains a closed subspace $\mathcal{M}$ which, apart from zero, consists of measures with everywhere infinite variation and everywhere noncompact range and is such that, for every $A \in \Sigma$ with $\lambda(A) > 0$, its subspace consisting of measures concentrated on A is infinite-dimensional.*

7. Sets of measures with everywhere infinite variation or everywhere noncompact range

In Theorem 7.4 below, we give analogues of Theorems 4.5 and 6.8 (a) for the closed subspaces of $ca(\Sigma, X)$ and $cca(\Sigma, X)$ consisting of nonatomic measures.

Actually, the main results of this section are Theorem 7.1 and its consequence, Corollary 7.3, concerning certain G_δ-sets in Banach spaces with Schauder decompositions. From Corollary 7.3, combined with Theorems 4.5 and 6.8 (a), Theorem 7.4 follows almost immediately. We are grateful to Professor Dan Mauldin who, in answer to our query, provided us with a proof that the subset of $\ell_1(I)$ consisting of points whose coordinates are irrational or zero is a G_δ-set. Although his argument used the fact that the set of rationals is countable, it was fairly easy to modify it so as to obtain more general results.

7.1. THEOREM. *Let Z be a normed space, and let $(P_i)_{i \in I}$ be an equicontinuous family of linear operators, where each P_i maps Z into a normed space Z_i. Assume that for every $z \in Z$ and $r > 0$ the set $\{i \in I : \|P_i(z)\| \geq r\}$ is finite. Moreover, for every $i \in I$ let H_i be a G_δ-subset of Z_i. Then*

$$D := \bigcap_{i \in I} P_i^{-1}(H_i)$$

is a G_δ-subset of Z.

Given a nonempty set E in a normed space, denote $d(E,0)=\inf\{\|x\|:x\in E\}$. In the proof of Theorem 7.1 we make use of the following

7.2. LEMMA. *Let Z, Z_i and P_i be as above and let F_i be a nonempty closed subset of Z_i for every $i\in J$, where $J\subset I$. If $r:=\inf\{d(F_i,0):i\in J\}>0$, then*

$$F:=\bigcup_{i\in J}P_i^{-1}(F_i)$$

is closed in Z.

Proof. Given $z\in Z$, we have $\|P_i(z)\|<r/2$ for all except finitely many $i\in I$. Set $c=2\sup_{i\in I}\|P_i\|$. It follows that the open (r/c)-ball in Z centered at z intersects only finitely many of the closed sets $P_i^{-1}(F_i)$, $i\in I$. This yields the assertion. ∎

Proof of Theorem 7.1. Since $\bigcap_{i\in J}P_i^{-1}(Z_i\setminus\{0\})$ is a G_δ-subset of Z for every $J\subset I$, we may assume that $0\in H_i$ for all $i\in I$. First consider the case when H_i is open in Z_i for every $i\in I$. We may also assume that $H_i\neq Z_i$ for every $i\in I$. Then

$$Z\setminus D=\bigcup_{k=1}^{\infty}\bigcup_{i\in I_k}P_i^{-1}(F_i),$$

where $F_i=Z_i\setminus H_i$ and $I_k=\{i\in I:d(F_i,0)\geq k^{-1}\}$, and the assertion follows from Lemma 7.2.

Now consider the general case. For every $i\in I$ write $H_i=\bigcap_{n=1}^{\infty}G_{in}$, where each G_{in} is open in Z_i. Then

$$D=\bigcap_{n=1}^{\infty}\bigcap_{i\in I}P_i^{-1}(G_{in}),$$

and we get the assertion by applying the result established in the first part of the proof. ∎

7.3. COROLLARY. *Let Z be a Banach space with an unconditional Schauder decomposition (or just a Schauder decomposition if $I=\mathbb{N}$), $Z=\sum_{i\in I}Z_i$, and let $P_i:Z\to Z_i$ $(i\in I)$ be the projections associated with the decomposition. For every $i\in I$ let H_i be a G_δ-subset of Z_i. Then the set D defined as in Theorem 7.1 is a G_δ-subset of Z.*

Remark. Obviously, the set D of Corollary 7.3 is dense in Z provided that $0\in H_i$ and H_i is dense in Z_i for every $i\in I$.

We now return to vector measures. Assume that the σ-algebra Σ is *of type* (NA), by which we mean that it admits a nonatomic probability measure. Denote by $ca_{na}(\Sigma,X)$ and $cca_{na}(\Sigma,X)$ the closed subspaces of $ca(\Sigma,X)$ and $cca(\Sigma,X)$, respectively, formed by nonatomic measures. Applying the Kuratowski–Zorn Lemma, we can find a maximal family $\{\lambda_i:i\in I\}$ consisting of mutually singular nonatomic probability measures on Σ. It is then clear that we have the following unconditional Schauder decompositions:

$$ca_{na}(\Sigma, X) = \sum_{i\in I} ca(\Sigma, \lambda_i, X) \quad \text{and} \quad cca_{na}(\Sigma, X) = \sum_{i\in I} cca(\Sigma, \lambda_i, X),$$

where, for each $i \in I$, the associated projection P_i assigns to every μ its λ_i-continuous part in the Lebesgue decomposition of μ with respect to λ_i. (See Mauldin [Ma, Thm. 1] for a similar decomposition of the space $bvca(\Sigma, X)$; the idea of such decompositions goes back to Artemenko [Ar].) It is now important to note that a measure μ is everywhere of infinite variation [has everywhere noncompact range] if and only if the measure $P_i(\mu)$ is everywhere of infinite variation [has everywhere noncompact range] or $P_i(\mu) = 0$ for every $i \in I$. Therefore, the following result follows immediately from Corollary 7.3 and Theorems 4.5 and 6.8 (a). (Part (b) below is, essentially, a reformulation of Theorem 6.8 (b).)

7.4. THEOREM. *Assume that Σ is of type (NA).*

(a) *The measures that are everywhere of infinite variation form dense G_δ-sets in each of the spaces $ca_{na}(\Sigma, X)$ and $cca_{na}(\Sigma, X)$.*

(b) *If X has the (NCRP), then the measures that have everywhere noncompact range form a dense G_δ-set in $ca_{na}(\Sigma, X)$.*

(c) *If X has the (NCRP), then the measures that are everywhere of infinite variation and have everywhere noncompact range form a dense G_δ-set in $ca_{na}(\Sigma, X)$.*

The next result is an extension of Proposition 4.6.

7.5. PROPOSITION. *If Σ is of type (NA), then $cca_{na}(\Sigma, X)$ contains a closed subspace $\mathcal{M}$ which, apart from zero, consists of measures with everywhere infinite variation and is such that, for every nonatomic probability measure τ on Σ, the subspace $\mathcal{M} \cap cca_{na}(\Sigma, \tau, X)$ is infinite-dimensional.*

Proof. For every $i \in I$ choose a closed subspace $\mathcal{M}_i \subset cca(\Sigma, \lambda_i, X)$ provided by Proposition 4.6. Then, as easily verified,

$$\mathcal{M} := \{\mu \in cca_{na}(\Sigma, X) : P_i(\mu) \in \mathcal{M}_i \text{ for all } i \in I\}$$

is as required. ■

We leave it to the reader to formulate the corresponding extension of Proposition 6.10.

In connection with Theorems 5.1 and 7.4 it is worth while to point out the following simple

7.6. PROPOSITION. (a) *If Σ is infinite, then the measures of infinite variation form dense G_δ-sets in each of the spaces $ca(\Sigma, X)$ and $cca(\Sigma, X)$.*

(b) *If $L_1(\lambda)$ is infinite-dimensional, then the measures of infinite variation form dense G_δ-sets in each of the spaces $ca(\Sigma, \lambda, X)$ and $cca(\Sigma, \lambda, X)$.*

Proof. (a): Let $\mathcal{W}_\infty = \{\mu \in ca(\Sigma, X) : |\mu|(S) = \infty\}$. Since the function $\mu \to |\mu|(S)$ is lower semicontinuous, the set $\{\mu \in ca(\Sigma, X) : |\mu|(S) > n\}$ is open for every $n \in \mathbb{N}$. Hence $\mathcal{W}_\infty$ is a G_δ-set in $ca(\Sigma, X)$, and so is $\mathcal{W}_\infty \cap cca(\Sigma, X)$ in $cca(\Sigma, X)$.

By the Dvoretzky–Rogers theorem, $\mathcal{W}_\infty \cap cca(\Sigma, X) \neq \emptyset$. Now, if $\mu \in \mathcal{W}_\infty$ and $\nu \in ca(\Sigma, X)$, then $t\mu + \nu \notin \mathcal{W}_\infty$ for at most one value of t, and we are done.

(b): In view of (a) and its proof, we only need to show that the sets in question are nonempty. But this follows by considering the measure $\mu : \Sigma \to X$ defined by

$$\mu(A) = \sum_{n=1}^{\infty} \frac{\lambda(A \cap S_n)}{\lambda(S_n)} x_n,$$

where (S_n) is a sequence of pairwise disjoint sets in Σ with $\lambda(S_n) > 0$ and $\sum_{n=1}^{\infty} x_n$ is an unconditionally but not absolutely convergent series in X. ■

7.7. Corollary. *If X is separable, then $L_1(\lambda, X)$ is an $F_{\sigma\delta}$-subset of $ca(\Sigma, \lambda, X)$.*

This follows from Theorem 5.6 and Proposition 7.6 (b).

The next result is an analogue of Corollary 2.5 for the situation considered in Proposition 7.6.

7.8. Corollary. (a) *If Σ is infinite, then there exists a closed infinite-dimensional subspace of $cca(\Sigma, X)$ which, apart from zero, consists of measures of infinite variation.*

(b) *If $L_1(\lambda)$ is infinite-dimensional, then there exists a closed infinite-dimensional subspace of $cca(\Sigma, \lambda, X)$ which, apart from zero, consists of measures of infinite variation.*

Proof. We shall only establish (a); the proof of (b) is analogous.

By Proposition 7.6(a), $bvca(\Sigma, X) \cap cca(\Sigma, X)$ is a proper subspace of $cca(\Sigma, X)$. The variation norm defines a Banach-space topology on this subspace which is strictly stronger than that induced by the uniform norm. The latter is seen by considering measures of the form

$$\mu(A) = \sum_{i=1}^{n} \delta_i(A) z_i, \quad A \in \Sigma,$$

where $z_1, \ldots, z_n \in X$ are as in the proof of Proposition 2.2 and $\delta_1, \ldots, \delta_n$ are disjointly supported probability measures on Σ. Thus, the assertion follows from [D2, Thm. 5.6 (c)].

8. Positive vector measures with λ-everywhere infinite variation

We establish here a Banach-lattice version of Theorem 2.3. To this end we need the following result which is a special case of [M-N, Cor. 2.8.10] (see also [S, Thm. IV.2.7] for a weaker version due to V. Schlotterbeck).

8.1. Proposition. *For a Banach lattice X the following conditions are equivalent.*

(i) *X is not isomorphic to an AL-space.*

(ii) *There exists a sequence (x_n) of pairwise disjoint elements in X_+ such that the series $\sum_{n=1}^{\infty} x_n$ is unconditionally but not absolutely convergent.*

8.2. Theorem. *Suppose X is a Banach lattice which is not isomorphic to an AL-space. Then there exists a sequence of λ-simple measures $\mu_n : \Sigma \to X_+$ with the following properties.*

(a) *The sequence (μ_n) is an unconditional basic sequence in each of the Banach spaces $cca(\Sigma, \lambda, X)$ and $bvca(\Sigma, \lambda, X)$.*

(b) *The series $\sum_{n=1}^{\infty} \mu_n$ is absolutely* (*hence subseries*) *convergent in $cca(\Sigma, \lambda, X)$.*

(c) *For every infinite subset M of $\mathbb{N}$ the measure $\mu_M = \sum_{n\in M} \mu_n$ is λ-everywhere of infinite variation.*

(d) *For every sequence (M_k) of disjoint infinite subsets of $\mathbb{N}$, the measures μ_{M_k} (as defined in* (c)*) form an unconditional basic sequence in $cca(\Sigma, \lambda, X)$, and every nonzero measure in its closed linear span in $cca(\Sigma, \lambda, X)$ is λ-everywhere of infinite variation.*

Moreover,

$$\mu_n(A) \wedge \mu_m(A) = 0 \quad \text{for all } A \in \Sigma \text{ and } n \neq m.$$

The proof consists in a simple adaptation of the proof of Theorem 2.3. First, we choose a sequence (x_n) with nonzero positive terms according to Proposition 8.1, and note that it is an unconditional basic sequence in X with the basis constant one. Next, we select $0 = k_0 < k_1 < \ldots$ so that

$$\Big\| \sum_{i=k_{n-1}+1}^{k_n} x_i \Big\| \le 2^{-n} \quad \text{and} \quad \sum_{i=k_{n-1}+1}^{k_n} \|x_i\| \ge 2^n,$$

and for each of them define a (positive) measure μ_n as described in Lemma 2.1. The rest is a repetition of the proof of Theorem 2.3, and the verification of the (now stronger) assertion (d) is even simpler because of the unconditionality of (x_n). For the "moreover" part, it is enough to note that $|u| \wedge |v| = 0$ for all $u \in X_n$, $v \in X_m$ and $n \neq m$.

Remark. As easily seen by inspecting the proofs, many of the other results of the previous sections, in particular 2.5, 2.10, 4.1, 4.5, 6.3, 6.4, 6.5, 6.8, 6.9, 7.4, 7.5 and 7.6, have corresponding "positive versions". Roughly speaking, they are obtained by assuming that X is a Banach lattice and that, depending on the case, X is not isomorphic to an AL-space and/or $ca_+(\Sigma, \lambda, X) \setminus cca(\Sigma, \lambda, X) \neq \emptyset$, and changing the assertions appropriately so as to concern (sets of) positive measures. Here $ca_+(\Sigma, \lambda, X)$ denotes the closed cone in $ca(\Sigma, \lambda, X)$ consisting of measures with range contained in X_+.

Postscript. 1. The idea due to A. Szankowski, mentioned in the remark after the proof of Theorem 2.3, has been recently rediscovered and used to prove the incompleteness of $\mathcal{P}(\lambda, X)$ in a paper by S. J. Dilworth and M. Girardi, *Bochner vs. Pettis norm: examples and results*, in: *Banach Spaces*, Bor-Luh Lin and W. B. Johnson (eds.), Contemporary Math. 144, Amer. Math. Soc., Providence, R.I., 1993, 69–80.

2. Another example of a Banach space and its infinite-dimensional closed subspace consisting, apart from zero, of "pathological" elements is contained in a preprint by V. P. Fonf, V. I. Gurarij and M. I. Kadec, *Infinite-dimensional subspace in C consisting of nowhere differentiable functions*.

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