MATHEMATICAL CONTROL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 14
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

SEMI-GROUP METHODS IN STOCHASTIC CONTROL

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INTRODUCTION

We present in this paper some stochastic control problems, which are formulated in terms of semi-groups.

Actually we have two things in mind. On the one hand, the dynamic system is represented by a Markov semi-group for which we formulate several control problems. On the other hand, we consider non-linear semi-groups which are themselves derived from stochastic control.

This second idea has been introduced by M. Nisio [9] (see also S. R. Pliska [10] and J. Zabczyk [12]). In [2], [3] A. Bensoussan considered a semi-group approach to variational inequalities and quasi variational inequalities (or to stopping time and impulse control problems) where purely analytic techniques were used (see A. Bensoussan–J.-L. Lions [4] and M. Robin [11] for earlier work, using partly probabilistic and partly analytic techniques). In [6], A. Bensoussan and M. Robin used discretization to study the same problems. In [5], A. Bensoussan and J.-L. Lions considered non linear semi-groups corresponding to stopping time and impulse control problems. This was motivated by an earlier work of L. Barthelemy [1] (see also J. Zabczyk [12]).

The objective of this article is to review the main results obtained by the author (himself or in cooperation with J.-L. Lions or M. Robin) on these semi-group methods.

1. THE PROBLEM OF SEMI-GROUP ENVELOPE

1.1. Setting the problem and assumptions

Let E be a Polish space provided with the Borel σ-algebra $\mathcal{E}$. We denote by B the space of Borel bounded functions on E, and by C the space of bounded uniformly continuous functions on E. We consider a family

$\Phi^v(t)$, $v \in V$, of operators such that:

(1.1) V is a finite set,

$$\Phi^v(t) \in \mathscr{L}(B; B), \quad \Phi^v(0) = I,$$

(1.2) $$\|\Phi^v(t)\| \leqslant 1,$$

$$\Phi^v(t+s) = \Phi^v(t)\Phi^v(s),$$

$$\Phi^v(t)\varphi \geqslant 0 \quad \text{when} \quad \varphi \geqslant 0.$$

A semi-group of operators on B satisfying (1.2) is called a *Markov semi-group*.

We will assume that

(1.3) $$\Phi^v(t)\colon C \to C,$$

(1.4) $t \to \Phi^v(t)\varphi(x)$ is continuous from $(0, \infty) \to R$ $\forall x$ fixed $\forall \varphi \in C$.

Next let $L(x, v)$ be a function such that

(1.5) $$L_v(x) \equiv L(x, v) \in B,$$
$$\int_0^\infty e^{-\alpha t}\Phi^v(t)L_v\,dt \in C,$$

where α is a positive number.

The first problem we formulate is the following. Consider the set

(1.6) $$u \in B,$$
$$u \leqslant \int_0^t e^{-\alpha s}\Phi^v(s)L_v\,ds + e^{-\alpha t}\Phi^v(t)u \qquad \forall t \geqslant 0\ \forall v.$$

We have the following

THEOREM 1.1. *We assume* (1.1), ..., (1.5); *then the set of u satisfying* (1.6) *is not empty and has a maximum element.*

To prove Theorem 1.1 one relies on the following discretization scheme. Let $h > 0$; one considers u_h to be the unique solution of

(1.7) $$u_h = \operatorname*{Min}_v \Big[\int_0^h e^{-\alpha s}\Phi^v(s)L_v\,ds + e^{-\alpha h}\Phi^v(h)u_h\Big], \qquad u_h \in C.$$

Then one proves that

(1.8) $$u_{1/2^q} \downarrow u \quad \text{as} \quad q \uparrow \infty,$$

where u is the maximum element of (1.6). For details see A. Bensoussan–M. Robin [6].

1.2. Regularity results

We now assume the following regularity properties:

(1.9) E is a Banach space,

(1.10) $|L(x, v)-L(y, v)| \leqslant K|x-y|^{\delta}, \quad 0 \leqslant \delta \leqslant 1,$

(1.11) $\forall g \in C^{0,\delta}(E)$ (i.e., $|g(x)-g(y)| \leqslant \|g\|_{\delta}|x-y|^{\delta}$),

we have

$$|\Phi^{v}(t)g(x)-\Phi^{v}(t)g(y)| \leqslant e^{\lambda t}\|g\|_{\delta}|x-y|^{\delta},$$

with $\lambda \geqslant 0$,

(1.12) $t \to \Phi^{v}(t)\varphi(x)$ is (Lebesgue) measurable from $(0, \infty)$ into R

$$\forall \varphi \in B \ \forall x \text{ fixed}.$$

We can then state the following

THEOREM 1.2. *We make the assumptions of Theorem* 1.1 *and* (1.9), (1.10), (1.11), (1.12). *Then the maximum element u of* (1.6) *belongs to C and $u_{1/2^q}$ converges to u uniformly on every compact subset of E.*

An intermediary result, used in the proof of Theorem 1.2, is that if $a > \lambda$ then actually $u \in C^{0,\delta}(E)$.

1.3. Probabilistic interpretation

We give here the interpretation of the maximum element of the set (1.6). We assume

(1.13) $\Phi(t)1 = 1.$

Consider $\Omega = E^{[0,\infty)}$, $x(t, \omega)$ to be the canonical process, $M_t^s = \sigma(x(\lambda), t \leqslant \lambda \leqslant s)$, $M_t \doteq M_t^{\infty}$. For simplicity we take $V = \{1, 2, \ldots, m\}$. With $i \in V$ we associate a probability P_i^{xt} on (Ω, M_t) such that

(1.14) $E_i^{xt}\varphi(x(s)) = \Phi^{i}(s-t)\varphi(x) \quad \forall s \geqslant t.$

We denote by W the class of step processes adapted to M_0^t with values in V. More precisely, if $w \in W$, then there exists a sequence

$$\tau_0 = 0 < \tau_1 < \ldots < \tau_n < \ldots$$

which is *deterministic*, increasing and convergent to $+\infty$, and

(1.15) $w \equiv v(\cdot), \quad v(t, \omega) = v_n(\omega), \quad t \in [\tau_n, \tau_{n+1}),$

where v_n is $M_0^{\tau_n}$ measurable with values in V.

Then one can construct a probability P_w^x (for given x in E and w in W) on (Ω, M_0) such that the following property holds:

(1.16)

$$E_w^x[\varphi(x(t))|M_0^{\tau_n}] = \Phi^{v_n}(t-\tau_n)\varphi(x(\tau_n)) \quad \forall\varphi \in B \text{ and } \tau_n \leqslant t < \tau_{n+1}.$$

Next one defines the functional

$$J^x(w) = E_w^x \int_0^\infty e^{-\alpha t} L(x(t), v(t))\,dt. \tag{1.17}$$

Set

$$W_h = \{w \in W \mid \tau_n = nh\}. \tag{1.18}$$

We can state

THEOREM 1.3. *We make the assumptions of Theorem* 1.1 *and* (1.13). (*Then* u_h, *which is the unique solution of* (1.7), *satisfies*

$$u_h(x) = \operatorname*{Min}_{w \in W_h} J^x(w). \tag{1.19}$$

Moreover

$$u(x) = \operatorname*{Inf}_{w \in \bigcup_q W_{1/2^q}} J^x(w). \tag{1.20}$$

2. THE STOPPING TIME PROBLEM

2.1. Setting the problem

Let $(E, \mathscr{E})$ and B, C be as in § 1.1. We consider a Markov semi-group on B, $\Phi(t)$ (i.e., cf. (1.2)),

$$\begin{aligned} &\Phi(t) \in \mathscr{L}(B, B), \quad \Phi(0) = I, \quad \|\Phi(t)\| \leqslant 1,\\ &\Phi(t+s) = \Phi(t)\Phi(s),\\ &\Phi(t)\varphi \geqslant 0 \quad \text{if} \quad \varphi \geqslant 0. \end{aligned} \tag{2.1}$$

We will assume that

(2.1) $t \to \Phi(t)\varphi(x)$ is continuous from $(0, \infty) \to R$ $\forall x$ fixed $\forall\varphi \in B$.

Let also

$$\psi \in B, \tag{2.3}$$

(2.4)

$L \in B$ such that $t \to \Phi(t)L(x)$ is (Lebesgue) measurable $\forall x$ fixed.

Alternatively, if we make some regularity assumptions on ψ, L, namely

(2.5) $$\psi \in C, \quad \int_0^\infty e^{-\alpha t}\Phi(t) L\, dt \in C,$$

then we use a weaker form of (2.2), namely

(2.6) $$\Phi(t)\colon C \to C \quad \forall t > 0,$$

(2.7) $t \to \Phi(t)\varphi(x)$ is continuous from $(0, \infty) \to R$ $\quad \forall x$ fixed $\forall \varphi \in C$.

We define the following problem. Consider the set of functions

(2.8) $$u \in B, \quad u \leqslant \psi,$$
$$u \leqslant \int_0^t e^{-\alpha s}\Phi(s) L\, ds + e^{-\alpha t}\Phi(t) u \quad \forall t \geqslant 0;$$

then we have

THEOREM 2.1. *We assume* (2.1), (2.2), (2.3), (2.4) *or* (2.1), (2.5), (2.6), (2.7); *then the set of functions satisfying* (2.8) *is not empty and has a maximum element.*

2.2. Approximation schemes

There are two methods to prove Theorem 2.1, which are approximation methods of different kinds. One can use the penalty method:

(2.9) $$u_\varepsilon = \int_0^\infty e^{-\alpha t}\Phi(t)\left[L - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+\right] dt, \quad u_\varepsilon \in B,$$

or the discretization method:

(2.10) $$u_h = \operatorname{Min}\left[\psi, \int_0^h e^{-\alpha t}\Phi(t)\, dt + e^{-\alpha h}\Phi(h) u_h\right], \quad u_h \in B.$$

We can also obtain the continuity of the maximum element of (2.8) and the uniform convergence of u_ε, u_h towards u under slightly more stringent assumptions. We assume that $\Phi(t)$ satisfies (2.1), (2.6) and

(2.11) $t \to \Phi(t)\varphi$ is continuous from $[0, \infty)$ into C $\quad \forall \varphi \in C$.

We also assume (2.5) for the data; we then have

THEOREM 2.2. *We assume* (2.1), (2.6), (2.11), (2.5). *Then the maximum element of* (2.8) *belongs to* C *and we have*

$$u_\varepsilon \to u \quad in \quad C, \quad as \quad \varepsilon \to 0,$$

$$u_h \to u \quad in \quad C, \quad as \quad h \to 0.$$

For details, we refer to A. Bensoussan [3], and A. Bensoussan–M. Robin [6].

2.3. Probabilistic interpretation

We give the probabilistic interpretation of the maximum element u of (2.8) (under the assumptions of Theorem 2.1). Consider Ω, M_0 defined in § 1.3. For any fixed x in E, we construct P^x on (Ω, M_0) such that

$$E^x \varphi(x(t)) = \Phi(t)\varphi(x) \quad \forall \varphi \in B. \tag{2.12}$$

Let θ be a M^t stopping time; we define

$$J^x(\theta) = E^x\Big[\int_0^\theta e^{-\alpha t} L(x(t))\,dt + e^{-\alpha\theta}\psi(x(\theta))\Big]. \tag{2.13}$$

Consider stopping times of the form

$$\theta = \nu h, \tag{2.14}$$

where ν is a random integer such that $\{\nu = n\} \subset M^{nh}$ $\forall n$. We denote by Θ_h the set of stopping times satisfying (2.14). We have

THEOREM 2.3. *We make the assumptions of Theorem* 2.1; *then one has*

$$u_h(x) = \operatorname*{Min}_{\theta \in \Theta_h} J^x(\theta) \tag{2.15}$$

and

$$u(x) = \operatorname*{Inf}_{\theta \in \bigcup_q \Theta_{1/2^q}} J^x(\theta). \tag{2.16}$$

In the case of Theorem 2.2, one can prove the following

THEOREM 2.4. *We make the assumptions of Theorem* 2.2. *Then* u *satisfies*

$$u(x) = \operatorname*{Min}_{\theta} J^x(\theta). \tag{2.17}$$

The probabilistic set-up for Theorem 2.4 to hold true is actually slightly different from that of Theorem 2.3. One assumes that:

(2.18) $\qquad E, \mathscr{E}$ is a semi compact,

(2.19) $\qquad t \to \Phi(t)\varphi$ is continuous from $[0, \infty)$ into $\hat{C}$ $\quad \forall \varphi \in \hat{C}$, where

$$\hat{C} = \{\varphi \in C \mid \forall \varepsilon \exists K_\varepsilon \text{ compact such that } |\varphi(x)| < \varepsilon \ \forall x \notin K_\varepsilon\}.$$

Then one takes

$$\Omega = D([0, \infty); E), \qquad x(t; \omega) \equiv \omega(t),$$

$$M_0 = \sigma(x(t); t \geqslant 0), \qquad M^t = \sigma(x(s), s \leqslant t),$$

and by the general theory of Markov processes (cf. E. B. Dynkin [8]), there exists a unique probability P^x on Ω, M_0 such that if we consider

$$\bar{M}^t = M^{t+0} \text{ completed},$$

$$\bar{M}_0 = M_0 \quad \text{completed},$$

then $(\Omega, \bar{M}_0, P^x, \bar{M}^t, x(t))$ is a strong Markov process right continuous and quasi continuous from the left. This set-up permits us to obtain (2.17).

3. IMPLICIT OBSTACLES

We assume that $\Phi(t)$ satisfies (2.1), (2.6), (2.11) and

$$\text{(3.1)} \qquad L \in B, \quad \int_0^\infty e^{-\alpha t}\Phi(t)L\,dt \in C, \qquad L \geqslant 0.$$

Also, let M be an operator such that

(3.2) $M\colon C \to C$ is Lipschitz, concave and monotone increasing (i.e., $M\varphi_1 \leqslant M\varphi_2$ if $\varphi_1 \leqslant \varphi_2$), $\quad M(0) \geqslant k > 0$.

We consider the set of functions

$$\text{(3.3)} \qquad \begin{aligned} & u \in C, \quad u \leqslant Mu, \\ & u \leqslant \int_0^t e^{-\alpha s}\Phi(s)L\,ds + e^{-\alpha t}\Phi(t)u. \end{aligned}$$

Then we have

THEOREM 3.1. *We assume* (2.1), (2.6), (2.11) *and* (3.1), (3.2). *Then the set of functions u satisfying* (3.3) *is not empty and has a maximum element.*

One can approximate (3.3) by using the following discretization scheme:

$$\text{(3.4)} \qquad u_h = \operatorname{Min}\Big[Mu_h, \int_0^h e^{-\alpha t}\Phi(t)L\,dt + e^{-\alpha h}\Phi(h)u_h\Big], \qquad u_h \in C.$$

In particular, one can prove

(3.5) $$u_h \to u \quad \text{in} \quad C.$$

4. NON-LINEAR SEMI-GROUP

4.1. Assumptions – The equation

In this section we assume that $\Phi(t)$ satisfies (2.1), (2.6) and

(4.1) $$\sup_{0 \leqslant s \leqslant T} \|\Phi(t)\varphi(s) - \varphi(s)\|_C \to 0, \quad \text{as} \quad t \downarrow 0 \quad \forall \varphi \in C(0, T, C),$$

4.2) $\forall L \in B$, $t \to \Phi(t)L(x)$ is (Lebesgue) measurable $\forall x \in E$,

$$\int_0^\infty e^{-\alpha t}\Phi(t)L\,dt \in C, \quad \alpha \geqslant 0,$$

(4.3) $\bar{u} \in C$.

We first consider the problem

(4.4)
$$u(\cdot) \in C([0, T]; C), \quad u(0) = \bar{u},$$
$$u(t) = \int_0^{t-s} e^{-\alpha\sigma}\Phi(\sigma)L\,d\sigma + e^{-\alpha(t-s)}\Phi(t-s)u(s) \quad \forall s \leqslant t \in [0, T].$$

It is easy to check that (4.4) admits one and only one solution, namely:

(4.5) $$u(t) = \int_0^t e^{-\alpha\sigma}\Phi(\sigma)L\,d\sigma + e^{-\alpha t}\Phi(t)\bar{u}.$$

Then we set

(4.6) $$u(t) = T(t)\bar{u},$$

and $T(t)$ is a non-linear (in fact affine) semi-group of contractions on C.

An interesting problem is to prove Trotter's formula, (cf. M. G. Crandall–T. M. Liggett [7]). One considers for $\lambda > 0$

(4.7) $$R_\lambda(\bar{u}) = \int_0^\infty e^{-(1/\lambda+\alpha)t}\Phi(t)\,(\bar{u}/\lambda + L)\,dt.$$

Then we have

THEOREM 4.1. *We assume* (2.1), (2.6), (4.1), (4.2), (4.3); *then one has*

(4.8) $$\forall t > 0, \quad R^n_{t/n}(\bar{u}) \to T(t)\bar{u} \quad \textit{in} \quad C \quad \textit{as} \quad n \to \infty.$$

4.2. Evolution inequalities

We now consider a function $\psi(t)$ such that

$$\begin{aligned} &\psi \in C([0, T]; C), \\ &\bar{u} \leqslant \psi(0). \end{aligned} \tag{4.9}$$

We set the following problem:

$$\begin{aligned} &u(\cdot) \in C([0, T]; C), \quad u(0) = u, \\ &u(t) \leqslant \psi(t) \quad \forall t \in [0, T], \\ &u(t) \leqslant \int_0^{t-s} e^{-\alpha\sigma}\Phi(\sigma)L\,d\sigma + e^{-\alpha(t-s)}\Phi(t-s)u(s) \quad \forall s \leqslant t \in [0, T]. \end{aligned} \tag{4.10}$$

One has

THEOREM 4.2. *We make the same assumptions as in Theorem* 4.1, *and* (4.9). *Then the set of functions satisfying* (4.10) *is not empty and has a maximum element.*

If we consider next the case where $\psi(t)$ is constant,

$$\psi, \bar{u} \in C, \quad \bar{u} \leqslant \psi, \tag{4.11}$$

and denote by $u(t) = S(t)\bar{u}$ the maximum solution of (4.10), then $S(t)$ defines a non linear semi-group of contractions on

$$\mathscr{C} = \{\bar{u} \in C \mid \bar{u} \leqslant \psi\}.$$

Moreover,

$$S(t)\bar{u} \to \bar{u} \text{ in } C, \quad \text{as} \quad t \downarrow 0.$$

One then states Trotter's formula for this non-linear semi-group. We have to define the equivalent of the resolvent (as in (4.7)). This is done as follows. We write

$$R_\lambda(\bar{u}) = z_\lambda, \tag{4.12}$$

where z_λ is the maximum element of the set

$$\begin{aligned} &z \leqslant \psi, \quad z \in C, \\ &z \leqslant \int_0^t e^{-(\alpha+1/\lambda)s}\Phi(s)\,(L + \bar{u}/\lambda)\,ds + e^{-(\alpha+1/\lambda)t}\Phi(t)z. \end{aligned} \tag{4.13}$$

Then we obtain

THEOREM 4.3. *We assume* (2.1), (2.6), (4.2), (4.3), (4.11). *Then the following property holds:*

$$\forall t > 0 \quad R^n_{t/n}(\bar{u}) \to S(t)\bar{u} \quad in \quad C, \quad \forall \bar{u} \in \mathscr{C}, \quad as \quad n \to \infty. \tag{4.14}$$

In the proof of Theorem 4.3 an important role is played by the penalized semi-group. Define $u_\varepsilon(t)$ as the solution of

$$(4.15)\quad u_\varepsilon(t) = e^{-\alpha t}\Phi(t)\bar{u} + \int_0^t e^{-\alpha(t-\lambda)}\Phi(t-\lambda)\left[L - \frac{1}{\varepsilon}\big(u_\varepsilon(\lambda)-\psi\big)^+\right]d\lambda;$$

then

$$u_\varepsilon(t) = S_\varepsilon(t)\bar{u}.$$

With this penalized non-linear semi-group one associates the resolvent $R_{\lambda,\varepsilon}(\bar{u}) = z_\varepsilon$, defined as the solution of

$$(4.16)\qquad z_\varepsilon = \int_0^\infty e^{-(\alpha+1/\lambda)t}\Phi(t)\left(L + \bar{u}/\lambda - \frac{1}{\varepsilon}(z_\varepsilon-\psi)^+\right)dt.$$

The proof of Theorem 4.3 consists in obtaining a priori estimates, among them the following uniform estimate:

$$(4.17)\qquad \|R^n_{t/n,\varepsilon}\bar{u} - S_\varepsilon(t)\bar{u}\| \leqslant 2K\frac{t}{\sqrt{n}},$$

where K is a constant which does not depend on ε or t. For details see A. Bensoussan–J.-L. Lions [5].

4.3. Case of implicit obstacles

In this section we assume that $\Phi(t)$ satisfies (2.1), (2.6), (2.11) and we also assume (3.1), (3.2). Also, let $\bar{u}$ be such that

$$(4.18)\qquad \bar{u}\in C,\quad \bar{u}\geqslant 0,\quad \bar{u}\leqslant M\bar{u}.$$

One considers the following problem:

$$(4.19)\quad \begin{aligned} &u(\cdot)\in C([0,T];C),\quad u(0)=\bar{u},\\ &u(t)\leqslant Mu(t)\quad \forall t\in[0,T],\\ &u(t)\leqslant \int_0^{t-s} e^{-\alpha\sigma}\Phi(\sigma)L\,d\sigma + e^{-\alpha(t-s)}\Phi(t-s)u(s)\quad \forall s\leqslant t\in[0,T]. \end{aligned}$$

One obtains the following

THEOREM 4.4. *We assume* (2.1), (2.6), (2.11) *and* (3.1), (3.2), (4.18). *Then the set of functions u satisfying* (4.19) *is not empty and has a maximum element. If we set*

$$u(t) = S(t)\bar{u},$$

then $S(t)$ is a non-linear semi-group of contractions on

$$\mathscr{C} = \{\bar{u} \in C \mid \bar{u} \geqslant 0,\ \bar{u} \leqslant M\bar{u}\}.$$

One also can prove Trotter's formula, but only for $\bar{u}$ from a subset of $\mathscr{C}$. We define the resolvent $R_\lambda(\bar{u})\colon C \to C$ by setting

$$\begin{aligned} &z \leqslant Mz, \quad z \in C, \\ &z \leqslant \int_0^t e^{-(\alpha+1/\lambda)s}\Phi(s)\,(L+\bar{u}/\lambda)\,ds + e^{-(\alpha+1/\lambda)t}\Phi(t)z, \end{aligned} \tag{4.20}$$

and $z_\lambda = R_\lambda(\bar{u})$ is the maximum element of (4.20). One can then prove the following

THEOREM 4.5. *We make the assumptions of Theorem* 4.4 *Then one has*

$$\forall t > 0, \quad R^n_{t/n}(\bar{u}) \to S(t)\bar{u} \tag{4.21}$$

for any $\bar{u}$ such that $\bar{u} \in \mathscr{C}$ and

$$\bar{u} \leqslant \int_0^t e^{-\alpha\sigma}\Phi(\sigma)L\,d\sigma + e^{-\alpha t}\Phi(t)\bar{u} \qquad \forall t \geqslant 0.$$

In proving Theorem 4.5 one uses the following approximation:

$$\begin{aligned} u_\varepsilon(t) &= e^{-\alpha t}\Phi(t)\bar{u} + \int_0^t e^{-\alpha(t-s)}\Phi(t-s)\left[L - \frac{1}{\varepsilon}\big(u_\varepsilon(s) - Mu_\varepsilon(s)\big)^+\right]ds, \\ u_\varepsilon(t) &= S_\varepsilon(t)\bar{u}. \end{aligned} \tag{4.22}$$

For related results, see L. Barthelemy [1].

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