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Representations of Orlicz lattices

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Introduction

Representation theorems of vector lattices make it possible to identify abstract lattices with concrete function spaces, whose structure is usually better investigated. Bohnenblust's theorem on the representation of abstract L^p-spaces ($1 \leqslant p < \infty$) by $L^p(S, \Sigma, \mu)$ spaces is one of the best known representation theorems.

The present paper is devoted to studying Orlicz lattices (i.e., vector lattices with a modular defined on them) which are generalizations of abstract L^p-spaces. The main purpose of the paper is to present conditions for an Orlicz lattice to be mapped, with the preservation of all order and topological properties, into some known function space.

The paper consists of four parts. The first is of an introductory nature. Some theorems on order and order-topological properties of Musielak–Orlicz spaces are gathered together in the second part. The third part contains, among other things, the Main Representation Theorem, giving a representation of Orlicz lattices by super order dense sublattices in Musielak–Orlicz spaces. There are also theorems on representations of Orlicz lattices by Orlicz spaces, especially by L^p-spaces ($0 \leqslant p < \infty$). Bibliographical notes complete the paper.

The following notation will be used throughout the work:

R	the set of real numbers,
N	the set of natural numbers,
card A	the cardinality of a set A,
$\aleph_0$	the cardinality of N,
$f\|A$	the restriction of a function f to a set A.

Preliminaries

The basic terminology concerning vector lattices (Riesz spaces) and measure theory is taken mainly from [1], [9], [18], [31]. We recall some concepts needed for this paper.

The results included in sections A and B are well known, and so the

proofs are omitted (except for two simple Theorems A.9 and A.13 not formulated explicite in fundamental monographs), but we give precise information where these proofs are published.

A. Vector lattices

L will always denote a vector lattice. Its *positive cone* (= the set of positive elements) will be denoted by L_+. A subset $B \subset L$ is *order bounded* (briefly bounded) if it is both bounded from above and from below, in other words, B is contained in some order interval $[x, y] = \{z \in L: x \leqslant z \leqslant y\}$.

Elements $x, y \in L$ are *disjoint* if $|x| \wedge |y| = 0$ (in symbols $x \perp y$). In particular, the following properties of the disjointness relation will be useful:

(i) If $x \perp y$, then $x \perp ty$ for all $t \in \boldsymbol{R}$.

(ii) If $x \perp y$ and $z \perp y$, then $(x+z) \perp y$.

(iii) If $x \perp y$, then $|x \pm y| = |x| + |y| = |x| \vee |y|$.

The *disjoint complement* $A^\perp$ of a non-empty subset A of L is defined as $A^\perp = \{x \in L: x \perp a \text{ for all } a \in A\}$. We have the inclusion $A \subset A^{\perp\perp}$, where $A^{\perp\perp} = (A^\perp)^\perp$.

A vector lattice L is said to be *Archimedean* if:

$$x, y \in L_+ \text{ and } nx \leqslant y \text{ for all natural numbers } n \text{ imply } x = 0.$$

We say that a vector lattice has the CS-*property* (countable sup property) if every set $A \subset L$ having a least upper bound contains a countable subset B with $\sup A = \sup B$. We will often use the following theorem:

THEOREM A.1. *Let L be an Archimedean vector lattice. L has the* CS-*property if and only if every bounded set of positive disjoint elements is at most countable.*

Proof. See [9], Theorem 18 D, p. 31.

If L is *Dedekind complete* (*σ-Dedekind complete*) then this fact will be expressed as follows: L is DC (σ-DC). A vector lattice L is called *super Dedekind complete* if it is DC and has the CS-property. In this case we will say: L is SDC. In [18], p. 129, the following theorem is proved:

THEOREM A.2. *Every σ-*DC *vector lattice L having the* CS-*property is* SDC.

A net (x_α) of a vector lattice L is increasing (to x) if $x_\alpha \leqslant x_\beta$ for $\alpha \leqslant \beta$ (and x is the supremum of (x_α)). This fact will be denoted by $x_\alpha \uparrow (x_\alpha \uparrow x)$. The definition and notation for a decreasing net (decreasing to x) is analogous.

A linear subspace K of L is called:

a *sublattice* if $x \vee y \in K$ for all $x, y \in K$,

an *ideal* if it is *solid*, i.e., $|x| \leqslant |y|$ for some $y \in K$ imply $x \in K$,

a *band* if it is an ideal, and if A is a subset of K having a supremum in L then $\sup A \in K$.

Clearly, every ideal (band) is a sublattice. Moreover, $A^{\perp}$ is a band for an arbitrary set $A \subset L$.

If $I \subset L$ is an ideal then, introducing in the quotient space L/I the order relation

$$[x] \leqslant [y] \quad \text{iff} \quad \exists z_1, z_2 \in I \quad x+z_1 \leqslant y+z_2,$$

we conclude that $(L/I, \leqslant)$ is a vector lattice.

Let $K \subset L$ be a vector sublattice. We will say that

K is *full* in L if for every $x \in L$ there exists a $y \in K$ with $|x| \leqslant y$,

K is *order dense* in L if for every $0 < x \in L$ there exists a $y \in K$ with $0 < y \leqslant x$,

K is *super order dense* in L if for every $0 < x \in L$ there exists a sequence $(x_n) \subset K$ such that $0 \leqslant x_n \uparrow x$ in L,

K is *regular* in L if every subset of K having in K a least upper bound (a greatest lower bound) has the same least upper bound (greatest lower bound) in L.

Reducing our considerations to Archimedean vector lattices, we can introduce order density in an equivalent way:

THEOREM A.3. *For a vector sublattice K of an Archimedean vector lattice L the following statements are equivalent*:

(i) *K is order dense in L.*

(ii) *If $x \in L_+$ then $x = \sup\{y \in K: 0 \leqslant y \leqslant x\}$.*

(iii) *If $x \in L_+$ then there exists a net $(x_\alpha) \subset K$ with $0 \leqslant x_\alpha \uparrow x$.*

Proof. See [1] Theorem 1.14, p. 8.

A connection between the order density of a sublattice and its regularity gives

THEOREM A.4. *Every sublattice K order dense in a vector lattice L is regular in L. Moreover, if K is* DC *then K is an ideal in L.*

Proof. See [1], Theorem 1.10, p. 6 and Theorem 2.2, p. 10.

For every non-empty set $A \subset L$ there exists a smallest ideal (band) containing A. This ideal (band) is called an *ideal (band) generated by a set A*. The ideal generated by a set A consists of elements $x \in L$ such that

$$|x| \leqslant \sum_{i=1}^{n} t_i |a_i|,$$

where $t_i \in [0, \infty)$, $a_i \in A$ and n's are natural numbers. The form of a band generated by a subset of an Archimedean vector lattice is also known:

THEOREM A.5. *The following statements are equivalent*:

(i) *L is an Archimedean vector lattice.*

(ii) *The band generated by a set $A \subset L$ is $A^{\perp\perp}$.*

Proof. See [18], Theorem 22.3, p. 114.

A band $K \subset L$ is a *projection band* if $L = K \oplus K^{\perp}$. The projection P_K of L onto K has the form:

$$P_K(x) = \sup\{y \in K: 0 \leqslant y \leqslant x\} \quad \text{for} \quad x \in L_+.$$

In particular, if a projection band is generated by a single point set $\{y\}$, then the projection onto this band will be denoted by P_y and we obtain

$$P_y(x) = \sup_n x \wedge n|y| \quad \text{for} \quad x \in L_+.$$

A family $\{x_\alpha.\ \alpha \in \mathscr{A}\} \subset L_+ \setminus \{0\}$ having the properties:

(i) $x_\alpha \perp x_\beta$ for $\alpha \neq \beta$,

(ii) $x_\alpha \wedge |x| = 0$ for all $\alpha \in \mathscr{A}$ imply $x = 0$,

is called a *complete disjoint system* of L. Using the Kuratowski–Zorn Lemma, we can prove the existence of a complete disjoint system in every vector lattice.

THEOREM A.6. *Let* $\{x_\alpha: \alpha \in \mathscr{A}\}$ *be a complete disjoint system of* L. *Then for every* $x \in L_+$

$$x = \sup_\alpha P_{x_\alpha}(x).$$

Proof. See [1], Theorem 2.15, p. 16.

An element $x \in L_+ \setminus \{0\}$ is an *atom* if $|y| \leqslant x$ implies $y = tx$ for some real number t. It is evident that x is an atom iff the ideal (and band) generated by $\{x\}$ is the subspace $\operatorname{lin}\{x\}$. A *discrete vector lattice* is a vector lattice L having the following property: for every $x \in L_+ \setminus \{0\}$ there exists an atom y belonging to $[0, x]$. Note that L is discrete iff L has a complete disjoint system consisting of atoms. A vector lattice without atoms is called *non-atomic*. A band A generated by the set of all atoms of L is a discrete lattice. In particular, for DC vector lattices we have a decomposition $L = A \oplus A^{\perp}$ into a discrete and a non-atomic part.

In further considerations the concept of a weak unit will often be used. A *weak unit* is an element $e \in L_+ \setminus \{0\}$ such that $e \wedge |x| = 0$ implies $x = 0$. In other words, $e \in L_+$ is a weak unit if the band generated by the set $\{e\}$ is the whole vector lattice L. From the form of the projection onto a band generated by a single point set we can easily deduce that

$$x = \sup_n x \wedge ne \quad \text{for every} \quad x \in L_+.$$

If $x \in L_+$, then $C(x)$ will denote the set of *components* of x, i.e., $C(x) = \{p \in L_+: p \wedge (x - p) = 0\}$. The following approximation theorem is very useful:

THEOREM A.7 (Freudenthal's spectral theorem). *Let* L *be a vector lattice such that every band in* L *generated by a single point-set is a projection band.*

Let $e \in L_+$ be a weak unit, $([a_n, b_n])_{n=1}^{\infty}$ a sequence of intervals in $\boldsymbol{R}$ such that $0 \geqslant a_n \downarrow -\infty$, $0 \leqslant b_n \uparrow \infty$. Let $\Pi_n = \{\alpha_{n,0}, \alpha_{n,1}, \ldots, \alpha_{n,m_n}\}$ be a partition of the interval $[a_n, b_n]$ and let Π_{n+1} restricted to $[a_n, b_n]$ be finer than Π_n. Suppose moreover that the diameters $\delta(\Pi_n)$ of the partitions Π_n decrease to zero. For a fixed $x \in L$ and $r \in \boldsymbol{R}$ denote $P_r = P_{(re-x)^+}(e)$. Put

$$s_n = \sum_{k=1}^{m_n} \alpha_{n,k-1}(P_{\alpha_{n,k}} - P_{\alpha_{n,k-1}}).$$

Then there exists a sequence $v_n \downarrow 0$ with $|x - s_n| \leqslant v_n$. Moreover, $s_n^+ \uparrow x^+$

Proof. See [18], Theorem 40.3, p. 258.

The above theorem states in particular that every element $x \in L_+$ is the least upper bound of some increasing sequence $(s_n) \subset \operatorname{lin} C(e)$. It is worth while to note the following equality:

$$\operatorname{lin} C(e) = \{\sum_{i=1}^{n} a_i p_i\colon\ p_i \in C(e),\ p_i \perp p_j \text{ for } i \neq j,\ \bigvee_{i=1}^{n} p_i = e,\ a_i \in \boldsymbol{R},\ n \in \boldsymbol{N}\}.$$

Note also that $C(e)$ is a Boolean algebra under the operations $\vee$, $\wedge$, $-$, where $-p = e - p$. Moreover, if L is DC then $C(e)$ is a complete Boolean algebra, and so its Stone space is extremally disconnected (i.e., the closure of every open set is open).

Let L, M be vector lattices. An operator $T\colon L \to M$ is *positive* if $T(L_+) \subset M_+$. Every additive and positive homogeneous operator $T_0\colon L_+ \to M_+$ has a unique extension to a linear operator $T\colon L \to M$. This extension is defined by the formula $T(x) = T_0(x^+) - T_0(x^-)$.

THEOREM A.8. *For a linear mapping $T\colon L \to M$ the following statements are equivalent:*

(i) $T(x \wedge y) = T(x) \wedge T(y)$.
(ii) $T(x \vee y) = T(x) \vee T(y)$.
(iii) $T(x \vee y) = T(x) \vee T(y)$ *if* $x \wedge y = 0$.
(iv) $x \wedge y = 0$ *implies* $T(x) \wedge T(y) = 0$.
(v) $|T(x)| = T(|x|)$.

Proof. See [18], § 18.

A linear operator $T\colon L \to M$ satisfying any condition of the above theorem will be called a *homomorphism*. The quotient map $Q\colon L \to L/I$, where I is an ideal, is a homomorphism. A homomorphism T is a *normal homomorphism* (*σ-homomorphism*) if the equality $x = \sup_{\alpha} x_\alpha$ ($x = \sup_n x_n$) implies $T(x) = \sup_{\alpha} T(x_\alpha)$ ($T(x) = \sup_n T(x_n)$). For example, every projection onto a projection band is a normal homomorphism. An *isomorphism* is a one-to-one homomorphism (we do not assume that an isomorphism is onto).

An isomorphism mapping a vector lattice onto an ideal in another vector lattice is a normal homomorphism. Two vector lattices L and M are called *isomorphic* if there exists an isomorphism from L onto M.

The simple fact mentioned below will be useful for our purposes:

THEOREM A.9. *Let L_1, L_2, L_3 be vector lattices and operators T: $L_1 \to L_2$, U: $L_2 \to L_3$ be such isomorphisms that $T(L_1)$ and $U(L_2)$ are super order dense ideals in L_2 and L_3, respectively. Then $UT(L_1)$ is a super order dense ideal in L_3.*

Proof. Clearly $UT(L_1)$ is an ideal. Suppose $z \in L_3$, $z \geqslant 0$ and $(y_n) \subset L_2$ is a sequence with $U(y_n) \uparrow z$. From the assumption it follows that there exists a family of sequences $(x_n^k) \subset L_1$ such that $T(x_n^k) \uparrow y_n$ for all n. Putting $w_n = \bigvee_{k=1}^{n} x_k^n$, we obtain $UT(w_n) \uparrow z$.

We will often use the possibility of embedding an Archimedean vector lattice into some DC vector lattice. More precisely:

THEOREM A.10. *Let L be an Archimedean vector lattice. Then there exists a* DC *vector lattice $\hat{L}$ (unique up to an isomorphism) containing a full order dense sublattice isomorphic to L. This sublattice is identified with L.*

Proof. See [1], Theorem 2.3, p. 10.

The unique DC vector lattice $\hat{L}$ is called the *Dedekind completion* of L.

In the theory of vector lattices a special role is played by the space $C_\infty(S)$ consisting of continuous functions f mapping the topological space S into the extended real line for which the open sets $\{s \in S: |f(s)| < \infty\}$ are dense in S. If the space S is extremally disconnected, then $C_\infty(S)$ forms a DC vector lattice under the ordinary operations (for details see [34], p. 133–142). The following representation theorem holds:

THEOREM A.11. *Every* DC *vector lattice L is isomorphic to an order dense ideal in $C_\infty(S)$, where S is some compact and extremally disconnected topological space. Moreover, if L has a weak unit e then an isomorphism can be chosen in such way that it maps this unit onto the function identically one on S (and thus $\operatorname{lin} C(e)$ is mapped onto a subspace consisting of continuous simple functions).*

Proof. See [34], Theorem V.4.1, p. 151, and Theorem V.4.2, p. 155.

The preceding theorem will remain true also in a more general form: every Archimedean vector lattice is isomorphic to an order dense sublattice in $C_\infty(S)$, where S is a compact extremally disconnected topological space. Indeed, L is isomorphic to some order dense sublattice in the Dedekind completion $\hat{L}$ on L and $\hat{L}$ is isomorphic to an order dense ideal in $C_\infty(S)$.

A space $C_\infty(S)$, (where S is extremally disconnected), belongs to the class of so-called *universally complete vector lattices* (i.e., DC vector lattices with a property that every subset of disjoint positive elements has a supremum) — see [18], Theorem 47.4, p. 323.

A universal completion of a vector lattice L is a universally complete vector lattice M having an order dense sublattice isomorphic to L. Thus every Archimedean vector lattice has a universal completion. Since any two universal completions of a vector lattice are isomorphic ([1], Theorem 23.18, p. 173), we can talk about the *universal completion* of an Archimedean vector lattice. The universal completion of a vector lattice L will be denoted by L^u.

A subset $A \subset L_+$ is *dominable* if

$$\forall u \in L_+ \setminus \{0\}\ \exists v \in L_+ \setminus \{0\}\ \exists k \in \mathbf{N}\ \forall a \in A\ (ku-a)^+ \geqslant v.$$

We will need the following characterization of dominable sets:

THEOREM A.12. *Let L be an Archimedean vector lattice and let A be a non-empty subset of L_+. Then the following statements are equivalent:*

(i) *A is a dominable subset.*

(ii) *A is order bounded in L^u.*

Proof. See [1], Theorem 23.22, p. 176.

Presentation of some facts from the theory of locally solid vector lattices will end the first part of the preliminaries.

A linear topology τ on a vector lattice is said to be locally solid if τ has a basis Γ for zero consisting of solid sets (i.e., $V \in \Gamma$, $|x| \leqslant |y|$ and $y \in V$ imply $x \in V$). A *locally solid vector lattice* (L, τ) is a vector lattice L equipped with a locally solid topology τ. A locally solid vector lattice (L, τ) has the *σ-Levi property* if every increasing τ-bounded sequence of elements of L_+ has a supremum in L.

THEOREM A.13. *Let (L, τ), (M, η) be two locally solid vector lattices and let $T\colon L \to M$ be an isomorphism such that $T(L)$ is super order dense in M. If (L, τ) has the σ-Levi property and T is a linear homeomorphism then $T(L) = M$.*

Proof. First we will show that T is a normal homomorphism. To do this let $A \subset L$ and $x = \sup A$. Moreover, let z be an upper bound of $T(A)$. Suppose $T(x) - (T(x) \wedge z) > 0$. Then, according to the assumptions, we can choose an element $v \in L_+ \setminus \{0\}$ such that

$$0 < T(v) \leqslant T(x) - (T(x) \wedge z).$$

We have for an arbitrary $a \in A$:

$$0 < T(v) \leqslant T(x) - (T(x) \wedge z) \leqslant T(x) - T(a) = T(x-a),$$

and so $v + a \leqslant x$ for all a. Therefore $v + x \leqslant x$, a contradiction because $v > 0$. Thus $T(x) = T(x) \wedge z \leqslant z$. Finally, $T(x) = \sup\{T(a)\colon a \in A\}$. Since T is a normal homomorphism and a homeomorphism, it follows that $(T(L), \eta|T(L))$ has the σ-Levi property. Let $u \in M_+$. There exists a sequence $(u_n) \subset T(L)$ increasing to u. Thus (u_n) is η-bounded, and so the u_n's have the supremum u_0 in $T(L)$. We have $u = u_0$ because $T(L)$ is regular in M (see Theorem A.4).

A *monotone F-norm* is a functional $|||\cdot|||: L \to [0, \infty)$ satisfying the following conditions:

(i) $|||x||| = 0$ iff $x = 0$.

(ii) $|x| \leqslant |y|$ implies $|||x||| \leqslant |||y|||$.

(iii) $\lim_{t \to \infty} |||tx||| = 0$.

(iv) $|||x+y||| \leqslant |||x||| + |||y|||$.

A vector lattice L together with a monotone F-norm will be called an *F*-lattice* (or a *normed lattice* if $|||\cdot|||$ is a norm). A topologically complete F^*-lattice will be called an *F-lattice* (*Banach lattice*).

A monotone F-norm $|||\cdot|||$ is *σ-order continuous* (*order continuous*) if $x_n \downarrow 0$ implies $\inf_n |||x_n||| = 0$ ($x_\alpha \downarrow 0$ implies $\inf_\alpha |||x_\alpha||| = 0$).

THEOREM A.14. *For every F-lattice* $(L, |||\cdot|||)$ *the following statements are equivalent*:

(i) *L is* SDC *and* $|||\cdot|||$ *is order continuous.*

(ii) L is σ-DC and $|||\cdot|||$ is σ-order continuous.

(iii) Every increasing order bounded sequence is topologically convergent.

(iv) Every order bounded sequence of positive disjoint elements tends topologically to zero.

Proof. Repeating verbatim the proof of Theorem 3.4 from [4] p. 85, we will get the equivalence of conditions (i), (ii), (iii). The equivalence of conditions (iii) and (iv) is included in Theorem 10.1 [1] p. 64.

B. Measure theory

Let (S, Σ, μ) be a measure space. $N(\mu)$ will denote the family of μ-negligible sets, i.e., $N(\mu) = \{A \in \Sigma: \mu(A) = 0\}$.

A set $A \in \Sigma$ is an *atom* of a measure μ if $A \notin N(\mu)$ and for every $B \in \Sigma \cap A$ $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. If Σ does not contain atoms of μ then such a measure μ is called *non-atomic*. A measure μ is said to be *purely atomic* if every set from Σ with a positive measure contains an atom of μ. For every σ-finite measure μ there exists a set $S_0 \in \Sigma$ such that $\mu|\Sigma \cap (S \setminus S_0)$ is non-atomic and $\mu|\Sigma \cap S_0$ is purely atomic. It is well to remember that a real-valued measurable function is constant almost everywhere on every atom.

A measure μ is *semi-finite* if $\forall A \in \Sigma \setminus N(\mu)\ \exists B \in \Sigma \cap A\ 0 < \mu(B) < \infty$.

In the case of a finite measure the set S is a finite sum of disjoint sets which are either atoms or have a measure smaller than a fixed number. More precisely:

SAKS' LEMMA. *Let* $\mu(S) < \infty$. *For every number* $\varepsilon > 0$ *there exist disjoint sets* $A_1, \ldots, A_n$ *such that*:

(i) $\bigcup_{k=1}^{n} A_k = S$.

(ii) *For every* $i \in \{1, \ldots, n\}$ $\mu(A_i) < \varepsilon$ *or* A_i *is a* μ*-atom and* $\mu(A_i) \geqslant \varepsilon$.

Proof. See [31], p. 220.

LAPUNOV'S THEOREM (one-dimensional case). *Let* μ *be a finite non-atomic measure defined on a* σ*-algebra. Then*

$$\forall a \in [0, 1] \; \exists A \in \Sigma \quad \mu(A) = a\mu(S).$$

Proof. See [36], Theorem 5.6.2, p. 52.

Let $(S_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in \mathscr{A}$, be a family of measure spaces. We can assume that $S_\alpha \cap S_\beta = \emptyset$. Put $S = \bigcup_\alpha S_\alpha$, $\Sigma = \{A \subset S\colon A \cap S_\alpha \in \Sigma_\alpha \text{ for all } \alpha \in \mathscr{A}\}$ and define a function $\mu\colon \Sigma \to [0, \infty]$ by the formula

$$\mu(A) = \sum_\alpha \mu_\alpha(A \cap S_\alpha).$$

(S, Σ, μ) is a measure space called the *direct sum of measure spaces* $(S_\alpha, \Sigma_\alpha, \mu_\alpha)$. The space of all real-valued measurable functions μ-almost everywhere finite will be denoted by $L^0(S, \Sigma, \mu)$ (functions equal μ-almost everywhere are identified, of course). This space is a σ-DC vector lattice with respect to the natural "pointwise" order: $f \leqslant g$ if and only if $f(s) \leqslant g(s)$ μ-almost everywhere.

Moreover:

THEOREM B.1. *If* μ *is a finite measure then the space* $L^0(S, \Sigma, \mu)$ *– with the topology of convergence in measure – has the* σ*-Levi property.*

Proof. See [11], Lemma 1, p. 97.

A *support* of a measurable function f is the set $\operatorname{supp} f = \{s \in S\colon f(s) \neq 0\}$. The characteristic function of a set A will be denoted by 1_A.

Let $\mathscr{U}$ be a Boolean algebra. A function $\nu\colon \mathscr{U} \to [0, \infty]$ will be called a *Boolean measure* if ν satisfies the following conditions:

(i) $\nu(a) = 0$ iff $a = 0$.

(ii) $\nu(a \cup b) = \nu(a) + \nu(b)$ if $a \cap b = 0$.

(iii) If $a_n \uparrow a$, then $\nu(a) = \lim_{n \to \infty} \nu(a_n)$.

To every measure there corresponds some Boolean measure. Indeed, let (S, Σ, μ) be a measure space and $\sim$ the following equivalence relation in Σ:

$$A \sim B \quad \text{iff} \quad A \dot{-} B \in N(\mu)$$

($\dot{-}$ denotes the symmetric difference). Denote by $\Sigma/N(\mu)$ the set of equivalence classes and by $[A]$ the equivalence class of A. Then $\Sigma/N(\mu)$ with operations $\dot{\cup}$, $\dot{\cap}$, Δ given by

$$[A] \dot{\cup} [B] = [A \cup B], \quad [A] \dot{\cap} [B] = [A \cap B], \quad \Delta[A] = [S \setminus A]$$

is a Boolean algebra and the function $\dot{\mu}$: $\Sigma/N(\mu) \to [0, \infty]$ defined by the formula $\dot{\mu}([A]) = \mu(A)$ is a Boolean measure. The pair $(\Sigma/N(\mu), \dot{\mu})$ is called a *measure algebra* of the measure space (S, Σ, μ).

A measure space (S, Σ, μ) is *Maharam* if:

(i) μ is semi-finite.
(ii) $\Sigma/N(\mu)$ is a complete Boolean algebra.

In particular, every measure space with a σ-finite measure and the direct sum of spaces $(S_\alpha, \Sigma_\alpha, \mu_\alpha)$ where $\mu_\alpha(S_\alpha) < \infty$ are both Maharam.

THEOREM B.2. *The following statements are equivalent*:

(i) (S, Σ, μ) *is Maharam.*
(ii) $L^0(S, \Sigma, \mu)$ *is a* DC *vector lattice.*

Proof. See [9] Theorem 64 B p. 170.

Let G be a subset of $L^0(S, \Sigma, \mu)$ with (S, Σ, μ) being Maharam. Then $\sup\{\operatorname{supp} g\colon g \in G\}$ exists in $\Sigma/N(\mu)$ and this element will be called the *support of* G (the symbol $\operatorname{supp} g$ means here the equivalence class of elements $\operatorname{supp} h$, where h equals g μ-almost everywhere).

Let S be a compact extremally disconnected topological space, Σ_0 the algebra of closed-open subsets of S, and $\mathcal{N}$ the family of sets of the first category in S. The algebra Σ_0 with standard operations is a complete Boolean algebra and for every family $(A_\alpha) \subset \Sigma_0$ $\sup_\alpha A_\alpha = \overline{\bigcup_\alpha A_\alpha}$. Moreover, let μ_0 be a semi-finite Boolean measure on Σ_0. Then the family

$$(*) \qquad \Sigma = \{A \div N\colon A \in \Sigma_0,\ N \in \mathcal{N}\}$$

is a σ-algebra containing the σ-algebra of Borel sets and the function μ: $\Sigma \to [0, \infty]$ defined by the formula

$$(**) \qquad \mu(A \div N) = \mu_0(A)$$

is a complete measure on Σ with $N(\mu) = \mathcal{N}$ (these problems are presented in detail in [11], p. 373–375).

THEOREM B.3. *Let S be a compact extremally disconnected topological space, Σ_0 the algebra of closed-open subsets of S, and μ_0 a semi-finite Boolean measure on Σ_0. Let Σ and μ be defined by equalities (*) and (**), respectively. Then the spaces $C_\infty(S)$ and $L^0(S, \Sigma, \mu)$ are isomorphic by the mapping $f \to [f]$, where $[f]$ denotes the equivalence class of f.*

Proof. See [11], Corollary on page 375.

For measurable sets A_n $(n = 0, 1, 2, \ldots)$ the notation $A_n \uparrow A_0$ will have the following meaning: $\mu(A_n \setminus A_{n+1}) = 0$ for all $n \in N$ and $\mu(A_0 \setminus \bigcup_{n=1}^{\infty} A_n) = 0$.

Musielak–Orlicz spaces

1. Some properties of Musielak–Orlicz spaces

The Musielak–Orlicz spaces are of special importance in our considerations. Many papers, often difficult to get at, are devoted to these spaces, but there is no monograph containing fundamental information about Musielak–Orlicz spaces. This is the main reason why the order properties of Musielak–Orlicz spaces are presented here more precisely and most of the theorems are proved (though they are known). Another reason is the following: we admit Musielak–Orlicz functions taking infinite values and we consider Musielak–Orlicz spaces over arbitrary measure spaces (in general we drop the assumption of σ-finiteness of a measure, which is usually found in the literature).

Some theorems, after appropriate reformulation, remain true also for some classes of locally solid Riesz spaces, for example Lemma 2.1, however, restricting our considerations to Musielak–Orlicz spaces, we have often been able to simplify the proofs and make them clearer.

Let (S, Σ, μ) be a measure space. A function

$$\psi\colon [0, \infty) \times S \to [0, \infty]$$

is called a *Musielak–Orlicz function* if it satisfies the following two conditions:

(ψ1) For every $s \in S$ the function $\psi(\cdot, s)\colon [0, \infty) \to [0, \infty]$ is non-decreasing, left continuous, continuous at zero and $\psi(r, s) = 0$ iff $r = 0$.

(ψ2) For every $r \in [0, \infty)$ the function $\psi(r, \cdot)\colon S \to [0, \infty]$ is Σ-measurable.

A Musielak–Orlicz function ψ determines a functional $M_\psi\colon L^0(S, \Sigma, \mu) \to [0, \infty]$ defined by the formula

$$M_\psi(f) = \int_S \psi(|f(s)|, s)\, d\mu.$$

We will often write briefly M instead of M_ψ, and M will be called an *integral modular*.

The subspace $L^\psi(S, \Sigma, \mu)$ of $L^0(S, \Sigma, \mu)$ given in the following way:

$$L^\psi(S, \Sigma, \mu) = \{g \in L^0(S, \Sigma, \mu)\colon \exists\, t > 0 \;\; M_\psi(tg) < \infty\}$$

will be called a *Musielak–Orlicz space*. If μ is purely atomic, taking the value one on every atom, then the Musielak–Orlicz space associated with such a measure space will be denoted by $l^\psi(\Gamma)$ where Γ is a set having the same cardinal number as the set of all atoms of μ. If the case $\Gamma = N$ we will briefly write l^ψ.

If $\psi(r, s) = \psi(r, t)$ for all $s, t \in S$ then the function ψ is called an *Orlicz function*. An Orlicz function will usually be denoted by the letter φ.

A Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$ is an ideal in $L^0(S, \Sigma, \mu)$.

Moreover, $L^{\psi}(S, \Sigma, \mu)$ is SDC. Indeed, $L^{\psi}(S, \Sigma, \mu)$ is clearly σ-DC and every order bounded family of disjoint elements of $L^{\psi}(S, \Sigma, \mu)$ must be at most countable, because a function from $L^{\psi}(S, \Sigma, \mu)$ which is an upper bound of this family has a support of σ-finite measure. Now it is enough to apply Theorems A.1 and A.2.

The space $L^{\psi}(S, \Sigma, \mu)$ is complete with respect to the following F-norm:

$$|||g||| = \inf\{a > 0: M_{\psi}(g/a) \leqslant a\}.$$

We can prove this fact immediately or refer to the theorem mentioned below.

THEOREM 1.1. *Every Musielak–Orlicz space has the σ-Levi property.*

Proof. Let (g_n) be an increasing and topologically bounded sequence in $L^{\psi}_{+}(S, \Sigma, \mu)$. Since the set $A = \bigcup_{n=1}^{\infty} \operatorname{supp} g_n$ has a σ-finite measure, we can assume that μ is σ-finite because it is enough to show the existence of the least upper bound of our sequence in $L^{\psi}(A, \Sigma \cap A, \mu|\Sigma \cap A)$. Let $S = \bigcup_{i=1}^{\infty} S_i$, $\mu(S_i) < \infty$, $S_i \cap S_j = \emptyset$ for $i \neq j$. For every natural number k the sequence $(g_n 1_{S_k})$ is topologically bounded in the space $L^0(S_k, \Sigma \cap S_k, \mu|\Sigma \cap S_k)$ equipped with the topology of convergence in measure (see [7], Theorem 3.6, p. 802), and so the set $\{g_n 1_{S_k}: n \in \mathbb{N}\}$ has a least upper bound $\bar{g}_k$ in $L^0(S_k, \Sigma \cap S_k, \mu|\Sigma \cap S_k)$. Put $g = \sup_k \bar{g}_k$. Then $g = \sup_n g_n$ in $L^0(S, \Sigma, \mu)$.

The topological boundedness of the sequence (g_n) implies in particular that $c = \sup_n |||g_n||| < \infty$. Therefore, using the dominated convergence theorem, we have

$$M_{\psi}(g/2c) = \lim_{n \to \infty} M_{\psi}(g_n/2c) \leqslant 2c < \infty,$$

and so $g \in L^{\psi}(S, \Sigma, \mu)$.

Let $L^{\psi_\alpha}(S_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in \mathscr{A}$, be a family of Musielak–Orlicz spaces and let (S, Σ, μ) be the direct sum of measure spaces $(S_\alpha, \Sigma_\alpha, \mu_\alpha)$. This family generates a certain Musielak–Orlicz space. Indeed, put

$$\psi(r, s) = \psi_\alpha(r, s) \quad \text{for} \quad s \in S_\alpha$$

(we can of course assume that $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$). Then the Musielak–Orlicz space $L^{\psi}(S, \Sigma, \mu)$ is isomorphic to the space

$$\bigoplus_\alpha L^{\psi_\alpha}(S_\alpha, \Sigma_\alpha, \mu_\alpha) = \{(g_\alpha) \in \prod_\alpha L^{\psi_\alpha}(S_\alpha, \Sigma_\alpha, \mu_\alpha): \exists\, t > 0 \ \sum_\alpha M_{\psi_\alpha}(t g_\alpha) < \infty\}$$

with "coordinatewise" order. The operator $T: \bigoplus_\alpha L^{\psi_\alpha}(S_\alpha, \Sigma_\alpha, \mu_\alpha) \to L^{\psi}(S, \Sigma, \mu)$

(called the *union operator*) defined by the formula

$$(T(g_\alpha))(s) = g_\alpha(s) \quad \text{for} \quad s \in S_\alpha$$

is an isomorphism "onto". Moreover, $M_\psi(T(g_\alpha)) = \sum_\alpha M_{\psi_\alpha}(g_\alpha)$.

Let $L_f^\psi(S, \Sigma, \mu) = \{g \in L^\psi(S, \Sigma, \mu)\colon \forall t > 0\ M_\psi(tg) < \infty\}$. The subspace $L_f^\psi(S, \Sigma, \mu)$ is a closed ideal in $L^\psi(S, \Sigma, \mu)$ and the F-norm $|||\cdot|||$ restricted to $L_f^\psi(S, \Sigma, \mu)$ is order continuous. If ψ takes only finite values, then $L_f^\psi(S, \Sigma, \mu) = L_a^\psi(S, \Sigma, \mu)$ where $L_a^\psi(S, \Sigma, \mu) = \{g \in L^\psi(S, \Sigma, \mu)\colon |g| \geqslant g_n \downarrow 0 \Rightarrow |||g_n||| \to 0\}$ (see Corollary 1.4). In general the equality $L_f^\psi(S, \Sigma, \mu) = L_a^\psi(S, \Sigma, \mu)$ does not hold, as the following example shows:

EXAMPLE. Put $S = N$, $\Sigma = 2^N$, $\mu(\{n\}) = 1$,

$$\psi(r, n) = \begin{cases} r & \text{for} \quad (r, n) \in [0, \infty) \times \{2, 3, 4, \ldots\}, \\ r^2/(1-r) & \text{for} \quad (r, n) \in [0, 1) \times \{1\}, \\ \infty & \text{for} \quad (r, n) \in [1, \infty) \times \{1\}. \end{cases}$$

Then $l^\psi = l_a^\psi = \{x\colon x = (x(n))_{n \in N}, (x(n))_{n \geqslant 2} \in l^1\}$ and $l_f^\psi = \{x\colon x = (x(n))_{n \in N}, x(1) = 0, (x(n))_{n \geqslant 2} \in l^1\}$.

If a Musielak–Orlicz function ψ satisfies the *generalized* (Δ_2)*-condition*:

$$\exists k > 0\ \exists g \in L^1(S, \Sigma, \mu)\ \exists A \in N(\mu)\ \forall r \geqslant 0\ \forall s \in S \setminus A \quad \psi(2r, s) \leqslant k\psi(r, s) + g(s),$$

then $L_f^\psi(S, \Sigma, \mu) = L^\psi(S, \Sigma, \mu)$.

We will say that an Orlicz function φ satisfies:

(Δ_2^0)*-condition* if $\exists K > 0\ \exists r_0 > 0\ \forall r \in [0, r_0]\ \varphi(2r) \leqslant K\varphi(r)$,

(Δ_2^∞)*-condition* if $\exists K > 0\ \exists r_0 > 0\ \varphi(r_0) < \infty\ \forall r \geqslant r_0\ \varphi(2r) \leqslant K\varphi(r)$,

(Δ_2)*-condition* if $\exists K > 0\ \forall r \geqslant 0\ \varphi(2r) \leqslant K\varphi(r)$.

If φ satisfies the (Δ_2^0)-condition ((Δ_2^∞)-condition and $\mu(S) < \infty$), then $l_f^\varphi = l^\varphi$ ($L_f^\varphi(S, \Sigma, \mu) = L^\varphi(S, \Sigma, \mu)$). If (S, Σ, μ) is an arbitrary measure space and φ satisfies the (Δ_2)-condition, then $L_f^\varphi(S, \Sigma, \mu) = L^\varphi(S, \Sigma, \mu)$.

For an Orlicz function φ satisfying the (Δ_2)-condition the *majorizing function* $\Phi\colon [0, \infty) \to [0, \infty)$ is defined by

$$\Phi(t) = \sup_{r > 0} \frac{\varphi(rt)}{\varphi(r)}.$$

The function Φ is left continuous at 1 and non-decreasing. If φ is convex (concave), then Φ is also convex (concave).

In further considerations we will use the following lemma:

LEMMA 1.2. *Let* (S, Σ, μ) *be a finite measure space and* $\psi\colon [0, \infty) \times S \to [0, \infty)$ *a Musielak–Orlicz function. Then*

$$\forall c \in (0, \mu(S))\ \exists B \in \Sigma\ \mu(B) \geqslant \mu(S) - c \text{ and } 1_B \in L_f^\psi(S, \Sigma, \mu).$$

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Proof. Take a number $c \in (0, \mu(S))$ and denote $B_k^n = \{s\colon \psi(n, s) \leqslant k\}$. Since $B_k^n \uparrow S$ for every n, we can choose a sequence of sets $(B_{k_n}^n)$ such that $\mu(S \setminus B_{k_n}^n) < c2^{-n}$. Put $B = \bigcap_{n=1}^{\infty} B_{k_n}^n$. Then

$$\mu(B) = \mu(S) - \mu(S \setminus B) = \mu(S) - \mu\Big(\bigcup_{n=1}^{\infty} S \setminus B_{k_n}^n\Big)$$

$$\geqslant \mu(S) - \sum_{n=1}^{\infty} \mu(S \setminus B_{k_n}^n) \geqslant \mu(S) - c.$$

Let $r > 0$ be an arbitrary number let n be a natural number greater than r. Then

$$M_\psi(r1_B) \leqslant M_\psi(n1_{B_{k_n}^n}) = \int_{B_{k_n}^n} \psi(n, s)\, d\mu \leqslant k_n \mu(S) < \infty,$$

and so $1_B \in L_f^\psi(S, \Sigma, \mu)$.

Now we will present a theorem on an order property of $L_f^\psi(S, \Sigma, \mu)$.

THEOREM 1.3. *Let* $A_\infty = \bigcup_{n=1}^{\infty} \{s\colon \psi(n, s) = \infty\}$. *Then the following statements are equivalent*:

(i) $L_f^\psi(S, \Sigma, \mu)$ *is super order dense in* $L^\psi(S, \Sigma, \mu)$.
(ii) $\mu(A_\infty \cap \operatorname{supp} g) = 0$ *for every* $g \in L^\psi(S, \Sigma, \mu)$.
(ii') $\mu(A_\infty) = 0$ *or* $\mu(\Sigma \cap A_\infty) = \{0, \infty\}$.
(iii) $\forall g \in L^\psi(S, \Sigma, \mu)\ \exists (A_n) \subset \Sigma\ A_n \uparrow \operatorname{supp} g$ *and* $1_{A_n} \in L_f^\psi(S, \Sigma, \mu)$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $\mu(A_\infty \cap \operatorname{supp} g) > 0$ for some element $g \in L_+^\psi(S, \Sigma, \mu)$. Since there exists a sequence $(g_n) \subset L_f^\psi(S, \Sigma, \mu)$ increasing to g, we can find a number n_0 such that $\mu(A_\infty^{n_0} \cap \operatorname{supp} g_{n_0}) > 0$ where $A_\infty^n = \{s\colon \psi(n, s) = \infty\}$. Let $C_k = \{s \in \operatorname{supp} g_{n_0} \cap A_\infty^{n_0}\colon g_{n_0}(s) \geqslant n_0 k^{-1}\}$. Then $C_k \uparrow \operatorname{supp} g_{n_0} \cap A_\infty^{n_0}$; thus $\mu(C_{k_0}) > 0$ for some natural number k_0. We have

$$M_\psi(k_0 g_{n_0}) \geqslant M_\psi(k_0 g_{n_0} 1_{C_{k_0}}) \geqslant M_\psi(n_0 1_{C_{k_0}}) = \int_{C_{k_0}} \psi(n_0, s)\, d\mu = \infty,$$

and so $g_{n_0} \notin L_f^\psi(S, \Sigma, \mu)$ — a contradiction.

(ii) $\Rightarrow$ (iii). Let $g \in L^\psi(S, \Sigma, \mu)$. The set $\operatorname{supp} g$ has a σ-finite measure; therefore there exists a finite measure ν defined on $\Sigma \cap \operatorname{supp} g$ and equivalent to $\mu | \Sigma \cap \operatorname{supp} g$, i.e., $\nu(A) = 0$ iff $\mu(A \cap \operatorname{supp} g) = 0$. Since $g \in L^\chi(\operatorname{supp} g, \Sigma \cap \operatorname{supp} g, \nu)$, where $\chi = \psi\, d\mu/d\nu$, using Lemma 1.2 we will find sets $B_n \in \Sigma \cap \operatorname{supp} g$ having the following properties:

$$(+) \qquad \nu(B_n) \geqslant \nu(\operatorname{supp} g) - 2^{-n},$$

$$(++) \qquad 1_{B_n} \in L_f^\chi(\operatorname{supp} g, \Sigma \cap \operatorname{supp} g, \nu).$$

Put $A_k = \bigcup_{n=1}^{k} B_n$, $A_0 = \operatorname{supp} g \setminus \bigcup_{n=1}^{\infty} A_n$. We obtain $A_k \cup A_0 \uparrow \operatorname{supp} g$ and $\mu(A_0) = 0$. Moreover

$$\int_{A_k \cup A_0} \psi(r, s)\, d\mu = \int_{A_k} \psi(r, s)\, d\mu = \int_{A_k} \psi(r, s) \frac{d\mu}{dv}(s)\, dv < \infty.$$

Therefore $1_{(A_k \cup A_0)} \in L_f^{\psi}(S, \Sigma, \mu)$.

(iii) $\Rightarrow$ (i). Let $g \in L_+^{\psi}(S, \Sigma, \mu)$. By the assumption there exist measurable sets A_n increasing to $\operatorname{supp} g$ whose characteristic functions belong to $L_f^{\psi}(S, \Sigma, \mu)$. Thus, if (g_n) is a sequence of simple functions increasing to g then $g_n 1_{A_n} \uparrow g$ and $g_n 1_{A_n} \in L_f^{\psi}(S, \Sigma, \mu)$ for all $n \in N$.

(ii) $\Rightarrow$ (ii'). Suppose that A_∞ contains a subset B of positive and finite measure. We have $\mu(B \cap B_{n_0}) > 0$ for some $n_0 \in N$, where $B_n = \{s: \psi(n^{-1}, s) < \infty\}$, because $\psi(n^{-1}, s) \to 0$ if $n \to \infty$ for every s. Applying Egoroff's theorem, we obtain $\psi(n^{-1}, s) \to 0$ if $n \to \infty$ uniformly on some set $C \subset B \cap B_{n_0}$ of positive measure. Let n_1 be so large that $\psi(n_1^{-1}, \cdot)(C) \subset [0, 1)$. Then $M_{\psi}(n_1^{-1} 1_C) \leqslant \mu(C) < \infty$. Thus $1_C \in L^{\psi}(S, \Sigma, \mu) \setminus \{0\}$ and $\mu(A_\infty \cap C) = \mu(C) > 0$ — a contradiction.

(ii') $\Rightarrow$ (ii). Obvious.

COROLLARY 1.4. *If ψ is a Musielak–Orlicz function taking only finite values, then $L_f^{\psi}(S, \Sigma, \mu)$ is super order dense in $L^{\psi}(S, \Sigma, \mu)$ and $L_f^{\psi}(S, \Sigma, \mu) = L_a^{\psi}(S, \Sigma, \mu)$.*

Proof. We need only to show the inclusion $L_a^{\psi}(S, \Sigma, \mu) \subset L_f^{\psi}(S, \Sigma, \mu)$. Let $g \in L_a^{\psi}(S, \Sigma, \mu)$. According to the preceding theorem there exists a sequence $(g_n) \subset L_f^{\psi}(S, \Sigma, \mu)$ increasing to $|g|$. Therefore $\| |g| - g_n \| \to 0$ if $n \to \infty$. Thus $M_{\psi}(r(|g| - g_n)) \to 0$ if $n \to 0$ for every $r \geqslant 0$. Fixing $r > 0$ and taking a sufficiently large n, we will get

$$M_{\psi}(rg) \leqslant M_{\psi}(2r(|g| - g_n)) + M_{\psi}(2rg_n) < 1 + M_{\psi}(2rg_n) < \infty;$$

so $g \in L_f^{\psi}(S, \Sigma, \mu)$.

To formulate the next corollary we will need the following definition: a Hausdorff locally solid topology η on a vector lattice L is called *minimal* if η is coarser than any other Hausdorff locally solid topology τ on L.

COROLLARY 1.5. *Let ψ be a Musielak–Orlicz function taking only finite values. Then the topology of convergence in measure (on the measurable sets of finite measure) is a minimal topology on $L^{\psi}(S, \Sigma, \mu)$.*

Proof. Repeating almost verbatim the arguments of the proof of Theorem 7 ([2], p. 169), we find that the topology of convergence in measure is a minimal topology on $L_f^{\psi}(S, \Sigma, \mu)$. According to Theorem 1.3, $L_f^{\psi}(S, \Sigma, \mu)$ is order dense in $L^{\psi}(S, \Sigma, \mu)$. If A is a set of finite measure then $L^0(A, \Sigma \cap A, \mu|\Sigma \cap A)$ is the universal completion of $L^{\psi}(A, \Sigma \cap A, \mu|\Sigma \cap A)$. Indeed, for every $g \in L_+^0(A, \Sigma \cap A, \mu|\Sigma \cap A)$ there exists a simple function $\bar{g}$

such that $0 < \bar{g} \leqslant g$ and $\mu(\operatorname{supp}\bar{g}) > 2^{-1}\,\mu(\operatorname{supp} g)$. Using Lemma 1.2, we can find a set $B \in \Sigma$ such that $\mu(B) > 2^{-1}\,\mu(\operatorname{supp} g)$ and $1_B \in L_f^{\psi}(A, \Sigma \cap A, \mu|\Sigma \cap A)$. Therefore $\bar{g}1_B \in L_f^{\psi}(A, \Sigma \cap A, \mu|\Sigma \cap A)$ and $0 < \bar{g}1_B \leqslant g$.

To end our proof it suffices to repeat the proof of Theorem 11 ([2], p. 171).

Note that the σ-finiteness of a measure and the topological completeness of $L^{\psi}(S, \Sigma, \mu)$ imply the existence of a weak unit in $L^{\psi}(S, \Sigma, \mu)$. Indeed, every complete disjoint system (x_α) of $L^{\psi}(S, \Sigma, \mu)$ must be at most countable (because μ is σ-finite); so, choosing real numbers $t_\alpha > 0$ such that $\sum_\alpha |||t_\alpha x_\alpha||| < \infty$, we will get the convergence of the series $\sum_\alpha t_\alpha x_\alpha$ (a consequence of the topological completeness). Clearly the element $\sum_\alpha t_\alpha x_\alpha$ is a weak unit in $L^{\psi}(S, \Sigma, \mu)$.

2. Isomorphisms between Musielak–Orlicz spaces

If (S, Σ, μ) is a measure space then the symbol $N(\mu)$ denotes, as usual, the family of μ-negligible sets and $P(\Sigma)$ is a subspace of $L^0(S, \Sigma, \mu)$ consisting of simple functions.

THEOREM 2.1. *Let (S, Σ, μ) and (T, Θ, τ) be measure spaces. If there exists a Boolean isomorphism $\Delta\colon \Sigma/N(\mu) \to \Theta/N(\tau)$ then Δ generates an isomorphism $\bar{\Delta}\colon L^0(S, \Sigma, \mu) \to L^0(T, \Theta, \tau)$ which is defined on $P(\Sigma)$ by the equality*

$$\bar{\Delta}\Big(\sum_{i=1}^{n} a_i 1_{A_i}\Big) = \sum_{i=1}^{n} a_i 1_{\Delta(A_i)}$$

($1_{\Delta(A)}$ denotes the equivalence class of function 1_B where B is an arbitrary element from $\Delta(A)$).

Proof. See [9], p. 158.

In our further considerations we will always assume that the Musielak–Orlicz functions take only finite values and the measures are σ-finite.

Now we will present a theorem on a general form of isomorphisms between Musielak–Orlicz spaces.

THEOREM 2.2. *Let U be an isomorphism mapping $L^{\psi_1}(S, \Sigma, \mu)$ onto $L^{\psi_2}(T, \Theta, \tau)$. Then, for every weak unit $x_0 \in L^{\psi_1}(S, \Sigma, \mu)$, there exists a Boolean isomorphism $\Delta\colon \Sigma/N(\mu) \to \Theta/N(\tau)$ such that*

$$U(x) = U(x_0)\,\bar{\Delta}\left(\frac{x}{x_0}\right)$$

for every $x \in L^{\psi_1}(S, \Sigma, \mu)$, *where* $\bar{\Delta}$ *is the isomorphism between* $L^0(S, \Sigma, \mu)$ *and* $L^0(T, \Theta, \tau)$, *generated by* Δ.

Proof. Let x_0 be a weak unit in $L^{\psi_1}(S, \Sigma, \mu)$. The function $U(x_0)$ is clearly a weak unit in $L^{\psi_2}(T, \Theta, \tau)$. Define an operator $\Delta: \Sigma/N(\mu) \to \Theta/N(\tau)$ by

$$\Delta(A) = \operatorname{supp} U(x_0 1_A)$$

(a set and its equivalence class are denoted by the same letter, but this will not cause any misunderstanding). It is not difficult to verify that Δ is a Boolean isomorphism "onto" (Δ is "onto" because $C(U(x_0)) = U(C(x_0)) = U(\{x_0 1_A: A \in \Sigma\})$). According to Theorem 2.1, Δ generates an isomorphism $\bar{\Delta}$ mapping $L^0(S, \Sigma, \mu)$ onto $L^0(T, \Theta, \tau)$.

Let $x \in L^{\psi_1}_+(S, \Sigma, \mu)$. There exists a sequence (x_n) of simple functions increasing to x/x_0. The operators U and $\bar{\Delta}$ are isomorphisms and surjections, $x_n x_0 \uparrow x$ in $L^{\psi_1}(S, \Sigma, \mu)$, and so

$$\bar{\Delta}(x_n) \uparrow \bar{\Delta}\left(\frac{x}{x_0}\right) \quad \text{and} \quad U(x_n x_0) \uparrow U(x).$$

For every simple function $f = \sum_i a_i 1_{A_i} \in L^{\psi_1}(S, \Sigma, \mu)$ we have

$$U(fx_0) = \sum_i a_i U(x_0 1_{A_i}) = \sum_i a_i 1_{\Delta(A_i)} U(x_0) = \bar{\Delta}(f) U(x_0).$$

Thus

$$U(x) = \sup_n (Ux_n(x_0)) = \sup_n \bar{\Delta}(x_n) U(x_0) = U(x_0) \bar{\Delta}\left(\frac{x}{x_0}\right).$$

Operators U and $\bar{\Delta}$ are linear and every function from $L^{\psi_1}(S, \Sigma, \mu)$ is a difference of two positive functions; so the equality $U(x) = U(x_0) \bar{\Delta}\left(\frac{x}{x_0}\right)$ holds for every $x \in L^{\psi}(S, \Sigma, \mu)$.

Remark. Repeating the arguments of the above proof, we find that of the same form is every order isomorphism mapping a sublattice $M \subset L^0(S, \Sigma, \mu)$ (having a weak order unit and the property: $x \in M$ implies $x1_A \in M$ for all $A \in \Sigma$) onto a regular sublattice $N \subset L^0(T, \Theta, \tau)$; of course, the isomorphism Δ will act between measure algebras generated by supports of M and N.

THEOREM 2.3. *The following statements are equivalent*:

(i) *There exists an isomorphism "onto"* $U: L^{\psi_1}(S, \Sigma, \mu) \to L^{\psi_2}(T, \Theta, \tau)$ *such that* $M_{\psi_1}(x) = M_{\psi_2}(U(x))$ *for every* $x \in L^{\psi_1}(S, \Sigma, \mu)$.

(ii) *There exist: a Boolean isomorphism* Δ *mapping* $\Sigma/N(\mu)$ *onto* $\Theta/N(\tau)$, *a weak unit* $x_0 \in L^{\psi_1}(S, \Sigma, \mu)$ *and a function* $y_0 \in L^{\psi_2}_+(T, \Theta, \tau)$ *such*

that

$$(+)\qquad \exists B\in N(\tau)\ \forall t\in T\setminus B\ \forall r\geqslant 0\ \psi_2(ry_0(t), t) = \bar{\Delta}\big(\psi_1(rx_0(\cdot), \cdot)\big)(t)\frac{d\mu^*}{d\tau}(t),$$

where $\mu^*\colon \Theta \to [0, \infty]$ *is a measure given by the formula* $\mu^*(B) = \mu(A)$ *for* $B\in\Delta(A)$.

Proof. (i) $\Rightarrow$ (ii). Let U be an isomorphism mapping $L^{\psi_1}(S, \Sigma, \mu)$ onto $L^{\psi_2}(T, \Theta, \tau)$ and preserving integral modulars. According to the preceding theorem there exists a Boolean isomorphism Δ from $\Sigma/N(\mu)$ onto $\Theta/N(\tau)$ such that $U(x) = U(x_0)\bar{\Delta}\left(\frac{x}{x_0}\right)$ for every $x\in L^{\psi_1}(S, \Sigma, \mu)$ where x_0 is a weak unit in $L^{\psi_1}(S, \Sigma, \mu)$. Moreover, the definition of the measure μ^* gives the equality

$$\int_A f\,d\mu = \int_{\Delta(A)} \bar{\Delta}(f)\,d\mu^*,$$

which holds for $f\in L^0_+(S, \Sigma, \mu)$ (here $\Delta(A)$ has a double meaning – an equivalence class and an element of it). Therefore, for arbitrary $A\in\Sigma$ and $r\geqslant 0$

$$\begin{aligned}\int_{\Delta(A)} \bar{\Delta}\big(\psi_1(rx_0(\cdot), \cdot)\big)(t)\frac{d\mu^*}{d\tau}(t)\,d\tau &= \int_{\Delta(A)} \bar{\Delta}\big(\psi_1(rx_0(\cdot), \cdot)\big)(t)\,d\mu^*\\ &= \int_A \psi_1(rx_0(s), s)\,d\mu = M_{\psi_1}(rx_0 1_A) = M_{\psi_2}(rU(x_0 1_A))\\ &= \int_T \psi_2(rU(x_0 1_A)(t), t)\,d\tau = \int_{\Delta(A)} \psi_2(rU(x_0)(t), t)\,d\tau.\end{aligned}$$

If A runs over the σ-algebra Σ, then $\Delta(A)$ runs over the whole Boolean algebra $\Theta/N(\tau)$. Thus $\psi_2(rU(x_0)(t), t) = \bar{\Delta}\big(\psi_1(rx_0(\cdot), \cdot)\big)(t)\frac{d\mu^*}{d\tau}(t)$ τ-almost everywhere for every r separately, and so the above equality holds τ-almost everywhere for all rational numbers. Since ψ_1, ψ_2 are left-continuous and $\bar{\Delta}$ is an isomorphism "onto", this equality holds τ-almost everywhere for all $r\geqslant 0$.

(ii) $\Rightarrow$ (i). Define an operator $U\colon L^0(S, \Sigma, \mu) \to L^0(T, \Theta, \tau)$ by the formula

$$U(x) = y_0\bar{\Delta}\left(\frac{x}{x_0}\right).$$

Let $x\in L^{\psi_1}_+(S, \Sigma, \mu)$ and let (x_n) be a sequence of simple functions increasing to $\frac{x}{x_0}$, $x_n = \sum_{k=1}^{i(n)} a_k^n 1_{A_k^n}$. Using equality $(+)$ and the monotone convergence

theorem, we obtain

$$\int_S \psi_1(x(s), s)\,d\mu = \lim_{n\to\infty} \sum_{k=1}^{i(n)} \int_{A_k^n} \psi_1(a_k^n x_0(s), s)\,d\mu$$

$$= \lim_{n\to\infty} \sum_{k=1}^{i(n)} \int_{A(A_k^n)} \bar{A}\big(\psi_1(a_k^n x_0(\cdot), \cdot)\big)(t)\frac{d\mu^*}{d\tau}(t)\,d\tau$$

$$= \lim_{n\to\infty} \sum_{k=1}^{i(n)} \int_{A(A_k^n)} \psi_2(a_k^n y_0(t), t)\,d\tau = \lim_{n\to\infty} \int_T \psi_2\big(\bar{A}(x_n) y_0(t), t\big)\,d\tau$$

$$= \int_T \psi_2\left(y_0\,\bar{A}\left(\frac{x}{x_0}\right)(t), t\right)d\tau = \int_T \psi_2(U(x)(t), t)\,d\tau.$$

Thus U maps $L^{\psi_1}(S, \Sigma, \mu)$ into $L^{\psi_2}(T, \Theta, \tau)$ and U preserves integral modulars. A function y_0 satisfying equality $(+)$ is positive τ-almost everywhere, and so U is an isomorphism.

We need only to show that U is "onto" If $y \in L^{\psi_2}(T, \Theta, \tau)$ then $\frac{y}{y_0} = \bar{A}(z)$ for some $z \in L^0(S, \Sigma, \mu)$ because $\bar{A}$ is "onto". Moreover, $U(zx_0) = y_0\,\bar{A}(z) = y$ and $M_{\psi_2}(y) = M_{\psi_2}(U(zx_0)) = M_{\psi_1}(zx_0)$. Therefore $zx_0 \in L^{\psi_1}(S, \Sigma, \mu)$, and this means that U maps $L^{\psi_1}(S, \Sigma, \mu)$ "onto" $L^{\psi_2}(T, \Theta, \tau)$.

Remark. Let φ_1, φ_2 be finite-valued Orlicz functions, let μ be a finite measure. If there exists an isomorphism mapping $L^{\varphi_1}(S, \Sigma, \mu)$ onto $L^{\varphi_2}(T, \Theta, \tau)$ and preserving integral modulars, then

$$(++) \qquad \varphi_2(ar) = b\varphi_1(r)$$

for some constants $a, b > 0$. Indeed, according to the proof of the implication (i) $\Rightarrow$ (ii), for the function 1_S there exists a function $y_0 \in L^{\varphi_2}(T, \Theta, \tau)$ such that condition $(+)$ holds; in this case it has the form

$$\varphi_2(ry_0(t)) = \varphi_1(r)\frac{d\mu^*}{d\tau}(t)$$

τ-almost everywhere. Thus we obtain $(++)$.

3. Drewnowski–Orlicz Theorem

The Drewnowski–Orlicz Theorem ([8], Theorem 2.1, p. 810) will play a key role in the proof of the Main Representation Theorem (Theorem 5.1).

Let (S, Σ, μ) be a measure space with μ σ-finite, G a sublattice of $L^0(S, \Sigma, \mu)$ with the property: $g \in G$ implies $g1_A \in G$ for all $A \in \Sigma$. Moreover, let $\varrho\colon G \to [0, \infty]$ be a functional satisfying the following conditions:

1. $\varrho(g) = 0$ iff $g = 0$.
2. $|g| \leqslant |h|$ implies $\varrho(g) \leqslant \varrho(h)$.
3. $g_n, g \in G$ and $0 \leqslant g_n \uparrow g$ imply $\varrho(g_n) \uparrow \varrho(g)$.
4. If $g \perp h$ then $\varrho(g+h) = \varrho(g)+\varrho(h)$.

THEOREM 3.1 (Drewnowski–Orlicz). *There exists a function* $\chi\colon [0, \infty) \times S \to [0, \infty]$ *with the following properties*:

(i) *The function* $\chi(r, \cdot)\colon S \to [0, \infty]$ *is* Σ*-measurable for every* r.

(ii) *The function* $\chi(\cdot, s)\colon [0, \infty) \to [0, \infty]$ *is non-decreasing, left continuous, positive for* $r > 0$ *and* $s \in \operatorname{supp} G$, $\chi(\cdot, s) = 0$ *for* $s \in S \setminus \operatorname{supp} G$ *and* $\chi(0, s) = 0$ *for* $s \in S$.

(iii) $\varrho(g) = \int\limits_{\operatorname{supp} G} \chi(|g(s)|, s)\, d\mu$.

Proof. See [8], Theorem 2.1, p. 810.

Since μ is σ-finite, we can find a set $S_0 \in \Sigma$ such that $\mu|\Sigma \cap S_0$ is non-atomic and $S \setminus S_0 = \bigcup\limits_{n=1}^{\infty} S_n$, where (S_n) is a sequence of disjoint atoms. Assume additionally:

5. $1_S \in G$.

6. If $g, h \in G$ and for every real number r $\mu(\{s\colon g(s) > r\}) = \mu(\{s\colon h(s) > r\})$ then $\varrho(g) = \varrho(h)$.

Under these assumptions we obtain

THEOREM 3.2. *There exist functions* $\varphi\colon [0, \infty) \to [0, \infty]$ *and* $\chi\colon [0, \infty) \times N \to [0, \infty]$ *such that*:

(i) φ *is non-decreasing, left continuous and takes the value zero only at the point* $r = 0$.

(ii) *For every* $n \in N$ *the function* $\chi(\cdot, n)\colon [0, \infty) \to [0, \infty]$ *is non-decreasing, left continuous and* $\chi(r, n) = 0$ *iff* $r = 0$.

(iii) $\varrho(g) = \int\limits_{S_0} \varphi(|g(s)|)\, d\mu + \sum\limits_{n=1}^{\infty} \chi(|g_n|, n)$, *where* $\{g_n\} = g(S_n)$.

Proof. See [8], Theorem 2.2, p. 812.

Representations of Orlicz lattices

4. Basic properties of orthogonal additive modulars and examples of Orlicz lattices

The notion of a modular is fundamental in this paper, and so its properties will be presented in a more detailed way. The facts mentioned here can be found in [1], [3], [4], [21], [22], [23], [24].

Let L be a vector lattice. A function $\varrho\colon L \to [0, \infty]$ is called an *orthogonal additive modular* if ϱ satisfies the following five conditions:

(ϱ1) $\varrho(x) = 0$ iff $x = 0$.

(ϱ2) $\quad |x| \leqslant |y| \quad$ implies $\quad \varrho(x) \leqslant \varrho(y).$

(ϱ3) $\quad \varrho(tx) \to 0 \quad$ for $\quad t \to 0.$

(ϱ4) $\quad 0 \leqslant x_n \uparrow x \quad$ implies $\quad \varrho(x_n) \uparrow \varrho(x).$

(ϱ5) $\quad \varrho(x+y) = \varrho(x)+\varrho(y) \quad$ for $\quad x \perp y.$

We will usually say briefly "modular" instead of "orthogonal additive modular". Conditions (ϱ1)–(ϱ5) imply some order properties of a vector lattice.

THEOREM 4.1. *Let ϱ be a modular defined on L. Then the following holds:*

(i) *L is Archimedean.*

(ii) *L has the* CS*-property.*

(iii) *If L is σ-*DC *then it is* SDC.

Proof. (i). Assume $x \in L_+$ and $z \leqslant n^{-1}x$ for all $n \in N$. Using Birkhoff's inequality and the property (ϱ2), we get

$$\varrho(z - z \wedge 0) = \varrho(|z \wedge n^{-1}x - z \wedge 0|) \leqslant \varrho(n^{-1}x),$$

and therefore by (ϱ3) and (ϱ1) we have $z = z \wedge 0 \leqslant 0$; so $\inf_n n^{-1}x = 0$.

(ii). Note first that every order bounded set of positive disjoint elements is at most countable. Indeed, let $\{x_\alpha\colon \alpha \in \mathscr{A}\} \subset L_+$ be a set of disjoint elements and let $x_\alpha \leqslant x$ for all α. Choosing a number $t > 0$ such that $\varrho(tx) < \infty$, defining the set A_k by

$$A_k = \{x_\alpha\colon \varrho(tx_\alpha) > k^{-1}\},$$

and using the property (ϱ1), we have

$$\{x_\alpha\colon x_\alpha \neq 0\} = \{x_\alpha\colon \varrho(tx_\alpha) > 0\} = \bigcup_{k=1}^{\infty} A_k.$$

Every set A_k must be finite. Otherwise, for some natural number k_0 there would exist a sequence $(x_{\alpha_n}) \subset A_{k_0}$ such that $\varrho(tx_{\alpha_n}) > k_0^{-1}$ for all n. But for an arbitrary natural number m the following inequalities hold:

$$\varrho(tx) \geqslant \varrho\Big(t \bigvee_{n=1}^{m} x_{\alpha_n}\Big) = \varrho\Big(t \sum_{n=1}^{m} x_{\alpha_n}\Big) = \sum_{n=1}^{m} \varrho(tx_{\alpha_n}) > mk_0^{-1},$$

and so $\varrho(tx) = \infty$, a contradiction. Therefore card $\{x_\alpha\colon x_\alpha \neq 0\} \leqslant \aleph_0$. To end the proof it is sufficient to use Theorem A.1.

(iii). Apply part (ii) and Theorem A.2.

Since a vector lattice L with a modular is Archimedean, there exists a Dedekind completion $\hat{L}$ of L. Moreover, L can be identifiedd with some full and order dense vector sublattice in $\hat{L}$. The next theorem, about the extension of a modular on the Dedekind completion, allows us to reduce some investigations to DC lattices.

THEOREM 4.2. *Let ϱ be a modular defined on a vector lattice L. The modular ϱ has a unique extension to a modular $\hat{\varrho}$ defined on the Dedekind completion $\hat{L}$ of L.*

Proof. Let $Q(\hat{x}) = [0, |\hat{x}|] \cap L_+$ for each $\hat{x} \in \hat{L}$. It is obvious that the function $\hat{\varrho}$ given by the formula

$$\hat{\varrho}(\hat{x}) = \sup_{x \in Q(\hat{x})} \varrho(x)$$

has properties (ϱ1), (ϱ2) and $\hat{\varrho}|L = \varrho$. Since L is full in $\hat{L}$, $\hat{\varrho}$ also satisfies condition (ϱ3). Moreover, $\hat{L}$ has the CS-property because L has this property and it is a full and order dense sublattice of $\hat{L}$.

Let $0 \leqslant \hat{x}_n \uparrow \hat{x}$. For every $x \in Q(\hat{x})$ we can find sequences $(y_n^k) \subset L_+$ increasing to $\hat{x}_n \wedge x$, i.e., $y_n^k \uparrow_k \hat{x}_n \wedge x$. If $z_n = \bigvee_{j=1}^{n} y_j^n$ then $z_n \in L_+$, $z_n \uparrow$ and $z_n \leqslant \hat{x}_n \wedge x \leqslant x$. We have $z_{j+k} \geqslant y_j^{j+k}$ for arbitrary natural numbers j, k, and so $\sup_n z_n = \sup_k z_{k+j} \geqslant \sup_k y_j^{k+j} = \hat{x}_j \wedge x$ for every j. Therefore $\sup_n z_n \geqslant \sup_j \hat{x}_j \wedge x = x$. Since $z_n \leqslant x$ for all n, we have $z_n \uparrow x$ in L. Using property (ϱ5), we will get

$$\varrho(x) = \lim_{n\to\infty} \varrho(z_n) \leqslant \lim_{n\to\infty} \hat{\varrho}(\hat{x}_n \wedge x) \leqslant \lim_{n\to\infty} \hat{\varrho}(\hat{x}_n).$$

The above relations hold for every $x \in Q(\hat{x})$, so $\hat{\varrho}(\hat{x}) \leqslant \lim_{n\to\infty} \hat{\varrho}(\hat{x}_n)$. The inverse inequality is clear: $\hat{\varrho}$ has property (ϱ2). Finally, $\hat{\varrho}(\hat{x}_n) \uparrow \hat{\varrho}(\hat{x})$.

We will show now that $\hat{\varrho}$ has property (ϱ5). To do this, let $\hat{x} \perp \hat{y}$ and let (x_n), $(y_n) \subset L_+$ be sequences increasing to $|\hat{x}|$ and $|\hat{y}|$, respectively. Therefore

$$\begin{aligned}\hat{\varrho}(\hat{x}+\hat{y}) &= \hat{\varrho}(|\hat{x}|+|\hat{y}|) = \lim_{n\to\infty} \hat{\varrho}(x_n+y_n) = \lim_{n\to\infty} \varrho(x_n+y_n) \\ &= \lim_{n\to\infty} \varrho(x_n) + \lim_{n\to\infty} \varrho(y_n) = \lim_{n\to\infty} \hat{\varrho}(x_n) + \lim_{n\to\infty} \hat{\varrho}(y_n) = \hat{\varrho}(\hat{x}) + \hat{\varrho}(\hat{y}).\end{aligned}$$

Let $\tilde{\varrho}$ be another extension of the modular ϱ. If $\hat{x} \in \hat{L}$ then, choosing a sequence $(x_n) \subset L_+$ increasing to $|\hat{x}|$ and using (ϱ4), we will obtain

$$\tilde{\varrho}(\hat{x}) = \lim_{n\to\infty} \tilde{\varrho}(x_n) = \lim_{n\to\infty} \varrho(x_n) = \lim_{n\to\infty} \hat{\varrho}(x_n) = \hat{\varrho}(\hat{x}),$$

and so $\tilde{\varrho} = \hat{\varrho}$.

We will employ the previous theorem to prove the following property of a modular:

(ϱA) $\qquad \varrho(x \vee y) + \varrho(x \wedge y) = \varrho(x) + \varrho(y) \quad$ for all $x, y \in L_+$.

(J. Musielak and W. Orlicz note this property for modulars defined on DC vector lattices – see [23], p. 665). Let $\hat{L}$ and $\hat{\varrho}$ have the previous meaning. Since $\hat{L}$ is Dedekind complete, according to Freudenthal's theorem for every

$\hat{x}, \hat{y} \in \hat{L}_+$ there exist sequences $(\hat{x}_n), (\hat{y}_n)$ in $\operatorname{lin} C(\hat{x} \vee \hat{y})$ such that

(a) $\hat{x}_n \uparrow \hat{x}$, $\hat{y}_n \uparrow \hat{y}$,

(b) every x_n (y_n) has a form $\hat{x}_n = \sum_i a_n^i p_n^i$, where $a_n^i \geqslant 0$, $p_n^i \wedge p_n^j = 0$ for $i \neq j$, $\sup_i p_n^i = \hat{x} \vee \hat{y}$.

Applying property (b), we easily get the equality $\hat{\varrho}(\hat{x}_n \vee \hat{y}_n) + \hat{\varrho}(\hat{x}_n \wedge \hat{y}_n) = \hat{\varrho}(\hat{x}_n) + \hat{\varrho}(\hat{y}_n)$. Therefore property (ϱ4) gives $\hat{\varrho}(\hat{x} \vee \hat{y}) + \hat{\varrho}(\hat{x} \wedge \hat{y}) = \hat{\varrho}(\hat{x}) + \hat{\varrho}(\hat{y})$. According to Theorem 4.2 $\hat{\varrho}|L = \varrho$, and so ϱ has the (ϱA) property.

The following important property arises at once from (ϱA) and (ϱ2):

(ϱB) $\varrho(ax+by) \leqslant \varrho(x)+\varrho(y)$ for $x, y \in L$ and $a, b \geqslant 0$, $a+b=1$.

A modular ϱ is said to be *convex* if it satisfies additionally the condition

(ϱB′) $\varrho(ax+by) \leqslant a\varrho(x)+b\varrho(y)$ for $x, y \in L$ and $a, b \geqslant 0$, $a+b=1$

(or equivalently: the functions $\varphi_x(t) = \varrho(tx)$ are convex for all $x \in L_+$). Note that if ϱ is convex then its extension $\hat{\varrho}$ is also convex.

An F-norm is associated with every modular ϱ. This F-norm is given by the formula

$$|||x|||_\varrho = \inf\{a > 0: \varrho(x/a) \leqslant a\}.$$

This formula defines an F-norm if a function ϱ satisfies only conditions (ϱ1)–(ϱ3) and (ϱB) – see [22], p. 49 and p. 52. The F-norm $|||\cdot|||_\varrho$ has two important properties:

(+) $|x| \leqslant |y|$ implies $|||x|||_\varrho \leqslant |||y|||_\varrho$ (*monotonicity*)

(++) $0 \leqslant x_n \uparrow x$ implies $|||x_n|||_\varrho \uparrow |||x|||_\varrho$ (*the σ-Fatou property*).

Moreover, if $|||x|||_\varrho < 1$ then $\varrho(x) \leqslant |||x|||_\varrho$. A convex modular determines the norm $\|\cdot\|_\varrho$ on L:

$$\|x\|_\varrho = \inf\{a > 0: \varrho(x/a) \leqslant 1\}.$$

The norm $\|\cdot\|_\varrho$ and the F-norm $|||\cdot|||_\varrho$ are equivalent. The convergence with respect to the F-norm $|||\cdot|||_\varrho$ can be expressed as a "modular convergence":

$$|||x_n - x|||_\varrho \to 0 \quad \text{iff} \quad \varrho(t(x_n - x)) \to 0 \quad \text{for every } t > 0.$$

If a *modular ϱ is finite*, i.e., $\varrho(L) \subset [0, \infty)$, then the F-norm $|||\cdot|||_\varrho$ is order continuous. Note first that $|||x_n|||_\varrho \to 0$ for all order bounded sequences (x_n) of positive disjoint elements. Indeed, for $t > 0$ and a natural number m we have

$$\sum_{n=1}^{m} \varrho(tx_n) = \varrho\Big(t \bigvee_{n=1}^{m} x_n\Big) \leqslant \varrho(tx),$$

where x is an upper bound of $\{x_n: n\in N\}$. Therefore $\sum_{n=1}^{\infty} \varrho(tx_n)$ is convergent for every $t>0$, and so $\varrho(tx_n)\to 0$ for every $t>0$. In other words, $|||x_n|||_\varrho \to 0$. Moreover, the F-norm has the σ-Fatou property, which implies that every Cauchy sequence $y_n \downarrow 0$ is convergent ([1], Theorem 17.3, (ii), p. 116). Since in $(L, |||\cdot|||_\varrho)$ every order bounded sequence of positive disjoint elements is convergent to zero and every Cauchy sequence decreasing to zero is convergent, it follows that $|||\cdot|||_\varrho$ is order continuous ([1], Theorem 17.8, p. 118). The order continuity of $|||\cdot|||_\varrho$ will give at once the following relation:

$$x_\alpha \downarrow 0 \Rightarrow \inf_\alpha \varrho(x_\alpha) = 0.$$

A vector lattice L will be SDC if we assume that ϱ is a finite modular and L is complete with respect to the F-norm $|||\cdot|||_\varrho$ (ϱ is finite, and so condition (iv) of Theorem A.14 is satisfied).

Let β: $L\to[0,\infty)$ be a function satisfying conditions (ϱ1), (ϱ2), (ϱ5) and (ϱB′). In this case β automatically satisfies condition (ϱ3). If, moreover, L is $|||\cdot|||_\beta$-complete, then β also satisfies (ϱ4) (see [4], Theorem 5.1), and so β is a modular.

For a finite and convex modular ϱ the following inequality holds (see [4], p. 87):

$$(\varrho B'') \qquad \forall x, y\in L\ \forall a\in(0,1]\ \varrho(x)\leqslant \varrho(y)+\frac{a}{2}\left(\varrho(2y)+\varrho\left(\frac{2}{a}(x-y)\right)\right).$$

Among finite modulars there are modulars satisfying the (Δ_2)-condition:

$$\exists K>0\ \forall x\in L\ \varrho(2x)\leqslant K\varrho(x).$$

The (Δ_2)-condition in a slightly more general form:

$$(\Delta_2^\infty) \qquad \forall a>0\ \exists K>0\ \forall x\in L\ \left(\varrho(x)\geqslant a\Rightarrow \varrho(2x)\leqslant K\varrho(x)\right)$$

is, under some assumptions, equivalent to the finiteness of a modular.

THEOREM 4.3 (I. Amemiya). *Let L be non-atomic* DC *vector lattice. If ϱ is a finite modular such that $x_n\uparrow$ and $\sup_n \varrho(x_n)<\infty$ imply the existence of the supremum of $\{x_n: n\in N\}$, then ϱ satisfies the (Δ_2^∞)-condition.*

Proof. See [29].

Various so-called completeness conditions are investigated in the theory of modular spaces (see [12], [26], [30]). The following conditions are considered most frequently:

(i) $0\leqslant x_n\uparrow$ and $\sup_n \varrho(x_n)<\infty \Rightarrow \exists x\in L\ \ x_n\uparrow x$ (monotone completeness)

(ii) $0 \leqslant x_n \uparrow$ and $\{x_n\colon n \in N\}$ topologically bounded $\Rightarrow \exists\, x \in L\ x_n \uparrow x$ (the σ-Levi property).

(iii) $0 \leqslant x_n$ and $\sum_{n=1}^{\infty} \varrho(x_n) < \infty \Rightarrow \exists\, x \in L\ x = \sup_n x_n$.

(iv) $0 \leqslant x_n$, $x_n \wedge x_m = 0$ for $n \neq m$ and $\sum_{n=1}^{\infty} \varrho(x_n) < \infty \Rightarrow \exists\, x \in L\, x = \sup_n x_n$ (orthogonal completeness).

The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are obvious. It is not difficult to verify that the implication (ii) $\Rightarrow$ (iii) is also true. Indeed, if $x_n \geqslant 0$, $\sum_{n=1}^{\infty} \varrho(x_n) < \infty$, then putting $y_k = \bigvee_{n=1}^{k} x_n$ and using the property (ϱA) we obtain $\varrho(y_k) \leqslant \sum_{n=1}^{\infty} \varrho(x_n)$ for all $k \in N$. The sequence (y_k) is topologically bounded. To show this let $\varepsilon > 0$ be a fixed real number and let n_0 be a natural number such that $\sum_{n=n_0}^{\infty} \varrho(x_n) < \varepsilon/2$. We can find a number $t \in (0, 1)$ for which the following inequality holds: $\sum_{n=1}^{n_0} \varrho(tx_n) < \varepsilon/2$. Therefore, for every $k \in N$, $\varrho(ty_k) \leqslant \sum_{n=1}^{n_0} \varrho(tx_n) + \sum_{n=n_0+1}^{\infty} \varrho(x_n) < \varepsilon$, and so $\|\|(t\varepsilon) y_k\|\|_\varrho \leqslant \varepsilon$ for all k, but this means that $\{y_k\colon k \in N\}$ is topologically bounded. According to (ii) there exists an $x \in L$, which is the supremum of $\{y_k\colon k \in N\}$ and thus also the supremum of $\{x_n\colon n \in N\}$.

If L is DC then (iv) $\Rightarrow$ (ii). Indeed, every increasing topologically bounded sequence (x_n) is a dominable set ([1], Theorem 24.2, p. 182), and so $\{x_n\colon n \in N\}$ has the supremum x in L^u — the universal completion of L (see Theorem A.12). Since L is DC and order dense in L^u, then there exists a net $\{y_\alpha\}_{\alpha \in \mathscr{A}} \subset L_+$ of disjoint elements such that $x = \sup_\alpha y_\alpha$ ([1], Lemma 23.15, p. 172). The topological boundedness of (x_n) implies the finiteness of $\sup_n \varrho(tx_n)$ for some $t > 0$. Repeating the arguments used in the proof of Theorem 4.1, we can verify that the set $\{y_\alpha\colon \alpha \in \mathscr{A}\}$ is at most countable and $\sum_\alpha \varrho(ty_\alpha) < \infty$. By the asumption there exists an element $z \in L_+$ which is the supremum of the set $\{ty_\alpha\colon \alpha \in \mathscr{A}\}$. Therefore $x = t^{-1} z$, so $x \in L$.

The following example shows that the Dedekind completeness cannot be dropped.

Let $L_c = \{f \in L^1[0, 1]\colon f$ is at most countably valued$\}$, $\varrho(f) = \int_0^1 |f(s)|\, ds$. In this situation condition (iv) holds but (ii) does not hold (the function $f(s) = s$ is the supremum of some increasing sequence of simple functions which belong to L_c).

The validity of the implication (ii) $\Rightarrow$ (i) for convex modulars is evident. In general condition (ii) does not imply (i) (consider $L^0[0, 1]$ with $\varrho(f) = \int_0^1 |f(s)|(1+|f(s)|)^{-1} ds$ and the sequence $x_n = n1_{[0,1]}$). Using the Main Representation Theorem (see the next section), we will show that the implication (ii) $\Rightarrow$ (i) holds under the assumption $\forall x \neq 0 \ \lim_{t\to\infty} \varrho(tx) = \infty$.

The pair (L, ϱ), where ϱ is a modular defined on L, will be called an *Orlicz lattice* and will be denoted by L^ϱ.

Now we will give some examples of Orlicz lattices.

1. Let $(L, \|\cdot\|)$ be an abstract L-space, i.e., a Banach lattice with the norm satisfying the condition

$$\|x+y\| = \|x\|+\|y\| \quad \text{for} \quad x \perp y.$$

The norm $\|\cdot\|$ is a finite modular (condition (ϱ4) is a consequence of the order continuity of the norm $\|\cdot\|$).

2. A Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$ with the integral modular

$$M_\psi(x) = \int_S \psi(|x(s)|, s)\, d\mu.$$

3. Let (x_n) be an unconditional basis in a Banach space $(X, \|\cdot\|)$ with the unconditional constant equal to 1. X will be a vector lattice with respect to the order

$$x \leqslant y \quad \text{iff} \quad x_n^*(x) \leqslant x_n^*(y) \quad \text{for every } n,$$

where x_n^* are the biorthogonal functionals associated with the basis (x_n). It is not difficult to verify the following facts:

(a) $x \vee y = \sum_{n=1}^{\infty} \max(x_n^*(x), x_n^*(y))\, x_n$,

$x \wedge y = \sum_{n=1}^{\infty} \min(x_n^*(x), x_n^*(y))\, x_n$.

(b) X is DC.

(c) x_n^*'s are σ-homomorphisms.

(d) $\|\cdot\|$ is monotone (because for every bounded sequence (t_n)

$$\Big\|\sum_{n=1}^{\infty} t_n x_n^*(x)\, x_n\Big\| \leqslant \sup_n |t_n| \Big\|\sum_{n=1}^{\infty} x_n^*(x)\, x_n\Big\| \quad - \text{ see [17], p. 19).}$$

Let (S, Σ, μ) be a measure space and let $L^0(S, \Sigma, \mu, X) = \{f\colon S \to X,$ f is strongly measurable$\}$ (functions equal μ-almost everywhere are identified). The space $L^0(S, \Sigma, \mu, X)$ with the order $f \leqslant g$ iff $f(s) \leqslant g(s)$ μ-almost everywhere is a vector lattice. On the space $L^0(S, \Sigma, \mu, X)$ we define the

function ϱ by the formula

$$\varrho(f) = \sum_{n=1}^{\infty} \int_S \psi(|x_n^* f(s)|, s)\, d\mu,$$

where ψ is a Musielak–Orlicz function. Denote by $L^\psi(S, \Sigma, \mu, X)$ the following subspace of $L^0(S, \Sigma, \mu, X)$:

$$L^\psi(S, \Sigma, \mu, X) = \{f \in L^0(S, \Sigma, \mu, X): \varrho(tf) \to 0 \text{ if } t \to 0\}.$$

The pair $(L^\psi(S, \Sigma, \mu, X), \varrho)$ is an Orlicz lattice. We need only to check conditions (ϱ4) and (ϱ5) (the other conditions are evident). (ϱ5) is a simple consequence of the fact that the functionals x_n^* are homomorphisms. Since X is separable and DC, the norm $\|\cdot\|$ is order continuous ([27], p. 95). From this we have

$$f_n, f \in L^\psi(S, \Sigma, \mu, X),\ 0 \leqslant f_n \uparrow f \text{ implies } f_n(s) \uparrow f(s) \ \mu\text{-almost everywhere.}$$

The functionals x_n^* are σ-homomorphisms, and so $x_k^* f_n(s) \uparrow x_k^* f(s)$ μ-almost everywhere. To see that $\varrho(f_n) \uparrow \varrho(f)$ it is sufficient to use the monotone convergence theorem.

4. The modular direct sum. Let $L_a^{\varrho_a}$, $a \in A$, be a family of Orlicz lattices. Let

$$\bigoplus_a L_a^{\varrho_a} = \{x \in \prod_a L_a: \exists t > 0\ \varrho(tx) = \sum_a \varrho_a(tx(a)) < \infty\}.$$

According to the property (ϱB) $\bigoplus_a L_a^{\varrho_a}$ is a linear space. Moreover, this space is a vector lattice with respect to the natural "pointwise" ordering: $x \leqslant y$ iff $x(a) \leqslant y(a)$ for all $a \in A$. The reader can verify without any difficulty that ϱ is a modular on $\bigoplus_a L_a^{\varrho_a}$.

5. Ultraproducts of Orlicz lattices (details will be presented in Section 7).

Let $L_f^\varrho = \{x \in L: \forall t > 0\ \varrho(tx) < \infty\}$. L_f^ϱ is a closed ideal and the F-norm $|||\cdot|||_\varrho$ restricted to L_f^ϱ is order continuous (because ϱ is finite on L_f^ϱ). In general, L_f^ϱ is different from the subspace of elements of the absolute continuous norm $L_a^\varrho = \{x \in L: |x| \geqslant x_n \downarrow 0 \text{ implies } |||x_n|||_\varrho \to 0\}$ (see the example from the preceding section).

The Orlicz lattices L^ϱ and K^η are called *isomodular* if there exists an isomorphism ("onto") T: $L \to K$ which preserves modulars, i.e.,

$$(*) \qquad \varrho(x) = \eta(Tx) \quad \text{for all } x \in L.$$

For example, every Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$, where μ is σ-finite, is isomodular to some Musielak–Orlicz space $L^\chi(S, \Sigma, \nu)$, where ν is finite. Indeed, if $f \in L^1(S, \Sigma, \mu)$ is positive on S then the function ν given by the equality $\nu(A) = \int_A f d\mu$ is a finite measure equivalent to μ (i.e., $N(\mu) = N(\nu)$);

so, defining the function χ: $[0, \infty) \times S \to [0, \infty]$ by the formula $\chi(r, s) = \psi(r, s)(f(s))^{-1}$, we find that $L^{\psi}(S, \Sigma, \mu)$ and $L^{\chi}(S, \Sigma, \nu)$ are isomodular – the identity operator is a modular preserving isomorphism.

Every modular preserving isomorphism is an isometry between Orlicz lattices whose F-norms are generated by modulars. What can be said about the converse problem? Does every isomorphism, which is at the same time an isometry between Orlicz lattices, preserve modulars? The answer is no. The example given below presents isomorphic and isometric Orlicz lattices which are not isomodular.

Let ϱ_i: $\boldsymbol{R} \to [0, \infty)$, $i = 1, 2$, be defined by the equalities

$$\varrho_1(x) = |x|, \quad \varrho_2(x) = x^2.$$

The modulars ϱ_i are convex and $\|x\|_{\varrho_1} = \|x\|_{\varrho_2} = |x|$. Therefore the identity is an isomorphism and an isometry. Every one-to-one linear operator T: $\boldsymbol{R} \to \boldsymbol{R}$ has the form $T(x) = ax$ $(a \in \boldsymbol{R} \setminus \{0\})$. If the lattices $\boldsymbol{R}^{\varrho_1}$, $\boldsymbol{R}^{\varrho_2}$ were isomodular, there would exist a number $a \in \boldsymbol{R} \setminus \{0\}$ such that for every x

$$|x| = \varrho_1(x) = \varrho_2(ax) = a^2 x^2.$$

But this equality is true only for $x \in \{-a^{-2}, 0, a^{-2}\}$.

5. The Main Representation Theorem for Orlicz lattices

THEOREM 5.1 (Main Representation Theorem). *For every Orlicz lattice L^{ϱ} there exist: a measure space (S, Σ, μ) which is the direct sum of finite measure spaces, a Musielak–Orlicz function ψ: $[0, \infty) \times S \to [0, \infty]$ and a super order dense sublattice V in $L^{\psi}(S, \Sigma, \mu)$ such that L^{ϱ} is isomodular to $V^{M_{\psi}}$.*

Moreover, if L is DC *then V is an ideal.*

If L^{ϱ}_f is $\|\|\cdot\|\|_{\varrho}$-complete then $L^{\psi}_f(S, \Sigma, \mu) \subset V$.

$V = L^{\psi}(S, \Sigma, \mu)$ if and only if L^{ϱ} has the σ-Levi property.

Proof. I. First we will assume additionally that L is DC and $e \in L_+$ is a weak unit. An element te is a weak unit for every $t > 0$, and so according to the property (ϱ3) we can assume $\varrho(e) < \infty$.

By Stone's representation theorem for Boolean algebras there exist an extremally disconnected space S and a Boolean isomorphism onto T_0: $C(e) \to \Sigma_0$, where Σ_0 is the algebra of open-closed subsets of S. The operator T_0 determines an isomorphism T: $L \to C_{\infty}(S)$ with the property $T(p) = 1_{T_0(p)}$ for all $p \in C(e)$ and, moreover, $T(L)$ is an order dense ideal in $C_{\infty}(S)$ (see Theorem A.11). The linearity of T implies the equality $\operatorname{lin}\{1_A: A \in \Sigma_0\} = T(\operatorname{lin} C(e))$.

The function μ_0: $\Sigma_0 \to [0, \varrho(e)]$ given by the formula $\mu_0(T_0(p)) = \varrho(p)$ is a Boolean measure. If $\Sigma = \{A \div N: A \in \Sigma_0, N \text{ is a set of first category in } S\}$,

then Σ is a σ-algebra, and a function μ defined by

$$\mu(A \div N) = \mu_0(A)$$

is a measure on Σ (see the remarks after Theorem B.2).

The operator $f \to [f]$ is an isomorphism between $C_\infty(S)$ and $L^0(S, \Sigma, \mu)$ (see Theorem B.3). Therefore we can say that T maps L onto an order dense ideal in $L^0(S, \Sigma, \mu)$. Since μ is finite, the space $L^0(S, \Sigma, \mu)$ has the CS-property, and so $T(L)$ is a super order dense ideal.

Define a modular M on $T(L)$ putting for every $f \in T(L)$

$$M(f) = \varrho(T^{-1}(f)).$$

Moreover, if $f \in T(L)$ then $f1_A \in T(L)$ for every $A \in \Sigma$, because $T(L)$ is an ideal. Applying the Drewnowski–Orlicz Theorem (Theorem 3.1) to the space $T(L)$ and the modular M, we will find a function $\psi\colon [0, \infty) \times S \to [0, \infty]$ with the following properties:

(a) $\psi(r, \cdot)$ is Σ-measurable for every r,

(b) $\psi(r, s) = 0$ iff $r = 0$, $\psi(\cdot, s)$ is non-decreasing and left continuous for every s,

(c) $M(f) = \int_S \psi(|f(s)|, s)\, d\mu$ for every $f \in T(L)$.

The function ψ is continuous at zero for μ-almost all s. Indeed, since $\psi(\cdot, s)$ is monotonic, it is enough to show that $\psi(r_n, s) \to 0$ for μ-almost all s and for some sequence (r_n) decreasing to zero. Let $1 > r_n \downarrow 0$. Therefore

$$\int_S \psi(r_n, s)\, d\mu = M(r_n 1_S) = \varrho(r_n e) \to 0,$$

and so there exist a natural number n_0 and a set $A_0 \in N(\mu)$ such that $\psi(r_n, s) < \infty$ for $n > n_0$ and $s \in S \setminus A_0$. The sequence $(\psi(r_n, s))$ tends to zero in measure, and thus it is possible to choose a subsequence (r_{n_k}) and a set $B_0 \in N(\mu)$ such that $\psi(r_{n_k}, s) \to 0$ as $k \to \infty$ and $s \in S \setminus B_0$. Finally, $\psi(\cdot, s)$ is continuous at zero for all $s \in S \setminus (A_0 \cup B_0)$.

Thus for we have shown that if L is DC and L has a weak unit then L^ϱ is isomodular to a super order dense ideal in some Musielak–Orlicz space.

II. Let L be DC without a weak unit. Let (x_a), $a \in A$, be a complete disjoint system of L. We can assume $\varrho(x_a) < \infty$ for all $a \in A$. Every element $y \in L_+$ has the form $y = \sup_a y_a$, where y_a denotes the projection of y onto the band $L_a = \{x_a\}^{\perp\perp}$. Every Orlicz lattice has the CS-property, and so $\operatorname{card}\{y_a\colon y_a > 0\} \leqslant \aleph_0$. The bands L_a, with modulars $\varrho_a = \varrho | L_a$, are DC Orlicz lattices with weak units x_a. The operator $\bar{U}\colon L_+ \to \bigoplus_a L_a^{\varrho_a}$ given by

$$\bar{U}(y) = (y_a)_{a \in A}$$

is additive and positive homogeneous, $y \wedge z = 0$ implies $\bar{U}(y) \wedge \bar{U}(z) = 0$,

$\bar{U}(y) = 0$ iff $y = 0$. Moreover, $\bar{U}$ preserves modulars. Therefore $\bar{U}$ has a unique extension to an isomorphism U from L^ϱ into $\bigoplus_a L_a^{\varrho_a}$. In addition, U preserves modulars and $U(L^\varrho)$ is a super order dense ideal.

According to part I for every $a \in A$ there exists a Musielak–Orlicz space $L^{\psi_a}(S_a, \Sigma_a, \mu_a)$ where μ_a is finite and a modular preserving isomorphism T_a: $L_a \to L^{\psi_a}(S_a, \Sigma_a, \mu_a)$ mapping L_a onto some super order dense ideal in $L^{\psi_a}(S_a, \Sigma_a, \mu_a)$. Moreover, $T_a(\operatorname{lin} C(x_a))$ is a subspace of $L^{\psi_a}(S_a, \Sigma_a, \mu_a)$ consisting of simple functions.

Composing the operator U with the operator T: $\bigoplus_a L_a^{\varrho_a} \to \bigoplus_a L^{\psi_a}(S_a, \Sigma_a, \mu_a)$ defined by the formula

$$(**) \qquad T((z_a)) = (T_a(z_a))_{a \in A},$$

we will get a modular preserving isomorphism which maps L^ϱ onto some super order dense ideal in $\bigoplus_a L^{\psi_a}(S_a, \Sigma_a, \mu_a)$ (Theorem A.9). The space $\bigoplus_a L^{\psi_a}(S_a, \Sigma_a, \mu_a)$ can be identified with a Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$ where (S, Σ, μ) is the direct sum of measure spaces (S_a, Σ_a, μ_a) and $\psi(r, s) = \psi_a(r, s)$ for $s \in S_a$ (see the remarks after Theorem 1.1).

Thus we have shown that at a DC Orlicz lattice is isomodular to a super order dense ideal in some Musielak–Orlicz space.

III. If L^ϱ has the σ-Levi property then L is DC. Let symbols T, U have the same meaning as in part II and let $L^\psi(S, \Sigma, \mu)$ be the Musielak–Orlicz space defined in that part. The composition TU maps L^ϱ (isomorphically) onto a super order dense ideal in $L^\psi(S, \Sigma, \mu)$ and TU preserves modulars. According to Theorem A.13 we have $TU(L^\varrho) = L^\psi(S, \Sigma, \mu)$.

On the other hand, if L^ϱ is isomodular to some Musielak–Orlicz space, then L^ϱ has the σ-Levi property because every Musielak–Orlicz space has this property (see Theorem 1.1).

IV. Suppose that L^ϱ is a DC Orlicz lattice and L_f^ϱ is $|||\cdot|||_\varrho$-complete. Using the second part of our proof, we find a Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$ and a modular preserving isomorphism T between L^ϱ and $L^\psi(S, \Sigma, \mu)$ such that $T(L^\varrho)$ is a super order dense ideal in $L^\psi(S, \Sigma, \mu)$. Therefore, for every function $g \in L_f^\psi(S, \Sigma, \mu)$, $g \geqslant 0$, there exists a sequence $(z_n) \subset L$ with the property $T(z_n) \uparrow g$. Thus $z_n \in L_f^\varrho$ for every n. Since $|||g - T(z_n)|||_{M_\psi}$ tends to zero ($|||\cdot|||_{M_\psi}$ restricted to $L_f^\psi(S, \Sigma, \mu)$ is order continuous), (z_n) is a Cauchy sequence in L_f^ϱ. The $|||\cdot|||_\varrho$-completeness of L_f^ϱ implies the existence of an element z which is a limit of the sequence (z_n). The element z is also the supremum of $\{z_n\colon n \in N\}$ because (z_n) is increasing. T is a normal isomorphism, and so $T(z) = \sup_n T(z_n) = g$. Therefore $T(L_f^\varrho) = L_f^\psi(S, \Sigma, \mu)$.

Thus we have shown that if L^ϱ is DC and L^ϱ_f is $|||\cdot|||_\varrho$-complete then L^ϱ is isomodular to an ideal V in some Musielak–Orlicz space and V contains the subspace $L^\psi_f(S, \Sigma, \mu)$.

V. Let L^ϱ be an arbitrary Orlicz lattice, $\hat{L}$ the Dedekind completion of L, and $\hat{\varrho}$ the unique extension of the modular ϱ (see Theorem 4.2). $\hat{L}^{\hat{\varrho}}$ is isomodular to a super order dense ideal in some Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$, and so L^ϱ, as a super order dense sublattice in $\hat{L}^{\hat{\varrho}}$, is isomodular to some super order dense sublattice in $L^\psi(S, \Sigma, \mu)$.

If L^ϱ_f is $|||\cdot|||_\varrho$-complete then $\hat{L}^{\hat{\varrho}}_f = L^\varrho_f$. Indeed, for every $\hat{x} \in \hat{L}^{\hat{\varrho}}_f$, $\hat{x} \geqslant 0$, there exists a sequence $(z_n) \subset L^\varrho$ increasing to $\hat{x}$. Order continuity of the F-norm $|||\cdot|||_{\hat{\varrho}}$ restricted to $\hat{L}^{\hat{\varrho}}_f$ implies that (z_n) is a Cauchy sequence in $(L^\varrho_f, |||\cdot|||_\varrho)$, and so (z_n) converges to an element $z \in L^\varrho_f$ and $z_n \uparrow z$. We have $\hat{x} = z$ because L is a regular sublattice in $\hat{L}$. Using part IV, we find that L^ϱ is isomodular to a sublattice V in some Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$ and V contains $L^\psi_f(S, \Sigma, \mu)$.

We will briefly say that an Orlicz lattice *embeds* into a Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$ if there exists a modular preserving isomorphism T: $L^\varrho \to L^\psi(S, \Sigma, \mu)$ which maps L onto a super order dense sublattice in $L^\psi(S, \Sigma, \mu)$.

Some properties of a modular ϱ are shared by a function ψ generating a Musielak–Orlicz space which is isomodular to L^ϱ. More precisely:

COROLLARY 5.2. *Let L^ϱ be an Orlicz lattice with a modular which is convex (finite, satisfying the (Δ_2)-condition). Then L^ϱ embeds into some Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$ and ψ is convex (takes only finite values, satisfies the condition $\exists K > 0\ \forall r \geqslant 0\ \forall s\ \psi(2r, s) \leqslant K\psi(r, s)$).*

Proof. Let $\hat{L}$ be the Dedekind completion of L, $\hat{\varrho}$ the unique extension of ϱ. If $\hat{L}^{\hat{\varrho}}$ is isomodular to a super order dense ideal in some Musielak–Orlicz space $L^\chi(S, \Sigma, \mu)$ then L^ϱ is isomodular to some super order dense sublattice in $L^\chi(S, \Sigma, \mu)$. Moreover, if ϱ is convex, finite or satisfies the (Δ_2)-condition, then $\hat{\varrho}$ is convex, finite, or satisfies the (Δ_2)-condition, respectively (L is full and super order dense in $\hat{L}$). Therefore we can restrict our considerations to DC Orlicz lattices.

Let (x_a) be a complete disjoint system of a DC Orlicz lattice L^ϱ. Let the symbols L_a, ϱ_a, T_a, T, U have the same meaning as in the second part of the proof of the preceding theorem. Let $L^{\psi_a}(S_a, \Sigma_a, \mu_a)$, $L^\psi(S, \Sigma, \mu)$ be Musielak–Orlicz spaces constructed in the proof of the Main Representation Theorem. For every $B \in \Sigma_a$ there exists a $p \in C(x_a)$ such that $T_a(p) = 1_B$. If ϱ is convex, and $c \in [0, 1]$ then for some numbers r, t we have

$$\int_B \psi_a\big(cr+(1-c)t, s\big)\,d\mu_a = \varrho_a\big(\big(cr+(1-c)t\big)p\big) \leqslant c\varrho_a(rp)+(1-c)\varrho_a(tp)$$

$$= \int_B \big(c\psi_a(r, s)+(1-c)\psi_a(t, s)\big)\,d\mu_a.$$

This inequality holds for every $B\in\Sigma_a$, so $\psi_a(cr+(1-c)t, s)\leqslant c\psi_a(r, s)+$ $+(1-c)\psi_a(t, s)$ μ_a-almost everywhere. In other words, $\psi_a(cr+$ $+(1-c)t, s)\leqslant c\psi_a(r, s)+(1-c)\psi_a(t, s)$ for all $s\in S_a\setminus B_a$ where $B_a\in N(\mu_a)$ for all rational r, t and for all rational $c\in[0, 1]$.

Let $c\in(0, 1)$, $r, t>0$. We now choose non-decreasing sequences of rational numbers (c_n), (r_n), (t_n) converging to c, r, $(1-c)t$, respectively. The sequence (w_n), $w_n=t_n(1-c_n)^{-1}$, is non-decreasing and converges to t. Using the left continuity of ψ_a, we will get for all $s\in S_a\setminus B_a$

$$\psi_a(cr+(1-c)t, s)=\lim_{n\to\infty}\psi_a(c_nr_n+(1-c_n)w_n, s)$$

$$\leqslant\lim_{n\to\infty}(c_n\psi_a(r_n, s)+(1-c_n)\psi_a(w_n, s))=c\psi_a(r, s)+(1-c)\psi_a(t, s).$$

A function ψ'_a defined by the formula

$$\psi'_a(r, s)=\begin{cases}\psi_a(r, s) & \text{for} \quad s\in S_a\setminus B_a,\\ r & \text{for} \quad s\in B_a\end{cases}$$

is a convex Musielak–Orlicz function and $L^{\psi_a}(S_a, \Sigma_a, \mu_a)=L^{\psi'_a}(S_a, \Sigma_a, \mu_a)$. If we put $\psi'(r, s)=\psi'_a(r, s)$ for $s\in S_a$, we will find that ψ' is a convex Musielak–Orlicz function and $L^{\psi'}(S, \Sigma, \mu)=\bigoplus_a L^{\psi'_a}(S_a, \Sigma_a, \mu_a)=\bigoplus_a L^{\psi_a}(S_a, \Sigma_a, \mu_a)$ $=L^{\psi}(S, \Sigma, \mu)$. Therefore the composition TU is a modular preserving isomorphism mapping L^ϱ into $L^{\psi}(S, \Sigma, \mu)$.

The rest will be proved in a similar way.

COROLLARY 5.3. *An Orlicz lattice L^ϱ embeds into some Musielak–Orlicz space $L^{\psi}(S, \Sigma, \mu)$, where μ is non-atomic if and only if L is non-atomic.*

Proof. Assume first that L is non-atomic and let $\hat{L}$ be the Dedekind completion of L. $\hat{L}$ is non-atomic iff L is non-atomic (because of the order density of L in $\hat{L}$). Therefore we can reduce our considerations to DC Orlicz lattices.

According to Theorem 5.1 there exist a Musielak–Orlicz space $L^{\psi}(S, \Sigma, \mu)$ and an ideal $V\subset L^{\psi}(S, \Sigma, \mu)$ isomodular to L^ϱ. Let $T: L^\varrho\to V$ be an isomorphism onto. Since (S, Σ, μ) is a Maharam space, the space $L^0(S, \Sigma, \mu)$ is DC and the quotient algebra $\Sigma/N(\mu)$ is a complete Boolean algebra.

We have an obvious inclusion $V\subset L^{\psi}(A, \Sigma\cap A, \mu|\Sigma\cap A)$, where $A=\sup\{\operatorname{supp} f: f\in V\}$. To end this part of the proof it is enough to show that $\mu|\Sigma\cap A$ is non-atomic.

Assume that $B\in\Sigma\cap A$ is a μ-atom. The definition of the set A implies the existence a non-negative function $f\in V$ whose support has a non-negligible intersection with B, i.e., $\mu(B\cap\operatorname{supp} f)>0$. The function $1_{B\cap\operatorname{supp} f}\in V$ because V is an ideal and $1_{B\cap\operatorname{supp} f}\leqslant cf$ where $f(B)=\{c^{-1}\}$. Clearly, $1_{B\cap\operatorname{supp} f}$ is an atom in V, and so $T^{-1}(1_{B\cap\operatorname{supp} f})$ is an atom in L – a contradiction.

Suppose that L^ϱ embeds into a Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$ where μ is non-atomic and that $x \in L_+$ is an atom. If $g \in [0, T(x)]$ (T is an isomorphism between L^ϱ and $L^\psi(S, \Sigma, \mu)$) then there exists a sequence $(x_n) \subset [0, x]$ with a property $T(x_n) \uparrow g$. Every x_n has the form $x_n = t_n x$ for some $t_n \geqslant 0$, and so $T(x_n) = t_n T(x)$. Therefore $g = (\sup_n t_n) T(x)$ and thus $T(x)$ is an atom in $L^\psi(S, \Sigma, \mu)$ – a contradiction.

It is very easy to prove the following theorem without recalling the Main Representation Theorem:

THEOREM 5.4. *If L is a discrete vector lattice then, for every modular ϱ defined on L, an Orlicz lattice L^ϱ embeds in some Musielak–Orlicz space $l^\psi(\Gamma)$.*

Proof. Let $(x_\gamma)_{\gamma \in \Gamma}$ be a complete disjoint system consisting of atoms. We can clearly assume that $\varrho(x_\gamma) < \infty$ for all $\gamma \in \Gamma$. Since the band $\{x_\gamma\}^{\perp\perp} = \operatorname{lin}\{x_\gamma\}$ is a projection band (see [1], Theorem 2.16, p. 17) and the projection on this band is a normal homomorphism, it follows that for every $y \in L_+$ there exists a unique family (a_γ) of non-negative real numbers such that the following holds: $y = \sup_\gamma a_\gamma x_\gamma$. The relation $\operatorname{card}\{a_\gamma\colon a_\gamma \neq 0\} \leqslant \aleph_0$ is a consequence of the CS-property.

Define a function $\psi\colon [0, \infty) \times \Gamma \to [0, \infty]$ by the equality

$$\psi(r, \gamma) = \varrho(r x_\gamma).$$

An operator $T\colon L_+ \to l^\psi(\Gamma)$ given by the formula $T(y) = (a_\gamma)_{\gamma \in \Gamma}$ has a unique extension to a modular preserving isomorphism defined on the whole of L. The sublattice $T(L^\varrho)$ is super order dense in $l^\psi(\Gamma)$ because $\{1_{\{\gamma\}}\colon \gamma \in \Gamma\} = T(\{x_\gamma\colon \gamma \in \Gamma\}) \subset T(L^\varrho)$.

In the preceding section we considered some completeness condition and we observed that under the assumption that $\forall\, x \neq 0$, $\lim_{t \to \infty} \varrho(tx) = \infty$ the σ-Levi property implies monotone completeness. Now we will prove this fact.

Fix an increasing sequence (x_n) of positive elements such that $\sup_n \varrho(x_n) < \infty$. Let $T\colon L^\varrho \to L^\psi(S, \Sigma, \mu)$ be a modular preserving isomorphism. Put $A = \{s\colon \sup_n Tx_n(s) = \infty\}$ and suppose $\mu(A) > 0$. Since (S, Σ, μ) is a Maharam space, there exists a measurable set $B \subset A$ of positive and finite measure According to Egoroff's theorem, $Tx_n(s) \uparrow \infty$ uniformly on some set $C \subset B$ o. positive measure. Thus, for every natural number k,

$$M_\psi(k1_C) \leqslant \sup_n M_\psi(Tx_n) = \sup_n \varrho(x_n) < \infty.$$

The sublattice $T(L)$ is super order dense in $L^\psi(S, \Sigma, \mu)$, and so $Ty_n \uparrow 1_C$ for some sequence $(y_n) \subset L_+$. Taking arbitrary natural numbers k, j, we obtain

$$\varrho(ky_j) = M_\psi(kTy_j) \leqslant M_\psi(k1_C) < \infty.$$

Therefore we have a contradiction because $\lim_{k\to\infty} \varrho(ky_j) = \infty$ for every $y_j \neq 0$. Thus $\mu(A) = 0$ and $f_0(s) = \sup_n Tx_n(s) \in L^0(S, \Sigma, \mu)$. According to Levi's theorem there exists a function $f \in L^1(S, \Sigma, \mu)$ such that $\psi(Tx_n(s), s) \uparrow f(s)$ μ-almost everywhere. From this fact we get $f_0 \in L^\psi(S, \Sigma, \mu)$, and so the sequence (Tx_n) is topologically bounded. The continuity of T implies the topological boundedness of $\{x_n: n \in N\}$. Therefore this set has a supremum in L.

Let e be a weak unit in a Dedekind complete Orlicz lattice L^ϱ such that $\varrho(e) < \infty$. Define a function $F: R \times L \to [0, \infty)$ by the formula

$$F(r, x) = \varrho\left(P_{(x-re)^+}(e)\right),$$

where $P_{(x-re)^+}$ is the projection on the band generated by the element $(x-re)^+$ Let the symbols T, $L^0(S, \Sigma, \mu)$ have the same meaning as in part I of the proof of the Main Representation Theorem. Note that functions $f, g \in T(L^\varrho)$ are μ-equimeasurable if and only if $F(r, T^{-1}(f)) = F(r, T^{-1}(g))$ for all $r \in R$. Indeed, if $f = T(x)$ then $P_{(x-re)^+}(e) \in C(e)$ and

$$T\left(P_{(x-re)^+}(e)\right) = T\left(\sup_n e \wedge n(x-re)^+\right) = \sup_n 1_S \wedge n(f-r1_S)^+ = 1_{\mathrm{supp}(f-r1_S)^+}.$$

Directly from the definition of the measure μ we have $\varrho\left(P_{(x-re)^+}(e)\right) = \mu\left(\mathrm{supp}(f-r1_S)^+\right)$ so

$$\begin{aligned} F(r, x) &= \mu\left(\mathrm{supp}(f-r1_S)^+\right) = \mu\left(\{s: (f-r1_S)^+(s) \neq 0\}\right) \\ &= \mu\left(\{s: f(s) > r\}\right). \end{aligned}$$

The above reasoning and Theorems 5.1 and 3.2 give

COROLLARY 5.5. *Let L^ϱ be a* DC *Orlicz lattice with a weak unit e, $\varrho(e) < \infty$. If the following condition holds:*

$$\text{(E)} \qquad \left(\forall r \in R \; F(r, x) = F(r, y)\right) \Rightarrow \varrho(x) = \varrho(y),$$

then L^ϱ embeds into a space $L^\varphi(S, \Sigma, \mu) \oplus l^\psi$, where φ is an Orlicz function and μ is finite.

Condition (E) is closely related to the concept of component invariance:

Let ϱ be a modular on a vector lattice L and let $x \in L$. The modular ϱ is called *x-component invariant* if

(a) $\varrho(x) < \infty$.

(b) $\forall r \geqslant 0 \; \forall p \in C(|x|) \; \varrho(rp)\varrho(x) = \varrho(p)\varrho(rx)$.

THEOREM 5.6. *Let L^ϱ be a* DC *Orlicz lattice, and e a weak unit in L, and let $\varrho(e) < \infty$.*

(i) *If L is a non-atomic vector lattice and ϱ satisfies condition* (E), *then ϱ is e-component invariant.*

(ii) *If ϱ is e-component invariant, then ϱ satisfies condition* (E).

Proof. (i). Note first that

$$(*) \qquad \forall a \in [0, 1]\ \exists p \in C(e)\ \varrho(p) = a\varrho(e).$$

Using Corollary 5.3, we can embed L^ϱ in a Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$ where μ is non-atomic. Let $T\colon L^\varrho \to L^\psi(S, \Sigma, \mu)$ be an isomorphism and let T preserve modulars. The equality

$$\nu(A) = M_\psi(T(e)1_A)$$

defines a measure $\nu\colon \Sigma \cap \operatorname{supp} T(e) \to [0, \infty)$. Moreover, $N(\nu) = N(\mu|\Sigma \cap \cap \operatorname{supp} T(e))$ and ν is non-atomic. Fix a number $a \in [0, 1]$. On account of Lapunov's theorem it is possible to find a set $B \in \Sigma \cap \operatorname{supp} T(e)$ such that $\nu(B) = a\nu(\operatorname{supp} T(e))$. Since $T(C(e)) = C(T(e)) = \{T(e)1_A\colon A \in \Sigma\}$, we have $T(e)1_B = T(p)$ for some $p \in C(e)$ (we have the equality $T(C(e)) = C(T(e))$ because $T(L)$ is an ideal). Thus

$$\begin{aligned} a\varrho(e) &= aM_\psi(T(e)) = a\nu(\operatorname{supp} T(e)) = \nu(B) \\ &= M_\psi(T(e)1_B) = M_\psi(T(p)) = \varrho(p). \end{aligned}$$

For every $p \in C(e)$ and $t \geqslant 0$ we have

$$F(r, tp) = \begin{cases} \varrho(e) & \text{if} \quad r < 0, \\ \varrho(p) & \text{if} \quad 0 \leqslant r < t, \\ 0 & \text{if} \quad t \leqslant r; \end{cases}$$

so, if $\varrho(p) = \varrho(q)$ $(p, q \in C(e))$ then, in view of condition (E), $\varrho(tp) = \varrho(tq)$ for all $t \geqslant 0$.

Let $n \in N$. There exist positive disjoint elements $p_i \in C(e)$, $i = 1, \ldots, n$, such that $\bigvee_{i=1}^{n} p_i = e$ and $\varrho(p_i) = n^{-1}\varrho(e)$. Therefore $\varrho(rp_j) = n^{-1}\sum_{i=1}^{n}\varrho(rp_i)$ $= n^{-1}\varrho(re)$ for $j \in \{1, \ldots, n\}$. Using this fact, we will easily find that the relation $\varrho(p) = w\varrho(e)$, where $p \in C(e)$ and $w \in [0, 1]$ is a rational number, implies $\varrho(rp) = w\varrho(re)$ for all $r \geqslant 0$. Finally, let $a \in (0, 1)$ and $\varrho(p) = a\varrho(e)$. Let (w_n) be a sequence of rational numbers increasing to a. We choose (see $(*)$) a sequence $(p_n) \subset C(e)$ increasing to p such that $\varrho(p_n) = w_n\varrho(e)$. Since $rp_n \uparrow rp$ for all $r > 0$, we have

$$\varrho(rp) = \lim_{n\to\infty}\varrho(rp_n) = \lim_{n\to\infty} w_n\varrho(re) = a\varrho(re).$$

Thus $\varrho(rp)\varrho(e) = \varrho(p)\varrho(re)$ for all $p \in C(e)$ and all $r \geqslant 0$.

(ii). Define a function $F'\colon R \times L \to [0, \infty)$ by the formula

$$F'(r, x) = \varrho(P_{(re-x)^+}(e)).$$

The following relations can easily be verified in view of the remarks preceding Corollary 5.5:

(a) $F'(r, x) = \varrho(e) - \lim_{m\to\infty} F(t_m, x)$, where $t_m < t_{m+1}$ and $t_m \uparrow r$.

(b) $\forall r \geqslant 0\ F'(r, |x|) = F'(r, x) - \lim_{m\to\infty} F'(t_m, x)$, where $t_m \in (-r, r)$ and $t_m \downarrow -r$.

These relations give the implication

$$F(r, x) = F(r, y) \text{ for all } r \in \boldsymbol{R} \Rightarrow F'(r, |x|) = F'(r, |y|) \text{ for all } r \in \boldsymbol{R}.$$

Let $x, y \in L$ and let $F(r, x) = F(r, y)$ for all $r \in \boldsymbol{R}$. By Freudenthal's theorem there exists a family of numbers $a_{n,k}$, $n = 1, 2, \ldots$, $k = 0, 1, \ldots, m_n$, such that, putting

$$x_n = \sum_{k=1}^{m_n} a_{n,k-1}\left(P_{(a_{n,k}e-|x|)^+(e)} - P_{(a_{n,k-1}e-|x|)^+(e)}\right),$$

$$y_n = \sum_{k=1}^{m_n} a_{n,k-1}\left(P_{(a_{n,k}e-|y|)^+(e)} - P_{(a_{n,k-1}e-|y|)^+(e)}\right),$$

we get $x_n \uparrow |x|$ and $y_n \uparrow |y|$. Denote $p_{n,k} = P_{(a_{n,k}e-|x|)^+(e)}$ and $q_{n,k} = P_{(a_{n,k}e-|y|)^+(e)}$. Since $p_{n,k-1} \leqslant p_{n,k}$ and $q_{n,k-1} \leqslant q_{n,k}$, we have $p_{n,k} - p_{n,k-1}$, $q_{n,k} - q_{n,k-1} \in C(e)$ and

$$\begin{aligned}\varrho(p_{n,k} - p_{n,k-1}) &= \varrho(p_{n,k}) - \varrho(p_{n,k-1}) = F'(a_{n,k}, |x|) - F'(a_{n,k-1}, |x|)\\ &= F'(a_{n,k}, |y|) - F'(a_{n,k-1}, |y|) = \varrho(q_{n,k}) - \varrho(q_{n,k-1})\\ &= \varrho(q_{n,k} - q_{n,k-1}).\end{aligned}$$

The above equalities and the assumption of component invariance give

$$\varrho\left(a_{n,k-1}(p_{n,k} - p_{n,k-1})\right) = \varrho\left(a_{n,k-1}(q_{n,k} - q_{n,k-1})\right)$$

for $n = 1, 2, \ldots$, $k = 0, 1, \ldots, m_n$. The elements x_n, y_n are linear combinations of disjoint elements, and so $\varrho(x_n^+) = \varrho(y_n^+)$ for all n. Therefore $\varrho(x) = \varrho(y)$.

6. Representation of Orlicz lattices by Orlicz spaces

THEOREM 6.1. *If a* DC *Orlicz lattice* L^ϱ *has a complete disjoint system* $(x_a)_{a\in A}$ *with the following properties*:

(i) ϱ *is* x_a*-component invariant for every* $a \in A$.

(ii) $\forall r \geqslant 0\ \forall a, b \in A\ \varrho(rx_a)\ \varrho(x_b) = \varrho(x_a)\varrho(rx_b)$,

then L^ϱ *embeds into some Orlicz space* $L^\varphi(S, \Sigma, \mu)$, *where* $\varphi(r) = \varrho(rx_a) \times (\varrho(x_a))^{-1}$.

On the other hand, if L^ϱ *embeds into an Orlicz space* $L^\varphi(S, \Sigma, \mu)$ *and the*

range of L contains characteristic functions of sets of finite measure, then L has a complete disjoint system satisfying conditions (i), (ii).

Proof. Use the notation of the second part of the proof of the Main Representation Theorem. Put

$$\varphi(r) = \varrho(rx_a)\big(\varrho(x_a)\big)^{-1}$$

For every $F \in \Sigma_a$ there exists an element $p \in C(x_a)$ such that $\mu_a(F) = \varrho_a(p)$. Thus we have for every $r \geqslant 0$

$$\int_F \varphi(r)\, d\mu_a = \varphi(r)\,\mu_a(F) = \varphi(r)\,\varrho_a(p) = \varphi(r)\,\varrho(p)$$

$$= \varrho(rx_a)\,\varrho(p)\big(\varrho(x_a)\big)^{-1} = \varrho(rp) = \varrho_a(rp) = \int_F \psi_a(r, s)\, d\mu_a.$$

These equalities hold for all $F \in \Sigma_a$, and so $\varphi(r) = \psi_a(r, s)$ for all rational r and all $s \in S_a \setminus B_a$, where B_a is a set belonging to $N(\mu_a)$. Since both functions are left continuous, we have $\varphi(r) = \psi_a(r, s)$ for all $r \geqslant 0$, and for all $s \in S_a \setminus B_a$. Of course, $L^{\varphi}(S_a, \Sigma_a, \mu_a) = L^{\psi_a}(S_a, \Sigma_a, \mu_a)$, so $L^{\varphi}(S, \Sigma, \mu) = \bigoplus_a L^{\varphi}(S_a, \Sigma_a, \mu_a)$ $= \bigoplus_a L^{\psi_a}(S_a, \Sigma_a, \mu_a) = L^{\psi}(S, \Sigma, \mu)$, where (S, Σ, μ) is the direct sum of measure spaces (S_a, Σ_a, μ_a). Therefore the operator TU maps L^{ϱ} into $L^{\varphi}(S, \Sigma, \mu)$.

Let $T\colon L^{\varrho} \to L^{\varphi}(S, \Sigma, \mu)$ be a modular preserving isomorphism and let $T(L)$ contain characteristic functions of sets of finite measure. Choose a complete disjoint system $(f_a)_{a \in A}$ in $T(L)$. The sets $\operatorname{supp} f_a$ have σ-finite measure, and so $\operatorname{supp} f_a = \bigcup_{n=1}^{\infty} B_a^n$, where (B_a^n) is a sequence of disjoint sets of finite measure. From our assumption $1_{B_a^n} \in T(L)$, and thus $(t1_{B_a^n})$ is a complete disjoint system in L^{φ}, where t is a number such that $0 < \varphi(t) < \infty$. The family $\big(T^{-1}(t1_{B_a^n})\big)$ is a complete disjoint system in L satisfying conditions (i), (ii).

This theorem states in particular that if a DC Orlicz lattice L^{ϱ} has a weak unit e and ϱ is e-component invariant then L^{ϱ} embeds into an Orlicz space. It is therefore natural to look for answers to the following two questions:

(a) If an Orlicz lattice L^{ϱ} has a weak unit does L also have a weak unit such that ϱ is component invariant with respect to it?

(b) How many such units exist in L?

The solution of the first problem is very simple. The space $L^p[0, 1] \oplus L^q[1, 2]$ $(p \neq q)$ considered with a modular defined by $\varrho\big((x, y)\big)$ $= \int_0^1 |x(s)|^p\, ds + \int_1^2 |y(s)|^q\, ds$ has a weak unit – the element $(1_{[0,1]}, 1_{[1,2]})$ for example – but there exists no weak unit such that ϱ is component invariant

with respect to it. Indeed, for every weak unit (x, y) in $L^p \oplus L^q$ the elements $(x, 0)$, $(0, y)$ are its components, and so if ϱ were (x, y)-component invariant then we would have the following equality holding for all r:

$$r^p = \frac{\varrho(r(x, 0))}{\varrho((x, 0))} = \frac{\varrho(r(0, y))}{\varrho((0, y))} = r^q$$

– a contradiction.

On the other hand, the integral modular on $L^p(S, \Sigma, \mu)$ $(0 < p < \infty)$ is component invariant with respect to every $f \in L^p(S, \Sigma, \mu)$. If (S, Σ, μ) is a finite measure space and φ is an Orlicz function such that $\varphi(t_0) < \infty$ for some $t_0 > 0$, then the integral modular generated by φ is $t_0 1_S$-component invariant.

We can characterize simply the ranges of functions with respect to which the integral modular is component invariant. Let φ be an Orlicz function taking only finite values. A number $a > 0$ is called a *multiplier* of φ if $\varphi(ar)\,\varphi(1) = \varphi(a)\,\varphi(r)$ holds for all $r \geqslant 0$. It is not difficult to verify that a set $\mathcal{M}_\varphi$ of all multipliers of φ is a closed group under multiplication. Therefore $\mathcal{M}_\varphi = (0, \infty)$ or $\mathcal{M}_\varphi = \{1\}$ or $\mathcal{M}_\varphi$ is cyclic.

THEOREM 6.2. *Let (S, Σ, μ) be a finite measure space and let φ be an Orlicz function satisfying the (Δ_2^∞)-condition. If M_φ is f-component invariant and $f \geqslant 0$, then*

$$\exists\, B \in N(\mu)\ \forall\, s \in (\operatorname{supp} f) \setminus B\ f((\operatorname{supp} f) \setminus B) \subset f(s)\, \mathcal{M}_\varphi.$$

Proof. Since M_φ is f-component invariant, the equality

$$M(f) \int_A \varphi(rf(s))\, d\mu = M(rf) \int_A \varphi(f(s))\, d\mu$$

holds for all $A \in \Sigma$ and all $r \geqslant 0$. Thus, for every r separately,

$$(+) \qquad M(f)\, \varphi(rf(s)) = M(rf)\, \varphi(f(s))$$

μ-almost everywhere. The function φ is left continuous, and so there exists a set $B \in N(\mu)$ such that $(+)$ holds for all $r \geqslant 0$ and all $s \in S \setminus B$. Therefore, if $s_1, s_2 \in (\operatorname{supp} f) \setminus B$, then

$$(*) \qquad \varphi(rf(s_1))\, \varphi(f(s_2)) = \varphi(rf(s_2))\, \varphi(f(s_1)).$$

Let $s_0 \in (\operatorname{supp} f) \setminus B$ be fixed. Take $s \in (\operatorname{supp} f) \setminus B$. Assume that $c = f(s)/f(s_0) > 1$. Putting $r' = rf(s_0)$ in $(*)$, we get

$$\varphi(r'c)\, \varphi(f(s_0)) = \varphi(r')\, \varphi(f(s));$$

so $\varphi(1) = \varphi(f(s_0))\,(\varphi(f(s)))^{-1}\, \varphi(c)$. Therefore $\varphi(rc)\,\varphi(1) = \varphi(r)\,\varphi(c)$ for all $r \geqslant 0$. This means that $c \in \mathcal{M}_\varphi$. In other words, $f(s) \in f(s_0)\, \mathcal{M}_\varphi$. If $c' = f(s_0)/f(s) > 1$ then, putting $r' = rf(s)$ in $(*)$, we get $c' \in \mathcal{M}_\varphi$. But $\mathcal{M}_\varphi$ is a group, and so $c'^{-1} \in \mathcal{M}_\varphi$, i.e., $f(s) \in f(s_0)\, \mathcal{M}_\varphi$.

In the case $f(s)=f(s_0)$ we have $f(s)(f(s_0))^{-1}=1\in\mathcal{M}_\varphi$. Finally, for all $s\in(\operatorname{supp} f)\setminus B$ we obtain $f(s)\in f(s_0)\mathcal{M}_\varphi$.

Now we can characterize those Orlicz spaces generated by Orlicz functions which satisfy the (Δ_2^∞)-condition, where integral modulars are component invariant only with respect to functions of the form $r1_A$.

THEOREM 6.3. *Let (S, Σ, μ) be a finite measure space, and φ an Orlicz function satisfying the (Δ_2^∞)-condition. Assume additionally that S is not an atom of the measure μ. Then the following statements are equivalent:*

(i) $\forall a, b>0\ a\neq b\ \exists r>0\ \varphi(rb)\varphi(a)\neq\varphi(b)\varphi(ra)$.

(ii) *M_φ is component invariant only with respect to functions of the form $t1_A$.*

(iii) $\mathcal{M}_\varphi=\{1\}$.

Proof. (i) $\Rightarrow$ (iii). Suppose $c\in\mathcal{M}_\varphi\setminus\{1\}$. Then, by (i), $\varphi(r_0c)\varphi(1)\neq\varphi(c)\varphi(r_0)$ for some r_0, and so $c\notin\mathcal{M}_\varphi$ — a contradiction.

(iii) $\Rightarrow$ (ii). See the preceding theorem.

(ii) $\Rightarrow$ (i). Suppose now that condition (i) does not hold, i.e., $\exists a, b>0$ $a\neq b$ $\forall r>0$ $\varphi(rb)\varphi(a)=\varphi(b)\varphi(ra)$. Let $A\in\Sigma$ and $\mu(A)\in(0, \mu(S))$. Put $f=a1_A+b1_{(S\setminus A)}$. By elementary calculations we obtain the equality

$$M(rf1_C)M(f)=M(f1_C)M(rf),$$

which holds for every $C\in\Sigma$. Thus M is f-component invariant, but f is not equivalent to any function of the form $t1_C$ — a contradiction.

The functions

$$\varphi(r)=\sum_{k=1}^{n}a_kr^k,\qquad n>1,\ a_k>0 \text{ for } k=1, 2, \ldots, n,$$

$$\varphi(r)=\ln(r+1)$$

are examples of Orlicz functions satisfying conditions (Δ_2^∞) and (i) of Theorem 6.3.

Every Orlicz function φ taking only finite values which does not satisfy condition (i) has the form

$$(++)\qquad \varphi(r)=d^k\varphi_0(c^{-k}r)\quad \text{for}\quad r\in[c^{k-1}, c^k],$$

k runs over the set of integers and c, d are some numbers greater that 1, φ_0: $[c^{-1}, 1]\to(0, \infty)$ is non-decreasing, left continuous, $\varphi_0(1)=d\varphi_0(c^{-1})$.

Indeed, every function φ defined by formula $(++)$ and taking value zero at the point $r=0$ is an Orlicz function (the condition $d>1$ ensures its continuity at the point $r=0$). Clearly, φ does not satisfy condition (i) because $\varphi(rc)\varphi(1)=\varphi(c)\varphi(r)$ for all $r>0$.

Let φ be an Orlicz function not satisfying condition (i). Since $\varphi(rb)\varphi(a)=\varphi(b)\varphi(ra)$ for some $a, b>0$, $a\neq b$ for all $r>0$, assuming $a<b$ and

putting $c = b/a$, $d = \varphi(b)/\varphi(a)$, we get

$$\varphi(ct) = d\varphi(t)$$

for all $t \geqslant 0$ (thus it is evident that every Orlicz function not satisfying condition (i) satisfies the (Δ_2)-condition). We have $d > 1$. Indeed, if the equality $d = 1$ were true, it would follow that $\varphi(t) = \varphi(c)$ for $t > 0$, and by the continuity of φ at the point $t = 0$ we would get $\varphi(c) = 0$, and so $c = 0$ – a contradiction. Put $\varphi_0 = \varphi | [c^{-1}, 1]$. The function φ_0 is positive, non-decreasing, left continuous and formula $(++)$ holds.

THEOREM 6.4. *Let $L^\varphi(S, \Sigma, \mu)$ be an Orlicz space and let μ be finite. If there exist disjoint sets A, B of positive measure such that the integral modular is component invariant with respect to every function $f_a = 1_B + a1_A$ $(a > 0)$, then $\varphi(u) = \varphi(1)u^p$ for some $p \in (0, \infty)$.*

Proof. The component invariance of the modular with respect to functions f_a gives for arbitrary $a, r > 0$:

$$\begin{aligned}\big(\varphi(r)\cdot\mu(B) + \varphi(ra)\cdot\mu(A)\big)\varphi(1)\cdot\mu(B) &= M(rf_a)\,M(f_a 1_B)\\ &= M(rf_a 1_B)\,M(f_a) = \varphi(r)\mu(B)\big(\varphi(1)\mu(B) + \varphi(a)\mu(A)\big).\end{aligned}$$

Therefore $\varphi(1)\varphi(ra) = \varphi(r)\varphi(a)$. This equality together with continuity at zero and left continuity implies that the function φ has the form $\varphi(r) = \varphi(1)r^p$ or $\varphi = 0$.

Now we will present necessary and sufficient conditions for embedding an Orlicz lattice into the space $L^p(S, \Sigma, \mu)$ $(p \in [0, \infty))$.

THEOREM 6.5. *Let L^ϱ be a* DC *Orlicz lattice. Then the following statements are equivalent:*

(i) *ϱ is x-component invariant for every $x \in L$.*

(ii) *L^ϱ embeds into the space $L^p(S, \Sigma, \mu)$ for some $p \in (0, \infty)$.*

(iii) *ϱ is a finite modular and, for every x, $y \in L$ and every $r \geqslant 0$, $\varrho(rx)\varrho(y) = \varrho(x)\varrho(ry)$.*

Proof. Implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are obvious. We will show (i) $\Rightarrow$ (ii). To this end let $(x_a)_{a\in A}$ be a complete disjoint system of L. Denote $\varphi_a(r) = \varrho(rx_a)\big(\varrho(x_a)\big)^{-1}$. The function φ_a is an Orlicz function and φ_a takes only finite values. Elements x_a, x_b $(a \neq b)$ are components of the element $x_a + x_b$, and so $\varphi_a(r) = \varphi_b(r)$ for all $a, b \in A$. Moreover, for an arbitrary $c > 0$ the modular ϱ is $(cx_a + x_b)$-component invariant. Thus

$$\big(\varrho(rcx_a) + \varrho(rx_b)\big)\varrho(x_b) = \varrho(rx_b)\big(\varrho(cx_a) + \varrho(x_b)\big).$$

Therefore $\varrho(rcx_a)\varrho(x_b) = \varrho(rx_b)\varrho(cx_a)$. This equality gives at once

$$\varphi_a(rc) = \frac{\varrho(rcx_a)}{\varrho(x_a)} = \frac{\varrho(rx_b)}{\varrho(x_b)}\cdot\frac{\varrho(cx_a)}{\varrho(x_a)} = \varphi_b(r)\varphi_a(c) = \varphi_a(r)\varphi_a(c).$$

Since φ_a is left continuous and continuous at zero, we have $\varphi_a(r) = r^p$ for some $p \in (0, \infty)$. Now it is sufficient to refer to the proof of Theorem 6.1.

THEOREM 6.6. *Let (S, Σ, μ) denote a finite measure space and let ϱ be a modular on $L^0(S, \Sigma, \mu)$. If ϱ determines the topology of convergence in measure μ, then* $\sup\{\varrho(x1_{(S\setminus W)})\colon x \in L^0(S, \Sigma, \mu)\} < \infty$, *where W is a sum of some finite family of atoms.*

On the other hand, if for some set $W \subset S$ which is a sum of a finite family of atoms the modular $\varrho|L^0(S\setminus W, \Sigma \cap (S\setminus W), \mu|\Sigma \cap (S\setminus W))$ is finite, then ϱ determines on $L^0(S, \Sigma, \mu)$ the topology of convergence in measure μ.

Proof. Assume first that ϱ determines the convergence in measure. Therefore

$$\exists\, \delta > 0 \ \forall x \in L^0(S, \Sigma, \mu)\ \ \mu(\operatorname{supp} x) < \delta \Rightarrow \varrho(x) < 1.$$

Using Saks' lemma, we get a decomposition

$$S = B_1 \cup \ldots \cup B_m \cup A_1 \cup \ldots \cup A_k$$

where $\mu(B_i) < \delta$, A_j are atoms and $\mu(A_j) \geqslant \delta$. Putting $W = \bigcup_{j=1}^{k} A_j$, we have for every $x \in L^0(S, \Sigma, \mu)$

$$\varrho(x1_{(S\setminus W)}) = \varrho\Big(\sum_{i=1}^{m} x1_{B_i}\Big) = \sum_{i=1}^{m} \varrho(x1_{B_i}) \leqslant m.$$

Now we will proceed with the second part of the theorem. The implication $|||x_n|||_\varrho \to 0 \Rightarrow x_n \xrightarrow{\mu} 0$ has been proved in [7], p. 802.

If $x_n \xrightarrow{\mu} 0$ then of course $|||x_n 1_W|||_\varrho \to 0$ because $x_n 1_W = \sum_{j=1}^{k} x_n 1_{A_j}$ $= \sum_{j=1}^{k} a_j^n 1_{A_j}$, where $\{a_j^n\} = x_n(A_j)$ and $a_j^n \to 0$ if n tends to infinity.

Since a sequence (x_n) of measurable functions finite almost everywhere converging in measure to a measurable function x finite almost everywhere contains a subsequence (x_{n_k}) converging to x almost everywhere, it is enough to show that

$$0 \leqslant x_n 1_{(S\setminus W)} \to 0 \ \mu\text{-almost everywhere} \ \Rightarrow\ \varrho(x_n 1_{(S\setminus W)}) \to 0.$$

Denote $S_1 = S\setminus W$ and $L_1^0 = L^0(S_1, \Sigma \cap S_1, \mu|\Sigma \cap S_1)$. According to Egoroff's theorem there exists a decreasing sequence of sets (C_k) such that $\mu\big(\bigcap_{k=1}^{\infty} C_k\big) = 0$ and $x_n 1_{(S_1\setminus C_k)} \to 0$ if $n \to \infty$ uniformly on S_1. We can of course assume $\bigcap_k C_k = \emptyset$. We have

$$\varrho(x_n 1_{C_k}) \to 0 \text{ if } k \to \infty \text{ uniformly with respect to } n, \text{ i.e., } \sup_n \varrho(x_n 1_{C_k}) \to 0.$$

Indeed, otherwise we would have for some subsequence $(x'_n) \subset (x_n)$ and some $\varepsilon > 0$

$$\varrho(x'_k 1_{C_k}) > \varepsilon > 0 \quad \text{for} \quad k = 1, 2,$$

Since ϱ is finite on L_1^0, we have $\varrho(x 1_{C_k}) \to 0$ for every $x \in L_1^0$ because of the order continuity of $\|\|\cdot\|\|_\varrho$ restricted to L_1^0. Therefore we would find a sequence (k_m) such that

$$\varrho(x'_{k_m} 1_{(C_{k_m} \setminus C_{k_{m+1}})}) > \varepsilon, \quad m = 1, 2,$$

A function $y = \sup_m x'_{k_m} 1_{(C_{k_m} \setminus C_{k_{m+1}})} \in L_1^0$, and so $\varrho(y) < \infty$, but

$$\varrho(y) = \sum_{m=1}^{\infty} \varrho(x'_{k_m} 1_{(C_{k_m} \setminus C_{k_{m+1}})}) = \infty,$$

a contradiction.

Fix a number $\varepsilon > 0$. Using the fact that $\sup_n \varrho(x_n 1_{C_k}) \to 0$, we choose a natural number k_1 such that $\sup_n \varrho(x_n 1_{C_{k_1}}) \leqslant \varepsilon$. Since $x_n 1_{(S_1 \setminus C_k)} \to 0$ uniformly on S_1, there exist a sequence (a_n) tending to zero and a natural number n_0 such that

$$0 \leqslant x_n 1_{(S_1 \setminus C_{k_1})} \leqslant a_n 1_{S_1} \quad \text{for} \quad n \geqslant n_0.$$

Therefore $\varrho(x_n 1_{(S_1 \setminus C_{k_1})}) \leqslant \varrho(a_n 1_{S_1}) \to 0$ if $n \to \infty$. Finally

$$\varrho(x_n 1_{(S \setminus W)}) = \varrho(x_n 1_{S_1}) = \varrho(x_n 1_{(S_1 \setminus C_{k_1})}) + \varrho(x_n 1_{C_{k_1}}) \leqslant \varrho(a_n 1_{S_1}) + \varepsilon < 2\varepsilon$$

for sufficiently large n.

We say that an *Orlicz lattice L^ϱ is isomodular to a space $L^0(S, \Sigma, \mu)$* ($\mu(S) < \infty$) if there exists a modular M_1 defined on $L^0(S, \Sigma, \mu)$ and determining the topology of convergence in measure such that L^ϱ is isomodular to $(L^0(S, \Sigma, \mu))^{M_1}$.

THEOREM 6.7. *The following statements are equivalent:*

(i) *L^ϱ is isomodular to $L^0(S, \Sigma, \mu)$ where $\mu(S) < \infty$.*

(ii) *L has a weak unit e, L^ϱ has the σ-Levi property and* $\sup\{\varrho(x - P(x)): x \in L\} < \infty$, *where P is the projection on a band generated by some finite family of atoms $\{x_1, \ldots, x_n\}$.*

Proof. (i) $\Rightarrow$ (ii). Denote by M a modular on $L^0(S, \Sigma, \mu)$ determining the topology of convergence in measure μ and let T: $L^\varrho \to (L^0(S, \Sigma, \mu))^M$ be a modular preserving isomorphism.

The element $e = T^{-1}(1_S)$ is a weak unit in L. From the preceding theorem we know that $\sup\{M(f 1_{(S \setminus W)}): f \in L^0(S, \Sigma, \mu)\} < \infty$ where the set W is a sum of some finite family of atoms $S_1, \ldots, S_n$. Elements $T^{-1}(1_{S_i}) = x_i$ are atoms in L and they are also components of the unit e. Denoting

$p = x_1 \vee \quad \vee x_n$, we have

$$L = \{e-p\}^{\perp\perp} \oplus \operatorname{lin}\{x_1, \ldots, x_n\}.$$

Let P be the projection on a band $\operatorname{lin}\{x_1, \ldots, x_n\}$. Therefore for $x \in L_+$

$$T(x-Px) = T\big(\sup_n x \wedge n(e-p)\big) = \sup_n Tx \wedge n1_{(S\setminus W)} = Tx \cdot 1_{(S\setminus W)}.$$

Thus

$$\sup_{x\in L} \varrho(x-Px) = \sup_{x\in L} M\big(T(x-Px)\big) = \sup_{x\in L} M\big(Tx \cdot 1_{(S\setminus W)}\big) < \infty.$$

(ii) $\Rightarrow$ (i). The assumptions imply the existence of a Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$ with a finite measure μ and a modular preserving isomorphism "onto" T: $L^\varrho \to L^\psi(S, \Sigma, \mu)$ such that $T\big(\operatorname{lin} C(e)\big) = \operatorname{lin}\{1_A\colon A \in \Sigma\}$.

Functions Tx_i are atoms in $L^\psi(S, \Sigma, \mu)$, and thus sets $S_i = \operatorname{supp} Tx_i$ are atoms of the measure μ. Therefore the functions Tx_i have the form $T(x_i) = a_i 1_{S_i}$. For $x \in L_+$ we obtain

$$\begin{aligned} T(x-Px) &= T(x) - T\big(\sup_k x \wedge k(x_1 \vee \quad \vee x_n)\big) \\ &= T(x) - \sup_k T(x) \wedge k \sum_{i=1}^n T(x_i) \\ &= T(x) - T(x) \cdot 1_{\bigcup_{i=1}^n S_i} = T(x) \cdot 1_{(S\setminus \bigcup_{i=1}^n S_i)}, \end{aligned}$$

and so

$$\begin{aligned} \sup\{M(f1_{(S\setminus \bigcup_{i=1}^n S_i)})\colon f \in L^\psi(S, \Sigma, \mu)\} &= \sup\{M\big(T(x)\cdot 1_{(S\setminus \bigcup_{i=1}^n S_i)}\big)\colon x \in L\} \\ &= \sup\{\varrho(x-Px)\colon x \in L\} < \infty. \end{aligned}$$

To end the proof we need only to show the equality $L^\varrho(S, \Sigma, \mu) = L^\psi(S, \Sigma, \mu)$ and to use Theorem 6.6.

Let $g \in L^\varrho_+(S, \Sigma, \mu)$. Obviously, $g1_{\bigcup_{i=1}^n S_i}$ belongs to $L^\psi(S, \Sigma, \mu)$ as a simple function, because $\operatorname{lin}\{1_A\colon A \in \Sigma\} = T\big(\operatorname{lin} C(e)\big)$. If (g_m) is a sequence of simple functions increasing to $g1_{(S\setminus \bigcup_{i=1}^n S_i)}$, then, applying the left continuity of the function $\psi(\cdot, s)$ and the monotone convergence theorem, we will get

$$\begin{aligned} M(g \cdot 1_{(S\setminus \bigcup_{i=1}^n S_i)}) &= \sup_m M(g_m) \leqslant \sup\{M(f1_{(S\setminus \bigcup_{i=1}^n S_i)})\colon f \in L^\psi(S, \Sigma, \mu)\} \\ &= \sup\{\varrho(x-Px)\colon x \in L\} < \infty. \end{aligned}$$

Finally, $g \in L^\psi(S, \Sigma, \mu)$.

7. Ultraproducts of some Orlicz lattices

Let X be a compact topological space, $\mathcal{U}$ an ultrafilter on some set $I \neq \emptyset$ and $(x_i)_{i\in I}$ a family of elements of X. The element $x_0 \in X$ is called a *limit of* (x_i) *over the ultrafilter* $\mathcal{U}$ if $\{i\colon x_i \in V\} \in \mathcal{U}$ for every neighbourhood V of x_0. Since every ultrafilter on a compact space is convergent, it is not difficult to check that the family $(x_i)_{i\in I}$ always has a unique limit over $\mathcal{U}$. This limit will be denoted by $\lim_{\mathcal{U}} x_i$. In particular, if $(x_i)_{i\in I} \subset \boldsymbol{R}$ and $\sup|x_i| < \infty$, then $\lim_{\mathcal{U}} x_i$ exists in $\boldsymbol{R}$.

The following simple statements hold for families $(x_i)_{i\in I}$, $(y_i)_{i\in I}$ of real numbers with $\sup_i |x_i| + \sup_i |y_i| < \infty$:

1. $\lim_{\mathcal{U}}(x_i + y_i) = \lim_{\mathcal{U}} x_i + \lim_{\mathcal{U}} y_i$.

2. $\lim_{\mathcal{U}}(x_i y_i) = \lim_{\mathcal{U}} x_i \cdot \lim_{\mathcal{U}} y_i$.

3. *If* $x_i \in [-K, K]$ *for all* $i \in I$ *and a function* $f\colon [-K, K] \to \boldsymbol{R}$ *is continuous, then* $\lim_{\mathcal{U}} f(x_i) = f(\lim_{\mathcal{U}} x_i)$.

4. *If* $x_i \leqslant y_i$ *for* $i \in I$, *then* $\lim_{\mathcal{U}} x_i \leqslant \lim_{\mathcal{U}} y_i$.

5. $\{i\colon x_i = y_i\} \in \mathcal{U}$ *implies* $\lim_{\mathcal{U}} x_i = \lim_{\mathcal{U}} y_i$.

6. *If* $\{i\colon x_i \geqslant a\} \in \mathcal{U}$, *then* $\lim_{\mathcal{U}} x_i \geqslant a$ (*similarly for the converse inequality*), $\lim_{\mathcal{U}} x_i > a$ *implies* $\{i\colon x_i > a\} \in \mathcal{U}$ (*similarly for the converse inequality*).

Let $(E_i, |||\cdot|||_i)$, $i \in I$, be a family of F^*-spaces (= metrizable topological linear spaces). Define

$$l^\infty(E_i, I) = \Big\{(x_i) \in \prod_i E_i\colon \sup_i |||tx_i|||_i \to 0 \text{ for } t \to 0\Big\}.$$

The function $|||(x_i)||| = \sup_i |||x_i|||_i$ is an F-norm on $l^\infty(E_i, I)$. For an ultrafilter $\mathcal{U}$ on I we will denote by $N_{\mathcal{U}}$ the following subspace of $l^\infty(E_i, I)$

$$N_{\mathcal{U}} = \{(x_i) \in l^\infty(E_i, I)\colon \lim_{\mathcal{U}} |||x_i|||_i = 0\}.$$

The subspace $N_{\mathcal{U}}$ is closed because the transformation $\lim_{\mathcal{U}}\colon l^\infty(E_i, I) \to \boldsymbol{R}$ is continuous:

$$\big|\lim_{\mathcal{U}} |||x_i|||_i - \lim_{\mathcal{U}} |||y_i|||_i\big| \leqslant \lim_{\mathcal{U}} |||x_i - y_i|||_i \leqslant |||(x_i - y_i)|||.$$

The *ultraproduct* $(E_i)_{\mathcal{U}}$ of the family of F^*-spaces (E_i), $i \in I$, with respect to the ultrafilter $\mathcal{U}$ is the quotient space $l^\infty(E_i, I)/N_{\mathcal{U}}$ equipped with the quotient F-norm.

If $|||\cdot|||$ denotes the F-norm of an ultraproduct (i.e., the quotient F-norm), then

$$|||[(x_i)]||| = \lim_{\mathcal{U}} |||x_i|||_i$$

where $[(x_i)]$ is the corresponding equivalence class of $(x_i) \in l^\infty(E_i, I)$. Indeed, put $a = \lim_{\mathcal{U}} |||x_i|||_i$ and $I_n = \{i: |||x_i|||_i < a + n^{-1}\}$. Obviously, $I_n \in \mathcal{U}$. Moreover, define

$$y_i^n = \begin{cases} 0 & \text{for} \quad i \in I_n, \\ -x_i & \text{for} \quad i \in I \setminus I_n. \end{cases}$$

Therefore $\lim_{\mathcal{U}} |||y_i^n|||_i = 0$ for all $n \in N$. Thus

$$|||[(x_i)]||| \leqslant \sup_i |||x_i + y_i^n|||_i = \sup\{|||x_i|||_i: i \in I_n\} \leqslant a + n^{-1}$$

for every n, and so $|||[(x_i)]||| \leqslant a$. The converse inequality is clear.

If E_i are F^*-lattices, then $l^\infty(E_i, I)$ is a vector lattice with respect to the natural "pointwise" order: $(x_i) \leqslant (y_i)$ iff $x_i \leqslant y_i$ for all $i \in I$. The subspace $N_{\mathcal{U}}$ is an ideal in $l^\infty(E_i, I)$, and so $(E_i)_{\mathcal{U}}$ is a vector lattice with respect to the following order:

$$[(x_i)] \leqslant [(y_i)] \quad \text{iff} \quad \exists (u_i), (v_i) \in N_{\mathcal{U}} \ \forall i \ x_i + u_i \leqslant y_i + v_i.$$

Let $L_i^{\varrho_i}$ $(i \in I)$ be a family of Orlicz lattices with modulars satisfying the following two conditions:

(*) $$\exists K > 0 \ \forall x_i \in L_i \ \ \varrho_i(2x_i) \leqslant K\varrho_i(x_i).$$

(**) $$0 \leqslant a_k \uparrow 1 \Rightarrow \Big(\forall (x_i) \in \prod_i L_i \ \sup_i \varrho_i(x_i) < \infty \Rightarrow \varrho_i(a_k x_i) \uparrow \varrho_i(x_i) \text{ uniformly for } i \in I\Big).$$

If ϱ_i are convex or concave modulars (i.e., $\varrho(ax+by) \geqslant a\varrho(x) + b\varrho(y)$ for $a, b \geqslant 0$, $a+b=1$) satisfying condition (*), then condition (**) is also satisfied. Indeed, in the case of convex modulars, using the inequality (ϱB''), we get for an arbitrary $a \in (0; 1)$:

$$0 \leqslant \varrho_i(x_i) - \varrho_i(a_k x_i) \leqslant \frac{a}{2}\left(\varrho_i(2a_k x_i) + \varrho_i\left(\frac{2}{a}(a_k - 1)x_i\right)\right)$$

$$\leqslant \frac{Ka}{2}\left(\varrho_i(x_i) + (1-a_k)\varrho_i(a^{-1}x_i)\right) \leqslant \frac{Ka}{2}\sup_i \varrho_i(x_i) + \frac{Ka}{2}(1-a_k)\sup_i \varrho_i(a^{-1}x_i).$$

Fix $\varepsilon > 0$ and let a be so small that $Ka \sup_i \varrho_i(x_i) < \varepsilon$. For this choice of a we find $k_0 \in N$ such that $\left(Ka \sup_i \varrho_i(a^{-1}x_i)\right)(1-a_k) < \varepsilon$ if $k \geqslant k_0$. Finally, for every $k \geqslant k_0$ and all $i \in I$,

$$0 \leqslant \varrho_i(x_i) - \varrho_i(a_k x_i) < \varepsilon.$$

If ϱ_i are concave modulars, then $\varrho_i(ax_i) \geqslant a\varrho_i(x_i)$ for $a \in [0, 1]$, and so

$$0 \leqslant \varrho_i(x_i) - \varrho_i(a_k x_i) \leqslant (1-a_k) \sup_i \varrho_i(x_i) \to 0 \quad \text{if} \quad k \to \infty.$$

Condition (*) implies the following two equivalences:

(i) $(x_i) \in l^\infty(L_i^{\varrho_i}, I) \Leftrightarrow \sup_i \varrho_i(tx_i) \to 0$ *if* $t \to 0$.

(ii) *If* $(x_i) \in l^\infty(L_i^{\varrho_i}, I)$, *then* $\lim_{\mathcal{U}} |||x_i|||_{\varrho_i} = 0 \Leftrightarrow \lim_{\mathcal{U}} \varrho_i(x_i) = 0$.

THEOREM 7.1. *Let $L_i^{\varrho_i}$ be a family of Orlicz lattices complete with respect to F-norms $|||\cdot|||_{\varrho_i}$ and with modulars satisfying conditions (*) and (**). The function ϱ: $(L_i^{\varrho_i})_{\mathcal{U}} \to [0, \infty)$ defined by the formula*

$$\varrho([(x_i)]) = \lim_{\mathcal{U}} \varrho_i(x_i)$$

is a modular on $(L_i^{\varrho_i})_{\mathcal{U}}$ and the F-norm $|||\cdot|||_\varrho$ is equal to the original F-norm of an ultraproduct.

Proof. First we will verify that for $(y_i) \in N_{\mathcal{U}}$ and $(x) \in l^\infty(L_i^{\varrho_i}, I)$ we have

$$\lim_{\mathcal{U}} \varrho_i(x_i + y_i) = \lim_{\mathcal{U}} \varrho_i(x_i). \tag{+}$$

This equality guarantees that ϱ is well-defined. Thus, let $(y_i) \in N_{\mathcal{U}}$. The following inequality holds for every number $a \in (0, 1)$:

$$\varrho_i(a(x_i + y_i)) \leqslant \varrho_i(x_i) + \varrho_i\left(\frac{a}{1-a} y_i\right).$$

From this fact and equivalence (ii) we get

$$\lim_{\mathcal{U}} \varrho_i(a(x_i + y_i)) \leqslant \lim_{\mathcal{U}} \varrho_i(x_i).$$

Suppose that $b = \lim_{\mathcal{U}} \varrho_i(x_i) < \lim_{\mathcal{U}} \varrho_i(x_i + y_i)$. Take a number $\varepsilon > 0$ such that $b + \varepsilon < \lim_{\mathcal{U}} \varrho_i(x_i + y_i)$. Therefore $\{i\colon \varrho_i(x_i + y_i) > b + \varepsilon\} \in \mathcal{U}$. According to condition (**) there exists a number $a_0 \in (0, 1)$ such that $\{i\colon \varrho_i(a_0(x_i + y_i)) > b + \varepsilon/2\} \supset \{i\colon \varrho_i(x_i + y_i) > b + \varepsilon\} \in \mathcal{U}$, and so

$$\lim_{\mathcal{U}} \varrho_i(a_0(x_i + y_i)) \geqslant b + \varepsilon > b,$$

but this is impossible. Thus the inequality $\lim_{\mathcal{U}} \varrho_i(x_i + y_i) \leqslant \lim_{\mathcal{U}} \varrho_i(x_i)$ must hold. We will get the converse inequality if we interchange the elements $(x_i + y_i)$ and x_i in the above reasoning.

Now we will show that the function ϱ is a modular. The properties (ϱ1) and (ϱ2) are immediate consequences of equivalence (ii) and equality (+).

Suppose $\lim_{\mathcal{U}} \varrho_i(t_n x_i) > \varepsilon$ for some element $(x_i) \in l^\infty(L_i^{\varrho_i}, I)$, some number

$\varepsilon\in(0, 1)$ and some sequence (t_n) tending to zero. Then $\{i\colon \varrho_i(t_n x_i) > \varepsilon\}\in\mathcal{U}$ for every $n\in N$ and $\{i\colon |||t_n x_i|||_{\varrho_i} \geqslant \varepsilon\} \supset \{i\colon \varrho_i(\varepsilon^{-1} t_n x_i) > \varepsilon\} \supset \{i\colon \varrho_i(t_n x_i) > \varepsilon\}$; so $\lim_{\mathcal{U}} |||t_n x_i|||_{\varrho_i} \geqslant \varepsilon$ for all natural n. However, $\lim_{\mathcal{U}} |||t_n x_i|||_{\varrho_i} \leqslant \sup_i |||t_n x_i|||_{\varrho_i}$ $= |||t_n(x_i)||| \to 0$ for $n\to\infty$ because $(x_i)\in l^\infty(L_i^{\varrho_i}, I)$, a contradiction. Therefore the function ϱ satisfies condition (ϱ3).

If $[(x_i)]\perp[(y_i)]$ then $\lim_{\mathcal{U}} \varrho_i(|x_i|\wedge|y_i|) = 0$. Thus

$$\begin{aligned}\varrho([(x_i)]+[(y_i)]) = \varrho(|[(x_i)]-[(y_i)]|) &= \lim_{\mathcal{U}} \varrho_i(|x_i|-|y_i|)\\ &= \lim_{\mathcal{U}} \varrho_i((|x_i|-|y_i|)^+)+\lim_{\mathcal{U}} \varrho_i((|x_i|-|y_i|)^-)\\ &= \lim_{\mathcal{U}} \varrho_i(|x_i|-(|x_i|\wedge|y_i|))+\lim_{\mathcal{U}} \varrho_i(|y_i|-(|x_i|\wedge|y_i|))\\ &= \lim_{\mathcal{U}} \varrho_i(x_i)+\lim_{\mathcal{U}} \varrho_i(y_i) = \varrho([(x_i)])+\varrho([(y_i)]),\end{aligned}$$

and so ϱ satisfies condition (ϱ5). The formula

$$|||[(x_i)]|||_\varrho = \inf\left\{\varepsilon > 0\colon \varrho\left(\frac{[(x_i)]}{\varepsilon}\right)\leqslant\varepsilon\right\}$$

defines an F-norm because, as follows immediately from the definition, ϱ satisfies condition (ϱB). Now we will prove the equality $|||\cdot|||_\varrho = |||\cdot|||$, where $|||\cdot|||$ denotes the original F-norm of the ultraproduct. Clearly it is enough to show that $|||\cdot|||_\varrho = \lim_{\mathcal{U}} |||\cdot|||_{\varrho_i}$.

Let $a = |||[(x_i)]|||_\varrho$ and let $\eta > 0$ be fixed. We have

$$\lim_{\mathcal{U}} \varrho_i\left(\frac{x_i}{a+\eta}\right) = \varrho\left(\frac{[(x_i)]}{a+\eta}\right)\leqslant a+\eta < a+2\eta.$$

Thus

$$\left\{i\colon \varrho_i\left(\frac{x_i}{a+\eta}\right) < a+2\eta\right\}\in\mathcal{U}.$$

Moreover

$$\begin{aligned}\left\{i\colon \varrho_i\left(\frac{x_i}{a+\eta}\right) < a+2\eta\right\} &\subset \left\{i\colon \varrho_i\left(\frac{x_i}{a+2\eta}\right)\leqslant a+2\eta\right\}\\ &\subset \{i\colon |||x_i|||_{\varrho_i}\leqslant a+2\eta\};\end{aligned}$$

so $\lim_{\mathcal{U}} |||x_i|||_{\varrho_i}\leqslant a+2\eta$. Therefore $\lim_{\mathcal{U}} |||x_i|||_{\varrho_i}\leqslant |||[(x_i)]|||_\varrho$ because η can be taken arbitrarily small.

Denote $b = \lim_{\mathcal{U}} |||x_i|||_{\varrho_i}$ and take a number $\eta > 0$. Since $\{i\colon |||x_i|||_{\varrho_i} < b+\eta\} \in \mathcal{U}$, we have $\left\{i\colon \varrho_i\left(\frac{x_i}{b+\eta}\right) \leqslant b+\eta\right\} \in \mathcal{U}$ because

$$\{i\colon |||x_i|||_{\varrho_i} < b+\eta\} \subset \left\{i\colon \varrho_i\left(\frac{x_i}{b+\eta}\right) \leqslant b+\eta\right\},$$

and so

$$\varrho\left(\frac{[(x_i)]}{b+\eta}\right) = \lim_{\mathcal{U}} \varrho_i\left(\frac{x_i}{b+\eta}\right) \leqslant b+\eta.$$

Thus $|||[(x_i)]|||_\varrho \leqslant b+\eta$; in other words, $|||[(x_i)]|||_\varrho \leqslant \lim_{\mathcal{U}} |||x_i|||_{\varrho_i}$.

According to our assumptions Orlicz lattices are topologically complete, and so their ultraproduct is also complete. For every order bounded sequence (x_n) of positive disjoint elements we have $|||x_n|||_\varrho \to 0$ if $n \to \infty$ because ϱ satisfies conditions (ϱ2) and (ϱ5). This fact together with the topological completeness of the ultraproduct and Theorem A.14 gives the order continuity of the F-norm $|||\cdot|||_\varrho$.

To end the proof we have to verify that condition (ϱ4) is satisfied. Let $0 \leqslant [(x_i^n)] \uparrow [(x_i)]$. The order continuity of the F-norm $|||\cdot|||_\varrho$ implies $\varrho\left(\frac{a}{1-a}([(x_i)]-[(x_i^n)])\right) \to 0$ if $n \to \infty$ for every $a \neq 1$. If $a \in (0, 1)$ then

$$\varrho(a[(x_i)]) \leqslant \varrho([(x_i^n)]) + \varrho\left(\frac{a}{1-a}([(x_i)]-[(x_i^n)])\right).$$

Therefore $\varrho(a[(x_i)]) \leqslant \lim_{n\to\infty} \varrho([(x_i^n)])$ for all $a \in (0, 1)$. Using condition (**), we will get $\varrho([(x_i)]) \leqslant \lim_{n\to\infty} \varrho([(x_i^n)])$. The converse inequality is a consequence of condition (ϱ2), and so finally $\varrho([(x_i)]) = \lim_{n\to\infty} \varrho([(x_i^n)])$.

COROLLARY 7.2. *If $(L_i^{\varrho_i})$ is a family of topologically complete Orlicz lattices with modulars satisfying conditions (*) and (**), then there exists a Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$ such that $((L_i^{\varrho_i})_{\mathcal{U}}, \varrho)$ is isomodular to $L_f^\psi(S, \Sigma, \mu)$.*

Proof. It is enough to apply the preceding theorem and the Main Representation Theorem.

Now we will present some theorems about ultraproducts of Orlicz spaces. If $L^{\varphi_i}(S_i, \Sigma_i, \mu_i)$, $i \in I$, is a family of Orlicz spaces with integral modulars satisfying conditions (*) and (**), then the ultraproduct $(L^{\varphi_i}(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ will always be considered with the modular M given by the formula

$$M([(x_i)]) = \lim_{\mathcal{U}} M_{\varphi_i}(x_i).$$

We will usually write M_i instead of M_{φ_i}.

THEOREM 7.3. *Let (S_i, Σ_i, μ_i) be a family of probability measure spaces. Let (φ_i), $i \in I$, be a family of Orlicz functions equicontinuous at zero and having the property $\inf_i \varphi_i(1) > 0$. If the modulars M_i satisfy conditions (*) and (**), then there exists a finite measure space (S, Σ, μ) such that $\{[(1_{S_i})]\}^{\perp\perp}$ is a complemented subspace in $(L^{\varphi_i}(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ and isomodular to $L^{\varphi}(S, \Sigma, \mu)$ where $\varphi(r) = \lim_{\mathcal{U}} \varphi_i(r)(\varphi_i(1))^{-1}$.*

Proof. Since φ_i are equicontinuous at zero, we have $(1_{S_i}) \in l^{\infty}(L^{\varphi_i}, I)$ Moreover, the assumption $\inf_i \varphi_i(1) > 0$ implies $[(1_{S_i})] \neq 0$. The ultraproduct $(L^{\varphi_i}(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ is a DC vector lattice (see the preceding corollary), and so the band $\{[(1_{S_i})]\}^{\perp\perp}$ is a projection band. Of course, $[(1_{S_i})]$ is a weak unit in $\{[(1_{S_i})]\}^{\perp\perp}$.

Now we will show that

$$(++) \qquad C([(1_{S_i})]) = Q(C((1_{S_i}))),$$

where Q denotes the quotient homomorphism from $l^{\infty}(L^{\varphi_i}, I)$ onto $(L^{\varphi_i}(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$. The inclusion $Q(C((1_{S_i}))) \subset C([(1_{S_i})])$ is a direct consequence of the order properties of the operator Q. Let $[(x_i)] \in C([(1_{S_i})])$. Then $[(|x_i|)] \wedge [(|1_{S_i} - x_i|)] = 0$, and so $\lim_{\mathcal{U}} M_i(|x_i| \wedge |1_{S_i} - x_i|) = 0$. Moreover, let

$$A_i = \{s \in S_i : |x_i(s)| = |x_i(s)| \wedge |1_{S_i}(s) - x_i(s)|\}.$$

Thus

$$\lim_{\mathcal{U}} M_i(x_i 1_{A_i}) + \lim_{\mathcal{U}} M_i((1_{S_i} - x_i) 1_{(S_i \setminus A_i)}) = \lim_{\mathcal{U}} M_i(|x_i| \wedge |1_{S_i} - x_i|) = 0.$$

Therefore

$$\lim_{\mathcal{U}} M_i(x_i 1_{A_i}) = \lim_{\mathcal{U}} M_i((1_{S_i} - x_i) 1_{(S_i \setminus A_i)}) = 0.$$

In other words,

$$[(x_i 1_{A_i})] = [((1_{S_i} - x_i) 1_{(S_i \setminus A_i)})] = 0.$$

Finally

$$\begin{aligned}[(1_{(S_i \setminus A_i)})] &= [((1_{S_i} - x_i) 1_{(S_i \setminus A_i)})] + [(x_i 1_{(S_i \setminus A_i)})] \\ &= [(x_i 1_{(S_i \setminus A_i)})] + [(x_i 1_{A_i})] = [(x_i)],\end{aligned}$$

and so $C([(1_{S_i})]) \subset Q(C((1_{S_i})))$. By using equality $(++)$ it is not difficult to verify that the modular M is $[(1_{S_i})]$-component invariant. To end the proof it is enough to apply Theorem 6.1 to the Orlicz lattice $(\{[(1_{S_i})]\}^{\perp\perp}, M)$.

A non-decreasing and left continuous function $\chi: [0, \infty) \to [0, \infty)$ such that $\chi(r) > 0$ for $r > 0$ is said to be a *slowly varying function* if $\lim_{r \to \infty} (\chi(ar)/\chi(r)) = 1$ for all $a > 0$.

THEOREM 7.4. *Let (S_i, Σ_i, μ_i), $i \in I$, be a family of probability measure spaces. If $\varphi(r) = r^p \chi(r)$ where $p \in (0, \infty)$ and χ is a slowly varying function such that the modulars M_i satisfy conditions (*) and (**), then the ultraproduct $(L^\varphi(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ is isomodular to a space having the form $L^{\varphi'}(S, \Sigma, \mu) \oplus L^p(T, \Theta, \tau)$ where μ is finite and $\varphi' = (\chi(1))^{-1} \varphi$.*

Proof. All the assumptions of the preceding theorem hold here, and so $(L^\varphi(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ is isomodular to the space $L^{\varphi'}(S, \Sigma, \mu) \oplus \operatorname{Ker} P$, where P is the projection onto the band $\{[(1_{S_i})]\}^{\perp\perp}$. Fix $a, b > 0$ and $[(x_i)] \in \operatorname{Ker} P$. Moreover, put $A_i(a, b) = \{s \in S_i : a \leqslant |x_i(s)| \leqslant b\}$. Since $[(|x_i|)] \wedge [(1_{S_i})] = 0$, we have $\lim_{\mathcal{U}} M_i(|x_i| \wedge b 1_{S_i}) = 0$. Thus

$$0 \leqslant \varphi(a) \lim_{\mathcal{U}} \mu_i(A_i(a, b)) \leqslant \lim_{\mathcal{U}} M_i(|x_i| 1_{A_i(a,b)})$$
$$= \lim_{\mathcal{U}} M_i((|x_i| \wedge b 1_{S_i}) 1_{A_i(a,b)}) \leqslant \lim_{\mathcal{U}} M_i(|x_i| \wedge b 1_{S_i}) = 0$$

and so $\lim_{\mathcal{U}} \mu_i(A_i(a, b)) = 0$. Moreover,

$$\lim_{\mathcal{U}} M_i(x_i 1_{\{s:|x_i(s)|>b\}}) = \lim_{\mathcal{U}} M_i(x_i 1_{\{s:|x_i(s)|>1\}}).$$

Indeed,

$$|\lim_{\mathcal{U}} M_i(x_i 1_{\{s:|x_i(s)|>b\}}) - \lim_{\mathcal{U}} M_i(x_i 1_{\{s:|x_i(s)|>1\}})|$$
$$\leqslant \lim_{\mathcal{U}} M_i(x_i 1_{A_i(b \wedge 1, b \vee 1)}) \leqslant \varphi(b \vee 1) \lim_{\mathcal{U}} \mu_i(A_i(b \wedge 1, b \vee 1)) = 0.$$

There exists a number r_0 such that for every $r \geqslant r_0$

$$(1-b)\chi(r) < \chi(ar) < (1+b)\chi(r).$$

Therefore

$$(1-b) a^p M_i(x_i 1_{\{s:r_0<|x_i(s)|\}}) \leqslant M_i(a x_i 1_{\{s:r_0<|x_i(s)|\}}) \leqslant M_i(a x_i)$$
$$= M_i(a x_i 1_{\{s:|x_i(s)|<r_0 \wedge b\}}) + M_i(a x_i 1_{A_i(r_0 \wedge b, r_0 \vee b)}) + M_i(a x_i 1_{\{s:|x_i(s)|>r_0 \vee b\}})$$
$$\leqslant \varphi(a(r_0 \wedge b)) + \varphi(a(r_0 \vee b)) \mu_i(A_i(r_0 \wedge b, r_0 \vee b)) +$$
$$+ (1+b) a^p M_i(x_i 1_{\{s:|x_i(s)|>r_0 \vee b\}}).$$

Thus

$$(1-b) a^p \lim_{\mathcal{U}} M_i(x_i 1_{\{s:|x_i(s)|>1\}}) = (1-b) a^p \lim_{\mathcal{U}} M_i(x_i 1_{\{s:|x_i(s)|>r_0\}})$$
$$\leqslant M(a[(x_i)]) \leqslant \varphi(a(r_0 \wedge b)) + (1+b) a^p \lim_{\mathcal{U}} M_i(x_i 1_{\{s:|x_i(s)|>r_0 \vee b\}})$$
$$= \varphi(a(r_0 \wedge b)) + (1+b) a^p \lim_{\mathcal{U}} M_i(x_i 1_{\{s:|x_i(s)|>1\}}).$$

Since b can be taken arbitrarily small and φ is continuous at zero, we have for all numbers $a > 0$

$$M(a[(x_i)]) = a^p \lim_{\mathcal{U}} M_i(x_i 1_{\{s:|x_i(s)|>1\}}).$$

Finally, for every $[(x_i)] \in \operatorname{Ker} P$ we obtain

$$M(a[(x_i)]) = a^p M([(x_i)]).$$

According to Theorem 6.5 the Orlicz lattice $(\operatorname{Ker} P, M)$ embeds into $L^q(T, \Theta, \tau)$. The range of $\operatorname{Ker} P$ is super order dense sublattice in L^q, which together with the topological completeness of $\operatorname{Ker} P$ and the order continuity of the F-norm $|||\cdot|||_M$ shows that the embedding is onto. Since the embedding preserves modulars, we have $p = q$.

COROLLARY 7.5. *If (S_i, Σ_i, μ_i), $i \in I$, is a family of probability measure spaces and $p \in (0, \infty)$, then $(L^p(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ is isomodular to $L^p(S, \Sigma, \mu)$.*

COROLLARY 7.6. *If φ is a slowly varying Orlicz function such that M_φ is a modular satisfying conditions (*) and (**) and (S_i, Σ_i, μ_i) is a family of probability measure spaces, then the Orlicz lattice $(L^\varphi(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ is isomodular to $L^{\varphi'}(S, \Sigma, \mu)$ where μ is finite and $\varphi' = (\varphi(1))^{-1}\varphi$.*

Proof. Repeating the arguments of the proof of Theorem 7.4, we show that, for every $[(x_i)] \in \operatorname{Ker} P$ and every number $a > 0$, holds $M(a[(x_i)]) = M([(x_i)])$. Therefore $\operatorname{Ker} P = \{0\}$.

In particular, if spaces $L^0(S_i, \Sigma_i, \mu_i)$ $(\mu_i(S_i) = 1)$ are considered with integral modulars determined by the function $\varphi(r) = r/(1+r)$, then the ultraproduct $(L^0(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ is isomodular to the space $L^0(S, \Sigma, \mu)$ with an integral modular generated by the function 2φ.

Remarks. 1. If $\varphi(r) = r^p \chi(r)$, where χ is a slowly varying concave function, then the modulars M_φ satisfy conditions (*) and (**).

2. Restricting our considerations to ultraproducts of Orlicz spaces $L^\varphi(S_i, \Sigma_i, \mu_i)$, where μ_i are probability measures, we can easily prove that a function M defined on the ultraproduct $(L^\varphi(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ by $M([(x_i)]) = \lim_{\mathcal{U}} M_i(x_i)$ is a modular if (**) holds and φ satisfies the (Δ_2^∞)-condition. Indeed, we need condition (*) to show that

$$\lim_{\mathcal{U}} |||x_i|||_{\varrho_i} = 0 \quad \text{iff} \quad \lim_{\mathcal{U}} \varrho_i(x_i) = 0.$$

In the present situation the equivalence will remain true because the (Δ_2^∞)-condition implies the following property of φ:

$$\forall r > 0 \ \exists K_r \ \forall t \geqslant r \ \ \varphi(2t) \leqslant K_r \varphi(t),$$

and so

$$\lim_{\mathcal{U}} \varrho_i(x_i) = 0 \quad \text{iff} \quad \forall t \geqslant 0 \ \lim_{\mathcal{U}} \varrho_i(tx_i) = 0.$$

An ultrafilter $\mathcal{U}$ is said to be *countably incomplete* if there exists a sequence of decreasing sets from $\mathcal{U}$ with empty intersection. For example, every ultrafilter in N containing complements of finite sets is incomplete. If $\mathcal{U}$ is an incomplete ultrafilter $(A_n) \subset \mathcal{U}$ and $A_n \supset A_{n+1}$ for all n, $\bigcap_{n=1}^{\infty} A_n = \emptyset$ then, setting

$$a_i = \begin{cases} 1 & \text{for} \quad i \in I \setminus A_1, \\ n^{-1} & \text{for} \quad i \in A_n \setminus A_{n+1}, \end{cases}$$

we obtain $\lim_{\mathcal{U}} a_i = 0$, i.e., for every incomplete ultrafilter there exists a family of positive numbers (a_i) with $\lim_{\mathcal{U}} a_i = 0$.

THEOREM 7.7. *Let $\mathcal{U}$ be an incomplete ultrafilter, $L^{\varphi_i}(S_i, \Sigma_i, \mu_i)$, $i \in I$, a family of Orlicz spaces such that M_i satisfy conditions (*) and (**). If μ_i are non-atomic, φ_i are continuous and unbounded, and functions Φ_i majorizing φ_i are equicontinuous at zero, then the ultraproduct $(L^{\varphi_i}(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ has no weak unit (and thus the ultraproduct is not isomodular to any Musielak–Orlicz space $L^{\Psi}(S, \Sigma, \mu)$ where μ is σ-finite).*

Note that every family (φ_i) of convex Orlicz functions satisfying the (Δ_2)-condition with the same constant satisfies also the assumptions of the above theorem concerning Orlicz functions and functions Φ_i.

Proof. Suppose that $[(x_i)]$ is a weak unit in the ultraproduct. Without loss of generality we can assume $x_i \geqslant 0$ for all $i \in I$. Choose a family $(k_i)_{i \in I} \subset N$ such that

$$\mu_i(\{s \in S_i\colon x_i(s) < k_i\}) \geqslant 2^{-1} \min(1, \mu_i(\operatorname{supp} x_i)).$$

Since μ_i are non-atomic, we can find sets $C_i \subset \{s \in S_i\colon x_i(s) < k_i\}$ such that $0 < \varphi_i(k_i)\mu_i(C_i) < a_i$, where $(a_i)_{i \in I}$ is a family of positive real numbers with $\lim_{\mathcal{U}} a_i = 0$.

Functions φ_i are continuous, non-decreasing and unbounded, and so it is possible to find numbers $m_i \geqslant k_i$ satisfying the inequalities

$$1 \leqslant \varphi_i(m_i)\mu_i(C_i) \leqslant 2.$$

Let $w_i = m_i 1_{C_i}$. We have $(w_i) \in l^{\infty}(L^{\varphi_i}, I)$ because

$$\sup_i M_i(rw_i) = \sup_i \varphi_i(rm_i)\mu_i(C_i) \leqslant \sup_i \Phi_i(r)\varphi_i(m_i)\mu_i(C_i) \leqslant 2 \sup_i \Phi_i(r)$$

and $\sup_i \Phi_i(r) \to 0$ if $r \to 0$. Moreover, $\lim_{\mathcal{U}} M_i(w_i) \geqslant 1$; so $[(w_i)] \neq 0$. However,

$$M([(x_i)] \wedge [(w_i)]) = \lim_{\mathcal{U}} M_i(x_i \wedge w_i) \leqslant \lim_{\mathcal{U}} M_i(k_i 1_{C_i})$$

$$= \lim_{\mathcal{U}} \varphi_i(k_i)\, \mu_i(C_i) \leqslant \lim_{\mathcal{U}} a_i = 0.$$

Thus $[(x_i)]$ is not a weak unit, a contradiction.

We will end our considerations by asking the following question: What conditions must an Orlicz function φ and an ultrafilter $\mathcal{U}$ satisfy in order that the ultraproduct $(L^{\varphi}(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ be isomodular to some Orlicz space?

Notes and comments

Spaces $L^{\psi}(S, \Sigma, \mu)$, called later Musielak–Orlicz spaces, were investigated at the beginning as examples of the so-called modular spaces. The theory of such spaces was presented by H. Nakano in [24]. A modular space is a pair (L, ϱ) where L is a DC vector lattice and ϱ a function from L into $[0, \infty]$ such that:

1. $\forall t \geqslant 0\ \varrho(tx) = 0$ *implies* $x = 0$.
2. $\forall x\ \exists t > 0\ \varrho(tx) < \infty$.
3. For every $x \in L$ the function $\phi_x(t) = \varrho(tx)$ is 1/2 convex (i.e., $\phi_x(\frac{1}{2}r + \frac{1}{2}t) \leqslant \frac{1}{2}\phi_x(r) + \frac{1}{2}\phi_x(t)$).
4. $|x| \leqslant |y|$ implies $\varrho(x) \leqslant \varrho(y)$.
5. $x \perp y$ implies $\varrho(x+y) = \varrho(x) + \varrho(y)$.
6. $0 \leqslant x_\alpha \uparrow x$ implies $\varrho(x) = \sup_\alpha \varrho(x_\alpha)$.

In 1959 J. Musielak and W. Orlicz extended Nakano's concepts. Among other things they gave a more general definition of a modular space. Musielak and Orlicz define a modular space as a pair (X, ϱ) where X is a linear space and ϱ a mapping of X into $[0, \infty]$ satisfying the following conditions:

(A) $\varrho(x) = 0$ iff $x = 0$.
(B) $\varrho(x) = \varrho(-x)$ ($\varrho(x) = \varrho(e^{it}x)$ in the case of complex linear spaces).
(C) $\varrho(ax+by) \leqslant \varrho(x) + \varrho(y)$ for $a, b \geqslant 0$, $a+b = 1$.
(D) $\varrho(tx) \to 0$ if $t \to 0$.

One of the fundamental examples of such modular spaces is again the space $L^{\psi}(S, \Sigma, \mu)$ but now one does not need to assume the convexity of the function $\psi(\cdot, s)$.

In the late fifties intensive investigations of spaces $L^{\psi}(S, \Sigma, \mu)$ began. Many papers about Musielak–Orlicz spaces have been published within the

last twenty years. Investigations of vector Musielak–Orlicz spaces have also been initiated; functions with values in Banach spaces X (or even in topological vector spaces) are considered instead of real-valued functions, and a function ψ is defined on the product $X \times S$ (of course, the assumptions about ψ must be different from the assumptions made in this paper). For details see [13] and [35].

The first representation theorem for Orlicz lattices was proved by H. Nakano. In the final part of his book [24] Nakano notes that the order and topological structure of DC vector lattices, with a convex modular defined on them, is very similar to the structures of some function spaces (Musielak–Orlicz spaces). More precisely, H. Nakano shows the existence of a modular preserving isomorphism between a DC vector lattice with a convex modular and a sublattice of some Musielak–Orlicz space. Moreover, if a modular is monotone complete, then the above isomorphism is onto. Nakano's proof is based on rather complicated techniques and uses many concepts, and so we do not present a more detailed analysis of it.

In 1968 L. Drewnowski and W. Orlicz discovered the form of a modular defined on a sublattice of $L^0(S, \Sigma, \mu)$. Their Theorem 2.1, published in [8], is one of the most valuable representation theorems.

H. Nakano did not use the concepts of "Orlicz lattice", "isomodular lattices", "modular direct sum". These concepts appear in Claas' paper [3], who noticed the importance of component invariance. The most important of Class' results is the following (see [3], Theorem 15.1, p. 44, or [4], Theorem 5.4, p. 90):

Let L^ϱ be an Orlicz lattice with a convex modular ϱ satisfying the (Δ_2)-condition. If L is complete with respect to the norm $\|\cdot\|_\varrho$ and if L has a weak unit e such that ϱ is e-component invariant, then there exists an Orlicz space isomodular to L^ϱ.

The order continuity of the norm $\|\cdot\|_\varrho$ plays a basic role in the proof of the above theorem. Due to the order continuity and the topological density of the subspace $\operatorname{lin} C(e)$ in L it suffices to construct a modular preserving isomorphism mapping $\operatorname{lin} C(e)$ onto the subspace of simple functions dense in a suitable Orlicz space. W. J. Claas also observed that the assumptions about a complete disjoint system given in Theorem 6.1 are sufficient for the representation of an Orlicz lattice (with a convex modular satisfying the (Δ_2)-condition) by an Orlicz space (see [3], Theorem 15.2, p. 45). Moreover, Theorem 6.5 (more precisely, the equivalence of conditions (ii) and (iii)) was formulated (for convex modulars) and proved in [3] (Theorem 12.7, p. 38; see also [4], Theorem 5.3, p. 89).

In subsequent papers ([33], [14]) representation theorems for broader classes of Orlicz lattices were proved. In [33] the assumption of component invariance was removed and the following theorem was obtained:

Let L^ϱ be an Orlicz lattice with a convex modular satisfying the (Δ_2)-condition. If L has a weak unit and if L is $\|\cdot\|_\varrho$-complete, then there exist a measure space (S, Σ, μ) and a convex Musielak–Orlicz function $\psi\colon [0, \infty) \times S \to [0, \infty)$ satisfying the condition $\psi(2r, s) \leqslant K\psi(r, s)$ such that L^ϱ is isomodular to $L^\psi(S, \Sigma, \mu)$.

In [33] it is also shown that Claas' theorem can be deduced from the above theorem. The main result of [33] is a particular case of Nakano's representation theorem, but its proof is different than Nakano's proof.

The authors of [14] remove the assumption of the component invariance, the convexity of a modular and the (Δ_2)-condition. They show the following fact ([14], Theorem 3.1):

Let L^ϱ be an Orlicz lattice complete with respect to the F-norm $|||\cdot|||_\varrho$ and let ϱ be finite. Then there exists a space $L^\psi_f(S, \Sigma, \mu)$ isomodular to L^ϱ.

The proofs of the theorems presented in [33] and [14] are based on the Drewnowski–Orlicz theorem, and, as in Claas', the order continuity of the F-norm is used. W. J. Claas, as well as the authors of [14], observed that the finiteness of a modular implies the super Dedekind completeness of a vector lattice L.

Theorem 5.1 is the most general representation theorem. No assumptions about a vector lattice are made there and a necessary and sufficient condition for the isomodularity of an Orlicz lattice to some Musielak–Orlicz space is given (the σ-Levi property).

Investigations of ultraproducts of Orlicz spaces were initiated by D. Dacunha–Castelle and J. L. Krivine in [6]. They restricted their considerations to ultraproducts of families $L^\varphi(S_i, \Sigma_i, \mu_i)$ where φ is a convex Orlicz function satisfying the (Δ_2)-condition. Also they observed that a function M defined by $M([(x_i)]) = \lim_{\mathcal{U}} M_i(x_i)$ is a convex modular. D. Dacunha–Castelle and J. L. Krivine obtained a result similar to Theorem 7.3: an ultraproduct $(L^\varphi(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ has the form $L^\varphi(S, \Sigma, \mu) \oplus B$. The space B is described exactly only in the case of $\varphi(t) = t^p\chi(t)$ where χ is a slowly varying function. A theorem about the existence of a complemented subspace in an ultraproduct $(L^\varphi(S_i, \Sigma_i, \mu_i))_{\mathcal{U}}$ isomorphic and isometric to some Orlicz space $L^\varphi(S, \Sigma, \mu)$ was also proved by S. Heinrich in [10].

In [5] D. Dacunha–Castelle shows, among other results, that an ultraproduct of Musielak–Orlicz spaces generated by convex Musielak–Orlicz functions is isomodular to some Musielak–Orlicz space. This fact is proved under stronger assumptions than conditions (*) and (**).

A definition (not fully correct) of an ultraproduct of F-spaces was formulated by M. Schreiber (see [28]). He also noted that an ultraproduct of the spaces $L^p(S_i, \Sigma_i, \mu_i)$, where $p \in (0, 1)$ and μ_i are probability measures, is isomorphic and isometric to the space $L^p(S, \Sigma, \mu)$.

Theorem 7.4 presented in this paper is not new. It was taken (together

with the proof) from [6] and [10]. Similarly, the proofs of theorems about isomorphisms between Musielak–Orlicz spaces are slight modifications of the proof of Theorem 4.1 from [16] concerning the form of some modular preserving authomorphisms of Orlicz spaces.

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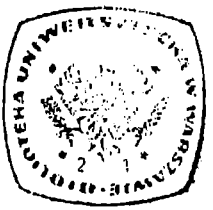
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