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QUALITATIVE CLUSTER SETS

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In [1] and [2] there was investigated the boundary behaviour of real functions defined in the upper half-plane. The authors have proved several theorems concerning the approximate limit numbers of continuous and measurable functions at points of the boundary of the upper half-plane. In this paper we prove similar theorems, but for so-called qualitative limit numbers.

We give now some main definitions and notation. Let H denote the open upper half-plane, and R — its boundary, i.e., the x-axis. Points $(x, 0)$ on the x-axis will be denoted, simply, by x.

Definition 1. If $E \subset H$, then $x \in R$ is called a *point of the second category* of E if, for every $r > 0$, the set $K(x, r) \cap E$ is of the second category in H (here $K(x, r)$ denotes the circle with the centre x and the radius r). $x \in R$ is called a *point of the first category* of E if it is not a point of the second category of E. The set of all points of the first (second) category of E will be denoted by E_{I} (E_{II}).

Definition 2. If $E \subset H$, then $x \in R$ is called a *directional point of the second category* of E in the direction θ if, for every $r > 0$, the set $L(x, \theta, r) \cap E$ is of the second category as a linear set (here $L(x, \theta, r)$ denotes the open segment contained in H, of the length r, with the end point x, in the direction θ). $x \in R$ is called a *directional point of the first category* of E in the direction θ if it is not a directional point of the second category of E in the direction θ. The set of all directional points of the first (second) category of E in the direction θ will be denoted by $E_{\mathrm{I}}(\theta)$ $(E_{\mathrm{II}}(\theta))$.

Definition 3. If $E \subset H$, then $x \in R$ is called a *sectorial point of the second category* of E with respect to the sector σ if, for every $r > 0$, the set $S(x, \sigma, r) \cap E$ is of the second category in H (here $S(x, \sigma, r)$ denotes the open part of the circle $K(x, r)$ contained between the segments $L(x, \theta_1, r)$ and $L(x, \theta_2, r)$, where θ_1, θ_2 determine the sector σ). $x \in R$ is called a *sectorial point of the first category* of E with respect to the sector σ if it is not a sectorial point of the second category oi E with respect to σ. The set of all sectorial points of the first (second) category with respect to the sector σ will be denoted by $E_{\mathrm{I}}(\sigma)$ $(E_{\mathrm{II}}(\sigma))$.

Now let f be a real function defined on H.

Definition 4. A real number y is called a *qualitative limit number* of f at $x \in R$ if $x \in \big(f^{-1}((y-\varepsilon, y+\varepsilon))\big)_{\mathrm{II}}$ for every $\varepsilon > 0$. Moreover, $+\infty$ $(-\infty)$ is called a *qualitative limit number* of f at $x \in R$ if $x \in \big(f^{-1}((A, +\infty))\big)_{\mathrm{II}}$ $\big(x \in \big(f^{-1}((-\infty, A))\big)_{\mathrm{II}}\big)$ for every A. The set of all qualitative limit numbers of f at x is called the *qualitative cluster set* and will be denoted by $C_q(f, x)$. A *qualitative directional cluster set* $(C_q(f, x, \theta))$ in the direction θ and a *qualitative sectorial cluster set* $(C_q(f, x, \sigma))$ in the sector σ are defined similarly in an obvious way.

Let us introduce the following notation:

$$q\limsup_{p\to x} f(p) = \sup C_q(f, x), \qquad q\liminf_{p\to x} f(p) = \inf C_q(f, x),$$

$$q\limsup_{p\to x,\,\theta} f(p) = \sup C_q(f, x, \theta), \qquad q\liminf_{p\to x,\,\theta} f(p) = \inf C_q(f, x, \theta),$$

$$q\limsup_{p\to x,\,\sigma} f(p) = \sup C_q(f, x, \sigma), \qquad q\liminf_{p\to x,\,\sigma} f(p) = \inf C_q(f, x, \sigma).$$

LEMMA 1. *If the set $E \subset H$ has the Baire property, then, for any two directions θ_1 and θ_2, the set $E_{\mathrm{I}}(\theta_1) \cap E_{\mathrm{II}}(\theta_2)$ is of the first category in R.*

Proof. Let

$$B_n(\theta) = \{x\colon E \cap L(x, \theta, (n\sin\theta)^{-1}) \text{ is of the first category}\}.$$

We have

$$E_{\mathrm{I}}(\theta_1) = \bigcup_{n=1}^{\infty} B_n(\theta_1),$$

and so it suffices to show that $B_n(\theta_1) \cap E_{\mathrm{II}}(\theta_2)$ is of the first category for every n. Let us observe that $B_n(\theta)$ has the Baire property for every θ and n (as a linear set). Indeed, $E = G \triangle P$, where G is open and P is of the first category (both as plane sets). If

$$D_1 = \{x\colon G \cap L(x, \theta, (n\sin\theta)^{-1}) \text{ is of the second category}\}$$

and

$$D_2 = \{x\colon P \cap L(x, \theta, (n\sin\theta)^{-1}) \text{ is of the second category}\},$$

then we have $D_1 - D_2 \subset R - B_n(\theta) \subset D_1 \cup D_2$. Simultaneously, D_1 is open and D_2 is of the first category (both as linear sets) in virtue of the Kuratowski-Ulam theorem (see [3], p. 56). Hence $R - B_n(\theta)$ and $B_n(\theta)$ have the Baire property. Obviously, also

$$E_{\mathrm{II}}(\theta) = R - \bigcup_{n=1}^{\infty} B_n(\theta)$$

has the Baire property, and so $B_n(\theta_1) \cap E_{II}(\theta_2) = G_n \triangle P_n$, where G_n is open and P_n is of the first category (both as linear sets).

Suppose that $B_n(\theta_1) \cap E_{II}(\theta_2)$ is of the second category. Then $G_n \neq \emptyset$. Let $x_0 \in G_n$. There exists an $\varepsilon_0 > 0$ such that $(x_0 - \varepsilon_0, x_0 + \varepsilon_0) \subset G_n$. Consider the open region O_n contained between the x-axis, the straight line $y = n^{-1}$ and the segments

$$L\Big(x_0 - \varepsilon_0, \min(\theta_1, \theta_2), \big(n \sin\big(\min(\theta_1, \theta_2)\big)\big)^{-1}\Big)$$

and

$$L\Big(x_0 + \varepsilon_0, \max(\theta_1, \theta_2), \big(n \sin\big(\max(\theta_1, \theta_2)\big)\big)^{-1}\Big).$$

For the set $O_n \cap E$ (having, of course, the Baire property) we have

(1) $\quad \big\{x\colon O_n \cap E \cap L\big(x, \theta_1, (n \sin\theta_1)^{-1}\big) \text{ is of the first category}\big\}$

is the residual set in $(x_0 - \varepsilon_0, x_0 + \varepsilon_0)$, and

(2) $\quad \big\{x\colon O_n \cap E \cap L\big(x, \theta_2, (n \sin\theta_2)^{-1}\big) \text{ is of the second category}\big\}$

is the residual set in $(x_0 - \varepsilon_0, x_0 + \varepsilon_0)$.

Hence, in virtue of the Kuratowski-Ulam inverse theorem (see [3], p. 57), it follows from (1) that $O_n \cap E$ is of the first category and it follows from (2) that $O_n \cap E$ is of the second category, a contradiction. So $G_n = \emptyset$ and $B_n(\theta_1) \cap E_{II}(\theta_2)$ is of the first category.

THEOREM 1. *If $f\colon H \to R$ has the Baire property, then, for any two directions θ_1 and θ_2, we have*

$$q\limsup_{p \to x,\, \theta_1} f(p) \geqslant q\liminf_{p \to x,\, \theta_2} f(p)$$

except for a set of the first category.

Proof. For each rational w, let

$$E^w = \{p\colon f(p) > w\} \quad \text{and} \quad A_w = E^w_{I}(\theta_1) \cap E^w_{II}(\theta_2).$$

By Lemma 1, A_w is of the first category. Simultaneously, the set

$$A = \bigcup_w A_w,$$

w — rational, contains the set of all $x \in R$ for which

$$q\limsup_{p \to x,\, \theta_1} f(p) < q\liminf_{p \to x,\, \theta_2} f(p),$$

and A is of the first category.

THEOREM 2. *If $f\colon H \to R$ is continuous, then, for any two directions θ_1 and θ_2, the set*

$$A = \{x\colon q\limsup_{p \to x,\, \theta_1} f(p) < q\liminf_{p \to x,\, \theta_2} f(p)\}$$

is at most denumerable.

Proof. For each rational w, let

$$A_w = \{x\colon \operatorname*{qlimsup}_{p\to x,\,\theta_1} f(p) < w < \operatorname*{qliminf}_{p\to x,\,\theta_2} f(p)\}$$

and let $E^w = \{p\colon f(p) > w\}$. For every rational w and every natural n, let

$$B_{w,n} = \{x\colon E^w \cap L(x, \theta_1, n^{-1}) \text{ is of the first category}\}.$$

Then

$$A = \bigcup_w A_w \quad \text{and} \quad A_w \subset \bigcup_{n=1}^{\infty} B_{w,n} \quad \text{for each } w,$$

whence

$$A = \bigcup_w \bigcup_{n=1}^{\infty} (A_w \cap B_{w,n}).$$

Suppose that $\theta_1 > \theta_2$. We prove that, for any w and n, $A_w \cap B_{w,n}$ has no point of the right-hand accumulation. Hence it easily follows that $A_w \cap B_{w,n}$ and A are denumerable. Let x_0 be a point of the right-hand accumulation of $A_w \cap B_{w,n}$. We prove that $x_0 \notin A_w \cap B_{w,n}$. Let $x_k \in A_w \cap B_{w,n}$ for $k = 1, 2, \ldots$ and $x_k \searrow x_0$. For each k, the set $E^w \cap L(x_k, \theta_1, n^{-1})$ is of the first category, and so it is empty (for E^w is open). Hence we have

$$f(p) \leqslant w \quad \text{for } p \in \bigcup_{k=1}^{\infty} L(x_k, \theta_1, n^{-1}).$$

It is also easy to see that

$$L(x_0, \theta_2, m^{-1}) \cap \bigcup_{k=1}^{\infty} L(x_k, \theta_1, n^{-1}) \neq \emptyset$$

for each natural m. Hence, for every real $r > w$ and for every natural m, the set $\{p\colon f(p) < r\} \cap L(x_0, \theta_2, m^{-1})$, being open and non-empty, is of the second category (as a linear set). Then

$$\operatorname*{qliminf}_{p\to x_0,\,\theta_2} f(p) \leqslant w \quad \text{and} \quad x_0 \notin A_w \cap B_{w,n}.$$

For $\theta_1 < \theta_2$, the set $A_w \cap B_{w,n}$ has no point of the left-hand accumulation and, for $\theta_1 = \theta_2$, the set A is empty.

LEMMA 2. *If $E \subset H$ is open, then, for every direction θ, the set $E_{\mathrm{II}} \cap E_{\mathrm{I}}(\theta)$ is of the first category.*

Proof. Let $B_n = \{x\colon E \cap L(x, \theta, n^{-1}) \text{ is of the first category}\}$. Then, for $x \in B_n$, we have $E \cap L(x, \theta, n^{-1}) = \emptyset$ (for E is open), and it is easy to see that the set B_n is closed. We have also

$$E_{\mathrm{II}} \cap E_{\mathrm{I}}(\theta) \subset \bigcup_{n=1}^{\infty} B_n.$$

Now, it is obvious that if $x \in E_{\mathrm{II}}$, then $x \notin \operatorname{Int} B_n$ for each n. Hence

$$E_{\mathrm{II}} \cap E_{\mathrm{I}}(\theta) \subset \bigcup_{n=1}^{\infty} (B_n - \operatorname{Int} B_n),$$

and so $E_{\mathrm{II}} \cap E_{\mathrm{I}}(\theta)$ is of the first category.

LEMMA 3. *If $E \subset H$ has the Baire property, then, for every direction θ, the set $E_{\mathrm{II}} \cap E_{\mathrm{I}}(\theta)$ is of the first category.*

Proof. $E = P \triangle Q$, where P is open and Q is of the first category. We have, obviously, $Q_{\mathrm{II}} = \emptyset$, whence $E_{\mathrm{II}} = P_{\mathrm{II}}$. It follows from the Kuratowski-Ulam theorem that $Q_{\mathrm{II}}(\theta)$ is of the first category. Hence the set

$$\big(E_{\mathrm{II}} \cap E_{\mathrm{I}}(\theta)\big) \triangle \big(P_{\mathrm{II}} \cap P_{\mathrm{I}}(\theta)\big) \subset Q_{\mathrm{II}}(\theta)$$

is of the first category and, in virtue of Lemma 2, $E_{\mathrm{II}} \cap E_{\mathrm{I}}(\theta)$ is of the first category.

LEMMA 4. *If $E \subset H$ is open, then $E_{\mathrm{II}}(\theta) \subset E_{\mathrm{II}}$ for every direction θ.*

The proof follows immediately from the definitions.

THEOREM 3. *If $f\colon H \to R$ has the Baire property and θ is a direction, then*

$$C_q(f, x, \theta) \supset C_q(f, x)$$

except for a set of the first category; and if f is continuous, then

$$C_q(f, x, \theta) = C_q(f, x)$$

except for a set of the first category and, for every x,

$$C_q(f, x, \theta) \subset C_q(f, x).$$

The proof is almost the same as that of Theorem 2 in [2] and follows by applying Lemmas 2, 3 and 4 to the inverse images of rational open intervals.

LEMMA 5. *If $E \subset H$ is an arbitrary set, then, for every sector σ, we have $E_{\mathrm{II}}(\sigma) \subset E_{\mathrm{II}}$.*

The proof follows immediately from the definitions.

LEMMA 6. *If $E \subset H$ is an arbitrary set, then, for every sector σ, the set $E_{\mathrm{II}} \cap E_{\mathrm{I}}(\sigma)$ is of the first category.*

Proof. Let $B_n = \{x\colon E \cap S(x, \sigma, n^{-1})$ is of the first category$\}$. As in the proof of Lemma 2, one can prove that B_n is closed and

$$E_{\mathrm{II}} \cap E_{\mathrm{I}}(\sigma) \subset \bigcup_{n=1}^{\infty} (B_n - \operatorname{Int} B_n).$$

THEOREM 4. *If $f: H \to R$ is an arbitrary function, then, for every sector σ,*

$$C_q(f, x, \sigma) = C_q(f, x)$$

except for a set of the first category and, for every x,

$$C_q(f, x, \sigma) \subset C_q(f, x).$$

The proof follows in the standard way from Lemmas 5 and 6.

It is not difficult to find examples showing that the set of asymmetry in Theorems 1, 3 and 4 need not be of measure 0.

REFERENCES

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