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Remote points

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CONTENTS

0. Conventions

All spaces considered are completely regular. We assume X is a subspace of βX. X^* is $\beta X - X$. If $f\colon X \to Y$ is a map, $\beta f\colon \beta X \to \beta Y$ is its Stone extension.

If f is a function and A is a set, $f^{\to} A$ and $f^{\leftarrow} A$ denote the image and preimage of A under f, respectively. If f is a bijection, f^{-1} is the inverse function.

As usual an ordinal is the set of smaller ordinals, and a cardinal is an initial ordinal. ω is ω_0.

We frequently use indexed families and indexed sets, and if Γ is the index set (usually a cardinal), we write $\langle U_\gamma\colon \gamma \in \Gamma\rangle$, or $\langle x_\gamma\colon \gamma \in \Gamma\rangle$. We call a family $\langle U_\gamma\colon \gamma \in \Gamma\rangle$ of subsets of a space

disjoint if $U_\gamma \cap U_{\gamma'} = \emptyset$ for distinct $\gamma, \gamma' \in \Gamma$,

discrete if every point of X has a neighborhood intersecting U_γ for at most one $\gamma \in \Gamma$

(Formally $\langle U_\gamma\colon \gamma \in \Gamma\rangle$ is a function with domain Γ and range $\{U_\gamma\colon \gamma \in \Gamma\}$.)

1. Introduction

1.1. A natural question about $X^* = \beta X - X$ is whether all points are the same. This is in fact two questions:

QUESTION 1. *Are all points of X^* the same from the point of view of X? In other words, if $p, q \in X^*$ are arbitrary, is there a homeomorphism h from X onto X such that $\beta h(p) = q$?*

[Note that βh will be a homeomorphism from βX onto βX.]

QUESTION 2. *Are all points of X^* the same from the point of view of X^*? In other words, is X^* homogeneous?*

It is easy to construct X from which the answer is in the negative, but it is not immediately clear what the answer is if X is one of the following spaces: $\boldsymbol{N}$, the integers, $\boldsymbol{P}$, the irrationals, $\boldsymbol{Q}$, the rationals, $\boldsymbol{R}$, the reals and S, the Sorgenfrey line.[1]

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[1] We will use these symbols with this meaning in the sequel.

An easy cardinality argument shows that for these X the answer to Question 1 is in the negative.(²) Using a more complicated cardinality argument, Frolík showed that the answer to Question 2 also is in the negative, and in fact X^* is not homogeneous if X is any nonpseudocompact space, [20].

1.2. In both cases the proof is not what I would like to call *effective*: the proof only shows *that* the answer is in the negative, and not *why* the answer is in the negative, for the proof does not exhibit two points which have different topological behavior that can easily be expressed. This is a nonmathematical concept, but an example makes clear what I mean: the closed unit interval $[0, 1]$ is not homogeneous because some but not all points are cut points. Similarly, I will call a proof that two spaces are not homeomorphic *effective* if it exhibits a topological property which only one of the two spaces has.(³) [An example of a non-effective proof that two spaces are not homeomorphic can be found in [14]: if X and Y are disjoint stationary sets in ω_1, the countable ordinals, then X and Y are nonhomeomorphic subspaces; no effective proof is known.]

1.3. We first tackle Question 1. Think of the points of X^* as the *infinite* points, then we will show that if X is one of $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ or $\boldsymbol{S}$, then some infinite points of X are more infinite than others, cf. [30]. There are two ways to make this precise.

1.4. DEFINITION. A point p of βX will be called a *far* point of X if $p \in X^*$, but $p \notin \mathrm{Cl}_{\beta X} D$ for every closed (in X) discrete subset D of the space X.

A point p of X will be called a *remote* point of X if $p \in X^*$ but $p \notin \mathrm{Cl}_{\beta X} A$ for every nowhere dense subset A of X.

The set of far/remote points of X will be denoted by $\varphi(X)/\varrho(X)$.

We use *special points* as a generic term for points in one of $\varrho(X)$, $X^* - \varrho(X)$, $\varphi(X)$, $\varphi(X) - \varrho(X)$ and $X^* - \varphi(X)$.

Note that if X has no isolated points, then every remote point of βX also is a far point.

The basic existence theorem is the following; it should be emphasized that the theorem is true in ZFC, i.e. no additional set theoretic axioms are needed. A more technical (and more useful) theorem will be stated as Theorem (4.2).

1.5. THEOREM. *If X is a nonpseudocompact space with countable π-weight,*(⁴) *then X has a remote point.* [See (4.3).]

(²) For X is separable and $|X| \leqslant \mathfrak{c}$, so there are no more than $\mathfrak{c}$ homeomorphisms from X onto itself. But $|X^*| = 2^{\mathfrak{c}}$.

(³) There should be no confusion with the other meaning of "effective" (= without using the axiom of choice).

(⁴) A space has *countable π-weight* if it has a countable π-base: a *π-base* for a space X is a family $\mathscr{B}$ of nonempty open sets such that every nonempty open set in X includes a member of $\mathscr{B}$.

So if X is one of P, Q, R and S, then some points of X^* are remote points of X, but clearly not all points of X^* are remote points of X, for this X. This, then, is an effective answer to Question 1 (see also (1.15)). Note that the method does not work for $X = N$, I do not know an effective way to handle N in a similar way in ZFC.

1.6. In order to answer Question 2 we need another concept. Recall that a space is called *extremally disconnected* if every two disjoint open sets have disjoint closures, or, equivalently, if the closure of every open set is open. We need a local version of this concept.

1.7. DEFINITION. The space X is said to be *extremally disconnected at the point* p if

$$\text{for every two disjoint open sets } U \text{ and } V \text{ in } X,\ p \notin \bar{U} \cap \bar{V},$$

or, equivalently

$$\text{for every open set } U \text{ in } X, \text{ if } p \in \bar{U} \text{ then } p \in \bar{U}^0.$$

Moreover, X is said to be *somewhere/nowhere extremally disconnected* if it is extremally disconnected at some/no point.

We will show that βX is extremally disconnected at each remote point of X, (5.2), and that X^* is extremally disconnected at each point βX, provided X is nowhere locally compact.(5) Hence Theorem (1.5) implies

1.8. THEOREM. *Let X be a nonpseudocompact space with countable π-weight.*

(a) *βX is extremally disconnected at some point of X^*.* (See (5.3).)

(b) *If X is nowhere locally compact,*(5) *then X^* is somewhere extremally disconnected.* (See (6.3).)

1.9. Although this theorem is closely related to Theorem (1.5), it is much more important; it strikes me as a fundamental fact about Čech–Stone compactifications, even though the condition on the π-weight is an inconvenient limitation. I do not know if the condition of the π-weight can be omitted in either Theorem (1.5) or (1.8).* It suffices to answer this question for (1.5), but the question for (1.8) is more important.

QUESTION. *If X is not pseudocompact, then is βX extremally disconnected at some point of X^*?*

Theorem (1.8) has several applications. First of all, it can be used to answer Question 2 effectively, at least for P, Q and S.

1.10. THEOREM. *If X is one of P, Q and S, then X^* is not homogeneous because it is extremally disconnected at some but not all points.* [See (6.6).]

(5) A space is called *nowhere locally compact* if no point has a compact neighborhood.
* Added in print: It can not.

1.11. Another application deals with products of Čech–Stone compactifications. Recall that Glicksberg proved that if X and Y are infinite then $\beta(X\times Y) = \beta X\times\beta Y$ iff $X\times Y$ is pseudocompact, [26].(6) As is shown in [22] this does not tell us when $\beta(X\times Y) \approx \beta X\times\beta Y$:(7) if $X = N+\beta N$ then $\beta(X\times X) \approx \beta X\times\beta X$, though clearly $X\times X$ is not pseudocompact.

1.12. THEOREM. *If X and Y are spaces with countable π-weight, neither of which has an isolated point, then the following conditions are equivalent:*

(a) $X\times Y$ *is pseudocompact;*

(b) $\beta(X\times Y) = \beta X\times\beta Y$;

(c) $\beta(X\times Y) \approx \beta X\times\beta Y$,

because if $X\times Y$ is not pseudocompact, then $\beta(X\times Y)$ is somewhere extremally disconnected and $\beta X\times\beta Y$ is nowhere extremally disconnected. [See (7.4).]

1.13. The condition that neither space has an isolated point is essential, but the condition on the π-weight can be dropped, [9]; however then the proof is not as effective. In a similar way we have

1.14. THEOREM. *Q^* and $Q^*\times Q^*$ are not homeomorphic because Q^* is somewhere extremally disconnected and $Q^*\times Q^*$ is nowhere extremally disconnected.*

Other applications of Theorem (1.8) are in (5.5), (10.1), (11.1), (12.2), (12.4), (13.1), (13.2), §§ 17, 18, and (20.2).

1.15. It also is worth pointing out that Theorem (1.8) also can be used to answer Question 1, at least for $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$ and $\boldsymbol{S}$. Indeed, if X is one of these spaces then βX is extremally disconnected at some point of X^* but not at all points, see (6.5) or (8.2), and (1.8); note that this way of distinguishing points of X^* does not mention X.

We also pay some attention to remote points in their own right, see Chapter 4, and to far points, see § 15.

NOTES TO § 1

1.16. Eberlein constructed a far point in $\boldsymbol{R}^*$, i.e. a point that is not in the closure of any closed discrete subset of $\boldsymbol{R}$ ([18], footnote 4). This motivated Fine and Gillman, [18], to investigate the question of whether there are points in $\boldsymbol{R}^*$ that are not in the closure of any discrete subset of $\boldsymbol{R}$, whether closed or not.

If X has no isolated points, then trivially a point in X^* that is not in the closure of any nowhere dense subset of X is not in the closure of any discrete subset of X. If X is metrizable then the converse is true,(8) as is well known (the special case $X = \boldsymbol{R}$ is mentioned explicitly in [18]). Metrizability is essential, as we show below, but even if it were not essential one should define a remote point to be a point that is not in the

(6) $\beta(X\times Y) = \beta X\times\beta Y$ means that there is a homeomorphism from $\beta(X\times Y)$ onto $\beta X\times\beta Y$ which leaves $X\times Y$ pointwise fixed.

(7) $A \approx B$ means that A and B are homeomorphic.

(8) For in a metrizable space every nowhere dense set is in the closure of a discrete set.

closure of any nowhere dense (rather than discrete) set, simply because this is the property which makes remote points useful.

1.17. We now describe the example mentioned above. There is a countable regular space X without isolated points in which every relatively discrete subset is closed, and X has countable π-weight, [11]. Let $p \in X$ be arbitrary. Retopologize $X \times X$ as follows: $U \subseteq X \times X$ is open iff

1) $\{y \in X: \langle x, y\rangle \in U\}$ is open in X for every $x \in X$, i.e. U intersects every vertical line in an open set; and

2) $\{x \in X: \langle x, p\rangle \in U\}$ is open in X, i.e. U intersects the horizontal line $X \times \{p\}$ in an open set.

Call the resulting space E, let $H = X \times \{p\}$. Then E is normal and H is closed in E, so we can identify βH with $\mathrm{Cl}_{\beta E} H$. Since H has countable π-weight, H has a remote point, a say. Then a is not a remote point of E since H is nowhere dense in E. But one easily checks that every relatively discrete subset of E is closed in E, so if $D \subseteq E$ is relatively discrete, then $a \notin \mathrm{Cl}_{\beta X}(D-H)$ since $D-H$ and H are disjoint closed sets in the normal space D, and $a \notin \mathrm{Cl}_{\beta X}(D \cap H)$, since $D \cap H$ is nowhere dense in H and a is a remote point of H. Consequently $a \notin \mathrm{Cl}_{\beta X} D$.

1.18. The results of Sections 5, 6, 10, 12, 13 and 14 appear in [6] (which will not be published in that form) (in [6] we had to assume MA in order to guarantee the existence of remote points).

Some of our results about remote points have been discovered independently by others, but the applications in Chapters 2 and 3 are new. Other papers dealing with remote points are [18], [21], [27], [28], [32], [33], [46].

I am indebted to Grant Woods for bringing several references to my attention.

Chapter I

Tools and the construction of special points

2. **Tools.** Here we collect some tools which make the study of βX and X^* easy. The facts we quote are well known, or at least easy to prove.

2.1. X is not pseudocompact iff some nonempty G_δ-subset of βX is included in X^* iff X has a (necessarily closed) C-embedded copy of N iff there is a discrete countably infinite family consisting of open subsets. In particular, if X is not pseudocompact, it is not countably compact.

2.2. X is realcompact iff every point of X^* is contained in a nonempty G_δ-subset of βX which is included in X^*. Lindelöf spaces apparently are realcompact, hence so are $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$ and S.

2.3. X is nowhere locally compact iff for every (or, equivalently, for some) compactification bX of X, $bX-X$ is dense in bX (and then $bX-X$ also is nowhere locally compact). We frequently use the evident fact that a nowhere locally compact space has no isolated points.

2.4. If Y is a dense subspace of (the regular) space X, then X and Y have the same π-weight (= smallest cardinality of a π-base). Consequently every dense subspace of βX (in particular X^* if X is nowhere locally compact) has the same π-weight as X.

2.5. If $X \subseteq Y \subseteq \beta X$, then $\beta Y = \beta X$.

2.6. If the (Hausdorff) space X is separable, $|X| \leqslant 2^{\mathfrak{c}}$.

2.7. Remark. In all our theorems "realcompact" can be weakened to "nearly realcompact", where a space X is called *nearly realcompact* if $\beta X - vX$ is dense in X^*. The class of nearly realcompact is much more extensive than the class of realcompact spaces. Indeed, the product of Q and any space (whether nearly realcompact or not) is nearly realcompact, [4].

3. **Extension of open sets.** The results of this section are not needed for the construction of remote points, but are useful when we study extremal disconnectedness at points.

For every space X we define a function

$$\mathrm{Ex}_X\colon \{\text{open sets of } X\} \to \{\text{open sets of } \beta X\}$$

by

$$\mathrm{Ex}_X U = \beta X - \mathrm{Cl}_{\beta X}(X-U).$$

By a straightforward computation one shows

3.1. LEMMA. *Let U, V be open in X, then*

(a) $X \cap \mathrm{Ex}_X U = U$, *hence* $\mathrm{Cl}_{\beta X}\,\mathrm{Ex}_X U = \mathrm{Cl}_{\beta X} U$;

(b) $\mathrm{Ex}_X(U \cap V) = (\mathrm{Ex}_X U) \cap (\mathrm{Ex}_X V)$.

The following lemma is the key in the proof that βX is extremally disconnected at all remote points, for all X.

3.2. LEMMA. *For every space X, if U is open in X then*

$$\mathrm{Bd}_{\beta X}\,\mathrm{Ex}_X U = \mathrm{Cl}_{\beta X}\,\mathrm{Bd}_X U.$$

□ It suffices to prove

$$\mathrm{Ex}_X U = \mathrm{Cl}_{\beta X}\,\mathrm{Ex}_X U - \mathrm{Cl}_{\beta X}\,\mathrm{Bd}_X U$$

Now $\mathrm{Ex}_X U$ is open in βX and is disjoint from $\mathrm{Bd}_X U$ since, by (3.1a)

$$(\mathrm{Ex}_X U) \cap (\mathrm{Bd}_X U) = (\mathrm{Ex}_X U) \cap X \cap (\mathrm{Bd}_X U) = U \cap \mathrm{Bd}_X U = \emptyset.$$

Hence $\mathrm{Ex}_X U \subseteq \mathrm{Cl}_{\beta X}\,\mathrm{Ex}_X U - \mathrm{Cl}_{\beta X}\,\mathrm{Bd}_X U$. It remains to prove the reverse inclusion.

Let $p \in \mathrm{Cl}_{\beta X}\,\mathrm{Ex}_X U - \mathrm{Cl}_{\beta X}\,\mathrm{Bd}_X U$ be arbitrary. There is a continuous $f\colon X \to \boldsymbol{R}$ such that

$$f(p) = 1, \quad f(x) = 0 \text{ for } x \in \mathrm{Cl}_{\beta X}\,\mathrm{Bd}_X U, \quad 0 \leqslant f(x) \leqslant 1 \text{ for all } x \in \beta X.$$

Define a function $g\colon X \to \boldsymbol{R}$ by

$$g(x) = \begin{cases} f(x) & x \in U, \\ -f(x) & x \in X - U. \end{cases}$$

Then both $g \restriction \mathrm{Cl}_X U$ and $g \restriction (X-U)$ are continuous, hence so is g. Since g is bounded, it follows that $\beta g\colon \beta X \to \boldsymbol{R}$ exists. Now $f(x) = |g(x)| = |\beta g(x)|$ for all $x \in X$, hence $f(x) = |\beta g(x)|$ for all $x \in \beta X$. It follows that either $\beta g(p) = 1$ or $\beta g(p) = -1$. But $p \in \mathrm{Cl}_{\beta X} U$ $(= \mathrm{Cl}_{\beta X}\,\mathrm{Ex}_X U$ by (3.1a)), and $g(x) \geqslant 0$ for $x \in U$, hence $\beta g(p) = 1$. Since

$$g(x) \leqslant 0 \text{ for } x \in X-U, \text{ hence } \beta g(x) \leqslant 0 \text{ for } x \in \mathrm{Cl}_{\beta X}(X-U)$$

it follows that $p \in \beta X - \mathrm{Cl}_{\beta X}(X-U) = \mathrm{Ex}_X U$, as required. □

NOTES TO § 3

3.3. The function Ex_X is apparently due to Šanin. [37]. Lemma (3.1) is known of course. It was known before that Lemma (3.2) holds for normal X, cf. [15], 7.1.14; it really is something exceptional that a formula as in (3.2) can be true for *all* open U in a not necessarily normal X.

Added in print: (3.2) is known already, see [51], p. 218.

4. Construction of special points in X^*. We first prove the existence of remote points for normal spaces, from which we deduce the fundamental existence theorem for remote points.

4.1. LEMMA. *Let X be a normal space with countable π-weight which is not pseudocompact. Then X has $2^{\mathfrak{c}}$ remote points.*

□ Let $^-$ be the closure operator in X. Since X is not pseudocompact, there is a *discrete* family

$$\langle I_n\colon n\in\omega\rangle$$

consisting of nonempty open sets (with $I_m \neq I_n$ if $m\neq n$ of course). Let

$$\mathscr{D} = \{D \subseteq X\colon D \text{ is nowhere dense in } X\}.$$

We will construct for each $D\in\mathscr{D}$ a closed F_D in X in such a way that if

$$\mathscr{F} = \{F_D - I_n\colon D\in\mathscr{D},\ n\in\omega\}$$

then

(1) $F_D\cap\bar{D} = \emptyset$;

(2) $\mathscr{F}$ has the finite intersection property;

(3) $F \subseteq \bigcup_{n\in\omega} I_n$ for $F\in\mathscr{F}$.

Then

$$T = \bigcap\{\mathrm{Cl}_{\beta X} F\colon F\in\mathscr{F}\}$$

is nonempty by (2), and $T\subseteq X^*$ by (3). Given any $D\in\mathscr{D}$, it follows from (1) and the fact that X is normal that $\mathrm{Cl}_{\beta X} F_D\cap \mathrm{Cl}_{\beta X} D = \emptyset$. Consequently all points of T are remote. We will see from the construction that $|T|\geqslant 2^{\mathfrak{c}}$.

Enumerate some countable π-base $\mathscr{B}$ for X as $\langle B_n\colon n\in\omega\rangle$. Recall that $\emptyset\notin\mathscr{B}$.

Our plan. Because the main difficulty is how to achieve (2), we will motivate a construction that makes sure that the following weaker condition holds:

(2′) $\{F_D\colon D\in\mathscr{D}\}$ has the 2-intersection property.[9]

Each F_D is going to be the union of finitely many sets $\bar{B}_n$; each B_n which is used this way will be called a *building block* for F_D. Suppose we know the building blocks for F_D, then define

$$k_D = \min\{n\in\omega\colon B_n \text{ is a building block for } F_D\}.$$

Let $D, E\in\mathscr{D}$ be arbitrary. We want $F_D\cap F_E\neq\emptyset$. If $k_D = k_E$ this inequality holds, so without loss of generality $k_E < k_D$. Instead of $F_D\cap F_E$

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[9] A family $\mathscr{A}$ of sets has the *$\varkappa$-intersection property* if every nonempty subfamily with at most $\varkappa$ members has nonempty intersection.

$\neq \emptyset$ we require the stronger $F_D \cap B_{k_E} \neq \emptyset$. Since k_E could be any integer $< k_D$, it is natural to require the even stronger condition that $F_D \cap B_n \neq \emptyset$ for all $n < k_D$. Since this requirement does not impose any condition on k_D, we let k_D be the first integer k for which B_k can be a building block. Technical details are given below.

A similar analysis shows what we have to do to give $\{F_D: D \in \mathscr{S}\}$ the n-intersection property, for a fixed integer n.

Construction of the F_D's: We know how to get the n-intersection property for a fixed n. In order to construct the F_D's we will construct a family

$$\{F_{D,n}: D \in \mathscr{D}, n \in \omega\}$$

of closed sets satisfying

(1*) $F_{D,n} \cap \bar{D} = \emptyset$;

(2*) $\{F_{D,n}: D \in \mathscr{D}\}$ has the n-intersection property; and

(3*) $F_{D,n} \subseteq I_n$;

and then define

$$F_D = \bigcup_{n \in \omega} F_{D,n}.$$

Each F_D is closed since $\langle I_n: n \in \omega \rangle$ is a discrete sequence. It is clear that (1), (2) and (3) hold.

To construct the $F_{D,n}$'s, define

$$K(D, n) = \{i \in \omega: \bar{B}_i \cap \bar{D} = \emptyset,\ \bar{B}_i \subseteq I_n\} \quad (D \in \mathscr{D},\ n \in \omega);$$

this set lists indices of candidates for building blocks for $F_{D,n}$. Since the members of $\mathscr{D}$ are nowhere dense we can construct integers $k(D, n, m)$ $(D \in \mathscr{D}, 0 \leqslant m \leqslant n \leqslant \omega)$ as follows:

$$k(D, n, 0) = \min K(D, n);$$

$k(D, n, m+1) = \min \{i \in \omega: i \geqslant k(D, n, m)$, and for all $s \in \omega$ with $\bar{B}_s \subseteq I_n$ and $s \leqslant k(D, n, m)$ there is a $t \in K(D, n)$ with $t \leqslant i$ and $B_t \subseteq B_s\}$.

Define

$$F_{D,n} = \bigcup \{\bar{B}_i: i \in K(D, n),\ i \leqslant k(D, n, n)\}.$$

Proof that this works. We only have to check (2*). Let $\mathscr{E}$ be a subfamily of $\mathscr{D}$ with $1 \leqslant |\mathscr{E}| \leqslant n$. Denote $|\mathscr{E}|$ by e. With recursion on j pick $E_j \in \mathscr{E} - \{E_i: 0 \leqslant i$ and $i < j\}$, for $0 \leqslant j < e$ in such a way that

(4) $k(E_j, n, j) \leqslant k(E, n, j)$ for all $E \in \mathscr{E} - \{E_i: 0 \leqslant i$ and $i < j\}$.

Next define $s(j) \in \omega$, for $0 \leqslant j < e$, by

$$s(0) = k(E_0, n, 0);$$

$$s(j+1) = \min \{t \in K(E_{j+1}, n): B_t \subseteq B_{s(j)}\} \quad (0 \leqslant j < e);$$

this is possible since the E_j's are nowhere dense. Using the facts that $k(D, n, m) \leqslant k(D, n, m+1)$ $(D \in \mathscr{D}, 0 \leqslant m < n)$, and that $k(E_j, n, j) \leqslant k(E_{j+1}, n, j)$ for $0 \leqslant j < e-1$, by (4), one can check with induction that $s(j) \leqslant k(E_j, n, j)$, i.e. that $B_{s(j)}$ is a building block for F_{E_j}, for $0 \leqslant j < e$. It follows that

$$\bigcap_{j<e} F_{E_j,n} \supseteq \bigcap_{j<e} B_{s(j)} = B_{s(e-1)} \neq \emptyset$$

since $\emptyset \notin \mathscr{B}$.

The number of remote points. For each free ultrafilter p on ω the set

$$J_p = \bigcap \{\mathrm{Cl}_{\beta X}(\bigcup_{n \in A} I_n)\colon A \in p\}$$

intersects T, by (2*). But $J_p \cap J_q = \emptyset$ for distinct ultrafilters on ω: if $A \in p-q$ then $\{I_n\colon n \in A\}$ and $\{I_n\colon n \in \omega - A\}$ have disjoint closures in X, hence in βX since X is normal. Since there are $2^{\mathfrak{c}}$ free ultrafilters on ω ([23], 9.2), it follows that $|T| \geqslant 2^{\mathfrak{c}}$. But $|\beta X| \leqslant 2^{\mathfrak{c}}$ by (2.6), hence there are precisely $2^{\mathfrak{c}}$ remote points. □

It now is easy to prove the following

4.2. THEOREM. *Let X be any space with countable π-weight. Then for every nonempty G_δ-subset G of βX, if $G \subseteq X^*$ then G contains $2^{\mathfrak{c}}$ remote points of X.*

□ Let $Y = \beta X - G$. Then Y is an F_σ in βX, hence Y is normal, and Y is not pseudocompact by (2.1). Also, Y has countable π-weight by (2.4), hence βY has $2^{\mathfrak{c}}$ remote points by Theorem (4.1). Now $\beta Y = \beta X$, by (2.5) and hence every remote point of Y is a remote point of X, for every nowhere dense subset of X is nowhere dense in Y. □

4.3. Theorem (1.5) is an immediate consequence of (2.1) and (4.2).

Similarly, from (2.2) and (4.2) one easily deduces the following

4.4. THEOREM. *Let X be a realcompact space with countable π-weight. Then every nonempty open set of the subspace X^* contains $2^{\mathfrak{c}}$ remote points.* □

We now consider points which are not far or not remote.

4.5. THEOREM. *Let X be any space. Then for every nonempty G_δ-subset G of βX, if $G \subseteq X^*$, then G contains (at least) $2^{\mathfrak{c}}$ points which are not far (hence are not remote if X has no isolated points).*

□ Let $Y = \beta X - G$. Then $\beta Y = \beta X$ since $X \subseteq Y \subseteq \beta X$. Hence G is a G_δ in βY. It follows from (2.1) that Y is not pseudocompact. Since X is dense in Y, this enables us to construct a countably infinite $D \subseteq X$ such that D is discrete and C-embedded in Y. Then $\mathrm{Cl}_{\beta Y} D = |\beta N| = 2^{\mathfrak{c}}$. But D is closed in Y, and $\beta X = \beta Y$, so $|G \cap \mathrm{Cl}_{\beta X} D| = |(\mathrm{Cl}_{\beta Y} D) - D| = 2^{\mathfrak{c}}$. Clearly no point of $G \cap \mathrm{Cl}_{\beta X} D$ is far. □

4.6. We leave it to the reader to formulate a theorem analogous to (4.4).

We finally turn attention to points which are far but not remote. We need the following Lemma.

4.7. LEMMA. *Let T be a first countable space without isolated points. Then T has a countable nowhere dense subspace F which has no isolated points.*

□ Choose a decreasing neighborhood base $\langle B(x,n): n \in \omega \rangle$ for each $x \in T$. With recursion on $n \in \omega$ construct finite sets F_n and finite collections $\mathscr{U}_n$ of nonempty open sets such that

(1) $F_0 \neq \emptyset$;

(2) $F_n \subseteq F_{n+1}$, $\mathscr{U}_n \subseteq \mathscr{U}_{n+1}$;

(3) $(F_{n+1} - \{x\}) \cap B(x,n) \neq \emptyset$ for all $x \in F_n$;

(4) for all $x \in F_n$ there is a $U \in \mathscr{U}_n$ with $U \subseteq B(x,n)$; and

(5) $F_n \cap \bar{U} = \emptyset$ for all $U \in \mathscr{U}_n$.

Conditions (2) and (5) guarantee that the construction does not stop prematurely. Put $F = \bigcup_{n \in \omega} F_n$. Then F has no isolated points by (2) and (3), and F is nowhere dense by (2), (4) and (5). □

4.8. THEOREM. *Let X be a first countable normal space without isolated points. Then for every nonempty G_δ-subset G of βX, if $G \subseteq X^*$ then G contains (at least) $2^{\mathfrak{c}}$ far points which are not remote.*

□ Put $Y = \beta X - G$. Then $\beta Y = \beta X$ since $X \subseteq Y \subseteq \beta X$, hence G is a G_δ in βY. Then Y is not pseudocompact, by (2.1), hence there is a discrete open family $\langle I_n: n \in \omega \rangle$ consisting of (nonempty) open sets in Y (with $I_m \neq I_n$ if $m \neq n$ of course). For each $n \in \omega$ the subspace $X \cap I_n$ of X has no isolated points, hence has a countable, dense in itself, nowhere dense (in $X \cap I_n$) subspace K_n, by Lemma (4.7). Then $\bigcup_{n \in \omega} K_n$ is nowhere dense in X since $\langle X \cap I_n: n \in \omega \rangle$ is a disjoint open family in X. It follows that

$$K = \mathrm{Cl}_X \Big(\bigcup_{n \in \omega} K_n \Big)$$

is a separable nowhere dense closed subspace of X. Hence $\mathrm{Cl}_{\beta X} K = \beta K$.

If we pick $p_n \in I_n \cap K_n$ for each $n \in \omega$, then every limit point of $\langle p_n: n \in \omega \rangle$ in $\beta X = \beta Y$ is in $Y^* = G$, hence $G \cap \beta K$ is a nonempty G_δ in βK which misses K. It follows from (4.2) that G contains $2^{\mathfrak{c}}$ remote points of K, since K has countable π-weight, being separable and first countable. Obviously no such point is a remote point of βX. But if p is a remote point of βK, then p is a far point of βX. Indeed, let D be a closed discrete subset of X. Then $K \cap D$ is nowhere dense in K since K has no isolated points, hence $p \notin \mathrm{Cl}_{\beta X} K \cap D$. But also $p \notin \mathrm{Cl}_{\beta X} D - K$ since the disjoint closed subsets $D-K$ and K of the normal space X have disjoint closures in βX, and $p \in \mathrm{Cl}_{\beta X} K$. □

4.9. COROLLARY. *If X is a noncompact metrizable space without isolated points, then X has a far point.* □

4.10. We leave it to the reader to formulate a theorem analogous to (4.4).

NOTES TO § 4

4.11. That Theorem (4.2) is true under CH follows from a result by Fine and Gillman ([18], 2.3); in fact CH implies that X has remote points if X is separable (or even: $\pi(X) \leqslant \mathfrak{c}$ and X has countable cellularity) and nonpseudocompact. Theorem (4.2) was proved from Martin's Axiom (or rather $P(\mathfrak{c})$) in [6]. Hechler has the existence of remote points for $\boldsymbol{R}$ under a slightly weaker hypothesis.

Corollary (4.9) was proved in [7]. I originally used ideas from [7] to show only that $\boldsymbol{Q}$ has remote points. I am indebted to Mary Ellen Rudin for showing me how my ideas can be used to show that $\boldsymbol{R}$ has remote points, [34]; Theorem (4.2) is a further improvement (and simplification of the construction).

Chapter II

Effective proofs using remote points

We show that βX is extremally disconnected at all remote points in § 5, and use this fact in §§ 6, 7, 8, 9. In § 9 we also use far points.

5. Remote points and extremal disconnectedness at points. The following lemma is the fundamental characterization of remote points, see also (14.2).

5.1. LEMMA. *Let X be any space. Then the following conditions on a point p of X^* are equivalent:*

(a) *p is a remote point of βX;*

(b) *for all open U in X, if $p \in \mathrm{Cl}_{\beta X} U$, then $p \in \mathrm{Ex}_X U$;*

(c) *for all open V in βX, if $p \in \mathrm{Cl}_{\beta X} V$, then $p \in \mathrm{Ex}_X (X \cap V)$.*

□ (a) → (b): Let U be open in X and assume $p \in \mathrm{Cl}_{\beta X} U$. From (3.1a), $\mathrm{Cl}_{\beta X} U = \mathrm{Cl}_{\beta X} \mathrm{Ex}_X U$. Since p is remote, $p \notin \mathrm{Cl}_{\beta X} \mathrm{Bd}_X U$. It follows from (3.2) that

$$p \in \mathrm{Cl}_{\beta X} \mathrm{Ex}_X U - \mathrm{Bd}_{\beta X} \mathrm{Ex}_X U = \mathrm{Ex}_X U.$$

(b) → (c): $\mathrm{Cl}_{\beta X} V = \mathrm{Cl}_{\beta X} (X \cap V)$.

(c) → (b): $\mathrm{Ex}_X (U) = \mathrm{Ex}_X (X \cap \mathrm{Ex}_X U)$ and $\mathrm{Cl}_{\beta X} U = \mathrm{Cl}_{\beta X} \mathrm{Ex}_X U$ by (3.1a).

(b) → (a): Let $A \subseteq X$ be nowhere dense, and put $U = X - \mathrm{Cl}_X A$. Then $\mathrm{Cl}_{\beta X} U = \beta X$; hence $p \in \mathrm{Ex}_X U$. But

$$A \cap \mathrm{Ex}_X U = A \cap X \cap \mathrm{Ex}_X U = A \cap U = \emptyset$$

by (3.1a). It follows that $p \notin \mathrm{Cl}_{\beta X} A$. □

5.2. COROLLARY. *βX is extremally disconnected at each remote point, for every space X.*

□ If p is remote, and V is open in βX, then $p \in \mathrm{Cl}_{\beta X} V$ implies $p \in \mathrm{Int}_{\beta X} \mathrm{Cl}_{\beta X} V$ because

$$\mathrm{Ex}_X(X \cap V) \subseteq \mathrm{Int}_{\beta X} \mathrm{Cl}_{\beta X} \mathrm{Ex}_X(X \cap V)$$
$$= \mathrm{Int}_{\beta X} \mathrm{Cl}_{\beta X}(X \cap V) = \mathrm{Int}_{\beta X} \mathrm{Cl}_{\beta X} V. \quad \square$$

The converse of (5.2) is false, as βX is extremally disconnected whenever X is ([23], 6M.1), and there are extremally disconnected spaces X for which βX has nonremote points, see Remark (10.3). But see (8.2).

5.3. Theorem (1.8a) is an immediate consequence of (1.5) and (5.2).

5.4. We leave it to the reader to formulate stronger versions, analogous to (4.2) and (4.4).

A space is called *(weakly) zero-dimensional* if the family of clopen sets is a base. Our next result is an easy consequence of (5.1).

5.5. THEOREM. *Let X be a realcompact space with countable π-weight. Then X^* has a dense zero-dimensional subspace.*

□ Let E be the set of remote points of βX. E is dense in X^* by (4.4). Let $x \in E$, and let U be an open set in βX containing x. Let V be an open set in βX with $x \in V$ and $\mathrm{Cl}_{\beta X} V \subseteq U$, and define $W = \mathrm{Ex}_X V$. Then $x \in W$, and $\mathrm{Cl}_{\beta X} W \subseteq U$ by (3.1a). Also, $E \cap \mathrm{Bd}_{\beta X} W = \emptyset$ by (5.1b). □

I do not know an example of a space which does not have a dense zero-dimensional subspace.

See Section 8 for Notes.

6. Remote points and nonhomogeneity. In this section we prove a Theorem, (6.6), which implies Theorem (1.10) (which tells why X^* is not homogeneous for suitable X).

We need the following definition, which is not new.

6.1. DEFINITION. Let X be a space, and let γ be a cardinal. Then $F \subseteq X$ is called a *γ-set* if there is a pairwise disjoint open family $\mathscr{U}$ with $|\mathscr{U}| = \gamma$ and $F \subseteq \mathrm{Bd}_X U$ for all $U \in \mathscr{U}$. Also, $p \in X$ is called a *γ-point* if $\{p\}$ is a γ-set. Finally, p is called a *strict γ-point* if it is a γ-point but not a γ^+-point.

The proof of the following simple observation is omitted.

6.2. LEMMA. *Let X be a space, and let γ be a cardinal.*

(a) *Every point of a γ-set is a γ-point.*

(b) *If F is a γ-set in X, so is $\mathrm{Cl}_X F$.*

(c) *If Y is a dense subspace of X, and $F \subseteq Y$, then F is a γ-set in Y iff F is a γ-set in X.* □

6.3. Theorem (1.8b) is an immediate consequence of (1.5), (2.3) and (6.2c).

6.4. Again we leave it to the reader to formulate theorems analogous to (4.2) and (4.4).

In order to apply Theorem (1.8b) (which says that X^* is somewhere extremally disconnected for suitable X) for effective nonhomogeneity proofs, we'll have to show that X^* is not everywhere extremally disconnected for suitable X (see Remark 6.7). The following lemma gives an easily checked condition.

6.5. LEMMA. *Let γ be a cardinal, and let X be a space with a dense set of γ-points. Then for every nonempty G_δ G in βX, if $G \subseteq X^*$ then G contains (at least) 2^c γ-points of βX.*

□ As is the proof of (4.8), put $Y = \beta X - G$, and let $\langle I_n: n \in \omega \rangle$ be a countably infinite discrete family of (nonempty) open sets in Y. Choose a γ-point $a_n \in I_n$ for $n \in \omega$, and let $A = \{a_n: n \in \omega\}$. Since $\langle I_n: n \in \omega \rangle$ is disjoint one easily checks that A is a γ-set in X. Using in turn (6.2c), (6.2b) and (6.2a) we see that every point of $G \cap \mathrm{Cl}_{\beta X} A$ is a γ-point of βX. But $G \cap \mathrm{Cl}_{\beta X} A = (\mathrm{Cl}_{\beta X} A) - A$, which is homeomorphic to N^*, hence $|G \cap \mathrm{Cl}_{\beta X} A| = 2^c$. □

From (1.8b), (2.3), (6.2c) and (6.5) we deduce the following effective nonhomogeneity result, which implies Theorem (1.10).

6.6. THEOREM. *Let X be a nowhere locally compact nonpseudocompact space with countable π-weight which fails to be extremally disconnected at a dense set of points (e.g. if X is first countable).*

Then X^* is not homogeneous because X^* is extremally disconnected at some but not at all points. □

6.7. Remark. If X is extremally disconnected, so is βX ([23], 6M.1). If X is in addition nowhere locally compact, it follows from (6.2c) that X^* is extremally disconnected at all points. Hence the above argument does not work for extremally disconnected spaces.

6.8. Remark. If X is noncompact but realcompact and locally compact, then every nonempty G_δ of X^* has nonempty interior in X^* ([17], 3.1). If in addition, $|X^*|$ is not Ulam-measurable, e.g. if $X = \mathbf{R}$, then by a recent result of Szymański, [41], every point of X^* is an ω-point of X^*. So then the argument of (6.6) cannot be used to show that X^* is nonhomogeneous although X^* certainly is nonhomogeneous, [20].

I do not know if a homeomorphism of X^* onto itself can send a nonremote point to a remote point for $X = \mathbf{R}$, or, more generally, for realcompact X. If X is not realcompact, there is a counterexample.

6.9. EXAMPLE. *There is a nonpseudocompact space E (with countable π-weight) such that some homeomorphism from E^* onto E^* maps a nonremote point onto a remote point (and E^* and $\mathbf{Q}^*$ are homeomorphic).*

□ Let p be a point in Q^* that is not far. Let

$$H = (\beta Q \times \beta Q) - (\{p\} \times Q^*)$$

then $E = Q+H$ is as required. For Q has remote points, and H has remote points, but $E^* = Q^*+H^*$, so it suffices to show that H^* and are homeomorphic.

Indeed, since $(\beta Q - \{p\}) \times \beta Q$ is pseudocompact, being the product a pseudocompact factor, by (2.1), and a compact factor,

$$\beta((\beta Q - \{p\}) \times \beta Q) = \beta(\beta Q - \{p\}) \times \beta Q = \beta Q \times \beta Q$$

by Glicksberg's Theorem, [26], quoted in (1.11), and (2.5). Using (2.5) o more, we see that $H^* = \{p\} \times Q^*$.

(E^* and Q^* are homeomorphic since $Q^* + Q^*$ and Q^* are.) □

NOTES TO § 6

6.10. W. Rudin gave an effective proof that X^* is not homogeneous, for nonpse compact locally compact X, assuming CH: some but not all points of X^* are P-points,(10) [As mentioned earlier, Frolík proved that X^* is not homogeneous for all nonpseudocompa without using additional axioms, [20], see also [19]; that proof is not effective. I gav effective proof that X^* is not homogeneous if X is nowhere locally compact and e metrizable or normal but not Lindelöf in [7], see also § 9. In [13] I will give an effe proof that R^* is not homogeneous.

7. Čech-Stone compactifications and products

7.1. In this section we use the concept of extremal disconnected at a point to effectively prove that certain spaces cannot be factored a nontrivial way. Theorems 1.12 and 1.14 are immediate consequen An attractive feature of our proofs is that they do not explicitly men C^*-embedding, like proofs in [26], [22] or [9].

We need the following observation. (Since we need countable π-we for showing that βX is somewhere extremally disconnected, it does not m sense to consider nonseparable spaces.)

7.2. LEMMA. *Let X and Y be separable, if x is not isolated in X y is not isolated in Y, then $X \times Y$ is not extremally disconnected at* $\langle x,$

□ There are countable dense subsets A of X and B of Y with $x \notin A$ $y \notin B$. Imitating the proof that a countable regular space is normal, can find disjoint open U and V in $X \times Y$ such that $A \times \{y\} \subseteq U$ $\{x\} \times B \subseteq V$. Then $\langle x, y\rangle \in \bar{U} \cap \bar{V}$. □

7.3. COROLLARY. *If X and Y are separable spaces without isolated po then $X \times Y$ is nowhere extremally disconnected.* □

7.4. THEOREM. *Let X be a nonpseudocompact space with countable π-weight. If Y and Z are any two spaces without isolated points, then βX and $Y \times Z$ are not homeomorphic, because βX is separable and somewhere extremally disconnected, but $Y \times Z$ is either nonseparable or nowhere extremally disconnected (or both).*

□ βX is separable because X is, and if $Y \times Z$ is separable, so are Y and Z, hence the theorem follows from (1.8a) and (7.3). □

7.5. COROLLARY. *Let $1 \leqslant \varkappa \leqslant \omega$. Let $\langle X_n : n \in \varkappa \rangle$ be a family of spaces each with countable π-weight. If either two X_n's have no isolated points or infinitely many X_n's have at least two points, then the following are equivalent:*

(1) $\prod_{n \in \varkappa} X_n$ *is pseudocompact;*

(2) $\beta(\prod_{n \in \varkappa} X_n) = \prod_{n \in \varkappa} \beta X_n$,

(3) $\beta(\prod_{n \in \varkappa} X_n) \approx \prod_{n \in \varkappa} \beta X_n$.

□ (1) ↔ (2): This is a special case of a theorem of Glicksberg, [26].

(2) → (3): Trivial.

(3) → (1): $\prod_{n \in \varkappa} \beta X_n$ is homeomorphic to $Y \times Z$, for certain subproducts Y and Z which have no isolated points. □

Our next result is an analogue of (7.4).

7.6. THEOREM. *Let X be a nowhere locally compact nonpseudocompact space with countable π-weight. If Y and Z are any two spaces without isolated points, then X^* and $Y \times Z$ are not homeomorphic because X^* is separable and somewhere extremally disconnected, but $Y \times Z$ is either nonseparable or nowhere extremally disconnected (or both).*

□ X^* is separable because it has countable π-weight, by (2.4), hence the theorem follows from (1.8b) and (7.3). □

7.7. COROLLARY. (a) *If X and Y are nowhere locally compact spaces with countable π-weight, and $X \times Y$ is not pseudocompact, then $(X \times Y)^*$ and $X^* \times Y^*$ are not homeomorphic.*

(b) *If $n > 1$, then Q^* and $(Q^*)^n$ are not homeomorphic.* □

7.8. Unfortunately there is no analogue of (7.5), for there is a nowhere locally compact space K with countable π-weight, such that $(K \times K)^*$ and $K^* \times K^*$ are not homeomorphic (which is not surprising), see (21.1). In fact even the following question is open.

7.9. QUESTION. *Do there exist noncompact X and Y such that $(X \times Y)^*$ and $X^* \times Y^*$ are homeomorphic? Can $X \times Y$ be pseudocompact?*

Corollary (7.7.b) suggests the following

7.10. QUESTION. *Is it true that $(Q^*)^k$ and $(Q^*)^n$ are homeomorphic (if and)* *only if* $k = n$*? Can one at least find for each* $k \in \omega$ *an* $m \in \omega$ *such that* $(Q^*)^k$ *and* $(Q^*)^n$ *are not homeomorphic if* $n > m$*?*

A natural way to attack this question would be to show that $(Q^*)^k$ contains a strict m-point for some $m < \omega$, since one can easily generalize (7.3) to the effect that every point of $(Q^*)^n$ is an n-point. Unfortunately our next result shows that this method does not work.

7.11. PROPOSITION. *Every point of* $Q^* \times Q^*$ *is an* ω*-point.*

□ Denote $\beta Q \times \beta Q$ by Π, and let $^-$ be the closure operator in Π. Since $Q^* \times Q^*$ is dense in Π, it suffices to show that every point of Π is an ω-point, by (6.2c). We prove a little bit more, see Lemma (6.2), by showing that $\beta Q \times \{y\}$ is an ω-set in Π for every $y \in \beta Q$.

Fix $y \in \beta Q$. List Q as $\langle q_n : n \in \omega \rangle$, with $q_n \neq q_{n'}$ if $n \neq n'$. For each $n \in \omega$ we will construct a sequence $\langle p_{n,k} : k \in \omega \rangle$ in $Q \times Q$, and families $\langle V_{n,k} : k \in \omega \rangle$ and $\langle W_{n,k} : k \in \omega \rangle$ of clopen sets in βQ such that

(1) $\langle q_n, y \rangle \in \{p_{n,k} : k \in \omega\}^-$;

(2) $p_{n,k} \in V_{n,k} \times W_{n,k}$;

(3) every point of $\beta Q - \{q_n\}$ has a neighborhood which intersects at most one $V_{n,k}$;

(4) $y \notin W_{n,k}$; and

(5) $(V_{n,k} \times W_{n,k}) \cap (V_{n',k'} \times W_{n',k'}) = \emptyset$ if $n \neq n'$ or $k \neq k'$.

Suppose for a moment that we have constructed this. Every point of $Q \times Q$ is an ω-point in $Q \times Q$, hence in Π, by (6.2). It follows from (2) that we can find for each $n, k \in \omega$ a disjoint open family $\langle U_{n,k,i} : i \in \omega \rangle$ in Π such that

(6) $p_{n,k} \in \bar{U}_{n,k,i}$ for all $i \in \omega$;

(7) $\bar{U}_{n,k,i} \subseteq V_{n,k} \times W_{n,k}$ for all $i \in \omega$.

Define

$$\mathscr{U} = \langle \bigcup \{U_{n,k,i} : n, k \in \omega\} : i \in \omega \rangle.$$

Then $\mathscr{U}$ is a countably infinite open family, which is disjoint by (5) and (7). And $Q \times \{y\} \subseteq \bar{U}$, hence $\beta Q \times \{y\} \subseteq \bar{U}$, for every $U \in \mathscr{U}$, by (1) and (6). So $\mathscr{U}$ witnesses that $\beta Q \times \{y\}$ is an ω-set.

We now proceed to the construction.

Suppose we have constructed $\langle p_{n,k} : k \in \omega \rangle$, $\langle V_{n,k} : k \in \omega \rangle$ and $\langle W_{n,k} : k \in \omega \rangle$ for $n < m$. It follows from (3) that there is a neighborhood A of q_n in βQ which intersects at most one $V_{n,k}$ ($k \in \omega$), for every $n < m$. It then follows from (4) and the fact that the $W_{n,k}$'s are clopen that there is a neighborhood B of y in βQ such that $A \times B$ does not intersect any $V_{n,k} \times W_{n,k}$ with $n < m$, $k \in \omega$.

Since βQ is first countable at q_m, we can choose a neighborhood base $\{A_k : k \in \omega\}$ of clopen sets for q_m in βQ with $A = A_0$, and $A_k \supset A_{k+1}$ for $k \in \omega$, and then pick $a_k \in Q \cap (A_k - A_{k+1})$ and put $V_{m,k} = A_k - A_{k+1}$, for $k \in \omega$.

We also can list $(Q\cap B)-\{y\}$ as $\langle b_k: k\in\omega\rangle$, and choose a clopen set $W_{m,k}$ in βQ with $b_k\in W_{m,k}\subseteq B-\{y\}$, for $k\in\omega$.

Put $p_{m,k}=\langle a_k, b_k\rangle$. To show that this works, we only have to verify (1), since (2), (3), (4) and (5) are evident. Let $C\times D$ be a neighborhood of $\langle q_n, y\rangle$. Then D will contain infinitely many b_k's. But C contains all but finitely many a_k's, hence $p_{m,k}=\langle a_k, b_k\rangle\in C\times D$ for some $k\in\omega$. □

7.12. Corollary. *If $\varkappa>1$, then every point of $(Q^*)^\varkappa$ is an ω-point.* □

We finish by giving an effective proof that $\beta Q \not\approx \beta Q\times\beta N$. This does not follow from (7.3), and in fact $\beta Q\times\beta N$ is, like βQ, somewhere extremally disconnected.

7.13. Theorem. *βQ and $\beta Q\times\beta N$ are not homeomorphic because every nonempty G_δ in βQ which does not contain a point with a countable neighborhood base, has a point at which βQ is extremally disconnected, but $\beta Q\times\beta N$ has a nonempty G_δ without points with a countable neighborhood base and without points at which $\beta Q\times\beta N$ is extremally disconnected.*

□ The statement about βQ follows from (4.2) and the fact that $x\in\beta Q$ has a countable neighborhood base in βQ iff $x\in Q$.

The G_δ in $\beta Q\times\beta N$ is $\beta Q\times(\beta N-N)$, because of (7.2). □

NOTES TO § 7

7.14. It is a famous result of Glicksberg, [26], that $\beta X\times\beta Y=\beta(X\times Y)$ iff either one of X and Y is finite or $X\times Y$ is pseudocompact. Gillman and Jerison, [22], give an example of an E such that $\beta E\times\beta E\approx\beta(E\times E)$ but $\beta E\times\beta E\neq\beta(E\times E)$. In [9] we give an example of spaces X and Y, with X without isolated points, such that $\beta X\times\beta Y\approx\beta(X\times Y)$ but $\beta X\times\beta Y\neq\beta(X\times Y)$, inspired by E.

In [9] it is shown that the restriction that X have countable π-weight can be omitted in Theorem (7.4), hence Corollary (7.5) is true even if the conditions "$\varkappa\leqslant\omega$" and "each with countable π-weight" are omitted. I do not know if the condition on the π-weight is essential in (7.6) and (7.7a).

Other conditions under which $\beta X\times\beta Y\approx\beta(X\times Y)$ implies that $X\times Y$ is pseudocompact can be found in [22]. One of their conditions is that X and Y both are first countable ((7.13) is a special case), and that proof is effective: if $X\times Y$ is not pseudocompact then $\beta(X\times Y)\not\approx\beta X\times\beta Y$ since the subspace of points with a countable neighborhood base in $\beta(X\times Y)$ is C^*-embedded, but the corresponding statement of $\beta X\times\beta Y$ is false. (The reason for including (7.13) is that it does not mention C^*-embedding.)

8. When "extremally disconnected at" implies remote

8.1. We have seen that βX is extremally disconnected at remote points, (5.2), and that under natural conditions βX is not extremally disconnected at all points, (6.5), and that in fact some points of X^* can be ω-points, e.g. if $X=Q$. The main result of this section is that βX is not extremally disconnected at nonremote points of X^*, and in fact that all nonremote points of βX in X^* are ω-points, for nice X like Q and R. Consequently, if p is a point of Q^* at which Q^* is not extremally disconnected, then p is an ω-point of Q^*, by (6.2c). This is unfortunate: it would have been nice if for every n, $2 \leqslant n < \omega$, there was a strict n-point in Q (i.e. an n-point that is not an $(n+1)$-point). The following result applies to $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and also $\boldsymbol{S}$ (this justifies the somewhat technical condition on X).

8.2. THEOREM. *Let X be a first countable space in which every closed subspace is separable. Then the following conditions on a point p of X^* are equivalent*:

(a) *p is a remote point;*

(b) *βX is extremally disconnected at p;*

(c) *p is not an ω-point of βX.*

□ (a) → (b). This is (5.2).

(b) → (c). This is trivial.

(c) → (a). Suppose p is not a remote point. Then p is in the closure of a nowhere dense subset A of X. Without loss of generality A is closed in X, hence is separable. Because of (6.2) it now suffices to show that A is an ω-set in X. That we state as a separate lemma since we need it later. □

8.3. LEMMA. *Let A be a nowhere dense separable subset of a first countable space X. Then A is an ω-set in X.*

□ Enumerate some countable dense subset of A as $\langle a_n \colon n \in \omega \rangle$. For each $n \in \omega$ choose a decreasing neighborhood base $\langle B(n,k) \colon k \in \omega \rangle$ of a_n in X. With recursion on $k \in \omega$ choose pairwise disjoint nonempty open subsets $U(n,k,l)$, for $n \leqslant k, l \leqslant k$, in such a way that

$$U(n,k,l) \subseteq B(n,k), \quad A \cap \mathrm{Cl}_X U(n,k,l) = \emptyset \quad (n, l \leqslant k).$$

Then

$$\left\langle \bigcup \{U(n,k,l) \colon k \geqslant l \text{ and } n \leqslant k\} \colon l \in \omega \right\rangle$$

witnesses that A is an ω-set. □

This can also be proved for all metrizable spaces, so (8.2) is true for metrizable X.

8.4. COROLLARY. *Let X be a nowhere locally compact first countable space in which every closed subspace is separable. Then the following conditions on $p \in X^*$ are equivalent*:

(a) *p is a remote point,*

(b) *X^* is extremally disconnected at p,*

(c) *p is not an ω-point of X^*.*

□ Use (2.3) and Lemma (6.2c). □

Again, this also is true for metrizable X.

NOTES TO § 8.

8.5. Porter and Woods ([33], 2.6), characterize, independently, the set of remote points of a separable metrizable nowhere locally compact space X as the largest extremally disconnected subspace X^*; this is a special case of (8.4).

9. Far points, remote points, and nonhomogeneity

9.1. The usefulness of far points is that they can be used for showing why certain Čech–Stone remainders are not homogeneous. The following easy lemma summarizes the relevant information from [7], § 2.

9.2. Lemma. *Let X be a nowhere locally compact space.*

(a) *If $p \in X^*$ is not in the closure (in βX) of a (countable) closed discrete subset of X, then p is not in the closure (in βX^*) of a (countable) closed discrete subset of X^{**}.*

(b) *If X is not countably compact, some points of X^* are in the closure (in βX^*) of a countable closed discrete subset of X^{**}.*

(c) *Let h be a homeomorphism of X^* onto itself. If $x \in X^*$ is in the closure (in βX^*) of a (countable) closed discrete subset of X^{**}, so is $h(x)$.* □

9.3. In other words, for certain X the remainder X^* is not homogeneous because some but not all points of X^* are "far from X^{**}". In [7] I showed that βX has a far point if X is a noncompact metrizable space without isolated points. That also follows from (4.8). I also observed that if X is a normal nonLindelöf space, then some point of X^* is not in the closure (in βX^*) of any *countable* closed discrete subset of X^{**}; this is enough for telling *why* X^* is not homogeneous if X is in addition nowhere locally compact and is not countably compact: some but not all points of X^* are in the closure (in βX^*) of a countable closed discrete subset of X^{**}.

It is unknown if βX must have a far point if X is a noncompact Lindelöf space without isolated points, but (4.8) shows that X has such a point if X is in addition first countable. [A noncompact Lindelöf space is not pseudocompact of course.]

In (6.6) we showed effectively that Q^* has two points with different topological behavior. Frolík's nonhomogeneity proof shows that there are $2^{\mathfrak{c}}$ points in Q^* which are topologically distinct, [20]; that proof is not effective. We now show that we can effectively get three points in Q^*, and also in P^* and S^*, with different topological behavior. Recall that in § 8 we showed that a natural way to get countably many points does not work.

9.4. Theorem. *Let X be a first countable, separable, normal, nowhere locally compact, nonpseudocompact space. Define $A, B \subseteq X$ by*

$$A = \{x \in X^*\colon\ x \notin \mathrm{Cl}_{\beta X^*} D \text{ for every countable closed discrete subset } D \text{ of } X^{**}\},$$
$$B = \{x \in X^*\colon\ X^* \text{ is extremally disconnected at } x\},$$

then $B \subseteq A$, and $B, A-B$ and X^*-A are nonempty.

Consequently there are three points in X^* such that no homeomorphism of X^* onto itself maps one of them onto another.

□ First we note that (9.1a) implies that $A = \varphi(X)$. The proof of Theorem (4.8) really shows that the subset

$$C = \{x \in X^*\colon\ x \in \mathrm{Cl}_{\beta X} K \text{ for some separable nowhere dense (closed) } K \subseteq X\}$$

of $X^* - \varrho(X)$ intersects $\varphi(X)$. But every point of C is an ω-point by (8.3) and (6.2). Hence

$$B \cap C = \emptyset.$$

Since $X^* - A = X^* - \varphi(X) \subseteq C$, it follows that $B \subseteq A$, and since $A = \varphi(X)$ intersects C, it follows that $A - B \neq \emptyset$. Furthermore, $B \neq \emptyset$ by (5.2) and $X - A = X - \varphi(X) \neq \emptyset$ by (4.8).

If h is a homeomorphism from X^* onto itself, then $h^{\rightarrow}A = A$ by (9.2c), and trivially $h^{\rightarrow}B = B$, hence $h^{\rightarrow}(A-B) = A-B$. □

Chapter III

Other applications of remote points

One of the other applications has been given already: we used remote points in the proof of Theorem (5.5). Another, potential, application of remote points will be given in § 20.

Recall that $\varrho(X)$ is the set of remote points of X and $\varphi(X)$ is the set of far points of X.

10. Two examples on extremal disconnectedness

10.1. Example. *A countable extremally disconnected space E without isolated points which has countable π-weight.*

□ Q^* has countable π-weight, (2.4), hence there is a countable dense subset E of Q^* consisting of remote points. Then E is also dense in βQ, hence

E is extremally disconnected, since it is extremally disconnected at each of its points, by (5.2) and (6.2c);

E has no isolated points, being dense in βQ; and

E has countable π-weight, again by (2.4). □

10.2. Remark. βE is the absolute, [25], of βQ, hence of I, the closed unit interval. For βE is extremally disconnected since E is ([23], 6M.1), and there is an irreducible map from E onto βQ: since βQ is a compactification of E there even is a map f from βE onto βQ such that $f{\restriction}E = \mathrm{id}_E$. [$f$ is irreducible since $f^{\leftarrow}E^* = \beta Q - E$, by [23], 6.12, or [15], 3.5.7.]

10.3. Remark. Since E is countable but not compact, it is not pseudocompact. Hence βE has both remote and nonfar points, by (4.4) and (4.5).

10.4. EXAMPLE. *A countable space without isolated points which is extremally disconnected at precisely one point.*

□ By (1.8a) there is a point q at which βQ is extremally disconnected. Then the subspace $Q \cup \{q\}$ of βQ is extremally disconnected at q, by (6.2c), but not at any other point of course. □

That such an example exists answers a question of Telgársky, [43].

11. A partition of R^*. Recall that a space Y is called *ω-bounded* (or $\aleph_0$-bounded) if every countable subset of Y has compact closure in Y.

11.1. THEOREM. *Let X be a σ-compact locally compact space without isolated points with countable π-weight.*

(a) *$\{\varrho(X), X^*-\varrho(X)\}$ is a partition of X^* into two dense ω-bounded subspaces.*

(b) *If X also is first countable, then $\{\varrho(X), \varphi(X)-\varrho(X), X^*-\varphi(X)\}$ is a partition of X^* into three dense ω-bounded subspaces.*

□ It is clear that the families mentioned are partitions. Since X is realcompact, being σ-compact, (2.2), it follows from (4.4), (4.6) and (4.10) that the members of the partitions are dense in X^*. That they are ω-bounded follows from the lemma below. □

The following lemma is recorded separately because we will need it later, in § 17.

11.2. LEMMA. *Let X be a Lindelöf space. Let $^-$ denote the closure operator in βX. Put*

$$\Omega = \{Y \subseteq X^*: \text{for every countable } A \subseteq Y, \text{ if } \bar{A} \subseteq X^* \text{ then } \bar{A} \subseteq Y\}.$$

Then $\varrho(X), \varphi(X), X^*-\varrho(X), X^*-\varphi(X)$ and $\varphi(X)-\varrho(X)$ belong to Ω.

□ $\varrho(X) \in \Omega$: Let $A \subseteq \varrho(X)$ be countable, and assume $\bar{A} \subseteq X^*$. Let D be any nowhere dense subset of X, we want to show that $\bar{A} \cap \bar{D} = \emptyset$. Without loss of generality D is closed in X. Let $Y = X \cup A$. Then D is closed in Y since $\bar{D} \cap A = \emptyset$, and A is closed in Y since $\bar{A} \subseteq X^*$; so D and A are disjoint closed sets in Y. But Y is normal, being Lindelöf, so A and D have disjoint closures in βY. Now $\beta Y = \beta X$ by (2.5), since $X \subseteq Y \subseteq \beta X$. It follows that $\bar{A} \cap \bar{D} = \emptyset$. Since D was arbitrary, it follows that $\bar{A} \subseteq \varrho(X)$.

$\varphi(X) \in \Omega$: The same proof works.

$X^*-\varrho(X) \in \Omega$: Let $A \subseteq \varrho(X)$ be countable, and assume $\bar{A} \subseteq X^*$. Since X is Lindelöf and $X \cap \bar{A} = \emptyset$, there is an open F_σ-subset U of βX with $X \subseteq U \subseteq \beta X - A$. There is a sequence $\langle U_n: n \in \omega\rangle$ of open subsets of βX with $\bigcup_{n\in\omega} U_n = U$, and $\bar{U}_n \cap A = \emptyset$ and $U_n \subseteq U_{n+1}$ for all $n \in \omega$.

Enumerate A as $\langle a_n: n \in \omega\rangle$, and choose for each n a nowhere dense subset D_n of X with $a_n \in \bar{D}_n$. Then also $a_n \in \overline{D_n - U_n}$, because $\beta X - \bar{U}_n$ is a neighborhood of a_n in βX, for $n \in \omega$. Hence if $D = \bigcup_{n\in\omega} (D_n - U_n)$ then

$A \subseteq \bar{D}$, hence $\bar{A} \subseteq \bar{D}$. But D is a nowhere dense subset of X: if $x \in X$, then $x \in U_k$ for some k, but $U_k \cap (D_n - U_n) = \emptyset$ for $n \geqslant k$. Since $\bar{A} \subseteq X^*$, it follows that $\bar{A} \subseteq X^* - \varrho(X)$.

$X^* - \varphi(X) \in \Omega$: The same proof works.

$\varphi(X) - \varrho(X) \in \Omega$: Ω clearly is closed under arbitrary intersections, and $\varphi(X)$ and $X^* - \varrho(X)$ belong to Ω. □

NOTES TO § 11

11.3. After proving (11.1) I discovered that Woods also proved that $\varrho(X)$ and $X^* - \varrho(X)$ are ω-bounded if X is σ-compact and locally compact, [47]. His proof argument for $\varrho(X)$ is somewhat different, and his argument for $X^* - \varrho(X)$ is similar, but not entirely correct.

Woods also shows that if X is σ-compact and locally compact, and has $\mathfrak{c}$ zero-sets, then there is a partition of X^* into $2^{\mathfrak{c}}$ dense ω-bounded subspaces under CH. So we have a long way to go.

12. Extremal disconnectedness and C^*-embedding. $\boldsymbol{Q}$ is not C^*-embedded in $\beta\boldsymbol{R}$, so $\beta\boldsymbol{R}$, which is a compactification of $\boldsymbol{Q}$, is not $\beta\boldsymbol{Q}$. In fact $\beta\boldsymbol{R}$ is pretty far from being $\beta\boldsymbol{Q}$. In spite of this we will show that $\boldsymbol{Q}^* \cap \boldsymbol{R}^* \neq \emptyset$, in a sense to be made precise in Theorem (12.2). The key to this result is the following lemma, which also will be useful in (13.2).

12.1. Lemma. *Let X be a dense subspace of Y. If Y is extremally disconnected at each point of $Y - X$, then X is C^*-embedded in Y.*

□ Let $\boldsymbol{I}$ be the closed unit interval. It suffices to show how to extend continuous functions from X to $\boldsymbol{I}$. Let $f\colon X \to \boldsymbol{I}$ be continuous. In order to show that f can be extended to a continuous function $f\colon Y \to \boldsymbol{I}$ it is sufficient (and necessary) to show

(1) for any two disjoint closed F, G in $\boldsymbol{I}$, the sets $f^{\leftarrow}F$ and $f^{\leftarrow}G$ have disjoint closures in Y,

([42], [15], Thm. 3.2.1). So let F and G be disjoint closed sets in $\boldsymbol{I}$. Clearly no point of X belongs to $(\mathrm{Cl}_Y f^{\leftarrow}F) \cap (\mathrm{Cl}_Y f^{\leftarrow}G)$. Let U and V be disjoint open sets in $\boldsymbol{I}$ with $F \subseteq U, G \subseteq V$. Then $f^{\leftarrow}U$ and $f^{\leftarrow}V$ are disjoint open sets in X. If $y \in Y - X$ then $X \cup \{y\}$ is extremally disconnected at y, by (6.2c), hence $y \notin (\mathrm{Cl}_Y f^{\leftarrow}U) \cap (\mathrm{Cl}_Y f^{\leftarrow}V)$. Consequently $y \notin (\mathrm{Cl}_Y f^{\leftarrow}F) \cap (\mathrm{Cl}_Y f^{\leftarrow}G)$. This completes the proof of (1). □

12.2. Theorem. *Let Z be a space with countable π-weight which is not pseudocompact. Let X be a dense subspace, and let $f\colon \beta X \to \beta Z$ be the (unique) map such that $f \restriction X = \mathrm{id}_X$. Then there is a subset E of Z^* with $|E| = 2^{\mathfrak{c}}$ such that the restriction*

$$r = f \restriction (X \cup f^{\leftarrow}E)$$

is a homeomorphism.

□ Let E be the set of points of Z^* at which βX is extremally disconnected.

By (5.4), $|E| = 2^{\mathfrak{c}}$. We give two proofs that this works. Put $Y = X \cup E$. Note that $X \cup f^{\leftarrow}E = f^{\leftarrow}Y$.

First proof. Y is extremally disconnected at each point of E, because of (6.2c). Hence $\beta X = \beta Y$ by (12.1), i.e. there is a homeomorphism h: $\beta X \to \beta Y$ with $h \restriction Y = \mathrm{id}_Y$. There also is a map g: $\beta Y \to \beta Z$ with $g \restriction Y = \mathrm{id}_Y$. Then

$$\begin{array}{ccc} \beta X & \xrightarrow{h} & \beta Y \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle g} \\ & & \beta Z \end{array}$$

$f \restriction X = \mathrm{id}_X = g \circ h \restriction X$, hence $f = g \circ h$. Hence f embeds $f^{\leftarrow}Y = (g \circ h)^{\leftarrow}Y$ into βZ.

Second proof. Since $f \restriction f^{\leftarrow}Y$ is a closed continuous map (because f is closed and continuous), it suffices to show that $|f^{\leftarrow}\{y\}| = 1$ for $y \in Y$. Since $f^{\rightarrow}(\beta X - X) = \beta Z - X$ ([15], 3.5.7 or [23], 6.12), it suffices to prove this for $y \in Y$. But this is the same as Gleason's proof that if φ: $S \to T$ is an irreducible closed continuous surjection, and T is extremally disconnected, then φ is one-to-one ([25], 2.3). □

See (18.4) for a similar result.

The following is an amusing corollary to (12.1) (use (2.3) and (6.2)).

12.3. Proposition. *Let bX be a compactification of a nowhere locally compact space X. If $bX - X$ is extremally disconnected, then $\beta X = bX$.* □

The same sort of argument can be used to show that Q^* has an unusual property.

12.4. Proposition. *Let X be a nowhere locally compact realcompact space with countable π-weight. Then every compactification of X^* is the Čech–Stone compactification of some nowhere locally compact (in particular: proper) subspace.*

□ Given a compactification bX^* of X^*, consider the subspace

$$Y = (bX^* - X^*) \cup \{x \in X^*\colon X^* \text{ is not extremally disconnected at } x\}.$$

Y is dense in bX^* since $bX^* - X^*$ is, (2.3), and is nowhere locally compact since $bX^* - Y$ is dense in X^*, by (6.4), hence in bX^*. □

NOTES TO § 12

12.5. Lemma (12.1) is an easy extension of [23], 6M.2, where the case that Y is extremally disconnected (= at *all* points) is considered. Theorem (12.2) is vaguely related to (independent) ideas of Woods, [46].

13. Connected compactifications. Since extremal disconnectedness and connectedness are extreme opposites, the following example is unexpected.

13.1. Example. *There is an extremally disconnected space which has a connected compactification.*

□ Let X be any nowhere locally compact connected separable metrizable space, e.g. $\boldsymbol{R}\times\boldsymbol{R}-\boldsymbol{Q}\times\boldsymbol{Q}$ or $\boldsymbol{R}^{\omega}$. Then βX is connected. Also, βX has a dense extremally disconnected subspace by (5.4) and (6.2c). □

A space with an isolated point cannot have a connected compactification of course. More generally, a space which has a nonempty proper compact open subset cannot have a connected compactification. In view of this the following Example gives a nontrivial example of a space without connected compactification.

13.2. EXAMPLE. *There is a nowhere locally compact space which has no connected compactification.*

□ We need the following facts about extremally disconnected spaces, cf. [23], 6M.1:

(1) X is extremally disconnected iff βX is;

(2) if X is extremally disconnected, every dense subspace of X is C^*-embedded in X (this also follows from (12.1)).

Let E be any countable extremally disconnected space without isolated points, e.g. Example (10.1). Then E is nowhere locally compact, hence $A = E^*$ is dense in βE, by (2.3). It follows from (1) and (2) that

(3) $\beta A = \beta E$ and A is extremally disconnected.

Let bX be any compactification of A, let $B = bA-A$. Then B is dense in bA since A is nowhere locally compact, and bA is extremally disconnected at each point of A by (6.2c). It follows from (12.3) that

(4) $bA = \beta B$.

The (unique) map $f\colon \beta A\to bA$ with $f\restriction A = \mathrm{id}_A$ maps A^* onto B ([15], 3.5.7, or [23], 6.12), hence $|B|\leqslant |A^*| = |E| = \omega$, by (3). It now follows from (4) that bA is not connected since B is not connected. (In fact bA is strongly zero-dimensional of course.) □

13.3. Remark. A totally different proof that bA is not connected would use the fact that if $f\colon \beta A\to bA$ is the unique map with $f\restriction A = \mathrm{id}_A$, then $|f^{\leftarrow}\{x\}| > 1$ for only finitely many $x\in bA-A$, cf. [45].

If A is the space of Example (13.2), it is not to surprising that A has no connected compactification: since A^* is countable, we do not have enough opportunity to manifacture a connected compactification. A very natural class of spaces X for which X^* has everywhere many points is the class of realcompact spaces. This leads to the following

13.4. QUESTION. *Let X be a realcompact space which has no nonempty open compact proper subset. (In particular, let X be nowhere locally compact.) Does X have a connected compactification?*

Smirnow has shown that the answer is yes if X is in addition separable and metrizable, [39]. I believe that the answer to (13.4) is in the negative, and offer the following candidate for a counter example:

13.5. CONJECTURE. *The Sorgenfrey line does not have a connected (Hausdorff!) compactification.*

(Mike Starbird has observed that S can be densely embedded into a connected Hausdorff space, [40], the construction is similar to the one in [3].)

Chapter IV

Properties of remote points

Recall that $\varrho(X)$ is the set of remote points of X and $\varphi(X)$ is the set of far points of X.

14. The structure of remote points. Remote points are defined externally: they have to avoid the closures of certain sets. We now show that the traces on X of the neighborhood filter of a point of X^* determine whether or not the point is remote; since a point of βX can be identified with the family of traces on X of its neighborhood filter, this means that remote points can be internally characterized. The following definition is recorded for definiteness.

14.1. DEFINITION. An *open ultrafilter* on a space X is a centered family of open sets of X not properly included in another such family.

Closed ultrafilters are defined similarly.

14.2. THEOREM. *Let X be a space. Then the following conditions on a point p of X^* are equivalent:*

(a) *p is a remote point;*

(b) *the family*

$$\mathscr{U} = \{X \cap V\colon V \text{ is an open neighborhood of } p \text{ in } \beta X\}$$

is an open ultrafilter on X;

(c) *the family*

$$\mathscr{K} = \{X \cap F\colon F \text{ is a closed neighborhood of } p \text{ in } \beta X\}$$

is a closed ultrafilter on X.

□ (a) → (b): Let U be an open set in X which intersects every member of $\mathscr{U}$. Then $p \in \mathrm{Cl}_{\beta X} U$, hence $p \in \mathrm{Ex}_X(U)$ by (5.1b). But $\mathrm{Ex}_X(U)$ is open in βX and $X \cap \mathrm{Ex}_X(U) = U$, by (3.1a). Consequently $U \in \mathscr{U}$.

(b) → (a): Let $A \subseteq X$ be nowhere dense. Then $U = X - \mathrm{Cl}_X A$ is dense in X, hence U intersects every member of $\mathscr{U}$, so $U \in \mathscr{U}$. The definition of $\mathscr{U}$ then tells that some neighborhood of p misses A.

(a) → (c): Let K be a closed set of X which intersects every member of $\mathscr{K}$ Then $p \in \mathrm{Cl}_{\beta X} K$. Let $U = \mathrm{Int}_X K$. Then $K - U$ is nowhere dense, hence $p \in \mathrm{Cl}_{\beta X} U$ since $p \notin \mathrm{Cl}_{\beta X}(K-U)$. It follows from (5.1b) that $p \in \mathrm{Ex}_X(U)$.

But $\mathrm{Ex}_X(U) \subseteq \mathrm{Cl}_{\beta X} U \subseteq \mathrm{Cl}_{\beta X} K$, hence $\mathrm{Cl}_{\beta X} K$ is a closed neighborhood of p. As $K = X \cap \mathrm{Cl}_{\beta X} K$, it follows that $K \in \mathscr{K}$

(c) $\to$ (a): Let $A \subseteq X$ be nowhere dense in X. Since $\mathrm{Cl}_{\beta X} A$ is nowhere dense in βX, it is not a neighborhood of p, hence $\mathrm{Cl}_X A = X \cap \mathrm{Cl}_{\beta X} A$ does not belong to $\mathscr{K}$ By maximality, there is an element of $\mathscr{K}$ which misses $\mathrm{Cl}_X A$, hence misses A. The definition of $\mathscr{K}$ now implies that some neighborhood of p misses A, hence $p \notin \mathrm{Cl}_{\beta X} A$. □

14.3. THEOREM. *There is a filter base $\mathscr{K}$, consisting of clopen subsets of Q, such that for every open or closed set U in Q the following holds:*

if U intersects every member of $\mathscr{K}$, then U includes a member of $\mathscr{K}$

□ Q is strongly zero-dimensional, by [15], Thm. 6.2.7, hence so is βQ, ([15], Thm. 6.2.12). Thus each remote point has a neighborhood base consisting of clopen sets. Now use (14.2). □

The question of whether or not such a filter base $\mathscr{K}$ exists was my original motivation for getting interested in remote points: in [11] I will use $\mathscr{K}$ to construct certain unusual examples.

NOTES TO § 14

14.4. Grant Woods has pointed out that ideas closely related to (but not identical with) Theorem (14.2) are in Mandelker's paper [28]. (Mandelker shows that $p \in R^*$ is remote iff $\{p\}$ is a round subset of βR (this we do not define). However, if ω_1+1 has the order topology, then the point ω_1 in $\omega_1+1 = \beta(\omega_1)$ is not remote, yet $\{\omega_1\}$ is a round subset of $\beta(\omega_1)$.)

15. Nonhomogeneity of spaces of special points

15.1. W. Rudin gave under CH an effective proof that N^* is not homogeneous by showing that under CH N^* has both P-points and non-P-points ([35], 4.4). He also showed that under CH the subspace of P-points of N^* is homogeneous ([35], 4.7). This suggests the question of whether one of the subspaces of special points of Q^*, P^* or S^* is homogeneous. We show that this is not the case.

15.2. THEOREM. *Let X be a nowhere locally compact, realcompact space with countable π-weight.*

(a) *$\varrho(X)$, $X^*-\varrho(X)$, $\varphi(X)$ and $X^*-\varphi(X)$ are not homogeneous.*

(b) *If X also is first countable and normal, $\varphi(X)-\varrho(X)$ is not homogeneous either.*

□ Let T be one of the spaces considered. Then T is dense in X^* and has cardinality 2^c by (4.4), (4.6) and (4.9). So T has countable π-weight by (2.3). However, if Y is a homogeneous (Hausdorff) space with π-weight $\varkappa$, then $|Y| \leqslant 2^{\varkappa}$ ([8], 1.1). □

This proof is noneffective since the proof of [8], 1.1, is noneffective. For $\varphi(X)$ we can do better:

15.3. PROPOSITION. *Let X be a first countable nowhere locally compact realcompact normal space. Then $\varphi(X)$ is not homogeneous because it is extremally disconnected at some but not at all points.*

□ Since $\varphi(X)$ is dense in X^* (because $\varphi(X)-\varrho(X)$ is, by (4.9)), it suffices in view of (6.2c) to show that X^* is extremally disconnected at some but not at all points of $\varphi(X)$. Since $\emptyset \neq \varrho(X) \subseteq \varphi(X)$ by (1.5), X^* is extremally disconnected at some point of $\varphi(X)$ by (5.2), namely at all points of $\varrho(X)$. In (4.8) we showed that $\varphi(X)-\varrho(X)$ is not empty by finding a point $p \in \varphi(X)$ with $p \in \mathrm{Cl}_{\beta X} K$ for some *separable* nowhere dense subset K of X. K is an ω-set in X by (8.3), hence p is an ω-point of X^* by (6.2) and (2.3). In particular, X^* is not extremally disconnected at p. □

15.4. If we would have a means to effectively prove that one of the other spaces of special points is not homogeneous, then we could effectively find more than three points in X^* which are mutually topologically distinct.

16. Special points and Baire spaces. In this section we give an effective proof that $\varrho(P)$ and $\varrho(Q)$ are not homeomorphic.

16.1. PROPOSITION. *Let X be a realcompact space with countable π-weight.*

(a) *$\varrho(X)$ is a Baire space iff X^* is a Baire space.*

(b) *$\varrho(X)$ is meager*[11] *iff X^* is meager.*

□ Recall from (4.4) that $\varrho(X)$ intersects every nonempty G_δ in X^*, in particular is dense in X^*. This reduces the proof to a triviality. □

16.2. THEOREM. *$\varrho(P)$ and $\varrho(Q)$ are not homeomorphic because $\varrho(P)$ is meager, while $\varrho(Q)$ is a Baire space.*

□ P is a G_δ in βP, being completely metrizable ([15], Thm. 4.3.26), hence P^* is an F_σ in βP. As P^* is dense in βP, (2.3), it follows that P^* is meager, hence so is $\varrho(P)$ by (16.1).

Q is countable, hence Q^* is an absolute G_δ ($\equiv$ Čech-complete). Therefore Q^* is a Baire space, hence so is $\varrho(Q)$. □

16.3. We leave it to the reader to do similar things for the other special points in X^*.

NOTES TO § 16

16.4. That $\varrho(P)$ and $\varrho(Q)$ are not homeomorphic was first proved, using a very long cardinality argument, by Gates, [21].

17. Pseudocompact spaces between X and βX. We begin with an easy observation.

17.1. THEOREM. *Let X be a space. Let Y be*

(a) $X \cup (X^* - \varphi(X))$; *or*

---

[11] $\equiv$ first category.

(b) $X \cup (X^* - \varrho(X))$, *and assume X has no isolated points; or*

(c) $X \cup \varphi(X)$ *or* $X \cup (\varphi(X) - \varrho(X))$, *and assume X is a first countable normal space without isolated points; or*

(d) $X \cup \varrho(X)$, *and assume X has countable π-weight.*

Then Y is pseudocompact.

□ $X \subseteq Y \subseteq \beta X$, hence $\beta Y = \beta X$ by (2.5). Therefore Y intersects every nonempty G_δ in βY: use (4.5) for (a) and (b), (4.8) for (c), and (4.2) for (d). Consequently Y is pseudocompact by (2.1). □

17.2. In particular, $X \cup \varrho(X)$ and $X \cup (X^* - \varrho(X))$ are pseudocompact, if $X = Q$ or R. Since $A \times B$ is not pseudocompact if A and B are pseudocompact subspaces of βX with $A \cap B = X$ if $X = N$ (such A and B exist, see [44] or [29]), this suggest the question of whether $(X \cup \varrho(X)) \times (X \cup (X^* - \varrho(X)))$ also fails to be pseudocompact if $X = Q$ or R. We will show that this is not the case. (See also Remark (17.12).) Although the results are of a technical nature, they are of interest since they show how abundant each class of special points is.

There are two ways of attacking this problem. The first, which works for all Y as in (17.1), provided X is first countable, is based on a weakening of the concept of $\mathscr{G}$-pseudocompactness, [24]. The second, which only works for $X \cup (X^* - \varphi(X))$ for first countable Lindelöf spaces, is based on a weakening of the notion of ω-boundedness.

17.3. Definitions. Let X be a space, and let $\langle A_n : n \in \omega \rangle$ be a sequence of subsets of X. Let $p \in X$.

(a) p is called a *limit point of* $\langle A_n : n \in \omega \rangle$ if every neighborhood of p intersects A_n for infinitely many n's.

Let $\mathscr{G}$ be a free ultrafilter on ω.

(b) p is called a *$\mathscr{G}$-limit point of* $\langle A_n : n \in \omega \rangle$ if for every neighborhood U of p, $\{n \in \omega : U \cap A_n \neq \emptyset\} \in \mathscr{G}$, cf. [2], [24].

17.4. Definition. Let $\mathscr{G}$ be a free ultrafilter on ω. Then the space X is called *$\mathscr{G}$-pseudocompact* if every sequence of nonempty open sets has a $\mathscr{G}$-limit point, [24].

The relevant information for appreciating these notions is summarized in the next proposition.

17.5. Proposition. (a) *If a sequence of nonempty sets has a limit point then it has a $\mathscr{G}$-limit point for a suitable free ultrafilter $\mathscr{G}$ on ω.*

(b) *A space is pseudocompact iff every sequence of nonempty open sets has a limit point,* [26]; *hence*

(c) *A product is pseudocompact iff every countable subproduct is,* [26].

(d) *For each free ultrafilter $\mathscr{G}$ on ω the product of any family of $\mathscr{G}$-pseudocompact spaces is $\mathscr{G}$-pseudocompact,* [24]. □

So for a fixed $\mathscr{G}$ the class of $\mathscr{G}$-pseudocompact spaces is a well-behaved subclass of the class of pseudocompact spaces. Unfortunately we have the following proposition.

17.6. PROPOSITION. *Let X be a space with countable π-weight. Then X is $\mathcal{G}$-pseudocompact for every free ultrafilter $\mathcal{G}$ on ω if and only if X is compact.* □

17.7. This does not exclude the possibility that for some $\mathcal{G}$ both $X \cup \varrho(X)$ and $X \cup (X^* - \varrho(X))$ are $\mathcal{G}$-pseudocompact, if $X = \mathbf{Q}$ or $\mathbf{R}$. Indeed, if $\mathcal{G}$ is a P-point, this is the case. Since it is unlikely that the existence of P-points is true in ZFC, and since I do not know another sort of ultrafilter that works and that exists in ZFC, I will follow a different method.

17.8. DEFINITION. Let $\mathcal{G}$ be a free ultrafilter on ω. Then the space X is called *weakly $\mathcal{G}$-pseudocompact* if every sequence of nonempty open sets has a $\mathcal{G}$-limit point or has a limit point which has a countable neighborhood base.

17.9. THEOREM. *Let $\mathcal{G}$ be a free ultrafilter on ω. Then any product of weakly $\mathcal{G}$-pseudocompact spaces is pseudocompact.*

□ Because of (17.5c) it suffices to prove this for countable products.

Let X_n be a weakly $\mathcal{G}$-pseudocompact space, for $n \in \omega$. Denote $\prod_{n\in\omega} X_n$ by Π. Let $\langle U_k : k \in \omega \rangle$ be any sequence of nonempty open sets. Since we want to show that $\langle U_k : k \in \omega \rangle$ has a limit point, we may assume without loss of generality that each U_k is a basic open set in Π, so we assume

$$U_k = \prod_{n\in\omega} U_{k,n}, \quad U_{k,n} \text{ open in } X_n \text{ for each } n \in \omega.$$

(We don't need the condition that $U_{k,n} = X_n$ for all but finitely many $n \in \omega$.) If a sequence $\langle A_k : k \in \omega \rangle$ of sets in a space Y clusters at a point which has a countable neighborhood base, then for some infinite $K \subseteq \omega$ every neighborhood of y intersects A_k for all but finitely many $k \in K$. Hence we can construct a sequence $\langle K_n : n \in \omega \rangle$ of infinite subsets of ω satisfying

(1) either there is a point of X_n every neighborhood of which intersects $U_{k,n}$ for all but finitely many $k \in K_n$,
or no point of X_n with a countable neighborhood base, is a cluster point of $\langle U_{k,n} : k \in K_n \rangle$;

(2) $K_n \supseteq K_{n+1}$.

Let $\langle k(i) \rangle_{i\in\omega}$ be any sequence of integers such that

(3) $k(i) < k(i+1)$;

(4) $k(i) \in K_i$.

Then $k(i) \in K_n$ for all but finitely many i's, for every $n \in \omega$. It follows from (1) that for each $n \in \omega$ we can choose a $p_n \in X_n$ which is a $\mathcal{G}$-limit of $\langle U_{k(i),n} : i \in \omega \rangle$: just observe that if every neighborhood of $p_n \in X_n$ intersects

$U_{k(i),n}$ for all but finitely many i's, then p_n is a $\mathscr{D}$-limit point of $\langle U_{k(i),n}: i \in \omega\rangle$. But then $p = \langle p_n: n \in \omega\rangle$ is a $\mathscr{D}$-limit point of $\langle U_{k(i)}: i \in \omega\rangle$, hence is a limit point of $\langle U_k: k \in \omega\rangle$.

It now follows from (17.5b) that Π is pseudocompact. □

17.10. THEOREM. *If X and Y are as in* (17.1), *and if X is first countable, then Y is weakly $\mathscr{D}$-pseudocompact for every free ultrafilter $\mathscr{D}$ on ω.*

□ Let $\langle U_n: n \in \omega\rangle$ be a sequence of nonempty open sets in Y, let $I_n = X \cap U_n$, for $n \in \omega$. If $\langle I_n: n \in \omega\rangle$ has a limit point in X then so has $\langle U_n: n \in \omega\rangle$. If not, then $\langle I_n: n \in \omega\rangle$ is a discrete sequence of nonempty open sets in βX. The proof that this sequence has a $\mathscr{D}$-limit point is the same as the proof that every nonempty G_δ in βX which misses X contains at least $2^{\mathfrak{c}}$ points of $Y - X$, see (4.1), (4.2), (4.5) and (4.8). □

17.11. COROLLARY. *If $X = Q$ or R, then $(X \cup \varrho(X)) \times (X \cup (X^* - \varrho(X)))$ is pseudocompact.* □

17.12. Remark. It is not true that if T is a space without isolated points, and X and Y are pseudocompact spaces with $T \subseteq X$, $Y \subseteq \beta T$ then $X \times Y$ is pseudocompact. In fact one can show that for every nonpseudocompact T there are (even) countably compact X and Y between T and βT such that $X \times Y$ is not pseudocompact.

We now turn our attention to $X \cup (X^* - \varphi(X))$. We need another definition.

17.13. DEFINITION. Let X be a space.

(a) X is called *ω-bounded* ($\aleph_0$-bounded) if every countable subset of X has compact closure.

(b) X is called *weakly ω-bounded* (some authors: strongly ω-compact) if every infinite subset of X has an infinite subset whose closure is compact.

Since a separable space is ω-bounded iff it is compact (cf. (17.5)), one cannot show that $X \cup (X^* - \varphi(X))$ is ω-bounded, this is why we have to consider weakly ω-bounded spaces. The following proposition summarizes all essential information.

17.14. PROPOSITION. (a) *Any countable product of weakly ω-bounded spaces is weakly ω-bounded.*

(b) *Any product of at most ω_1 weakly ω-bounded spaces is countably compact.*

(c) *Any product of weakly ω-bounded spaces is pseudocompact.*

(d) *$\{\beta N - \{p\}: p \in N^*\}$ is a family of weakly ω-bounded spaces whose product is not weakly ω-bounded.*

□ (a) Same proof as for sequential compactness.

(b) [36].

(c) This follows from (a) and (17.4c).

(d) ([24], 4.4). □

17.15. THEOREM. *Let X be a first countable Lindelöf space. Let Y be $X \cup (X^* - \varphi(X))$, or $X \cup (X^* - \varrho(X))$ and assume X has no isolated points. Then Y is weakly ω-bounded.*

□ We give the proof for $Y = X \cup (X^* - \varphi(X))$. Let A be a countably infinite subset of Y. If some point of X is a cluster point of A, then A has an infinite subset whose closure is compact, since every point of X has a countable neighborhood base in Y. So assume that no point of X is a cluster point of A.

Case 1. $A \cap X$ is infinite. Now $A \cap X$ is closed discrete in X, by assumption, so $\mathrm{Cl}_{\beta X}(A \cap X) \subseteq Y$.

Case 2. $A \cap X$ is finite. Then $A - X$ is infinite. But $X \cap \mathrm{Cl}_{\beta X}(A - X) = \emptyset$ by assumption, so $\mathrm{Cl}_{\beta X}(A - X) \subseteq Y$ by (11.2). □

17.16. Remark. Let $\mathscr{D}$ be a free ultrafilter on $\mathscr{D}$. A space is called *$\mathscr{D}$-compact* if every sequence of points has a $\mathscr{D}$-limit point, [2]. A separable space is $\mathscr{D}$-compact for every $\mathscr{D}$ iff it is compact, cf. (17.5). The previous proof can easily be modified to show that Y is *weakly* $\mathscr{D}$-compact (defined as in (17.6)), for every free ultrafilter $\mathscr{D}$ on ω. One can show that $\mathbf{R} \cup (\mathbf{R}^* - \varphi(\mathbf{R}))$ is $\mathscr{D}$-compact if and only if $\mathscr{D}$ is a P-point, but $\mathbf{R} \cup (\mathbf{R}^* - \varrho(\mathbf{R}))$ can be $\mathscr{D}$-compact for non-P-points $\mathscr{D}$, although I do not know if one can find such a $\mathscr{D}$ in ZFC.

NOTES TO § 17

17.17. That $X \cup \varrho(p)$ and $X \cup (X^* - \varrho(X))$ are pseudocompact for $X = \mathbf{R}$ is due to Fine and Gillman, who point out that their method does not work for $X = \mathbf{Q}$, [18].

(17.9) generalizes the fact that a product of $\mathscr{D}$-pseudocompact spaces, [24], or of first countable pseudocompact spaces, [26], is pseudocompact; the proof doesn't use any new ideas.

Chapter V

Miscellaneous remarks

18. Zero-dimensional spaces

18.1. A space is *zero-dimensional* if it has a base consisting of clopen (= closed and open) sets. Every zero-dimensional space X has a maximal zero-dimensional compactification, ζX, [2]; ζX is characterized by the property that every continuous function $X \to 2$ (= $\{0, 1\}$ with the discrete topology) can be extended to a continuous function $\zeta X \to 2$, or, equivalently, by the property that for every compact zero-dimensional space Y every continuous function $X \to Y$ can be extended to a continuous function $\zeta X \to Y$.

If X is zero-dimensional, we will denote $\zeta X - X$ by $X^{\cdot}$ A point p of ζX is called a remote point of X if $p \in X^{\cdot}$ but $p \notin \mathrm{Cl}_{\zeta X} D$ for every nowhere dense $D \subseteq X$.

Most of our results about remote points in βX can be transformed in a straightforward manner into results about remote points in ζX. We only give the key steps, and leave the details to the reader.

18.2. THEOREM. *Let X be a zero-dimensional space with countable π-weight which is not pseudocompact. Then X has a remote point in ζX.*

□ We first show that there is a nonempty G_δ G in ζX with $G \subseteq X^{\cdot}$ Let $\omega+1$ have the order topology. Since X is zero-dimensional but not pseudocompact, one can find a partition $\langle U_n : n \in \omega \rangle$ of X into open sets (with $U_m \neq U_n$ if $m \neq n$ of course). Then

$$f = \bigcup_{n \in \omega} U_n \times \{n\}$$

is a continuous function from X to $\omega+1$. Let $\zeta f : \zeta X \to \omega+1$ be its continuous extension. Then $G = (\zeta f)^{\leftarrow}\{\omega\}$ is a nonempty G_δ in ζX with $G \subseteq X^{\cdot}$

Let $Y = \zeta X - G$. Then $X \subseteq Y \subseteq \zeta X$, hence $\zeta Y = \zeta X$, cf. (2.5). But Y is zero-dimensional and Lindelöf, hence $\zeta Y = \beta Y$. Since Y is not pseudocompact, and since Y has countable π-weight, (2.4), and since every nowhere dense subset of X is nowhere dense in Y, it follows from (1.5) that ζX has a remote point. □

18.3. One can use the same argument and (1.8a) to show that ζX is extremally disconnected at some point of $X^{\cdot}$, with X as in (18.2). However, there is another proof for this fact, similar to the one given for βX:

If X is zero-dimensional, define a function

$$\mathrm{Ez}_X : \{\text{open sets of } X\} \to \{\text{open sets of } \zeta X\}$$

by

$$\mathrm{Ez}_X\, U = \zeta X - \mathrm{Cl}_{\zeta X}(X - U).$$

Then one can easily prove analogues of (3.1), (3.2), and then of (5.1) and (5.2) (with X as in (18.2) of course).

18.4. One consequence of the fact that ζX is extremally disconnected at some (in fact 2^{c}) points of $X^{\cdot}$ should be pointed out: $X^{\cdot} \cap X^* \neq \emptyset$, cf. (12.2).

19. Q, Q^*, Q^{**}, Q^{***}, ... Define a sequence $\langle Q_n : n \in \omega \rangle$ of spaces by $Q_0 = Q$, $Q_{n+1} = (Q_n)^*$. With induction on n one can prove the following

19.1. PROPOSITION. (a) *Q_n is nowhere locally compact and has countable π-weight, for $n \in \omega$.*

(b) *Q_{2n} admits a perfect map onto Q, and Q_{2n+1} admits a perfect map onto P, for $n \in \omega$.*

(c) *Q_n has a dense set of ω-points, and also has a dense set of points at which Q_n is extremally disconnected, for $n \in \omega - \{0\}$.* □

It follows from (19.1b) that Q_{2n} is σ-compact that Q_{2n+1} is not, for $n \in \omega$, hence Q_n and Q_k are not homeomorphic if $|n-k|$ is odd. I don't know if Q_n and Q_{n+2} can ever be homeomorphic. If not, it might be interesting to define Q_α's, for $\alpha \geqslant \omega$, using inverse limits at limit stages. One can show that there must be an α for which the natural map from $Q_{\alpha+2}$ onto Q_α is a homeomorphism.

In (8.1) we saw that every point of $Q_1 = Q^*$ is either an ω-point or a point at which Q_1 is extremally disconnected. I don't know if the corresponding statement for Q_n is true if $n > 1$.

20. Retractions from βX onto X^*. In this section we show how the statement

(?) If X is not pseudocompact, then βX is extremally disconnected at some point of X^*,

can be used to give an elegant proof of a special case of the following

20.1. THEOREM. *If X^* is a retract of βX, then X is locally compact and pseudocompact.*

20.2. Comfort proved this theorem assuming CH, [5]. After proving that a special case of (20.1) follows from (?) I succeeded in proving (20.1) outright, [10]. So even if (?) is false, it is at least useful in finding theorems.

20.3. We will prove (20.1) from (?) under the additional hypothesis that $|X|$ is not Ulam-measurable. We will use the following facts:

(1) If X is locally compact, then for every nonempty G_δ G in βX, if $G \subseteq X^*$ then

(a) $\mathrm{Int}_{X^*} G \neq \emptyset$;

(b) $|G| > 1$.

(2) If Y is a Hausdorff space without isolated points in which every nonempty G_δ has nonempty interior, and if the cellularity of Y is not Ulam-measurable, then every point of Y is an ω-point.

(1a) is essentially due to Fine and Gillman, [17], 3.1, (1b) is well known, and follows e.g. from (4.5), and (2) is a recent result of Szymański, [41].

Assume that X^* is a retract of βX. Then X is locally compact since X^* is compact. Suppose X is not pseudocompact. Then there is a nonempty G_δ G in βX with $G \subseteq X^*$. Since $\mathrm{Int}_{X^*} G$ is a G_δ in βX, we may assume that G is open in X^*.

We claim that every point of G is an ω-point in X^*. Indeed, every G_δ-subset of G, in particular every open subset of G, is a G_δ in βX, hence G has no isolated points, and every nonempty G_δ in G has nonempty interior, by (1). Also, the cellularity of G is at most $|G| \leqslant |\beta X| \leqslant 2^{2^{|X|}}$, hence is not Ulam-measurable ([23], 12.5). So every point of G is an ω-point of G, by (2). Since G is open in X^*, this proves our claim.

Since X^* is a retract of βX, it follows that every point of G is an ω-point of βX. However, this is impossible: Consider $Y = \beta X - G$, then $\beta Y = \beta X$ since $X \subseteq Y \subseteq \beta X$, hence Y is not pseudocompact. It now follows from (?) that $\beta X = \beta Y$ is extremally disconnected at some point of G. This contradiction completes the proof.

21. Products of remainders. Here we give the example promised in (7.7). Recall that a space X is weakly ω-bounded if every infinite subset has an infinite subset whose closure in X is compact, (17.13), and that the product of at most countably many weakly ω-bounded spaces is weakly ω-bounded, hence pseudocompact, (17.14).

21.1. Example. *There is a nowhere locally compact weakly ω-bounded space K with countable π-weight such that $(K \times K)^*$ and $K^* \times K^*$ are not homeomorphic.*

□ Since βQ has countable π-weight, (2.4), and Q^* is dense in βQ, (2.3), there is a countable dense L in βQ with $L \subseteq Q^*$. Let $K = \beta Q - L$. Then $Q \subseteq K \subseteq \beta Q$, hence $\beta K = \beta Q$, (2.5). It also follows that K is nowhere locally compact, (2.3), and has countable π-weight.

We first show that K is weakly ω-bounded. Let $A \subseteq K$ be an infinite set. Without loss of generality A is countable, $A \subseteq Q$ or $A \subseteq Q^*$, and A is relatively discrete. If some point of Q is a limit point of A then A has an infinite subset whose closure in K is compact because every point of Q has a countable neighborhood base in K. So assume no point of Q is a limit point of A. We claim that A is C^*-embedded in βQ. Indeed, if $A \subseteq Q$ this is clear (since A will be closed in Q), so assume $A \subseteq Q^*$. Then A is closed in $A \cup Q$, which is normal. Since $\beta(A \cup Q) = \beta Q$ by (2.5), it follows that A is C^*-embedded in $A \cup Q$, hence in βQ.

Hence $\bar{A}$ (closure in βQ) and βN are homeomorphic. But N^* is not separable, hence some point p of $\bar{A} - A$ is not in the closure of $\bar{A} \cap L = (\bar{A} \cap L) - A$. Let U be a clopen neighborhood in $\bar{A}$ of p which misses L. Then $A \cap U$ is an infinite subset of A which has compact closure in K.

This completes the proof that K is weakly ω-bounded.

Then $K \times K$ is pseudocompact, as observed above, so $\beta(K \times K) = \beta K \times \beta K$, by a well-known result of Glicksberg, [26]. Consequently $(K \times K)^* = L \times \beta Q \cup \beta Q \times L$, while $K^* \times K^* = L \times L$. Hence $(K \times K)^*$ and $K^* \times K^*$ are not homeomorphic because $|(K \times K)^*| = 2^{c} > \omega = |K^* \times K^*|$. □

22. Questions

22.1. *If X is not pseudocompact, is βX extremally disconnected at some point of X^*?*
(YES if (21.2) has an affirmative answer.)

22.2. *If X is not pseudocompact, does X have a remote point?*
(YES if X has countable π-weight, or if X is separable and CH holds.)

The answer to (21.2) is open even for metrizable X, so the easiest open case is $D(\omega_1)^\omega$, the product of countably many copies of the discrete space with ω_1 points.

An important application of remote points is the effective proof that P^*, Q^* and S^* are not homogeneous, (1.10). I don't know if the points of $\varrho(R)$ can be distinguished internally in R^* from the points of $R^*-\varrho(R)$ (or at least from some points of $R^*-\varrho(R)$), whether or not in an effective manner.

22.3. *If $X = R$, or, more generally, if X is realcompact (and first countable if you wish), is it true that $h^{\to}\varrho(X) = \varrho(X)$ for every homeomorphism of X^* onto itself?*
(YES for suitable X, see (8.2). NO without realcompact, see (6.9).)

All remote points we constructed, and in fact all special points we constructed, lie in $\beta X - vX$. One can easily give examples showing that $vX - X$ can intersect $\varphi(X)$ or $X^* - \varphi(X)$, but the following is open.

22.4. *Can $vX - X$ have a remote point of X?*

22.5. *Does every space have a dense zero-dimensional subspace?*
(YES for metrizable spaces. See also (5.5).)

22.6. *If X is nowhere locally compact and nonpseudocompact, and if Y and Z are spaces without isolated points, is it true that X^* and $Y \times Z$ are not homeomorphic?*
(YES if X has countable π-weight, see (7.5).)

22.7. *Are $(Q^*)^n$ and $(Q^*)^k$ homeomorphic iff $n = k$?*
(YES if $\min\{n, k\} = 1$, see (7.6).)

22.8. *If X is a noncompact Lindelöf space without isolated points, does X have a far point?*
(YES if X first countable, see (4.8) and also (9.3), or if answer to (22.2) is affirmative.)

22.9. *Is it true that the Sorgenfrey line has no connected compactification?*

22.10. *Does there exists in* ZFC *a free ultrafilter $\mathscr{G}$ on ω such that for every sequence $\langle q_n\colon n \in \omega\rangle$ in Q (equivalently: in R) there is a $D \in \mathscr{G}$ such that $\{q_n\colon n \in \mathscr{G}\}$ is nowhere dense?*

This is related to § 17. The sequence $\langle Q_n: n \in \omega \rangle$ is defined by $Q_0 = Q$, $Q_{n+1} = (Q_n)^*$, see § 19.

22.11. *Are Q_n and Q_k homeomorphic iff $n = k$?*

NOTES TO § 22

22.12. After this paper has been written, the following has come to my attention.

Under CH there is a separable metrizable locally compact X such that for some homeomorphism h of X^* onto itself, $h^{\to} \varrho(X) \neq \varrho(X)$. Indeed, let $X = \omega \times 2^{\omega}$, then X^* and ω^* are homeomorphic because of CH, [31], and there are P-points p and q in X^* with $p \in \varrho(X)$ but $q \notin \varrho(X)$, also because of CH, [31]. But by a theorem of Rudin there is under CH for any two P-points s and t in ω^* a homeomorphism h of ω^* onto itself with $h(s) = t$ and $h(t) = s$ ([35], 4.4). This example is due to Grant Woods, [48].

Emeryk and Kulpa have answered (22.9) affirmatively, [16].

Shelah has shown that it is consistent with ZFC that there are no P-points in N^*.

Also, in a forthcoming paper (*Prime mappings, number of factors, and binary operations*, Diss. Math., to appear) 22.6 (even without the condition that X be nowhere locally compact) and 22.7 are answered affirmatively. The proofs are noneffective.

Added in print. The answers to 22.1 and 22.2 are no, but are yes in the special case X is metrizable. 22.4 should have been asked for X without isolated points; the answer is no in the special case the cellularity of X is not Ulam-measurable (see papers [49], [50], [52]).

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