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Singular arc-like continua

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Introduction

In 1959 R. D. Anderson and G. Choquet constructed an example of a plane continuum no two of whose nondegenerate subcontinua are homeomorphic [2]. By a slight modification of this example J. J. Andrews [4] obtained an example of an arc-like continuum with the same property. The arc-likeness implies immediately the planability and thus all troubles with the embeddability into the plane of the constructed inverse limit can be omitted. Those examples are continua for which the identity is the only homeomorphism into itself.

In 1967 applying Anderson's and Choquet's method of the condensation of singularities H. Cook builded two examples [6]. The first one is a continuum which admits only the identity mapping onto nondegenerate subcontinua. The second one is a continuum with the property that no two different nondegenerate subcontinua of it are comparable by continuous mappings. Cook's examples are nonplanable (the construction uses solenoids) and the second example is an hereditarily indecomposable continuum.

In [15] I have obtained, again using Anderson's and Choquet's method, an hereditarily decomposable arc-like continuum which possesses only the identity and constant mappings as continuous mappings into itself. Unfortunately, this continuum does not have the above-mentioned property of the second Cook's example.

In this paper I just have constructed an hereditarily decomposable arc-like continuum with this property. The reader can observe that the construction is intuitively quite clear, proofs are long and technical, but elementary.

It is worth to note that the method of the condensation of singularities was applied in 1912 by Janiszewski to produce a curve containing no arcs. But his description was rather intuitive than precise.

This method is still used to construct new examples. For example, recently, using it L. Mohler and L. Oversteegen [19] have constructed a hereditarily indecomposable hereditarily non-arclike planar tree-like continuum.

In this paper to produce an hereditarily decomposable arc-like continuum admitting no nondegenerate maps on subcontinua we find firstly

an arc-like continuum Σ which is hereditarily nondivisible by points (additionally Σ is homeomorphic to every layer of Σ). Next we use Krasinkiewicz's and Minc's function ξ_0 to produce a continuum which is not an image of Σ. Bellamy's pseudosuspensions from [4] permit us to construct countable collection of arc-like continua no one of which is an image (under some set-valued maps) of a subcontinuum of the another. Finally, by the inserting of obtained continua "everywhere" we find the main example.

1. Preliminaries

A. Mappings. All considered spaces are compacta, i.e. compact and metric. A mapping from a space X onto a space Y is said to be

(i) *monotone*, if the inverse image of any subcontinuum of Y is a continuum (see [12], p. 142),

(ii) *atomic*, if for each subcontinuum K of X such that the set $f(K)$ is nondegenerate we have $K = f^{-1}(f(K))$ (see [1]),

(iii) *weakly confluent*, if for each subcontinuum Q in Y there exists a continuum C in X such that $f(C) = Q$ (see [13]).

The following is known (see [7], Theorem 1, p. 49)

(1.1) *Any atomic mapping of a continuum is monotone.*

A subcontinuum K of X is called *terminal* if every subcontinuum of X which intersects K and its complement contains K. It follows from Proposition 4 in [12] (cf. [1]) that

(1.2) *A mapping f from X onto Y is atomic if and only if the inverse image of every point in Y (of any terminal subcontinuum of Y) is a terminal subcontinuum in X.*

Moreover (see [15], Propositions 5 and 6; compare [21]),

(1.3) *The composition of any finite number of atomic mappings is atomic.*

(1.4) *If f_n: $X_{n+1} \to X_n$, g_n: $Y_{n+1} \to Y_n$ are surjections and h_n: $X_n \to Y_n$ is an atomic (monotone) surjection such that $h_n f_n = g_n h_{n+1}$ for $n = 1, 2, \ldots$, then the induced mapping h from $X = \operatorname{invlim}\{X_n, f_n\}$ onto $Y = \operatorname{invlim}\{Y_n, g_n\}$ is atomic (monotone, respectively).*

An easy consequence of definitions is the following

(1.5) *If f: $X \to Y$, g: $Y \to Z$, f is weakly confluent and gf is atomic, then g is atomic.*

The assumption that f is weakly confluent is essential. Moreover, Problem (5.22) from [14] has the negative answer; namely we have

EXAMPLE. There are mapping f from a continuum X onto Y and a monotone mapping from Y onto an arc such that gf is atomic and g is not atomic.

Let X be a copy of Knaster's irreducible continuum (see [10]) which can be mapped onto $[0, 1]$ under an open and monotone mapping h such that $h^{-1}(t)$ is nondegenerate for every $t \in [0, 1]$. The properties of X and h imply that h is atomic. Choose an open set U in X such that $h(\operatorname{cl} U)$ is an arc and $h^{-1}(t) \cap \operatorname{cl} U$ is a proper subset of $h^{-1}(t)$ for every $t \in [0, 1]$. Consider a mapping f from X such that $f|X \setminus \operatorname{cl} U$ is a homeomorphism and f identifies points from $h^{-1}(t) \cap \operatorname{cl} U$ for every $t \in [0, 1]$. Put $Y = f(X)$ and $g(y) = hf^{-1}(y)$ for $y \in Y$. Then $gf = h$ and g is not atomic.

Now we will prove

(1.6) *If f is an atomic mapping from a continuum X onto Y, A is a zerodimensional closed subset of Y such that $f|f^{-1}(A)$ is open, then for every component K of $f^{-1}(A)$ there is a sequence of continua K_n such that* $\operatorname{Lim} K_n = K$ *and* $K_n \subset X \setminus f^{-1}(A)$ *for* $n = 1, 2, \ldots$

Indeed, according to the assumptions we can choose an open neighbourhood U of K in X such that $\operatorname{dist}(\operatorname{cl} U, K) < \varepsilon$, $(\operatorname{cl} U) \setminus U \cap f^{-1}(A) = \emptyset$ and $\operatorname{dist}(C, K) < \varepsilon$ for every component C of $f^{-1}(A)$ contained in $\operatorname{cl} U$, where dist denotes the Hausdorff distance and ε is a positive number. Let V_n be open neighbourhoods of $f^{-1}(A) \cap \operatorname{cl} U$ such that $V_{n+1} \subset \operatorname{cl} V_n \subset U$ and $\bigcap_{n=1}^{\infty} V_n = f^{-1}(A) \cap \operatorname{cl} U$. Since X is a continuum, there is a component L_n of $(\operatorname{cl} U) \setminus V_n$ such that $L_n \cap ((\operatorname{cl} U) \setminus U) \neq \emptyset \neq L_n \cap ((\operatorname{cl} V_n) \setminus V_n)$. We can assume that the sequence $\{L_n\}$ is convergent and denote its limit by L. Then $L \subset \operatorname{cl} U$, and $L \cap f^{-1}(A) \neq \emptyset$. Therefore there is a component C of $f^{-1}(A) \cap U$ which is contained in L, because f is atomic. Hence $\operatorname{dist}(L, K) \leqslant 2\varepsilon$. This implies that we can find L_n such that $\operatorname{dist}(L_n, K) < 3\varepsilon$. Consequently for every positive number ε we have a continuum P such that $\operatorname{dist}(P, K) < 3\varepsilon$ and $P \subset X \setminus f^{-1}(A)$.

The assumption that $f|f^{-1}(A)$ is open is essential in (1.6). One can easily find a suitable example.

Moreover, we have (compare [13], Lemma 1.3)

(1.6') *If f is an atomic mapping from a continuum X onto Y, $\{K_n\}$ is a sequence of nondegenerate subcontinua of Y converging to a nondegenerate continuum K, then the sequence $\{f^{-1}(K_n)\}$ converges to $f^{-1}(K)$.*

Suppose that the sequence $\{f^{-1}(K_n)\}$ is not convergent to $f^{-1}(K)$, then some subsequence $\{f^{-1}(K_{n_i})\}$ converges to a proper subcontinuum P of $f^{-1}(K)$. Since f is continuous, we infer $f(P) = K$. Therefore $P = f^{-1}f(P) = f^{-1}(K)$, because f is atomic. This contradiction completes the proof.

B. Arc-like continua. Recall that a continuum X is called *arc-like* if for any number $\varepsilon > 0$ there exists an ε-mapping f from X onto $[0, 1]$ (this means that $\operatorname{diam} f^{-1}(t) < \varepsilon$ for $t \in [0, 1]$). The following is known (see [5], [17]).

(1.7) *Arc-like continua are imbeddable into the plane and they are hereditarily unicoherent.*

Moreover (see [5], compare [8]),

(1.8). *A hereditarily decomposable continuum is arc-like if and only if it is hereditarily unicoherent and atriodic.*

It follows immediately from definitions (compare [15], Proposition 10) that

(1.9) *The inverse limit of arc-like continua is an arc-like continuum.*

We have the following (see [22], compare [14], (6.16), p. 56).

(1.10) *Any mapping of a continuum onto an arc-like continuum is weakly confluent.*

By the applying the results from [8] it is obtained in [15], Proposition 11,

(1.11) *If f is an atomic mapping from a continuum X onto an hereditarily decomposable arc-like continuum Y and sets $f^{-1}(y)$ are arc-like (hereditarily decomposable) continua, then X is an arc-like (hereditarily decomposable) continuum.*

Obviously, we have

(1.12) *If f is a monotone mapping from an atriodic continuum X onto Y and K is a subcontinuum of X, then $f^{-1}(y) \subset K$ for each point y belonging to $f(K)$ except at most two points.*

C. Pseudosuspensions. Applying results from [16] we easily obtain (see [15], Lemma 36)

(1.13) *If X is an onedimensional continuum, then there is a sequence A_1, $A_2, \ldots$ of closed zerodimensional disjoint sets in X such that* $\lim \operatorname{diam} A_n = 0$ *and for each n the set $A_n \cup A_{n+1} \cup \ldots$ intersects every nondegenerate subcontinuum of X.*

Theorem 15 in [15] gives us the following

(1.14) *If X is a continuum, M is a compactum, A is a zerodimensional compact subset of X and A is a decomposition space of M into components, then there is a continuum $\tilde{X}$ containing M such that there is an atomic map r from $\tilde{X}$ onto X such that $r|\tilde{X} \setminus M$ is a homeomorphism onto $X \setminus A$.*

Every $\tilde{X}$ determined by (1.14) will be called a *pseudosuspension* of M over X at A and the mapping r will be called a *natural projection* from $\tilde{X}$ onto X. We have

(1.15). *If f is a monotone mapping from a continuum X onto Y, A is a*

zerodimensional compact subset of X such that $f|A$ is a homeomorphism and $A = f^{-1}f(A)$ and if $\tilde{Y}$ is a pseudosuspension of M over Y at $f(A)$ with a natural projection r_Y, then there is a pseudosuspension $\tilde{X}$ of M over X at A with a natural projection r_X and a monotone mapping $\tilde{f}$ from $\tilde{X}$ onto $\tilde{Y}$ such that $r_Y\tilde{f} = fr_X$.

$$\begin{tikzcd} \tilde{Y} \arrow[d, "r_Y"'] & \tilde{X} \arrow[l, "\tilde{f}"'] \arrow[d, "r_X"] \\ Y & X \arrow[l, "f"'] \end{tikzcd}$$

Proof. Put $\tilde{X} = \{(x, y) \in X \times \tilde{Y}: f(x) = r_Y(y)\}$, $\tilde{f}(x, y) = y$ and $r_X(x, y) = x$ for $(x, y) \in \tilde{X}$. Then $r_X^{-1}(x) = \{x\} \times r_Y^{-1}(f(x))$, $\tilde{f}^{-1}(y) = f^{-1}r_Y(y) \times \{y\}$. Hence $r_X|\tilde{X} \setminus \{(x, y): y \in M\}$ is a homeomorphism onto $X \setminus A$ and $\tilde{f}$ is monotone. Thereby $\tilde{X}$ is a continuum. Moreover, $\tilde{f}|\tilde{X} \cap \{(x, y): y \in M\}$ is a homeomorphism onto M. Now, if $x \in X$ and $r_X^{-1}(x)$ is nondegenerate, then $\tilde{f}|r_X^{-1}(x)$ is a homeomorphism onto a component of M which is terminal in $\tilde{Y}$. Hence $r_X^{-1}(x)$ is terminal in $\tilde{X}$ (otherwise we find a continuum K in Y such that $\emptyset \neq r_X^{-1}(x) \cap K \neq r_X^{-1}(x)$ and $K \cap (\tilde{X} \setminus r_X^{-1}(x)) \neq \emptyset$ and the similar properties has $\tilde{f}(K)$). This means, according to (1.2) that r_X is atomic.

D. Set-valued maps. By a *set-valued* map $F: X \to Y$ we mean a correspondence such that $F(x)$ is a closed nonempty subset of Y for all $x \in X$. We say that F is *upper semi-continuous* (*u.s.c.*) provided $F^{-1}(A) = \{x: F(x) \cap A \neq \emptyset\}$ is closed for each closed $A \subset Y$. F is said to be *continuum-valued* in case $F(x)$ is a subcontinuum of Y for each $x \in X$. It is known (see [24])

(1.16) *If $F: X \to Y$ is an u.s.c. continuum-valued map and A is a subcontinuum of X, then $F(A)$ is a subcontinuum of Y.*

Recall (see [25]) that a continuum-valued map $F: X \to Y$ is called *pseudo-monotone* provided for every two continua $A \subset X$ and $B \subset Y$ with $B \subset F(A)$ there is a continuum $C \subset A$ such that $B \subset F(C)$ and $F(x) \cap B \neq \emptyset$ for all $x \in C$. We claim that (cf. (1.10))

(1.17) *If F is an u.s.c. continuum-valued map from a continuum X onto an arc-like continuum Y, then F is pseudo-monotone.*

Proof. Assume firstly that $Y = [0, 1]$, $A = X$ and $B = [b_1, b_2]$. If there is a point $a \in X$ such that $B \subset F(a)$, then $C = \{a\}$ has the required properties. Therefore we can assume that sets $F^{-1}([0, b_1])$ and $F^{-1}([b_2, 1])$ are disjoint, because F is continuum-valued. Let C be a subcontinuum of X such that $F^{-1}([0, b_1]) \cap C \neq \emptyset \neq F^{-1}([b_2, 1])$ and no proper subcontinuum of C has the same property. According to Theorem 5 in [12], p. 220, the set $D = C \setminus (F^{-1}([0, b_1]) \cup F^{-1}([b_2, 1]))$ is connected and dense in C. If $x \in D$,

then $F(x) \subset (b_1, b_2)$. Therefore $F(x) \cap B \neq \emptyset$ for all $x \in C$, because F is u.s.c. Obviously $F(C)$ is a continuum (by (1.16)) containing b_1 and b_2; thus $B \subset F(C)$.

Now, assume that $A = X$, $B \subset Y$, $B \subset F(A)$ and Y is an arbitrary arc-like continuum. For $n = 1, 2, \ldots$ we find $1/n$-mappings from Y onto $[0, 1]$. As above for each $n = 1, 2, \ldots$ we find a continuum C_n in X such that $f_n F(C_n) \supset f_n(B)$ and $f_n F(x) \cap f_n(B) \neq \emptyset$ for all $x \in C_n$. Put $C = \operatorname{Lim}_{n \to \infty} C_n$, then $B \subset F(C)$ and $F(x) \cap B \neq \emptyset$ for all $x \in C$, i.e. (1.17) holds.

An u.s.c. map $F\colon X \to Y$ such that sets $F(x)$ are continua in Y of the diameter smaller than ε is called an *ε-valued* map.

Put

$$\varepsilon(X) = \tfrac{1}{4} \sup \{\varepsilon \geqslant 0\colon \text{every subcontinuum of } X \text{ of the diameter smaller than } \varepsilon \text{ is path-connected}\}.$$

E. Cook continua. We will say (see [15]) that a continuum X is a *Cook continuum* provided for every two nondegenerate disjoint subcontinua of X there is no continuous mapping from one of them onto another. We have (see [15], Proposition 29)

(1.18) *If X is a Cook continuum, then*

(i) *if f is a continuous mapping from a subcontinuum K onto a nondegenerate $f(K)$, then $f(K) \subset K$ and f is a monotone retraction,*

(ii) *the identity is the only surjection from X onto itself.*

F. Countable-to-one mappings and admissible compacta. A continuous mapping $f\colon X \to Y$ from X onto Y is called *countable-to-one* if every point inverse set $f^{-1}(y)$ is a countable set for $y \in Y$. Obviously we have

(1.19) *If X possesses a countable-to-one mapping onto $[0, 1]$, Y is a countable space, then $X \times Y$ possesses a countable-to-one mapping.*

Moreover,

(1.20) *If X is a compactum which has a countable-to-one mapping f onto $[0, 1]$, $A \subset [0, 1]$ and A is a decomposition space of X into components, then for every pseudosuspension Y of X over $[0, 1]$ at A there is a countable-to-one mapping f' from Y onto $[0, 1]$ such that $f'|X = f$.*

In fact, let r be a natural projection from Y onto $[0, 1]$ such that $r^{-1}(A) = X$ and suppose thet f is a countable-to-one mapping from X onto $[0, 1]$. Let f^* be an arbitrary extension of f to a mapping from Y onto $[0, 1]$ (the existence of f^* follows from Tietze's extension theorem). Consider an arbitrary countable set $B \subset Y \setminus X$ such that B has no accumulation point, $(\operatorname{cl} B) \setminus B = X$ and if ab is an arc in $Y \setminus X$ with endpoints a and b and $ab \cap B = \{a, b\}$, then $\operatorname{diam}(ab) < 2 \min\{\varrho(a, X), \varrho(b, X)\}$. A mapping f' from Y

onto $[0, 1]$ is defined by the conditions: $f'|X \cup B = f^*$, if ab is an arc in $Y \setminus X$ such that $ab \cap B = \{a, b\}$ then $f'|ab$ is an arbitrary homeomorphism from ab onto the straight line interval $[f^*(a), f^*(b)]$ with $f^*(a) = f'(a)$ and $f^*(b) = f'(b)$ provided $f^*(a) \neq f^*(b)$, and if $f^*(a) = f^*(b)$, then there is a point $c \in ab$ such that $f'|ac$ and $f'|cb$ are homeomorphisms onto the interval $[f^*(a), c']$ with $f'(a) = f^*(a) = f'(b)$ and $|f(a) - c'| < \min\{\varrho(a, X), \varrho(b, X)\}$. One can easily check that f' is a continuous countable-to-one mapping.

A countable-to-one mappings have the following property:

(1.21) *If f: $X \to [0, 1]$ is a countable-to-one mapping from a compactum X onto $[0, 1]$, then for each $\varepsilon > 0$ there is a finite set $E \subset [0, 1]$ such that if C is a subcontinuum of X with* diam $C \geqslant \varepsilon$, *then* $\operatorname{card}(f(C) \cap E) \geqslant 3$.

Indeed, there is a positive number δ such that if C is a subcontinuum of X with $\operatorname{diam} C \geqslant \varepsilon$, then $\operatorname{diam} f(C) > \delta$. It suffices to take $E = \{x_0, x_1, \ldots, x_n\}$, where $x_0 = 0 < x_1 < \ldots < x_n = 1$ and $x_i - x_{i-1} < \frac{1}{4}\delta$ for $i = 1, 2, \ldots, n$.

We will say that a compactum X is *admissible* if there are a countable-to-one mapping f from X onto $[0, 1]$ and a finite set $E \subset [0, 1]$ such that every subcontinuum C of $X \setminus f^{-1}(E)$ is an arc.

From (1.19) and (1.20) we obtain

(1.22) *If X is an admissible compactum, Y is a countable compact set, then $X \times Y$ is an admissible compactum.*

(1.23) *If X is an admissible compactum with nondegenerate components, A is a subset of* $[0, 1]$ *and A is a decomposition space of the decomposition of X into components, then every pseudosuspension of X over* $[0, 1]$ *at A is an admissible compactum.*

In fact, let f be a countable-to-one mapping from X onto $[0, 1]$ and let E be a finite set in $[0, 1]$ such that every subcontinuum C of $X \setminus f^{-1}(E)$ is an arc. Applying (1.20), we find a countable-to-one extension f' of f. One can check that if E' is a finite set in $[0, 1]$ containing E and such that $(f')^{-1}(E')$ intersects every component of X, then every subcontinuum of $Y \setminus (f')^{-1}(E')$ is an arc.

2. Continua nondivisible by points

A. V-projections. We will say that a continuous mapping f: $X \to [0, 1]$ is a *λ-projection* provided X is irreducible between $f^{-1}(0)$ and $f^{-1}(1)$ and for $t \in [0, 1]$ the set $f^{-1}(t)$ is a boundary subcontinuum of X. It follows from Kuratowski's theorem (see [12], p. 200)

(2.1) *A continuum X is irreducible and does not contain an indecomposable continuum with nonempty interior if and only if there is*

a λ-projection from X onto $[0, 1]$. *Moreover, if f: $X \to [0, 1]$ is a λ-projection and g: $X \to [0, 1]$ is a monotone mapping, then there is a monotone mapping h: $[0, 1] \to [0, 1]$ such that $g = hf$.*

A subcontinuum A of X will be called an *endlayer* (a *layer*) of X if there is a λ-projection f: $X \to [0, 1]$ such that $A \in \{f^{-1}(0), f^{-1}(1)\}$ ($A \in \{f^{-1}(t)$: $t \in [0, 1]\}$). Remark that this definition is independent of the choice of a λ-projection by (2.1).

If β: $X \to Y$ and $A \subset Y$, then we write $K \in V(\beta, A)$ if there are continua L and R such that $K = L \cup R$, $\beta|L$ and $\beta|R$ are homeomorphisms onto Y and $\beta(L \cap R) = A$. We will say that β: $X \to Y$ is a *V-projection* provided for every layer K of X we have $K \in V(\beta, A_0) \cup V(\beta, A_1) \cup V(\beta, Y)$ where A_0 and A_1 are endlayers of Y and if for every convergent sequence $\{K_n\}$ of different layers of X and $y \in Y$ we have that the set $\mathrm{Ls}(\beta^{-1}(y) \cap K_n)$ is degenerate.

We have

(2.2) *If α is a λ-projection on X and β: $X \to Y$ is a V-projection onto Y with endlayers A_0 and A_1, then*

(i) *β is open,*

(ii) *if $B_i = \{t \in [0, 1]$: $\alpha^{-1}(t) \in V(\beta, A_i)\}$, then B_i is a countable dense subset of* $(0, 1)$ *for* $i = 0, 1$,

(iii) *if $t \in [0, 1] \setminus (B_0 \cup B_1)$, then $\alpha^{-1}(t)$ is a continuity layer of X,*

(iv) *mappings $\beta|\mathrm{cl}(\alpha^{-1}[0, t)) \setminus \alpha^{-1}[0, t)$ and $\beta|\mathrm{cl}(\alpha^{-1}(t, 1]) \setminus \alpha^{-1}(t, 1]$ for $t \in [0, 1]$ are homeomorphisms.*

Proof. Since $\beta(U) = \bigcup \{\beta(U \cap K)$: $\beta|K$ is a homeomorphism from K onto $Y\}$ for $U \subset X$, we conclude the openess of β, i.e. (i) holds. Observe now that

(2.2.1) *for $i = 0, 1$ the set B_i is dense in* $[0, 1]$.

Suppose that $B_0 \cap [t, t'] = \emptyset$ and $t < t'$. Then $\alpha^{-1}[0, t) \cup (\alpha^{-1}[t, t'] \cap \beta^{-1}(A_1)) \cup \alpha^{-1}[t', 1]$ is a proper subcontinuum of X joining $\alpha^{-1}(0)$ with $\alpha^{-1}(1)$, a contradiction, i.e. B_0 is dense in $[0, 1]$. The same arguments show the density of B_1 in the interval $[0, 1]$.

(2.2.2) *If K_n are subcontinua of X such that $\beta(K_n) = Y$ and $\beta|K_n$ is a homeomorphism for $n = 1, 2, \ldots$ and $K_n \to K$, then $\beta(K) = Y$ and $\beta|K$ is a homeomorphism.*

According to the assumptions we find $t_n, t \in [0, 1]$ such that $t_n \to t$, $K_n \subset \alpha^{-1}(t_n)$ and $K \subset \alpha^{-1}(t)$. Let b and b' be different points in K such that $\beta(b) = \beta(b')$. Then either $\{b, b'\} \cap \beta^{-1}(A_0) = \emptyset$ or $\{b, b'\} \cap \beta^{-1}(A_1) = \emptyset$. Assume that $\{b, b'\} \cap \beta^{-1}(A_0) = \emptyset$. Hence every subcontinuum of K containing $\{b, b'\}$ intersects $\beta^{-1}(A_0)$. We find points $b_n, b'_n \in K_n \setminus \beta^{-1}(A_0)$ such that $b_n \to b$ and $b'_n \to b'$. Then $\beta(b_n) \to \beta(b)$ and $\beta(b'_n) \to \beta(b)$. Hence we can choose continua L_n in Y such that $\{\beta(b_n), \beta(b'_n)\} \cup A_1 \subset L_n \subset Y \setminus A_0$ and $L_n \to L \subset Y \setminus A_0$. For $n = 1, 2, \ldots$ the set $\beta^{-1}(L_n) \cap K_n$ is a continuum. If we

denote the limit of $\{\beta^{-1}(L_n)\cap K_n\}$ by L, then L is a continuum containing h and h' and L is contained in $K\setminus\beta^{-1}(A_0)$, a contradiction.

It follows from (2.2.2) that if $t\notin B_0\cup B_1$, then $\alpha^{-1}(t)$ is a layer of the continuity of X, i.e. (iii) holds.

Suppose now that $t\in B_0\cup B_1$. By the symmetry we can assume $t\in B_0$. This means that there are continua L and R such that $\alpha^{-1}(t)=L\cup R$, $\beta|L$ and $\beta|R$ are homeomorphisms and $\beta(L\cap R)=A_0$. If $t_n\neq t$, $t_n\to t$ and $\{\alpha^{-1}(t_n)\}$ is convergent, then for each $y\in Y$, the set $\mathrm{Ls}(\beta^{-1}(y)\cap\alpha^{-1}(t_n))$ is degenerate. Therefore, by (2.2.2), we obtain either $\alpha^{-1}(t_n)\to L$ or $\alpha^{-1}(t_n)\to R$. Fix $a\in A_1$, $b_L\in L$ and $b_R\in R$ such that $\beta(b_L)\in\beta(b_R)=a$. If $\varepsilon=\frac{1}{3}\varrho(b_L,b_R)$ (ϱ denotes the metric in X), U_L and U_R are open ε-neighbourhoods of b_L and b_R in X, then there is a positive number δ such that if $|t'-t|<\delta$, $t\neq t'$, then either $\alpha^{-1}(t')\cap U_L=\emptyset$ and $\alpha^{-1}(t')\cap U_R\neq\emptyset$ or $\alpha^{-1}(t')\cap U_R=\emptyset$ and $\alpha^{-1}(t')\cap U_L\neq\emptyset$. This conclusion gives us a separation of the set $(t-\delta,t+\delta)\setminus\{t\}$ into two disjoint closed sets. The only possibility of such a separation is a separation into sets $(t-\delta,t)$ and $(t,t+\delta)$. Hence (iv) holds.

Moreover, for $t\in B_0\cup B_1$ the layer $\alpha^{-1}(t)$ of X is not a layer of the cohesion of X. From Remarks in [12], p. 201 we conclude that sets B_0 and B_1 are countable. Hence, by (2.2.1), the proof is complete.

Now, we will prove

(2.3) THEOREM. *For $i=0,1$ let $\alpha_i\colon X_i^*\to[0,1]$ be a λ-projection and $\beta_i\colon X_i^*\to X_i$ be a V-projection onto X_i with endlayers A_0^i and A_1^i. If f is a continuous mapping from X_0 onto X_1 such that $f(A_0^0)=A_0^1$ and $f(A_1^0)=A_1^1$, then there are a continuous surjection $f^*\colon X_0^*\to X_1^*$ and a homeomorphism $h\colon[0,1]\to[0,1]$ such that*

(i) *for $t\in[0,1]$ the mapping $\beta_0|\alpha_0^{-1}(t)$ is a homeomorphism if and only if $\beta_1|\alpha_1^{-1}(h(t))$ is a homeomorphism,*

(ii) *the following diagrams commute*

$$\begin{tikzcd}
X_0 \arrow[d, "f"] & X_0^* \arrow[l, "\beta_0"'] \arrow[r, "\alpha_0"] \arrow[d, "f^*"] & {[0,1]} \arrow[d, "h"] \\
X_1 & X_1^* \arrow[l, "\beta_1"'] \arrow[r, "\alpha_1"] & {[0,1]}
\end{tikzcd}$$

Proof. Put $B_i^j=\{t\in[0,1]\colon \alpha_j^{-1}(t)\in V(\beta_j,A_i^j)\}$ for $i,j=0,1$. According to (2.2), sets B_0^j and B_1^j are countable, dense and disjoint subsets of $(0,1)$ for $j=0,1$. An easy inductive construction implies the existence of a homeomorphism $h\colon[0,1]\to[0,1]$ such that $h(0)=0$, $h(B_0^0)=B_0^1$ and $h(B_1^0)=B_1^1$. Define a mapping $f^*\colon X_0^*\to X_1^*$ by the formula

$$f^*(x)=\begin{cases}\beta_1^{-1}f\beta_0(x)\cap\alpha_1^{-1}h\alpha_0(x) & \text{if } \alpha_0(x)\notin B_0^0\cup B_1^0,\\ \lim f^*(x_i) & \text{if } x=\lim x_i \text{ and } \alpha_0(x_i)\notin B_0^0\cup B_1^0.\end{cases}$$

If $\alpha_0(x) \notin B_0^0 \cup B_1^0$, then the set $\beta_1^{-1} f \beta_0(x) \cap \alpha_1^{-1} h\alpha_0(x)$ is an one-point set, because the mapping $\beta_1 | \alpha_1^{-1} h\alpha_0(x)$ is a homeomorphism. If $x = \lim x_i$, $\alpha_0(x) \in B_0^0 \cup B_1^0$, $\alpha_0(x_i) \notin B_0^0 \cup B_1^0$ and $\alpha_0(x_i) < \alpha_0(x)$, then $\lim f^*(x_i) = \lim(\beta_1^{-1} f\beta_0(x_i) \cap \alpha_1^{-1} h\alpha_0(x_i)) = (\beta_1^L)^{-1} f\beta_0(x) \cap \alpha_1^{-1} h\alpha_0(x)$ by (2.2) (iv), where $\beta_1^L = \beta_1 | \mathrm{cl}(\alpha_1^{-1}[0, h\alpha_0(x)) \setminus \alpha_1^{-1}[0, h\alpha_0(x))$; and similarly from the right side. Therefore the mapping f^* is well defined. One can easily check that f^* has all required properties.

(2.4) THEOREM. *Let X be an irreducible continuum with endlayers A_0 and A_1. There is a unique continuum X^* which possesses a λ-projection α: $X^* \to [0, 1]$ and a V-projection β: $X^* \to X$ onto X. Moreover, if X is an hereditarily decomposable arc-like continuum, then X^* is an hereditarily decomposable arc-like continuum.*

Proof. The uniqueness immediately follows from Theorem (2.3). Now, we will construct X^*, α and β.

Let F be the Cantor ternary set situated on the interval $[0, 1]$, φ be the step-function from F onto $[0, 1]$ and let φ_i be a mapping identifying every two endpoints of the contiguous intervals to F with lengths $1/3^{2n-i}$ where $i = 0, 1$ and $n = 1, 2, \ldots$ Denote the projections from $X \times F$ onto X and F by π_1 and π_2, respectively. Consider a mapping σ from $X \times F$ such that two different points (x, t), (x', t') from $X \times F$ are identified if and only if either $x = x' \in A_0$ and $\varphi_0(t) = \varphi_0(t')$ or $x = x' \in A_1$ and $\varphi_1(t) = \varphi_1(t')$. Put $X^* = \sigma(X \times F)$, $\alpha = \varphi\pi_2\sigma^{-1}$ and $\beta = \pi_1\sigma^{-1}$.

It is quite clear that

(2.4.1) *If K is a subcontinuum of X^*, $\{t\} = \varphi^{-1}\varphi(t)$ and $\alpha^{-1}(t) \cap K \neq \emptyset \neq (X \setminus \alpha^{-1}(t)) \cap K$, then $\alpha^{-1}(t) \subset K$.*

Condition (2.4.1) implies that X^* is an irreducible continuum with layers $\alpha^{-1}(t)$ for $t \in [0, 1]$ (because α is continuous and $\alpha^{-1}(t)$ are boundary subcontinua of X^*; compare (2.1)). Moreover, from (2.4.1) we conclude that if X is hereditarily unicoherent, hereditarily decomposable and atriodic, then X^* has the same properties, i.e. the additional implication in (2.4) holds by (1.8).

If X is an hereditarily decomposable arc-like continuum, then the uniquely determined continuum and mappings by Theorem (2.4) will be denoted in this section by X^*, α_X and β_X, respectively.

B. The construction of Σ. Put $\Sigma_1 = [0, 1]$, $\Sigma_{n+1} = (\Sigma_n)^*$, $\alpha_1 = \mathrm{id}_{[0,1]}$, $\alpha_n = \alpha_{\Sigma_n}$, $\beta_n = \beta_{\Sigma_n}$, $\sigma_1 = \alpha_{\Sigma_1}$, $\sigma_{n+1} = (\sigma_n)^*$ (compare (2.3)) for $n = 1, 2, \ldots$ (see Figure 1).

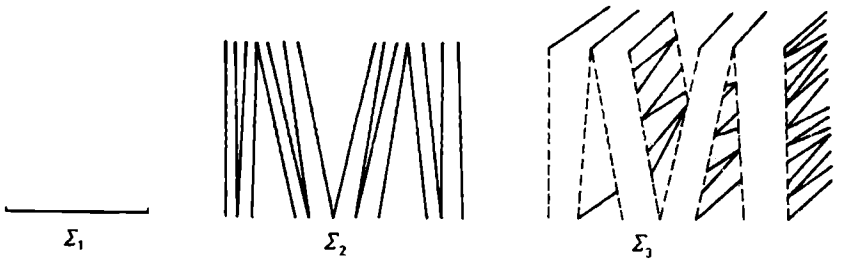

Fig. 1

The following diagrams commute for $n > 1$

$$\begin{tikzcd}
\Sigma_{n-1} & \Sigma_n \arrow[l, "\sigma_{n-1}"'] \\
\Sigma_n \arrow[u, "\beta_{n-1}"] \arrow[d, "\alpha_{n-1}"'] & \Sigma_{n-1} \arrow[l, "\sigma_n"'] \arrow[u, "\beta_n"'] \arrow[d, "\alpha_n"] \\
{[0,1]} & {[0,1]} \arrow[l, "\mathrm{id}_{[0,1]}"']
\end{tikzcd}$$

Define $\Sigma = \operatorname{inv}\lim\{\Sigma_n, \sigma_n\}$ and let π_n denote the projection from Σ onto Σ_n. Above diagrams and (2.4) give us

(2.5) *Σ is homeomorphic to Σ^*.*

In particular, Σ and each layer of Σ are homeomorphic.

Theorem (2.3) and an easy induction imply

(2.6) *For $n = 1, 2, \ldots$ σ_n is a monotone mapping from Σ_{n+1} onto Σ_n such that $\sigma_n^{-1}(y)$ are nondegenerate for $y \in \Sigma_n$ and if A and B are subcontinua of Σ_{n+1} such that $\sigma_n(A) \neq \sigma_n(A \cup B) \neq \sigma_n(B)$, then the set $A \cap B$ is nondegenerate provided it is nonempty.*

Every continuum Σ_n is an hereditarily decomposable arc-like continuum by (2.4). Therefore, according to (1.9), we have

(2.7) *Σ is an arc-like continuum.*

Moreover,

(2.8) *Σ is hereditarily decomposable.*

In fact, let K be a subcontinuum of Σ. There is a positive integer n such that the set $\pi_n(K)$ is nondegenerate. Since Σ_n is hereditarily decomposable,

we find continua A and B such that $\pi_n(K) = A \cup B$ and $A \neq \pi_n(K) \neq B$. Sets $\pi_n^{-1}(A)$ and $\pi_n^{-1}(B)$ are continua by (1.4) and (2.6) (the monotoneity of π_n). According to (1.7) and (2.7) sets $\pi_n^{-1}(A) \cap K$ and $\pi_n^{-1}(B) \cap K$ are continua. Since $K = (\pi_n^{-1}(A) \cap K) \cup (\pi_n^{-1}(B) \cap K)$ and $\pi_n^{-1}(A) \cap K \neq K \neq \pi_n^{-1}(B) \cap K$, we infer that K is decomposable.

The continuum Σ which we have constructed is hereditarily nondivisible by points, namely

(2.9) *If K is a subcontinuum of Σ, $p \in K$, then p does not separate K.*

Indeed, suppose that $K = A \cup B$ where A and B are proper subcontinua of K. According to (2.6) we find a positive integer m such that the set $\pi_m(A) \cap \pi_m(B)$ is nondegenerate. From (1.12) we conclude that there is a point $c \in \pi_m(A) \cap \pi_m(B)$ such that $\pi_m^{-1}(c) \subset A \cap B$. Once again by (2.6), the set $A \cap B$ is nondegenerate.

Note here that every hereditarily decomposable continuum contains an irreducible continuum with degenerate tranches (see [20]).

3. The function ξ_0

A. The construction of D_β and Σ_β. For our further considerations we will now modify the construction from Section 3 in [11]. Definitions which are not recalled here can be found in [11].

For two points a and b belonging to the plane denote by ab the straight line segment between these points and put

$$p = (0, 0),\ r = (0, 1),\ p_n = (1/n, 0),\ r_n = (1/n, 1) \quad \text{for} \quad n = 1, 2, \ldots$$

Denote $L = pr$ and $L_n = p_{2n+1} r_{2n+1} \cup r_{2n+1} p_{2n+2} \cup p_{2n+2} r_{2n+2}$ for $n = 0, 1, 2, \ldots$ and define $D = L \cup \bigcup_{n=0}^{\infty} L_n$. Take a mapping $\pi\colon D \to L$ which is a restriction of a projection from the plane onto the second axis. Then

(3.1) *π is an open retraction onto L and $\pi | L_n$ is a finite-to-one mapping for each $n = 0, 1, 2, \ldots$*

Fix a homeomorphism h_n from L onto L_n. Assume that for $n = 0, 1, 2, \ldots$ X_n is a compactum containing D and ω_n is an open retraction from X_n onto L such that $\omega_n | A$ is a finite-to-one open mapping from every component A of X_n onto L. Let $S = \{(X_n, \omega_n)\colon n = 0, 1, 2, \ldots\}$. We will now define a compactum Γ_S and a mapping γ_S with the same properties as every X_n and ω_n (compare Figure 2).

Consider the set $X = D \cup \bigcup_{n=0}^{\infty} (X_n \times \{n\})$ with an equivalence relation σ which identifies every point $\{x\} \times \{n\}$ where $x \in L$ with the point $h_n(x)$ belonging to D.

Denote the quotient space X/σ by Γ_S and the canonical mapping from X onto Γ_S by φ. The topology on Γ_S is defined as follows: neighbourhoods of points from $\Gamma_S \setminus \varphi(L) = \varphi\big(\bigcup_{n=0}^{\infty} (X_n \times \{n\})\big)$ are determined by neighbourhoods in $\bigcup_{n=0}^{\infty} (X_n \times \times \{n\})$ in the ordinary sense, for $x \in \varphi(L)$ and an open set U in D containing $\varphi^{-1}(x)$ we take the set of the form $\bigcup_{n=0}^{\infty} \varphi\omega_n^{-1} h_n^{-1}(L_n \cap U) \cup \varphi(U)$ as a neighbourhood of x.

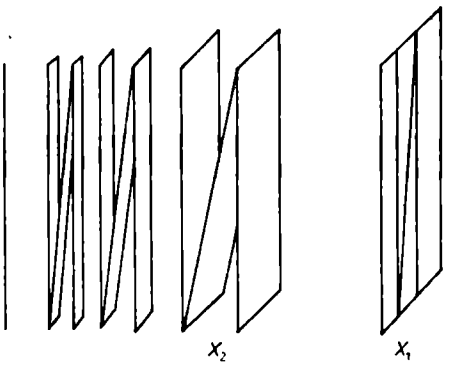

Fig. 2

A mapping γ_S is defined by the formulas $\gamma_S(x) = \varphi\pi\omega_n(z \times \{n\})$ if $z \times \{n\} \in \varphi^{-1}(x)$ and $\gamma_S(x) = \varphi\big(\pi(\varphi^{-1}(x) \cap D)\big)$ provided $\varphi^{-1}(x) \cap D \neq \emptyset$. If we identify x with $\varphi(x)$ for $x \in D$, then Γ_S and γ_S have the required properties.

One can observe that the above abstract construction can be realized geometrically, but then the description of it is more complicated and the properties are not so obvious as in this abstract approach.

Now we pass to the description of Γ_α's and γ_α's when α is a countable ordinal. The compacta Γ_α and mappings γ_α will be defined recursively in the following way:

Let $\Gamma_0 = D$, $\gamma_0 = \pi$ and assume the compacta Γ_α and mappings γ_α have been constructed for all $\alpha < \beta$.

Consider two cases:

(i) $\beta = \alpha + 1$.

Define $\Gamma_\beta = \Gamma_S$ and $\gamma_\beta = \gamma_S$ where $S = \{(X_n, \omega_n)\colon n = 0, 1, 2, \ldots, X_n = \Gamma_\alpha, \omega_n = \gamma_\alpha\}$.

(ii) β is a limit ordinal.

In this case arrange all ordinals $\alpha < \beta$ in a sequence $\alpha_0, \alpha_1, \ldots$ and set $\Gamma_\beta = \Gamma_S$ and $\gamma_\beta = \gamma_S$ where $S = \{(\Gamma_{\alpha_n}, \gamma_{\alpha_n})\colon n = 0, 1, 2, \ldots\}$.

From (3.1) and the above construction we easily obtain the following.

(3.2) *Every component A of Γ_β is an arc and $\gamma_\beta | A$ is a finite-to-one open mapping from A onto $L \subset \Gamma_\beta$.*

(3.3) *There is a positive number $\eta_\beta > 0$ such that* $\operatorname{diam} A > \eta_\beta$ *for every component A of Γ_β.*

Fix a countable ordinal β. Let α_Σ be a λ-projection from Σ onto L. Put $\Sigma_\beta = \{(x, y) \in \Sigma \times \Gamma_\beta\colon \alpha_\Sigma(x) = \gamma_\beta(y)\}$. Using (3.2), one can easily check that

(3.4) *Every component of Σ_β is homeomorphic to Σ and there is an open mapping $\bar{\gamma}_\beta$ from Σ_β onto Σ such that $\bar{\gamma}_\beta$ is finite-to-one on every component of Σ_β.*

(3.5) *There is a monotone mapping σ_β from Σ_β onto Γ_β.*

Denote by $\tilde{\Sigma}_\beta^\nabla$ an arbitrary pseudosuspension of Σ_β over $[0, 1]$ (at an arbitrary set being a decompositon space of Σ_β). The mapping σ_β induces a natural monotone mapping $\tilde{\sigma}_\beta^\nabla$ from $\tilde{\Sigma}_\beta^\nabla$ onto a space D_β being a pseudosuspension of Γ_β over $[0, 1]$. Applying (1.15) and (3.4) we find a continuum $\tilde{\Sigma}_\beta$ and a monotone mapping $\tilde{\sigma}_\beta$ from $\tilde{\Sigma}_\beta$ onto D_β ($\tilde{\Sigma}_\beta$ is a pseudosuspension of Σ_β over Σ in which layers corresponding to the decomposition space of Σ_β lying in $[0, 1]$ are contracted to points) in such a manner that

(3.6) *If A is a subcontinuum of $\tilde{\Sigma}_\beta$ such that $\tilde{\sigma}_\beta(A)$ is an arc in D_β, then A is homeomorphic to a subcontinuum of Σ.*

From (3.3) we conclude that (see § 1,D)

(3.7) $\varepsilon(D_\beta) > 0$.

Moreover,

(3.8) *The continuum $\tilde{\Sigma}_\beta$ is a hereditarily decomposable arc-like continuum no nondegenerate subcontinuum of which is separated by a point.*

B. The crucial property and its consequences. Now we are going to prove the crucial fact about continua D_β (compare [19], Section 3; the proof of Theorem 3.1)

(3.9) *If F is $\varepsilon(D_\alpha)$-valued map from an hereditarily decomposable continuum X onto D_α, then $\xi_0(X) \geqslant \alpha$.*

Proof. Clearly, it is enough to prove that if $\beta \leqslant \alpha$ and h is a homeomorphism from Γ_β into D_α, then

$$\big(X, F^{-1}(h(r)), F^{-1}(h(p))\big) \in T_0(X)_{(\beta)}.$$

This will be proved by transfinite induction. First we prove this for $\beta = 0$. Let $h\colon \Gamma_\beta \to D_\alpha$ be a homeomorphism. Let U be an open neighbourhood of $F^{-1}(h(r))$ and V an open neighbourhood of $F^{-1}(h(p))$ in X. Then $U' = D_\alpha \setminus F(X \setminus U)$ and $V' = D_\alpha \setminus F(X \setminus V)$ are neighbourhoods of $h(r)$ and $h(p)$, respectively. The assumptions guarantee that we can find U and V such that $d(U', V') > 2\varepsilon(D_\alpha)$ where d is a metric in D_α. Then there is an index n such that $\big(h(L_n), \{h(r_n)\}, \{h(p_n)\}\big) \in \mathrm{cr}(U', V')$. By 2.1 from [19] we conclude that $\big(D_\alpha, \{h(r_n)\}, \{h(p_n)\}\big) \in \mathrm{cr}(U', V')$. This means that there are closed sets W_1, W_2, W_3 such that $D_\alpha = W_1 \cup W_2 \cup W_3$, $h(r_n) \in W_1 \cap U'$, $h(p_n) \in W_3 \cap V'$, $W_1 \cap W_2 \subset V'$, $W_2 \cap W_3 \subset U'$ and $W_1 \cap W_2 = \emptyset$. Obviously $X = F^{-1}(W_1) \cup F^{-1}(W_2) \cup F^{-1}(W_3)$, $F^{-1}(h(r_n)) \subset F^{-1}(W_1) \cap U$, $F^{-1}(h(p_n)) \subset F^{-1}(W_3) \cap V$, $F^{-1}(W_1) \cap F^{-1}(W_2) \subset V$, $F^{-1}(W_2) \cap F^{-1}(W_3) \subset U$ and $F^{-1}(W_1) \cap F^{-1}(W_2) = \emptyset$ (if $x \in F^{-1}(W_1) \cap F^{-1}(W_2)$, then $F(x) \cap W_1 \neq \emptyset \neq F(x) \cap W_2$; therefore $F(x) \cap U' \neq \emptyset \neq F(x) \cap V'$, because $F(x)$ is a

continuum; thus diam $F(x) > \varepsilon(D_\alpha)$, a contradiction). Those relations mean that $(X, F^{-1}(h(r_n)), F^{-1}(h(p_n))) \in \mathrm{cr}(U, V)$. Hence $(X, F^{-1}(h(r)), F^{-1}(h(p))) \in T_0(X)_{(0)}$.

Assume that $\delta < \beta$ and h is a homeomorphism from Γ_δ into D_α, then $(X, F^{-1}(h(r)), F^{-1}(h(p))) \in T_0(X)_{(\delta)}$. Let g be a homeomorphism from Γ_β into D_α. Consider two cases.

(a) $\beta = \delta + 1$. Let U be an open neighbourhood of $F^{-1}(g(r))$ and V be an open neighbourhood of $F^{-1}(g(p))$ in X. The assumptions guarantee that U and V can be chosen in such a manner that sets $U' = D_\alpha \setminus F(X \setminus U)$ and $V' = D_\alpha \setminus F(X \setminus V)$ are open neighbourhoods of $g(r)$ and $g(p)$, respectively and $d(U', V') > 2\varepsilon(D_\alpha)$. There is an index n such that $(g(L_{n+1}), \{g(r_{n+1})\}, \{g(p_{n+1})\}) \in \mathrm{cr}(U', V')$. By the induction assumption, we conclude that $(X, F^{-1}(gh_n(r)), F^{-1}(gh_n(p))) \in T_0(X)_{(\delta)}$ (observe that $h_n(r) = r_{n+1}$, $h_n(p) = p_{n+1}$). As above we obtain $(X, F^{-1}(g(r_{n+1})), F^{-1}(g(p_{n+1}))) \in \mathrm{cr}(U, V)$ and $(X, F^{-1}(g(r_{n+1})), F^{-1}(g(p_{n+1}))) \in T_0(X)_{(\delta)}$. This shows that $(X, F^{-1}(g(r)), F^{-1}(g(p))) \in T_0(X)_{(\beta)}$.

(b) β is a limit ordinal. By an argument similar to the above one can prove

$$(X, F^{-1}(g(r)), F^{-1}(g(p))) \in T_0(X)_{(\delta)} \quad \text{for} \quad \delta < \beta.$$

Since

$$T_0(X)_{(\beta)} = \bigcap_{\delta < \beta} T_0(X)_{(\delta)},$$

the proof is completed.

(3.10) Corollary. *For every hereditarily decomposable continuum X there exists an countable ordinal α such that no subcontinuum of X can be mapped onto D_α by an $\varepsilon(D_\alpha)$-valued map.*

Proof. Take a countable ordinal $\alpha > \xi_1(X)$. Suppose there is a continuum $Y \subset X$ which can be mapped onto D_α by $\varepsilon(D_\alpha)$-valued map. By (3.9) we have $\xi_0(Y) \geqslant \alpha$. By (2.8) in [11] we have $\xi_1(X) \geqslant \xi_0(Y)$. Combining these inequalities we obtain $\xi_1(X) \geqslant \alpha$, contrary to the choice of α.

(3.8) and Corollary (3.10) imply the following conclusions (compare [11], Corollaries 4.2 and 4.3)

(3.11) Corollary. *There is no hereditarily decomposable continuum containing all hereditarily decomposable (hereditarily nondivisible by points) arc-like continua.*

(3.12) Corollary. *There is no hereditarily decomposable continuum which can be mapped onto every hereditarily decomposable (hereditarily nondivisible by points) arc-like continuum.*

Applying (3.10) we find D_α such that no subcontinuum of Σ can be mapped onto D_α by an $\varepsilon(D_\alpha)$-valued map. Hence, from (2.5), (2.7), (2.8), (2.10), (3.6), (3.7) and (3.8) we obtain the following conclusion.

(3.13) COROLLARY. *There are hereditarily decomposable arc-like continua D, Σ, $\tilde{\Sigma}$ and monotone surjections α: $\tilde{\Sigma} \to [0, 1]$ and σ: $\tilde{\Sigma} \to D$ such that*

(i) *Σ and $\tilde{\Sigma}$ are hereditarily nondivisible by points,*

(ii) *$\varepsilon(D) > 0$ and layers of every subcontinuum of D are arcs,*

(iii) *$\alpha^{-1}(t)$ is homeomorphic to Σ for $t \in [0, 1]$,*

(iv) *no subcontinuum of Σ can be mapped onto D by an $\varepsilon(D)$-valued map,*

(v) *if A is a subcontinuum of $\tilde{\Sigma}$ and $\sigma(A)$ is an arc, then A is homeomorphic to a subcontinuum of Σ.*

Henceforth only Σ, $\tilde{\Sigma}$, D, α and σ denote always fixed continua and mappings with properties mentioned in (3.11).

C. Pseudosuspensions with D and Σ. Firstly, we will prove

(3.14) *If an arc-like continuum M contains D, S is a pseudosuspension of M over $[0, 1]$ at $\{0\}$ and $\varepsilon(S) > 0$, then no subcontinuum of $\tilde{\Sigma}$ can be mapped onto S under an $\varepsilon(S)$-valued map.*

Indeed, let F be an $\varepsilon(S)$-valued map from a subcontinuum K of $\tilde{\Sigma}$ onto S. We can assume that K is a minimal continuum with respect to the property $F(K) = S$. Denote by A and B the endlayers of $\sigma(K)$. From (3.11) (ii) and (v) we conclude that sets $\sigma^{-1}(A)$ and $\sigma^{-1}(B)$ are homeomorphic to Σ. If r denotes the natural projection from S onto $[0, 1]$, then $F^{-1}r^{-1}(\{0, 1\}) \subset \sigma^{-1}(A) \cup \sigma^{-1}(B)$ by the minimality of K. In particular, $D \subset F(\sigma^{-1}(A) \cup \sigma^{-1}(B))$. Since F is pseudomonotone (by (1.17)), we find a subcontinuum L of $\sigma^{-1}(A) \cup \sigma^{-1}(B)$ which is mapped onto D under an $\varepsilon(D)$-valued map G defined by $G(x) = F(x) \cap D$ for $x \in L$. This is impossible by (3.11) (iv), because L is homeomorphic to a subcontinuum of Σ.

Fix $A_0 = \{0\}$, $A_1 = \{\frac{1}{2}\}$, $A_2 = \{0, \frac{1}{2}, 1\}$ and $A_3 = \{0\} \cup \{1/(n+1)$: $n = 1, 2, \ldots\}$. For $i = 0, 1, 2, 3$ by Σ^i we denote the space obtained from Σ by the identifying the points of every component of $\alpha^{-1}(A_i)$ and by α_i we denote the λ-projection from Σ^i onto $[0, 1]$ such that $\alpha_i^{-1}|A_i$ is a homeomorphism. Put $A^i = \alpha_i^{-1}(A_i)$ for $i = 0, 1, 2, 3$.

We have

(3.15) *If S_1 is a pseudosuspension of a continuum M over Σ^1 at A^1, S_2 is a pseudosuspension of a continuum K containing D over $[0, 1]$ at A_1 and F is an $\varepsilon(S_2)$-valued map from S_1 onto S_2, then $F(M) = K$ and $F(S_1 \setminus M) = S_2 \setminus K$.*

In fact, let r_i denote a natural projection from S_i onto Σ^1 and $[0, 1]$ respectively for $i = 1, 2$. Put $R = r_1^{-1}(\alpha_1^{-1}[0, 1] \setminus A_1)$. Then the set R is an open and dense subset of S_1. Therefore the set $F(R)$ intersects both path-components K_1 and K_2 of $S_2 \setminus K$. From (1.17) and (3.14) we conclude that K is not contained in $F(R)$, because $D \subset K$. Hence $K \cap F(R) = \emptyset$. This implies inclusions $K \subset F(M)$ and $F(R) \subset K_1 \cup K_2$. Suppose that $F(M) \cap K_1 \neq \emptyset$

and $F(r_1^{-1}\alpha_1^{-1}[0, \frac{1}{2})) \subset K_1$ and $F(r_1^{-1}\alpha_1^{-1}(\frac{1}{2}, 1]) \subset K_2$. Then $F(x) \subset K_1$ for some $x \in M$, because F is an $\varepsilon(S_2)$-valued map. Since F is upper semicontinuous and there is a sequence $\{x_n\}$ converging to x with $\alpha_1 r_1(x_n) > \frac{1}{2}$ for $n = 1, 2, \ldots$ we conclude that $F(r_1^{-1}\alpha_1^{-1}(\frac{1}{2}, 1]) \cap K_1 \neq \emptyset$, a contradiction. Therefore $F(M) \cap K_1 = \emptyset$ and $F(M) \cap K_2 = \emptyset$ by the symmetry; thus $F(M) = K$ and $F(S_1 \setminus M) = S_2 \setminus K$.

(3.16) *If S_1 is a pseudosuspension of a continuum M over Σ^1 at A^1, S_2 is a pseudosuspension of a continuum K containing D over $[0, 1]$ at A_1, S_1^* is a pseudosuspension of S_1 over Σ^0 at A^0, S_2^* is a pseudosuspension of S_2 over $[0, 1]$ at A_0 and if F is an $\varepsilon(S_2^*)$-valued map from S_1^* onto S_2^* such that $F(S_1 \setminus M) \cap S_2 \neq \emptyset$, then $F(S_1) = S_2$.*

In fact, denote the components of $S_1 \setminus M$ by M_1 and M_2, and the components of $S_2 \setminus K$ by K_1 and K_2. Firstly as above, we obtain that $F(S_1^* \setminus S_1) \cap (S_2^* \setminus S_2) \neq \emptyset$ and then $F(S_1^* \setminus S_1) \subset S_2^* \setminus S_2$. The same arguments give $F(M_1 \cup M) \subset K_1 \cup K$ if $F(M_1) \cap K_1 \neq \emptyset$. Therefore $F(M_2) \cap K_2 \neq \emptyset$, thus $F(M_2) \subset K_2$. Thereby $F(S_1) = S_2$. Analogously the proof can be done in all other cases.

4. Continua M_k and N_k

A. The construction and properties of M_k and $\tilde{M}_k$. We will define two sequences of continua M_k and $\tilde{M}_k$ and a sequence of mappings σ_k: $\tilde{M}_k \to M_k$ such that

(4.1) *M_0 is a pseudosuspension of $D \times A_2$ over $[0, 1]$ at A_2 and for $k > 1$ M_k is a pseudosuspension of M_{k-1} over $[0, 1]$ at A_1.*

(4.2) *$\tilde{M}_0$ is a pseudosuspension of $\tilde{\Sigma} \times A^2$ over Σ^2 at A^2 and for $k > 1$, $\tilde{M}_k$ is a pseudosuspension of $\tilde{M}_{k-1}$ over Σ^1 at A^1.*

(4.3) *σ_k is a monotone surjection from $\tilde{M}_k$ onto M_k.*

Let M_0 be an arbitrary pseudosuspension of $D \times A_2$ over $[0, 1]$ at A_2 (compare (1.14)). To find $\tilde{M}_0$ and σ_0 we apply (1.15). Suppose M_i, $\tilde{M}_i$ and σ_i are defined for $i < k$. Firstly we take an arbitrary pseudosuspension M of $\tilde{M}_{k-1}$ over $[0, 1]$ at A_1. The mapping σ_{k-1} gives us a monotone mapping φ from M onto some pseudosuspension M_k of M_{k-1} over $[0, 1]$ at A_1. Now, we apply again (1.15) and we find a pseudosuspension $\tilde{M}_k$ of $\tilde{M}_{k-1}$ over Σ^1 at A^1 with a monotone mapping φ' from M_k onto M. Then M_k, $\tilde{M}_k$ and $\sigma_k = \varphi\varphi'$ have all required properties (Figure 3 gives the intuitive image of M_k and of $\tilde{M}_k$).

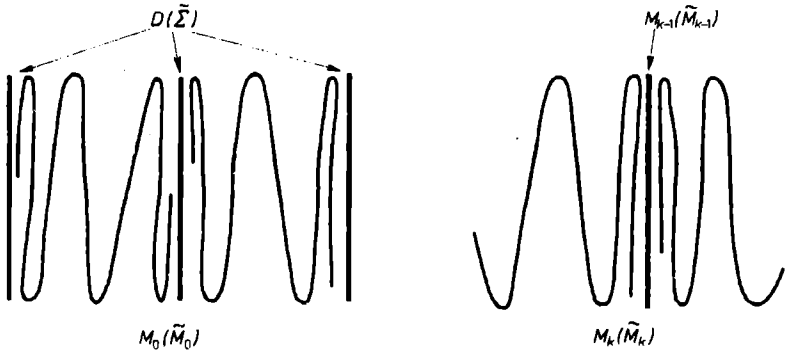

Fig. 3

Obviously, by (1.11) and (3.13) we have

(4.4) *M_k in an arc-like hereditarily decomposable continuum with* $\varepsilon(M_k) > 0$.

(4.5) *$\tilde{M}_k$ is an arc-like hereditarily decomposable continuum which is hereditarily nondivisible by points. Moreover, if C is a subcontinuum of $\tilde{M}_k$* such that *$\sigma_k(C)$ is an arc, then C is homeomorphic to a subcontinuum of Σ.*

We assume in the sequel that $\tilde{M}_k \subset \tilde{M}_{k+1}$ and $M_k \subset M_{k+1}$ for $k = 0, 1, 2, \ldots$; $\tilde{\Sigma} \subset \tilde{M}_0$ and $D \subset M_0$ ($\tilde{\Sigma}$ and D denote here layers which are not endlayers). Denote arbitrary λ-projections from $\tilde{M}_k$ and M_k onto $[0, 1]$ by $\tilde{\mu}_k$ and μ_k, respectively. We assume that $\tilde{\mu}_k(\tilde{M}_{k-1}) = \frac{1}{2} = \mu_k(M_{k-1})$ for $k > 0$ and $\tilde{\mu}_0(\tilde{\Sigma}) = \frac{1}{2} = \mu_0(D)$ and $\mu_k \sigma_k = \tilde{\mu}_k$.

We have

(4.6) *$\tilde{M}_k$ cannot be mapped onto M_0 under an $\varepsilon(M_0)$-valued map for $k > 0$.*

In fact, let F be an $\varepsilon(M_0)$-valued map from $\tilde{M}_k$ onto M_0 where $k > 0$. If K is a component of $\tilde{M}_k \setminus \tilde{\mu}_k^{-1}(\frac{1}{2})$ and $F(K) \cap \mu_0^{-1}(0) \neq \emptyset$, then $F(K) \subset \mu_0^{-1}(0)$ by (3.14). Since F is upper semi-continuous, we obtain that $F(x) \cap \mu_0^{-1}(0) \neq \emptyset$ for each $x \in \tilde{\mu}_k^{-1}(\frac{1}{2})$. Therefore $F(\tilde{\mu}_k^{-1}(\frac{1}{2})) \subset \mu_0^{-1}(0)$, because $F(x)$ are path-connected. Once again the upper semi-continuity of F implies that $L \cap \mu_0^{-1}[0, \frac{1}{2}) \neq \emptyset$ where $L = F(\tilde{M}_k \setminus (\tilde{\mu}_k^{-1}(\frac{1}{2}) \cup K))$. Since the set L is connected and intersects $\mu_0^{-1}[0, \frac{1}{2})$ and since F is a surjection, we obtain that $\mu_0^{-1}(\frac{1}{2}) \subset L$. According to (1.17) we obtain that the set $\mu_0^{-1}(\frac{1}{2})$ is an image under an $\varepsilon(M_0)$-valued map of a subcontinuum of Σ contrary to (3.14), because $\mu_0^{-1}(\frac{1}{2})$ is homeomorphic to D. Thereby $F(K) \cap \mu_0^{-1}(0) = \emptyset$. Similarly $F(K) \cap \mu_0^{-1}(1) = \emptyset$; thus $\mu_0^{-1}(0) \cup \mu_0^{-1}(1) \subset F(\tilde{\mu}_k^{-1}(\frac{1}{2}))$; but then the image of every component of $\tilde{M}_k \setminus \tilde{\mu}_k^{-1}(\frac{1}{2})$ intersects simultaneously sets $\mu_0^{-1}(0, \frac{1}{2})$ and $\mu_0^{-1}(\frac{1}{2}, 1)$ and we obtain a contradiction as previously, because the set $\mu_0^{-1}(\frac{1}{2})$ is homeomorphic to a continuum D.

Finally, we have (compare [4], Theorem 3.6, p. 300 and [15], Theorem 24)

(4.7) THEOREM. *$\tilde{M}_k$ cannot be mapped onto M_n under an $\varepsilon(M_n)$-valued map for $k \neq n$.*

Indeed, suppose that F is an $\varepsilon(M_n)$-valued map from $\tilde{M}_k$ onto M_n. Consider two cases:

(a) $k < n$. The mapping F, by (3.15), (4.1) and (4.2), implies the existence of an $\varepsilon(M_{n-k-1})$-valued map from $\tilde{\Sigma}$ onto M_{n-k-1}. This is impossible by (3.14).

(b) $k > n$. The mapping F, by (3.15), (4.1) and (4.2), implies the existence of an $\varepsilon(M_0)$-valued map from $\tilde{M}_{k-n}$ onto M_0. This is impossible by (4.6).

B. The construction and properties of N_k and $\tilde{N}_k$. Recall that $A_3 = \{0\} \cup \left\{\frac{1}{n+1}: n = 1, 2, \ldots\right\}$ and define

$$A_3^0 = \bigcup_{n=2}^{\infty} \left(\frac{1}{2n}, \frac{1}{2n-1}\right) \cup \left(\frac{1}{2}, 1\right] \quad \text{and} \quad A_3^1 = \bigcup_{n=1}^{\infty} \left(\frac{1}{2n+1}, \frac{1}{2n}\right).$$

For $i = 0, 1$ and $k = 0, 1, 2, \ldots$ let M_k^i and $\tilde{M}_k^i$ denote the homeomorphic copies of M_k and $\tilde{M}_k$ respectively. By the identification of one endlayer of M_k^0 with the corresponding endlayer of M_k^1 we obtain a continuum M_k^*. Similarly, we define $\tilde{M}_k^*$.

Next, as in the construction of M_k, $\tilde{M}_k$ and σ_k we define continua N_k^i and $\tilde{N}_k^i$ and mappings η_k^i: $\tilde{N}_k^i \to N_k^i$ such that

(4.8) *N_k^i is a pseudosuspension of $M_k^i \times A_3$ over $[0, 1]$ at A_3.*

(4.9) *$\tilde{N}_k^i$ is a pseudosuspension of $\tilde{M}_k^i \times A_3$ over Σ_3 at A_3.*

(4.10) *η_k^i is a monotone surjection from $\tilde{N}_k^i$ onto N_k^i.*

Denote arbitrary λ-projections from $\tilde{N}_k^i$ and N_k^i onto $[0, 1]$ by $\tilde{v}_k^i$ and v_k^i respectively and assume $\tilde{v}_k^i = v_k^i \eta_k^i$, $\tilde{v}_k^i(M_k^i \times A_3) = A_3$ and $v_k^i(M_k^i \times A_3) = A_3$.

Consider the disjoint union $\tilde{N}^k(N^k)$ of $\tilde{M}_k^* \times A_3$, $\tilde{N}_k^0 \setminus (\tilde{v}_k^0)^{-1}(A_3^0)$ and $\tilde{N}_k^1 \setminus (\tilde{v}_k^1)^{-1}(A_3^1)$ (of $M_k^* \times A_3$, $N_k^0 \setminus (v_k^0)^{-1}(A_3^0)$ and $N_k^1 \setminus (v_k^1)^{-1}(A_3^1)$) and the mapping $\tilde{\omega}_k$ (ω_k, respectively) which identifies every point (x, y) from the set $\tilde{M}_k^i \times A_3$ (from the set $M_k^i \times A_3$) with the corresponding point from the set $\tilde{M}_k^* \times A_3$ ($M_k^* \times A_3$, respectively).

Put $N_k = \omega_k(N^k)$ and $\tilde{N}_k = \tilde{\omega}_k(\tilde{N}^k)$ (Figure 4 gives the intuitive image of N_k and $\tilde{N}_k$).

The construction implies the existence of mappings η_k: $\tilde{N}_k \to N_k$, $\tilde{v}_k$: $\tilde{N}_k \to [0, 1]$ and v_k: $N_k \to [0, 1]$ such that

(4.11) *η_k is a monotone surjection from $\tilde{N}_k$ onto N_k.*

(4.12) *$\tilde{v}_k$ is a λ-projection from $\tilde{N}_k$ onto $[0, 1]$ such that every component*

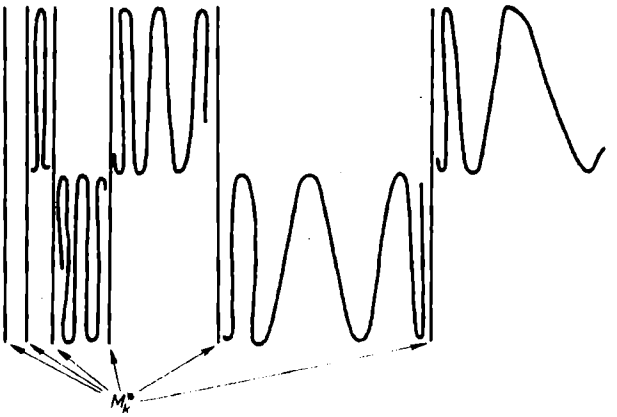

Fig. 4

of the set $\tilde{v}_k^{-1}(A_3)$ *is a homeomorphic copy of* $\tilde{M}_k^*$ *and for each closed subinterval* J *of* $[0, 1]\setminus A_3$ *the set* $\tilde{v}_k^{-1}(J)$ *is contained in a homeomorphic copy of* Σ *lying in* $\tilde{N}_k$.

(4.13) v_k *is a* λ*-projection from* N_k *onto* $[0, 1]$ *such that every component of the set* $v_k^{-1}(A_3)$ *is a homeomorphic copy of* M_k^* *and* $v_k | v^{-1}([0, 1]\setminus A_3)$ *is a homeomorphism.*

(4.14) *For each* n *the set* $E_n^k = \mathrm{cl}\big(v_k^{-1}(1/(n+2), 1/(n+1))\big) \cap \cap \mathrm{cl}\big(v_k^{-1}(1/(n+1), 1/n)\big)$ *is degenerate and* $E^k = \operatorname{Lim} E_n^k$ *is degenerate.*

(1.8), (4.4) and (4.5) imply that

(4.15) N_k *is an arc-like hereditarily decomposable continuum with* $\varepsilon(N_k) > 0$.

(4.16) $\tilde{N}_k$ *is an arc-like hereditarily decomposable continuum which is hereditarily nondivisible by points. Moreover, if* C *is a subcontinuum of* $\tilde{N}_k$ *such that* $\eta_k(C)$ *is an arc, then* C *is homeomorphic to a subcontinuum of* Σ.

C. Mappings onto N_k. We have

(4.17) *No subcontinuum of* $\tilde{M}_n$ *can be mapped onto* $N = v_k^{-1}[\frac{1}{2}, 1]$ *under an* $\varepsilon(N_k)$*-valued map (even for* $n = k$*).*

We proceed by the induction. Firstly suppose that F is an $\varepsilon(N_k)$-valued map from a subcontinuum K of $\tilde{M}_0$ onto N. According to (1.17) and (3.14) the set $\tilde{\mu}_0(K)$ is nondegenerate and $\tilde{\mu}_0(K) \cap A_2$ is nonempty. We have either $F(L) \cap M \neq \emptyset$ or $F(R) \cap M \neq \emptyset$ where $M = v_k^{-1}(\frac{1}{2}, 1]$, $L = \tilde{\mu}_0^{-1}[0, \frac{1}{2}) \cap K$ and $R = \tilde{\mu}_0^{-1}(\frac{1}{2}, 1] \cap K$. Since F is a surjection and F is upper semicontinuous, we obtain either $F(R) \cap S \neq \emptyset$ or $F(L) \cap S \neq \emptyset$ where $S = N\setminus \mathrm{cl}(M)$. By (3.14) and the symmetry we can assume that $F(L) \cap M \neq \emptyset$, $F(L) \cap S = \emptyset$, $F(R) \cap M = \emptyset$ and $F(R) \cap S \neq \emptyset$. In this case we obtain that

$\tilde{\mu}_0^{-1}(\frac{1}{2}) \subset K$ and $F(\mu_0^{-1}(\frac{1}{2})) \supset \mathrm{cl}(M) \setminus M \supset D$. This is a contradiction once again by (3.14).

Assume now, by the induction, that no subcontinuum of $\tilde{M}_n$ can be mapped onto N under an $\varepsilon(N_k)$-valued map and suppose that F is an $\varepsilon(N_k)$-valued map from a subcontinuum K of $\tilde{M}_{n+1}$ onto N. The induction assumption and the equality $\tilde{\mu}_{n+1}^{-1}(\frac{1}{2}) = \tilde{M}_n$ imply that $\tilde{\mu}_{n+1}(K)$ is nondegenerate and $\tilde{\mu}_{n+1}(K) \cap A_1 \neq \emptyset$. Since $K \setminus \tilde{\mu}_{n+1}^{-1}(A_1)$ has at most two components and it is dense in K, by the standard here arguments we obtain that the image of one of them is contained in the set $M = v_k^{-1}(\frac{1}{2}, 1]$ and the second is contained in the set $\mathrm{cl}(S)$ (the definition above in the proof). Those inclusions imply immediately a contradiction.

Furthermore,

(4.18) *Let φ be a λ-projection from a continuum X onto* $[0, 1]$ *and let L be a finite subset of* $[0, 1]$. *If for each closed sub-interval J of* $[0, 1] \setminus L$ *the set $\varphi^{-1}(J)$ is lying in a homeomorphic copy of Σ contained in K, then there is no $\varepsilon(N_k)$-valued map from K onto N_k.*

Suppose that F is an $\varepsilon(N_k)$-valued map from K onto N_k, C is a component of $N_k \setminus v_k^{-1}(A_3)$ and $y \in C$. Since F is a surjection we find $x \in K$ with $y \in F(x)$. The set $F(x)$ is path-connected and C is a path-connected subset of N_k (exactly C is a path- component of N_k); thus $F(x) \subset C$. Since C is open, F upper semi-continuous and the set $K \setminus \varphi^{-1}(L)$ is dense in K, we infer that $F(K \setminus \varphi^{-1}(L)) \cap C \neq \emptyset$.

Let P be a component of $K \setminus \varphi^{-1}(L)$ such that $F(P) \cap C \neq \emptyset$ and $F(P) \setminus C \neq \emptyset$. Then $F(P)$ contains a homeomorphic copy R of D ($D \subset M_k$) by (4.13). We can find a sequence $\{J_n\}$ of closed intervals in $[0, 1] \setminus A_3$ such that $P = \bigcup_{n=1}^{\infty} J_n$, $J_n \subset J_{n+1}$ and $\varphi^{-1}(J_n)$ is a homeomorphic copy of Σ in K for $n = 1, 2, \ldots$ Therefore $R \subset F(J_n)$ for some n. Since F is pseudomonotone (compare (1.17)) we find a continuum homeomorphic to a subcontinuum of Σ which can be mapped onto D (homeomorphic to R) under an $\varepsilon(D)$-valued map contrary to (3.13) (iv). Therefore $F(P) \subset C$.

Thereby the set $K \setminus \varphi^{-1}(L)$ has at least as many components as $N_k \setminus v_k^{-1}(A_3)$ does, a contradiction.

We will say that a continuum K is *of type* $\tilde{N}$ if there is a λ-projection $\tilde{v}_K$ from K onto $[0, 1]$ such that

(i) for each closed subinterval J of $[0, 1] \setminus A_3$ the set $\tilde{v}_K^{-1}(J)$ is lying in a homeomorphic copy of Σ contained in K;

(ii) $\operatorname{Li}_{n \to \infty} \tilde{v}_K^{-1}\left(\frac{1}{n+1}, \frac{1}{n}\right) \neq \emptyset$. (Li denotes a topological limes inferior).

We have (compare [15], Proposition 27)

(4.19) *If F is an $\varepsilon(N_k)$-valued map from a continuum K of type $\tilde{N}$ onto N_k, then for each positive integer n there are: a positive integer m and positive numbers δ_1 and δ_2 such that $\frac{1}{m+1} < \delta_1 < \frac{1}{m}$, $\frac{1}{n+1} < \delta_2 < \frac{1}{n}$ and the set $\operatorname{cl} \tilde{v}_K^{-1}\left(\frac{1}{m+1}, \delta_1\right]$ can be mapped onto $\operatorname{cl} v_k^{-1}\left(\frac{1}{n+1}, \delta_2\right]$ under an $\varepsilon(N_k)$-valued map. Moreover, $F(\tilde{v}_K^{-1}(0)) = v_k^{-1}(0)$.*

Proof. The same arguments as in the proof of (4.18) imply

(4.19.1) *The set $F(K \setminus \tilde{v}_K^{-1}(A_3))$ intersects every component of $N_k \setminus v_k^{-1}(A_3)$.*

(4.19.2) *If C is a component of $N_k \setminus v_k^{-1}(A_3)$, P is a component of $K \setminus \tilde{v}_K^{-1}(A_3)$ and $C \cap F(P) \neq \emptyset$, then $F(P) \subset C$.*

For each $t > 0$ the set $F(\tilde{v}_K^{-1}[t, 1] \setminus A_3)$ intersects only a finite number of components of $N_k \setminus v_k^{-1}(A_3)$ by (4.19.2). Therefore, by (4.19.1) and the upper semi-continuity of F we conclude that

(4.19.3) *For each $t > 0$ we have $F(\tilde{v}_K^{-1}[t, 1]) \subset v_k^{-1}(0, 1]$.*

We claim that

(4.19.4) $$F(\tilde{v}_K^{-1}(0)) = v_k^{-1}(0).$$

From (4.19.3) we obtain that $v_k^{-1}(0) \subset F(\tilde{v}_K^{-1}(0))$. Suppose that there is $x \in \tilde{v}_K^{-1}(0)$ such that the set $F(x)$ is contained in some path-component C of $N_k \setminus v_k^{-1}(A_3)$ (compare (4.19.2)). Since F is upper semi-continuous, we find a sequence $\{n_i\}$ such that $F\left(\tilde{v}_K^{-1}\left(\frac{1}{n_i+1}, \frac{1}{n_i}\right)\right) \subset C$ for $i = 1, 2, \ldots$ by (4.19.2). Since $\operatorname{Li}_{n\to\infty} \tilde{v}_K^{-1}\left(\frac{1}{n+1}, \frac{1}{n}\right) \neq \emptyset$, we find $t > 0$ such that $F((K \setminus \tilde{v}_K^{-1}(A_3)) \cap \tilde{v}_K^{-1}[0, t]) \subset \operatorname{cl} C$. This is impossible by (4.19.1). Thereby $F(\tilde{v}_K^{-1}(0)) \subset v_k^{-1}(0)$, i.e. (4.19.4) holds.

Fix $n = 1, 2, \ldots$ It follows from (4.19.1) and (4.19.4) that there is $m = 1, 2, \ldots$ such that m is maximal with respect to the property that the set $F\left(\tilde{v}_K^{-1}\left(\frac{1}{m+1}, \frac{1}{m}\right)\right) \cap v_k^{-1}\left(\frac{1}{n+1}, \frac{1}{n}\right)$ is nonempty. Then, by (4.19.2), the inclusion $F\left(\tilde{v}_K^{-1}\left(\frac{1}{m+1}\right)\right) \subset v_k^{-1}\left(\frac{1}{n+1}, \frac{1}{n}\right)$ holds. The choice of m implies that $F\left(\tilde{v}_K^{-1}\left(\frac{1}{m+1}\right)\right) \cap v_k^{-1}\left(\frac{1}{n+1}\right) \neq \emptyset$ and $F\left(\tilde{v}_K^{-1}\left[0, \frac{1}{m+1}\right)\right) \cap v_k^{-1}\left(\frac{1}{n+1}, 1\right] = \emptyset$. In particular, for each $x \in \operatorname{cl}\left(\tilde{v}_K^{-1}\left(\frac{1}{m+1}, \delta\right]\right)$ we have that the set $G(x)$ is nonempty, where $G(x) = F(x) \cap \left(\operatorname{cl}\left(v_k^{-1}\left(\frac{1}{n+1}, \frac{1}{n}\right)\right) \setminus v_k^{-1}\left(\frac{1}{n}\right)\right) \cup$

$\cup\, v_k^{-1}\left(\frac{1}{n+1}, \frac{1}{n}\right)$ and $\frac{1}{m+1} < \delta < \frac{1}{m}$. G is an $\varepsilon(N_k)$-valued map and now it is no problem to find δ_1 and δ_2. The proof of (4.19) is complete.

From (3.16), (4.6), (4.17), (4.18) and (4.19) we conclude

(4.20) THEOREM. *No subcontinuum of $\tilde{N}_n$ can be mapped onto N_k under an $\varepsilon(N_k)$-valued map for $k \neq n$ and $k, n > 0$.*

D. The slight modifications of $\tilde{N}_k$. Applying (1.22), (1.23) and (3.2) and going through the definitions of D, M_k and N_k we obtain immediately

(4.21) *For $k = 1, 2, \ldots$ the continuum N_k is admissible.*

According to (1.21) and (4.21) we conclude

(4.22) *For $k = 1, 2, \ldots$ there are a countable-to-one mapping β_k from N_k onto $[0, 1]$ and a finite set E_k in $[0, 1]$ such that:*

(i) *if C is a subcontinuum of N_k with* $\operatorname{diam} C \geqslant \varepsilon(N_k)$, *then* $\operatorname{card}(\beta_k(C) \cap E_k) \geqslant 3$,

(ii) *if C is a subcontinuum of $N_k \setminus \beta_k^{-1}(E_k)$, then C is an arc.*

Now, we contract every component of the set $\eta_k^{-1}\beta_k^{-1}(E_k)$ to a point by a mapping ψ_k and we obtain the space $\tilde{N}_k'$ from $\tilde{N}_k$. Next we take an arbitrary pseudosuspension N_k' of $\tilde{\Sigma} \times \beta_k^{-1}(E_k)$ over N_k at $\beta_k^{-1}(E_k)$. The mapping $\sigma\colon \tilde{\Sigma} \to D$ gives us a monotone mapping φ_k'' from N_k' onto some pseudosuspension N_k^* of $D \times \beta_k^{-1}(E_k)$ over N_k at $\beta_k^{-1}(E_k)$. Applying (1.15) we find a pseudosuspension $\tilde{N}_k^*$ of $\tilde{\Sigma} \times \psi_k(\eta_k^{-1}\beta_k^{-1}(E_k))$ over $\tilde{N}_k'$ at $\psi_k(\eta_k^{-1}\beta_k^{-1}(E_k))$ with a monotone mapping φ_k' from $\tilde{N}_k^*$ onto N_k'. Put $\eta_k^* = \varphi_k'\varphi_k''$, denote the natural projections from $\tilde{N}_k^*$ onto $\tilde{N}_k'$ and from N_k^* onto N_k by $\tilde{\varphi}_k$ and φ_k, respectively and set $\delta_k = \eta_k^*\varphi_k\beta_k$. Then

(4.23) *The following diagram commutes*

$$\begin{tikzcd}
\tilde{N}_k^* \arrow[rr, "\delta_k"] \arrow[dr, "\eta_k^*"] \arrow[dd, "\tilde{\varphi}_k"'] & & {[0,1] \supset E_k} \\
& N_k^* \arrow[d, "\varphi_k"] & \\
\tilde{N}_k' \arrow[r, "\eta_k\psi_k^{-1}"] \arrow[d, "\psi_k"'] & N_k \arrow[uur, "\beta_k"'] & \\
\tilde{N}_k \arrow[ur, "\eta_k"'] & &
\end{tikzcd}$$

(4.24) *The mappings η_k^*, $\eta_k\psi_k^{-1}$, η_k and ψ_k are monotone.*

(4.25) *The mappings $\tilde{\varphi}_k$ and φ_k are atomic and the mapping $\tilde{\varphi}_k|\,\delta_k^{-1}(E_k)$ is an open map from $\delta_k^{-1}(E_k)$ onto $\tilde{\varphi}_k(\delta_k^{-1}(E_k))$.*

The same arguments as for (4.15) and (4.16) show

(4.26) $\varepsilon(N_k^*) > 0$ *and* $\tilde{N}_k^*$ *is an hereditarily decomposable arc-like continuum which is hereditarily nondivisible by points.*

Moreover,

(4.27) *If C is a subcontinuum of $\tilde{N}_k^* \setminus \delta_k^{-1}(E_k)$, then C is homeomorphic to a subcontinuum of Σ.*

In fact, let C be a subcontinuum of $\tilde{N}_k^* \setminus \delta_k^{-1}(E_k)$. Then $\psi_k^{-1}\tilde{\varphi}_k|C$ is a homeomorphism and the set $\eta_k(\psi_k^{-1}\tilde{\varphi}_k(C))$ is an arc by (4.22) (ii). By (4.16), we infer that the continuum $\psi_k^{-1}\tilde{\varphi}_k(C)$ is homeomorphic to a subcontinuum of Σ.

It follows from (1.6), (4.25) and (4.27) that

(4.28) *If C is a subcontinuum of $\tilde{N}_k^*$ and K is a component of $\delta_k^{-1}(E_k)$ such that $K \subset C \neq K$, then there is a sequence $\{K_n\}$ of subcontinua of $C \setminus \delta_k^{-1}(E_k)$ such that $\operatorname{Lim}_{n\to\infty} K_n = K$ and for $n = 1, 2, \ldots$ the continuum K_n is homeomorphic to a subcontinuum of Σ.*

We can assume that if C is a continuum in N_k^* with $\operatorname{diam} C < \varepsilon(N_k^*)$, then $\operatorname{diam} \varphi_k(C) < \varepsilon(N_k)$.

E. The collections $\mathcal{N}(n)$. Now, consider the collection $\mathcal{N}(n)$ of all continua which can be obtained from a nondegenerate subcontinua C of $\tilde{N}_n^*$ by the replacing every member B of a finite union of closed disjoint zerodimensional subsets of C and such that B is contained in a homeomorphic copy of Σ lying either in $C \setminus \delta_n^{-1}(E_n)$ or in $\delta_n^{-1}(E_n) \cap C$ by a product $B \times K_B$, where K_B is an arbitrary continuum which depends only from B (the replacing here means always a suitable pseudosuspension).

The crucial property of the above modification $\tilde{N}_n^*$ of $\tilde{N}_n$ is the following

(4.29) THEOREM. *Any $N \in \mathcal{N}(n)$ cannot be mapped onto N_k^* under an $\varepsilon(N_k^*)$-valued map provided $k \neq n$.*

Proof. Suppose ω is a natural projection from N onto $\omega(N) \in C(\tilde{N}_n)$. Since $N \in \mathcal{N}(n)$ there is a finite collection $S_1, S_2, \ldots, S_m$ of homeomorphic copies of Σ lying either in $\omega(N) \setminus \delta_n^{-1}(E_n)$ or in $\delta_n^{-1}(E_n) \cap C$ and such that the mapping $\omega_0 = \omega|\ \omega^{-1}(\omega(N) \setminus (S_1 \cup S_2 \cup \ldots \cup S_m))$ is a homeomorphism. Suppose that F is an $\varepsilon(N_k^*)$-valued map from N onto N_k^*. Consider two cases

(a) There is $e \in E_n$ such that $\omega(N) \subset \delta_n^{-1}(e)$. Consider the mapping G: $\omega(N) \to N_k$ from $\omega(N)$ into N_k defined by $G(x) = \varphi_k F \omega^{-1}(x)$. We will show that G is an $\varepsilon(N_k)$-valued map.

Since G is an upper semi-continuous and continuum-valued map, we can assume, on the contrary that $\operatorname{diam} G(x) \geqslant \varepsilon(N_k)$ for some $x \in \omega(N)$. From (4.22) (i) we conclude that there are points y, $y' \in \omega^{-1}(x)$ such that $\beta_k \varphi_k(F(y)) < e' < \beta_k \varphi_k(F(y'))$ for some $e' \in E_k$. It follows from (1.6) that there is a sequence $\{K_i\}$ of subcontinua of N such that $\operatorname{Lim}_{i\to\infty} K_i = \omega^{-1}(x)$, and for $i = 1, 2, \ldots$ K_i is homeomorphic to a subcontinuum of Σ. Therefore there

is a positive integer i_0 such that $F(K_i) \cap \varphi_k^{-1} \beta_k^{-1}(e') \neq \emptyset$ for $i \geqslant i_0$. Since φ_k is atomic, we conclude that $F(K_{i_0})$ contains a homeomorphic copy of D. This is impossible by (3.13) (iv).

According to (3.14), the mapping G is not a surjection, thus F is not a surjection, a contradiction.

(b) For each $e \in E_n$, $\delta_n^{-1}(e) \subset \omega(N) \neq \delta_n^{-1}(e)$ provided $\delta_n^{-1}(e) \cap \omega(N) \neq \emptyset$. Put $W = \tilde{\varphi}_n \omega(N)$ and $G(x) = \varphi_k F \omega^{-1} \varphi_n^{-1}(x)$ for $x \in W$. According to (4.20) it suffices to show that the mapping G is an $\varepsilon(N_k)$-valued map. The mapping G is upper semi-continuous and continuum-valued. We will prove that $\operatorname{diam} G(x) < \varepsilon(N_k)$ for every $x \in K$.

If $e \in E_n$, $\delta_n^{-1}(e) \cap \omega(N) \neq \emptyset$, then $\delta_n^{-1}(e) \subset \omega(N) \neq \delta_n^{-1}(e)$. Therefore there is a sequence $\{K_i^e\}$ of subcontinua of $C \setminus \delta_n^{-1}(E_n)$ such that $\operatorname{Lim}_{i \to \infty} K_i^e = \delta_n^{-1}(e)$ and for $i = 1, 2, \ldots$ the continuum K_i^e is homeomorphic to a subcontinuum of Σ by (4.28). We may assume that $K_i^e \cap (S_1 \cup \ldots \cup S_m) = \emptyset$. According to (1.6′) we have $\operatorname{Lim}_{i \to \infty} \omega^{-1}(K_i^e) = \omega^{-1}(\delta_n^{-1}(e))$. Since ω_0 is a homeomorphism we infer that continua $\omega^{-1}(K_i^e)$ are homeomorphic to subcontinua of Σ.

If $S_i \cap \delta_n^{-1}(E_n) = \emptyset$, then for every $x \in S_i$ we find a sequence $\{K_i^x\}$ of subcontinua of S_i such that $\operatorname{Lim}_{i \to \infty} \omega^{-1}(K_i^x) = \omega^{-1}(x)$ and for $i = 1, 2, \ldots$ the continuum $\omega^{-1}(K_i^x)$ is homeomorphic to a subcontinuum of Σ.

Therefore, for every $x \in W$, we have found a sequence of subcontinua $\{K_i^x\}$ of N converging to $\omega^{-1} \varphi_n^{-1}(x)$ and such that continua K_i^x can be embedded into Σ.

Proceeding now in the same manner as in the case (a), one can show that G is an $\varepsilon(N_k)$-valued map. The proof of Theorem (4.29) is complete.

5. Zerodimensional sets in $\tilde{N}_k^*$

A. Nice sequences. A sequence $\mathscr{B} = \{B_1, B_2, \ldots\}$ of closed subsets of X will be called a *nice* sequence in X provided

(i) elements of $\mathscr{B}$ are zerodimensional and disjoint,

(ii) $\lim_{i \to \infty} \operatorname{diam} B_i = 0$,

(iii) for every positive integer n and for every nondegenerate subcontinuum K of X there is a positive integer $m \geqslant n$ such that $B_m \cap K \neq \emptyset$,

(iv) for every positive integer n, the set B_n is contained in a homeomorphic copy of Σ lying in X.

Since Σ is an onedimensional continuum, from (1.13) we obtain

(5.1) *Σ contains a nice sequence.*

Similarly, we have

(5.2) *If A is a closed zerodimensional subset of* $[0, 1]$, *then the set* $\alpha^{-1}([0, 1]\setminus A)$ *contains a nice sequence.*

Let $\mathscr{B}$ be a nice sequence in Σ. For each $B\in\mathscr{B}$ we consider $G_B = B\setminus\alpha^{-1}(A)$. Since $\alpha(G_B)$ is an open subset of the compact zerodimensional set B we find a countable collection $\mathscr{B}_B$ of compact zerodimensional sets which are pairwise disjoint and the union of elements of $\mathscr{B}_B$ gives G_B. Then $\bigcup\{\mathscr{B}_B\colon B\in\mathscr{B}\}$ ordered in a sequence is a nice sequence.

Now, applying (5.2) and going through the definitions of $\tilde{\Sigma}$, $\tilde{M}_k$ and $\tilde{N}_n$, we obtain that

(5.3) *For each $k = 1, 2, \ldots$ the continuum $\tilde{N}_k$ contains a nice sequence.*

Moreover,

(5.4) *For each $k = 1, 2, \ldots$ the continuum $\tilde{N}_k^*$ contains a nice sequence $\mathscr{B}_k = \{B_1^k, B_2^k, \ldots\}$ such that every B_i^k is contained in a homeomorphic copy Σ_i^k of Σ in $\tilde{N}_k^*$ such that either $\delta_k(\Sigma_i^k)$ is a point in E_k or $\Sigma_i^k\cap\delta_k^{-1}(E_k) = \emptyset$.*

In fact, let $\mathscr{B}$ be a nice sequence in $\tilde{N}_k$ (compare (5.3)). For each $B\in\mathscr{B}$ we consider $G_B = \tilde{\varphi}_k^{-1}\psi_k(B)\setminus\delta_k^{-1}(E_k)$ and a homeomorphic copy Σ_B of Σ in $\tilde{N}_k$ containing B. The set $\tilde{\varphi}_k^{-1}\psi_k(\Sigma_B)\setminus\delta_k^{-1}(E_k)$ has a countable collection $\mathscr{C}(B)$ of components. Since for each $C\in\mathscr{C}(B)$ the set $\psi_k^{-1}\tilde{\varphi}_k(C\cap G_B)$ is an open subset of the compact zerodimensional set B we find a countable collection $\mathscr{B}_{B,C}$ of compact zerodimensional sets which are pairwise disjoint and such that the union of elements of $\mathscr{B}_{B,C}$ gives $C\cap G_B$ for $C\in\mathscr{C}(B)$. Next, we add a nice sequence in $\tilde{\Sigma}\times\psi_k(\eta_k^{-1}\beta_k^{-1}(E_k))$ (the set $\psi_k(\eta_k^{-1}\beta_k^{-1}(E_k))$ is countable) to the union $\bigcup\{\mathscr{B}_{B,C}\colon B\in\mathscr{B},\ C\in\mathscr{C}(B)\}$, then we obtain a nice sequence in $\tilde{N}_k^*$ for which we were looking.

B. The dispersion of zerodimensional sets. Let B be an arbitrary compact zerodimensional set. A sequence $S = (S_1, S_2, \ldots)$ of finite collections S_n of closed and open subsets of B will be called a *disperion* of B provided

(i) members of S_n are pairwise disjoint,

(ii) $S_1 = \{B\}$,

(iii) every member of S_n is a union of some members of S_{n+1},

(iv) if $x\in B_n\in S_n$ for $n = 1, 2, \ldots$, then $\{x\} = \bigcap_{n=1}^{\infty} B_n$.

6. The main construction

Now, we will prove

(6.1) Main Theorem. *There exists a Cook continuum X which is an hereditarily decomposable arc-like continuum and which is hereditarily nondivisible by points.*

Proof. The continuum X will be obtained as an inverse limit of an inverse sequence $\{X_n, f_n\}$ where f_n is a mapping from X_{n+1} onto X_n. The continuum X_{n+1} we will construct as a suitable pseudosuspension of $(\tilde{N}^*_{k_1} \times B_1) \cup \ldots \cup (\tilde{N}^*_{k_m} \times B_m)$ over X_n at $B_1 \cup \ldots \cup B_n$ where B_i are closed disjoint and zerodimensional subsets of X_n. The mapping f_n will be a natural projection from X_{n+1} onto X_n.

Intuitively, we do changes in X_n to obtain X_{n+1} as follows. Let $X_1 = \tilde{N}^*_1$ and let B be a zerodimensional set in X_1 which is "nicely" embedded in X_1. We replace every point from B by a copy of $\tilde{N}^*_m$ where $m \neq 1$ by the taking a suitable pseudosuspension. So obtained continuum is X_2. To obtain X_3 similarly we do changes (by replacing points by copies of $\tilde{N}^*_w$ with $w \notin \{1, m\}$) either on a closed zerodimensional set (contained in some homeomorphic copy of Σ) which is disjoint with the set in which we have done changes previously or on a set B' which is contained in the product $B \times \tilde{N}^*_m$ which was taken to obtain X_2 previously and B' is such that B' is a product of some closed and open "small" set in B with a closed zerodimensional set lying in a homeomorphic copy of Σ contained in $\tilde{N}^*_m$; and so on.

We construct continuum X_{n+1} by changing previously constructed continuum X_n and we still remember that the changes should be done only on a closed zerodimensional set which is either disjoint with sets in which we have done changes previously (and is contained in a homeomorphic copy of Σ) or it is contained as a product of a closed and open set with "nicely" embedded closed zerodimensional set in every previously done change. The finite collection of closed zerodimensional sets in which we shall do changes to obtain X_{n+1} from X_n should be "better" packed in X_n than changes in X_{n-1} which we have done to obtain X_n.

Now we are going to give a precise description of our construction. According to (5.4) for each $k = 1, 2, \ldots$ we can take a nice sequence $\mathscr{B}_k = \{B^k_1, B^k_2, \ldots\}$ in $\tilde{N}^*_k$ such that every B^k_i is contained in a homeomorphic copy Σ^k_i of Σ in $\tilde{N}^*_k$ such that either $\delta_k(\Sigma^k_i)$ is a point in E_k or $\Sigma^k_i \cap \delta_k^{-1}(E_k) = \emptyset$. For $i, k = 1, 2, \ldots$ fix an arbitrary dispersion $S^k_i = (S^k_{i,1}, S^k_{i,2}, \ldots)$ of B^k_i (see § 5, B).

If $X_1, X_2, \ldots, X_n$ and $f_i\colon X_{i+1} \to X_i$ for $i = 1, \ldots, n-1$ will be constructed, then we denote

$$f_{i,j} = f_j \ldots f_i \quad \text{for } 1 \leqslant j < i \leqslant n,$$

$$H^k_m = \{x \in X_m\colon f_m^{-1}(x) \text{ is homeomorphic to } \tilde{N}^*_k\},$$

$$H_m = \{x \in X_m\colon f_m^{-1}(x) \text{ is nondegenerate}\},$$

$$\tilde{H}_m = \{k\colon H^k_m \neq \emptyset\} \quad \text{for } m = 1, \ldots, n-1.$$

We construct f_n and X_n by the induction and in such a manner that

(6.1.1) *X_n is an hereditarily decomposable arc-like continuum which is hereditarily nondivisible by points,*

(6.1.2) *f_n is an atomic map from X_{n+1} onto X_n,*

(6.1.3) *sets H_n^k are closed and zerodimensional, and $H_n = \bigcup H_n^k$,*

(6.1.4) *$\tilde{H}_n$ is finite and $\tilde{H}_n \cap \tilde{H}_m = \emptyset$ for $n \neq m$,*

(6.1.5) *$f_n^{-1}(H_n^k) = \tilde{N}_k^* \times H_n^k$ provided $H_n^k \neq \emptyset$,*

(6.1.6) *if $r \in \tilde{H}_j$, and $j < n$, then*

$$\bigcup_{w=1}^{n} S_{w,n}^r = \bigcup \{\pi_j^r f_{n-1,j}(H_n^k)\colon H_j^r \cap f_{n-1,j}(H_n^k) \neq \emptyset \text{ and } k \in \tilde{H}_n\}$$

where π_j^r is a projection from $\tilde{N}_r^* \times H_j^r$ onto $\tilde{N}_r^*$,

(6.1.7) *$f_n\big(\bigcup \{H_{n+1}^s\colon s \in \tilde{H}_{n+1}$ and $H_{n+1}^s \cap (B \times H_n^k) \neq \emptyset\}\big) = H_n^k$ for $B \in S_{i,n+1}^k$ and $1 \leqslant i \leqslant n+1$.*

Let X be the inverse limit of the inverse sequence $\{X_n, f_n\}$ and denote the projection from X onto X_n by g_n. By (1.4) and (6.1.2), the mappings g_n are atomic. Proposition (1.9) and (6.1.1) imply that X is an arc-like continuum. Since X_n are hereditarily decomposable (hereditarily nondivisible by points) and since mappings g_n are atomic, we infer that X has the same property.

It remains to prove that X is a Cook continuum. Suppose that K and L are nondegenerate disjoint subcontinua of X and f is a continuous mapping from K onto L. There are positive integers n and m and a homeomorphism h' such that $h'(\tilde{N}_m^*) \subset g_n(L)$. The set $g_n^{-1}(h'(\tilde{N}_m^*))$ is a subcontinuum of L, because g_n is atomic. Since any mapping onto an arc-like continuum is weakly confluent (compare (1.10)), we infer that there is a continuum $C \subset K$ with the property $g_n^{-1}(h'(\tilde{N}_m^*)) = f(C)$. Observe that there is an integer k_0 such that for each $k \geqslant k_0$ the set $g_k(C)$ belongs to $\mathcal{N}(m)$ (see § 4.E for the definition). Let h be a monotone mapping from $\tilde{N}_m^*$ onto N_m^*. Let δ be a positive number such that if $x, x' \in C$ and $\varrho(x, x') < \delta$ then $\varrho(hg_n f(x), hg_n f(x')) < \varepsilon/2$ where $\varepsilon = \varepsilon(N_m^*)$. There is a positive integer k_1 greater than k_0 such that g_{k_1} is a δ-mapping from X onto X_{k_1}. The set-valued map $F(x) = hg_n f(g_{k_1}^{-1}(x))$ is an ε-valued map from $g_{k_1}(C)$ (which belongs to $\mathcal{N}(m)$) onto N_m^*. This is impossible by Theorem (4.29). The proof of (6.1) is complete.

It follows from (1.18) and (6.1)

(6.2) Corollary. *There exists an arc-like continuum X such that if a continuous mapping maps a subcontinuum K of X onto $f(K) \subset X$, then either f is the identity on K or $f(K)$ is degenerate.*

A continuum X, the existence of which follows from Corollary (6.2) has the same property as the continuum M_2 in Theorem 11 from [6], but M_2 is not planable; our continuum X is planable by (1.17).

Moreover, X has properties of continua constructed in [1], [2], [3] and [15]. Corollary (6.2) solves Problem 33 in [15]. One can observe that similarly to Cook's results from [6] it answers positively the questions asked by J. de Groot in [7] and R. D. Anderson (compare [6]).

Finally, every continuum having properties mentioned in Corollary (6.2) (more exactly, every continuum X such that every subcontinuum of X is a Cook continuum) is hereditarily nondivisible by points.

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