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Topological and approximation methods of degree theory of set-valued maps

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Wojciech Kryszewski
Instytut Matematyki
Uniwersytet Mikołaja Kopernika
Chopina 12/18
87-100 Toruń
Poland
E-mail: wkrysz@mat.uni.torun.pl

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Summary

The theory of topological degree of set-valued maps determined by morphisms, i.e. maps with values which are continuous images of almost acyclic sets, is presented, together with some of its applications.

In the first part, morphisms defined on finite-dimensional Euclidean manifolds are considered and the integer-valued degree is introduced by means of the Eilenberg–Montgomery–Górniewicz method based on the Vietoris–Begle–Sklyarenko theorem and using the approach of Dold in terms of the Alexander–Spanier cohomology theory.

In the second part, the degree theory is extended to a wide class of noncompact set-valued maps determined by morphisms acting in infinite-dimensional spaces. This new class, of the so-called A-morphisms, generalizing the compact set-valued vector fields, is studied from the general approximation viewpoint.

Topological and approximation methods are also used in the context of another class of set-valued maps whose values satisfy some geometrical, rather than topological, conditions. Moreover, the degree theory is extended to the class of Petryshyn's A-proper maps with nonconvex values. The class of A-proper maps contains many different types of maps considered elsewhere.

The degree theory is compared with the notion of essentiality, another useful tool in nonlinear analysis.

The paper proposes several results extending the well-known theorems on single-valued maps, such as the Borsuk antipodal theorem, the Bourgin–Yang theorem, nonlinear alternative, invariance of domain and others.

Dedicated to my Parents

Introduction

The great interest in the theory of set-valued mappings is caused by many reasons. This intensely developing branch of mathematics has a lot in common with topology [73], [7], [41], nonlinear analysis [88], [26], [5], the theory of functions and ordinary differential equations [27], [94], and others. Many results of this theory have found interesting applications in game theory (the results of J. von Neumann, Ky Fan), mathematical economics and control theory.

A substantial part of the theory of set-valued maps is the study of the existence of fixed points and the solvability of generalized equations involving these maps.

In this paper we present some methods allowing us to investigate those questions. The most important approach, which can be called topological or, more precisely, homotopic, is based on the homotopy properties of a map. As in the single-valued case (see e.g. [26]), it can be shown, under appropriate assumptions, that a map which is homotopic to a sufficiently regular one (called essential) has a fixed point, or the equation involving this map admits a solution.

However, one should have a convenient tool for deciding whether a given map (or one that a given map can be homotopically deformed to) is essential. Such a tool is provided by a homotopy invariant called the topological degree.

The degree of a single-valued continuous map allows several descriptions. Some of them, having an intrinsically geometric nature or being purely analytic (see [79]), have a very clear geometric meaning.

In the set-valued case, one is forced to apply different techniques. In the first instance, sometimes there are sufficiently close single-valued approximations (understood in an appropriate sense) of a map whose topological behaviour reflects the properties of the map. Thus approximation techniques may interact with topological methods (see [21], [46], [47], [68]). On the other hand, many set-valued maps can be studied only by methods arising from algebraic topology (see [42]). For example, it is not clear at all whether any map with acyclic values admits single-valued approximations. Therefore only the homological apparatus seems to be applicable for maps with geometrically nontrivial values. However, since homological methods are convenient for maps of finite dimension, one has to employ the approximation approach once again. Namely, one approximates a given map of infinite-dimensional spaces by finite-dimensional (single- or set-valued) maps which, in turn, can be studied by homological tools.

We present a self-contained approach combining methods of algebraic topology with approximation techniques. We give a further and systematic development of the theory of morphisms introduced by Górniewicz and Granas in [44]. This theory provides a well-designed tool in studying the general coincidence problems as well as fixed problems of Lefschetz type for not necessarily acyclic set-valued maps. It turns out that this theory provides also an appropriate setting for the treatment of the topological questions mentioned above, i.e., for example, homotopy properties of set-valued maps, extension problems and Borsuk type theorems. Extended in this manner, the theory of morphisms embraces numerous fixed point results previously obtained by several authors (cf. [30], [49], [51], [20], [44], [45], [42]). Our approach is based on a homological method initiated by Eilenberg and Montgomery [30], and essentially developed by Górniewicz [42], which relies on the study of the homological properties of a set-valued map with sufficiently regular values by the use of the Vietoris–Begle theorem (see [102]). We also apply some ideas of Dold [24] used in the single-valued context.

Next we extend the theory of morphisms to a wide new class of maps of infinite-dimensional spaces (studied already in the single-valued case: [62], [63], [69], 106]) by an appropriate use of the approximation techniques.

The paper is organized as follows:

• In the first chapter we recall the most important definitions and properties of set-valued maps; we define and study Vietoris maps and discuss the notion of a morphism which is a modified version of the notion introduced in [44]. Moreover, we establish the properties of essential and inessential morphisms in a general abstract setting.

• In the second chapter we study the cohomological properties of morphisms, introduce the notion of the fundamental cohomology class of a compact connected subset of an oriented finite-dimensional Euclidean manifold; next, using these results, we present a definition of the topological degree of a morphism defined on a manifold over a free subset and study its properties. We generalize some results of Borsuk type concerning the parity of the degree and prove an extension of the Bourgin–Yang theorem. Moreover, several applications such as the nonlinear alternative and invariance of domain are given.

• The third chapter is devoted to approximation-admissible maps (morphisms). We give definitions of some auxiliary objects and study the general approximation properties of A-maps.

• In the fourth chapter we present an approximation degree theory of A-morphisms together with some applications and examples.

• The last (fifth) chapter is devoted to some other approximation methods in the theory of set-valued degree theory. We define a class of maps with values satisfying a condition of more geometric nature which generalizes in a simple way the class of convex-valued maps, and show the existence of arbitrarily close homotopy approximations. The constructed degree theory enables us to prove other versions of the Borsuk and Bourgin–Yang theorems. Finally, we discuss the so-called A-proper maps (in the sense of Petryshyn) with not necessarily convex values.

Preliminaries

We use only standard set-theoretical notations. For a set X, $P(X)$ denotes the family of all nonempty subsets of X. By the Latin letters $f, g, h, \ldots$ we denote only single-valued maps, by the Greek letters $\phi, \psi, \chi, \ldots, \Phi, \Psi, \ldots$ we denote set-valued maps, while α, β denote also single-valued maps. If X is a set, $A \subset X$, then $i_A : A \to X$ denotes the (identity) embedding. If Y is another set and $f : X \to Y$, then $f|A$ denotes the restriction of f to A; if $B \subset Y$ then f_B is the restriction $f|f^{-1}(B)$. If $g : A \to Y$, then by an extension of g onto X we mean a map $f : X \to Y$ such that $f|A = g$. The symbol $f : (X, A) \to (Y, B)$ denotes a map $f : X \to Y$ such that $f(A) \subset B$.

If $X = \bigtimes_{j\in J} X_j$ is the Cartesian product of a family $\{X_j\}_{j\in J}$, then the projection onto X_j of X is denoted by pr_j (or pr_{X_j}).

We consider only Hausdorff topological spaces. As a rule, we consider only one topology on a space. In particular, this concerns operations on spaces. That is, on a subspace, Cartesian product, topological sum, quotient space we study the natural topology. If X is a space, $A \subset Y \subset X$, then the symbols $\mathrm{cl}_Y A$, $\mathrm{int}_Y A$ and $\mathrm{bd}_Y A$ denote the closure, the interior and the boundary of A with respect to the topology of a subspace in Y. If $Y = X$, then we omit the subscript Y. If X and Y are homeomorphic, then we write $X \approx_{\mathrm{top}} Y$.

By a Euclidean m-dimensional manifold we understand a topological space satisfying the second countability axiom and such that any of its points has a neighbourhood homeomorphic to $\mathbb{R}^m$. Here $\mathbb{R}^m$ denotes the m-dimensional Euclidean space. The scalar product of $x, y \in \mathbb{R}^m$ is denoted by $(x|y)$ and the norm of $x \in \mathbb{R}^m$ by $|x| = (x|x)^{1/2}$. Moreover, we use the following symbols: $N^m_\varepsilon(x) = N^m(x, \varepsilon) = \{y \in \mathbb{R}^m : |x - y| < \varepsilon\}$, $B^m_\varepsilon(x) = B^m(x, \varepsilon) = \mathrm{cl}\, N^m_\varepsilon(x) = \{y \in \mathbb{R}^m : |x - y| \le \varepsilon\}$ and $S^{m-1}(x, \varepsilon) = S^{m-1}_\varepsilon(x) = \mathrm{bd}\, B^m(x)$. We write N^m, B^m and S^{m-1} instead of $N^m(0, 1), B^m(0, 1)$ and $S^{m-1}(0, 1)$. The unit interval $[0, 1] \subset \mathbb{R}^1$ is denoted by I. If X is a topological space, then, for $t \in I$, $i_t : X \to X \times I$ is the map given by $i_t(x) = (x, t)$ for $x \in X$, $t \in I$.

If X is a uniform space, then its uniform structure is denoted by $\mathrm{unf}\, X$ and elements of $\mathrm{unf}\, X$ are called *vicinities*. If $U, V \in \mathrm{unf}\, X$, $A \subset X$, then $U(A) = \{y \in X : (x, y) \in U$ for some $x \in A\}$, $U \circ V = \{(x, z) \in X \times X : (x, y) \in V,\ (y, z) \in U$ for some $y \in X\}$. In particular, if X is a metric space with metric d, and if $\varepsilon > 0$, then $N(A, \varepsilon) = N_\varepsilon(A) = \{x \in X : d(x, A) = \inf_{a\in A} d(x, a) < \varepsilon\}$ and $B(A, \varepsilon) = B_\varepsilon(A) = \{x \in X : d(x, A) \le \varepsilon\}$.

The family of F_σ-subsets of a topological space X is denoted by $F_\sigma(X)$; by

$C(X)$, $K(X)$ we denote the families of all closed and all compact subsets of X, respectively.

The category of topological spaces and continuous maps of these spaces is denoted by TOP and the category of pairs of spaces is denoted by TOP^2.

A partially ordered set $(T, \leq)$ is called *directed* if for any $t_1, t_2 \in T$ there is $t_0 \in T$ such that $t_1, t_2 \leq t_0$.

We consider only rings which are principal ideal domains, i.e. commutative rings with unit whose all ideals are principal. The direct sum of R-modules $\{M_j\}_{j\in J}$ is denoted by $\bigoplus_{j\in J} M_j$ and the direct product by $\prod_{j\in J} M_j$. If R-modules M, N are isomorphic, then we write $M \simeq N$.

We consider several (co)homology theories. By the symbols $H_*, H_s^*, H^*, \widehat{H}^*$ we denote singular homology, singular cohomology, Alexander–Spanier cohomology and Čech cohomology, respectively. By $\widetilde{H}$ we denote reduced (co)homology.

If E is a vector space, then $\dim E$ denotes the (algebraic) dimension of E. If $A \subset E$, then $\mathrm{span}(A)$ denotes the set of all linear combinations of elements of A. If E is a metrizable locally convex topological vector space, then E admits a translation-invariant metric d compatible with the topological and convex structure of E, i.e. d generates the topology of E and balls (with respect to d) are convex. By a Fréchet space we understand a complete metric locally convex topological vector space. If A is a compact subset of a Fréchet space, then the closed convex envelope $\mathrm{clconv}(A)$ of A is compact. If E is a locally convex space, then the space of all continuous linear forms on E is denoted by E'. If $x \in E$, $f \in E'$, then $(f, x) = f(x)$. Any topological vector space E is a uniform space. The family of neighbourhoods of 0 in E may be identified with a uniform structure of E. For a neighbourhood U of 0 in E, we then have $U(A) = U + A$ where $A \subset E$. Moreover, $C_V(E)$ denotes the family of closed convex nonempty subsets of E.

Other notation:

- $\mathbb{Z}$ — the set of integers;
- $\mathbb{Z}_2$ — the field of integers modulo 2;
- $\mathbb{C}$ — the field of complex numbers;
- $L^\infty(I, \mathbb{R}^m)$ — the space of essentially bounded functions (with respect to the Lebesgue measure);
- $C(X, Y)$ — the space of continuous mappings between topological spaces X, Y.

A. Elements of homology theory

1. Products. We use some (co)homology theories as presented e.g. in [104] with coefficients in some R-module.

(1.1) Let X be a topological space. If $A_i \subset X$, $i = 1, 2$, and a pair $\{A_1, A_2\}$ is excisive w.r.t. H_*, then there are graded homomorphisms $\alpha_*, \beta_*, \Delta_*$ such that the Mayer–Vietoris sequence

$$\ldots \to H_n(X, A_1 \cap A_2) \xrightarrow{\alpha_*} H_n(X, A_1) \oplus H_n(X, A_2) \xrightarrow{\beta_*} H_n(X, A_1 \cup A_2) \xrightarrow{\Delta_*} H_{n-1}(X, A_1 \cap A_2) \to \ldots$$

is exact and functorial (see e.g. [55, (17.4), (17.11)]).

(1.2) The homomorphism β_* is defined by the formula

$$\beta_*(\gamma_1, \gamma_2) = H_n(j_1)(\gamma_1) + H_n(j_2)(\gamma_2)$$

where $\gamma_i \in H_n(X, A_i)$, $i = 1, 2$, and the homomorphism α_* by the formula $\alpha_*(\gamma) = (H_n(i_1)(\gamma), -H_n(i_2)(\gamma))$ for $\gamma \in H_n(X, A_1 \cap A_2)$, where $j_k : (X, A_k) \to (X, A_1 \cup A_2)$ and $i_k : (X, A_1 \cap A_2) \to (X, A_k)$, $k = 1, 2$, are inclusions.

(1.3) If $(X, A) \in \mathrm{TOP}^2$, then we have a scalar product (Kronecker duality)

$$\langle\, ,\, \rangle : H_s^*(X, A) \times H_*(X, A) \to R$$

(see [25, VII.1.1], [104]) such that, for $f : (X, A) \to (Y, B)$, $b \in H_s^*(Y, B)$, $\alpha \in H_*(X, A)$ and $a \in H_s^{*-1}(A)$ we have

$$\langle H_s^*(f)(b), \alpha\rangle = \langle b, H_*(f)(\alpha)\rangle \quad \text{and} \quad \langle a, \partial_*(\alpha)\rangle = \langle \delta^*(a), \alpha\rangle.$$

(1.4) If $(X, A), (Y, B) \in \mathrm{TOP}^2$, then we have a cross product

$$\times : H_n(X, A) \times H_m(Y, B) \to H_{n+m}(X \times Y, A \times Y \cup X \times B)$$

(see [25, VII.2.1], [104]) such that if $f : (X, A) \to (Y, B)$ and $f' : (X', A') \to (Y', B')$, $\alpha \in H_n(X, A)$ and $\alpha' \in H_m(X', A')$, then

$$H_{n+m}(f \times f')(\alpha \times \alpha') = H_n(f)(\alpha) \times H_m(f')(\alpha').$$

(1.5) If $(X, A), (Y, B) \in \mathrm{TOP}^2$ and the pair $\{X \times Y, A \times Y \cup X \times B\}$ is excisive w.r.t. H_s, then we have a cross product

$$\times : H_s^n(X, A) \times H_s^m(Y, B) \to H_s^{n+m}(X \times Y, A \times Y \cup X \times B)$$

(see [25, VII.7.1], [104]) such that, under the same assumptions concerning f, f' as in (1.4), $a \in H_s^n(Y, B)$, $b \in H_s^m(Y', B')$, we have

$$H_s^{n+m}(f \times f')(a \times b) = H_s^n(f)(a) \times H_s^m(f')(b).$$

(1.6) If sets A, B are open in X, Y, respectively, then we can define a cross product in H^* having the same properties (see [83, 8.7]).

(1.7) There is a natural transformation ξ of the theory H^* into H_s^* such that, for any manifold X and its compact subset K,

$$\xi_{XK} = \xi(X, X\backslash K) : H^*(X, X\backslash K) \to H_s^*(X, X\backslash K)$$

is an isomorphism. To see this it is enough to recall [104] and use the five-lemma.

(1.8) There is a functorial isomorphism $\widehat{H}^*(X) \simeq H^*(X)$ where X is a paracompact space (see [104]).

(1.9) If (X,A), (Y,B) are compact pairs, then, for any ring R, there is an exact sequence

$$0 \to \bigoplus_{\substack{i,j\\ i+j=n}} H^i(X,A)\otimes_R H^j(Y,B) \to H^n(X\times Y, A\times Y\cup X\times B)$$
$$\to \bigoplus_{\substack{i,j\\ i+j=n}} H^i(X,A) *_R H^j(Y,B) \to 0$$

where $*_R$ stands for the torsion product of R-modules. The exactness of this so-called Künneth sequence follows from the exactness of the Künneth sequence for $\widehat{H}^*$ (on compact pairs) and from (1.8).

2. Orientation of manifolds. We use [25, VIII]. Let X be an m-dimensional manifold. For any $x\in X$, $H^n(X,X\backslash\{x\})$ is an infinite cyclic group for $n=m$ and vanishes for $n\neq m$ (we consider integer-valued singular homology). Let $\widetilde{X} = \bigvee_{x\in X} H_m(X,X\backslash\{x\})$ and let $p:\widetilde{X}\to X$ be the map defined by the condition

$$p^{-1}(x)\in H_m(X,X\backslash\{x\}).$$

One can introduce a topology on X such that p becomes a covering map. Let $q:\widetilde{X}\to\mathbb{Z}$ be defined by $q(k\alpha)=|k|$ where $\alpha\in H_m(X,X\backslash\{x\})$ is a generator. If $A\subset X$, then let ΓA be the set of all sections of the sheaf $p:\widetilde{X}\to X$. Obviously, ΓA has the structure of an abelian group (see [25, VIII.2.4]). Define a homomorphism $J_A: H_m(X,X\backslash A)\to \Gamma A$ by $J_A(\gamma)(x) = H_m(j_x^A)(\gamma)$ for $x\in A$, $\gamma\in H_m(X,X\backslash A)$ (1). If $B\subset A$, then the diagram

$$\begin{array}{ccc} H_m(X,X\backslash A) & \xrightarrow{J_A} & \Gamma A\\ {\scriptstyle H_m(j_B^A)}\downarrow & & \downarrow{\scriptstyle \varrho}\\ H_m(X,X\backslash B) & \xrightarrow[J_B]{} & \Gamma B \end{array} \tag{2.1}$$

where ϱ is the restriction of a section over A to B, is commutative (see [25, VIII.2.7 and 2.8]).

A section $\mu\in\Gamma A$ is called an *orientation* (more precisely, $\mathbb{Z}$-orientation) along A if $q(\mu(x))=1$ for $x\in A$. If $A=X$, then we say that μ is an *orientation of* X and the pair (X,μ) is called an *oriented manifold*.

(2.2) If $\alpha\in H_m(S^m)$ is a generator, then $J_{S^m}(\alpha)\in\Gamma S^m$ is an orientation of the sphere S^m.

(2.3) If U is an open subset of a manifold X, μ is an orientation of X, then the restriction $\mu|U$ is an orientation of the submanifold U (more precisely: for

---

(1) If $B\subset A$, then j_B^A denotes the inclusion $(X,X\backslash A)\to(X,X\backslash B)$.

$x \in U, (\mu|U)(x) = J^{-1}(\mu(x))$ where $J : H_m(U, U\backslash\{x\}) \to H_m(X, X\backslash\{x\})$ is the excision isomorphism).

(2.4) If manifolds X, X' of dimensions m, m', respectively, are oriented by μ and μ', then $X \times X'$ is oriented by the section $\mu \times \mu'$ given by $(\mu \times \mu')(x, x') = \mu(x) \times \mu'(x')$ for $x \in X,\ x' \in X'$ (see (1.4) and [25, VIII.2.13]).

(2.5) If X is an m-dimensional manifold with boundary $\operatorname{bd} X$ and μ is an orientation of $\operatorname{int} X$, then there is a homomorphism $b : \Gamma \operatorname{int} X \to \Gamma \operatorname{bd} X$ such that the diagram

$$\begin{array}{ccc} H_m(X, X\backslash \operatorname{int} X) & \xrightarrow{J_{\operatorname{int} X}} & \Gamma \operatorname{int} X \\ {\scriptstyle \delta_*}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ H_{m-1}(\operatorname{bd} X) & \xrightarrow[J_{\operatorname{bd} X}]{} & \Gamma \operatorname{bd} X \end{array}$$

is commutative and $b(\mu)$ is an orientation of $\operatorname{bd} X$ (see [25, VIII.2.19]).

Let $A \subset X$ and consider on $\mathbb{Z}$ the discrete topology. Since $C(A, \mathbb{Z})$ is a Specker group for A compact, we have:

(2.6) The group $C(A, \mathbb{Z})$ is free for any compact $A \subset X$.

(2.7) If a manifold X is orientable along $A \subset X$, then there is an isomorphism $\Phi_A : \Gamma A \to C(A, \mathbb{Z})$.

P r o o f. Let $\mu \in \Gamma A$ be an orientation of X along A. If $s \in \Gamma A$, then the correspondence $\Phi_A : s \to f_s$, where $f_s(x) = q(s(x))$ for $x \in A$, is an isomorphism. ■

(2.8) If K is a compact subset of a manifold X, then $J_K : H_m(X, X\backslash K) \to \Gamma K$ is an isomorphism (see [25, VIII.3.3]).

Let (X, μ) be an oriented m-manifold. By (2.8), there is a unique class $\mu_K \in H_m(X, X\backslash K)$ such that $J_K(\mu_K) = \mu|K$ provided K is compact. This class is called the *fundamental homology class* of the set K. By (2.1), we have

(2.9) If $K \subset N$, N is compact, then $H_m(j_K^N)(\mu_N) = \mu_K$.

(2.10) Under the assumptions of (2.4), if K is compact in X and K' is compact in X', then

$$(\mu \times \mu')_{K \times K'} = \mu_K \times \mu'_{K'} \in H_{m+m'}(X \times X', X \times X'\backslash K \times K').$$

(2.11) Let $K = K_1 \cup \ldots \cup K_n$, where K_i is compact connected, $i = 1, \ldots, n$, be a decomposition of a compact set K into the union of its components. In view of (1.1), there is an isomorphism

$$\alpha_* : H_m(X, X\backslash K) \simeq \bigoplus_{i=1}^{n} H_m(X, X\backslash K_i).$$

Moreover, $\alpha_*(\mu_K) = \{\mu_{K_i}\}_{i=1}^n$ (up to sign) in view of (2.9). Let

$$\gamma_j = \{\gamma_{ji}\}_{i=1}^n \in \bigoplus_{i=1}^n H_m(X, X\backslash K_i), \qquad j = 1, \ldots, n,$$

be given by $\gamma_{ji} = \delta_{ji}\mu_{K_i}$ (δ_{ji} denotes the Kronecker delta). The system $\{\alpha_*^{-1}(\gamma_j)\}_{j=1}^n$ is called the *fundamental system of homology classes* of the set K and is denoted by $\{\mu_K\}$. Clearly, $\{\mu_K\}$ depends only on μ.

The role of the fundamental class is reflected in the following result.

(2.12) (Comp. [25, VIII.3.4].) If (X, μ) is an oriented m-dimensional manifold,
$K \subset X$ is compact (resp. compact connected), then $H_m(X, X\backslash K)$ is a free (resp. infinite cyclic) group and the system $\{\mu_K\}$ is its base (resp. μ_K is its generator).

Proof. Follows from the definition, (2.6), (2.7), (2.8) and (2.11). ■

I. Topology of morphisms

In the first section of this chapter we recall the basic definitions and properties of set-valued maps. In the second section we study properties of Vietoris maps. In the next three sections we give the definition of a morphism which plays a crucial role in the sequel and establish some simple properties. Moreover, we consider the most important constructions in the category of morphisms. Next we study the notion of essentiality of morphisms.

1. Set-valued maps. Let X, Y be sets. By a *set-valued mapping* we understand a map $\psi : X \to P(Y)$. Observe that any relation $G \subset X \times Y$ with domain X determines a set-valued map $\psi_G : X \to P(Y)$ given by $\psi_G(x) = \{y \in Y : (x, y) \in G\}$. On the other hand, any set-valued map ψ defines a relation $G_\psi = \{(x, y) \in X \times Y : y \in \psi(x)\}$ having domain equal to X and called the *graph* of ψ. Clearly, the above correspondence is bijective. Hence it is sometimes convenient to identify a map ψ with its graph G_ψ and study ψ by means of the properties of G_ψ.

If $\psi : X \to P(Y)$, $A \subset X$ and $B \subset Y$, then we define sets

$$\psi(A) = \bigcup_{x\in A} \psi(x),$$
$$\psi_-^{-1}(B) = \{x \in X : \psi(x) \cap B \neq \emptyset\},$$
$$\psi_+^{-1}(B) = \{x \in X : \psi(x) \subset B\},$$

called the *image* of A, the *large preimage* of B and the *small preimage* of B under ψ, respectively. Moreover, we define a set-valued map $\psi^{-1} : \psi(X) \to P(X)$ by the formula $\psi^{-1}(y) = \psi_-^{-1}(\{y\}) = \{x \in X : y \in \psi(x)\}$.

Observe that

$$(\psi^{-1})^{-1}_{-}(A) = \psi(A) \tag{1.1}$$

for any $A \subset X$.

Given $A \subset X$, a map $\psi|A : A \to P(Y)$ defined by $(\psi|A)(x) = \psi(x)$ for $x \in A$ is called the *restriction* of ψ to A.

Let Z be a set and $\psi' : Y \to P(Z)$ a set-valued map. The map $X \ni x \mapsto \psi'(\psi(x)) \in P(Z)$ is denoted by $\psi' \circ \psi$ and called the *composition* of ψ and ψ'. Clearly, $\psi|A = \psi \circ i_A$.

Now, let X, Y be topological spaces. We say that a map $\psi : X \to P(Y)$ is *lower-semicontinuous* (abbr. l.s.c.) (resp. *upper-semicontinuous* (abbr. u.s.c.)) provided the set $\psi^{-1}_{-}(U)$ (resp. $\psi^{-1}_{+}(U)$) is open for every open subset U of Y. Observe that, in the single-valued case, the upper and lower semicontinuity are equivalent to the ordinary continuity.

After [7] we recall some properties of set-valued maps.

(1.2) PROPOSITION. (i) *The composition of l.s.c.* (*resp. u.s.c.*) *maps is l.s.c.* (*resp. u.s.c.*).

(ii) *If* $\psi : X \to K(Y)$ (*i.e.* $\psi : X \to P(Y)$ *and* $\psi(x) \in K(Y)$ *for any* $x \in X$) *is u.s.c., then* G_ψ *is a closed subset of* $X \times Y$.

(iii) *If* $\psi : X \to P(Y)$, $\Psi : X \to K(Y)$, *the graph* G_ψ *is closed*, Ψ *is u.s.c. and* $\psi(x) \cap \Psi(x) \neq \emptyset$ *for any* $x \in X$, *then* $\psi \cap \Psi : x \mapsto \psi(x) \cap \Psi(X)$ *is u.s.c.*

(iv) *If* $\psi : X \to K(Y)$ *is u.s.c. and* $A \subset X$ *is compact*, *then* $\psi(A)$ *is also compact.*

We say that a map $\psi : X \to P(Y)$ is *compact* (*locally compact*) if $\mathrm{cl}\,\psi(X) \in K(X)$ (resp. each point $x \in X$ has a neighbourhood U such that $\psi|U$ is compact).

(1.3) PROPOSITION. *A locally compact map with closed graph is u.s.c.*

(1.4) LEMMA. *Let* E *be a Fréchet space. If* $\psi : X \to P(E)$ *is an l.s.c. map with convex values and* $f : X \to E$, $\varepsilon : X \to [0, \infty)$ *are continuous maps such that* $f(x) \in \psi(x)$ *if* $\varepsilon(x) = 0$ *and* $N(f(x), \varepsilon(x)) \cap \psi(x) \neq \emptyset$ *if* $\varepsilon(x) > 0$, *then* $\Psi : x \mapsto B(f(x), \varepsilon(x)) \cap \psi(x)$ *is l.s.c.*

Proof. Let $K \in C(E)$. We show that $\Psi^{-1}_{+}(K) \in C(X)$. Consider a net $\{x_\mu : \mu \in M\} \subset \Psi^{-1}_{+}(K)$ converging to $x_0 \in X$. Assume that $x_0 \notin \Psi^{-1}_{+}(K)$ and $\varepsilon(x_0) > 0$. It is easily seen that there is $y_0 \in N(f(x_0), \varepsilon(x_0)) \cap \psi(x_0) \cap (E \backslash K)$. Let $\delta > 0$ be such that $d(f(x_0), y_0) < \varepsilon(x_0) - 3\delta$. There is $\mu_1 \in M$ such that $N(y_0, \delta) \subset N(f(x_\mu), \varepsilon(x_\mu))$ for any $\mu \geq \mu_1$. By the lower semicontinuity of ψ, there exists $\mu_2 \in M$ such that $\psi(x_\mu) \cap N(y_0, \delta) \cap (E \backslash K) \neq \emptyset$ for $\mu \geq \mu_2$. For any $\mu \geq \mu_1, \mu_2$, we have $\Psi(x_\mu) \cap (E \backslash K) \supset \psi(x_\mu) \cap N(y_0, \delta) \cap (E \backslash K) \neq \emptyset$, which is a contradiction. If $\varepsilon(x_0) = 0$, then a contradiction follows almost at once. ∎

Let X, Y be topological spaces. A map $\psi : X \to P(Y)$ is *perfect* if it is u.s.c., $\psi^{-1}(y) \in K(X)$ for any $y \in Y$ and $\psi(B) \in C(Y)$ for any $B \in C(X)$. A map ψ is *proper* if it is u.s.c. and $\psi^{-1}_{-}(K) \in K(X)$ for any $K \in K(Y)$.

(1.5) PROPOSITION. (i) *The composition of perfect maps is perfect.*

(ii) *If a u.s.c. map* $\psi : X \to P(Y)$ *is perfect, then the map* $\psi^{-1} : \psi(X) \to P(X)$ *has compact values and is u.s.c. If* $\psi(X)$ *is closed, then this condition is also sufficient.*

(iii) *Any perfect map is proper.*

(iv) *If* $\psi : X \to P(Y)$ *is proper, the graph* G_ψ *is closed and* Y *is a* k*-space, then* ψ *is perfect.*

Proof. (i) follows from (1.2)(i). To see (ii), it is enough to use (1.1), and (iii) follows from (ii) and (1.2)(iv). To prove (iv), it is sufficient to show that $\psi(B) \in C(Y)$ for any closed $B \subset X$. The set $\psi(B)$ is closed in the k-space Y if and only if $\psi(B) \cap K$ is compact for any compact $K \subset Y$. This last condition is satisfied since G_ψ is closed and ψ is proper. ■

One easily proves

(1.6) PROPOSITION. *Let* $A \in C(X)$ *and* $B \subset Y$. *If a map* $\psi : X \to P(Y)$ *is perfect, then so are the restrictions* $\psi|A : A \to P(Y)$ *and* $\psi_B : \psi_-^{-1}(B) \to P(B)$ *given by* $\psi_B(x) = \psi(x) \cap B$.

The existence of selections of set-valued maps is an important problem. Let X, Y be sets and $\psi : X \to P(Y)$. A map $\chi : X \to P(Y)$ is a *selection* of ψ if $\chi(x) \subset \psi(x)$ for any $x \in X$. There are many results concerning the existence of selections. The best known is the following result of Michael.

(1.7) THEOREM (see [84]). *Let* X *be a paracompact space and* E *a Fréchet space. Any l.s.c. convex-valued map* $\psi : X \to C_V(E)$ *has a continuous single-valued selection.*

(1.8) Let $\mathcal{F}$ be a class of maps (single- or set-valued ones). We write

$$\mathcal{F}(X, Y) = \{\psi : X \to P(Y) : \psi \text{ belongs to } \mathcal{F}\}.$$

Let $\mathcal{F}, \mathcal{G}$ be classes of maps and let $\mathcal{F} \subset \mathcal{G}$. We say that maps $\psi_0, \psi_1 \in \mathcal{F}(X, Y)$ are *homotopic* in the class $\mathcal{G}$ if there exists a map $\psi \in \mathcal{G}(X \times I, Y)$ such that $\psi \circ i_j = \psi_j$ for $j = 0, 1$. The map ψ is called a *homotopy* (in $\mathcal{G}$) joining ψ_0 and ψ_1. The problem of existence of a homotopy in $\mathcal{G}$ between given maps from $\mathcal{F}$ is a complex one and will be discussed later in more definite situations.

2. Vietoris maps. Let X be a topological space. If $A \subset X$, $A \neq \emptyset$, then by the *relative dimension* of the set A (in X) we mean the (possibly infinite) number

$$\mathrm{rd}_X(A) = \sup\{\dim(C) : C \in C(X),\ C \subset A\}$$

where by $\dim(C)$ we denote the covering dimension of the set C (see [2]). Additionally we let $\mathrm{rd}_X(\emptyset) = -\infty$.

(2.1) PROPOSITION. (i) *Let* $Y \subset X$, $A \subset Y$ *and* $A \subset B \subset X$. *If* $Y \in C(X)$ (*or* $Y \in F_\sigma(X)$ *and* X *is a normal space*), *then* $\mathrm{rd}_Y(A) \leq \mathrm{rd}_X(B)$.

(ii) *If a topological space* X' *is compact*, $X \times X'$ *is normal and* $A \subset X$, *then* $\mathrm{rd}_{X\times X'}(A \times X') \le \mathrm{rd}_X(A) + \dim(X')$.

Proof. (i) If Y is closed, then the assertion follows from the definition. Suppose that $Y \in F_\sigma(X)$ and that X is normal. Let $C \subset A$ be closed. Obviously, $C \subset F_\sigma(X)$, hence C is normal (as a subspace). Let $C = \bigcup_{i=1}^{\infty} C_i$ where $C_i \in C(X)$. For any $i \ge 1$, $C_i \subset A \subset B$ and $\dim(C_i) \le \mathrm{rd}_X(B)$. By Menger's theorem (see [2]), $\dim(C) \le \mathrm{rd}_X(B)$.

(ii) Let $C \in C(X \times X')$, $C \subset A \times X'$. Since X' is compact, by the Kuratowski theorem (see [32]), $\mathrm{pr}_X(C)$ is closed. Hence $\dim(\mathrm{pr}_X(C)) \le \mathrm{rd}_X(A)$. On the other hand, $C \subset \mathrm{pr}_X(C) \times X'$, so by [2, p. 166] and [33, p. 191], $\dim(C) \le \mathrm{rd}_X(A) + \dim(X')$. ■

Let G be a topological space, $p : G \to X$ a continuous map and let R be a principal ideal domain. For integers $k \ge 0$ and $N > 0$, we define

$$\begin{aligned} s_p^k(R) &= \{x \in X : \widetilde{H}^k(p^{-1}(x); R) \ne 0\}, \\ m_p^N(R) &= 1 + \max_{0\le k\le N-1} \{\mathrm{rd}_X(s_p^k(R)) + k\}, \\ m_p(R) &= \sup_{N>0} \{m_p^N(R), 0\}. \end{aligned}$$

To simplify the notation, in the whole Chapter I we write $H(\cdot)$ instead of $H(\cdot\,; R)$, s_p^k, m_p^N and m_p instead $s_p^k(R), m_p^N(R)$ and $m_p(R)$, respectively, unless this leads to ambiguity (see Section 7).

(2.2) Proposition. (i) $m_p < \infty$ (*resp.* $m_p = 0$) *if and only if* $\mathrm{rd}_X(s_p^k) < \infty$ *for all integers* $k \ge 0$ *and* $s_p^k = \emptyset$ *for almost all* $k \ge 0$ (*resp.* $s_p^k = \emptyset$ *for all* $k \ge 0$).

(ii) *If* $m_p = 0$, $B \subset X$, $p_B = p|p^{-1}(B) : p^{-1}(B) \to B$, *then* $m_{p_B}=0$. *If* $m_p > 0$, $B \in C(X)$ (*or* $B \in F_\sigma(X)$ *and* X *is normal*), *then* $m_{p_B} \le m_p$.

Proof. For $x \in B$, $p_B^{-1}(x) = p^{-1}(x)$. Thus $s_{p_B}^k = s_p^k \cap B$ for any integer k. If $m_p = 0$, then by (i), $s_{p_B}^k = \emptyset$; hence $m_{p_B} = 0$. If $m_p > 0$ and $B \in C(X)$ (or $B \in F_\sigma(X)$ where X is normal), then in view of (2.1)(i), we deduce that $\mathrm{rd}_B(s_{p_B}^k) \le \mathrm{rd}_X(s_p^k)$ for any $k \ge 0$. ■

The following fundamental result (see [102], comp. [104]) is a generalization of the well-known Vietoris–Begle theorem (see [108], [6]).

(2.3) Theorem. *Let* G, X *be paracompact spaces and* $p : G \to X$ *a continuous closed surjection. If there is an integer* $N > 0$ *such that* $m_p^N < N$, *then the induced homomorphism*

$$p^* : H^n(X) \to H^n(G)$$

is an epimorphism for $n = m_p^N$, *an isomorphism for* $m_p^N < n < N$ *and a monomorphism for* $n = N$.

Let $A \subset X$, $T \subset G$ and let $p : (G,T) \to (X,A)$. Given an integer $n > 0$, the map p is called a $\mathcal{V}_n$-*map* (with respect to R) — written $p \in \mathcal{V}_n((G,T),(X,A))$ — if

(i) p is a perfect surjection, $p^{-1}(A) = T$;
(ii) $m_p < n$.

We say that p is a $\mathcal{V}$-*map* (written $p \in \mathcal{V}((G,T),(X,A))$) provided

$$p \in \bigcup_{n=1}^{\infty} \mathcal{V}_n((G,T),(X,A)).$$

For historical reasons $\mathcal{V}_1$-maps are called *Vietoris maps* (comp. [42], [17], [44]; see also Section 7). It is clear that $\mathcal{V}_n \subset \mathcal{V}_m$ for $n \le m$.

(2.4) A perfect surjection $p : G \to X$ is a $\mathcal{V}_n$-map if and only if $\mathrm{rd}_X(s_p^k) < n - k - 1$ for any integer $k \ge 0$.

It follows that, for any $p \in \mathcal{V}_n(G,X)$, the preimages of points are nonacyclic sets, but the nonacyclicity occurs only in low dimensions and on sets of low dimension.

Some properties of $\mathcal{V}$-maps are collected in the following:

(2.5) PROPOSITION. *Let* $B \subset X$.

(i) *If* $p \in \mathcal{V}_1(G,X)$, *then* $p_B \in \mathcal{V}_1(p^{-1}(B),B)$.
(ii) *If* $B \in C(X)$ (*or* $B \in F_\sigma(X)$ *where* X *is normal*), $p \in \mathcal{V}_n(G,X), n \ge 1$, *then* $p_B \in \mathcal{V}_n(p^{-1}(B),B)$.

Proof. This follows immediately from (1.6), (2.2)(ii). ∎

(2.6) PROPOSITION. *Let* G', X' *be topological spaces.*

(i) *If* $p \in \mathcal{V}_1(G,X)$, $p' \in \mathcal{V}_1(G',X')$, *then* $p \times p' \in \mathcal{V}_1(G \times G', X \times X')$.
(ii) *If* X' *is compact*, $X \times X'$ *is normal*, $p \in \mathcal{V}_n(G,X)$, $n \ge 1$ *and* $p' \in \mathcal{V}_1(G',X')$, *then* $p \times p' \in \mathcal{V}_{n+m}(G \times G', X \times X')$ *where* $m = \dim(X')$.

Proof. (i) follows immediately from the Künneth theorem and [32, p. 232]. Let $P = p \times p'$. By the Künneth formula for Alexander–Spanier cohomology (see A.(1.9)), $s_P^k = s_p^k \times X'$ for any integer $k \ge 0$. In view of (2.1)(ii), the proof is complete. ∎

We shall need the following auxiliary notions. Consider a triad

$$(*) \qquad G_1 \underset{q}{\to} X \underset{p}{\leftarrow} G_2.$$

The *fibre product* of this triad is a map $f : G_1 \boxtimes_X G_2 \to X$ where $G_1 \boxtimes_X G_2 = \{(g_1,g_2) \in G_1 \times G_2 : q(g_1) = p(g_2)\}$ and $f(g_1,g_2) = q(g_1)$ $(= p(g_2))$ for $(g_1,g_2) \in G_1 \boxtimes_X G_2$. By the *pull-back* of $(*)$ we mean a cotriad $G_1 \xleftarrow{\overline{p}} G_1 \boxtimes_X G_2 \xrightarrow{\overline{q}} G_2$ where $\overline{p}(g_1,g_2) = g_1$, $\overline{q}(g_1,g_2) = g_2$ for $(g_1,g_2) \in G_1 \boxtimes_X G_2$.

If X, Y are spaces, then we say that a continuous map $q : X \to Y$ is an F_σ-*map* if there is a sequence $(F_n)_{n=1}^{\infty}$ of closed sets in X such that $X = \bigcup_{n=1}^{\infty} F_n$ and

$q|F_n$ is a closed map. Clearly, any closed map is an F_σ-map, any homeomorphism onto an F_σ-set is an F_σ-map and, in particular, the inclusion of an F_σ-set is an F_σ-map.

(2.7) PROPOSITION. *If $p \in \mathcal{V}_1(G_2, X)$, then $\overline{p}$ is a $\mathcal{V}_1$-map. If X is paracompact, G_1 is normal, $p \in \mathcal{V}_n(G_2, X)$, $n \geq 1$, and q is an F_σ-map such that $\sup\{\dim(q^{-1}(x)) : x \in X\} \leq m$, then $\overline{p} \in \mathcal{V}_{n+m}(G_1 \boxtimes_X G_2, G_1)$.*

Proof. Clearly, $\overline{p}$ is a continuous surjection. For any $g_1 \in G_1$, $\overline{p}^{-1}(g_1) = \{g_1\} \times p^{-1}(q(g_1)) \in K(G_1 \boxtimes_X G_2)$. Let $B \subset G_1 \boxtimes_X G_2$ be closed and let $g_1 \notin \overline{p}(B)$. Thus $\{g_1\} \times p^{-1}(q(g_1)) \cap B = \emptyset$ and there are open sets $V_1 \subset G_1, V_2 \subset G_2$ such that $g_1 \in V_1$, $p^{-1}(q(g_1)) \subset V_2$ and $V_1 \times V_2 \cap B = \emptyset$. Hence, $V_1 \cap \overline{p}(B) = \emptyset$, which shows that $\overline{p}(B)$ is closed and, therefore, $\overline{p}$ is a perfect map. For any integer $k \geq 0$, $q : s^k_{\overline{p}} \to s^k_p$. Thus if $m_p = 0$, then $s^k_p = \emptyset = s^k_{\overline{p}}$ and $m_{\overline{p}} = 0$. Suppose that $0 < m_p < n$ and let C be a closed set in G_1, $C \subset s^k_{\overline{p}}$. Since q is an F_σ-map, $G_1 = \bigcup_{n=1}^\infty F_n$, where F_n is closed and $q|F_n$ is a closed map. The set $C_n = C \cap F_n$ is closed for any $n \geq 1$ and $q(C_n)$ is closed in s^k_p; hence $\dim(q(C_n)) \leq \mathrm{rd}_X(s^k_p)$ for any $n \geq 1$. Since C_n is normal and $q(C_n)$ is paracompact, we have $\dim(C_n) \leq \dim(g(C_n)) + \sup\{\dim(q^{-1}(x)) : x \in C_n\} \leq \mathrm{rd}_X(s^k_p) + m$. Hence $\dim(C) \leq \mathrm{rd}_X(s^k_p) + m$. ∎

By (2.3), Vietoris maps induce isomorphisms of cohomology groups. More precisely:

(2.8) THEOREM. *Let X, G be topological spaces, $T \subset G$ and $A \subset X$. If X is paracompact, $A \in F_\sigma(X)$ and $p \in \mathcal{V}_n((G, T), (X, A))$, $n \geq 1$, then the induced homomorphism $p^* : H^k(X, A) \to H^k(G, T)$ is an isomorphism for any $k \geq n+1$. If $n = 1$, then p^* is an isomorphism for any $k \geq 0$ (in this case, we only require A to be paracompact).*

Proof. Since X is normal, by (2.5)(ii), $p_A : T \to A$ is a $\mathcal{V}_n$-map. Moreover, the spaces G, T are paracompact since the preimages of paracompact spaces under perfect maps are paracompact (see e.g. [32]). Define $\overline{p} : G \to X$ by $\overline{p}(g) = p(g)$ for $g \in G$ and let $k \geq n+1$. For any $m \geq k-1$, there is an integer $N > m \geq k-1 \geq n > m^N_p = m^N_{\overline{p}} \geq m^N_{p_A}$. Thus, by (2.3), $H^m(\overline{p})$ and $H^m(p_A)$ are isomorphisms. If $n = 1$, then $m_{\overline{p}} = m_{p_A} = 0$; hence $H^0(\overline{p})$ and $H^0(p_A)$ are isomorphisms as well. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
H^{k-1}(G) & \to & H^{k-1}(T) & \to & H^k(G,T) & \to & H^k(G) & \to & H^k(T) \\
\uparrow{\scriptstyle \overline{p}^*} & & \uparrow{\scriptstyle p_A^*} & & \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle \overline{p}^*} & & \uparrow{\scriptstyle p_A^*} \\
H^{k-1}(X) & \to & H^{k-1}(A) & \to & H^k(X,A) & \to & H^k(X) & \to & H^k(A)
\end{array}$$

The use of the five-lemma completes the proof. ∎

(2.9) Remark. If X is metrizable and A is open, then the assertion of (2.8) holds.

(2.10) PROPOSITION. *Let Y be a topological space. If $p_1 \in \mathcal{V}_1(G,X)$, $p_2 \in \mathcal{V}_n(X,Y)$, $n \geq 1$, then $p_2 \circ p_1 \in \mathcal{V}_n(G,Y)$.*

Proof. Obviously, by (1.5)(i), $p = p_2 \circ p_1$ is a perfect surjection. Let $y \in Y$; then $p^{-1}(y) = p_1^{-1}(p_2^{-1}(y))$. By (2.5), $p_1|p^{-1}(y) : p^{-1}(y) \to p_2^{-1}(y)$ is a $\mathcal{V}_1$-map. In view of (2.8), $H^*(p^{-1}(y)) \simeq H^*(p_2^{-1}(y))$, which ends the proof. ∎

(2.11) Remark. Clearly, if $p_1 \in \mathcal{V}_n(G,X)$, $n \geq 1$, and p_2 is a homeomorphism then $p_2 \circ p_1 \in V_n(G,Y)$.

3. Category of morphisms. Let (X,A) and (Y,B) be pairs of topological spaces. By $D((X,A),(Y,B))$ we denote the family of all pairs (p,q) where $p \in \mathcal{V}((G,T),(X,A))$ and $q : (G,T) \to (Y,B)$ is a continuous map. In the sequel we identify a pair $(p,q) \in D((X,A),(Y,B))$ with the cotriad

$$(X,A) \xleftarrow[p]{} (G,T) \xrightarrow[q]{} (Y,B).$$

We say that pairs $(p,q),(p',q') \in D((X,A),(Y,B))$ are *equivalent* (written $(p,q) \approx (p',q')$) if there is a homeomorphism $f : (p^{-1}(X),p^{-1}(A)) \to (p'^{-1}(X),p'^{-1}(A))$ such that $p' \circ f = p$ and $q' \circ f = q$. Observe that "$\approx$" is an equivalence relation in $D((X,A),(Y,B))$; moreover, if $(p,q) \approx (p',q')$ then $s_p^k = s_{p'}^k$ for any integer $k \geq 0$. This implies that p is a $\mathcal{V}_n$-map, for some $n \geq 1$, if and only if p' is a $\mathcal{V}_n$-map.

The quotient set $D((X,A),(Y,B))/{\approx}$ is denoted by $M((X,A),(Y,B))$ and its elements, denoted by the Greek letters $\varphi,\chi,\ldots$, are called *morphisms* (defined on (X,A) with values in (Y,B)). The notation and terminology will be justified below.

(3.1) Remark. The notion of morphism was introduced in [44] in a slightly different manner (see [7]). Namely, the definition of the relation "$\approx$" was less restrictive. However, equivalence classes of our relation have somewhat better properties.

If $(p,q) \in \varphi \in M((X,A),(Y,B))$, then we say that the pair (p,q) *represents* (or *determines*) the morphism φ. A morphism φ determined by a pair (p,q) is called an *n-morphism*, $n \geq 1$, if p is a $\mathcal{V}_n$-map. The set of all n-morphisms from $M((X,A),(Y,B))$ is denoted by $M_n((X,A),(Y,B))$. Obviously, $M = \bigcup_{n=1}^{\infty} M_n$ and $M_n \subset M_m$ if $n \leq m$.

Morphisms are closely related to set-valued maps. Any morphism $\varphi \in M((X,A),(Y,B))$ determines a set-valued map (denoted by the same letter φ)

$$X \ni x \mapsto \varphi(x) = q(p^{-1}(x))$$

where $(p,q) \in \varphi$. The set $\varphi(x)$ for $x \in X$, called the *value* of φ at $x \in X$, is well-defined since it does not depend on the choice of the representing pair of φ.

Moreover, in view of (1.5)(ii) and (1.2)(i), $\varphi : X \to K(Y)$ and is u.s.c. Observe that, for any $x \in A$, $\varphi(x) \subset B$, and, for any $C \subset Y$, $\varphi_-^{-1}(C) = p(q^{-1}(C))$ provided $(p, q) \in \varphi$.

(3.2) Remark. Having a morphism $\varphi \in M(X, Y)$ and sets $A \subset X, B \subset Y$ such that, for $x \in A$, $\varphi(x) \subset B$, we have a morphism $\varphi \in M((X, A), (Y, B))$. In particular, if $\varphi_-^{-1}(B) \subset A$, we have a morphism $\varphi_{AB} \in M((X, X\backslash A), (Y, Y\backslash B))$.

We say that a set-valued map $\psi : X \to K(Y)$ is *determined by a morphism* if there exists a morphism $\varphi \in M(X, Y)$ such that $\psi(x) = \varphi(x)$ for any $x \in X$. Clearly, any map determined by a morphism is u.s.c. A map may be determined by different morphisms at the same time.

The most important examples of maps determined by morphisms are provided by the following constructions. Given an integer $n \geq 1$, we say that a map $\psi : X \to K(Y)$ is *n-acyclic* (with respect to Alexander–Spanier cohomology with coefficients in the ring R) — written $\psi \in \mathcal{A}_n(X, Y)$ — if ψ is u.s.c. and $\mathrm{rd}_X\{x \in X : \widetilde{H}^k(\psi(x); R) \neq 0\} < n - k - 1$ for any $k \geq 0$ (see [17]).

(3.3) Proposition. *Any n-acyclic map is determined by an n-morphism.*

Proof. Let $\psi : X \to K(Y)$ be an n-acyclic map and let $p_\psi : G_\psi \to X$, $q_\psi : G_\psi \to Y$ be the restrictions to G_ψ of the projections onto X and Y, respectively. Obviously, for any $x \in X$, $\psi(x) = q_\psi(p_\psi^{-1}(x))$ and $p_\psi^{-1}(x) = \{x\} \times \psi(x) \in K(G_\psi)$. Let $B \subset G_\psi$ be closed and let $x \notin p_\psi(B)$. Hence $\{x\} \times \psi(x) \cap B = \emptyset$. It is easy to see that there are open sets $U' \subset X$ and $V \subset Y$ such that $x \in U'$, $\psi(x) \subset V$ and $U' \times V \cap B = \emptyset$. Since ψ is u.s.c., the set $U'' = \psi_+^{-1}(V)$ is open in X. Let $U = U' \cap U''$. Clearly, $x \in U$ and $U \cap p_\psi(B) = \emptyset$. This shows that p_ψ is a perfect map. Hence $p_\psi \in \mathcal{V}_n(G_\psi, X)$ because $\mathrm{rd}_X(s^k_{p_\psi}) < n - k - 1$ for each nonnegative integer k. ∎

(3.4) Remark. (i) The pair (p_ψ, q_ψ) representing the morphism determining an n-acyclic map will be called the *generic pair*. In spite of the fact that an n-acyclic map may be determined by different morphisms, the following relation holds. Assume that a morphism represented by a pair (p, q) determines an n-acyclic map ψ. The diagram

$$\begin{tikzcd} & G \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "f"] & \\ X & & Y \\ & G_\psi \arrow[ul, "p_\psi"] \arrow[ur, "q_\psi"'] & \end{tikzcd}$$

is commutative, where $f(g) = (p(g), q(g))$ for any $g \in G$.

(ii) By (3.3), in particular, any continuous single-valued map $f : X \to Y$ is determined by a 1-morphism. In this case, it is convenient to identify f with the 1-morphism represented by the pair (id_X, f) which is equivalent to its generic pair (see (i)).

Now, we define and study the composition of morphisms.

(3.5) THEOREM. *Let* (X,A), (Y,B) *and* (Z,C) *be pairs of spaces. If*

$$\varphi_1 \in M_n((X,A),(Y,B)) \quad \text{and} \quad \varphi_2 \in M_1((Y,B),(Z,C)) \quad \text{for } n \geq 1,$$

then there exists a morphism $\varphi \in M_n((X,A),(Z,C))$ *such that, for any* $x \in X$, $\varphi(x) = \varphi_2(\varphi_1(x))$.

Proof. Assume that $(p_i, q_i) \in \varphi_i$, $i = 1,2$, and $p_1 \in \mathcal{V}_n((G_1,T_1),(X,A))$, $p_2 \in \mathcal{V}_1((G_2,T_2),(Y,B))$. Define a cotriad

$$G_1 \xleftarrow{\overline{p}_2} G_1 \boxtimes_Y G_2 \xrightarrow{\overline{q}_1} G_2$$

to be the pull-back of the triad $G_1 \xrightarrow{q_1} Y \xleftarrow{p_2} G_2$, and let $G = G_1 \boxtimes_Y G_2$, $T = \overline{p}_2^{-1}(T_1)$. Obviously, $\overline{q}_1(T) \subset T_2$. By (2.7), $\overline{p}_2 \in \mathcal{V}_1((G,T),(G_1,T_1))$ and, by (2.10), $p = p_1 \circ \overline{p}_2 \in \mathcal{V}_n((G,T),(X,A))$. Let $q = q_2 \circ \overline{q}_1 : (G,T) \to (Z,C)$. It is easy to verify that if we let φ be the morphism represented by the pair (p,q), then $\varphi(x) = \varphi_2(\varphi_1(x))$ for each $x \in X$. ∎

In the sequel, the morphism φ defined above is denoted by $\varphi_2 \circ \varphi_1$ and called the *composition* of the morphisms φ_1 and φ_2. This definition is correct since it does not depend on the choice of representatives.

Finally, observe that composing single-valued maps $f_1 : X \to Y$, $f_2 : Y \to Z$ (or rather the morphisms represented by the pairs (id_X, f_1) and (id_Y, f_2) — see (3.4)(ii)) according to the above definition, we get the morphism φ represented by the pair (p,q) where $p : G \to X$, $q : G \to Z$, $G = \{(x, f_1(x)) : x \in X\}$, $p(x, f_1(x)) = x$ and $q(x, f_1(x)) = f_2(f_1(x))$ for $x \in X$, which is equivalent to the one represented by the pair $(\mathrm{id}_X, f_2 \circ f_1)$. Morever, if morphisms φ_1 and φ_2 determine maps f_2 and f_1, respectively, then $\varphi_2 \circ \varphi_1$ determines the map $f_2 \circ f_1$.

In view of the above considerations, one verifies easily the following:

(3.6) THEOREM (comp. [44]). *There exists a category* MOR_1^2 *with pairs of spaces as objects*, $M_1((X,A),(Y,B))$ *as the set of morphisms between objects* (X,A) *and* (Y,B), *and* "$\circ$" *as composition. The topological category* TOP^2 *is a subcategory of* MOR_1^2.

The full subcategory in MOR_1^2 with objects from the class of topological spaces is denoted by MOR_1.

The following proposition provides further information concerning compositions of morphisms.

(3.7) PROPOSITION. *Let* X, Y, Z *be topological spaces and let* $B \subset X$.

(i) *If* $B \in C(X)$ (*or* $B \in F_\sigma(X)$ *and* X *is normal*), $\varphi \in M_n(X,Y)$, $n > 1$, *then* $\varphi \circ i_B \in M_n(B,Y)$.

(ii) *If* X *is normal*, Y *paracompact*, $\varphi \in M_n(Y,Z)$, $n > 1$, *and* $f : X \to Y$ *is a continuous* F_σ*-map such that* $\dim(f) = \sup\{\dim(f^{-1}(y)) : y \in Y\} \leq m$, *then* $\varphi \circ f \in M_{n+m}(X,Z)$.

Proof. (i) follows from the definition of composition and (2.1)(i); (ii) is a consequence of (2.7). ∎

It is easy to see that the composition of acyclic maps may fail to be acyclic.

(3.8) EXAMPLE. Let $\psi : S^1 \to K(S^1)$ be given by $\psi(x) = \{y \in S^1 : |x-y| \le \sqrt{3}\}$. Obviously, $\psi \in \mathcal{A}_1(S^1, S^1)$. However, $(\psi \circ \psi)(x) = S^1$ for any $x \in S^1$.

The following proposition justifies the attempt to study maps determined by morphisms rather than only n-acyclic ones and proves that this class is more convenient to deal with.

(3.9) PROPOSITION. *If* $\psi_1 \in \mathcal{A}_n(X, Y)$, $\psi_2 \in \mathcal{A}_1(Y, Z)$, *then the composition* $\psi_2 \circ \psi_1$ *is determined by an* n*-morphism.*

4. Operations in the category of morphisms. It appears that morphisms have numerous properties which are similar to those of maps.

(4.1) Let X, Y be topological spaces and $\varphi \in M_n(X, Y)$, $n \ge 1$. Assume that a pair (p, q) represents φ. If

(i) $n = 1$, $B \subset X$, or

(ii) $n > 1$, $B \in C(X)$, or $B \in F_\sigma(X)$ and X is normal,

then $p_B = p|p^{-1}(B) \in V_n(p^{-1}(B), B)$ in view of (2.2)(ii). Thus in those cases one may define the *restriction* of φ to the set B as the morphism $\varphi|B \in M_n(B, Y)$ represented by the pair $(p_B, q|p^{-1}(B))$. Observe that, for all $x \in B$, $(\varphi|B)(x) = \varphi(x)$ and $\varphi|B = \varphi \circ i_B$ (comp. (3.7)(i)). Clearly, if $\varphi(B) \subset C$, then we can write $\varphi|B \in M_n(B, C)$.

Now, we discuss piecing morphisms together. Let X, Y be topological spaces and $X_j \in C(X)$, $j = 1, 2$, $X_1 \cup X_2 = X$.

(4.2) PROPOSITION. *If* $\varphi_j \in M_1(X_j, Y)$, $j = 1, 2$, $\varphi_1|X_1 \cap X_2 = \varphi_2|X_1 \cap X_2$, *then there exists a morphism* $\varphi \in M_1(X, Y)$ *such that* $\varphi|X_j = \varphi_j$. *If* X *is normal and* $\varphi_j \in M_n(X_j, Y)$, $j = 1, 2$, *where* $n \ge 1$, *then* $\varphi \in M_n(X, Y)$.

Proof. Let $(p_j, q_j) \in \varphi_j$ where $X_j \xleftarrow[p_j]{} G_j \xrightarrow[q_j]{} Y$, $j = 1, 2$. Assume that $M_j = p_j^{-1}(X_1 \cap X_2)$. There exists a homeomorphism $f : M_1 \to M_2$ such that the diagram

$$\begin{tikzcd}
 & M_1 \arrow[dl, "p_1"'] \arrow[dd, "f"] \arrow[dr, "q_1"] & \\
X_1 \cap X_2 & & Y \\
 & M_2 \arrow[ul, "p_2"] \arrow[ur, "q_2"'] &
\end{tikzcd}$$

is commutative. Let $G = G_1 \cup_f G_2$ be the result of piecing together the spaces G_1, G_2 along the map f and let $h_j : G_j \to G$ be the natural quotient map, $j = 1, 2$. We easily see that h_j is a homeomorphic embedding of G_j into G ($j = 1, 2$). Define $p_j : G \to X$ by the formula

$$p_j(h_j(g_j)) = p_j(g_j)$$

for $g_j \in G_j,\ j = 1, 2$. This definition is correct. In fact, $h_1(G_1) \cup h_2(G_2) = G$ and if $g_j \in G_j,\ h_1(g_1) = h_2(g_2)$, then $g_j \in M_j$; hence $g_2 = f(g_1)$ and $p_1(g_1) = p_2(g_2)$. Clearly, p is a continuous surjection. Moreover, for any $B \in C(G)$, $p(B) = p_1(h_1^{-1}(B)) \cup p_2(h_2^{-1}(B)) \in C(X)$ and $p^{-1}(x) = h_j(p_j^{-1}(x)) \in K(G)$ for any $x \in X_j,\ j = 1, 2$. Therefore, p is a perfect surjection. For any integer $k \geq 0$, $s^k_{p_j} = s^k_p \cap X_j$. If $m_{p_j} = 0$, then $m_p = 0$. Suppose that $0 < m_{p_j} < n$. If X is normal, $C \in C(X)$ and $C \subset s^k_p$, then $C_j = C \cap X_j \subset s^k_{p_j}$. Hence $\dim(C) \leq \max\{\dim(C_1), \dim(C_2)\} \leq \max\{\mathrm{rd}_{X_1}(s^k_{p_1}), \mathrm{rd}_{X_2}(s^k_{p_2})\} < n - k - 1$ in view of [2, p. 241]. Defining analogously a map $q : G \to Y$, we get the pair (p, q) which represents the required morphism φ. It is not difficult to see that the above construction does not depend on the choice of $(p_j, q_j) \in \varphi_j,\ j = 1, 2$. ∎

(4.3) Let X, Y_1, Y_2 be topological spaces and let $\varphi_1 \in M_1(X, Y_1)$, $\varphi_2 \in M_n(X, Y_2)$, $n \geq 1$. There exists a morphism $\varphi \in M_n(X, Y_1 \times Y_2)$ (denoted by (φ_1, φ_2)) such that $\varphi(x) = \varphi_1(x) \times \varphi_2(x)$ for any $x \in X$. Indeed, let $(p_j, q_j) \in \varphi_j$ where $X \underset{p_j}{\longleftarrow} G_j \underset{q_j}{\longrightarrow} Y_j$, $j = 1, 2$. We have the diagram

$$\begin{tikzcd}
 & & G_2 \boxtimes G_1 \arrow[r, "q"'] \arrow[dl, "\bar{p}_1"] \arrow[dr, "\bar{p}_2"'] & Y_1 \times Y_2 \\
X & G_2 \arrow[l, "p_2"] & & G_1 \\
 & & X \arrow[ul, "p_2"] \arrow[ur, "p_1"'] &
\end{tikzcd}$$

with the cotriad $G_2 \xleftarrow{\bar{p}_1} G_2 \boxtimes_X G_1 \xrightarrow{\bar{p}_2} G_1$ being the pull-back of the triad $G_2 \underset{p_2}{\longrightarrow} X \underset{p_1}{\longleftarrow} G_1$ and $q(g_2, g_1) = (q_1(g_1), q_2(g_2))$. The morphism (φ_1, φ_2) is represented by the pair (p, q) where $p = p_2 \circ \bar{p}_1 \in \mathcal{V}_n(G_2 \boxtimes_X G_1, X)$ in view of (2.7) and (2.10).

(4.4) Suppose that X is a topological space, and that E is a topological vector space. Let $\varphi \in M_n(X, E)$, $\psi \in M_1(X, E)$ and $\alpha \in M_1(X, \mathbb{R})$. If the maps $+ : E \times E \to E$, $\bullet : \mathbb{R} \times E \to E$ are defined by $+(x, y) = x + y$ and $\bullet(\lambda, x) = \lambda x$, then we define morphisms $\alpha \bullet \varphi$ and $\psi + \varphi$ as the compositions $\bullet \circ (\alpha, \varphi)$ and $+ \circ (\psi, \varphi)$, respectively. Clearly, $\alpha \bullet \varphi(x) = \{\lambda y \in E : \lambda \in \alpha(x),\ y \in \varphi(x)\}$ and $(\psi + \varphi)(x) = \{y + z : y \in \psi(x),\ z \in \varphi(x)\}$ for $x \in X$. The above definitions are correct by (4.3) and (3.5).

(4.5) Remark. In particular, if $f : X \to E$ is a continuous map, then we have an n-morphism $f + \varphi \in M_n(X, E)$ since $f \in M_1(X, E)$ (see (3.4)).

Obviously, if $\psi_1 \in A_1(X, E)$, $\psi_2 \in A_n(X, E)$, then the set-valued map $\psi_1 + \psi_2 : x \mapsto \psi_1(x) + \psi_2(x) \in K(E)$ is determined by an n-morphism.

(4.6) Let $X_j,\ Y_j,\ j = 1, 2$, be topological spaces. If $\varphi_j \in M_1(X_j, Y_j)$, then we may define a morphism $\varphi_1 \times \varphi_2 \in M_1(X_1 \times X_2, Y_1 \times Y_2)$ which determines the map $(x_1, x_2) \mapsto \varphi_1(x) \times \varphi_2(x)$ for $(x_1, x_2) \in X_1 \times X_2$. It suffices to define $\varphi_1 \times \varphi_2$

by the pair $(p_1 \times p_2, q_1 \times q_2)$, where $(p_j, q_j) \in \varphi_j,\ j = 1, 2$, and use (2.6)(i).

5. Homotopy and extension properties of morphisms. Now, we define the homotopy of morphisms. Let $(X, A),\ (Y, B)$ be pairs of topological spaces and let $\varphi_j \in M((X, A), (Y, B))$ for $j = 0, 1$. We say that φ_0 and φ_1 are *n-homotopic* $(n \geq 1)$ (or *homotopic in* M_n) — written $\varphi_0 \approx_n \varphi_1$ — if there exists a morphism $\varphi \in M_n((X \times I, A \times I), (Y, B))$ such that $\varphi \circ i_j = \varphi_j,\ j = 0, 1$. The morphism φ is called a *homotopy* joining φ_0 and φ_1. By (3.7)(i), this definition is correct and if $\varphi_0 \approx_n \varphi_1$, then $\varphi_j \in M_n((X, A), (Y, B))$ for $j = 0, 1$. Observe that if $\varphi_0 \approx_n \varphi_1$ and φ is a homotopy joining them, then $\varphi_j(x) = \varphi(x, j)$ for $x \in X,\ j = 0, 1$.

Assume that $\varphi_0, \varphi_1 \in M(X, Y)$ are joined by a homotopy φ in M_n and let $A \in C(X)$. If, for any $t \in I,\ \varphi \circ i_t | A = \varphi_0 | A$, then we write $\varphi_0 \approx_n \varphi_1$ (rel A).

(5.1) Example. Consider a morphism $\varphi \in M_1(S^1, S^1)$ represented by the generic pair (p_ψ, q_ψ) of the acyclic map ψ introduced in (3.8). We show that φ is homotopic in M_1 to the morphism φ' represented by (p_ψ, p_ψ). Let $P : G_\psi \times I \to S^1 \times I$ and $Q : G_\psi \times I \to S^1$ be given by $P(x, y, t) = (x, t)$ and $Q(x, y, t) = |(1-t)y + tx|^{-1}[(1-t)y + tx]$. By (2.6)(i), $P \in \mathcal{V}_1(G_\psi \times I, S^1 \times I)$ and the morphism represented by (P, Q) joins φ to φ'. Geometrically this fact is obvious: for any $x \in S'$ the set $\varphi(x) = \psi(x)$ is contractible to x.

(5.2) Proposition. *If $\varphi_0, \varphi_1 \in M(X, Y)$ are n-homotopic and $\psi_0, \psi_1 \in M(Y, Z)$ are 1-homotopic, then $\psi_0 \circ \varphi_0 \approx_n \psi_1 \circ \varphi_1$.*

Proof. Let $\varphi \in M_n(X \times I, Y),\ \psi \in M_1(Y \times I, Z)$ be homotopies joining φ_0, φ_1 and ψ_0, ψ_1, respectively. Define $h : X \times I \to I,\ T : I \times X \to X \times I$ by $h(x, t) = t,\ T(t, x) = (x, t)$ for $x \in X,\ t \in I$. It is easy to verify that the morphism $\psi \circ T \circ (h, \varphi) \in M_n(X \times I, Z)$ joins $\psi_0 \circ \varphi_0$ to $\psi_1 \circ \varphi_1$ (see (3.5), (4.3)). ∎

(5.3) Example. Let φ be the 1-morphism from (5.1). By (5.2), $\varphi \circ \varphi$ is homotopic to id_{S^1} (more precisely: to the corresponding morphism determining id_{S^1}). From the geometric point of view this is not so obvious since, for $x \in S^1,\ (\varphi \circ \varphi)(x) = S^1$. In Example (6.8) we shall see that the notion of homotopy of morphisms is closely related to the structure of morphisms and does not depend on the maps determined by these morphisms.

(5.4) Example. Let $X = I$ and $Y = [0, 1] \cup [2, 3]$. Let $f_0 : X \to [0, 1],\ f_1 : X \to [2, 3]$ be continuous maps. These functions are not homotopic in $M_1(X, Y)$. However, the 4-morphism determining the 4-acyclic map $\psi : X \times I \to Y$ given by

$$\psi(x, t) = \begin{cases} f_0(x) & \text{for } t \in \left[0, \frac{1}{3}\right], \\ \{f_0(x), f_1(x)\} & \text{for } t \in \left(\frac{1}{3}, \frac{2}{3}\right), \\ f_1(x) & \text{for } t \in \left[\frac{2}{3}, 1\right] \end{cases}$$

is a homotopy joining f_0 to f_1 in M_4.

(5.5) Remark. Homotopy of maps in $\mathcal{A}_n,\ n \geq 1$ (see (1.8)), is compatible with homotopy in M_n. Precisely: if $\psi, \psi' \in \mathcal{A}_n(X, Y)$ are homotopic in $\mathcal{A}_n$, then the morphisms φ, φ' represented by the generic pairs of ψ and ψ', respectively (see (3.4)(i)), are n-homotopic.

(5.6) Theorem. *The relation "$\approx_1$" of homotopy of morphisms from $M_1(X,Y)$ is an equivalence relation. Similarly, for $n > 1$, the relation "$\approx_{n+1}$" restricted to $M_n(X,Y)$ is an equivalence relation provided X is a binormal space.*

Actually, we prove that "$\approx_n$" is symmetric and transitive in M_n. But, for a morphism $\varphi_0 \in M_n(X,Y)$ $(n > 1)$, we are only able to find a homotopy joining φ_0 to itself which is merely an $(n+1)$-morphism.

Proof of (5.6). Let $\varphi_0 \in M_n(X,Y)$ be a morphism represented by a cotriad $X \underset{p}{\longleftarrow} G \underset{q}{\longrightarrow} Y$. By (2.6)(ii), $P = p \times \mathrm{id}_I \in \mathcal{V}_{n+1}(G \times I, X \times I)$. Clearly, the pair (P,Q) where $Q : G \times I \to Y$, $Q(g,t) = q(g)$ for $g \in G$, $t \in I$, represents a morphism which is a homotopy joining φ_0 to itself. If $n = 1$, then $P \in \mathcal{V}_1(G \times I, X \times I)$ (see (2.6)(i)).

The symmetry of "$\approx_n$" is obvious.

Now we prove transitivity. Let $\varphi_j \in M_n(X,Y)$, $j = 0,1,2$, and let $\varphi', \varphi'' \in M_n(X \times I, Y)$ join φ_0 to φ_1 and φ_1 to φ_2, respectively. Define $g : X \times \left[0, \frac{1}{2}\right] \to X \times I$, $h : X \times \left[\frac{1}{2}, 1\right] \to X \times I$ and $j : X \to X \times \left\{\frac{1}{2}\right\}$ by $g(x,t) = (x, 2t)$, $h(x,t) = (x, 2t-1)$ and $j(x) = \left(x, \frac{1}{2}\right)$. Clearly, $g \circ j = i_1$, $h \circ j = i_0$. In view of (3.7)(ii), $\varphi' \circ g \in M_n\left(X \times \left[0, \frac{1}{2}\right], Y\right)$, $\varphi'' \circ h \in M_n\left(X \times \left[\frac{1}{2}, 1\right], Y\right)$ and $\varphi' \circ g\big|X \times \left\{\frac{1}{2}\right\} = \varphi' \circ g \circ j = \varphi' \circ i_1 = \varphi_1 = \varphi'' \circ i_0 = \varphi'' \circ h \circ j = \varphi'' \circ h\big|X \times \left\{\frac{1}{2}\right\}$. By (4.2), there exists $\varphi \in M_n(X \times I, Y)$ such that $\varphi\big|X \times \left[0, \frac{1}{2}\right] = \varphi' \circ g$ and $\varphi\big|X \times \left[\frac{1}{2}, 1\right] = \varphi'' \circ h$. Hence φ is a homotopy joining φ_0 to φ_2 in M_n. ∎

The following notion plays an auxiliary role in the sequel. Let (X,A), (Y,B) be pairs of topological spaces. We say that pairs $(p,q), (p',q') \in D((X,A),(Y,B))$, where $p \in \mathcal{V}((G,T),(X,A))$ and $p' \in \mathcal{V}((G',T'),(X,A))$, are *h-linked* if there is a continuous map $f : (G,T) \to (G',T')$ (or $f' : (G',T') \to (G,T)$) such that the diagram

$$\begin{tikzcd} & (G,T) \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "f"] & \\ (X,A) & & (Y,B) \\ & (G',T') \arrow[ul, "p'"] \arrow[ur, "q'"'] & \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & (G',T') \arrow[dl, "p'"'] \arrow[dr, "q'"] \arrow[dd, "f'"] & \\ (X,A) & & (Y,B) \\ & (G,T) \arrow[ul, "p"] \arrow[ur, "q"'] & \end{tikzcd}$$

is homotopy commutative, i.e. q and p are homotopic to $q' \circ f$ and $p' \circ f$, respectively (or q' and p' are homotopic to $q \circ f'$ and $p \circ f'$, respectively).

Analogously, we say that morphisms $\varphi, \varphi' \in M((X,A),(Y,B))$ are *h-linked* if there are pairs $(p,q) \in \varphi$, $(p',q') \in \varphi'$ which are h-linked.

(5.7) Remark. By (3.4)(i), any morphism determining an n-acyclic map ψ is h-linked with the morphism represented by (p_ψ, q_ψ).

It is easy to see that homotopy has a lot in common with h-linking. Nevertheless, the two relations are different.

(5.8) Example. Let X be a topological space and let Y be a subset of a topological vector space. Assume that $\varphi_0 \in M_1(X,Y)$, $\varphi_1 \in M_n(X,Y)$, $n \geq 1$, and let $(1-t)\varphi_0(x) + t\varphi_1(x) \subset Y$ for any $x \in X$, $t \in I$. This means that the maps determined by φ_0 and φ_1 are homotopic in the class of all set-valued maps. However, it is not clear whether these morphisms are homotopic in M_m for some integer $m > 0$. But there exists an n-morphism Φ such that the morphisms φ_0, Φ and φ_1, Φ are h-linked. Indeed, suppose that cotriads $X \xleftarrow[p_j]{} G_j \xrightarrow[q_j]{} Y$ represent φ_j, $j = 0,1$. Define $G = \{(g_0, g_1) \in G_0 \times G_1 : p_0(g_0) = p_1(g_1)\}$ and $P : G \to X$, $Q : G \to Y$ by $P(g_0, g_1) = p_1(g_1)$ and $Q(g_0, g_1) = q_0(g_0)$. It is easy to see that $P \in \mathcal{V}_n(G, X)$. Moreover, define maps $f_j : G \to G_j$ by $f_j(g_0, g_1) = g_j$ for $j = 0, 1$. The diagram

$$\begin{tikzcd}
& G \arrow[dl, "P"'] \arrow[dr, "Q"] \arrow[dd, "f_j"] & \\
X & & Y \\
& G_j \arrow[ul, "p_j"] \arrow[ur, "q_j"'] &
\end{tikzcd}$$

is commutative for $j = 0$ and homotopy commutative for $j = 1$. This proves that the n-morphism Φ represented by the pair (P, Q) is h-linked with φ_0 and φ_1 simultaneously.

Now, we study extension of morphisms and its relation to homotopy.

Let X, Y be topological spaces and let $A \subset X$. We say that a morphism $\varphi^* \in M(X,Y)$ is an *extension* of a morphism $\varphi \in M(A,Y)$ if $\varphi^*|A = \varphi$. If φ^* is an extension of φ, then obviously $\varphi^*(x) = \varphi(x)$ for any $x \in A$; if $A \in C(X)$ and φ^* is an n-morphism, then by (4.1), $\varphi \in M_n(A,Y)$. As is readily seen, the problem of existence of an extension for morphisms is more complex than the one concerning single- or set-valued maps.

We begin with the simplest criterion for the existence of extensions.

(5.9) Proposition. *Let A be a retract of X.*

(i) *If $\varphi \in M_1(A,Y)$, then there is an extension $\varphi^* \in M_1(X,Y)$.*

(ii) *If $\varphi \in M_n(A,Y)$, $n > 1$, X is paracompact and there is a retraction $r : X \to A$ which is an F_σ-map with $\dim(r) \leq m$, then there is an extension $\varphi^* \in M_{n+m}(X,Y)$.*

Proof. It is sufficient to put $\varphi^* = \varphi \circ r$ and use (3.7)(ii). ∎

(5.10) Example. Let E be a normed space and let $D = \{x \in E : \|x\| \leq 1\}$. Then D is a retract of E, the radial retraction r is an F_σ-map and $\dim(r) = 1$. Hence, for any $\varphi \in M_n(D,Y)$, there is an extension $\varphi^* \in M_{n+1}(E,Y)$ for any space Y.

In the case when A and X are absolute neighbourhood retracts, the existence of extensions is a homotopy property. More generally:

(5.11) PROPOSITION. *Let X be compact, $A \in C(X)$ and the pair (X, A) be a cofibration. If $\varphi_0, \varphi_1 \in M_1(A, Y)$ are 1-homotopic and $\varphi \in M_1(A \times I, Y)$ is a homotopy joining them and there is an extension $\varphi_0^* \in M_1(X, Y)$ of φ_0, then there is $\varphi^* \in M_1(X \times I, Y)$ such that $\varphi^*|A \times I = \varphi$, $\varphi^* \circ i_0 = \varphi_0^*$ and $\varphi^* \circ i_1$ is an extension of φ_1.*

The assertion holds, in particular, when X, A are absolute neighbourhood retracts.

Proof. The proof is standard and goes as in the case of single-valued maps since, by assumption, $X \times \{0\} \cup A \times I$ is a retract of $X \times I$ and the relevant retraction is a closed map in view of the compactness of X. ■

Below, we give examples of the above situation.

(5.12) LEMMA. *Let E be a normed space, D its closed unit ball and $A = \operatorname{bd} D$. Assume additionally that Y is an absolute extensor for the class of normal spaces. If $\varphi \in M_1(A, Y)$ (or $\varphi \in M_n(A, Y)$, $n > 1$), then there exists an extension $\varphi^* \in M_1(D, Y)$ (or $\varphi^* \in M_{n+1}(D, Y)$) of φ.*

Proof. Assume that a cotriad $A \xleftarrow{p} G \xrightarrow{q} Y$ represents a morphism $\varphi \in M(A, Y)$. Let $f : G \times \{0\} \to X$ be given by the formula $f(g, 0) = p(g)$ and put $G^* = G \times [0, 1] \cup_f D$ (i.e. we identify $x \in D$ with the set $f^{-1}(x)$). If $i_0 : G \times [0, 1] \to G \times [0, 1] \oplus D$, $i_1 : D \to G \times [0, 1] \oplus D$ are embeddings and $v : G \times [0, 1] \oplus D \to G^*$ is the quotient map, then $j_k = v \circ i_k$, $k = 0, 1$, is continuous, $G = j_0(G \times [0, 1]) \cup j_1(D)$, $j_0|G \times (0, 1]$ and j_1 are homeomorphisms and j_0 is a closed map (comp. [32]). Define $p^* : G^* \to D$ by the formula

$$\begin{aligned} p^*(j_0(g, t)) &= \left(\tfrac{1}{2} + \tfrac{1}{2}t\right)p(g) && \text{for } g \in G,\ t \in I, \\ p^*(j_1(x)) &= \tfrac{1}{2}x && \text{for } x \in D. \end{aligned}$$

The map p^* is well-defined, continuous and $p^*(G^*) = D$. For any $C \in C(G^*)$, we have $p^*(C) = s(j_0^{-1}(C)) \cup \frac{1}{2}j_1^{-1}(C)$ where $s : G \times I \to D$ is given by $s(g, t) = \left(\frac{1}{2} + \frac{1}{2}t\right)p(g)$. Hence, p^* is a closed map. For any $x \in D$,

$$p^{*-1}(x) = \begin{cases} j_1(2x) & \text{for } \|x\| \le \frac{1}{2}, \\ j_0(p^{-1}(\|x\|x) \times \{2\|x\| - 1\}) & \text{for } \|x\| > \frac{1}{2}; \end{cases}$$

therefore, p^* is a perfect surjection.

Now, we show that $\operatorname{rd}_D(s_{p^*}^k) < n + 1 - k - 1$ for any $k \ge 0$ and $n > 1$. First, observe that $s_{p^*}^k \subset T = \left\{x \in E : \|x\| > \frac{1}{2}\right\}$ since, for $x \in D$ with $\|x\| \le \frac{1}{2}$, $\widetilde{H}^k(p^{*-1}(x)) = 0$, $k \ge 0$. Next, A is a retract of T; the radial retraction $r : T \to A$ is an F_σ-map. Since $x \in s_{p^*}^k$ if and only if $r(x) \in s_p^k$, it follows that r maps $s_{p^*}^k$ onto s_p^k. Let $C \subset C(D)$, $C \subset s_{p^*}^k$. Then $r(C) \subset s_p^k$. Since A, T are metric spaces and $\dim(r) = 1$, we have $\dim(C) \le \dim(r(C)) + 1 \le \operatorname{rd}_A(s_p^k) + 1 < n + 1 - k - 1$ (see [2]).

Clearly, if $\varphi \in M_1$, then $s^k_{p^*} = \emptyset$ for any $k \geq 0$. It is easily seen that $p^{*-1}(A) = j_0(G \times \{1\}) \approx_{\mathrm{top}} G \times \{1\} \approx_{\mathrm{top}} G$. Since G is paracompact (as the preimage of a paracompact space under the perfect map p) and Y is an absolute extensor for the class of normal spaces, there is a continuous map $q^* : G^* \to Y$ such that $q^*(j_0(g,1)) = q(g)$ for $g \in G$. Now, it suffices to represent a morphism φ^* by (p^*, q^*). ∎

(5.13) COROLLARY. *If S is an m-dimensional simplex in $\mathbb{R}^k$, $k \geq m$, Y is an absolute extensor for the class of normal spaces, $\varphi \in M_1(\partial S, Y)$ (or $\varphi \in M_n(\partial S, Y)$, $n > 1$), then there is an extension $\varphi^* \in M_1(S, Y)$ (or $\varphi^* \in M_{n+1}(S, Y)$) of φ (where ∂S denotes the geometric boundary of S).*

(5.14) THEOREM. *Let (P, P_0) be a (finite) polyhedral pair. If Y is an absolute extensor for the class of normal spaces, $\varphi \in M_1(P_0, Y)$ (or $\varphi \in M_n(P_0, Y)$, $n > 1$), then there is an extension $\varphi^* \in M_1(P, Y)$ (or $\varphi^* \subset M_{n+m}(P, Y)$ where $m = \max\{\dim(S) : S \in T,\ S \notin T_0\}$ and (T, T_0) is a triangulation of the pair (P, P_0)) of φ.*

Proof. Let $\dim(P) = N \geq N_0 = \dim(P_0)$. For any $0 \leq k \leq N$, by T^k we denote the k-dimensional skeleton of T, $P^k = |T^k|$; and similarly, for $0 \leq k \leq N_0$, T_0^k denotes the k-dimensional skeleton of T_0, $P_0^k = |T_0^k|$. Then, for $0 \leq k \leq N_0$, T_0^k is a subcomplex of T^k, and $P^N = P$, $P_0^{N_0} = P_0$. We shall construct a family of morphisms $\varphi_k \in M(P^k, Y)$, $k = 0, 1, \ldots, N$, such that

(i) $\varphi_k | P^{k-1} = \varphi_{k-1}$ for $1 \leq k \leq N$;
(ii) $\varphi_k | P_0^k = \varphi | P_0^k$ for $0 \leq k \leq N_0$.

The morphism $\varphi^* = \varphi_N$ will satisfy the assertion.

Let $P_0 \xleftarrow{p} G \xrightarrow{q} Y$ represent φ. Assume that $P^0 = \{x_1, \ldots, x_r\}$ and the vertices are ordered in such a manner that $x_1, \ldots, x_s$ ($s \leq r$) belong to P_0 and $x_{s+1}, \ldots, x_r \notin P_0$. Put $G_0 = p^{-1}(\{x_1, \ldots, x_s\}) \cup \{x_{s+1}, \ldots, x_r\}$ and define a map $p_0 : G_0 \to P^0$ by the formula $p_0(g) = p(g)$ for $g \in p^{-1}(\{x_1, \ldots, x_s\})$ and $p_0(x_j) = x_j$ for $s+1 \leq j \leq r$. Define $q_0 : G_0 \to Y$ similarly, i.e. $q_0(g) = q(g)$ for $g \in p^{-1}(\{x_1, \ldots, x_s\})$ and $q_0(x_j) = y_j$, $s+1 \leq j \leq r$, where y_j are arbitrarily chosen points of Y. Clearly, the morphism φ_0 represented by the pair (p_0, q_0) satisfies condition (ii).

Assume that, for $0 \leq k \leq N-1$, we have defined morphisms $\varphi_k \in M(P^k, Y)$ such that conditions (i), (ii) are satisfied. Now, it suffices to define φ_{k+1} on an arbitrary simplex S from T^{k+1} in such a way that $\varphi_{k+1} | \partial S = \varphi_k | \partial S$; next use (4.2) in order to piece together the given morphisms and obtain the required $\varphi_{k+1} \in M(P^{k+1}, Y)$.

Let $S \in T^{k+1}$. If $k+1 \leq N_0$ and $S \in T_0^{k+1}$, then we put $\varphi_{k+1} = \varphi | S$. If $k+1 > N_0$ or $S \notin T_0^{k+1}$, then, in view of (5.13), there is $\varphi_{k+1} \in M(S, Y)$ such that $\varphi_{k+1} | \partial S = \varphi_k | \partial S$.

From the above construction it follows that φ_{k+1} satisfies conditions (i), (ii). Moreover, if $\varphi \in M_n(P_0, Y)$, then $\varphi^* = \varphi_N \in M_{n+m}(P, Y)$ and $\varphi^* \in M_1(P, Y)$ provided $\varphi \in M_1(P_0, Y)$. ∎

As an immediate corollary we get:

(5.15) THEOREM. *Let* $X \subset \mathbb{R}^N$ *and* $A \subset X$ *be a compact neighbourhood retract in* $\mathbb{R}^N$. *If* $\varphi \in M_1(A, \mathbb{R}^M)$, *then there exists an extension* $\varphi^* \in M_1(X, \mathbb{R}^M)$ *of* φ.

Obviously, in view of (5.14), the above assertion generalizes to n-morphisms.

P r o o f. There are a compact neighbourhood U of A in $\mathbb{R}^N$ and a retraction $r : U \to A$. There is a polyhedron P such that $A \subset P \subset U$. By (5.9), there is a morphism $\varphi' \in M_1(U, \mathbb{R}^M)$ such that $\varphi'|A = \varphi$. Let $\varphi'' = \varphi'|P$. In view of (5.14), we can find $\varphi''' \in M_1(\mathrm{conv}P, \mathbb{R}^M)$ such that $\varphi'''|P = \varphi''$ and, in view of (5.9), there is $\varphi'^{\vee} \in M_1(\mathbb{R}^N, \mathbb{R}^M)$ such that $\varphi'^{\vee}|\mathrm{conv}P = \varphi'''$. Now, it suffices to define $\varphi^* = \varphi'^{\vee}|X$. ■

In particular, we get the following criterion for the existence of extensions of acyclic maps.

(5.16) COROLLARY. *Under the assumptions of* (5.15), *if* $\psi \in \mathcal{A}_1(A, \mathbb{R}^M)$, *then there exists a map* $\psi^* : X \to K(\mathbb{R}^M)$ *determined by a* 1*-morphism and such that* $\psi^*|A = \psi$.

Other facts concerning the existence of extensions of convex-valued maps can be found e.g. in [80], [94].

6. Essentiality of morphisms. The notion of essentiality of single-valued continuous maps into spheres was considered by K. Borsuk and other authors; next, it was thoroughly studied by A. Granas in a more general setting and used in fixed-point theory (see [51], [52], [26] and others). Some aspects of this notion for set-valued maps were presented in [46], [68].

Below, we present several results concerning the essentiality of maps determined by morphisms. First, we give a relatively general and abstract setting, and next, we confine ourselves to morphisms acting in finite-dimensional spaces.

Let X, Y be topological spaces, let $A \subset X$ be closed and $K \subset Y$ compact. Consider a morphism $\varphi \in M_1((X, A), (Y, Y \backslash K))$. We shall be interested whether there exists $x \in X$ such that $\varphi(x) \cap K \neq \emptyset$.

We say that a morphism φ is *inessential on* A *over* K if it is 1-homotopic to a morphism $\widetilde{\varphi} \in M_1((X, A), (Y, Y \backslash K))$ such that $\widetilde{\varphi}(X) \cap K = \emptyset$ (i.e. there is a homotopy $\Phi \in M_1((X \times I, A \times I), (Y, Y \backslash K))$ such that $\Phi \circ i_0 = \varphi$ and $\Phi \circ i_1 = \widetilde{\varphi}$). The morphism φ is *weakly inessential on* A *over* K if there exists an extension $\widetilde{\varphi} \in M_1((X, A), (Y, Y \backslash K))$ of $\varphi|A$ such that $\widetilde{\varphi}(X) \cap K = \emptyset$. Conversely, we say that φ is (*strongly*) *essential* (on A over K) if it is not (weakly) inessential.

It is quite obvious, in view of the results of Section 5, that on this level of generality we are forced to consider the notion of essentiality in the class of 1-morphisms (see the comments and examples at the end of the present section).

(6.1) PROPOSITION. *1-homotopic morphisms are either both essential or both inessential.*

Proof. This follows from the definition and the transitivity of homotopy. ∎

The next result states that inessential (resp. strongly essential) morphisms are actually weakly inessential (resp. essential).

(6.2) THEOREM. *Let X be a normal space. If $\varphi_0 \in M_1((X,A),(Y,Y\backslash K))$ is inessential on A over K, then there exists a morphism $\varphi_1 \in M_1((X,A),(Y,Y\backslash K))$ such that $\varphi_0 \simeq_1 \varphi_1$ (rel A) (mod A) and $\varphi_1(X)\cap K=\emptyset$.*

Proof. By definition there is a morphism $\varphi^* \in M_1((X\times I, A\times I,(Y,Y\backslash K))$ such that $\varphi^*\circ i_0=\varphi_0$ and $\varphi_1^*(X)\cap K=\emptyset$ where $\varphi_1^*=\varphi^*\circ i_1$. Let $B=\{x\in X: \varphi^*(x,t)\cap K\neq\emptyset$ for some $t\in I\}$. Clearly, $B=\mathrm{pr}_X(\varphi^{*-1}_{-}(K))$ where $\mathrm{pr}_X: X\times I\to X$ is the projection. Since the map determined by φ^* is u.s.c. and pr_X is a closed map, B is closed and $A\cap B=\emptyset$. The Urysohn Lemma gives a continuous map $f: X\to I$ such that $f|A=0$ and $f|B=1$. Let $\varphi=\varphi^*\circ g$ where $g: X\times I\to X\times I$ is given by $g(x,t)=(x,f(x)t)$. Clearly, $\varphi\in M_1(X\times I,Y)$ and, for $x\in A$ and $t\in I$, $\varphi(x,t)\subset Y\backslash K$. If we put $\varphi_1=\varphi\circ i_1$, then $\varphi_0\simeq_1\varphi_1$ (rel A) and $\varphi_1(X)\cap K=\emptyset$. ∎

(6.3) Remark. In contrast to (6.1), the question of homotopy invariance of weak inessentiality (or strong essentiality) remains open. Of course, if (X,A) is a cofibration and X is compact, then in view of (5.10), these notions are equivalent. Similarly, in the class of compact convex-valued maps in place of maps determined by morphisms (e.g. acyclic maps), we get the full homotopy invariance of weak inessentiality (for single-valued continuous maps — see [26]). Compare also the discussion in [68].

The essentiality of morphisms has the following property of localization.

(6.4) THEOREM. *Let $B\subset A$ be a closed set and $L\subset Y$ a compact superset of K. Assume that $\varphi\in M_1(X,Y)$ and $\varphi(A)\subset Y\backslash L$.*

(i) *If φ is (weakly) inessential on A over L, then φ is (weakly) inessential on B over K.*

(ii) *Additionally, assume that X is compact, Y is a locally convex topological vector space and L is convex. If φ is (weakly) inessential on A over K, then it is (weakly) inessential on B over L.*

Proof. (i) is immediate. To prove (ii), it is enough to show the assertion when $A=B$ and then use (i). Let φ be inessential on A over K. By (6.2), there is a morphism $\widetilde{\varphi}\in M_1((X,A),(Y,Y\backslash K))$ such that $\varphi\simeq_1\widetilde{\varphi}$ (rel A) and $\widetilde{\varphi}(X)\cap K=\emptyset$. In particular, $\Phi'(A\times I)\subset Y\backslash L$ where Φ' is a homotopy (rel A) joining φ to $\widetilde{\varphi}$. Without any loss of generality we may assume that $0\in K$. There is a closed convex neighbourhood U of 0 in Y such that $\widetilde{\varphi}(X)\cap(U+K)=\emptyset$. Since L is compact there is $\lambda>0$ such that $L\subset \mathrm{int}\,\lambda U$. Let $V=\lambda U$ and let $v: Y\to[0,\infty)$ be the Minkowski gauge of V. Define a retraction $r: Y\backslash\{0\}\ni y\mapsto r(y)=[v(y)]^{-1}y\in$

$\operatorname{bd} V$. Now, assume that $(\widetilde{p}, \widetilde{q}) \in \widetilde{\varphi}$ where $\widetilde{p} \in \mathcal{V}_1(\widetilde{G}, X)$ and $\widetilde{q} : \widetilde{G} \to Y$. Consider the morphism $\overline{\varphi}$ represented by the pair $(\widetilde{p}, \overline{q})$ where $\overline{q} = r \circ \widetilde{q} : \widetilde{G} \to \operatorname{bd} V$. Obviously, $\overline{\varphi} \in M_1((X, A), (Y, Y \backslash L))$ and $\overline{\varphi}(X) \cap L = \emptyset$. The morphism Φ'' represented by the pair (P, Q), where $P : \widetilde{G} \times I \to X \times I$, $Q : \widetilde{G} \times I \to Y$ and $P(g, t) = (\widetilde{p}(g), t)$, $Q(g, t) = (1 - t)\widetilde{q}(g) + t\overline{q}(g)$ for $g \in \widetilde{G}$, $t \in I$, is such that $\Phi''(A \times I) \subset Y \backslash L$ and furnishes a homotopy from $M_1((X, A), (Y, Y \backslash L))$ joining $\widetilde{\varphi}$ to $\overline{\varphi}$. Hence, in view of (5.6), $\varphi \in M_1((X, A), (Y, Y \backslash L))$ is inessential on A over L. The proof that weak inessentiality over K implies weak inessentiality over L is similar. ∎

(6.5) Corollary (Excision property). *Let $\varphi \in M_1((X, A), (Y, Y \backslash K))$, let U be open in X and $U \cap A = \emptyset$. If $\varphi_-^{-1}(K) \subset U$ and φ is (strongly) essential on A over K, then $\varphi|\operatorname{cl} U$ is (strongly) essential on $\operatorname{bd} U$ over K.*

Proof. Suppose that $\varphi' = \varphi|\operatorname{cl} U$ is inessential on $\operatorname{bd} U$ over K. By (6.2), there exists $\widetilde{\varphi}' \in M_1((\operatorname{cl} U, \operatorname{bd} U), (Y, Y \backslash K))$ such that $\varphi' \simeq_1 \widetilde{\varphi}'(\operatorname{rel} \operatorname{bd} U)$ and $\widetilde{\varphi}'(\operatorname{cl} U) \cap K = \emptyset$. Let $\widetilde{\varphi}$ be the morphism obtained by piecing together $\varphi|X \backslash U$ and $\widetilde{\varphi}'$ (see (4.2)). Clearly, $\varphi \simeq_1 \widetilde{\varphi}(\operatorname{rel}(X \backslash U))$ and $\widetilde{\varphi}(X) \cap K = \emptyset$; hence φ is inessential on $X \backslash U$ over K. By (6.4)(i), we get a contradiction. ∎

(6.6) Corollary. *Under the assumptions of* (6.4)(ii), *if $\varphi \in M_1((X, A), (Y, Y \backslash L))$ is essential on A over L and W is the component of $Y \backslash \varphi(A)$ containing L, then W is bounded, $W \subset \varphi(X)$ and $Y \backslash \varphi(A)$ is not connected.*

Proof. Let $w \in W$ and let x be an arbitrary point of L. There is an arc P which is the union of linear segments $P_i = [w_{i-1}, w_i], 1 \le i \le n$, where $w_0 = w$ and $w_n = x$, joining w to x in W, i.e.

$$P = P_1 \cup \ldots \cup P_n \subset W.$$

Suppose that $w \notin \varphi(X)$. This means that φ is inessential on A over $\{w\}$. By (6.4)(ii), φ is inessential over P_1 and, hence, over $\{w_1\}$ as well. By induction, φ is inessential over $\{x\}$ and, once again by (6.4)(ii), it is inessential over L — a contradiction. Obviously, W is bounded as a subset of the compact set $\varphi(X)$ (see (1.2)(iv)), and since $Y \backslash \varphi(A)$ is not bounded, it cannot be connected. ∎

Below, we give some examples of essential and inessential morphisms. For simplicity, we confine ourselves to the special case $X = B^n$, $A = S^{n-1}$, $A = \mathbb{R}^n$ and $K = \{0\}$, $n \ge 1$. In view of (5.10), in this case the notions of essentiality and strong essentiality are equivalent (see also (6.3)).

(6.7) Example (Normalization property). (i) Any morphism

$$\overline{\varphi} \in M_1((B^n, S^{n-1}), (\mathbb{R}^n, \mathbb{R}^n \backslash \{0\}))$$

such that $\overline{\varphi}|S^{n-1}$ determines the inclusion $j : S^{n-1} \to \mathbb{R}^n \backslash \{0\}$ is essential (on S^{n-1} over $\{0\}$). Indeed, suppose that there is a morphism $\varphi \in M_1(B^n, \mathbb{R}^n \backslash \{0\})$ such that $\varphi|S^{n-1} = \overline{\varphi}|S^{n-1}$. Let φ be represented by a pair (p, q), where $p \in$

$\mathcal{V}_1(G, B^n)$ and $q : G \to \mathbb{R}^n \backslash \{0\}$, and let $i : S^{n-1} \to B^n,\ w : p^{-1}(S^{n-1}) \to G$ be the inclusions. It is easy to see that the diagram

$$\begin{array}{ccccc} & & \mathbb{R}^n\backslash\{0\} & & \\ & {}^{a}\swarrow & & \nwarrow^{q} & \\ S^{n-1} & \underset{p}{\leftarrow} & p^{-1}(S^{n-1}) & \underset{w}{\rightarrow} & G \\ & {}_{i}\searrow & & \swarrow_{p} & \\ & & B^n & & \end{array}$$

is commutative. Therefore, in cohomology, we have

$$H^{n-1}(i) \circ [H^{n-1}(p)]^{-1} \circ H^{n-1}(q) = H^{n-1}(j),$$

which leads to a contradiction. One can easily show that there is no $(n-1)$-morphism $\varphi \in M_{n-1}(B^n, \mathbb{R}^n\backslash\{0\})$ which is an extension of a morphism determining j. However:

(ii) There is a $(2n+1)$-morphism which is an extension of j onto B^n. Let $G = \big\{(x, y) \in B^n \times (\mathbb{R}^n\backslash\{0\}) : |x| \geq \frac{1}{2},\ |y| = |x|$ and $|y - x| \leq 4 - 4|x|$ or $|x| \leq \frac{1}{2},\ |y| = \frac{1}{2}\big\}$. Let $p : G \to B^n$ and $q : G \to \mathbb{R}^n\backslash\{0\}$ be the projections. The pair (p, q) represents the required $(2n+1)$-morphism.

(iii) J. Jezierski has built (Zeszyty Nauk. Uniw. Gdań. 6 (1987)) a (1-2-3)-mapping $\psi : B^2 \to K(S^1)$ (i.e. a u.s.c. map such that, for any $x \in B^2$, $\psi(x)$ is a set of 1, 2 or 3 elements) such that, for $x \in S^1$, $\psi(x) = x$. Clearly, this map ψ is determined by a 4-morphism. The construction from (ii) indicates the existence of a 5-morphism with this property. In view of the results from [28], it follows that a (1-2)-map with this property does not exist. It would be interesting to find out whether there exists a morphism $\varphi \in M_k(B^n, \mathbb{R}^n\backslash\{0\})$ which is an extension of j onto B^n for $n - 1 < k < 2n + 1$.

The above examples (ii), (iii) show that it is rather senseless to discuss the notion of essentiality in M_k for $k > 1$. In the special case $X = B^n$, one can try to do that in M_{n-1} but there would be difficulties in obtaining the homotopy invariance of this new notion.

Below we provide the simplest example of an inessential morphism.

(6.8) Example. (i) An arbitrary m-morphism determining a constant k-acyclic map $\psi : S^{n-1} \to K(S^{n-1})$ (i.e. such that, for each $x, y \in S^{n-1}$, $\psi(x) = \psi(y)$) has an extension in $M_{m+1}(B^n, \mathbb{R}^n\backslash\{0\})$ provided $k \leq 2n - 1$. Indeed, let $T = \psi(x)$ for $x \in S^{n-1}$. There is $y_0 \in S^{n-1}\backslash T$ since otherwise φ would be a $2n$-acyclic map. Let a pair (p, q), where $p \in \mathcal{V}_m(G, S^{n-1})$, $q : G \to S^{n-1}$, represent an m-morphism that determines ψ. Consider the map $Q : G \times I \to S^{n-1}$ given by $Q(g, t) = |tq(g) - (1-t)y_0|^{-1}(tq(g) - (1-t)y_0)$ for $g \in G$, $t \in I$. The map Q furnishes a homotopy between q and $q_0 : G \to S^{n-1}$ given by $q_0(g) = -y_0$ for any $g \in G$. In view of (5.12), the m-morphism φ_0 represented by the pair (p, q_0) has an extension $\varphi_0^* \in M_{m+1}(B^n, \mathbb{R}^n\backslash\{0\})$. Assume that a pair (p^*, q_0^*), where $p^* \in \mathcal{V}_{m+1}(G^*, B^n)$, $q_0^* : G^* \to \mathbb{R}^n\backslash\{0\}$, represents φ_0^*. Without any loss of generality we may assume that $G \subset G^*$ and $p^*|G = p$. The space G^* is normal and G is its closed subset. Since $\mathbb{R}^n\backslash\{0\}$ is an absolute neighbourhood extensor for the class of normal spaces, the pair (G^*, G) has the homotopy extension property. Therefore, there is a continuous map $q^* : G^* \to \mathbb{R}^n\backslash\{0\}$ such that $q^*|G = q$. Clearly, the pair (p^*, q^*) represents an $(m+1)$-morphism φ^* such that $\varphi^*|S^{n-1} = \varphi$.

In particular, any morphism $\varphi \in M_1((B^n, S^{n-1}), (\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}))$ such that $\varphi|S^{n-1}$ determines a constant 1-acyclic map $\psi : S^{n-1} \to K(S^{n-1})$ is inessential.

(ii) In the case when $k > 2n - 1$, a constant k-acyclic map may or may not have extensions onto B^n and into $\mathbb{R}^n\backslash\{0\}$ determined by morphisms. For example, consider the 1-morphism φ defined in (5.1). By (5.3), $\varphi \circ \varphi$ is homotopic in M_1 to id_{S^1}, hence, by (6.1), (5.11), it has no extensions onto B^2 with values in $\mathbb{R}^2\backslash\{0\}$ belonging to M_1. On the other hand, the map ψ determined by $\varphi \circ \varphi$ is constant 4-acyclic. The 4-morphism represented by the generic pair (p_ψ, q_ψ) determines ψ. However, one easily sees that this morphism has an extension in $M_4(B^2, \mathbb{R}^2\backslash\{0\})$.

The above examples show that the homotopy and extension properties of morphisms depend deeply on the internal structure of a morphism and not on a map determined by it. Moreover, we think that studying the generalized equation

$$\varphi(x) \cap K \neq \emptyset, \qquad x \in X,$$

it is better to consider morphisms with their structure rather than merely maps (e.g. acyclic ones).

7. Concluding remarks. In [44], L. Górniewicz and A. Granas gave another definition of a morphism. First, by a Vietoris map $p : G \to X$, where G, X are spaces, they understood a perfect surjection such that $\widetilde{\widehat{H}}_*(p^{-1}(x); \mathbb{Q}) = 0$ for any $x \in X$, where $\mathbb{Q}$ stands for the field of rational numbers and $\widehat{H}_*$ is Čech homology with compact carriers (see [41]). In the set $D(X, Y)$ of all pairs $X \xleftarrow{p} G \xrightarrow{q} Y$, where p is a Vietoris map (in the above sense) and q is continuous, they considered a relation "$\sim$" such that $(p, q) \sim (p', q')$, where $X \xleftarrow{p'} G' \xrightarrow{q'} Y$, if and only if there are maps $f : G \to G'$ and $g : G' \to G$ such that $q' \circ f = q$, $p' \circ f = p$ and $q \circ g = q'$, $p \circ g = p'$. Equivalence classes of this relation were called morphisms. We easily see that our relation "$\approx$" is more restrictive. However, we believe that, having the relation introduced in Section 3, we will be able to obtain more interesting results concerning morphisms.

Moreover, in [41] (comp. [17]), acyclic maps with respect to the Čech homology with rational coefficients were considered. A map $\psi : X \to K(Y)$ is said to be acyclic (w.r.t. $\widehat{H}_*$) if, for any $x \in X$, $\widetilde{\widehat{H}}_*(\psi(x); \mathbb{Q}) = 0$.

(7.1) PROPOSITION. *An acyclic* (*w.r.t.* H) *map* $\psi : X \to K(Y)$ *is* 1-*acyclic w.r.t.* H^* *with integer coefficients.*

Proof. First, it is easy to see that $\widetilde{H}^*(\psi(x); \mathbb{Q}) = 0$ (see [41] and A.(1.8)). The assertion now follows from the universal coefficients theorem for Alexander–Spanier cohomology. ∎

In view of the above proposition we see that morphisms (or rather maps determined by them) embrace other classes of set-valued maps considered elsewhere (cf. e.g. [11], [39], [42], [17]).

As another consequence of the universal coefficients theorem we get:

(7.2) Proposition. *Let R be an arbitrary principal ideal domain. A map $p : G \to X$ is a $\mathcal{V}_1$-map with respect to $\mathbb{Z}$ if and only if p is a $\mathcal{V}_1$-map w.r.t. R. If p is a $\mathcal{V}_n$-map, $n \geq 1$, w.r.t. R, then p is a $\mathcal{V}_n$-map w.r.t. $\mathbb{Z}$.*

Proof. Observe that $s_p^k(\mathbb{Z}) \subset s_p^k(R)$ for any integer $k \geq 0$. Indeed, if $x \notin s_p^k(R)$, then $\widetilde{H}^k(p^{-1}(x); R) = 0$. Hence, the exactness of the sequence (see [104])

$$0 \to H^k(p^{-1}(x); \mathbb{Z}) \otimes R \to H^k(p^{-1}(x); R) \to H^{k+1}(p^{-1}(x); \mathbb{Z}) * R \to 0$$

entails that $\widetilde{H}^k(p^{-1}(x); \mathbb{Z}) = 0$ and $x \in s_p^k(\mathbb{Z})$. Therefore

$$\mathrm{rd}_X(s_p^k(\mathbb{Z})) \leq \mathrm{rd}_X(s_p^k(R)) < n - k - 1. \ ■$$

The above result shows that morphisms represented by pairs (p, q), where p is a $\mathcal{V}$-map with respect to $\mathbb{Z}$, play the central role in our investigations. Hence, in the next sections, we shall be mainly interested in these morphisms.

II. The topological degree theory of morphisms

In this chapter we present the theory of the topological degree of morphisms (and, in particular, of maps determined by morphisms) defined on finite-dimensional manifolds. It is our aim to obtain an integer-valued degree by applying Alexander–Spanier cohomology with integer coefficients.

The fixed-point index (or the degree) of set-valued maps defined on open subsets of the Euclidean space $\mathbb{R}^n$ or on S^n (resp. on open subsets of Banach spaces or absolute neighbourhood retracts) was considered and intensively studied by many authors:

1. Convex-valued maps – [49], [21], [10], [80];
2. Acyclic maps – [10], [41], [107], [39], [44], [71], [9], [20], [101], [103];
3. n-Acyclic maps – [10], [11], [17], [16];
4. Maps with proximally ∞-connected values – [10], [46], [47];
5. Morphisms (in the sense of [44]) – [44], [41], [43], [99];
6. So-called admissible maps – [41], [17], [71].

In [99], 1-morphisms on compact manifolds were studied.

In the above-mentioned papers, Čech homology theory with compact carriers was used and the rational-valued degree or index was defined (see e.g. [41], [71], [44], [99], [43]), while e.g. [11], [10], [17], [16] use Čech cohomology. Moreover, in these papers (except for [21], [80], [9], [46], [47], [68], [101], [103] where suitable approximation techniques were used) the Eilenberg–Montgomery method [30] based on the Vietoris theorem and developed by Górniewicz (e.g. [43], [45], [17]) together with ideas of Dold [24], [25] were applied (see also [90], [93]).

In the case of morphisms defined on open subsets of $\mathbb{R}^n$ (or on S^n) the homological approach does not differ too much from the cohomological one (for example compare [41] and [17]). We are going to define an integer-valued degree of morphisms of manifolds. Therefore we are forced to use cohomology theory. However, it is to be observed that in the case of manifolds in order to adapt the method of Dold (used in the homological setting for continuous single-valued maps) to morphisms and to cohomological context one should overcome several technical difficulties.

In [24] (comp. [25]), A. Dold introduces the following definition of the degree of continuous single-valued maps. Let $(X,\mu),(Y,\omega)$ be oriented m-dimensional manifolds and let $F: X\to Y$ be a continuous map. If $L\subset Y$ is a compact connected set and the set $K=F^{-1}(L)$ is compact, then $H_m(F): H_m(X,X\backslash K)\to H_m(Y,Y\backslash L)$ and $H_m(F)(\mu_K)=c\omega_L$ where $c\in\mathbb{Z}$. The degree of F over L is defined by the equality $\deg_L F=c$. The above approach cannot be adapted to the situation when one considers cohomology (and we have to do that in order to get an integer-valued degree).

In the first section of this chapter we study the cohomological properties of morphisms. In Section 2 we prepare some additional objects and notions necessary in the sequel. Next sections are devoted to the definition of the degree and its properties. The last section presents some results generalizing the well-known Borsuk theorem on antipodes.

1. Cohomological properties of morphisms. Let X be a paracompact space, $A\in F_\sigma(X)$; let (Y,B) be a pair of topological spaces and let $\varphi\in M_n((X,A),(Y,B))$, $n\ge 1$. Assume that a pair (p,q) represents φ, where $p\in\mathcal{V}_n((G,T),(X,A))$ and $q:(G,T)\to(Y,B)$ is continuous. In view of I.(2.8), $H^k(p): H^k(X,A;R)\to H^k(G,T;R)$ is an isomorphism for any integer $k\ge n+1$ (for all $k\ge 0$ if $n=1$) and for any principal ideal domain R. Hence, for $k\ge n+1$ (or $k\ge 0$ when $n=1$), we may define $H^k(\varphi): H^k(Y,B;R)\to H^k(X,A;R)$ by

$$H^k(\varphi)=[H^k(p)]^{-1}\circ H^k(q).$$

This definition is correct: if $(p,q)\approx(p',q')$, where $p'\in\mathcal{V}_n((G',T'),(X,A))$, $q':(G',T')\to(Y,B)$, then $[H^k(p')]^{-1}\circ H^k(q')=[H^k(p)]^{-1}\circ H^k(q)$, $k\ge n+1$.

Let $\varphi_1\in M_n((X,A),(Y,B))$, $n\ge 1$, $\varphi_2\in M_1((Y,B),(Z,C))$.

(1.1) LEMMA. *For any $k\ge n+1$ ($k\ge 0$ when $n=1$),*

$$H^k(\varphi_2\circ\varphi_1)=H^k(\varphi_1)\circ H^k(\varphi_2).$$

P r o o f. For simplicity, let $A=B=C=\emptyset$. Let $(p_j,q_j)\in\varphi_j$, $j=1,2$, where $p_1\in\mathcal{V}_n(G_1,X)$, $p_2\in\mathcal{V}_1(G_2,Y)$ and $q_1: G_1\to Y$, $q_2: G_2\to Z$ are continuous. By the definition of composition, the n-morphism $\varphi_2\circ\varphi_1$ is represented by the pair $(p_1\circ\overline{p}_2, q_2\circ\overline{q}_1)$ where $(\overline{p}_2,\overline{q}_1)$ is the pull-back of (q_1,p_2). We have the commutative diagram

$$\begin{tikzcd}
 & & G_1 \boxtimes_Y G_2 \arrow[dl, "\bar{p}_2"'] \arrow[dr, "\bar{q}_1"] \arrow[dd, "f"] & & \\
X \arrow[r, "p_1"'] & G_1 \arrow[dr, "q_1"'] & & G_2 \arrow[dl, "p_2"] \arrow[r, "q_2"'] & Z \\
 & & Y & &
\end{tikzcd}$$

where $f : G_1 \boxtimes_Y G_2 \to Y$ is the fibre product of (q_1, p_2). For $k \geq n+1$,

$$\begin{aligned}
H^k(\varphi_2 \circ \varphi_1) &= [H^k(p_1 \circ \overline{p}_2)]^{-1} \circ H^k(q_2 \circ \overline{q}_1) \\
&= [H^k(p_1)]^{-1} \circ H^k(q_1) \circ [H^k(p_2)]^{-1} \circ H^k(q_2) = H^k(\varphi_1) \circ H^k(\varphi_2). \blacksquare
\end{aligned}$$

(1.2) LEMMA. *Under the assumptions of* (1.1), *the diagram*

$$\begin{tikzcd}
H^k(Y,B) \arrow[r, "H^k(\varphi)"] & H^k(X,A) \\
H^{k-1}(B) \arrow[u, "\delta"] \arrow[r, "H^{k-1}(\psi)"'] & H^{k-1}(A) \arrow[u, "\delta"]
\end{tikzcd}$$

is commutative, *where* $\varphi \in M_n((X,A),(Y,B))$, $\psi = \varphi|A$ *and* $k \geq n+1$ ($k \geq 0$ *when* $n = 1$).

(1.3) LEMMA. *Under the above assumptions on* X, A *and* Y, B, *if* $\varphi_0, \varphi_1 \in M_n((X,A),(Y,B))$ *are* n-*homotopic*, *then*

$$H^k(\varphi_0) = H^k(\varphi_1)$$

for $k \geq n+1$ ($k \geq 0$ *when* $n = 1$).

(1.4) THEOREM. *The cofunctor* H^* *of Alexander–Spanier cohomology extends from the category* TOP^2 (*restricted to pairs of paracompact spaces*) *to a cofunctor defined on* MOR_1^2 *and satisfying all axioms of cohomology theory.*

Proof. This follows easily from the following lemma.

(1.5) LEMMA. *If* n-*morphisms* φ', φ'' *are* h-*linked*, *then*

$$H^k(\varphi') = H^k(\varphi'')$$

for $k \geq n+1$ ($k \geq 0$ *when* $n = 1$). *As a consequence*, *if* $\psi : X \to K(Y)$ *is an* n-*acyclic map*, *then* $H^k(\varphi') = H^k(\varphi'')$ *for* $k \geq n+1$ ($k \geq 0$ *when* $n = 1$) *and for any morphisms* φ', φ'' *determining* ψ.

Proof. By I.(5.7), φ' and φ'' are h-linked with the morphism φ represented by (p_ψ, q_ψ). $\blacksquare$

2. The fundamental cohomology class. From now on we assume that $R = \mathbb{Z}$, that is, we consider (co)homology with integer coefficients, and $\mathcal{V}$-maps and morphisms with respect to $\mathbb{Z}$.

Let (Y,ω) be an oriented m-dimensional manifold. We say that a subset L of Y is *free* (or precisely: $\mathbb{Z}$-free) if the group $H_{m-1}(Y,Y\backslash L)$ is free. It can be proved (see [25, VIII.3.5]) that, for any compact and connected set $L\subset Y$ (hence, by Mayer–Vietoris, for any compact L) the group $H_{m-1}(Y,Y\backslash L)$ is torsion-free. Thus, L is free whenever the group $H_{m-1}(Y,Y\backslash L)$ is finitely generated. The Alexander duality [104] immediately gives the following:

(2.1) PROPOSITION. *If L is free, then $H^1(L)$ is free, and if $H^1(L)$ is finitely generated, then L is free.*

(2.2) EXAMPLE. In view of (2.1), any compact contractible set or compact absolute neighbourhood retract is free (recall that any compact absolute neighbourhood retract has the homotopy type of a finite polyhedron).

Below we show that the family of compact free subsets of Y is cofinal in the family of all compact subsets of Y directed by inclusion. We prove even more.

(2.3) THEOREM. *For any compact connected subset $N\subset Y$ and an open neighbourhood U of N, there exists a compact connected and free set $L\subset Y$ such that $N\subset L\subset U$.*

First we prove the following:

(2.4) LEMMA. *If L_1, L_2 are compact connected and free, then $L_1\cup L_2$ is also free.*

Proof. The pair $\{Y\backslash L_1, Y\backslash L_2\}$ is excisive for H_*. Consider the Mayer–Vietoris sequence of this pair:

$$\ldots\xrightarrow{\beta_*} H_m(Y,Y\backslash(L_1\cap L_2))\xrightarrow{\Delta_*} H_{m-1}(Y,Y\backslash(L_1\cup L_2))\xrightarrow{\alpha_*}$$
$$\xrightarrow{\alpha_*} H_{m-1}(Y,Y\backslash L_1)\oplus H_{m-1}(Y,Y\backslash L_2)\xrightarrow{\beta_*}\ldots$$

It gives us the short exact sequence

$$(*)\qquad 0\to \operatorname{coker}(\beta_*)\to H_{m-1}(Y,Y\backslash(L_1\cup L_2))\to \operatorname{im}(\alpha_*)\to 0.$$

Clearly, $\operatorname{im}(\alpha_*)$ is free as a subgroup of a free group. We show that $\operatorname{coker}(\beta_*)$ is also free. To this end observe that $\operatorname{im}(\beta_*)$ is a direct summand of a free group $H_m(Y,Y\backslash(L_1\cap L_2))$ (comp. A.(2.12)). Indeed, if $\gamma\in H_m(Y,Y\backslash L_1)$, $\delta\in H_m(Y,Y\backslash L_2)$, then in view of A.(2.12), $\gamma=c\omega_{L_1}$, $\delta=d\omega_{L_2}$ where $c,d\in\mathbb{Z}$. Thus (see A.(1.2)), $\beta_*(\gamma,\delta)=H_m(j^{L_1}_{L_1\cap L_2})(\gamma)+H_m(j^{L_2}_{L_1\cap L_2})(\delta)=(c+d)(\omega_{L_1\cap L_2})$ in view of A.(2.9). Therefore the isomorphism $\Phi_{L_1\cap L_2}\circ J_{L_1\cap L_2}$ (see A.(2.8), A.(2.7)) maps $\operatorname{im}(\beta_*)$ onto the subgroup of constant functions in $C(L_1\cap L_2,\mathbb{Z})$, which is a direct summand.

The sequence $(*)$ is split and has outer terms free; hence the group $H_{m-1}(Y,Y\backslash (L_1\cup L_2))$ is free. ∎

Proof of (2.3). Obviously (see (2.1)), the closed ball B^m is a free subset of $\mathbb{R}^m$. First, assume that N is such that there exists an open V, $N\subset V\subset U$, and a homeomorphism $h : V \to \mathbb{R}^m$. By excision,

$$H_{m-1}(Y, Y\backslash N) \simeq H_{m-1}(V, V\backslash N) \simeq H_{m-1}(\mathbb{R}^m, \mathbb{R}^m\backslash h(N)).$$

Take an arbitrary ball B such that $h(N) \subset \operatorname{int} B$ and let $L = h^{-1}(B)$. Obviously, the set L is compact connected and free. Now, let N be arbitrary compact connected. We can decompose N into a union $N = N_1 \cup \ldots \cup N_r$ where N_j, for $1 \le j \le r$, is as above. For any $1 \le j \le r$, there is a compact connected and free set L_j such that $N_j \subset L_j \subset U$. In view of Lemma (2.4), $L = L_1 \cup \ldots \cup L_r$ has the required properties. ■

The main property of free sets is given in the following:

(2.5) LEMMA. *Let $L \subset Y$ be free. There exists a natural isomorphism*

$$I_L : H_s^m(Y, Y\backslash L) \to \operatorname{Hom}(H_m(Y, Y\backslash L), \mathbb{Z}).$$

Proof. By the universal coefficient theorem for singular cohomology we have a functorial short exact sequence

$$0 \to \operatorname{Ext}_{\mathbb{Z}} H_{m-1}(Y, Y\backslash L) \to H_s^m(Y, Y\backslash L) \xrightarrow[I_L]{} \operatorname{Hom}(H_m(Y, Y\backslash L), \mathbb{Z}) \to 0,$$

where the homomorphism I_L is given by the Kronecker duality

$$I_L(a)(\alpha) = \langle a, \alpha\rangle$$

for $a \in H_s^m(Y, Y\backslash L)$, $\alpha \in H_m(Y, Y\backslash L)$. Since $H_{m-1}(Y, Y\backslash L)$ is free, we deduce that $\operatorname{Ext}_{\mathbb{Z}} H_{m-1}(Y, Y\backslash L) = 0$ and, therefore, I_L is an isomorphism. ■

(2.6) Remark. If in (2.5), instead of $\mathbb{Z}$, we consider another ring, then the assertion may hold without any assumption on L. This is the case, for example, if the ring is a field.

For later convenience, we say that a set $L \subset Y$ is *admissible* if it is compact connected and free.

By the *fundamental cohomology class* of an admissible set $L \subset Y$ we understand the unique element $\omega^L \in H_s^m(Y, Y\backslash L)$ such that

$$I_L(\omega^L)(\omega_L) = 1 \tag{2.7}$$

where ω_L denotes the fundamental homology class of L (for the given orientation ω on Y).

In view of A.(1.3), A.(2.9) and (2.7), we have

(2.8) PROPOSITION. *If N, L are admissible sets and $L \subset N$, then*

$$H_s^m(j_L^N)(\omega^L) = \omega^N.$$

(2.9) Remark. Similarly to A.(2.11), we can define the fundamental system of cohomology classes of a compact and free set $L \subset Y$. In particular, the fundamental cohomology class ω^L of an admissible set $L \subset Y$ is a free generator of the infinite cyclic group $H_s^m(Y, Y\backslash L)$.

Assume that (Z,ν) is an oriented k-dimensional manifold and let $L\subset Y$ and $N\subset Z$ be admissible sets.

(2.10) Lemma. *If, for all $n\geq 1$, the groups $H_{m-n}(Y,Y\backslash L), H_{k-n}(Z,Z\backslash N)$ are free, then so is the set $L\times N$.*

Proof. This is an immediate consequence of the Künneth formula. ∎

(2.11) Proposition. *If the set $L\times N$ is free, then*

$$(\omega\times\nu)^{L\times N}=(-1)^{mk}\omega^L\times\nu^N\in H_s^{m+k}(Y\times Z,Y\times Z\backslash L\times N).$$

Proof. By the natural properties of the products involved we have

$$\langle(-1)^{mk}\omega^L\times\nu^N,(\omega\times\nu)_{L\times N}\rangle=\langle\omega^L,\omega_L\rangle\langle\nu^N,\nu_N\rangle=1.$$

The assertion follows from the definition (2.7). ∎

(2.12) Remark. It is well known that any manifold Y is $\mathbb{Z}_2$-orientable (see [25]). On the other hand, by (2.6), any compact set $L\subset Y$ is $\mathbb{Z}_2$-free. Hence, in this situation, we may repeat the constructions from (2.7), (2.8), (2.9) and (2.11).

The notion of free set was suggested to the author by Prof. A. Dold (private communication) and plays an auxiliary role. However, it is related to some other interesting problems. Let, as above, (Y,ω) be an oriented m-dimensional manifold and let u be the Thom class of the orientation. In [67], it was shown that, for any compact path-connected $L\subset Y$ and $T_L: H_0(L)\to H_s^m(Y,Y\backslash L)$ given by

$$T_L(\alpha)=H_s^m(j_L)(u)/\alpha,$$

where $j_L:(Y,Y\backslash L)\times L\to(Y\times Y,Y\times Y\backslash\varDelta_Y)$ is the inclusion, we have

$$\langle T_L(1),\omega_L\rangle=1$$

(recall that $H_0(L)\simeq\mathbb{Z}$). Hence, $T_L(1)\in H_s^m(Y,Y\backslash L)$ plays the role of the fundamental class of L. But is not clear whether T_L is an isomorphism, and so whether the analogue of (2.5) holds.

In [67], it was also proved that $\{T_L\}$ (where L ranges over the family of compact path-connected subsets of Y) is a morphism of direct systems $\{H_0(L), H_0(i_{KL})\}_{K\subset L}$ and $\{H_s^m(Y,Y\backslash L),H_s^m(j_K^L)\}_{K\subset L}$, where $i_{KL}:K\to L$ is the inclusion, and that $\varinjlim\{T_L\}$ is the inverse to the Poincaré isomorphism $D_Y:H_c^m(Y)\to H_0(Y)$ (see [55]). Thus under the assumption of the connectedness of Y, a generator of the group $H_c^m(Y)$ (which was called by Greenberg the fundamental cohomology class of the connected manifold Y) can be represented as a "limit" of fundamental cohomology classes of admissible sets $L\subset Y$ (comp. [55]).

3. The topological degree of morphisms. Let X and Y be topological spaces, $B\subset Y$ and let $\varphi\in M_n(X,Y)$. Assume that $\varphi_-^{-1}(B)\subset A$. If a pair (p,q) represents φ, where $p\in\mathcal{V}_n(G,X)$, $q:G\to Y$, then $p(q^{-1}(B))\subset A$. Hence, for $T=p^{-1}(A)$, we deduce that $p:(G,G\backslash T)\to(X,X\backslash A)$ is a $\mathcal{V}_n$-map and $q:(G,G\backslash T)\to(Y,Y\backslash B)$. Therefore we have an n-morphism $\varphi_{AB}\in$

$M_n(X, X\backslash A), (Y, Y\backslash B)$ (independent of the choice of (p, q)). Obviously, for $x \in X$, $\varphi_{AB}(x) = \varphi(x)$ and, for $x \in A$, $\varphi(x) \subset Y\backslash B$ (comp. I.(3.2)).

Let $(X, \mu), (Y, \omega)$ be oriented m-dimensional manifolds and let $\varphi \in M_n(X, Y)$ where $n = 1$ for $m = 1$ and $n \leq m - 1$ if $m \geq 2$. Next assume that L is an admissible subset of Y and that the set $K = \varphi_-^{-1}(L)$ is compact.

The morphism $\varphi_{KL} \in M_n((X, X\backslash K), (Y, Y\backslash L))$ satisfies the assumptions of Section 1; thus, we may define a homomorphism

$$H^m(\varphi_{KL}) : H^m(Y, Y\backslash L) \to H^m(X, X\backslash K).$$

In view of A.(1.7), there are natural isomorphisms

$$\xi_{XK} : H^m(X, X\backslash K) \to H_s^m(X, X\backslash K), \qquad \xi_{YL} : H^m(Y, Y\backslash L) \to H_s^m(Y, Y\backslash L).$$

We define the *topological degree of the morphism φ over the admissible set L* by

$$\deg(\varphi, L) = \langle \xi_{XK} \circ H^m(\varphi_{KL}) \circ \xi_{YL}^{-1}(\omega^L), \mu_K \rangle \in \mathbb{Z}. \tag{3.1}$$

Identifying $H^m(X, X\backslash K)$ with $H_s^m(X, X\backslash K)$ and $H^m(Y, Y\backslash L)$ with $H_s^m(Y, Y\backslash L)$ (through ξ_{XK} and ξ_{YL}, respectively), we can write

$$\deg(\varphi, L) = \langle H^m(\varphi_{KL})(\omega^L), \mu_K \rangle, \tag{3.2}$$

keeping in mind the above-mentioned identifications. Certainly, the naturality of the isomorphisms ξ_{XK} and ξ_{YL} plays a crucial role here.

Observe that our definition of the degree, in the case of a map $F : X \to Y$ such that $K = F^{-1}(L)$ is compact, agrees with the classical one of Dold. For, if $\deg_L F = c \in \mathbb{Z}$, then, for any morphism $\varphi \in M_n(X, Y)$ determining F, in view of (1.5), we have

$$\langle H^m(\varphi_{KL})(\omega^L), \mu_K \rangle = \langle H_s^m(F)(\omega^L), \mu_K \rangle = \langle \omega^L, H_m(F)(\mu_K) \rangle = c\langle \omega^L, \omega_L \rangle = c.$$

As we see the only difference is that we need to take an admissible set L in place of an arbitrary compact connected one. Under some additional (but altogether natural) assumptions even this circumstance will be avoided.

Now, we collect the most important properties of the above-defined degree generalizing those given in [25, VIII.4].

(3.3) Proposition. *Let X, Y, φ, L, K be as above.*

(i) *If* $\deg(\varphi, L) \neq 0$, *then* $K = \varphi_-^{-1}(L) \neq \emptyset$.

(ii) *If* $M \subset X$ *is compact and* $K \subset M$, *then*

$$\deg(\varphi, L) = \langle H^m(\varphi_{ML})(\omega^L), \mu_M \rangle.$$

(iii) *If* $M \subset X$ *is admissible and* $K \subset M$, *then*

$$H^m(\varphi_{ML})(\omega^L) = \deg(\varphi, L)\mu^M.$$

(iv) *If* $N \subset Y$ *is admissible and* $N \subset L$, *then*

$$\deg(\varphi, L) = \deg(\varphi, N).$$

(v) *If* $\deg(\varphi, L) \neq 0$, *then* $L \subset \varphi(X)$.

P r o o f. (i) is immediate.

(ii) Clearly, the homomorphism $H^m(\varphi_{ML})$ is correctly defined. Moreover, $\varphi_{KL} \circ j_K^M$ is an n-morphism and $\varphi_{KL} \circ j_K^M = \varphi_{ML}$. Hence, in view of A.(1.3), (2.8), A.(2.9),

$$\begin{aligned}\deg(\varphi, L) &= \langle H^m(\varphi_{KL})(\omega^L), \mu_K\rangle \\ &= \langle H^m(\varphi_{KL})(\omega^L), H_m(j_K^M)(\mu_M)\rangle = \langle H^m(\varphi_{ML})(\omega^L), \mu_M\rangle.\end{aligned}$$

(iii) In view of (2.9), μ^M is a generator of $H_s^m(X, X\backslash M)$. Hence, $H^m(\varphi_{ML})(\omega^L) = c\mu^M$, $c \in \mathbb{Z}$. In view of (ii), we have the assertion.

(iv) First, observe that $\deg(\varphi, N)$ is defined because $M = \varphi_-^{-1}(N) \subset K$ and, hence, M is compact. By (ii) and (2.8),

$$\begin{aligned}\deg(\varphi, N) &= \langle H^m(\varphi_{KN})(\omega^N), \mu_K\rangle \\ &= \langle H^m(j_N^L \circ \varphi_{KL})(\omega^N), \mu_K\rangle = \langle H^m(\varphi_{KL})(\omega^L), \mu_K\rangle.\end{aligned}$$

(v) Indeed, suppose that there is $y \in L\backslash\varphi(X)$. Hence, $\deg(\varphi, y) = \deg(\varphi, L) = 0$, a contradiction. ∎

Property (v) is a strong extension of the existence property (i) (comp. I.(6.6)). Moreover, observe that a set $M \subset X$ from (iii) always exists provided X is connected. This follows from (2.3) (if K is not connected, then we can make it connected by joining the components by paths).

Assume now that $\psi \in M(X, Y)$ and that the morphisms φ and ψ are n-homotopic. Then ψ is an n-morphism. There exists a homotopy $\Phi \in M_n(X \times I, Y)$ such that $\Phi \circ i_0 = \varphi$, $\Phi \circ i_1 = \psi$.

(3.4) Proposition (Homotopy invariance). *If $Z = \Phi_-^{-1}(L)$ is compact then*

$$\deg(\varphi, L) = \deg(\psi, L).$$

P r o o f. Observe that $T = \bigcup_{t\in I}\{x \in X : \Phi(x, t) \cap L \neq \emptyset\} = \mathrm{pr}_X(Z)$ is compact. Hence $K = \varphi_-^{-1}(L) \subset T$ and $\psi_-^{-1}(L) \subset T$ and $\Phi_{T\times I, L} \in M_n((X \times I, X \times I\backslash T \times I), (Y, Y\backslash L))$ is a homotopy joining φ_{TL} and ψ_{TL}. Therefore, by (1.3), $H^m(\varphi_{TL}) = H^m(\psi_{TL})$ and, by (3.3)(ii),

$$\deg(\varphi, L) = \langle H^m(\varphi_{TL})(\omega^L), \mu_T\rangle = \langle H^m(\psi_{TL})(\omega^L), \mu_T\rangle = \deg(\psi, L). \ ∎$$

(3.5) Proposition *Let $\varphi, \psi \in M_n(X, Y)$. If there exists a compact set $M \subset X$ such that $\varphi_-^{-1}(L), \psi_-^{-1}(L) \subset M$ and the morphisms φ_{ML} and ψ_{ML} are h-linked, then* $\deg(\varphi, L) = \deg(\psi, L)$.

P r o o f. This is obvious in view of (1.5) and (3.3)(ii). ∎

(3.6) Example. Let $\psi : X \to K(Y)$ be an n-acyclic map. Using (3.5) and (1.5), we define the *degree* of ψ over L by

$$\deg(\psi, L) = \deg(\varphi, L),$$

where φ is an arbitrary n-morphism determining ψ, provided that $\psi_-^{-1}(L)$ is compact in X. Moreover, observe that if $\varphi' \in M_n(X,Y)$ is a morphism determining any selection of ψ, then $\deg(\varphi', L)$ is defined and $\deg(\varphi', L) = \deg(\varphi, L)$.

Suppose that $X = \bigcup_{j\in J} X^j$ where each X^j is an open subset of X. For any $j \in J$, the set X^j is an oriented (by the section $\mu^j = \mu|X^j$) m-dimensional submanifold of X. Suppose that $K^j = \varphi_-^{-1}(L) \cap X^j$ are pairwise disjoint (and hence compact). In view of I.(4.1), $\varphi^j = \varphi|X^j \in M_n(X^j, Y)$. Thus, $\deg(\varphi^j, L)$ is defined for $j \in J$. Clearly, for almost all $j \in J$, $\deg(\varphi^j, L) = 0$ (because $K^j = \emptyset$ for almost all $j \in J$).

(3.7) Proposition (Additivity).

$$\deg(\varphi, L) = \sum_{j\in J} \deg(\varphi^j, L).$$

Proof. It is sufficient to consider $J = \{1, 2\}$. Let $x \in K$ and let $w^i : (X^i, X^i\backslash K^i) \to (X, X\backslash K)$, $i = 1, 2$, be the inclusion. The composition (see A.(1.1), A.(1.2), A.(2.9))

$$H_m(X^1, X^1\backslash K^1) \oplus H_m(X^2, X^2\backslash K^2) \xrightarrow{\beta_*} H_m(X, X\backslash K) \to H_m(X, X\backslash\{x\})$$

maps $(\mu^1_{K_1}, \mu^2_{K_2})$ onto $\mu(x) = \mu_x$. Thus, by A.(2.1),

$$\mu_K = \beta_*(\mu^1_{K_1}, \mu^2_{K_2}) = H_m(w^1)(\mu^1_{K_1}) + H_m(w^2)(\mu^2_{K_2}).$$

Hence, using A.(1.3), we have

$$\begin{aligned}\deg(\varphi, L) &= \langle H^m(\varphi_{KL})(\omega^L), \mu_K\rangle \\ &= \langle H^m(\varphi_{KL})(\omega^L), H_m(w^1)(\mu^1_{K_1})\rangle + \langle H^m(\varphi_{KL})(\omega^L), H_m(w^2)(\mu^2_{K_2})\rangle \\ &= \langle H^m(\varphi_{K_1L})(\omega^L), \mu^1_{K_1}\rangle + \langle H^m(\varphi_{K_2L})(\omega^L), \mu^2_{K_2}\rangle. \ \blacksquare\end{aligned}$$

(3.8) Corollary (Excision). *If U is an open subset of X such that $K \subset U$, then* $\deg(\varphi, L) = \deg(\varphi|U, L)$.

There is another consequence of (3.7). If $X = \bigcup_{j\in J} X^j$ is a decomposition of X into the union of its connected components, then formula (3.7) holds. The (at most countable) system of integers $\{\deg(\varphi^j, L)\}_{j\in J}$ is a homotopy invariant more subtle than $\deg(\varphi, L)$. It is obvious that $\deg(\varphi, L)$ may vanish despite that $\{\deg(\varphi^j, L)\}_{j\in J}$ is different from zero (in $\mathbb{Z}^J$). To see that take $X = (-2, 0) \cup (0, 2)$, $L = \{0\}$ and let φ be determined by the single-valued map $x \mapsto x^2 - 1$. Then the problem $\varphi(x) \cap L \neq \emptyset$ may have a solution even when $\deg(\varphi, L) = 0$.

Assume that $\varphi \in M_n(X, Y)$ determines a proper set-valued map. We easily see that in order for φ to determine a proper map it is necessary that, for any pair $(p, q) \in \varphi$, the map q is proper, and it is sufficient that, for some $(p, q) \in \varphi$, q is proper. Moreover, if X is compact, then any morphism determines a proper map.

(3.9) PROPOSITION. *Suppose that* $\varphi \in M_n(X,Y)$ *determines a proper set-valued map. For any admissible sets* $L_1, L_2 \subset Y$,

$$\deg(\varphi, L_1) = \deg(\varphi, L_2)$$

provided L_1, L_2 *lie in the same component of* Y.

Proof. In view of (2.3), there is an admissible set L lying in the same component of Y as L_1, L_2 and such that $L_1, L_2 \subset L$. By (3.3)(iv), $\deg(\varphi, L_i) = \deg(\varphi, L)$, for $i = 1, 2$. ∎

(3.10) Remark. (i) Suppose that a morphism $\varphi \in M_n(X,Y)$ determines a proper map and that $L \subset Y$ is only compact connected. In view of (2.3), there exists an admissible set $L' \subset Y$ such that $L \subset \operatorname{int} L'$. Let L'' be another admissible set such that $L \subset \operatorname{int} L''$. Once again, by (2.3), there is an admissible N such that $L \subset N \subset \operatorname{int} L' \cap \operatorname{int} L''$. Hence $\deg(\varphi, L') = \deg(\varphi, N) = \deg(\varphi, L'')$. The above reasoning allows us to make the following definition:

$$\deg(\varphi, L) = \deg(\varphi, L').$$

(ii) Even if φ does not determine a proper map but L is path-connected, then we can define $\deg(\varphi, L)$ as $\deg(\varphi, y)$ where y is an arbitrary point of L.

(iii) Under the assumptions of (3.9) and assuming additionally that Y is connected we can speak about the *degree* of φ putting

$$\deg(\varphi) = \deg(\varphi, L)$$

where $L \subset Y$ is an arbitrary admissible set. We shall study the special case of this situation more carefully in the next section.

(3.11) PROPOSITION. *If* Y *is connected*, (Z, ν) *is an oriented* m*-dimensional manifold*, $\varphi \in M_n(X,Y)$ *determines a proper map*, $\psi \in M_1(Y,Z)$, $L \subset Z$ *is admissible and* $K = \psi_-^{-1}(L)$ *is compact*, *then* $\deg(\psi \circ \varphi, L) = \deg(\varphi)\deg(\psi, L)$.

Proof. Let $M \subset Y$ be an admissible superset of K. In view of Proposition (3.3)(iii), $H^m(\psi_{ML})(\nu^L) = \deg(\psi, L)\omega^M$. Hence

$$\begin{aligned}\deg(\psi \circ \varphi, L) &= \langle H^m(\varphi_{NM}) H^m(\psi_{ML})(\nu^L), \mu_N \rangle \\ &= \deg(\psi, L)\langle H^m(\varphi_{NM})(\omega^M), \mu_N\rangle = \deg(\psi, L)\deg(\varphi)\end{aligned}$$

where $N = \varphi_-^{-1}(M)$. ∎

Now, we establish the multiplicativity property of our degree. Let (X_1, μ_1), (Y_1, ω_1) be oriented k-dimensional manifolds, and $L_1 \subset Y_1, L \subset Y$ be admissible sets such that $L \times L_1$ is admissible in $Y \times Y_1$. Suppose that $\varphi \in M_1(X,Y)$, $\varphi_1 \in M_1(X_1, Y_1)$ and that $K = \varphi_-^{-1}(L)$, $K_1 = \varphi_{1-}^{-1}(L_1)$ are compact. The manifolds $X \times X_1$ and $Y \times Y_1$ are oriented by the sections $\mu \times \mu_1$ and $\omega \times \omega_1$, respectively. By A.(2.10), $\mu_K \times \mu_{1_{K_1}}$ is the fundamental homology class of $K \times K_1$ and, by (2.11), $(-1)^{mk}\omega^L \times \omega_1^{L_1}$ is the fundamental cohomology class of $L \times L_1$. According to

I.(4.6), $\varphi\times\varphi_1\in M_1(X\times X_1, Y\times Y_1)$. It is easy to see that $(\varphi\times\varphi_1)_{K\times K_1, L\times L} = \varphi_{KL}\times\varphi_{K_1L_1}$. Therefore, after easy calculations we conclude that

$$\deg(\varphi\times\varphi_1, L\times L_1) = \deg(\varphi, L)\deg(\varphi_1, L_1). \tag{3.12}$$

The following example shows that the degree of a morphism depends on its structure and not on the map determined by it.

(3.13) Example. (i) Consider the 1-morphism $\varphi\in M_1(S^1, S^1)$ represented by the pair (p,q), where $p,q : \{(x,y)\in S^1\times S^1 : |x-y|\le\sqrt{2}\}\to S^1$, $p(x,y)=x$, $q(x,y)=y^2$ (multiplication in $\mathbb{C}$). Then $\deg(\varphi)=2$. If φ' is represented by (p,q'), where $q'(x,y)=y^3$, then $\deg(\varphi')=3$. But it is easy to see that the morphisms φ and φ' determine the same map $x\mapsto\varphi(x)=\varphi'(x)=S^1$.

(ii) Let $\varphi\in M_1(S^1,S^1)$ be represented by (p,q), where $p: G\to S^1$, $q: G\to S^1$, $G=S^1\times I$ and $p(e^{2\pi it}, s)=e^{2\pi i(s+t)}$, $q(e^{2\pi it})=e^{2\pi kit}$ for $t,s\in I$. We see that $\deg(\varphi)=k$.

To end this section we study the relation between the essentiality of morphisms and their degree.

(3.14) Theorem. *For any closed $A\subset X$ such that $X\backslash A$ is relatively compact, a morphism $\varphi\in M_1((X,A),(Y,Y\backslash L))$ is essential on A over an admissible set L provided* $\deg(\varphi,L)\ne 0$.

Proof. First, observe that $\deg(\varphi,L)$ is defined. In fact, the closed set $K=\varphi_-^{-1}(L)$ is compact since $K\subset X\backslash A$. Suppose that φ is inessential. Then there exists $\Phi\in M_1((X\times I, A\times I),(Y,Y\backslash L))$ such that $\varphi=\Phi\circ i_0$ and, for $\varphi_1=\Phi\circ i_1$, we have $\varphi_1(X)\cap L=\emptyset$. Since $\Phi_-^{-1}(L)\subset(X\backslash A)\times I$, by (3.4), $0\ne\deg(\varphi,L)=\deg(\varphi_1,L)$. In view of (3.3)(i), we get a contradiction. ∎

Since, for any closed A, there exists a closed absolute retract B such that $A\subset B$, if $X\backslash A$ is relatively compact, then $\deg(\varphi,L)\ne 0$ implies that φ is strongly essential on B over L because X is an absolute neighbourhood retract and because of I.(6.3).

4. The degree of morphisms of spheres and open subsets of Euclidean space. Let ω be an orientation of $\mathbb{R}^m$ and let U be an open subset of $\mathbb{R}^m$. If $\mu=\omega|U$, then according to Section 3, we can define $\deg(\varphi,L)$ for any $\varphi\in M_n(U,\mathbb{R}^m)$, where $n\le m-1$ (or $n=1$ if $m=1$) and an admissible $L\subset\mathbb{R}^m$, provided $\varphi_-^{-1}(L)$ is compact in U. Observe that if φ has an extension $\varphi^*\in M(\operatorname{cl}U,\mathbb{R}^m)$ and U is bounded, then $\varphi_-^{-1}(L)$ is compact in U if and only if $\varphi^*(\operatorname{bd}U)\cap L=\emptyset$.

Below, we also deal with the degree of morphisms from $M(S^m,S^m)$, $m\ge 1$. Following (3.9), (3.10)(iii), since any $\varphi\in M(S^m,S^m)$ determines a proper map, one may define $\deg(\varphi)$. Precisely, if $\varphi\in M_m(S^m,S^m)$, then a homomorphism $H^m(\varphi): H^m(S^m)\to H^m(S^m)$ is defined and if μ is an orientation of the sphere S^m, we can consider the fundamental cohomology class $a\in H_s^m(S^m)$ and the fundamental homology class $\alpha\in H_m(S^m)$ with respect to the orientation μ.

According to (3.10)(iii),

$$\deg(\varphi) = \langle H^m(\varphi)(a), \alpha\rangle. \tag{4.1}$$

Then, clearly, $H^m(\varphi)(a) = \deg(\varphi)a$ (comp. (3.3)(iii)). Recall also (see A.(2.2)) that $\mu = J_{S^m}(\alpha)$.

Now, we prove that in some cases $\deg(\varphi, L)$, for $\varphi \in M_n(U, \mathbb{R}^m)$ with U open in $\mathbb{R}^m$ and $n \le m-1$, depends only on the behaviour on $\operatorname{bd} U$ of an extension φ^* of φ onto $\operatorname{cl} U$. For simplicity, let $U = N^m = N^m(0,1)$. Then $\operatorname{cl} U = B^m$ and $\operatorname{bd} U = S^{m-1}$.

(4.2) Proposition. *If* $\varphi, \varphi' \in M_n(B^m, \mathbb{R}^m)$, $n \le m-1$ *where* $m \ge 2$, *and* $\varphi|S^{m-1} = \varphi'|S^{m-1}$, *then*

$$\deg(\varphi|N^m, L) = \deg(\varphi'|N^m, L)$$

for any admissible set $L \subset \mathbb{R}^m$ *such that* $L \cap \varphi(S^{m-1}) = \emptyset$.

This proposition may be easily generalized for any open convex and bounded subset $U \subset \mathbb{R}^m$ (since any such $U \approx_{\mathrm{top}} N^m$).

Proof. Without any loss of generality we can assume that $0 \in L$. In view of (3.3)(iv), it is sufficient to show that

$$\deg(\varphi|N^m, 0) = \deg(\varphi'|N^m, 0).$$

Let $f : \mathbb{R}^m \to I$ be continuous with $f(0) = 0$ and $f|\varphi(S^{m-1}) = 1$. Next, let $r : \mathbb{R}^m \backslash \{0\} \to S^{m-1}$ be the radial retraction. Define $g : \mathbb{R}^m \to \mathbb{R}^m$ by

$$g(x) = \begin{cases} 0 & \text{for } x = 0, \\ x - f(x)(x - r(x)) & \text{for } x \ne 0. \end{cases}$$

Obviously, g is continuous. Let $\psi = g \circ \varphi$ and $\psi' = g \circ \varphi'$. Then $\psi, \psi' \in M_n((B^m, S^{m-1}), (\mathbb{R}^m, S^{m-1}))$ and $\psi|S^{m-1} = \psi'|S^{m-1}$.

We now show that

$$\begin{aligned} \deg(\varphi|N^m, 0) &= \deg(\psi|N^m, 0) = \deg(\psi|S^{m-1}) = \deg(\psi'|S^{m-1}) \\ &= \deg(\psi'|N^m, 0) = \deg(\varphi'|N^m, 0). \end{aligned} \tag{4.3}$$

The third equality is obvious. We prove the first and the second one; the remaining are deduced analogously.

Let a pair (p, q) represent φ, where $p \in \mathcal{V}_n(G, B^m)$, $q : G \to \mathbb{R}^m$. Then $(p, g \circ q)$ represents ψ. For the set $T = \{p(y) \in B^m : y \in G$ and $(1-t)q(y) + tg \circ q(y) = 0,\ t \in I\}$, we have $T \cap S^{m-1} = \emptyset$. Moreover, the diagram

$$\begin{tikzcd}
 & (G, G\backslash p^{-1}(T)) \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "\mathrm{id}"] & \\
(B^m, B^m\backslash T) & & (\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \\
 & (G, G\backslash p^{-1}(T)) \arrow[ul, "p"] \arrow[ur, "g\circ q"'] &
\end{tikzcd}$$

is homotopy commutative, i.e. the morphisms φ_{TO} and ψ_{TO}, where $O = \{0\}$, are h-linked, hence in view of (3.5), $\deg(\varphi|N^m, 0) = \deg(\psi|N^m, 0)$.

Let $\nu = b(\mu)$ be the orientation of S^{m-1} induced by μ where $\mu = \omega|N^m$ and ω is an orientation of $\mathbb{R}^m$ (see A.(2.5)). Let α be the fundamental homology class of S^{m-1} (with respect to ν) and let $\beta = \partial^{-1}(\alpha)$ where $\partial : H_m(B^m, S^{m-1}) \to H_{m-1}(S^{m-1})$ is the connecting homomorphism. If $x \in N^m$ and $i_x : (B^m, S^{m-1}) \to (\mathbb{R}^m, \mathbb{R}^m \backslash \{x\})$ is the inclusion, then $H_m(i_x)(\beta) = \omega_x$ (ω_x stands for the fundamental homology class of $\{x\}$). Thus, for any compact $K \subset N^m$ and the inclusions $i : (N^m, N^m \backslash K) \to (B^m, B^m \backslash K)$ and $j : (B^m, S^{m-1}) \to (B^m, B^m \backslash K)$, in view of A.(2.5) we have

$$H_m(i)(\mu_K) = H_m(j)(\beta).$$

On the other hand, if $\delta : H^{m-1}(S^{m-1}) \to H^m(B^m, S^{m-1})$ is the connecting homomorphism, then

$$\delta(a) = H^m(i_0)(\omega^0)$$

where a is the fundamental cohomology class of S^{m-1} (with respect to ν) and ω^0 is the fundamental cohomology class of $\{0\} \subset \mathbb{R}^m$ and $i_0 : (B^m, S^{m-1}) \to (\mathbb{R}^m, \mathbb{R}^m \backslash \{0\})$ is the inclusion. In fact

$$\langle \delta(a), \beta \rangle = \langle a, \partial(\beta) \rangle = 1$$

and

$$\langle H^m(i_0)(\omega^0), \beta \rangle = \langle \omega^0, H^m(i_0)(\beta) \rangle = \langle \omega^0, \omega_0 \rangle = 1.$$

Now, we are ready for the proof of the second required equality. Without any loss of generality, we can assume that $\psi(B^m) \subset B^m$. Put $K = \psi^{-1}(0)$. Then K is a compact set in N^m. The diagram

$$\begin{tikzcd}
(N^m, N^m \backslash K) \arrow[r, "i"] & (B^m, B^m \backslash K) & (G, G \backslash p^{-1}(K)) \arrow[l, "p"'] \arrow[r, "g \circ q"] & (\mathbb{R}^m, \mathbb{R}^m \backslash \{0\}) \\
 & (B^m, S^{m-1}) \arrow[u, "j"] & (G, p^{-1}(S^{m-1})) \arrow[l, "p"] \arrow[r, "g \circ q"'] \arrow[u, "\cup"'] & (B^m, S^{m-1}) \arrow[u, "i_0"]
\end{tikzcd}$$

is commutative. If we denote by $\overline{\psi}$ the morphism represented by the pair from the lower row, then

$$\begin{aligned}
\deg(\psi|N^m, 0) &= \langle H^m(i) H^m(\psi_{KO})(\omega^0), \mu_K \rangle \\
&= \langle H^m(\psi_{KO})(\omega^0), H_m(j)(\beta) \rangle = \langle H^m(j) H^m(\psi_{KO})(\omega^0), \beta \rangle \\
&= \langle H^m(\overline{\psi}) H^m(i_0)(\omega^0), \beta \rangle = \langle H^m(\overline{\psi}) \delta(a), \beta \rangle.
\end{aligned}$$

One the other hand, in view of (1.2),

$$H^m(\overline{\psi})\delta(a) = \delta \circ H^{m-1}(\psi|S^{m-1})(a);$$

hence

$$\begin{aligned}
\deg(\psi|N^m, 0) &= \langle \delta \circ H^{m-1}(\psi|S^{m-1})(a), \beta \rangle = \langle H^{m-1}(\psi|S^{m-1})(a), \partial(\alpha) \rangle \\
&= \langle H^{m-1}(\psi|S^{m-1})(a), \alpha \rangle = \deg(\psi|S^{m-1}). \quad \blacksquare
\end{aligned}$$

(4.4) Corollary. *If* $\varphi \in M_n(B^m, \mathbb{R}^m)$, $n \le m-1$, *and for an admissible set* $L \subset \mathbb{R}^m$ *such that* $0 \in L$, $\varphi(S^{m-1}) \cap L = \emptyset$, *then*

$$\deg(\varphi|N^m, L) = \deg(\psi)$$

where $\psi = r \circ (\varphi|S^{m-1})$ *and* r *has the same meaning as in the proof of* (4.2).

As another consequence we have the following simple criterion of essentiality.

(4.5) Corollary. *If* $\varphi \in M_1((B^m, S^{m-1}), (\mathbb{R}^m, \mathbb{R}^m \backslash \{0\}))$ *and* $\deg(r \circ \varphi) \neq 0$, *then* φ *is strongly essential on* S^{m-1} *over* $\{0\}$.

Using I.(5.14), we can easily generalize (4.2) to 1-morphisms. Observe that (4.2) was valid only for convex, bounded and open sets $U \subset \mathbb{R}^m$.

(4.6) Proposition. *Suppose that* U *is a polyhedral domain in* $\mathbb{R}^m$ (*i.e.* U *is open and* $\operatorname{cl} U$ *is a finite polyhedron*). *If* $\varphi, \varphi' \in M_1((\operatorname{cl} U, \operatorname{bd} U), (\mathbb{R}^m, \mathbb{R}^m \backslash L))$, $L \subset \mathbb{R}^m$ *is admissible and* $\varphi|\operatorname{bd} U = \varphi'|\operatorname{bd} U$, *then* $\deg(\varphi|U, L) = \deg(\varphi'|U, L)$. *In particular*, $\deg(\varphi, L)$ *depends only on the behaviour of* φ *on* $\operatorname{bd} U$.

Proof. Let $j_k : \operatorname{cl} U \times \{k\} \to \operatorname{cl} U$, for $k = 0, 1$, be given by $j_k(x, k) = x$ for $x \in \operatorname{cl} U$ and $\operatorname{pr} : \operatorname{bd} U \times I \to \operatorname{cl} U$ be the projection. Define a morphism $\Phi \in M_1(\operatorname{cl} U \times \{0\} \cup \operatorname{bd} U \times I \cup \operatorname{cl} U \times \{1\}, \mathbb{R}^m)$ by piecing together the morphisms $\varphi \circ j_0$, $\varphi \circ \operatorname{pr}$ and $\varphi' \circ j_1$. Since $(\operatorname{cl} U \times I, \operatorname{cl} U \times \{0\} \cup \operatorname{bd} U \times I \cup \operatorname{cl} U \times \{1\})$ is a polyhedral pair, in view of I.(5.13), there exists an extension $\Phi^* \in M_1(\operatorname{cl} U \times I, \mathbb{R}^m)$ of Φ. Obviously, $\Phi^{*-1}_{-}(L)$ is compact in $U \times I$ and $\Phi^*|U \times I$ is a 1-homotopy joining $\varphi|U$ to $\varphi'|U$. By (3.4), we get the assertion. ∎

Now, let us introduce some notation necessary to formulate the contraction property of the degree. Let $k < m$ be positive integers. We identify $\mathbb{R}^m = \mathbb{R}^{m-k} \times \mathbb{R}^k$ and let $i : \mathbb{R}^{m-k} \to \mathbb{R}^m$, $r : \mathbb{R}^m \to \mathbb{R}^{m-k}$ be given by the formulae $i(x) = (x, 0)$, $r(x, y) = x$. In $\mathbb{R}^m$ we choose an orientation ω, in $\mathbb{R}^{m-k}$ an orientation $\overline{\omega}$ and in $\mathbb{R}^k$ — an orientation ν such that $\overline{\omega} \times \nu = \omega$.

Let U be an open and bounded subset of $\mathbb{R}^m$. We put $\overline{U} = i^{-1}(U) \subset \mathbb{R}^{m-k}$ and let $\bar{i} = i|\overline{U} : \overline{U} \to U$. Assume that $\mu = \omega|U$ and $\overline{\mu} = \overline{\omega}|\overline{U}$. We consider an admissible set $\overline{L} \subset \mathbb{R}^{m-k}$. Then the set $L = i(\overline{L})$ is admissible in $\mathbb{R}^m$. Finally, assume that $\varphi \in M_n(U, \mathbb{R}^m)$ where $n \le m - (k+1)$ for $m > k+1$ and $n = 1$ for $m = k+1$.

(4.7) Proposition. *If* $x - y \in i(\mathbb{R}^{m-k})$ *for any* $x \in U$ *and* $y \in \varphi(x)$, *then*

$$\deg(\overline{\varphi}, \overline{L}) = \deg(\varphi, L)$$

where $\overline{\varphi} = r \circ \varphi \circ \bar{i} \in M_n(\overline{U}, \mathbb{R}^{m-k})$, *provided* $\varphi_{-}^{-1}(L)$ *is compact*.

Proof. Let $\overline{K} = \overline{\varphi}_{-}^{-1}(L)$. Then $K = \varphi_{-}^{-1}(L) = \bar{i}(K)$. Let (p, q) represent φ, where $p \in \mathcal{V}_n(G, U)$, $q : G \to \mathbb{R}^m$. The morphism $\overline{\varphi}$ is then represented by $(\overline{p}, \overline{q})$, where $\overline{q} = r \circ q \circ j$, $\overline{p} \in \mathcal{V}_n(\overline{G}, \overline{U})$ and the cotriad $\overline{U} \xleftarrow{\overline{p}} \overline{G} = \overline{U} \boxtimes_U G \xrightarrow{j} G$ arises as the pull-back of the triad $\overline{U} \xrightarrow{\bar{i}} U \xleftarrow{p} G$. If we define a map $f : \overline{G} \to G$ by $f(\overline{u}, g) = g$ for $(\overline{u}, g) \in \overline{G}$ (i.e. $p(g) = \overline{u}$), then the diagram

$$\begin{tikzcd} U & G \arrow[l, "p"'] \arrow[r, "q"] & \mathbb{R}^m \\ \overline{U} \arrow[u, "\bar{i}"] & \overline{G} \arrow[l, "\bar{p}"] \arrow[r, "\bar{q}"'] \arrow[u, "f"'] & \mathbb{R}^{m-k} \arrow[u, "i"'] \end{tikzcd}$$

is commutative. There exists a neighbourhood $\overline{V}$ of $\overline{K}$ in $\overline{U}$ and $\varepsilon > 0$ such that $V = \overline{V} \times H \subset \overline{V} \times \operatorname{cl} H \subset U$ where $H = N^k(0, \varepsilon)$. On H the orientation is given by $\overline{\nu} = \nu | H$. Let $G' = p^{-1}(V)$, $p' = p|G'$, $q' = q|G'$, $\overline{G}' = \overline{p}^{-1}(V)$, $\overline{p}' = \overline{p}|\overline{G}'$ and $\overline{q}' = \overline{q}|\overline{G}'$. Clearly the diagram

$$\begin{tikzcd} V & G' \arrow[l, "p'"'] \arrow[r, "q'"] & \mathbb{R}^m \\ \overline{V} \arrow[u, "\bar{i}"] & \overline{G}' \arrow[l, "\overline{p}'"] \arrow[r, "\overline{q}'"'] \arrow[u, "f'"'] & \mathbb{R}^{m-k} \arrow[u, "i"'] \end{tikzcd}$$

is commutative, where $f' = f|\overline{G}'$. Moreover, put $T = p^{-1}(K)$, $\overline{T} = \overline{p}^{-1}(K)$ and consider the diagram

$$\begin{tikzcd} (V, V\backslash K) & (G', G'\backslash T) \arrow[l, "p'"'] \arrow[r, "q'"] & (\mathbb{R}^m, \mathbb{R}^m\backslash L) \\ & (\overline{G}' \times H, \overline{G}' \times H\backslash \overline{T} \times \{0\}) \arrow[ul, "\overline{p}' \times \mathrm{id}"] \arrow[u, "h"] \arrow[ur, "\overline{q}' \times \mathrm{id}"'] & \end{tikzcd}$$

where $h(g, s) = f'(g)$ for $g \in \overline{G}'$, $s \in H$. In view of I.(2.6), I.(2.5), $\overline{p}' \times \mathrm{id}$ is a $\mathcal{V}_{n+k}$-map. Observe that $p' \circ h(g, s) = p' \circ f'(g) = \bar{i} \circ \overline{p}(g)$ and $q' \circ h(g, s) = q' \circ f'(g) = i \circ \overline{q}'(g)$ for $g \in \overline{G}'$, $s \in H$. Therefore, (p', q') and $(\overline{p}' \times \mathrm{id}, \overline{q}' \times \mathrm{id})$ are h-linked. Hence, in view of (1.5),

$$[H^m(p')]^{-1} \circ H^m(q') = [H^m(\overline{p}' \times \mathrm{id})]^{-1} \circ H^m(\overline{q}' \times \mathrm{id}).$$

By (3.8), we have

$$\begin{aligned} \deg(\varphi, L) &= \deg(\varphi|V, L) = \langle [H^m(p')]^{-1} \circ H^m(q')(\omega^L), (\mu|V)_K \rangle \\ &= (-1)^{(m-k)k} \langle [H^m(\overline{p}' \times \mathrm{id})]^{-1} \circ H^m(\overline{q}' \times \mathrm{id})(\overline{\omega}^{\overline{L}} \times \overline{\nu}^0), \overline{\mu}_K \times \overline{\nu}_0 \rangle \\ &= \langle [H^{m-k}(\overline{p}')]^{-1} \circ H^m(\overline{q}')(\overline{\omega}^L), \overline{\mu}_K \rangle \\ &= \deg(\overline{\varphi}|\overline{V}, L) = \deg(\overline{\varphi}, \overline{L}). \ \blacksquare \end{aligned}$$

(4.8) Remark. Under the notation of (4.7), assume that $\psi \in M_n(U, \mathbb{R}^{m-k})$ and let $\varphi = i_U - \varphi$. Then the morphism φ satisfies the assumptions of (4.7) provided the set of fixed points of ψ (i.e. the set $\mathrm{Fix}(\psi) = \{x \in U : x \in \psi(x)\}$) is compact.

To end this section we study more carefully the degree of morphisms of spheres. It appears that many classical facts concerning the degree of continuous maps of spheres can be extended to morphisms.

(4.9) PROPOSITION. *Let $\psi : S^m \to K(S^m)$ be an acyclic map.*

(i) *If, for any $x \in S^m$, $x \in \psi(x)$, then* $\deg(\psi) = 1$.
(ii) *If, for any $x, y \in S^m$, $\psi(x) = \psi(y)$, then* $\deg(\psi) = 0$.

P r o o f. (i) follows from (3.6), and (ii) follows from I.(6.8) and (4.5). ■

(4.10) THEOREM. *Let $\varphi \in M_m(S^m, S^m)$.*

(i) *If $\psi \in M_1(S^m, S^m)$, then* $\deg(\psi \circ \varphi) = \deg(\psi)\deg(\varphi)$.
(ii) *If $\varphi' \in M(S^m, S^m)$ and φ, φ' are m-homotopic, then* $\deg(\varphi) = \deg(\varphi')$.
(iii) *If, for any $x \in S^m$, $x \notin \varphi(x)$, then* $\deg(\varphi) = (-1)^{m+1}$.
(iv) *If, for any $x \in S^m$, $-x \notin \varphi(x)$, then* $\deg(\varphi) = 1$.
(v) *If $m = 0 \pmod 2$, then there is $x \in S^m$ such that $x \in \varphi(x)$ or $-x \in \varphi(x)$.*

P r o o f. (i) follows from (3.11), and (ii) from (3.4).

(iii) Let (p, q) represent φ, where $p \in \mathcal{V}_m(G, S^m)$, $q : G \to S^m$. In view of our assumption, the diagram

$$\begin{tikzcd}
& G \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "\mathrm{id}"] & \\
S^m & & S^m \\
& G \arrow[ul, "p"] \arrow[ur, "-p"'] &
\end{tikzcd}$$

is homotopy commutative.

For (iv) we reason analogously to (iii), and (v) follows from (iii) and (iv). ■

In the next section we give some other results concerning the degree of morphisms of spheres.

5. Borsuk type theorems.

Let X be a topological space. Assume that we are given two triples (t, s, h), (t', s', h') of continuous maps, where $s, t : Z \to X$, $h : Z \to Z$ and $s', t' : Z' \to X$, $h' : Z' \to Z'$ (Z, Z' are topological spaces) such that h, h' are involutions. We say that these triples are *equivalent* if there is a homeomorphism $f : Z \to Z'$ such that the diagram

$$\begin{tikzcd}
Z \arrow[rr, "f"] \arrow[dr, "t"'] \arrow[dd, "h"'] & & Z' \arrow[dl, "t'"] \arrow[dd, "h'"] \\
& X & \\
Z \arrow[rr, "f"'] \arrow[ur, "s"] & & Z' \arrow[ul, "s'"']
\end{tikzcd}$$

is commutative. Assume, moreover, that t, t' are $\mathcal{V}$-maps (with respect to some ring R). Any equivalence class of this relation will be called an *involutive morphism*. Clearly, the pairs (t,s) and (t',s') determine the same morphism in $M(X,X)$. The involutions h, h' give the additional structure.

Observe that if $\alpha \in M(X,X)$ is a morphism and, for some pair (t,s) representing α, where $t \in \mathcal{V}(Z,X)$, $s : Z \to X$, there is an involution $h : Z \to Z$, then, for any pair $(t',s') \in \varphi$, where $t' \in \mathcal{V}(Z',X)$ and $s' : Z' \to X$, one can find an involution $h' : Z' \to Z'$ such that the triples (t,s,h) and (t',s',h') are equivalent in the above sense.

(5.1) Example. (i) A 1-morphism $\varphi \in M_1(S^1,S^1)$ defined in I.(5.1) is involutive.

(ii) In [41], L. Górniewicz defines a set-valued involution as an u.s.c. map $\psi : X \to K(X)$ such that $(x,y) \in G_\psi$ if and only if $(y,x) \in G_\psi$. Therefore any n-acyclic involution is determined by an involutive n-morphism.

(iii) It is clear that if $\alpha \in M(X,X)$ is an involutive morphism, then, for any $x \in X$, $x \in \alpha \circ \alpha(x)$.

The following theorem is a generalization of the well-known Borsuk theorem (see [26], [58], [41]).

(5.2) Theorem. *Let $\varphi \in M_1(S^m,S^m)$, $m \geq 1$, and let $\alpha \in M_1(S^m,S^m)$ be an involutive morphism.*

(i) *If $\varphi(x) \cap \varphi(\alpha(x)) = \emptyset$ for any $x \in S^m$, then*

$$\deg(\varphi) = 1 \pmod 2.$$

(ii) *If α has no fixed points and $(-\varphi(x)) \cap \varphi(\alpha(x)) = \emptyset$ for any $x \in S^m$, then*

$$\deg(\varphi) = 0 \pmod 2.$$

In [41], it was proved that under assumption (i) and a stronger assumption on α the degree $\deg(\varphi)$ does not vanish.

In order to prove the above theorem we need several auxiliary facts.

(5.3) Lemma. *Let X be a compact space such that*

$$H^*(X;\mathbb{Z}_2) \simeq H^*(S^m;\mathbb{Z}_2).$$

If $h : X \to X$ is a continuous involution and $f : X \to S^m$ is a continuous map such that, for any $x \in X$, $f(h(x)) \neq f(x)$, then

$$H^m(f;\mathbb{Z}_2) : H^m(S^m;\mathbb{Z}_2) \to H^m(X;\mathbb{Z}_2)$$

is an isomorphism.

This follows from [23, Cor. 4, p. 299] (comp. [34], [57]).

In the same spirit we have:

(5.4) Lemma. *If X is a compact space such that $H^*(X;\mathbb{Z}_2) \simeq H^*(S^m;\mathbb{Z}_2)$, $h : X \to X$ is a continuous involution without fixed points and $f : X \to S^m$ is a* *continuous map such that* $f(h(x)) \neq -f(x)$ *for any* $x \in X$, *then the homomorphism*

$$H^m(f;\mathbb{Z}_2) : H^m(S^m;\mathbb{Z}_2) \to H^m(X;\mathbb{Z}_2)$$

is trivial.

Proof. In the proof we use the theory of special Čech–Smith cohomology groups; we mainly follow the monograph of Bredon [12]. Let $G = \mathbb{Z}_2$; in the group ring $\mathbb{Z}_2 G$ we let $\sigma = 1 + g$ where g is the generator of G. For an arbitrary paracompact pair (Z, A), where Z is a G-space and $A \subset Z$ is G-invariant, the Čech–Smith graded cohomology group of Z modulo A is defined. Let us recall several properties of these groups. All the notation is taken from [12] (the ordinary Čech groups have $\mathbb{Z}_2$-coefficients as well).

(i) [12, (7.5)] If $Z^G = \{z \in Z : G(z) = z\}$, then the Smith sequence

$$\ldots \longrightarrow \widehat{H}^n_\sigma(Z) \xrightarrow{\sigma^*} \widehat{H}^n(Z) \oplus \widehat{H}^n(Z^G) \xrightarrow{\delta^*} \widehat{H}^{n+1}_\sigma(Z) \longrightarrow \ldots$$

is exact, where σ^* is defined in [12], i^* is induced by the sum of inclusions and δ^* is the connecting homomorphism.

(ii) [12, (7.6)] $\widehat{H}^*_\sigma(Z) \simeq \widehat{H}^*(Z/G, Z^G)$ (observe that Z^G may be identified with a subspace of Z/G).

(iii) [12, (7.8)] The diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \widehat{H}^n(Z/G, Z^G) \arrow[r] \arrow[d, "\simeq"'] & \widehat{H}^n(Z/G) \arrow[r] \arrow[d, "\pi_Z"'] & \widehat{H}^n(Z^G) \arrow[r] \arrow[d, "{(0,1)}"] & \widehat{H}^{n+1}(Z/G, Z^G) \arrow[r] \arrow[d, "\simeq"] & \ldots \\
\ldots \arrow[r] & \widehat{H}^n_\sigma(Z) \arrow[r, "\sigma^*"] & \widehat{H}^n(Z) \arrow[r, "i^*"'] & \widehat{H}^n_\sigma(Z) \oplus \widehat{H}^n(Z^G) \arrow[r, "\delta^*"'] & \widehat{H}^{n+1}_\sigma(Z) \arrow[r] & \ldots
\end{tikzcd}$$

is commutative (in the upper row we have the exact cohomology sequence of the pair $(Z/G, Z^G)$ and in the lower row the Smith sequence, and $\pi_Z : Z \to Z/G$ is the quotient map).

(iv) [12, (7.9)] If Z is a compact G-space and $A \subset Z$ is G-invariant, then

$$\operatorname{rank} \widehat{H}^n_\sigma(Z, A) + \sum_{i \ge n} \operatorname{rank} \widehat{H}^i(Z^G, A^G) \le \sum_{i \ge n} \operatorname{rank} \widehat{H}^i(Z, A)$$

for any positive integer n.

(v) If Z is a compact G-space such that $\widehat{H}^*(Z) = \widehat{H}^*(S^m)$ and $Z^G = \emptyset$, then $i^* : H^m(Z) \to H^m_\sigma(Z)$ is an isomorphism.

The above property follows easily from (iv) and (i).

Now, we are ready to prove (5.4). Let an action of G on X and S^m be defined by the involution h and the antipodal map, respectively. There is $\varepsilon \in (0, 1)$ such that, for $x \in X$, $|f(h(x)) + f(x)| \ge \varepsilon$. Let $Y = \{(y, y') \in S^m \times S^m : |y - y'| \ge \varepsilon\}$. Then Y is a compact G-space provided we define an action of G on Y by the involution $k : Y \to Y$ given by the formula $k(y, y') = (-y', -y)$. Then $Y^G =$

$\{(y,-y) \in Y : y \in S^m\}$. Let $\overline{p} : Y \to Y^G$ be given by $\overline{p}(y,y') = (y,-y)$ and let $j : Y^G \to Y$ by the inclusion. Observe that $\overline{p}$ is a $\mathcal{V}_1$-map (with respect to $\mathbb{Z}_2$); hence, by the Vietoris–Begle theorem for Čech cohomology (see e.g. [41]), $\widehat{H}^*(\overline{p})$ is a (graded) isomorphism. Since $\overline{p} \circ j = \mathrm{id}_{Y^G}$, we deduce that $\widehat{H}^*(j)$ is an isomorphism as well. In view of (iv), $\widehat{H}^*_\sigma(Y) = 0$, so by (ii), $\widehat{H}^*(Y/G, Y^G) = 0$. The exactness of the cohomology sequence of the pair $(Y/G, Y^G)$ implies that $\overline{H}^*(Y/G) \simeq \overline{H}^*(Y^G)$. Next, if $\pi_Y : Y \to Y/G$ is the quotient map, then the commutativity of the diagram

$$\begin{tikzcd} \widehat{H}^*(Y/G) \arrow[rr, "\text{iso}"] \arrow[dr, "\pi_Y^*"'] & & \widehat{H}^*(Y^G) \\ & \widehat{H}^*(Y) \arrow[ur, "\widehat{H}^*(j)"'] & \end{tikzcd}$$

entails that π_Y^* is an isomorphism.

In view of (iii) (the action of G on X is free), the diagram

$$\begin{tikzcd} \ldots \arrow[r] & \widehat{H}^m(X/G) \arrow[r] \arrow[d, "\simeq"] & \widehat{H}^m(X/G) \arrow[r] \arrow[d, "\pi_X^*"] & 0 \arrow[r] \arrow[d] & \widehat{H}^{m+1}(X/G) \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & \widehat{H}^m_\sigma(X) \arrow[r, "\sigma^*"] & \widehat{H}^m(X) \arrow[r, "i^*"] & \widehat{H}^m_\sigma(X) \arrow[r] & 0 \arrow[r] & \ldots \end{tikzcd}$$

(the last term in the lower row is trivial because $\widehat{H}^*(X) \simeq H^*(X) = H^*(S^m) \simeq \widehat{H}^*(S^m)$, see A.(1.8) and use (iv)) is commutative. By (v), we find that $\pi_X^* = 0$.

Let $F : X \to Y$ be given by the formula $F(x) = (f(x), -f(h(x)))$. The map F is correctly defined and $k \circ F = F \circ h$. Hence the diagram

$$\begin{tikzcd} H^m(Y/G) \arrow[r, "H^m(F/G)"] \arrow[d, "\pi_Y^*"] & H^m(F/G) \arrow[d, "\pi_X^*"] \\ H^m(Y) \arrow[r, "H^m(F)"] & H^m(X) \end{tikzcd}$$

is commutative and $\widehat{H}^m(F) = 0$. Let $\mathrm{pr} : Y \to S^m$ be the projection onto the first factor. Clearly, $f = \mathrm{pr} \circ F$; hence $\widehat{H}^m(f) = 0$ and $H^m(f) = 0$ because $\widehat{H}^*$ and H^* are naturally isomorphic. ■

(5.5) Lemma. *Let X be a topological space. Assume that we are given an exact sequence*

$$0 \to \mathbb{Z} \xrightarrow{a} \mathbb{Z} \xrightarrow{b} \mathbb{Z}_2 \to 0$$

where $a(c) = 2c$ for $c \in \mathbb{Z}$ and $b(c) = c \pmod 2$. Then there is an exact sequence

$$(*) \qquad \ldots \to H^n(X) \xrightarrow{a^*} H^n(X) \xrightarrow{b^*} H^n_2(X) \xrightarrow{e^*} H^{n+1}(X) \to \ldots$$

(from now on by $H^*(\cdot)$ and by $H_2^*(\cdot)$ we denote cohomology with integer and $\mathbb{Z}_2$-coefficients, respectively).

Proof. Let $\{C^*(X), \delta\}$, $\{C_2^*(X), \delta_2\}$ denote the Alexander–Spanier cochain complexes over $\mathbb{Z}$ and $\mathbb{Z}_2$, repectively. If $d : X^{n+1} \to \mathbb{Z}$ is an $(n+1)$-function, then $b^\# d : X^{n+1} \to \mathbb{Z}_2$, given by the formula $(b^\# d)(x_0, x_1, \ldots, x_n) = b(d(x_0, \ldots, x_n))$, is also an $(n+1)$-function. Moreover, it is easily seen that if d is locally zero, then so is $b^\# d$. Hence, we have a well-defined homomorphism $b^\# : C^*(X) \to C_2^*(X)$. Analogously we define $a^\# : C(X) \to C(X)$. One verifies that the sequence

$$0 \to C^n(X) \xrightarrow{a} C^n(X) \xrightarrow{b} C_2(X) \to 0$$

is exact, and moreover, $a^\#$ and $b^\#$ are cochain maps. Therefore, there exists a connecting homomorphism $e^* : H_2^n(X) \to H^{n+1}(X)$ such that $(*)$ is exact. ∎

(5.6) Remark. The above construction may be generalized without change to an arbitrary short sequence of groups (or R-modules) and in that case e^* yields an equivalent of the Bockstein homomorphism for singular cohomology (see [104]). Moreover, observe that with the aid of (5.5) we may provide a different proof of the fact that $\mathcal{V}_1$-maps with respect to $\mathbb{Z}$ are $\mathcal{V}_1$-maps with respect to $\mathbb{Z}_2$ (see I.(7.2)).

Proof of Theorem (5.2). Let (p, q) represent φ, where $p \in \mathcal{V}_1(G, S^m)$, $q : G \to S^m$, and let (t, s, h) represent α, where $t \in \mathcal{V}_1(Z, S^m)$, $s : Z \to S^m$ and $h : Z \to Z$ is an involution. Since p, t are $\mathcal{V}_1$-maps with respect to $\mathbb{Z}$, they are also $\mathcal{V}_1$-maps with respect to $\mathbb{Z}_2$.

By the definition of an involutive morphism, $t \circ h = s$. In view of assumption (i), for any $z \in Z$,

$$\varphi(t(z)) \cap \varphi(s(z)) = \emptyset;$$

and, if (ii) holds, then for any $z \in Z$,

$$(-\varphi(t(z))) \cap \varphi(s(z)) = \emptyset.$$

Let $X = \{(z, g, g') \in Z \times G \times G : t(z) = p(g),\ s(z) = p(g')\}$. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, "v"] & S^m \\ S^m & G \arrow[l, "p"] \arrow[u, "q"'] \end{tikzcd}$$

where $f(z, g, g') = q(g)$, $u(z, g, g') = t(z)$ and $v(z, g, g') = g$. This diagram is commutative and u is a $\mathcal{V}_1$-map (with respect to $\mathbb{Z}_2$) in view of I.(2.10) since it is the composition of three $\mathcal{V}_1$-maps:

$$u : (z, g, g') \mapsto (z, g) \mapsto z \mapsto t(z).$$

Define a map $N : X \to X$ by the formula $N(z, g, g') = (h(z), g', g)$. This map is well-defined and is an involution. If (ii) is satisfied, then N has no fixed points and, for any $(z, g, g') \in X$, $f(N(z, g, g')) \neq -f(z, g, g')$; if (i) is satisfied, then $f(N(z, g, g')) \neq f(z, g, g')$ for any $(z, g, g') \in X$.

The commutativity of the diagram

$$\begin{tikzcd}
H_2^m(S^m) \arrow[r, "H_2^m(f)"] \arrow[d, "H_2^m(q)"'] & H_2^m(X) \\
H_2^m(G) & H_2^m(S^m) \arrow[l, "H_2^m(p)"'] \arrow[u, "H_2^m(u)"']
\end{tikzcd}$$

implies that $[H_2^m(p)]^{-1} \circ H_2^m(q) = [H_2^m(u)]^{-1} \circ H_2^m(f)$. Moreover, $H_2^*(X) \simeq H_2^*(S^m)$. Therefore, by (5.3) (resp. by (5.4)), $H_2^m(f)$ is an isomorphism when (i) is satisfied (resp. $H_2^m(f) = 0$ when assumption (ii) holds). In view of the functoriality of the sequence (5.5)($*$) (here in place of X we take S^m), we get the commutative diagram

$$\begin{tikzcd}
H^m(S^m) \arrow[r, "{[H^m(p)]^1 \circ H^m(q)}"] \arrow[d, "b^*"'] & H^m(S^m) \arrow[d, "b^*"] \\
H_2^m(S^m) \arrow[r, "{[H_2^m(p)]^{-1} \circ H_2^m(q)}"] & H_2^m(S^m)
\end{tikzcd}$$

Moreover, (5.5) implies that b^* is an epimorphism. If assumption (i) is satisfied, then in the lower row we have an isomorphism, and if (ii) is satisfied, then in the lower row we have a trivial homomorphism. In both cases this entails the assertion since otherwise the diagram would not be commutative. ∎

Theorem (5.2) has several interesting consequences.

(5.7) COROLLARY. *Let $\alpha \in M_1(S^m, S^m)$ be an involutive morphism and $\varphi \in M_1((B^{m+1}, S^m), (\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}))$. If there exists no ray emanating from 0 that meets $\varphi(x)$ and $\varphi(\alpha(x))$ simultaneously, then φ is essential on S^m over 0.*

(5.8) COROLLARY. *Let α be as above.*

(i) *If $\varphi \in M_1(S^m, S^m)$ and $\varphi(x) \cap \varphi(\alpha(x)) = \emptyset$ for $x \in S^m$, then $\varphi(S^m) = S^m$.*

(ii) *If $\varphi \in M_1(S^m, \mathbb{R}^m)$, then there is $x \in S^m$ such that $\varphi(x) \cap \varphi(\alpha(x)) \neq \emptyset$.*

Proof. (i) follows from (5.2), (3.3)(v). Now, (ii) follows from (i). Indeed, if, for all $x \in S^m$, $\varphi(x) \cap \varphi(\alpha(x)) = \emptyset$, then treating S^m as a one-point compactification of $\mathbb{R}^m$ we would get a contradiction with (i). ∎

(5.9) Remark. The proofs of (5.2) strongly relied on the assumption that α and φ are 1-morphisms. It would be very interesting to relax this condition and show that these theorems remain valid for any n-morphisms, $n > 1$.

Proposition (5.8)(ii) is an extension of the well-known Borsuk–Ulam theorem (comp. [26, (5.2), p. 44]). The theorem of Yang [109] is another strong generalization of the classical result of Borsuk–Ulam. In his paper Yang introduced the index $\operatorname{ind}(Z, h)$ of a pair (Z, h), where Z is a compact space and $h : Z \to Z$ is a continuous involution without fixed points, and proved

THEOREM A. *If* $\operatorname{ind}(Z, h) = n$ *and* $f : Z \to \mathbb{R}^k$ $(0 < k \le n)$ *is a continuous map*, *then the set*

$$A^f = \{z \in Z \ : \ f(z) = f(h(z))\}$$

is compact, *h-invariant and* $\operatorname{ind}(A^f, h) = n - k$. *In particular*, $\dim A^f \ge n - k$.

In [98], W. Segiet showed

THEOREM B. *If* $\operatorname{ind}(Z, h) = n$ *and* $\psi : Z \to K(\mathbb{R}^k)$ *is a set-valued map having an acyclic* (*with respect to Čech homology with rational coefficients*) *selector*, *then*

$$A^{\psi} = \{z \in Z : \psi(z) \cap \psi(h(z)) \ne \emptyset\} \ne \emptyset.$$

Next, for $Z = S^n$ and $h = a$ the antipodal map, K. Gęba and L. Górniewicz in [40] proved

THEOREM C. *If* $\psi : S^n \to K(\mathbb{R}^k)$ *is a set-valued map having an acyclic selector*, *then the set*

$$A^{\psi} = \{z \in S^n : \psi(z) \cap \psi(-z) \ne \emptyset\}$$

is invariant, *compact and* $\operatorname{ind}(A^{\psi}, a) \ge n - k$. *In particular*, $\dim(A^{\psi}) \ge n - k$.

Below, we give an extension of Theorems A, B, C using the notion of involutive morphism.

We start by recalling some properties of the Yang index (see [22]).

(5.10) PROPOSITION. (i) *If there exists an invariant continuous map* $f : (Z, h) \to (Z', h')$, *then* $\operatorname{ind}(Z, h) \le \operatorname{ind}(Z', h')$.

(ii) *If* $A, B \subset Z$ *are h-invariant subsets of* Z *and* $A \cup B = Z$, *then*

$$\operatorname{ind}(Z, h) \le \operatorname{ind}(A, h) + \operatorname{ind}(B, h) + 1.$$

(iii) *If a set* $A \subset Z$ *is h-invariant*, *then* $\operatorname{ind}(A, h) = \operatorname{ind}(\operatorname{cl} U, h)$ *where* U *is an invariant neighbourhood of* A.

(iv) $\operatorname{ind}(S^n, a) = n$.

Moreover, in [40], it was observed that

$$\dim(Z) \ge \operatorname{ind}(Z, h). \tag{5.11}$$

Let X be a compact space and let $\alpha \in M(X, X)$ be an involutive morphism (with respect to $\mathbb{Z}_2$) without fixed points. We define

$$\operatorname{Ind}(X, \alpha) = \operatorname{ind}(Z, h)$$

where α is represented by a triple (t, s, h) such that $t, s : Z \to X$ and $h : Z \to Z$ is an involution (t is a $\mathcal{V}$-map). This definition is correct since h has no fixed points and it does not depend on the choice of the triple representing α in view of (5.10)(i).

(5.12) EXAMPLE. In [98], Segiet proposed the following definition of the index of the pair (X,ψ) where ψ is a set-valued acyclic involution (see (5.1)(ii)): $\mathrm{Ind}(X,\psi)=\mathrm{ind}(G_\psi,T)$ where the involution T is given by $T(x,y)=(y,x)$. In view of (5.10)(i), if an involutive morphism α determines ψ, then $\mathrm{Ind}(X,\alpha)\le \mathrm{Ind}(X,\psi)$. It suffices to consider the diagram from I.(3.4)(i).

(5.13) THEOREM. *Let X be a compact space and $\alpha\in M(X,X)$ be an involutive morphism* (*with respect to $\mathbb{Z}_2$*) *having no fixed points. If* $\mathrm{Ind}(X,\alpha)=n$ *and $\varphi\in M_1(X,\mathbb{R}^k)$, $0<k\le n$, then*

$$\mathrm{ind}(A(\varphi),T)\ge n-k$$

where $A(\varphi)=\{(x,y)\in G_\alpha:\varphi(x)\cap\varphi(y)\ne 0\}$ and the involution $T:A(\varphi)\to A(\varphi)$ is given by $T(x,y)=(y,x)$.

Proof. Let a triple (t,s,h) represent α, where $t:Z\to X$ is a $\mathcal{V}$-map, $s:Z\to X$ and $h:Z\to Z$ is an involution. Observe that s is also a $\mathcal{V}$-map. Let

$$A_Z=\{z\in Z:\varphi(t(z))\cap\varphi(s(z))\ne\emptyset\}.$$

Clearly, A_Z is closed and h-invariant. We shall prove that

$$\mathrm{ind}(A_Z,h)\ge n-k.$$

If so, then considering a map $f:A_Z\to A(\varphi)$ given by $f(z)=(t(z),s(z))$, for $z\in A_Z$, we see that $f\circ h=T\circ f$ and, by (5.10)(i),

$$n-k\le \mathrm{ind}(A_Z,h)\le \mathrm{ind}(A(\varphi),T).$$

Assume that (p,q) represents φ, where $p\in\mathcal{V}_1(G,X)$ (according to our assumption p is a $\mathcal{V}_1$-map with respect to $\mathbb{Z}$, but in view of I.(7.2), p is also a $\mathcal{V}_1$-map with respect to $\mathbb{Z}_2$) and $q:Z\to\mathbb{R}^k$. Let

$$Y=\{(z,g,g')\in Z\times G\times G:p(g)=t(z),\ p(g')=s(z)\}$$

and let $u:Y\to Z$ be given by $u(z,g,g')=z$. The map u, being the composition of two $\mathcal{V}_1$-maps, is a $\mathcal{V}_1$-map. Clearly, Y is compact. Let $H:Y\to Y$ be given by $H(z,g,g')=(h(z),g',g)$. It can easily be seen that H is a fixed-point free involution and $u:(Y,H)\to(Z,h)$ is invariant.

In [98], Segiet sketches a proof of the fact that if (Z',h'), (Z'',h'') are compact spaces with involutions and $v:Z'\to Z''$ is an invariant Vietoris map, then $\mathrm{ind}(Z',h')=\mathrm{ind}(Z'',h'')$. He uses Čech–Smith homology (because Vietoris maps are considered in terms of Čech homology). However, since the Smith theory does hold in the Čech cohomology setting and the theories of Čech and Alexander–Spanier cohomologies are naturally isomorphic for compact spaces (see A.(1.8)), we conclude that also in our case

$$\mathrm{ind}(Y,H)=\mathrm{ind}(Z,h)=\mathrm{Ind}(X,\alpha).$$

Let $Y_1 = u^{-1}(A_Z)$. The set Y_1 is closed and H-invariant. Moreover, by I.(2.5), $u|Y_1 : Y_1 \to A_Z$ is an invariant $\mathcal{V}_1$-map. Therefore $\operatorname{ind}(Y_1, H) = \operatorname{ind}(A_Z, h)$ and it suffices to prove that $\operatorname{ind}(Y_1, H) \geq n - k$.

By (5.10)(iii), there is an H-invariant neighbourhood U of Y_1 such that $\operatorname{ind}(Y_1, H) = \operatorname{ind}(\operatorname{cl} U, H)$. We prove that $\operatorname{ind}(Y \setminus U, H) \leq k-1$. Let $Q : Y \setminus U \to \mathbb{R}^k$ be given by $Q(z, g, g') = q(g) - q(g')$ for $(z, g, g') \in (Y \setminus U)$. Observe that $Q(Y \setminus U) \subset \mathbb{R}^k \setminus \{0\}$. In fact, if $q(g) = q(g')$ for $(z, g, g') \in Y \setminus U$, then

$$q(g) = q(g') \in \varphi(t(z)) \cap \varphi(s(z))$$

and hence, $z \in A_Z$. Thus $(z, g, g') \in u^{-1}(\{z\}) \subset Y_1 \subset U$, a contradiction. The map $r \circ Q : Y \setminus U \to S^{k-1}$, where $r : \mathbb{R}^k \setminus \{0\} \to S^{k-1}$ is the radial retraction, satisfies

$$r \circ Q \circ H = a \circ r \circ Q;$$

hence the map $r \circ Q : (Y \setminus U, H) \to (S^{k-1}, a)$ is invariant. By (5.10)(i) and (iv), $\operatorname{ind}(Y \setminus U, H) \leq k - 1$. On the other hand, we see that $Y = (Y \setminus U) \cup \operatorname{cl} U$, so in view of (5.10)(ii),

$$n = \operatorname{ind}(Y, H) \leq \operatorname{ind}(Y \setminus U, H) + \operatorname{ind}(Y_1, H) + 1.$$

Therefore, we get at once $\operatorname{ind}(Y_1, H) \geq n - k$. ∎

(5.14) Corollary. *Under the assumptions of* (5.13), $\dim(A(\varphi)) \geq n - k$.

Finally, observe that if α is single-valued and $\psi : X \to K(\mathbb{R}^k)$ is determined by φ, then the pair (A^ψ, α) is invariantly equivalent to $(A(\varphi), T)$.

6. Applications. Let X, Y be topological spaces. In the spirit of Górniewicz [41] and adapting his definition we say that a set-valued map $\psi : X \to P(Y)$ is *n-admissible* ($n \geq 1$) if there exists a morphism $\varphi \in M_n(X, Y)$ determining a selector of ψ. It seems that the quite general class of n-admissible maps is particularly well-designed for applications. For example, admissible maps arise quite naturally in the theory of ordinary differential equations (see e.g. [27]).

(6.1) Example. (i) Any map determined by an n-morphism and, in particular, any n-acyclic map is n-admissible.

(ii) If X is a paracompact space, Y is a Fréchet space, then any l.s.c. map $\psi : X \to C_V(Y)$ is 1-admissible (see I.(1.7)).

(iii) The problem of whether a given map $\psi : X \to P(Y)$ is n-admissible is closely related to the selection problem. Observe that a map ψ is n-admissible if and only if there exists a $\mathcal{V}_n$-map $p : G \to X$ such that $\psi \circ p : G \to P(Y)$ has a continuous selection.

The class of admissible maps has nice functorial properties. In view of I.(3.5), the composition of an n-admissible map with a 1-admissible one is again n-admissible. Moreover, by I.(4.1), I.(4.3), I.(4.4) and I.(4.6) some natural operations in this class are canonical.

Now, we give several applications of our degree theory to solvability of set-valued equations involving admissible maps. The following two assertions may be treated as extensions of the well-known nonlinear and Leray–Schauder alternatives (see e.g. [26] for the single-valued case).

If Y is a vector space, $x \in Y$ and $A \subset Y$, then we put

$$\mathrm{conv}(x, A) = \{y \in Y : y = (1-t)x + ta,\ a \in A\}.$$

(6.2) THEOREM. (i) *Let U be an open neighbourhood of 0 in $\mathbb{R}^m$, $\psi : \mathrm{cl}\, U \to K(\mathbb{R}^m)$ an n-admissible map, $n \le m-1$ if $m \ge 2$ and $n = 1$ if $m = 1$. If $0 \notin \mathrm{conv}(x, \psi(x))$ for any $x \in \mathrm{bd}\, U$ and W is the component of $\mathbb{R}^m \setminus \psi(\mathrm{bd}\, U)$ that contains 0, then W is bounded, $W \subset \psi(U)$ and $\mathbb{R}^m \setminus \psi(\mathrm{bd}\, U)$ is not connected.*

(ii) *Let $\psi : \mathbb{R}^m \to P(\mathbb{R}^m)$ be an n-admissible map ($n \le m-1$ if $m \ge 2$ and $n = 1$ if $m = 1$). Then either the set $\{x \in \mathbb{R}^m : 0 \in \mathrm{conv}(x, \psi(x))\}$ is unbounded or $0 \in \psi(\mathbb{R}^m)$.*

Proof. Let $\varphi \in M_n(\mathrm{cl}\, U, \mathbb{R}^m)$ be a morphism determining a selector of ψ. Clearly, $\varphi \in M_n((\mathrm{cl}\, U, \mathrm{bd}\, U), (\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}))$ and, for each $t \in I$, $x \in \mathrm{bd}\, U$,

$$(1-t)x + t\varphi(x) \subset \mathbb{R}^m \setminus \{0\}.$$

Using I.(5.8), we see that there exists an n-morphism $\Phi \in M_n((\mathrm{cl}\, U, \mathrm{bd}\, U), (\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}))$ such that φ, Φ and φ_0, Φ (where φ_0 is an arbitrary n-morphism determining the inclusion $\mathrm{cl}\, U \to \mathbb{R}^m$) are h-linked. Therefore, in view of (3.5),

$$\deg(\varphi_0|U, 0) = \deg(\varphi|U, 0).$$

But, by (4.9) and (4.4), $\deg(\varphi_0|U, 0) \ne 0$. Hence, by (3.3)(i), $0 \in \varphi(U)$. Next, let $w \in W$. By (3.3)(iv), if L is a path in W joining w to 0, then L is an admissible subset of $\mathbb{R}^m$ and

$$\deg(\varphi|U, L) = \deg(\varphi|U, 0) \ne 0;$$

hence, by (3.3)(v), $w \in L \subset \varphi(U)$. Obviously, W is bounded and $\mathbb{R}^m \setminus \varphi(\mathrm{bd}\, U)$ cannot be connected not being bounded. ∎

(6.3) COROLLARY. (i) *Let $U \subset \mathbb{R}^m$ and n be as above. If $\psi : \mathrm{cl}\, U \to P(\mathbb{R}^m)$ is an n-admissible map such that, for $x \in \mathrm{bd}\, U$ and $y \in \psi(x)$, $(y|x) \ge 0$, then $0 \in \psi(\mathrm{cl}\, U)$.*

(ii) *If $\psi : \mathbb{R}^m \to P(\mathbb{R}^m)$ is an n-admissible map such that, for $y \in \psi(x)$,*

$$\lim_{|x| \to \infty} |x|^{-1}(y|x) = \infty,$$

then $\psi(\mathbb{R}^m) = \mathbb{R}^m$.

As another corollary we have the following version of the Brouwer fixed point theorem.

(6.4) COROLLARY. *Any n-admissible map $\psi : B^m \to P(\mathbb{R}^m)$ such that $\psi(S^{m-1}) \subset B^m$ ($n \le m-1$ when $m \ge 2$ and $n = 1$ when $m = 1$) has a fixed point.*

P r o o f. Obviously, $\varphi = i_{B^m} - \psi$ is an n-admissible map. In view of (6.2) we deduce that either, for some $x_0 \in S^{m-1}$, $0 \in \operatorname{conv}(x_0, \varphi(x_0))$ and then $x_0 \in \psi(x_0)$, or, for any $x \in S^{m-1}$, $0 \notin \operatorname{conv}(x, \varphi(x))$ so, by (6.2), $0 \in \varphi(N^m)$. ■

Using (4.10), (5.8) and (5.11) we easily obtain

(6.5) THEOREM. (i) *Let* $\psi : S^m \to P(S^m)$ *be an* n*-admissible map. There is* $x \in S^m$ *such that* $x \in \psi(x)$ *or* $-x \in \psi(x)$ *provided* m *is an even integer.*

(ii) *If* ψ *is* 1*-admissible and* $\psi(-x) \cap \psi(x) = 0$ *for any* $x \in S^m$, *then* $\psi(S^m) = S^m$.

(iii) *If* ψ *is* 1*-admissible*, *then there is* $x \in S^m$ *such that* $\psi(x) \cap \psi(-x) \neq \emptyset$ *or* $-\psi(x) \cap \psi(-x) \neq \emptyset$.

(iv) *If* $\psi : S^m \to P(\mathbb{R}^k)$, $0 < k \leq m$, *is* 1*-admissible*, *then there is* $x \in S^m$ *such that* $\psi(x) \cap \psi(-x) \neq 0$. *Moreover*, $\dim(\{x \in S^m : \psi(x) \cap \psi(-x) \neq 0\}) \geq m - k$.

As another consequence of (5.2) we can prove the following version of the theorem on the invariance of domain.

(6.6) THEOREM. *Let* U *be an open subset of* $\mathbb{R}^m$, $\varphi \in M_1(U, \mathbb{R}^m)$. *If for any* $x, y \in U$, $x \neq y$, $\varphi(x) \cap \varphi(y) = \emptyset$, *then* $\varphi(U)$ *is an open set in* $\mathbb{R}^m$. *More generally*, *if* X *is an* n*-dimensional manifold*, $\varphi \in M_1(X, \mathbb{R}^m)$ *and* $\varphi(x) \cap \varphi(y) = 0$ *for any* $x, y \in X$, $x \neq y$, *then* $\varphi(X)$ *is open in* $\mathbb{R}^m$.

P r o o f. Let $y_0 \in \varphi(U)$. There is $x_0 \in U$ such that $y_0 \in \varphi(x_0)$. Without any loss of generality we can assume that $y_0 \equiv x_0 = 0$. Otherwise, consider the morphism $\overline{\varphi} = j - \varphi \circ i$ where $i : x \mapsto x_0 - x$ and $j : x \mapsto y_0$.

Since $x_0 = 0 \in U$, there is $\varepsilon > 0$ such that $B^m_\varepsilon = B^m(0, \varepsilon) \subset U$. Obviously, for $x \in S^{m-1}_\varepsilon = S^{m-1}(0, \varepsilon)$, $0 \notin \varphi(x)$. We shall show that $\deg(\varphi | N^m_\varepsilon, 0) \neq 0$ where $N^m_\varepsilon = N^m(0, \varepsilon)$. If so, then we are done since, in view of (3.3)(iv), $\varphi(N^m_\varepsilon)$ contains the whole component of $\mathbb{R}^m \setminus \varphi(S^{m-1}_\varepsilon)$ containing 0.

By (4.4), $\deg(\varphi | N^m_\varepsilon, 0) = \deg(r \circ \varphi | S^{m-1}_\varepsilon)$ where $r : \mathbb{R}^m \setminus \{0\} \to S^{m-1}$ is the radial retraction. Let (p, q) represent the morphism $\varphi | B^m_\varepsilon$, where $p \in \mathcal{V}_1(G, B^m_\varepsilon)$, $q : G \to \mathbb{R}^m$. Consider the commutative diagram

$$\begin{tikzcd}
& p^{-1}(S^{m-1}) \arrow[dl, "p"'] \arrow[dr, "r\circ q"] \arrow[dd, "k"] & \\
S^{m-1}_\varepsilon & & S^{m-1} \\
& Z \arrow[ul, "u"] \arrow[ur, "r\circ f"'] &
\end{tikzcd}$$

where $Z = \{(x, x', g, g') \in B^m_\varepsilon \times B^m_\varepsilon \times G \times G : |x - x'| = \varepsilon,\ p(g) = x,\ p(g') = x'\}$, $k(g) = (p(g), 0, g, g_0)$ for $g \in p^{-1}(S^{m-1})$ and $g_0 \in G$ such that $p(g_0) = q(g_0) = 0$, $f(x, x', g, g') = q(g) - q(g')$ for $(x, x', g, g') \in Z$ and u is defined as follows: For $(x, x', g, g') \in Z$, $u(x, x', g, g')$ is the point of S^{m-1}_ε which is the intersection of the ray emanating from x' passing through x with S^{m-1}_ε. It can easily be seen that u is a $\mathcal{V}_1$-map. Therefore $H^*_2(Z) \simeq H^*_2(S^{m-1})$. Define an involution $h : Z \to Z$ by $h(x, x', g, g') = (x', x, g', g)$. We see that, for any $z \in Z$, $r \circ f(z) = -r \circ f(h(z))$.

In view of (5.3), $H_2^*(r \circ f)$ is an isomorphism. Arguing similarly to the proof of (5.2)(i), we conclude that $\deg(r \circ \varphi | S^{m-1}) = 1 \pmod 2$.

To see the last assertion, recall that X is locally homeomorphic to $\mathbb{R}^m$. ∎

(6.7) COROLLARY. *If* $\varphi \in M_1(\mathbb{R}^m, \mathbb{R}^m)$ *and, for any* $x, y \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$, $\varphi(x) + \varphi(y) \subset \varphi(x+y)$ *and* $\varphi(\lambda x) \subset \lambda\varphi(x)$, *then either*

(i) *there is* $x \neq 0$ *such that* $0 \in \varphi(x)$, *or*
(ii) $\varphi(\mathbb{R}^m) = \mathbb{R}^m$.

Proof. First, in fact, for any $x, y \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$, $\varphi(x) + \varphi(y) = \varphi(x+y)$ and $\varphi(\lambda x) = \lambda\varphi(x)$. Next, if for $x, y \in \mathbb{R}^m$, $x \neq y$, $\varphi(x) \cap \varphi(y) = \emptyset$, then, by (6.6), $\varphi(\mathbb{R}^m) = \mathbb{R}^m$. Otherwise, there are points x, y, $x \neq y$, such that $\varphi(x) \cap \varphi(y) \neq \emptyset$. Hence $0 \in \varphi(x-y)$. ∎

For n-admissible maps of manifolds the following definition of degree seems to be the most satisfactory.

Let X, Y be m-dimensional oriented manifolds, $L \subset Y$ be an admissible set (i.e. compact connected and free). If $\psi : X \to P(Y)$ is an n-admissible map ($n \leq m-1$ when $m \geq 2$ and $n = 1$ when $m = 1$) such that $K = \psi_-^{-1}(L)$ is compact, then we put (comp. [41])

$$\mathrm{Deg}(\psi, L) = \{\deg(\varphi, L)\}$$

where $\varphi \in M_n(X, Y)$ ranges over the set of all n-morphisms determining selectors of ψ.

Observe that if $\mathrm{Deg}(\psi, L) \neq \{0\}$, then $L \subset \psi(X)$. Moreover, Deg has all the important properties. In view of (3.6), if $\psi : X \to K(Y)$ is an n-acyclic map, then $\mathrm{Deg}(\psi, L) = \{\deg(\psi, L)\}$. On the other hand, for instance, if ψ is determined by the 1-morphism $\varphi \circ \varphi$ where φ is as in (5.1), then $\mathrm{Deg}(\psi \circ L) = \mathbb{Z}$ since any continuous single-valued $f : S^1 \to S^1$ is a selector of ψ.

III. The class of approximation-admissible morphisms

The degree theory and its applications were successfully extended to set-valued maps of infinite-dimensional spaces. In many papers (e.g. [49], [21], [80], [10]) convex-valued compact maps were considered, in others (e.g. [35], [36], [105], [60]) convex-valued condensing maps were studied. Similarly, acyclic and admissible (in the sense of Górniewicz) compact maps, condensing maps or maps with compact attractor were considered (see [16], [71], [72], [107], [39], [20], [48], [68]). In these papers the degree theory was developed for maps of the form $i - \varphi$, where φ was a map from the class considered, using appropriate approximation procedures.

In this chapter we introduce a class of set-valued maps and study its properties. In the next chapter, we shall build the approximation degree theory for this class of maps.

1. Filtrations. In order to define the class of approximation n-admissible morphisms (and set-valued maps determined by them), we have to introduce several notions concerning the structure of underlying spaces.

Let X be a topological space and let $(T,\le)$ be a directed set. By a *filtration* in X we mean a family $\mathcal{X}=\{X_t\}_{t\in T}$ of closed subsets of X such that, for almost all $t\in T, X_t\ne\emptyset$ (i.e. there is $t_0\in T$ such that, for $t\ge t_0$, $X_t\ne\emptyset$), and $X_s\subset X_t$ for $s\le t$. We say that a filtration $\mathcal{X}$ is *countable* if the set T is countable, and that $\mathcal{X}$ is *dense* if

$$\operatorname{cl}\Big(\bigcup_{t\in T}X_t\Big)=X.$$

If E is a topological vector space and $\mathcal{E}=\{E_t\}_{t\in T}$ is its filtration, then we say that the filtration $\mathcal{E}$ is *linear* provided, for any $t\in T$, E_t is a finite-dimensional linear subspace of E.

(1.1) Example. (i) If $\mathcal{X}=\{X_t\}_{t\in T}$ is a filtration in a space X and $Y\subset X$ is such that, for almost all (a.a.) $t\in T$, $Y_t=Y\cap X_t\ne 0$, then $\mathcal{Y}=\{Y_t\}_{t\in T}$ is a filtration in Y. In particular, the filtration $\mathcal{Y}$ is dense if so is $\mathcal{X}$.

(ii) If $\mathcal{X}=\{X_t\}_{t\in T}$ is a (dense) filtration in X and $T'\subset T$ is cofinal in T, then $\mathcal{X}'=\{X_t\}_{t\in T'}$ is a (dense) filtration in X.

(iii) If $Y\subset X$ and $\mathcal{Y}=\{Y_t\}_{t\in T}$ is a (dense) filtration in Y, then the family $\{\operatorname{cl}Y_t\}$ is a (dense) filtration in $\operatorname{cl}Y$.

(iv) If E is a topological vector space, $\mathcal{E}=\{E_t\}_{t\in T}$ is a linear filtration, $U\subset E$ is an open set such that $U_t=U\cap E_t\ne 0$ for a.a. $t\in T$, then $\{U_t\}$ is a filtration in U and, for all $t\in T$, U_t is an oriented n_t-dimensional manifold where $n_t=\dim E_t$.

(v) If U is an open set in E such that $\operatorname{cl}U\ne E$, then the family $\{B_t\}_{t\in T}$, where $B_t=\operatorname{bd}U\cap E_t$, is a dense filtration in $B=\operatorname{bd}U$ provided $\mathcal{E}=\{E_t\}_{t\in T}$ is a dense filtration in E. In fact, let $x\in B$ and let V be a neighbourhood of x in B. Therefore, $V=\overline{V}\cap B$ where $\overline{V}$ is open in E. There is $t_0\in T$ and $\overline{x}\in V\cap E_{t_0}$. Let W be an $\overline{x}$-starshaped neighbourhood such that $\overline{x}\in W\subset\overline{V}$. By (i), the families $\{U\cap E_t\}$, $\{(E\setminus\operatorname{cl}U)\cap E_t\}$ are dense filtrations in U and $E\setminus\operatorname{cl}U$, respectively. There are $t_1,t_2\in T$ and points $\overline{x}_1\in W\cap E_{t_1}\cap U$ and $\overline{x}_2\in W\cap(E\setminus\operatorname{cl}U)\cap E_{t_2}$. For any $t\ge t_0,t_1,t_2$, the union l of linear segments joining $\overline{x}_1$ to $\overline{x}$ and $\overline{x}$ to $\overline{x}_2$ lies in E_t. Since L is a connected set, $\overline{x}_1\in U$ and $\overline{x}_2\in E\setminus\operatorname{cl}U$, there is $y\in l\cap B\subset W\subset B\subset E_t$.

Let X be a uniform space with uniformity $\operatorname{unf}(X)$. We say that a filtration $X=\{X_t\}_{t\in T}$ in X is *regular over a set* $Z\subset X$ such that $Z\cap\operatorname{cl}(\bigcup_{t\in T}X_t)\ne\emptyset$ if, for any vicinity $U\in\operatorname{unf}(X)$, there is $V\in\operatorname{unf}(X)$, $t_0\in T$ and a family of continuous maps $\{\pi_t:V(Z)\cap V(X_t)\to X_t\}_{t\ge t_0}$ satisfying the following conditions:

(R1) for $x\in V(Z)\cap X_t$, $\pi_t(x)=x$, i.e. $\pi_t\circ j_t=\operatorname{id}_{X_t}$ where $j_t:X_t\to X$ is the inclusion;

(R2) the maps $j_t \circ \pi_t$ and $j : V(Z) \cap V(X_t) \to X$ (j is the inclusion) are U-homotopic (i.e. there is a homotopy $h : V(Z) \cap V(X_t) \times I \to X$ such that $h(\cdot, 0) = j_t \circ \pi_t, h(\cdot, 1) = j$ and, for $x \in V(Z) \cap V(X_t)$, $h(\{x\} \times I) \times h(\{x\} \times I) \subset U$).

If $Z = X$, then we say that the filtration $\mathcal{X}$ is *regular* (comp. [63]).

(1.2) Theorem. *Let E be a metrizable locally convex topological vector space and let C be a closed subset of E. Assume $\mathcal{E} = \{\mathcal{E}_t\}_{t\in T}$ is a linear filtration in E.*

(i) *If, for a.a. $t \in T$, $C_t = C \cap E_t \neq \emptyset$, then the filtration $\{C_t\}$ in C is regular.*

(ii) *Let X be an open subset of C such that the family $\{X_t\}$, where $X_t = X \cap E_t$, is a filtration in X. If $Z \subset X$, $Z \cap \mathrm{cl}(\bigcup_{t\in T} X_t) \neq \emptyset$ and there is a neighbourhood W of 0 in E such that $(W + Z) \cap C \subset X$, then the filtration $\{X_t\}$ is regular over Z. In particular, the filtration $\{X_t\}$ is regular over any compact subset of X.*

(iii) *If U is a convex open and bounded subset of E and E is a normed space, $B = \mathrm{bd}\, U$, $\{B_t\} = \{B \cap E_t\}$ is a filtration in B, then $\{B_t\}$ is regular.*

Proof. (i) is a consequence of (ii).

(ii) Let d be a translation-invariant metric on E, compatible with the topology and convex structure of E. Let $\varepsilon > 0$ be such that $B(0, 4\varepsilon) \subset U \cap W$ and put $V = N(0, \varepsilon)$. Then $V(Z) = N_\varepsilon(Z)$. Choose $t_0 \in T$ such that $N_\varepsilon(Z) \cap X_{t_0} \neq \emptyset$. For any $t \in T$, $t \geq t_0$ and $x \in N_\varepsilon(Z) \cap N_\varepsilon(X_t)$, let $d_{x,t} = d(x, C_t) < \varepsilon$. We define a set-valued map

$$N_\varepsilon(Z) \cap N_\varepsilon(X_t) \ni x \mapsto \psi_t(x) = B(x, 2d_{x,t}) \cap C_t, \qquad t \geq t_0.$$

Cleary, ψ_t has nonempty closed convex values and, by I.(1.4), ψ_t is l.s.c. Therefore, in view of I.(1.7), there is a continuous map $\pi_t : N_\varepsilon(Z) \cap N_\varepsilon(X_t) \to E_t$ such that $\pi_t(x) \in \psi_t(x)$. Obviously, for $x \in N_\varepsilon(Z) \cap N_\varepsilon(X_t)$, $\pi_t(x) \in X_t$ and $\pi_t(x) = x$ for $x \in N_\varepsilon(Z) \cap X_t$. Moreover,

$$(1 - \lambda) j_t \circ \pi_t(x) + \lambda x \in N(x, 2d_{x,t}) \subset X \cap U(x),$$

hence $j_t \circ \pi_t$ and j are U-homotopic.

(iii) follows easily from (ii). ■

(1.3) Remark. (i) If we assume in Example (1.1)(iv) that E is locally convex and metrizable, then the filtration $\{U_t\}$ in U is regular over any set Z that lies "deeply" in U, i.e. such that $W + Z \subset U$ for some neighbourhood W of 0 in E.

(ii) In view of (1.1)(v), (1.2)(iii), if U is a bounded convex and open neighbourhood of 0 in a normed space E with a given linear filtration $\{E_t\}$, then $\{B_t\}$ is a regular filtration and, for any $t \in T$, B_t is a finite-dimensional manifold.

It is clear that in a uniform space with a dense regular filtration any point may be approximated in a controlled way by points from the sets of the filtration. There is a homological analogue of this fact — see [63].

Let $X, \overline{X}$ be topological spaces with filtrations $\mathcal{X} = \{X_t\}_{t\in T}$ and $\overline{\mathcal{X}} = \{\overline{X}_t\}_{t\in T}$, respectively. By the Cartesian product of these filtrations we mean the filtration $\mathcal{X} \times \overline{\mathcal{X}} = \{X_t \times \overline{X}_t\}_{t\in T}$ in $X \times \overline{X}$. In particular, in the space $X \times I$ we always consider the filtration $\mathcal{X} \times \mathcal{I}$ where $\mathcal{I} = \{I_t\}_{t\in T}$ with $I_t = I$ for all $t \in T$.

Let X be a topological space and let $\mathcal{X} = \{X_t\}_{t\in T}$ be a filtration in X. We say that a set $A \subset X$ is *bounded with respect to* $\mathcal{X}$ if, for a.a. $t \in T$, $A_t = A \cap X_t$ is relatively compact (in X_t). We say that the space X is *locally bounded with respect to* $\mathcal{X}$ if every point of X has a neighbourhood bounded with respect to $\mathcal{X}$. It is clear that in order for X to be locally bounded with respect to $\mathcal{X}$ it is sufficient and necessary that X can be represented as a union of open sets bounded with respect to $\mathcal{X}$. Moreover, any locally compact space is locally bounded with respect to any filtration.

(1.4) Example. Let E be an infinite-dimensional locally convex space. The following conditions are equivalent:

(i) there exists a dense linear filtration $\mathcal{E} = \{E_t\}_{t\in T}$ in E with respect to which E is locally bounded,

(ii) there is a continuous seminorm $p : E \to [0, \infty)$ with $\dim(E/p^{-1}(0)) = \infty$.

In fact, assume that (ii) does not hold. Let $\{E_t\}_{t\in T}$ be an arbitrary linear and dense filtration in E and let U be a neighbourhood of 0 in E. Choose a convex balanced neighbourhood V of 0 such that $V \subset U$. The Minkowski gauge p_V of V is a continuous seminorm and $\operatorname{codim} p_V^{-1}(0) = n < \infty$. For some $t \in T$, $\dim(E_t) > n$; hence $E_t \cap p_V^{-1}(0) \ni x \neq 0$. Thus the set $V \cap E_t$ is not bounded.

Assume that (ii) is satisfied. Let $\{y_a\}_{a\in A}$ be a dense subset of E and $\{V_b\}_{b\in B}$ a base of neighbourhoods of 0 in E. Make the set $A \times B$ well-ordered. Then it is well-ordered similarly to the set of all ordinals $\xi < \alpha\beta$ where α is the ordinal number of A and β is the ordinal of B. Let $\varphi : A \times B \to \{\xi : \xi < \alpha\beta\}$ be a given similarity. Using transfinite induction we define a transfinite sequence $\overline{E}_0, \overline{E}_1, \ldots, \overline{E}_\xi, \ldots$, $\xi < \alpha\beta + 1$, of finite-dimensional subspaces of E such that $\overline{E}_\xi \cap p^{-1}(0) = \{0\}$ and $\operatorname{cl}(\bigcup_{\xi<\alpha\beta+1} \overline{E}_\xi) = E$. Let $\overline{E}_0 = 0$ and let $\xi < \alpha\beta$. The space $\overline{E}_\xi + p^{-1}(0)$ is a nowhere dense subset of E, since otherwise $\operatorname{codim} p^{-1}(0) < \infty$. Let $(a, b) \in A \times B$ be such that $\xi = \varphi(a, b)$ and let $x_{\xi+1} \in y_a + V_b \setminus (\overline{E}_\xi + p^{-1}(0))$. We put $\overline{E}_{\xi+1} = \operatorname{span}(\{x_{\xi+1}\} \cup \overline{E}_\xi)$. If $\lambda \le \alpha\beta$ is an ordinal of the second type, then we put $\overline{E}_\lambda = \bigcap_{\xi<\lambda} \overline{E}_\xi = \{0\}$. Observe that $\overline{E}_{\xi+1} \cap (y_a + V_b) \neq \emptyset$ for $\xi = \varphi(a, b) < \alpha\beta$. Hence, if U is an open subset of E, then there exists (a, b) such that $y_a + V_b \subset U$, so $U \cap \overline{E}_{\xi+1} \neq 0$ where $\xi = \varphi(a, b)$. Therefore, the set $\bigcup_{\xi<\alpha\beta+1} \overline{E}_\xi$ is dense in E. Moreover, if $x \in \overline{E}_{\xi+1} \cap p^{-1}(0)$, then $x = sx_{\xi+1} + uy$ where $s, u \in \mathbb{R}$ and $y \in \overline{E}_\xi$. If $s = 0$, then $x \in \overline{E}_\xi \cap p^{-1}(0)$, so $x = 0$. If $s \neq 0$, then $x_{\xi+1} = s^{-1}(x - uy) \in \overline{E}_\xi + p^{-1}(0)$, a contradiction.

Now, let H be an (algebraic) base of $\overline{E} = \bigcup_{\xi<\alpha\beta+1} E$ and denote by T the family of all finite subsets of H (directed by inclusion). If $t \in T$, then we put

$E_t = \operatorname{span}(t)$. Then $\bigcup_{t\in T} E_t = \overline{E}$ is a dense subspace; hence $\{E_t\}_{t\in T}$ is a dense linear filtration. Moreover, for $t \in T$, $E_t \cap p^{-1}(0) = \{0\}$, thus E is locally bounded with respect to the filtration $\{E_t\}$, as required. ■

2. Approximation-admissible morphisms and maps. In this section we define a class of morphisms (set-valued maps) for which the approximation degree theory is available.

Let X, Y be topological spaces and let $\mathcal{X} = \{X_t\}_{t\in T}$, $\mathcal{Y} = \{Y_t\}_{t\in T}$ be filtrations in X, Y, respectively. Next, let $\mathcal{F}$ be a class of set-valued maps (e.g. $\mathcal{F}$ is the class of all upper-semicontinuous maps, n-acyclic maps (i.e. $\mathcal{F} = \mathcal{A}_n$), convex-valued maps etc.).

We say that a map $\phi : X \to P(Y)$ belonging to $\mathcal{F}$ is a *filtered map* with respect to the filtrations $\mathcal{X}$, $\mathcal{Y}$ (written $\phi \in \mathcal{F}^F((X,\mathcal{X}),(Y,\mathcal{Y}))$) if there is $t_0 \in T$ such that, for any $t \geq t_0$, $\phi(X_t) \subset Y_t$. Similarly, we say that an n-morphism $\varphi \in M_n(X,Y)$ is an F-n-morphism (written $\varphi \in M_n^F((X,\mathcal{X}),(Y,\mathcal{Y}))$) if φ determines a filtered map. Moreover, we let $M^F = \bigcup_{n\geq 1} M_n^F$ and

$$M_n^F((X,A;\mathcal{X}),(Y,B;\mathcal{Y})) = \{\varphi \in M_n^F((X,\mathcal{X},(Y,\mathcal{Y})) : \varphi(A) \subset B\}.$$

Suppose additionally that Y is a uniform space. A map $\phi : X \to P(Y)$ from the class $\mathcal{F}$ is called *approximation-admissible* relative to $\mathcal{X}$, $\mathcal{Y}$ (written $\phi \in \mathcal{F}^A((X,\mathcal{X}),(Y,\mathcal{Y}))$) if, for any vicinity $V \in \operatorname{unf}(Y)$, there is $t_0 \in T$ such that $\phi(X_t) \subset V(Y_t)$ for $t \geq t_0$.

Analogously we define the class $M_n^A((X,A;\mathcal{X}),(Y,B;\mathcal{Y}))$.

Remark. If the filtrations $\mathcal{X}$, $\mathcal{Y}$ of X, Y, respectively, are fixed, then we suppress the symbols $\mathcal{X}$, $\mathcal{Y}$ in the notation.

(2.1) Proposition. (i) $\mathcal{F}^F \subset \mathcal{F}^A$; *in particular*, $M_n^F \subset M_n^A$.

(ii) *If the class $\mathcal{F}$ is closed under composition, then so is $\mathcal{F}^F$. In particular, if Z is a space with a filtration $\{Z_t\}_{t\in T}$, $\varphi_1 \in M_n^F(X,Y)$, $\varphi_2 \in M_1^F(Y,Z)$, then $\varphi_2 \circ \varphi_1 \in M_n^F(X,Z)$.*

(iii) *If Y is a metric space with a metric d, then a map $\phi : X \to P(Y)$ from the class $\mathcal{F}$ is an A-map* (*i.e.* $\phi \in \mathcal{F}^A(X,Y)$) *if and only if*

$$\lim_{t\in T} \sup_{x\in X_t} \sup_{y\in\psi(x)} d(y, Y_t) = 0.$$

The same statement holds for morphisms.

Now, assume that X is also a uniform space. A map $\phi : X \to P(Y)$ from the class $\mathcal{F}$ is called *strongly approximation-admissible* relative to $\mathcal{X}$, $\mathcal{Y}$ (written $\phi \in \mathcal{F}^{sA}((X,\mathcal{X}),(Y,\mathcal{Y}))$) if, for any $V \in \operatorname{unf}(Y)$, there are $U \in \operatorname{unf}(X)$ and $t_0 \in T$ such that, for $t \geq t_0$, $\phi(U(X_t)) \subset V(Y_t)$. Analogously we define sA-n-morphisms.

(2.2) Proposition. (i) $\mathcal{F}^{sA} \subset \mathcal{F}^A$. *If $\phi \in \mathcal{F}^A(X,Y)$ and ϕ is uniformly upper-semicontinuous* (*i.e. for any $V \in \operatorname{unf}(Y)$, there is $U \in \operatorname{unf}(X)$ such that $\phi(U(x)) \subset V(\phi(x))$ for $x \in X$*), *then $\phi \in \mathcal{F}^{sA}(X,Y)$.*

(ii) *Let $\mathcal{F}$ be closed under composition. If $\phi_1 \in \mathcal{F}^A(X,Y)$, $\phi_2 \in \mathcal{F}^{sA}(Y,Z)$, then $\phi_2 \circ \phi_1 \in \mathcal{F}^A(X,Z)$. A similar statement holds for morphisms.*

(iii) *Let E be a topological vector space with a linear filtration $\mathcal{E} = \{E_t\}$, $\phi_1, \phi_2 \in \mathcal{F}^A(X,E)$ and $f_1, f_2 : X \to \mathbb{R}$ be continuous bounded functions*; *then $f_1\phi_1 + f_2\phi_2 \in \mathcal{F}^A(X,E)$ provided $f_1\phi_1 + f_2\phi_2 \in F$. Similarly for morphisms $\varphi_1 \in M_1^A$ and $\varphi_2 \in M_n^A$.*

(2.3) Remark. Using I.(4.1), I.(4.2) and I.(4.6), the reader will easily formulate and prove facts concerning restriction, piecing together and Cartesian products of A-morphisms.

We shall be mainly interested in A-morphisms (the class of A-morphisms or maps determined by them constitutes the most general class of those considered in this chapter).

Let X be a topological space with a filtration $\mathcal{X} = \{X_t\}_{t\in T}$ and Y a uniform space with a filtration $\mathcal{Y} = \{Y_t\}_{t\in T}$. Assume that $\varphi \in M(X,Y)$ and let a pair (p,q) represent φ, where $p \in \mathcal{V}(W,X)$, $q : W \to Y$. The map p determines a filtration $\mathcal{W} = \{W_t\}_{t\in T}$ in W where $W_t = p^{-1}(X_t)$. We easily see that if $(p',q') \in \varphi$, where $p' \in \mathcal{V}(W',X)$, and $f : W \to W'$ is a homeomorphism such that $p' \circ f = p$ and $q' \circ f = q$, then f is a filtered map relative to $\mathcal{W}, \mathcal{W}'$ (in W, W', respectively). Moreover, p is a filtered map as well. The following simple fact holds.

(2.4) Proposition. *A morphism $\varphi \in M(X,Y)$ is an A-morphism relative to $\mathcal{X}$ and $\mathcal{Y}$ if and only if*, *for any pair $(p,q) \in \varphi$, where $p \in \mathcal{V}(W,X)$ and $q : W \to Y$, the map q is an A-map relative to $\mathcal{W}, \mathcal{Y}$. If*, *for some $(p,q) \in \varphi$, $p^{-1} : X \to K(W)$ is uniformly upper-semicontinuous and q is a uniformly continuous A-map*, *then $\varphi \in M^{sA}(X,Y)$ provided X is a paracompact uniform space.*

Proof. The first part is obvious. Next observe that, since $p : W \to X$ is a perfect map, p^{-1} is u.s.c. and $W = p^{-1}(X)$ is paracompact, hence uniformizable. The further reasoning is evident. ∎

The next theorem gives sufficient conditions for a given morphism to be approximation-admissible.

Let W be a topological space and Z a normed space with norm $\|\cdot\|$. A continuous (single-valued) map $f : D \to Z$, where $D \subset W$, is called *uniformly finitely approximable* (u.f.a.; see [75]) if for any $\varepsilon > 0$, there is a continuous map $f' : D \to Z$ such that

$$\sup_{W\in D} \|f(w) - f'(w)\| < \varepsilon$$

and $f'(D) \subset L$ where L is a finite-dimensional subspace of Z. In view of the well-known Schauder projection lemma, it follows that any compact map $f : D \to Z$ is u.f.a.

Let E be a topological vector space and F be a normed space. Assume that we are given families $\{X_n\}_{n=1}^{\infty}$, $\{Y_n\}_{n=1}^{\infty}$, where X_n is a topological vector space and Y_n is a normed space for any $n = 1, 2, \ldots$, and families of linear operators $\{i_n : X_n \to E\}_{n=1}^{\infty}$, $\{j_n : Y_n \to F\}_{n=1}^{\infty}$. Suppose that the families $\{i_n\}$, $\{j_n\}$ satisfy the following conditions:

(∗) for any n, $i_n(X_n) \subset i_{n+1}(X_{n+1})$, $j_n(Y_n) \subset j_{n+1}(Y_{n+1})$;

(∗∗) $\sup\limits_n \|j_n\| < \infty$;

(∗∗∗) for any n, $\dim i_n(X_n) \geq \dim j_n(Y_n)$.

Let $B \subset E$ and $B_n = i_n(X_n) \cap B$ for any integer n. Finally, we assume that $\varphi : B \to P(W)$ is a set-valued map, $q : W \to F$, where W is a topological space, and $W_n = \varphi(B_n)$ for any n.

(2.5) Theorem. *If there exists a family $\{q_n : W_n \to Y_n\}_{n=1}^{\infty}$ of u.f.a. maps such that*

$$\lim_{n\to\infty} \sup_{w\in W_n} \|q(w) - j_n \circ q_n(w)\| = 0,$$

then there are linear filtrations $\{E_n\}_{n=1}^{\infty}$, $\{F_n\}_{n=1}^{\infty}$ in E and F, respectively, such that $\dim E_n = \dim F_n$ for any n and $q \circ \varphi : B \to P(F)$ is an A-map relative to $\{B \cap E_n\}, \{F_n\}$.

Proof. Let $q_1' : W_1 \to Y_1$ be a map such that

$$\sup_{w\in W_1} \|q_1(w) - q_1'(w)\| < 1$$

and $q_1'(W_1) \subset L_1$ where L_1 is a finite-dimensional linear subspace of Y_1. Let $F_1 = j_1(L_1)$. Clearly, $\dim F_1 < \infty$.

Assume that, for $k \leq n-1$, we have defined a map $q_k' : W_k \to Y_k$ such that

$$\sup_{w\in W_k} \|q_k(w) - q_k'(w)\| < 1/k, \qquad q_k'(W_k) \subset L_k,$$

where L_k is a finite-dimensional subspace of Y_k, and a finite-dimensional subspace $F_k \subset F$, $F_{k-1} \subset F_k$ and $j_k(L_k) \subset F_k$.

Let $q_n' : W_n \to Y_n$ be a map such that

$$\sup_{w\in W_n} \|q_n(w) - q_n'(w)\| < 1/n$$

and $q_n'(W_n) \subset L_n$ where L_n is a finite-dimensional subspace of Y_n. We let $F_n = F_{n-1} + j_n(L_n)$. Clearly, $F_{n-1} \subset F_n$, $j_n(L_n) \subset F_n$ and $\dim F_n < \infty$.

Let $\varepsilon > 0$ and let N be a positive integer such that

$$\sup_{w\in W_n} \|q(w) - j_n \circ q_n(w)\| < \varepsilon/2$$

for $n \geq N$ and such that, for any n and $y \in Y_n$, $\|j_n(y)\| < \varepsilon/2$ provided $\|y\| < 1/N$.

Let $n \geq N$ and $x \in B_n$. For any $w \in \varphi(x) \subset W_n$, we have

$$d(q(w), F_n) \leq \|q(w) - j_n \circ q_n'(w)\| \leq \|q(w) - j_n \circ q_n(w)\| + \|j_n(q_n(w) - q_n'(w))\| < \varepsilon.$$

Clearly, $\dim i_n(X_n) \geq \dim F_n$. Choose a subspace E_n in $i_n(X_n)$ such that $\dim E_n = \dim F_n$. Since, according to $(*)$, $i_n(X_n) \subset i_{n+1}(X_{n+1})$, one can choose E_n in such a manner that $E_n \subset E_{n+1}$ for any n. Then

$$\lim_{n \to \infty} \sup_{y \in q \circ \varphi(x)} d_F(y, F_n) = 0.$$

Therefore, in view of (2.1)(iii), the proof is complete. ■

(2.6) Remark. (i) If $E = F$ and $i_n = j_n$, then $q \circ \varphi$ is an A-map relative to $\{F_n\}, \{F_n\}$.

(ii) Under the assumptions of the above theorem, if in place of φ we take the map $p^{-1} : E \to K(W)$, where $p \in \mathcal{V}(W, E)$, and assume that $q : W \to F$ is continuous, then (2.5) gives a sufficient condition for a morphism represented by (p, q) to be an A-morphism.

Below we give several examples of A-morphisms (or maps determined by them). Obviously, we shall be interested in A-maps belonging to the class of u.s.c. maps, acyclic or convex-valued maps or at least those determined by morphisms. For these set-valued maps, we can construct the degree theory.

Let X, Y be topological spaces. Recall that a map $\psi : X \to P(Y)$ from the class $\mathcal{F}$ is called *compact* (or a *K-map*, written $\psi \in \mathcal{F}^K(X, Y)$) if $\operatorname{cl} \psi(X)$ is a compact subset of Y.

(2.7) Example. (i) If Y is a uniform space, $\mathcal{X}$, $\mathcal{Y}$ are filtrations in X, Y, respectively, and $\mathcal{Y}$ is a dense filtration, then $\mathcal{F}^K(X, Y) \subset \mathcal{F}^A((X, \mathcal{X}), (Y, \mathcal{Y}))$.

(ii) If X, Y are uniform spaces and $\mathcal{Y}$ is dense, then $\mathcal{F}^K(X, Y) \subset \mathcal{F}^{sA}(X, Y)$.

We shall show (ii) (the proof of (i) runs similarly). Let $\psi \in \mathcal{F}^K(X, Y)$ and $V \in \operatorname{unf}(Y)$. Since $\operatorname{cl} \psi(X)$ is compact and $\mathcal{Y}$ is a dense filtration, there are a finite number of points $y_1, \ldots, y_k \in \bigcup_{t \in T} Y_t$ such that $\operatorname{cl} \psi(X) \subset \bigcup_{i=1}^k V(y_i)$. Let $t_0 \in T$ be such that $y_i \in Y_{t_0}$, $i = 1, \ldots, k$.

For any vicinity $U \in \operatorname{unf}(X)$ and $t \geq t_0$, $\psi(U(X_t)) \subset V(Y_t)$.

Similarly, any compact morphism $\varphi \in M^K(X, Y)$ (i.e. a morphism determining a compact map) is an A-morphism.

(2.8) Example. Let X be a topological vector space and U its open subset. If a filtration $\mathcal{X}$ in X is dense, $\varphi \in M^K(U, X)$ and $i_U : U \to X$ is the inclusion, then the morphism $i_U - \varphi$ is an A-morphism relative to $\{U \cap X_t\}$ and $\mathcal{X} = \{X_t\}$. In particular, any set-valued compact field with convex or n-acyclic values is an A-map relative to any dense filtration in the range.

(2.9) Example. Let X be an arbitrary space and Y a topological vector space. If $\psi \in \mathcal{F}^K(X, Y)$, then there is a linear filtration $\mathcal{E} = \{Y_t\}_{t \in T}$ in Y such that $\operatorname{cl} \psi(X) \subset \operatorname{cl}(\bigcup_{t \in T} Y_t)$. Moreover, ψ is an A-map relative to an arbitrary filtration $\mathcal{X}$ in X and $\mathcal{E}$.

The above examples show that (even in the single-valued case) the class of A-maps generalizes the class of maps considered in the Leray–Schauder theory and its set-valued analogues.

There are several "concrete" examples of A-maps which are not compact (in the abstract setting it is rather easy to find this type of examples). We give one such example. Others are given for instance in [65], [66] together with applications to boundary value problems for ordinary differential equations.

(2.10) Example. Let Z be a compact acyclic (in the sense of Alexander–Spanier cohomology) metric space and let $f : Z \times \mathbb{R}^N \to \mathbb{R}^M$ be a continous map. Consider the set-valued map $\psi : L^\infty(I, \mathbb{R}^N) \to K(L^\infty(I, \mathbb{R}^M))$ given by

$$\psi(x) = \{y \in L^\infty(I, \mathbb{R}^M) : y(t) = f(z, x(t)),\ t \in I,\ \text{for some } z \in Z\}.$$

This map is well-defined and determined by a morphism. Indeed, let $p : Z \times L^\infty(I, \mathbb{R}^N) \to L^\infty(I, \mathbb{R}^N)$ be the projection. It is easy to see that p is a $\mathcal{V}_1$-map. Next, define $q : Z \times L^\infty(I, \mathbb{R}^N) \to L^\infty(I, \mathbb{R}^M)$ by $q(z, x)(t) = f(z, x(t))$, $t \in I$, $z \in Z$ and $x \in L^\infty(I, \mathbb{R}^N)$. Let $z \in Z$, $x \in L^\infty(I, \mathbb{R}^N)$ and $\|x\|_\infty \le R$. Since $f|Z \times B^N(0, R + 1)$ is uniformly continuous, for any $\varepsilon > 0$, there is $0 < \delta < 1$ such that $|f(z', u') - f(z'', u'')| < \varepsilon$ provided $z', z'' \in Z$, $u', u'' \in B^N(0, R+1)$ and $d_Z(z', z'') < \delta$, $|u' - u''| < \delta$. Hence, for $x' \in L^\infty(I, \mathbb{R}^N)$, $z' \in Z$, if $\|x - x'\|_\infty < \delta$ and $d_Z(z, z') < \delta$, then $|f(z, x(t)) - f(z', x'(t))| < \varepsilon$ almost everywhere, so $\|q(z, x) - q(z', x')\|_\infty < \varepsilon$.

We see that $\psi(x) = q(p^{-1}(x))$ for $x \in L^\infty(I, \mathbb{R}^N)$, therefore ψ is determined by the morphism represented by the pair (p, q). This type of maps is frequently encountered in control theory.

Consider a map $g : I \times \mathbb{R}^M \to \mathbb{R}^M$ and define the superposition operator $G : L^\infty(I, \mathbb{R}^M) \to L^\infty(I, \mathbb{R}^N)$ by $G(y)(t) = g(t, y(t))$ for $t \in I$ and $y \in L^\infty(I, \mathbb{R}^M)$. We now show that $G \circ \psi : L^\infty(I, \mathbb{R}^N) \to L^\infty(I, \mathbb{R}^N)$ (which is obviously determined by a 1-morphism) is an A-map relative to some linear filtrations in $L^\infty(I, \mathbb{R}^N)$.

First observe that the operator G is well-defined and continuous. For any positive integer n and $j = 0, 1, \ldots, 2^n - 1$, let $t_{nj} = j/2^n$ and, for $j = 0, \ldots, 2^n - 2$, let $I_{nj} = [t_{nj}, t_{nj+1})$, and $I_{n,2^n-1} = [(2^n - 1)/2^n, 1]$. If we denote by $\mathcal{X}_{nj}$ the characteristic function of the interval I_{nj}, then we put

$$E_n = \Big\{x \in L^\infty(I, \mathbb{R}^N) : x = \sum_{j=0}^{2^n-1} a_{nj}\mathcal{X}_{nj},\ a_{nj} \in \mathbb{R}^N\Big\}.$$

Clearly, $E_n \subset E_{n+1}$ and $\dim E_n = N2^n$ for any n.

We claim:

If B is a bounded subset of $L^\infty(I, \mathbb{R}^N)$ such that $\{B_n = B \cap E_n\}_{n=1}^\infty$ is a filtration in B, then $G \circ \psi|B$ is an A-map relative to $\{B_n\}, \{E_n\}$.

Fix $\varepsilon > 0$ and let $R = \sup_{x \in B} \|x\|_\infty$. The uniform continuity of $g|I \times B^N(0, R)$

implies the existence of $\delta > 0$ such that $|g(t,u) - g(t',u)| < \varepsilon$ provided $t, t' \in I$, $|t - t'| < \delta$ and $u \in B^N(0,R)$. Choose n_0 such that $2^{-n_0} < \delta$, let $n \ge n_0$, $x \in B_n$ and $y \in \psi(x)$. For $v \in L^\infty(I, \mathbb{R}^N)$ given by $v(t) = g(t_{nj}, y(t_{nj}))$ for $t \in I_{nj}$ we have $v \in E_n$ and

$$\begin{aligned}|G(y)(t) - v(t)| &= |g(t, y(t)) - g(t_{nj}, y(t_{nj}))|\\ &= |g(t, f(z, x(t))) - g(t_{nj}, f(z, x(t_{nj})))| < \varepsilon,\end{aligned}$$

where $z \in Z$, $0 \le j \le 2^n - 1$ is such that $t \in I_{nj}$. Thus, for $n \ge n_0$,

$$\sup_{x \in B_n} \sup_{y \in \psi(x)} d(G(y), E_n) < \varepsilon.$$

Finally, observe that, generally speaking, the considered map is not compact.

3. Approximation of A-maps. In this section we discuss some other properties of A-maps relating to approximation.

Denote by $\mathcal{F}_i$ the following classes of set-valued maps:

$i = 1$: the class of all uper-semicontinuous maps;
$i = 2$: the class of compact-valued u.s.c. maps;
$i = 3$: the class of compact convex-valued u.s.c. maps.

We start with the following simple result.

(3.1) Theorem. *Let X be a topological space with a filtration $\mathcal{X} = \{X_t\}_{t \in T}$.*

(i) *Let Y be a uniform space with a filtration $\mathcal{Y} = \{Y_t\}_{t \in T}$ regular over a set $Z \subset Y$. If $\psi \subset \mathcal{F}_i^A((X,\mathcal{X}),(Y,\mathcal{Y}))$, $i = 1, 2$, $\psi(X) \subset Z$, then for any $U \in \mathrm{unf}(Y)$ there is $t_1 \in T$ and a family $\{\psi_t : X_t \to P(Y_t)\}_{t \ge t_1}$ of maps from $\mathcal{F}_i$ such that $\psi_t(x) \subset U(\psi(x))$ and $\psi(x) \subset U(\psi_t(x))$ for $x \in X_t$, $t \ge t_1$. Moreover, the map $i_t \circ \psi_t$ is homotopic in $\mathcal{F}_i$ to $\psi|X_t$* (1).

(ii) *If E is a metrizable locally convex space with a linear filtration $\mathcal{E} = \{E_t\}_{t \in T}$ and $\psi \in \mathcal{F}_3^A((X,\mathcal{X}),(E,\mathcal{E}))$, then, for any neighbourhood $\overline{U}$ of 0 in E, there is $t_1 \in T$ and a family of maps $\{\overline{\psi}_t : X_t \to K(E_t)\}_{t \ge t_1}$ from the class $\mathcal{F}_3$ such that $\overline{\psi}_t(x) \subset \psi(x) + \overline{U}$, $\psi(x) \subset \overline{\psi}_t(x) + \overline{U}$ and $i_t \circ \overline{\psi}_t$, $\psi|X_t$ are homotopic in $\mathcal{F}_3$.*

Proof. (i) Let $U \in \mathrm{unf}(Y)$. We can assume that U is symmetric. By the definition of the regularity of a filtration over Z, we have $V \in \mathrm{unf}(Y)$, $t_0 \in T$ and a family $\{\pi_t : V(Z) \cap V(Y_t) \to Y_t\}_{t \ge t_0}$ such that, for $t \ge t_0$, $i_t \circ \pi_t$ and $i : V(Z) \cap V(Y_t) \to Y$ are U-homotopic. Since ψ is an A-map, there is $t_1 \ge t_0$ such that, for $t \ge t_1$, $\psi(X_t) \subset V(Y_t) \cap Z$. Define $\psi_t = \pi_t \circ \psi|X_t$. Clearly, if $\psi \in \mathcal{F}_i$, $i = 1, 2$, then $\psi_t \in \mathcal{F}_i$ as well. Moreover, for $x \in X_t$, $t \ge t_1$, $\psi(x) \in U(\psi_t(x))$ and $\psi_t(x) \subset U(\psi(x))$. If $h_t : V(Z) \cap V(Y_t) \times I \to Y$ denotes the homotopy joining $i_t \circ \pi_t$ to i, then $i_t \circ \psi_t$ and $\psi|X_t$ are joined by $\chi_t(x,s) = h_t(\psi(x) \times \{s\})$, $x \in X_t$, $s \in I$, $t \ge t_1$.

(1) Recall that by i_t we denote the inclusion $X_t \to X$.

(ii) In view of (1.2)(ii), the filtration $\mathcal{E}$ is regular. In the above proof put $Y = E$ and $\mathcal{Y} = \mathcal{E}$ and let U be a convex symmetric neighbourhood of 0 in E such that $U \subset \overline{U}$. Other notation comes from the above proof. Thus, we have a family $\{\psi_t : X_t \to K(Y_t)\}_{t \geq t_1}$ such that $\psi_t(x) \subset U(\psi(x)) = U + \psi(x)$ and $\psi(x) \subset U + \psi_t(x)$ for $x \in X_t$, $t \geq t_1$. Define $\overline{\psi}_t(x) = \operatorname{conv} \psi_t(x)$ for $x \in X_t$, $t \geq t_1$. The map $\overline{\psi}_t$ is u.s.c., has compact convex values and satisfies the other conditions of the assertion. ■

We easily see that if ψ is, for instance, an acyclic map, then the maps ψ_t may not be acyclic in general. Hence, from the topological point of view, the above result is not sufficient.

The next result justifies the use of morphisms in Chapter II instead of just acyclic maps. It appears that morphisms (or rather maps determined by them) can be conveniently approximated by maps of this class. Precisely:

(3.2) Corollary. *Let X, Y, $\mathcal{X}$, $\mathcal{Y}$ and Z have the same meaning as in* (3.1). *If $\varphi \in M_n^A((X,\mathcal{X}),(Y,\mathcal{Y}))$, $\varphi(X) \subset Z$, is represented by a pair (p,q), where $p \in \mathcal{V}_n(W,X)$, $q : W \to Y$, then, for any $U \in \operatorname{unf}(Y)$, there is $t_1 \in T$ and a family of morphisms $\{\varphi_t\}_{t \geq t_1}$, where $\varphi_t \in M_n(X_t, Y_t)$, $t \geq t_1$, is represented by a pair (p_t, q_t), such that $p_t = p|p^{-1}(X_t)$, $q_t : p^{-1}(X_t) \to Y_t$, and the maps $i_t \circ q_t$ and $q|p^{-1}(X_t)$ are U-homotopic. Moreover*, *for $t \geq t_1$ and $w \in p^{-1}(X_t)$*, *if $q(w) \in Y_t$, then $q_t(w) = q(w)$.*

The following definition will be frequently used in the sequel. Let X be a topological space with a filtration $\mathcal{X} = \{X_t\}_{t \in T}$ and let Y be a uniform space with a filtration $\mathcal{Y} = \{Y_t\}_{t \in T}$. Assume that $U \in \operatorname{unf}(Y)$ and $t_1 \in T$. By a *U-approximation system* of an A-morphism $\varphi \in M_n^A((X,\mathcal{X}),(Y,\mathcal{Y}))$ we mean a family $\{\varphi_t\}_{t \geq t_1}$, where $\varphi_t \in M_n(X_t, Y_t)$ is represented by a pair (p_t, q_t) with $p_t \in \mathcal{V}_n(W_t, X_t)$, $q_t : W_t \to Y_t$, provided the following condition is satisfied:

• there is a pair (p,q), $p \in \mathcal{V}_n(W,X)$, $q : W \to Y$, representing φ and a family $\{f_t : W_t \to p^{-1}(X_t)\}_{t \geq t_1}$ of homeomorphisms such that, for any $t \geq t_1$, the diagram

$$\begin{tikzcd} X_t & & p^{-1}(X_t) \arrow[ll, "p"'] \\ & W_t \arrow[ul, "p_t"] \arrow[ur, "f_t"'] & \end{tikzcd}$$

is commutative and the maps $i_t \circ q_t$, $q \circ f_t$ are U-homotopic.

(3.3) Remark. (i) Under the assumptions of (3.1)(i), in view of (3.2), any A-morphism $\varphi \in M_n^A((X,\mathcal{X}),(Y,\mathcal{Y}))$ has a U-approximation system for any $U \in \operatorname{unf}(Y)$.

(ii) If $\{\varphi_t\}_{t \geq t_1}$ is a U-approximation system of φ, then $\varphi(x) \subset U(\varphi_t(x))$, $\varphi_t(x) \subset U(\varphi(x))$ for $t \geq t_1$ and $x \in X_t$.

(iii) If $\{\varphi_t\}_{t\geq t_1}$ is a U-approximation system of φ, $t_2 \geq t_1$ and $U \subset V \in \mathrm{unf}(Y)$, then $\{\varphi_t\}_{t\geq t_2}$ is a V-approximation system of φ.

(iv) Observe that the definition of an approximation system does not depend on the choice of representing pairs of the morphisms φ_t and φ.

The next result shows that all U-approximation systems, for sufficiently small U, are homotopically equivalent.

(3.4) THEOREM. *Let X, Y, $\mathcal{X}$, $\mathcal{Y}$, Z satisfy the assumptions of* (3.1)(i) *and let $\varphi \in M_n^A((X,\mathcal{X}),(Y,\mathcal{Y}))$, $\varphi(X) \subset Z$. For any $R \in \mathrm{unf}(Y)$, there is $V \in \mathrm{unf}(Y)$ such that if $\{\varphi_t\}_{t\geq t_1}$ and $\{\overline{\varphi}_t\}_{t\geq \bar{t}_1}$ are V-approximation systems of φ, then*

(i) *there is $t_2 \in T$, $t_2 \geq t_1$, such that the morphisms φ_t and $\overline{\varphi}_t$ are h-linked for $t \geq t_2$;*

(ii) *moreover, there exists a family $\{\chi_t\}_{t\geq t_2}$, where $\chi_t \in M_{n+1}(X_t \times I, Y_t)$, of homotopies joining φ_t to $\overline{\varphi}_t$ such that $\chi_t(x,s) \subset R(\varphi(x))$ for $x \in X_t$, $s \in I$ and $t \geq t_2$.*

Proof. Fix $R \in \mathrm{unf}(Y)$ and take a symmetric vicinity $U \in \mathrm{unf}(Y)$ such that $U \circ U \subset R$. In view of the regularity of φ over Z, there is a symmetric vicinity $V \in \mathrm{unf}(Y)$, $V \subset U$, and a family $\{\pi_t : V(Z) \cap V(Y_t) \to Y_t\}_{t\geq t_0}$ such that

(*) $\quad \pi_t \circ i_t = \mathrm{id}_{Y_t}$, $t \geq t_0$;

(**) $\quad$ the maps $i_t \circ \pi_t$ and $i : V(Z) \cap V(Y_t) \to Y$ are U-homotopic.

Let $\{\varphi_t\}_{t\geq t_1}$ and $\{\overline{\varphi}_t\}_{t\geq \bar{t}_1}$ be V-approximation systems of φ. Assume that pairs (p_t, q_t), $t \geq t_1$, $p_t \in \mathcal{V}_n(W_t, X_t)$, $q_t : W_t \to Y_t$ and $(\overline{p}_t, \overline{q}_t)$, $t \geq \bar{t}_1$, $\overline{p}_t \in \mathcal{V}_n(\overline{W}_t, \overline{X}_t)$, $\overline{q}_t : \overline{W}_t \to Y_t$, represent φ_t and $\overline{\varphi}_t$, respectively. By the definition of approximation system, there are pairs (p,q) and $(\overline{p},\overline{q})$, where $p \in \mathcal{V}_n(W,X)$, $q : W \to Y$ and $\overline{p} \in \mathcal{V}_n(\overline{W},X)$, $\overline{q} : \overline{W} \to Y$, representing φ and families of homeomorphisms $\{f_t : W_t \to p^{-1}(X_t)\}_{t\geq t_1}$, $\{\bar{f}_t : \overline{p}^{-1}(X_t) \to \overline{W}_t\}_{t\geq \bar{t}_1}$ such that the diagrams

$$\begin{tikzcd}
& p^{-1}(X_t) \arrow[dl, "p"'] & \overline{p}^{-1}(X_t) \arrow[dr, "\bar{p}"] \arrow[dd, "\bar{f}_t"'] & \\
X_t & & & X_t \\
& W_t \arrow[uu, "f_t"'] \arrow[ul, "p_t"] & \overline{W}_t \arrow[ur, "\bar{p}_t"'] &
\end{tikzcd}$$

are commutative and the maps $i_t \circ q_t$, $q \circ f_t$ and $i_t \circ \overline{q}_t$, $\overline{q} \circ \overline{f}_t^{-1}$ are V-homotopic for $t \geq t_2 \geq t_1, \bar{t}_1$.

Since the pairs (p,q), $(\overline{p},\overline{q})$ are equivalent, there is a homeomorphism $f : W \to \overline{W}$ such that the diagram

$$\begin{tikzcd}
& W \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "f"] & \\
X & & Y \\
& \overline{W} \arrow[ul, "\bar{p}"] \arrow[ur, "\bar{q}"'] &
\end{tikzcd}$$

is commutative.

Fix $t \geq t_2$ and consider the diagram

$$\begin{tikzcd} & W_t \arrow[dl, "p_t"'] \arrow[dr, "q_t"] \arrow[dd, "g_t"] & \\ X_t & & Y_t \\ & \overline{W}_t \arrow[ul, "\overline{p}_t"] \arrow[ur, "\overline{q}_t"'] & \end{tikzcd}$$

where $g_t = \overline{f}_t \circ f \circ f_t$. We shall show that this diagram is homotopy commutative. Clearly, $\overline{p}_t \circ g_t = p_t$. Therefore, we need to prove that the maps $\overline{q}_t \circ g_t$ and q_t are homotopic.

There exists a V-homotopy $h_t : W_t \times I \to Y$ joining $i_t \circ q_t$ to $q \circ f_t$ and a V-homotopy $\overline{h}_t : \overline{W}_t \times I \to Y$ joining $i_t \circ \overline{q}_t$ to $\overline{q} \circ \overline{f}^{-1}$. It is easy to verify that $h_t(w, 1) = \overline{h}_t(g_t(w), 1)$ for any $w \in W_t$, and moreover, $h_t(w, 0) = i_t \circ q_t(w)$, $\overline{h}_t(g_t(w), 0) = i_t \circ \overline{q}_t \circ g_t(w)$. On the other hand, we easily show that, for any $w \in W_t$, $s \in I$, $h_t(w, s)$, $\overline{h}_t(g_t(w), s) \in V(Z) \cap V(Y_t)$. Define a map $H_t : W_t \times I \to Y_t$ by

$$H_t(w, s) = \begin{cases} \pi_t \circ h_t(w, 2s) & \text{for } s \in \left[0, \frac{1}{2}\right], \\ \pi_t \circ \overline{h}_t(g_t(w), 2 - 2s) & \text{for } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

This map is well-defined and continuous. Moreover, $H_t(w, 0) = q_t(w)$, $H_t(w, 1) = \overline{q}_t \circ g_t$ for any $w \in W_t$.

To prove (ii), observe that, for $w \in W_t$, $s \in \left[0, \frac{1}{2}\right]$, $H_t(w, s) \in U(h_t(w, 2s))$ and, for $s \in \left[\frac{1}{2}, 1\right]$, $H_t(w, s) \in U(\overline{h}_t(g_t(w), 2 - 2s))$. Hence, for $w \in W_t$, $s \in I$, $H_t(w, s) \in U \circ V(q \circ f_t(w)) \subset R(q \circ f_t(w))$. Now, define a morphism $\chi_t \in M_{n+1}(X_t \times I, Y_t)$ by the pair $(p_t \times \mathrm{id}, H_t)$, $t \geq t_2$. Then $\chi_t \circ i_0 = \varphi_t$ and $\chi_t \circ i_1 = \overline{\varphi}_t$. Therefore χ_t joins φ_t to φ_t in M_{n+1} and, for any $x \in X_t$, $s \in I$, $\chi_t(x, s) \subset R(\varphi(x))$. ■

Theorems (3.2) and (3.4) are the principal approximation results for A-morphisms.

Now we prove a result which needs stronger assumptions but whose assertion seems to be also stronger and interesting.

(3.5) Theorem. *Let X be a paracompact space with a countable filtration $\mathcal{X} = \{X_n\}_{n=1}^{\infty}$ and let E be a locally convex space with a linear filtration $\mathcal{E} = \{E_n\}_{n=1}^{\infty}$. If $X_0 = \bigcup_{n=1}^{\infty} X_n$ and $\varphi \in M_m^A((X, \mathcal{X}), (E, \mathcal{E}))$, then, for any neighbourhood U of 0 in E, there exists a morphism $\overline{\varphi} \in M_m^F((X_0, \mathcal{X}), (E, \mathcal{E}))$ such that $\varphi(x) \subset \overline{\varphi}(x) + U$, $\overline{\varphi}(x) \subset \varphi(x) + U$ for each $x \in X_0$.*

Proof. Let V, $\overline{V}$ be open symmetric and convex neighbourhoods of 0 in E such that $V + V \subset \overline{V} \subset U$. Let a pair (p, q) represent φ, where $p \in \mathcal{V}_m(W, X)$, $q : W \to Y$. By (2.4), q is an A-map relative to filtrations $\mathcal{W} = \{W_n = p^{-1}(X_n)\}$ in W and $\mathcal{E}$. Let $w \in W_0 = \mathrm{cl}(\bigcup_{n=1}^{\infty} W_n) = \mathrm{cl}\, p^{-1}(X_0)$. Then $q(w) \in \mathrm{cl}(\bigcup_{n=1}^{\infty}(E_n + V))$. Hence $(q(w) + V) \cap \bigcup_{n=1}^{\infty}(E_n + V) \neq \emptyset$. Define an integer $n(w)$ by $n(w) = \inf\{n : (q(w) + V) \cap (E_n + V) \neq \emptyset\}$ and let $a(w)$ be an arbitrary point of the set $(q(w) + \overline{V}) \cap E_{n(w)}$. Put $V(w) = q^{-1}((q(w) + V) \cap (a(w) + \overline{V}))$. Then $w \in V(w)$;

hence $\{V(w)\}_{w\in W_0}$ is an open covering of W_0. Since W is paracompact, so is W_0. Take a locally finite partition of unity $\{f_w\}_{w\in W_0}$ subordinate to the covering $\{V(w)\}$. Define a map $\overline{q}: W_0 \to E$ by

$$\overline{q}(v) = \sum_{w\in W_0} f_w(v)a(w)$$

for $v \in W_0$. There is n_0 such that $q(W_n) \subset E_n + V$ for $n \geq n_0$. Fix $n \geq n_0$ and $v \in W_n$. If $f_w(v) \neq 0$, then $v \in V(w)$, so $q(v) \in (q(w) + V) \cap (a(w) + \overline{V})$ and $(q(w) + V) \cap (E_n + V) \neq \emptyset$. This implies that $n(w) \leq n$ and hence $a(w) \in E_{n(w)} \subset E_n$. Therefore, $\overline{q}(v) \in E_n$. Moreover, for $v \in W_0$, $\overline{q}(v) - q(v) = \sum_{w\in W_0} f_w(v)(a(w) - q(v)) \in \overline{V} \subset U$. In place of $\overline{\varphi}$ we can take a morphism represented by the pair $(\overline{p}, \overline{q})$ where $\overline{p} = p|\bigcup_{n=1}^{\infty} W_n$. This definition is correct since in view of I.(2.5)(ii), $\overline{p}$ is a $\mathcal{V}_m$-map. ■

To end this section we introduce the notion of A-homotopy. Let $\mathcal{F}$, $\mathcal{G}$ be two classes of maps and let $\mathcal{F} \subset \mathcal{G}$. If X, Y are topological spaces with filtrations $\mathcal{X}$, $\mathcal{Y}$, respectively, and $\psi_0, \psi_1 \in \mathcal{F}^A(X, Y)$, then we say that the maps ψ_0 and ψ_1 are *A-homotopic* in $\mathcal{G}$ provided there is a map $\psi \in \mathcal{G}^A((X \times I, \mathcal{X} \times \mathcal{I}), (Y, \mathcal{Y}))$ such that ψ is a homotopy joining $\psi_0, \psi_1 \in \mathcal{F}(X, Y)$ in $\mathcal{G}$ (see I.(1.8)).

Analogously we define the notion of F-homotopy, sA-homotopy in the class $\mathcal{G}$ (of maps from $\mathcal{F}$), and the notions of A- (resp. F-, sA-) homotopy of morphisms in M_n.

By the definition and I.(5.6), we have an immediate corollary:

(3.6) Proposition. *If X is a space with a filtration $\mathcal{X}$ and Y is a uniform space with a filtration $\mathcal{Y}$, then A-homotopy of morphisms from $M_1((X, \mathcal{X}), (Y, \mathcal{Y}))$ is an equivalence relation. Similarly, when $n > 1$ and X is a binormal space, then A-homotopy in M_{n+1} of morphisms from $M_n((X, \mathcal{X}), (Y, \mathcal{Y}))$ is an equivalence relation.*

The reader will easily prove the following statement which extends (3.4).

(3.7) Theorem. *Let X, Y, $\mathcal{X}$, $\mathcal{Y}$, Z satisfy the assumptions of* (3.1)(i) *and let morphisms $\varphi, \overline{\varphi} \in M^A((X, \mathcal{X}), (Y, \mathcal{Y}))$ be A-homotopic in M_n. For any vicinity $U \in \mathrm{unf}(Y)$, there exists a vicinity $V \in \mathrm{unf}(Y)$, $V \subset U$, such that, for V-approximations systems $\{\varphi_t\}_{t\geq t_1}$, $\{\overline{\varphi}_t\}_{t\geq \overline{t}_1}$ of φ and $\overline{\varphi}$, respectively, there is $t_2 \geq t_1, \overline{t}_1$ such that the morphisms φ_t and $\overline{\varphi}_t$, $t \geq t_2$, are homotopic in M_{n+1}.*

(3.8) Remark. One easily shows that already for $k \geq n$, $H^k(\varphi_t) = H^k(\overline{\varphi}_t)$, $t \geq t_2$.

Having the notion of homotopy we are in a position to define the analogues of essentiality and strong essentiality for A-maps. All results from I.6 extend to this case. We leave the details to the reader (comp. [65]).

IV. Approximation degree theory for A-morphisms

In 1934, the celebrated paper [78] of Leray and Schauder was published. Since then, and particularly in the past three decades, various degree theories which extend or generalize the Brouwer and Leray–Schauder theories have been defined for various classes of maps of infinite-dimensional spaces. In general, constructions of generalized degree theories are accomplished by certain approximation methods; in an approximation framework it is possible to extend the given degree theory from a class of (single- or set-valued) maps to a wider or more general one. An abstract setting of an approximation process was presented in the single-valued context by Browder in [14]. Browder's approach can easily be generalized to the set-valued case. However, here we stick to an intuitive understanding of approximation. It should be stressed that almost any degree theory may be put into an approximation framework — for continuous single-valued maps see [79], [88], for condensing maps [97], for A-proper maps [15], [82] or finally for DC-maps (this is another name for continuous single-valued A-maps) see [89], [69], [90]; for others see [70].

1. The degree of A-morphisms. First, let us describe a group the constructed degree will take values in. Let T be a directed set. For any $t \in T$, let $G_t = \mathbb{Z}$ and $\overline{G}_t = \prod_{s\ge t} G_s$, $\overline{G} = \bigcup_{t\in T} \overline{G}_t$. We define an equivalence relation in $\overline{G}$ by setting:

$$(g_s)_{s\ge t} \sim (g'_s)_{s\ge t'}$$

if and only if there is $t'' \in T$ such that $t'' \ge t, t'$ and, for $s \ge t''$, $g_s = g'_s$. Let $G = \overline{G}/{\sim}$; we call G the *asymptotic product* of $G_0 = \mathbb{Z}$. The canonical projection $\overline{G} \to G$ will be denoted by ν. Observe that the neutral element of G is given by $(g_s)_{s\in T}$ where $g_s = 0$ for all $s \in T$. Additionally, put $\mathbf{1} = (g_s)_{s\in T}$ where $g_s = 1$ for any $s \in T$. It is easy to see that if T is a countable set, then G is isomorphic to the group

$$\prod_{n=1}^{\infty} \mathbb{Z} \Big/ \bigoplus_{n=1}^{\infty} \mathbb{Z}.$$

Now, we state the general hypotheses allowing us to construct the degree for A-morphisms.

(1.1) Assumptions. (i) *Let X be a topological space with a filtration $\mathcal{X} = \{X_t\}_{t\in T}$ and Y a uniform space with a filtration $\mathcal{Y} = \{Y_t\}_{t\in T}$, such that, for a.a. $t \in T$, X_t and Y_t are oriented manifolds of the same dimension.*

(ii) *Let $\varphi \in M_n^A((X,\mathcal{X}),(Y,\mathcal{Y}))$, $n \ge 1$ and $n+1 < \sup_{t\in T} \dim X_t$.*

(iii) *Let the filtration $\mathcal{Y}$ be regular over a set $Z \subset Y$ such that $\varphi(X) \subset Z$.*

(iv) *Let L be a compact connected subset of Y such that $L \subset \operatorname{cl} \bigcup_{t\in T} Y_t$.*

(v) *For any vicinity $U \in \operatorname{unf}(Y)$, there is $R \in \operatorname{unf}(Y)$ such that $R(L) \cap Y_t$ is connected for a.a. $t \in T$.*

(vi) *There is a vicinity* $U_0 \in \mathrm{unf}(Y)$ *such that* $\mathrm{cl}\,\varphi_-^{-1}(U_0(L))$ *is bounded with respect to the filtration* $\mathcal{X}$.

Below, we discuss the above assumptions.

(1.2) Remark. (i) Let E be a topological vector space with a linear filtration $\mathcal{E} = \{E_t\}_{t\in T}$ and let $U \subset E$ be an open set such that $U \cap E_t \neq 0$ for a.a. $t \in T$. If we put $X = U$, $X_t = U \cap E_t$, $Y = E$ and $Y_t = E_t$, then assumption (1.1)(i) is satisfied.

(ii) If E is a metrizable locally convex space with a linear filtration $\mathcal{E} = \{E_t\}$ and Y is a convex closed subset of E such that $Y_t = Y \cap E_t \neq 0$ for a.a. $t \in T$, then the filtration $\{Y_t\}$ is regular (see III.(1.2)(ii)), so assumption (1.1)(ii) is always satisfied regardless of the choice of φ.

(iii) If $Y = E$ is a locally convex space with a linear filtration $\{Y_t\}$, $L \subset Y$ is compact connected and (1.1)(iv) holds, then (1.1)(v) is also satisfied.

(iv) Assume that U is an open convex and bounded subset of a normed space E with a linear filtration $\mathcal{E} = \{E_t\}_{t\in T}$. If $Y = \mathrm{bd}\,U$ and $Y_t = Y \cap E_t \neq 0$ for a.a. $t \in T$ (see also III.(1.1)(v)), then the filtration $\{Y_t\}$ is regular (see III.(1.2)(iii)). Moreover, if (1.1)(iv) holds, then (1.1)(v) is satisfied as well. In particular, if in this case we put $X = Y$, $X_t = Y_t$ and assume that U is an open ball in E, then (1.1)(i),(iii),(v) are satisfied provided (1.1)(iv) holds.

(v) Observe that if (1.1)(vi) is satisfied, $V \in \mathrm{unf}(Y)$ and $V \subset U_0$, then, clearly, the set $\mathrm{cl}\,\varphi_-^{-1}(V(L))$ is bounded with respect to $\mathcal{X}$. Now, we discuss assumption (1.1)(vi). Let E be a locally convex space with a linear filtration $\mathcal{E} = \{E_t\}_{t\in T}$ and $X \subset E$ a set open and bounded with respect to $\mathcal{E}$ (see III.(1.4)). Assume that $\mathcal{X} = \{X_t = X \cap E_t\}$ is a filtration in X, (1.1)(ii) holds and there exists a morphism $\overline{\varphi} \in M(\mathrm{cl}_E X, Y)$ such that $\overline{\varphi}|X = \varphi$. We claim that (1.1)(vi) is satisfied whenever $\mathrm{cl}\,\overline{\varphi}(\mathrm{bd}_E X) \cap L = \emptyset$. Indeed, there is a vicinity $U_0 \in \mathrm{unf}(Y)$ such that $\overline{\varphi}(\mathrm{bd}_E X) \cap \mathrm{cl}\, U_0(L) = \emptyset$. Hence the set $\mathrm{cl}\,\varphi_-^{-1}(U_0(L)) = \mathrm{cl}\,\overline{\varphi}_-^{-1}(U_0(L)) \subset \overline{\varphi}_-^{-1}(\mathrm{cl}(U_0(L))) \subset X$ is also bounded with respect to $\mathcal{X}$.

Suppose now that X is a uniform space locally bounded with respect to some filtration $\{X_t\}$ of X. If $K = \varphi_-^{-1}(L)$ is compact and there exists a closed neighbourhood W of K such that $\varphi|W$ determines a perfect map then (1.1)(vi) holds. In fact, there exists a vicinity $V \in \mathrm{unf}(X)$ such that $\mathrm{cl}\, V(K)$ is bounded with respect to $\mathcal{X}$, and $V(K) \subset W$. Since $\varphi|W$ is perfect, $(\varphi|W)^{-1}$ is u.s.c., so there is $U_0 \in \mathrm{unf}(Y)$ such that $\varphi_-^{-1}(U_0(L)) = \varphi^{-1}(U_0(L)) \subset V(K)$.

(vi) In particular, if $Y = E$ is an infinite-dimensional normed space (or merely a locally bounded locally convex space), X is an open subset of E, $\mathcal{E} = \{E_t\}_{t\in T}$ is a dense linear filtration in E, $\mathcal{X} = \{X \cap E_t\}$, $\varphi = i_X - \Phi$ where $i_X : X \to E$ is the inclusion and $\Phi \in M_n^K(X, Y)$ (n is arbitrary) and $\mathrm{Fix}(\Phi)$ is compact in X, then, for $L = \{0\}$, all assumptions (1.1) are satisfied.

Now, we present an approximation process leading to an approximation degree theory for A-morphisms.

(1.3) Let $R \in \operatorname{unf}(Y)$ be a symmetric vicinity such that $R \circ R \subset U_0$ and, for $t \geq t_0$, the sets $R(L) \cap Y_t$ are connected. Without any loss of generality we may assume that, for $t \geq t_0$, the set $\operatorname{cl} \varphi_-^{-1}(U_0(L)) \cap X_t$ is compact in X_t and $\dim X_t = \dim Y_t > n$.

(1.4) Since all assumptions of Theorem III.(3.4) are satisfied, there is a vicinity $V \in \operatorname{unf}(Y)$, $V \subset R$, such that the assertion of III.(3.4) holds. Let $\{\varphi_t\}_{t \geq t_1}$, where $t_1 \geq t_0$, be an arbitrary V-approximation system of φ.

(1.5) For any $t \geq t_1$, let L_t be an admissible (see II.2) subset of $R(L) \cap Y_t$. For $t \geq t_1$, we have $\varphi_{t-}^{-1}(L_t) \subset \operatorname{cl} \varphi_-^{-1}(U_0(L))$, hence $\varphi_{t-}^{-1}(L_t)$ is compact in X_t.

(1.6) According to II.3, for $t \geq t_1$, we may define an integer $d_t = \deg(\varphi_t, L_t)$. This number does not depend on the choice of L_t. Indeed, if $L_t' \subset R(L) \cap Y_t$, $t \geq t_1$, is another admissible set, then, in view of II.(2.3), there is an admissible set L_t'' such that $L_t, L_t' \subset L_t'' \subset R(L) \cap Y_t$, $t \geq t_1$. By II.(3.3)(iv), $\deg(\varphi_t, L_t) = \deg(\varphi_t, L_t') = \deg(\varphi_t, L_t'')$ for $t \geq t_1$.

(1.7) We have defined an element $d = (d_t)_{t \geq t_1} \in \overline{G}_{t_1}$ and consider $D = \nu(d) \in G$. We show that D does not depend on the choice of R, V, L_t and $\{\varphi_t\}_{t \geq t_1}$. Let $\overline{R}$, $\overline{V}$, $\overline{L}_t$, $\{\overline{\varphi}_t\}_{t \geq \overline{t}_1}$ be as in (1.3)–(1.5). Clearly, we may assume that $\overline{R} \subset R$, $\overline{V} \subset V$ and $\overline{t}_1 \leq t_1$. Then $\{\overline{\varphi}_t\}_{t \geq t_1}$ is a V-approximation system of φ. According to III.(3.4), the systems $\{\varphi_t\}_{t \geq t_1}$ and $\{\overline{\varphi}_t\}_{t \geq t_1}$ are equivalent, i.e. there is a family $\{\chi_t\}_{t \geq t_1}$ of homotopies joining φ_t to $\overline{\varphi}_t$ and $\chi_t(x, s) \subset R(\varphi(x))$ for $x \in X_t$, $s \in I$ and $t \geq t_1$. Therefore, since $\overline{L}_t \subset \overline{R}(L) \cap Y_t \subset R(L) \cap Y_t$, we deduce that $\chi_{t-}^{-1}(\overline{L}_t)$ is compact in $X_t \times I$. By II.(3.4), $\deg(\varphi_t, \overline{L}_t) = \deg(\overline{\varphi}_t, \overline{L}_t)$, $t \geq t_1$. This equality, together with (1.6), ends our argument.

(1.8) We introduce the following definition of the *degree* of the morphism φ over L:

$$\operatorname{Deg}_{\mathcal{X}\mathcal{Y}}(\varphi, L) = D.$$

In the sequel, we shall write $\operatorname{Deg}(\varphi, L)$ if the filtrations $\mathcal{X}$ and $\mathcal{Y}$ are fixed.

(1.9) Remark. In view of the above construction, if assumptions (1.1) are satisfied, then there exists a vicinity $V \in \operatorname{unf}(Y)$ such that

$$\operatorname{Deg}(\varphi, L) = \nu((\deg(\varphi_t, L_t))_{t \geq t_1}) \in G$$

where $\{\varphi_t\}_{t \geq t_1}$ is a V-approximation system of φ, L_t is an admissible subset of a nonempty connected set $V(L) \cap Y_t$ and $\operatorname{cl} \varphi_-^{-1}(V(L))$ is bounded with respect to $\mathcal{X}$, $t \geq t_1$. We say that such a vicinity is *admissible for the construction of the degree of an A-morphism φ over L.*

Clearly, for any vicinity $U \in \operatorname{unf}(Y)$, there exists an admissible vicinity $V \subset U$.

2. Properties of the degree of A-morphisms. Suppose that all assumptions (1.1) hold, and moreover, let a compact connected set $N \subset L$ satisfy the same assumption as L does (see (1.1)(v)).

(2.1) THEOREM. (i) *If* $\operatorname{Deg}(\varphi, L) \neq 0 \in G$, *then* $L \subset \operatorname{cl} \varphi(X)$.
(ii) $\operatorname{Deg}(\varphi, L) = \operatorname{Deg}(\varphi, N)$.

Proof. (i) Suppose to the contrary that there is $y \in L$ such that $y \notin \operatorname{cl} \varphi(X)$. There is a symmetric vicinity $V \subset \operatorname{unf}(Y)$ admissible for the construction of the degree over L, such that $\varphi(X) \cap V \circ V(y) = \emptyset$. Take any V-approximation system $\{\varphi_t\}_{t \geq t_1}$ of φ and let, for $t \geq t_1$, L_t be an admissible subset of $V(L) \cap Y_t$ and let $y_t \in V(y) \cap Y_t$. By II.(2.3), there is an admissible set $\overline{L}_t$ such that $\{y_t\} \cup L_t \subset \overline{L}_t \subset V(L) \cap Y_t$, $t \geq t_1$. Clearly, for $t \geq t_1$, $\deg(\varphi_t, \overline{L}_t) = 0$ since $y_t \notin \varphi_t(X_t)$ (see II.(3.3)(v)). Therefore, $\deg(\varphi_t, L_t) = 0$ and $\operatorname{Deg}(\varphi, L) = 0$, a contradiction.

(ii) follows easily from II.(3.3)(iv). ∎

In order to formulate the next property, let us introduce the following notational convention. Let $t_0 \in T$ and let J be an arbitrary set. We say that an indexed family $(d_t^j)_{t \geq t_0, j \in J}$ of integers *satisfies condition* (SUM) if

$$\text{(SUM)} \qquad \begin{aligned} &\text{(i) for any } j \in J,\ (d_t^j)_{t \geq t_0} \in \overline{G}_{t_0};\\ &\text{(ii) for any } t \geq t_0,\ d_t^j = 0 \text{ for all but a finite number of } j \in J. \end{aligned}$$

If the family (d_t^j) satisfies condition (SUM), then we put

$$\sum_{j \in J} \nu((d_t^j)_{t \geq t_0}) = \nu\Big(\Big(\sum_{j \in J} d_t^j\Big)_{t \geq t_0}\Big).$$

(2.2) PROPOSITION. *Assume that* $X = \bigcup_{j \in J} X^j$ *where* X^j, $j \in J$, *is an open subset of* X *such that* $\mathcal{X}^j = \{X_t^j = X^j \cap X_t\}_{t \in T}$ *is a filtration in* X^j. *If*, *for some vicinity* $U \in \operatorname{unf}(Y)$, *the sets* $\operatorname{cl} \varphi_-^{-1}(U(L)) \cap X^j$ *are pairwise disjoint*, *then*

$$\operatorname{Deg}(\varphi, L) = \sum_{j \in J} \operatorname{Deg}_{\mathcal{X}^j \mathcal{Y}}(\varphi | X^j, L).$$

Proof. It is easy to see that, for any $j \in J$, the degree $\operatorname{Deg}_{\mathcal{X}^j \mathcal{Y}}(\varphi | X^j, L)$ is defined. Let V be a sufficiently small vicinity admissible for the construction of the degree of $\varphi | X^j$ over L, for any $j \in J$. If $\{\varphi_t\}_{t \geq t_1}$ is a V-approximation system of φ, and $L_t \subset V(L) \cap Y_t$, $t \geq t_1$, is an admissible set, then the sets $\varphi_{t-}^{-1}(L_t) \cap X^j$ are disjoint, and since $\varphi_{t-}^{-1}(L_t)$ is compact for each $t \geq t_1$, $\varphi_{t-}^{-1}(L_t) \cap X^j = \emptyset$ for all but a finite number of $j \in J$. Hence the family $(\deg(\varphi_t | X_t^j, L_t))_{t \geq t_1, j \in J}$ satisfies condition (SUM). Thus $\sum_{j \in J} \operatorname{Deg}(\varphi | X^j, L) = \nu((\sum_{j \in J} \deg(\varphi_t | X_t^j, L_t))_{t \geq t_1}) = \mathcal{V}(\deg(\varphi_t, L_t)_{t \geq t_1}) = \operatorname{Deg}(\varphi, L)$ by II.(3.7). ∎

(2.3) Remark. Observe that if $T = J$ is the set of all positive integers, then the family $(\delta_t^j)_{t \in T, j \in J}$ (where $\delta_t^j = 1$ for $j = t$ and 0 for $j \neq t$) satisfies condition (SUM). We have $\nu((\sum_{j \in J} \delta_t^j)_{t \in T}) = \mathbf{1} \in G$ but, for all $j \in J$, $\mathcal{V}((\delta_t^j)_{t \in T}) = 0$. Therefore our definition of $\sum_{j \in J}$ may lead to paradoxes. In [95], an example is given of a single-valued A-map for which, in the situation of (2.2), the paradox described above occurs.

(2.4) Corollary. *Let $\overline{X}$ be an open subset of X such that $\overline{\mathcal{X}} = \{\overline{\mathcal{X}}_t = \overline{\mathcal{X}} \cap \mathcal{X}_t\}$ is the filtration of X. If* $\mathrm{cl}\,\varphi(X \setminus \overline{X}) \cap L = \emptyset$, *then* $\mathrm{Deg}(\varphi, L) = \mathrm{Deg}_{\overline{\mathcal{X}}\mathcal{Y}}(\varphi|\overline{X}, L)$.

Proof. For some vicinity $U \in \mathrm{unf}(Y)$, $\varphi(X \setminus \overline{X}) \cap \mathrm{cl}\,U(L) = \emptyset$. Hence $\mathrm{cl}_X \varphi_-^{-1}(U(L)) \subset \varphi_-^{-1}(\mathrm{cl}\,U(L)) \subset \overline{X}$. Therefore $\mathrm{cl}_{\overline{X}}\overline{\varphi}_-^{-1}(U(L))$ is bounded with respect to $\overline{\mathcal{X}}$, where $\overline{\varphi} = \varphi|\overline{X}$, provided U is sufficiently small, and hence $\mathrm{Deg}_{\overline{\mathcal{X}}\mathcal{Y}}(\overline{\varphi}, L)$ is defined. The assertion follows from II.(3.8). ■

Now, let $\chi \in M_n^A((X \times I, \mathcal{X} \times \mathcal{I}), (Y, \mathcal{Y}))$ be an A-homotopy joining $\varphi_0, \varphi_1 \in M_n^A((X, \mathcal{X}), (Y, \mathcal{Y}))$. If, for some $U_0 \in \mathrm{unf}(Y)$, the set $\mathrm{cl}\,\chi_-^{-1}(U_0(L))$ is bounded with respect to $\mathcal{X} \times \mathcal{I}$, then the sets $\mathrm{cl}(\chi \circ i_s)_-^{-1}(U_0(L))$ are bounded with respect to $\mathcal{X}$ for any $s \in I$ (as usual, $i_s : X \to X \circ I$, $i_s(x) = (x, s)$, $x \in X$, $s \in I$). Hence, for each $s \in I$, $\mathrm{Deg}(\chi \circ i_s, L)$ is defined.

(2.5) Proposition. *Under the above assumptions*,

$$\mathrm{Deg}(\varphi_0, L) = \mathrm{Deg}(\varphi_1, L).$$

Proof. As in (1.4), (1.5), we take $V \in \mathrm{unf}(Y)$ such that, for any V-approximation system $\{\chi_t\}_{t \geq t_1}$ of χ and a family $\{L_t \subset V(L) \cap Y_t\}_{t \geq t_1}$ of admissible sets, the sets $\chi_{t-}^{-1}(L_t)$ are compact in X_t, $t \geq t_1$. Then one can define $\deg(\chi_t \circ i_s^t, L_t)$ where $i_s^t : X_t \to X_t \times I$, $i_s^t(x) = (x, s)$ for $x \in X_t$, $s \in I$, for any $t \geq t_1$. To this end, it is sufficient to observe that, for each $s \in I$, $\{\chi_t \circ i_s^t\}_{t \geq t_1}$ is a V-approximation system of $\chi \circ i_s$ and recall II.(3.4). ■

Now, suppose that assumptions (1.1)(i)–(v) are satisfied.

(2.6) Proposition. *If*

(i) *X is locally bounded with respect to $\mathcal{X}$ and φ determines a perfect map*, *or*

(ii) *X is bounded with respect to $\mathcal{X}$*,

then $\mathrm{Deg}(\varphi, L)$ *is defined. If*, *moreover*, *for a.a. $t \in T$, Y_t is connected*, *then*, *for each compact connected L' which satisfies* (1.1)(iv)–(v), *we have* $\mathrm{Deg}(\varphi, L) = \mathrm{Deg}(\varphi, L')$.

Proof. The first assertion was already discussed in (1.2)(v). The second follows easily from the construction in Section 1 and II.(3.9). ■

(2.7) Remark. If, for a.a. $t \in T$, Y_t is a connected manifold and (2.6)(i) or (ii) holds, then we can define $\mathrm{Deg}(\varphi)$ by putting

$$\mathrm{Deg}(\varphi) = \nu(\deg(\varphi_t)_{t \geq t_1})$$

(comp. II.(3.10)(iii)), where $\{\varphi_t\}_{t \geq t_1}$ is a V-approximation system of X and V is a sufficiently small vicinity in $\mathrm{unf}(Y)$. Indeed, when (2.6)(i) holds, φ_t determines a perfect map provided V is sufficiently small.

(2.8) Example. If E is a normed space furnished with a linear filtration $\mathcal{E} = \{E_t\}_{t \in T}$ and $X = Y$ is the boundary of a convex bounded neighbourhood of 0 in E, and $\mathcal{X} = \mathcal{Y} = \{Y \cap E_t\}$, then, for each $\varphi \in M_n^A((X,\mathcal{X}),(Y,\mathcal{Y}))$ where $n+2 < \sup_{t\in T} \dim E_t$, assumptions (1.1)(i)–(v) (see (1.2)) and (2.6)(ii) are satisfied. Hence, we can consider $\mathrm{Deg}(\varphi)$.

3. Further properties of the degree. Applications. Assume that E is an infinite-dimensional normed space with a given dense linear filtration $\mathcal{E} = \{E_t\}_{t\in T}$. Of course, $\sup_{t\in T} \dim E_t = \infty$. Let X be an open subset of E and let $\Phi \in M_n^K(X,E)$, $n \geq 1$. If $\mathcal{X} = \{X_t\}$, where $X_t = X \cap E_t$, $t \in T$, then $\mathcal{X}$ is a filtration in X and, by III.(2.7), $\Phi \in M_n^A((X,\mathcal{X}),(E,\mathcal{E}))$. Consider the morphism $\varphi = i_X - \Phi$ (as usual, i_X denotes a morphism determining the inclusion $X \to E$). In view of I.(4.4) and III.(2.8), $\varphi \in M_n^A((X,\mathcal{X}),(E,\mathcal{E}))$. Take a compact connected set $L \subset E$ and suppose that $\varphi_-^{-1}(L)$ is compact in X (observe that if $L = \{0\}$, then $\varphi_-^{-1}(L) = \mathrm{Fix}(\Phi)$). There exists a closed neighbourhood of $\varphi_-^{-1}(L)$ on which φ determines a perfect map. Since X is locally bounded with respect to L, by (1.2), all assumptions (1.1) are satisfied and $\mathrm{Deg}_{\mathcal{X}\mathcal{E}}(\varphi, L)$ is defined. We shall prove that this degree stabilizes to the value of the Leray–Schauder degree (or the fixed-point index if $L = \{0\}$) — see [71], [16], [11].

(3.1) Proposition. *Under the above assumptions*,

$$\mathrm{Deg}(\varphi, L) = \nu((d_t)_{t\geq t_0})$$

where $d_t = d$ *for each* $t \geq t_0$. *If* $L = \{0\}$, *then* $d = \mathrm{Ind}(\Phi)$ (see [71]).

Proof. Let V be a neighbourhood of 0 in E admissible for the construction of the degree of φ over L. Then $\mathrm{Deg}(\varphi, L) = \nu((\deg(\varphi_t, L_t))_{t\geq t_0})$ where $\{\varphi_t\}_{t\geq t_0}$ is any V-approximation system of φ and $L_t \subset V(L) \cap E_t$ is any admissible set, $t \geq t_0$. We may assume that V is open and convex and such that $\mathrm{cl}\, \Phi(X) \subset V + E_{t_0}$ where $\dim E_{t_0} > n$. For any $t \geq t_0$, assume that $\Phi_t = p \circ \Phi$ where $p : V + E_{t_0} \to E_{t_0}$ is a Schauder projection. For $t \geq t_0$, we put $\varphi_t = i_t - \Phi_t$ where i_t is a 1-morphism determining the inclusion $X_t \to E_t$. It is easily seen that the family $\{\varphi_t\}_{t\geq t_0}$ constructed above is a V-approximation system of φ. Moreover, put $L_t = \overline{L}$ for any $t \geq t_0$, where $\overline{L} \subset V(L) \cap E_{t_0}$ is any admissible set. In this situation, for any $t \geq t_0$, $x \in X_t$ and $y \in \varphi_t(x)$, we have $x - y \in E_{t_0}$. Therefore, for $t \geq t_0$, $\deg(\varphi_t, L_t) = \deg(\varphi_{t_0}, L_{t_0})$ in view of II.(4.7). ∎

The above proposition shows that our degree, when applied to the so-called compact set-valued vector fields, is compatible with the usual Leray–Schauder degree.

Let E be a normed space with a linear filtration $\mathcal{E} = \{E_t\}_{t\geq T}$ and X an open subset bounded with respect to $\mathcal{E}$ and such that $\mathcal{X} = \{X_t = X \cap E_t\}$ is a filtration in X. If $D = \mathrm{cl}_E X$, then $\mathcal{D} = \{D_t = D \cap E_t\}$ is a filtration in D. Consider a compact connected set $L \subset \mathrm{cl} \bigcup_{t\in T} E_t$ and $\varphi \in M^A((D,\mathcal{D}),(E,\mathcal{E}))$ such that $\mathrm{cl}\, \varphi(C) \cap L = \emptyset$ where $C = \mathrm{bd}_E X$. Let $\varphi|X \in M_n^A((X,\mathcal{X}),(E,\mathcal{E}))$, $n \geq 1$, and $\sup_{t\in T} \dim E_t > n+1$. Then $\mathrm{Deg}(\varphi|X, L)$ is defined.

(3.2) PROPOSITION. *If D is a convex set, then for any morphism $\varphi' \in M_n^A((D,\mathcal{D}),(E,\mathcal{E}))$ such that $\varphi|C = \varphi'|C$, we have* $\mathrm{Deg}(\varphi'|X,L) = \mathrm{Deg}(\varphi|X,L)$.

Proof. This follows immediately from II.(4.2). Namely, it is easy to find (arbitrarily close) approximation systems $\{\varphi_t\},\{\varphi'_t\}$ of φ,φ', respectively, such that $\varphi_t|C\cap E_t = \varphi'_t|C\cap E_t$. ∎

(3.3) Remark. If D is bounded convex, $\varphi \in M_n^A((D,\mathcal{D}),(E,\mathcal{E}))$, $0 \notin \mathrm{cl}\,\varphi(C)$ and $r : E\setminus\{0\} \to S = \{x \in E : \|x\| = 1\}$ is the radial retraction, then $\psi = r\circ(\varphi|C) \in M_n^A((C,\mathcal{C}),(S,\mathcal{S}))$ where $\mathcal{C} = \{C\cap E_t\}$, $\mathcal{S} = \{S\cap E_t\}$ are filtrations in C and S, respectively. By (2.7), $\mathrm{Deg}(\psi)$ is defined. By II.(4.4), $\mathrm{Deg}(\psi) = \mathrm{Deg}(\varphi|X,0) = \mathrm{Deg}(\varphi|X,L)$ provided $L \subset \mathrm{cl}\bigcup_{t\in T} E_t$ is a compact connected set lying in the same component of $E\setminus \mathrm{cl}\,\varphi(C)$ as $\{0\}$ does.

Now, we show several applications of the presented degree theory (comp. II.6).

(3.4) THEOREM. *Under the above assumptions, if $x_0 \in X\cap \mathrm{cl}\bigcup_{t\in T} E_t$ and*

$$x_0 \notin \mathrm{cl\ conv}(x,\varphi(x))$$

for all $x \in C$, then $x_0 \in \mathrm{cl}\,\varphi(D)$.

Proof. Let a pair (p,q), $p : W \to X$, $q : W \to E$, represent the morphism $\psi = \varphi|X$. We define an A-homotopy $\chi \in M_{n+1}^A((X\times I,\mathcal{X}\times\mathcal{I}),(E,\mathcal{E}))$ by the pair (P,Q) where $P : W\times I \to X\times I$, $Q : W\times I \to E$ are given by $P(w,s) = (p(w),s)$, $Q(w,s) = (1-s)p(w) + sq(w)$ for $w \in W$, $s \in I$. In view of our assumption, $x_0 \notin \mathrm{cl}\,\chi(C\times I)$. Since $\mathrm{Deg}(\chi\circ i_0|X,x_0) = \mathbf{1} \in G$, $\mathrm{Deg}(\chi\circ i_1|X,x_0) = \mathrm{Deg}(\varphi|X,x_0) \neq 0$. ∎

(3.5) COROLLARY. *Under the same assumptions, suppose additionally that X is bounded and $x_0 = 0$. Suppose that either*

(i) *D is convex and, for all $x \in D$ and $y \in \varphi(x)$, we have $x - y \in D$, or*

(ii) *$\|y - x\| \le \|x\|$ for all $x \in C$ and $y \in \varphi(x)$, or*

(iii) *$\|y - x\| \le \|y\|$ for all $x \in C$ and $y \in \varphi(x)$, and $\sup\{\|y - x\| : x \in C, y \in \varphi(x)\} < \infty$, or*

(iv) *for some $r > 1$, $\|y - x\|^r \le \|y\|^r + \|x\|^r$ and $\sup\{\|y - x\| : x \in C, y \in \varphi(x)\} < \infty$.*

Then $0 \in \mathrm{cl}\,\varphi(D)$.

Proof. (i) Let p be the Minkowski gauge of the set D. Then, for any $x \in C$ and $y \in \varphi(x)$, $p(y - x) \le p(x) = 1$. Next, the reasoning goes as in (ii) below.

(ii) Let $m = \inf\{\|x\| : x \in C\}$, $M = \sup\{\|x\| : x \in C\}$. Suppose that $0 \notin \mathrm{cl}\,\varphi(D)$. There exists $\varepsilon_0 > 0$ such that $\varepsilon_0 < 2m$ and, for $x \in C$ and $y \in \varphi(x)$, $\|y\| > \varepsilon_0$. By (3.4), there are sequences (x_n) in C and (y_n) in E, $y_n \in \varphi(x_n)$, (s_n) in I, such that $\|s_n y_n + (1 - s_n)x_n\| \to 0$ as $n \to \infty$. Therefore, in case (ii), for some positive integer N and $n \geq N$, we have

$$\begin{aligned}\varepsilon_0 < \|y_n\| &\leq \|s_n y_n + (1-s_n)x_n\| + (1-s_n)\|y_n - x_n\| \\ &< \varepsilon_0/2 + (1-s_n)\|x_n\| \leq \varepsilon_0/2 + (1-s_n)M,\end{aligned}$$

and thus $s_n < 1 - \varepsilon_0/2M = \beta < 1$. Hence

$$\|x_n\| \leq \|s_n y_n + (1-s_n)x_n\| + s_n\|y_n - x_n\| \leq \|s_n y_n + (1-s_n)x_n\| + \beta\|x_n\|$$

and $\|s_n y_n + (1-s_n)x_n\| \geq (1-\beta)m$, a contradiction.

In case (iii) or (iv), put $\overline{M} = \sup\{\|y-x\| : x \in C,\ y \in \varphi(x)\}$. For $n \geq N$, $\|x_n\| - s_n\|y_n - x_n\| \leq \|s_n y_n + (1-s_n)x_n\| < \varepsilon_0/2$. Hence $s_n \geq \|y_n - x_n\|^{-1}(m - \varepsilon_0/2) > (m-\varepsilon_0/2)\overline{M}^{-1} = \alpha > 0$. Therefore, when (iii) holds,

$$\|y_n\| \leq \|s_n y_n + (1-s_n)x_n\| + (1-s_n)\|y_n - x_n\| \leq \|s_n y_n + (1-s_n)x_n\| + (1-\alpha)\|y_n\|,$$

so $\alpha\varepsilon_0 < \alpha\|y_n\| \leq \|s_n y_n + (1-s_n)x_n\|$, a contradiction.

Consider (iv). Put $a_n = \|y_n - x_n\|$, $b_n = \|x_n\|$ and $c_n = \|y_n\|$. Then $|b_n - s_n a_n| \to 0$ and $|c_n - (1-s_n)a_n| \to 0$. Since the function $z \mapsto z^r$ is uniformly continuous on bounded sets, $|b_n^r - s_n^r a_n^r| \to 0$ and $|c_n^r - (1-s_n)^r a_n^r| \to 0$. In view of the assumption (iv), $a_n^r \leq b_n^r + c_n^r$, hence

$$a_n^r((1-s_n^r) - (1-s_n)^r) \leq c_n^r - (1-s_n)^r a_n^r + b_n^r - s_n^r a_n^r \to 0.$$

Since $\inf\{((1-s^r)-(1-s)^r) : s \in [\alpha,\beta]\} > 0$, $a_n \to 0$. However, $a_n = \|y_n - x_n\| \geq (1-s_n)^{-1}\varepsilon_0/2 \geq (1-\alpha)^{-1}\varepsilon_0/2 > 0$, a contradiction. ■

(3.6) Corollary. *Let E, $\mathcal{E}$ be as above. If $\varphi \in M_n(E,E)$ is such that, for any closed bounded neighbourhood D of 0 in E, $\varphi|D \in M_n^A((D,\mathcal{D}),(E,\mathcal{E}))$ where $\mathcal{D} = \{D \cap E_t\}$, then either the set*

$$S_\delta(\varphi) = \{x \in E : \inf\{\|z\| : z \in \operatorname{conv}(x,\varphi(x))\} \leq \delta\}$$

is unbounded for any $\delta > 0$, or $0 \in \operatorname{cl}\varphi(E)$.

Proof. If $S_\delta(\varphi)$ is bounded for some $\delta > 0$, then $S_\delta(\varphi) \subset \{x \in E : \|x\| < R\}$ where $R > 0$. Putting $D = \{x : \|x\| \leq R\}$ and using (3.4), we get the assertion. ■

(3.7) Corollary. *Assume that E is furnished with a scalar product and D is a closed bounded neighbourhood of 0 in E.*

(i) *If $\varphi \in M_n^A((D,\mathcal{D}),(E,\mathcal{E}))$ and, for each $x \in C$ and $y \in \varphi(x)$,*

$$\operatorname{Re}(y|x) \geq 0,$$

then $0 \in \operatorname{cl}\varphi(D)$.

(ii) *If $\varphi \in M_n^A((E,\mathcal{E}),(E,\mathcal{E}))$ and*

$$\lim_{\|x\|\to\infty} \|x\|^{-1} \inf_{y\in\varphi(x)} \operatorname{Re}(y|x) = \infty,$$

then $\operatorname{cl}\varphi(E) = E$.

Proof. (i) Similarly to (3.5)(ii), assuming that $0 \notin \operatorname{cl}\varphi(D)$, we can find sequences (x_n) in C, (y_n) in E, $y_n \in \varphi(x_n)$ and (s_n) in $(0,\beta]$, where $0 < \beta < 1$, such that $\|s_n y_n + (1-s_n)x_n\| \to 0$. Hence $(1-\beta)m^2 \le (1-s_n)\|x_n\|^2 = \mathrm{Re}\,(s_n y_n + (1-s_n)x_n|x_n) - \mathrm{Re}\,(s_n y_n|x_n) \le \mathrm{Re}\,(s_n y_n + (1-s_n)x_n|x_n)$, a contradiction.

(ii) is a straightforward corollary of (i). ■

(3.8) Remark. (i) Observe that, for $r = 2$ in (3.5)(iv), (3.7c)(i) and (3.5)(i) are equivalent.

(ii) If in (3.5), (3.7)(i) we take a morphism $\varphi = i_D - \Phi$ where Φ is an arbitrary A-morphism, then we obtain sufficient conditions for the existence of approximate fixed points of Φ. These conditions correspond to the well-known criteria of Rothe ((3.5)(ii)), Altmann ((3.5)(iv)) and Krasnosel'skiĭ ((3.7)(i)) (comp. [26], [88]). Similarly, (3.6) corresponds to the so-called Leray–Schauder alternative (comp. [26]).

Now, we shall confine ourselves to the particular situation when $X = B = \{x \in E : \|x\| < 1\}$, $D = \{x \in E : \|x\| \le 1\}$ and $C = S = \{x \in E : \|x\| = 1\}$. Clearly, $\mathcal{B} = \{B \cap E_t\}$, $\mathcal{D} = \{D \cap E_t\}$ and $\mathcal{S} = \{S \cap E_t\}$ are filtrations in B, D and S, respectively. Obviously, $S_t = S \cap E_t \approx_{\mathrm{top}} S^{m_t}$ where $m_t = \dim E_t - 1$. For any morphism $\varphi \in M_n((S,\mathcal{S}),(S,\mathcal{S}))$, according to (2.7), $\mathrm{Deg}(\varphi)$ is defined.

Let us collect several properties of this degree.

(3.9) Proposition. (i) *If* $\mathrm{Deg}(\varphi) \ne 0 \in G$, *then* $\mathrm{cl}\bigcup_{t\in T} S_t \subset \mathrm{cl}\,\varphi(S)$.

(ii) *If* φ *determines a constant* n*-acyclic map*, *then* $\mathrm{Deg}(\varphi) = 0$. *If* $\sup_{t\in T} \dim E_t = \infty$ *and* φ *determines a constant map*, *then* $\mathrm{Deg}(\varphi) = 0$ *as well.*

(iii) *If* φ *determines an* n*-acyclic map and*, *for any* $x \in S$, $x \in \varphi(x)$, *then* $\mathrm{Deg}(\varphi) = \mathbf{1} \in G$.

(iv) *If there is* $\varepsilon > 0$ *such that*, *for any* $x \in S$, $x \notin N_\varepsilon(\varphi(x))$, *then* $\mathrm{Deg}(\varphi) = \nu((d_t)_{t\ge t_0})$ *where* $d_t = (-1)^{m_t+1}$.

(v) *If there is* $\varepsilon > 0$ *such that*, *for any* $x \in S$, $-x \notin N_\varepsilon(\varphi(x))$, *then* $\mathrm{Deg}(\varphi) = \mathbf{1} \in G$.

Proof. Only (iii) requires some comments. By III.(1.2)(iii), the filtration $\mathcal{S}$ is regular. For an approximation system that defines $\mathrm{Deg}(\varphi)$, we may take the system given in III.(3.2). Then, for $t \ge t_0$ and $x \in S_t$, $x \in \varphi_t(x)$. The assertion follows from II.(4.9)(i) (observe that here φ_t is no longer n-acyclic). However, we have the factorization

$$\begin{tikzcd}
 & p^{-1}(S_t) \arrow[dl, "p"'] \arrow[r, "q"] & S \arrow[dd, "\pi_t"] \\
S_t & & \\
 & S_t \arrow[ul, "\mathrm{id}"] \arrow[uu, "f"] \arrow[r, "\mathrm{id}"'] & S_t
\end{tikzcd}$$

for $t \ge t_0$, where $f(x) = (x,x)$ and p, q are projections. ■

(3.10) Corollary. (i) *The set* S *is not* n*-acyclically* A*-contractible*, *i.e. there exists no morphism* $\chi \in M_n^A((S \times I, \mathcal{S} \times \mathcal{I}), (S,\mathcal{S}))$ *such that* $\chi \circ i_j$, $j = 0, 1$, *determine an n-acyclic map and such that $x \in \chi \circ i_0(x)$ and $\chi \circ i_1$ is a constant map.*

(ii) *There exists no acyclic A-retraction of D onto S. Precisely, there is no map $\psi \in A_1^A((D,\mathcal{D}),(S,\mathcal{S}))$ such that $x \in \psi(x)$ for $x \in S$.*

Proof. (i) In view of (3.9)(iii), if such an A-homotopy existed, then $\operatorname{Deg}(\chi \circ i_0) = \mathbf{1}$, while $\operatorname{Deg}(\chi \circ i_1) = 0$ in view of (3.9)(ii). By homotopy invariance, we get a contradiction.

(ii) If such a retraction ψ existed, then the A-1-morphism $\chi = \varphi \circ h$, where φ determines ψ and $h : S \times I \to D$ is given by $h(x,s) = (1-s)x$, would be of the kind whose existence was excluded by (i). ∎

The above simple fact has a consequence in the theory of Banach spaces (see V.2).

As a straightforward corollary of the construction of $\operatorname{Deg}(\varphi)$ (comp. (2.7), (2.8)) and II.(5.2) we have

(3.11) Theorem. *Let $\alpha \in M_1^F((S,\mathcal{S}),(S,\mathcal{S}))$ be a fixed-point free involutive morphism and $\varphi \in M_1^A((S,\mathcal{S}),(S,\mathcal{S}))$.*

(i) *If there exists $\varepsilon > 0$ such that, for $x \in S$,*

$$N_\varepsilon(\varphi(x)) \cap N_\varepsilon(\varphi(\alpha(x))) = \emptyset,$$

then $\operatorname{Deg}(\varphi) = \nu((d_t)_{t \ge t_0})$ *where* $d_t = 1 \pmod 2$ *for each* $t \ge t_0$.

(ii) *If there exists $\varepsilon > 0$ such that, for $x \in S$,*

$$(-N_\varepsilon(\varphi(x))) \cap N_\varepsilon(\varphi(\alpha(x))) = \emptyset,$$

then $\operatorname{Deg}(\varphi) = \nu((d_t)_{t \ge t_0})$ *where* $d_t = 0 \pmod 2$ *for each* $t \ge t_0$.

Assume now, additionally, that the filtration $\mathcal{E}$ is dense.

(3.12) Theorem. *Let α be as above. If E^k is a vector subspace in E of codimension $k \ge 1$, $\mathcal{E}^k = \{E^k \cap E_t\}_{t \in T}$ and $\varphi \in M_1^A((S,\mathcal{S}),(E^k,\mathcal{E}^k))$, then, for any $\varepsilon > 0$,*

(i) *there is $x \in S$ such that $N_\varepsilon(\varphi(x)) \cap N_\varepsilon(\varphi(\alpha(x))) \ne \emptyset$, and*

(ii) $\operatorname{rd}_S A(\varphi) \ge k-1$ *where* $A_\varepsilon(\varphi) = \{x \in S : N_\varepsilon(\varphi(x)) \cap N_\varepsilon(\varphi(-x)) \ne \emptyset\}$.

Proof. Fix $\varepsilon > 0$ and let $\{\varphi_t\}_{t \ge t_0}$ be an ε-approximation system of φ. Since α determines a filtered map and the filtration is dense, we may assume that, for $t \ge t_0$, $\alpha(S_t) \subset S_t$ and $\dim(E^k \cap E_t) = \dim E_t - k$. Since $\varphi_t \in M_1(S_t, E^k \cap E_t)$, $t \ge t_0$, in view of II.(5.8), for some $x \in S_t$, $\varphi_t(x) \cap \varphi_t(\alpha(x)) \ne \emptyset$, which proves (i). In order to prove (ii), put $A(\varphi_t) = \{x \in S_t : \varphi_t(x) \cap \varphi_t(-x) \ne \emptyset\}$. Clearly, $A(\varphi_t)$ is closed in S and $A(\varphi_t) \subset A_\varepsilon(\varphi)$; moreover, by II.(5.13), $\dim A(\varphi_t) \ge \dim E_t - 1 - \dim(E^k \cap E_t) \ge k-1$. ∎

We also have the following partial generalization of (3.11).

(3.13) Proposition. *If a morphism* $\varphi \in M_n^A((D,\mathcal{D}),(E,\mathcal{E}))$ *is represented by a pair* (p,q) *(with* $p \in \mathcal{V}_n(W,D)$, $q : W \to E$*),* $h : W \to W$ *is a continuous involution such that, for* $w \in p^{-1}(S_t)$, $t \in T$, $p \circ h(w) \in S_t$, $0 \notin \operatorname{cl}\varphi(S)$ *and* $-q(w) = q(h(w))$ *for* $w \in p^{-1}(S)$, *then* $\operatorname{Deg}(\varphi|B, 0) \neq 0$.

Proof. Obviuosly, $\inf\{\|q(w)\| : w \in p^{-1}(S)\} > 0$. Let $r : E \setminus \{0\} \to S$ be the radial retraction. An A-morphism $\psi = r \circ (\varphi|S)$ is represented by the pair $(\overline{p}, Q)$ where $\overline{p} = p|p^{-1}(S)$, $Q = r \circ (q|p^{-1}(S))$. For any $w \in p^{-1}(S)$, $Q(h(w)) = -Q(w)$. Let $\{Q_t'\}_{t \geq t_0}$ be an η-approximation system for Q, $\eta < 1/2$, i.e. for $t \geq t_0$, $Q_t' : p^{-1}(S_t) \to S_t$ and $\|Q(w) - Q_t'(w)\| < \eta$ for $w \in p^{-1}(S_t)$. Define a map $Q_t : p^{-1}(S_t) \to S_t$, $t \geq t_0$, by $Q_t(w) = r(Q_t'(w) - Q_t'(h(w)))$ for $w \in p^{-1}(S_t)$. Then, for $t \geq t_0$ and $w \in p^{-1}(S_t)$, we have $\|Q_t(w) - Q(w)\| < 2\eta$ and $-Q_t(w) = Q_t(h(w))$. A family $\{\psi_t\}_{t \geq t_0}$, where the morphism ψ_t is represented by the pair $(p|p^{-1}(S_t), Q_t)$, is a 2η-approximation system of ψ. Therefore, if η is sufficiently small, $\operatorname{Deg}(\psi) = \nu((\deg(\psi_t)))$. ■

Below we give an example showing that the theory of A-maps may provide a useful tool for studying classical problems for compact maps.

(3.14) Proposition (The Birkhoff–Kellogg theorem, comp. [26], [62]). *Let* $\mathcal{E}$ *be a dense filtration. If* $\varphi \in M_n^K(S,E)$ *and* $0 \notin \operatorname{cl}\varphi(S)$, *then there exist* $\lambda > 0$ *and* $x \in S$ *such that* $\lambda x \in \varphi(x)$.

Proof. Clearly, $\operatorname{Deg}(\psi)$ is defined where $\psi = r \circ \varphi$ and $r : E \setminus \{0\} \to S$ is the radial retraction. Since $\psi(S)$ is a compact subset of S, there are $t_0 \in T$ and $y_0 \in E_{t_0}$ and $\delta > 0$ such that $N_\delta(y_0) \cap \psi(S) = \emptyset$. Let $\{\psi_t\}_{t \geq t_1}$, $t_1 \geq t_0$, be an ε-approximation system of ψ, with $0 < \varepsilon \leq \delta$, such that $\operatorname{Deg}(\psi) = \nu((\deg(\psi_t)))$. It is easy to see that, for any $t \geq t_1, y_0 \notin \psi_t(S_t)$, therefore $\deg(\psi_t) = 0$. Thus $\operatorname{Deg}(\psi) = 0$. On the other hand, $\operatorname{Deg}(\mathrm{id}_S) = \mathbf{1}$. Hence, for the morphism $\chi \in M_n^A(S \times I, E)$ given by $\chi = r \circ (f_1 \ \mathrm{id}_S + f_2\varphi)$, where $f_1(x,s) = (1-s)$, $f_2(x,s) = s$ for $x \in S$, $s \in I$, one must have $\operatorname{cl}\chi(S \times I) \ni 0$, which ends the proof. ■

V. Other classes of set-valued maps

In this chapter we present other approximation methods available for the construction of the degree theory of other classes of set-valued maps.

1. Single-valued approximations. In Chapter II we gave a description of the degree theory of morphisms (or maps determined by them) based on methods of algebraic topology, while in Chapter IV, the degree was built with the use of an appropriate approximation of the original set-valued map by finite-dimensional maps.

On the other hand, there is another useful technique allowing one to construct the degree; namely that of single-valued approximation.

Let X, Y be metric spaces, and $\psi : X \to K(Y)$ an upper-semicontinuous map. If $\varepsilon > 0$, then a continuous single-valued map $f : X \to Y$ is said to be an *ε-graph-approximation* of ψ (written $f \in a_\varepsilon(\psi)$) if

$$G_f \subset N_\varepsilon(G_\psi)$$

where the ε-neighbourhood of G_ψ is considered in the product $X \times Y$ furnished with the ordinary max-metric. By straightforward calculation we get

(1.1) PROPOSITION. (i) *$f \in a_\varepsilon(\psi)$ if and only if $f(x) \in N_\varepsilon(\psi(N_\varepsilon(x)))$ for each $x \in X$.*

(ii) *If $X, L \subset Y$ are compact, $U \subset X$ is open and $\psi_-^{-1}(L) \subset U$, then there is $\delta > 0$ such that, for any $f \in a_\delta(\psi)$, $f^{-1}(L) \subset U$.*

In order to provide examples of maps having sufficiently close continuous graph-approximations, we recall the following definition. By an *R_δ-set* we mean a metric space which may be represented as the intersection of a decreasing sequence of compact contractible metric spaces. Any R_δ-set is acyclic and, for example, any convex compact set in a metric vector space is an R_δ-set.

(1.2) THEOREM. *Let X, Y be absolute neighbourhood retracts, X be compact, and let $\psi : X \to K(Y)$ be a u.s.c. map such that, for any $x \in X$, $\psi(x)$ is an R_δ-set.*

(i) *For any $\varepsilon > 0$, $a_\varepsilon(\psi) \neq \emptyset$.*

(ii) *For any $\delta > 0$, there is $\varepsilon > 0$ such that any $f, g \in a_\varepsilon(\psi)$ are joined by a homotopy $h : X \times I \to Y$ such that $h(\cdot, t) \in a_\delta(\psi)$ for any $t \in I$.*

The above result was proved in [47] (comp. [46]). It enables us to build the degree theory of maps having R_δ-values.

Let X, Y be two oriented manifolds of the same dimension and let $\psi : X \to K(Y)$ be a u.s.c. map such that, for each $x \in X$, $\psi(x)$ is an R_δ-set. Let L be a connected compact subset of Y such that $K = \psi_-^{-1}(L)$ is compact. Obviously, there exists a compact absolute neighbourhood retract $A \subset X$ such that $K \subset \operatorname{int} A$. Combining (1.1)(ii) and (1.2), we get

(1.3) COROLLARY. *There is $\varepsilon > 0$ such that, for any $f, f' \in a_\varepsilon(\psi|A)$, there is a map $h : A \times I \to Y$ such that the set $\{x \in A : h(x,t) \in K$ for some $t \in I\}$ is compact in $\operatorname{int} A$, and $h(\cdot, 0) = f$, $h(\cdot, 1) = f'$.*

We define

$$d(\psi, L) = \deg(f|X', L)$$

where $X' = \operatorname{int} A$ and $f \in a_\varepsilon(\psi|A)$. Here, on the right-hand side we have the (ordinary) Brouwer degree of a map of manifolds. This definition is correct since it does not depend on the choice of f (in view of (1.3)) and on the choice of A in view of the excision property of the ordinary degree (comp. II.(3.8)). It is easily shown that the degree defined above satisfies all the standard properties of the topological degree.

Since, as has already been mentioned, any u.s.c. map with R_δ-values is 1-acyclic, a natural question arises whether $\deg(\psi, L) = d(\psi, L)$ provided L is an admissible subset of Y (see II.(3.6)). The answer is positive. To see this, in view of the above construction, it is enough to prove the following general theorem.

(1.4) THEOREM. *Let X, Y be oriented manifolds of the same dimension m and let $\psi : X \to K(Y)$ be an n-acyclic map ($n = 1$ if $m = 1$ and $n \le m-1$ for $m \ge 2$) such that, for any $\varepsilon > 0$, $a_\varepsilon(\psi) \ne \emptyset$. If L is an admissible subset of Y such that $K = \psi_-^{-1}(L)$ is compact, then $\deg(\psi, L) = \deg(f, L)$ where f is a sufficiently close graph-approximation of ψ.*

Proof. In view of (1.1)(ii), there are $\delta > 0$ and a compact $M \subset X$ such that $K \subset M$ and $f^{-1}(L) \subset M$ provided $f \in a_\delta(\psi)$. By II.(3.3)(iii),

$$\deg(\psi, L) = \langle H^m(\psi_{ML})(\omega^L), \mu_M\rangle$$

where μ, ω are orientations of X, Y, respectively. Put $\gamma = \omega^L$.

Let N be a positive integer such that $N^{-1} < \delta$. For any integer $i \ge N$, we define $U_i = N_{1/i}(G) \subset X \times Y$ where G is the graph of ψ. Moreover, let (p, q), with $p \in \mathcal{V}_n(G, X)$, $q : G \to Y$, be the generic pair representing ψ.

Clearly, $\bigcap_{i\ge N} U_i = G$ and $\bigcap_{i\ge N} U_i^M = p^{-1}(X \setminus M)$ where $U_i^M = U_i \cap ((X \setminus M) \times Y)$. For any $j \ge i \ge N$, let $\kappa_{ij} : H^m(U_i, U_i^M) \to H^m(U_j, U_j^M)$ be a homomorphism induced by the inclusion $U_j \to U_i$, and let $\kappa_i : H^m(U_i, U_i^M) \to H^m(G, G^M)$, where $G^M = p^{-1}(X \setminus M) = G \cap ((X \setminus M) \times Y)$, be the homomorphism induced by the inclusion $G \to U_i$. By the tautness of Alexander–Spanier cohomology (see [104]),

$$H^m(G, G^M) \simeq \varinjlim_{i\ge N} i \ge N H^m(U_i, U_i^M)$$

and this isomorphism is realized by the compatible system $\{\kappa_i : i \ge N\}$.

Let, for any $i \ge N$, $p_i : U_i \to X$, $q_i : U_i \to Y$ be the projections. Then $p_i : (U_i, U_i^M) \to (X, X \setminus M)$ and $q_i : (U_i, U_i^M) \to (Y, Y \setminus L)$. For any $j \ge N$, we have the following commutative diagram:

$$\begin{tikzcd}
 & H^m(G, G^M) & \\
H^m(X, X\setminus M) \arrow[ur, "p^*"] \arrow[dr, "p_j^*"'] & & H^m(Y, Y\setminus L) \arrow[ul, "q^*"'] \arrow[dl, "q_j^*"] \\
 & H^m(U_j, U_j^M) \arrow[uu, "\kappa_j"'] &
\end{tikzcd}$$

Moreover, for any $j \ge i \ge N$, $p_j^* = \kappa_{ij} \circ p_i^*$ and $q_j^* = \kappa_{ij} \circ q_i^*$. Fix $i \ge N$. Obviously, there exists $\alpha \in H^m(X, X \setminus M)$ such that $\kappa_{ij} \circ p_i^*(\alpha) = p^*(\alpha) = q^*(\gamma) = \kappa_i \circ q_i^*(\gamma)$. Thus there is $j \ge i$ such that $p_j^*(\alpha) = q_j^*(\gamma)$. Now let $f \in a_{1/j}(\psi)$. Hence $G' = G_f \subset U_j$. If $p' : G' \to X$, $q' : G' \to Y$ are projections and $\beta : H^m(U_j, U_j^M) \to H^m(G', p'^{-1}(X \setminus M))$ is a homomorphism induced by the inclusion $G' \to U_j$, then we have the following commutative diagram:

$$\begin{array}{ccccc} & \nearrow^{p_j^*} & H^m(U_j, U_j^M) & \nwarrow^{q_j^*} & \\ H^m(X, X\backslash M) & & \downarrow \beta & & H^m(Y, Y\backslash L) \\ & \searrow_{p'^*} & H^m(G', p'^{-1}(X\backslash M)) & \swarrow_{q'^*} & \end{array}$$

where p'^* is an isomorphism. Hence, after an easy computation,

$$p^{*-1} \circ q^*(\gamma) = \alpha = p'^{*-1} q'^*(\gamma).$$

Therefore $\deg(\psi, L) = \langle \alpha, \mu_M \rangle = \deg(f, L)$. ∎

(1.5) Remark. (i) It is rather clear that the above result does not hold if we replace the n-acyclic map ψ with a map determined by an n-morphism. For instance, it is easy to see that the constant map $\psi : S^1 \to K(S^1)$ with $\psi(x) = S^1$ for any $x \in S^1$ may be determined by a morphism φ such that $\deg(\varphi) = 1$; however, ψ has a selection $f : S^1 \to S^1$ with $\deg(f) = 2$ (selections are obviously ε-approximations for any $\varepsilon > 0$). On the other hand, applying the same reasoning as above, we can generalize (1.4) as follows. Let $\varphi \in M_n(X, Y)$ be a morphism represented by a pair (p, q) where $p \in \mathcal{V}_n(W, X)$, $q : W \to Y$. If, for any $\varepsilon > 0$, the map $\psi : X \to K(W)$ given by $\psi(x) = p^{-1}(x)$ has ε-approximations, then the assertion of (1.4) holds. In particular, if $\psi = \psi_k \circ \ldots \circ \psi_1$, ψ_j is an acyclic map, $j = 1, \ldots, k$, and for any $\varepsilon > 0$ and $j = 1, \ldots, k$, $a_\varepsilon(\psi_j) \neq \emptyset$, then the assertion of (1.4) holds as well.

(ii) In spite of the fact that the algebraic-topological approach can be applied to a far more general class of maps (it is known that the class of maps having sufficiently close approximations is only slightly more general than maps with R_δ-values — see [47]), it should be stressed that the approximation approach presented above is much simpler and sometimes gives better computational results.

Now, we shall consider a class of set-valued maps whose values satisfy a certain geometrical condition instead of topological ones. This class contains the class of convex-valued maps (see [68]).

Let X be a topological space and let E be a locally convex space. We say that a map $\psi : X \to P(E)$ is 0-*separating* if, for any $x \in X$, there is a continuous linear form $l_x \in E'$ such that either $l_x(y) > 0$ for $y \in \psi(x)$, or $0 \in \psi(x)$.

It is obvious that nothing can be said a priori about the topological structure of values of a 0-separating map. On the other hand, any closed convex-valued map is clearly 0-separating.

Let $\psi : X \to P(E)$ be a 0-separating map such that $0 \notin \psi(X)$. A continuous (single-valued) map $f : X \to E$ is a *homotopy approximation* of ψ if for any $x \in X$, there is a linear form $l_x \in E'$ such that $l_x(y) > 0$ for any $y \in \operatorname{conv}(f(x), \psi(x))$ (see II.(6.2)).

(1.6) THEOREM. *If X is paracompact and $\psi : X \to P(E)$ is a u.s.c. 0-separating map such that $0 \notin \psi(X)$, then there exists a homotopy approximation f of ψ such that $f(X) \subset \operatorname{conv} \psi(X)$.*

Proof. For any linear form $l \in E'$, let $U_l = \{x \in X : l(y) > 0 \text{ for } y \in \psi(x)\}$. Clearly, $\mathcal{U} = \{U_l\}_{l \in E'}$ is an open covering of X since ψ is u.s.c. Let $\mathcal{V} = \{V_j\}_{j \in J}$ be an open pointwise starshaped covering inscribed in $\mathcal{U}$ (see [32]) and let $\{f_j\}_{j \in J}$ be a locally finite partition of unity subordinate to $\mathcal{V}$. Choose arbitrary $y_j \in \psi(V_j)$, $j \in J$, and, for $x \in X$, define

$$f(x) = \sum_{j \in J} f_j(x) y_j.$$

Obviously, $f : X \to E$ is continuous and $f(X) \subset \operatorname{conv} \psi(X)$. We shall show that f is a homotopy approximation of ψ. Let $x \in X$ and let $\{j_1, \ldots, j_k\} = \{j \in J : f_j(x) \neq 0\}$. There is $l \in E'$ such that

$$\operatorname{St}(x, \mathcal{V}) = \bigcup \{V_j : j \subset J \text{ and } x \in V_j\} \subset U_l.$$

Thus, for $i = 1, \ldots, k$, $l(y_{j_i}) > 0$, and $l(y) > 0$ for $y \in \psi(x)$. Hence $l(z) > 0$ for $z \in \operatorname{conv}(f(x), \psi(x))$. ∎

Having the above result, we are in a position to define the topological degree for 0-separating maps. We shall provide a construction in a certain particular situation leaving a more general case to the reader. Let X be an open bounded subset of $\mathbb{R}^m$ and let $\psi : \operatorname{cl} X \to P(\mathbb{R}^m)$ be a u.s.c. 0-separating map such that $0 \notin \psi(\operatorname{bd} X)$. By (1.6), there is a homotopy approximation $f : \operatorname{bd} X \to \mathbb{R}^m$ of $\psi|\operatorname{bd} X$. In particular, $0 \notin f(\operatorname{bd} X)$. Define

$$\deg(\psi, 0) = \deg(f^*|X, 0)$$

where $f^* : \operatorname{cl} X \to \mathbb{R}^m$ is an arbitrary extension of f onto $\operatorname{cl} X$.

This definition is correct since it does not depend on the choice of f and f^*. Indeed, let $\widetilde{f}^*$ be another extension of f onto $\operatorname{cl} X$. Then the map $h : \operatorname{cl} X \times I \to \mathbb{R}^m$ given by $h(x,t) = (1-t) f^*(x) + t\widetilde{f}^*(x)$, $x \in \operatorname{cl} X$, $t \in I$, provides a homotopy from f^* to $\widetilde{f}^*$ and shows that $\deg(f^*|X, 0) = \deg(\widetilde{f}^*|X, 0)$ because $h(x,t) \neq 0$ for $x \in \operatorname{bd} X$, $t \in I$. Now, let $g : \operatorname{bd} X \to \mathbb{R}^m$ be another homotopy approximation of $\psi|\operatorname{bd} X$. According to the definition, the maps $\chi_1, \chi_2 : \operatorname{bd} X \to P(\mathbb{R}^m)$ given by $\chi_1(x,t) = (1-t) f(x) + t\psi(x)$, $\chi_2(x,t) = tg(x) + (1-t)\psi(x)$, $x \in \operatorname{bd} X$, $t \in I$, are 0-separating and $0 \notin \chi_i(\operatorname{bd} X \times I)$, $i = 1, 2$. Define $\chi : \operatorname{bd} X \times I \to P(\mathbb{R}^m)$ by

$$\chi(x,t) = \begin{cases} \chi_1(x, 2t) & \text{for } t \in \left[0, \frac{1}{2}\right], \\ \chi_2(x, 2t-1) & \text{for } t \in \left[\frac{1}{2}, 1\right], \end{cases}$$

for $x \in \operatorname{bd} X$. The map χ is 0-separating and $0 \notin \chi(\operatorname{bd} X \times I)$. In view of (1.6), there is $\overline{h} : \operatorname{bd} X \times I \to \mathbb{R}^m$ such that, for each $x \in \operatorname{bd} X$, $t \in I$, there is a linear form $l_{x,t}$ over $\mathbb{R}^m$ such that $l_{x,t}(z) > 0$ for $z \in \operatorname{conv}(\overline{h}(x,t), \chi(x,t))$. Define

$h : \operatorname{bd} X \times I \to \mathbb{R}^m$ by

$$h(x,t) = \begin{cases} 3t\overline{h}(x,0) + (1-3t)f(x) & \text{for } t \in \left[0, \frac{1}{3}\right], \\ \overline{h}(x, 3t-1) & \text{for } t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ (3-3t)\overline{h}(x,1) + (3t-2)g(x) & \text{for } t \in \left[\frac{2}{3}, 1\right], \end{cases}$$

for $x \in \operatorname{bd} X$. Then $h(\cdot,0) = f$, $h(\cdot,1) = g$ and $0 \notin h(\operatorname{bd} X \times I)$. Using the Borsuk homotopy extension theorem, we end the proof. ■

The defined degree of 0-separating maps has all the properties of the topological degree. As an illustration we show three of them.

(1.7) Proposition. *Under the above assumptions*:

(i) *If* $0 \notin \psi(X)$, *then* $\deg(\psi, 0) = 0$.

(ii) *If* $\chi : \operatorname{cl} X \times I \to P(\mathbb{R}^m)$ *is u.s.c. and* 0*-separating*, $0 \notin \chi(\operatorname{bd} X \times I)$, *then* $\deg(\chi_0, 0) = \deg(\chi_1, 0)$ *where* $\chi_i = (\cdot, i)$, $i = 0, 1$.

Proof. (i) In view of (1.6), there is a homotopy approximation f of ψ such that $0 \notin f(\operatorname{cl} X)$ and $\deg(\psi, 0) = \deg(f, 0)$.

(ii) Repeat the arguments proving the independence of the definition on the choice of homotopy approximation and its extensions. ■

The following is a generalization of the Borsuk antipodal theorem to 0-separating maps.

(1.8) Theorem. *Let* $X = N^m = \{x \in \mathbb{R}^m : |x| < 1\}$ *and let* $\psi : B^m \to P(\mathbb{R}^m)$ *be a u.s.c.* 0*-separating map such that*, *for any* $x \in S^{m-1}$, *there is a linear form* l_x *over* $\mathbb{R}^m$ *such that if* $y \in \psi(x)$, $\overline{y} \in \psi(-x)$, *then* $l_x(y) > 0$ *and* $l_x(\overline{y}) < 0$. *Then* $\deg(\psi, 0) = 1 \pmod 2$.

Proof. First of all, observe that the degree is defined since $0 \notin \psi(S^{m-1})$. Now, we modify a bit the construction provided in the proof of (1.6). For any linear form l over $\mathbb{R}^m$, let

$$U_l = \{x \in S^{m-1} : l(y) > 0,\ l(\overline{y}) < 0 \text{ for } y \in \psi(x),\ \overline{y} \in \psi(-x)\}.$$

The family $\mathcal{U} = \{U_l\}$ is an open covering of S^{m-1}. Moreover, for any form l, $U_l = -U_{-l}$. In particular, if $x \in U_l$, then $-x \in U_{-l}$. Let $\mathcal{V} = \{V_j\}_{j \in J}$ be an open pointwise starshaped covering inscribed in $\mathcal{U}$. Consider the open covering $\mathcal{W} = \{V_i \cap -V_j\}_{i,j \in J}$. It is easy to see that if $W = V_i \cap -V_j \in \mathcal{W}$, then $-W = V_j \cap -V_i \in \mathcal{W}$. Moreover, $\mathcal{W}$ is a pointwise starshaped covering inscribed in $\mathcal{U}$ since, for each $x \in S^{m-1}$, $\operatorname{St}(x, \mathcal{W}) \subset \operatorname{St}(x, \mathcal{V})$. Assume that elements of $\mathcal{W}$ are indexed by a set S and let $\{f_s\}_{s \in S}$ be a locally finite partition of unity subordinate to $\mathcal{W}$. We define a map $f : S^{m-1} \to \mathbb{R}^m$ by

$$f(x) = \sum_{s \in S} f_s(x) y_s \quad \text{for } x \in S^{m-1}$$

where $y_s \in \psi(W_s)$, $s \in S$, is an arbitrarily chosen point. Clearly, f is a homotopy approximation of ψ. Observe that, for any $x \in S^{m-1}$, $f(x) \neq f(-x)$. In fact, if

$\mathrm{St}(x, \mathcal{W}) \subset U_l$, then $l(f(x)) > 0$ but, as can easily be verified, $l(f(-x)) < 0$. Therefore, the classical Borsuk theorem shows the assertion. ■

In order to state the next result, we introduce the following notation. For any positive integer m, let L_m be the set of all linear forms over $\mathbb{R}^m$. If $\psi : S^n \to P(\mathbb{R}^m)$ is a 0-separating map and $l \in L_m$, then we put

$$U(l, \psi) = \{x \in S^n : l(\psi(x)) > 0,\ l(\psi(-x)) < 0\},$$
$$A(\psi) = S^n \setminus \bigcup_{l \in L_m} U(l, \psi).$$

(1.9) Remark. If $\psi : S^n \to P(\mathbb{R}^{n+1} \setminus \{0\})$ is a u.s.c. and 0-separating map, then we define

$$\deg(\psi) = \deg(r \circ f)$$

where $f : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ is a homotopy approximation of ψ and $r : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ is the radial retraction.

(1.10) Corollary. (i) *If $\psi : S^n \to P(\mathbb{R}^{n+1})$ is u.s.c. and 0-separating, $A(\psi) = \emptyset$, then, for any $y \in \mathbb{R}^{n+1}$, $\psi(S^n) \cap \{ty : t \geq 0\} \neq \emptyset$.*

(ii) *If $\psi : S^n \to P(\mathbb{R}^n)$ is u.s.c. and 0-separating, then $A(\psi) \neq \emptyset$.*

Proof. (i) If for some $y_0 \in \mathbb{R}^{n+1}$, $\psi(S^n) \cap \{ty_0 : t \geq 0\} = \emptyset$, then $\deg(\psi) = 0$, a contradiction.

(ii) follows directly from (i). ■

Let $\psi : S^n \to P(\mathbb{R}^m)$ be a u.s.c. and 0-separating map. The set $A(\psi)$ is clearly compact and symmetric. By its *genus* we understand (as usual) the number

$$\gamma(A(\psi)) = \min\{r : \exists f : A(\psi) \to S^r \text{ odd continuous}\}.$$

It is well known that $\dim A(\psi) \geq \gamma(A(\psi))$ (see e.g. [40]). We have the following extension of (1.10)(ii).

(1.11) Theorem. *If $m \leq n$, then, under the above assumptions, $\gamma(A(\psi)) \geq n - m$.*

Proof. In view of (1.10), we may assume whithout any loss of generality that $n - m \geq 1$. Now, suppose that $\gamma(A(\psi)) = k < n - m$. There is a continuous odd map $f : A(\psi) \to S^k$; let $f' : S^n \to \mathbb{R}^{k+1}$ be an extension of f onto S^n. Consider a map $\psi' : S^n \to P(\mathbb{R}^n)$ given by $\psi'(x) = \psi(x) \times \{f'(x)\} \subset \mathbb{R}^m \times \mathbb{R}^{k+1} \subset \mathbb{R}^n$. This map is u.s.c. and 0-separating, so, by (1.10)(ii), $A(\psi') \neq \emptyset$; hence there is $x' \notin \bigcup_{l \in L_n} U(l, \psi')$. We claim that $x' \in A(\psi)$. Suppose that $x' \notin A(\psi)$; there exists a form $l \in L_m$ such that $x' \in U(l, \psi)$. Take a form $l' \in L_n$ such that $l' | \mathbb{R}^m = l$ and $\mathbb{R}^{n-m} \subset \ker l'$. Then $l'(\psi'(x')) = l(\psi(x')) > 0$ and $l'(\psi'(-x')) < 0$; hence $x' \in U(l', \psi')$, a contradiction. Since $x' \in A(\psi)$, it follows that $-f'(x') = -f(x') = f(-x') = f'(-x')$. Thus there must exist a linear form $l'' \in L_{n-m}$ with an extension $\widetilde{l''}$ such that $x' \in U(\widetilde{l''}, \psi')$, a contradiction. ■

As before, we now study the compatibility of the above-defined degree with the one introduced in Chapter II. Assume that a 0-separating map $\psi : \mathrm{cl}\, X \to K(\mathbb{R}^m)$, where X is an open bounded subset of $\mathbb{R}^m$, is determined by an n-morphism $\varphi \in M_n(\mathrm{cl}\, X, \mathbb{R}^m)$ ($n = 1$ if $m = 1$ and $n \le m-1$ if $m \ge 2$), and that $0 \notin \psi(\mathrm{bd}\, X)$. According to II.3, $\deg(\varphi|X, 0)$ is defined.

(1.12) PROPOSITION. *Under the above assumptions*,

$$\deg(\varphi|X, 0) = \deg(\psi, 0).$$

Proof. Let a pair (p, q), where $p \in \mathcal{V}_n(W, \mathrm{cl}\, X)$, $q : W \to \mathbb{R}^m$, represent the morphism φ and let $f : \mathrm{bd}\, X \to \mathbb{R}^m$ be a homotopy approximation of ψ, and $f^* : \mathrm{cl}\, X \to \mathbb{R}^m$ its extension onto $\mathrm{cl}\, X$. Define $Q : W \to \mathbb{R}^m$ by $Q(w) = f^*(p(w))$ for $w \in W$. By Φ we denote the n-morphism represented by the pair (p, Q). Let $L = \{0\}$ and $N = \{w \in W : (1-t)Q(w) + tq(w) = 0 \text{ for some } t \in I\}$. If $M = p(N)$, then $M \cap \mathrm{bd}\, X = \emptyset$. Therefore the morphisms Φ_{ML} and ψ_{ML} are h-linked. This, together with II.(3.5), ends the proof. ∎

We end this section by showing how to generalize the notion of 0-separating map and its degree to the infinite-dimensional setting.

Assume now that E is a metrizable and complete locally convex space, and let $\psi : \mathrm{cl}\, X \to C(E)$, where X is an open subset of E, be u.s.c. compact and such that the field $\Psi = i - \psi$ is 0-separating. Let $\mathrm{Fix}(\psi) \cap \mathrm{bd}\, X = \emptyset$.

(1.13) PROPOSITION. *For any $x \in \mathrm{bd}\, X$, there are a linear form $l_x \in E'$ and a neighbourhood V_x of x such that, for $z \in V_x$ and $y \in \psi(V_x \cap \mathrm{cl}\, X)$, $l_x(z-y) > 0$.*

Proof. Let $x \in \mathrm{bd}\, X$. Since $x \notin \psi(x)$, there is a form $l_x \in E'$ such that $l_x(x-y) > 0$ for $y \in \psi(x)$. Suppose that, for any positive integer n, there are $z_n \in V_n = \{z \in E : d(z, x) < n^{-1}\}$ and $y_n \in \psi(V_n \cap \mathrm{cl}\, X)$ such that $l_x(z_n - y_n) \le 0$. Of course, $x_n \to x$ as $n \to \infty$, and $y_n \in \psi(x_n)$ where $x_n \in V_n \cap \mathrm{cl}\, X$. By the compactness of ψ, we may assume that $y_n \to y \in E$. Since the graph of ψ is closed, $y \in \psi(x)$. The continuity of l_x yields a contradiction. ∎

(1.14) THEOREM. *Under the above assumptions, there exists a compact continuous map $f : \mathrm{bd}\, X \to E$ such that the map $F = i - f$ is a homotopy approximation of Ψ. In particular*, $\mathrm{Fix}(f) = \emptyset$.

Proof. For any $y \in \mathrm{bd}\, X$, there is a pair (V_y, l_y) where V_y is a neighbourhood of y and $l_y \in E'$ is such that $l_y(z-z') > 0$ for $z \in V_y$, $z' \in \psi(V_y \cap \mathrm{cl}\, X)$. The family $\mathcal{V} = \{V_y\}$ is an open covering of $\mathrm{bd}\, X$. Consider an open covering $\mathcal{W} = \{W_j\}_{j \in J}$ which is pointwise starshaped and inscribed in $\mathcal{V}$ and a locally finite partition of unity $\{f_j\}_{j \in J}$ subordinate to $\mathcal{W}$. Choose $z_j \in \psi(W_j)$ and define $f : \mathrm{bd}\, X \to E$ by

$$f(x) = \sum_{j \in J} f_j(x) z_j$$

for $x \in \mathrm{bd}\, X$. The map f is continuous and compact since $f(\mathrm{bd}\, X) \subset \mathrm{conv}\, \psi(\mathrm{cl}\, X)$. From the definition we see that $l_y(x - f(x)) > 0$ and $l_y(y - z) > 0$ where $z \in$

$\psi(x)$ and l_y is a form associated with a neighbourhood V_y such that $\mathrm{St}(x,\mathcal{W}) \subset V_y$. ∎

Now, we can easily build the degree $\deg(\Psi, 0)$. Precisely, modifying the proof of [26, (2.5)] by use of the Dugundji Extension Formula instead of Lemma (2.4) from [26], we obtain a compact extension $f^* : \mathrm{cl}\, X \to E$ of f provided by (1.14). Then it is enough to put $\deg(\Psi, 0) = \deg(i - f^*, 0)$. The correctness of this definition can be verified in exactly the same manner as before. It is rather easily seen that the degree defined has all the good properties. In particular, one can formulate a result concerning the oddness of the degree of odd fields of spheres and its consequences.

Let us remark that the procedure described above for compact infinite-dimensional maps extends quite easily to A-set-valued maps. We leave the details to the reader. We only remark that, in order to adapt the reasoning for 0-separating A-maps, one should assume a bit more about the separation properties of the map in question. Namely, we should consider maps which are strongly 0-separating; a map $\psi : X \to P(E)$, where X is a topological space and E is a locally convex space, is *strongly* 0-*separating* if, for each neighbourhood U of 0 in E, there is a neighbourhood $V \subset U$ such that, for any $x \in X$, either

(i) $\psi(X) \cap V = \emptyset$, or

(ii) there is a linear form $l_x \in E'$ such that

$$l_x(y) \geq \sup_{v \in V} l_x(v) \quad \text{for } y \in \psi(x).$$

One can observe at once that any map $\psi : X \to P(E)$ with convex values is strongly 0-separating.

The only remaining difficulty is to prove that when an A-map is strongly 0-separating, then it has homotopy approximations (which are single-valued A-maps). The necessary extension result for (single-valued) A-maps can be found in [65].

2. Linear filtrations. AP-maps of Petryshyn. In the last section of this paper, we present several results concerning linear filtrations, projection schemes, and we briefly discuss the class of approximation proper (AP) maps which, in the single-valued case, were introduced by W. V. Petryshyn and, in the convex-valued case, were studied by P. S. Milojević [85], [86] (comp. [82]).

As we have seen before, linear filtrations are particularly important in the approximation approach to degree theory.

Let E be a linear metric space and let $\mathcal{E} = \{E_n\}_{n=1}^{\infty}$ be a linear filtration of E. We say that the filtration *approximates rapidly* (comp. [1]) if there is a translation-invariant metric d in E such that

$$\lim_{n\to\infty} \sup_{x\in E} d(x, E_n) = 0.$$

(2.1) EXAMPLE. Let E be the space of all real sequences. Obviously, E is metrizable, locally convex and complete when its topology is generated by the family of seminorms $p_n : (x_k)_{k=1}^\infty \mapsto |x_n|$. Let $E_n = \{x = (x_k) : x_k = 0 \text{ for } k > n\}$. The filtration $\mathcal{E} = \{E_n\}$ is linear, dense and approximates rapidly. Indeed, for the metric d given by

$$d(x,y) = \sup_n 2^{-n}(1 + p_n(x-y))^{-1} p_n(x-y)$$

where $x, y \in E$, we have $d(x, E_n) < 2^{-n-1}$ for any $x \in E$. Observe that d is uniformly equivalent to the well-known metric

$$d'(x,y) = \sum_{n=1}^{\infty} 2^{-n}(1 + p_n(x-y))^{-1} p_n(x-y).$$

It appears that the existence of rapidly approximating filtrations is a very rare phenomenon. In fact, in [1] it was shown that the space E (from (2.1)) is the only, up to isomorphism, infinite-dimensional metrizable space which admits rapidly approximating filtrations.

As is easy to see, if a filtration $\mathcal{E}$ approximates rapidly, then any continuous map is an A-map. This fact has an interesting consequence.

(2.2) THEOREM. *If E is an infinite-dimensional normed space, then E does not admit rapidly approximating filtrations.*

P r o o f. The proof is very easy. If such a filtration existed, then no retraction of the unit ball would exist (see IV.(3.10)). On the other hand, it is well known that there are maps of the ball into itself without fixed points. Hence, such a retraction exists. ∎

The notion of a projection scheme is closely related to that of a linear filtration. Let $\mathcal{E} = \{E_n\}_{n=1}^\infty$ be a linear filtration in a normed space E. Assume that we are given a family $\mathcal{P} = \{P_n : E \to E_n\}_{n=1}^\infty$ of continuous linear projections such that

$$(*) \qquad P_n x \to x \quad \text{as } n \to \infty, \qquad \text{for any } x \in E.$$

Such a pair $(\mathcal{E}, \mathcal{P})$ will be called an (*internal*) *projection scheme* in E. Obviously, if $(\mathcal{E}, \mathcal{P})$ is a projection scheme, then the filtration $\mathcal{E}$ is dense and regular.

Observe that, if the family $\mathcal{P}$ satisfies the condition

$$(\pi_\alpha) \qquad \sup_n \|P_n\| \le \alpha \quad \text{and} \quad \mathcal{E} \text{ is dense},$$

then $(*)$ holds. In this case, we have, for each $x \in E$, $\|x - P_n x\| \le (1+\alpha) d(x, E_n)$. Conversely, if E is a Banach space, then, by the Banach–Steinhaus theorem, we see that $(*)$ implies (π_α) for some α.

The problem of the existence of projection schemes is rather complicated (see [59], [96]). We shall restrict ourselves to one case which is a source of many examples.

Assume that E has a Schauder basis $\{e_n\}_{n=1}^{\infty}$. Then the formulae

$$E_n = \operatorname{span}(e_1, \ldots, e_n),$$
$$P_n(x) = \sum_{i=1}^{n} (e_i', x)e_i, \quad x \in E,$$

where the linear forms $e_i' \in E'$ are given by $(e_i', e_j) = \delta_{ij}$, determine a projection scheme in E for which condition (π_α) is satisfied for some α and $P_n \circ P_m = P_{\min(m,n)}$. Conversely, if $(\{E_n\}, \{P_n\})$ is a projection scheme in a normed space E and $P_n \circ P_m = P_{\min(m,n)}$, then E admits a Schauder basis which determines the given scheme in the way described above.

Similar result can be formulated for more general spaces (i.e. complete metric — see [87], or ultrabarrelled — [96]), that is, it is possible to construct and consider projection schemes in those spaces. As a general reference concerning projection schemes we recommend the survey by Petryshyn [92].

AP-maps (originally A-proper maps) appeared after many modifications of previous definitions in papers by Petryshyn — comp. [92] (and the rich bibliography therein).

(2.3) Assume that X is a normed space with a filtration $\{X_n\}_{n=1}^{\infty}$ and $D \subset X$ such that $\mathcal{D} = \{D_n = D \cap X_n\}_{n=1}^{\infty}$ is a filtration in D. Let a normed space E be furnished with a projection scheme $(\mathcal{E}, \mathcal{P})$ and let $\mathcal{F}$ be a class of maps. We say that a map $\psi : D \to P(E)$ from the class $\mathcal{F}$ is an *AP-map* if the following condition is satisfied:

(AP) for any increasing sequence (n_k) of positive integers and a bounded sequence (x_{n_k}) such that $x_{n_k} \in D_{n_k}$, if

$$P_{n_k} y_{n_k} \to y \in E \quad \text{where } y_{n_k} \in \psi(x_{n_k}),$$

then there exists a subsequence $(x_{n_{k(j)}})$ such that $x_{n_{k(j)}} \to x \in D$ and $y \in \psi(x)$.

AP-morphisms are defined similarly. Precisely, if $\varphi \in M_m(D, E)$, we say that φ is an *AP-m-morphism* if φ determines a map which is an AP-map.

Observe that the very definition of AP-map is not convenient in the sense that, in order to verify whether a given map is AP using the definition, one is led to the solvability of the inclusion $y \in \psi(x)$.

Since, in what follows, we shall be primarily concerned with the degree theory for AP-maps, we shall deal with AP-morphisms. The approach to different types of AP-maps is similar.

Now, we give several examples of AP-morphisms. Assume that $X = E$, $X_n = E_n$ for a.a. n and that the set D is closed.

(2.4) Example. Assume additionally that a projection scheme $(\mathcal{E}, \mathcal{P})$ satisfies condition (π_1). If a morphism $\varphi \in M_m(D, E)$ determines a β-condensing map, where β is the Hausdorff measure of noncompactness, then $i - \varphi$ is an

AP-morphism (i, as usual, denotes the inclusion $D \to E$). This kind of morphisms were already studied in [48]. Recall that, for a bounded set $A \subset E$, $\beta(A) = \inf\{r > 0 : \text{in } E \text{ there is a finite } r\text{-net of } A\}$ and a map φ is β-condensing if $\beta(\varphi(A)) < \beta(A)$ for any set A such that $\operatorname{cl} A$ is not compact.

To prove that $i - \varphi$ is an AP-morphism, take a bounded sequence (x_{n_k}) such that $x_{n_k} \in D_{n_k}$. Let $y_{n_k} = x_{n_k} - z_{n_k}$ where $z_{n_k} \in \varphi(x_{n_k})$ and suppose that $P_{n_k} y_{n_k} = x_{n_k} - P_{n_k} z_{n_k} \to y \in E$. Using the properties of β (see e.g. [97]), we have

$$\beta(\{x_{n_k}\}) \le \beta(\{P_{n_k} z_{n_k}\}) \le \beta(\{z_{n_k}\})$$

because, for any k, $\|P_{n_k}\| \le 1$ (here it is important that we consider the Hausdorff measure of noncompactness, and not the Kuratowski measure). Suppose that $\{x_{n_k}\}$ has no limit points; then $\beta(\{z_{n_k}\}) < \beta(\{x_{n_k}\})$, a contradiction.

Therefore $x_{n_{k(j)}} \to x \in D$ for some subsequence. Since φ determines a u.s.c. map with compact values, $(z_{n_{k(j)}})$ has a subsequence (denoted by the same symbol) such that $z_{n_{k(j)}} \to z \in \varphi(x)$. Hence $y = x - z \in x - \varphi(x)$.

(2.5) Example. If a morphism $\varphi \in M_m^K(D, E)$, then φ is an AP-morphism. The proof of this fact is obvious.

(2.6) Example. If $\varphi_1 \in M_1(E, E)$ determines a map which is a k-contraction ($k < 1$), i.e. for any $x, y \in E$, $\varrho(\varphi(x), \varphi(y)) \le k\|x - y\|$, where ϱ is the Hausdorff distance, and $\varphi_2 \in M_m^K(D, E)$, then the morphism $\Phi = \varphi_1 | D + \varphi_2 \in M_m(D, E)$ determines a k-β-contraction, i.e. for any bounded $A \subset D$, $\beta(\Phi(A)) \le k\beta(A)$. Hence, if a projection scheme $(\mathcal{E}, \mathcal{P})$ satisfies condition (π_α) with $\alpha \le k^{-1}$, then the morphism $i - \Phi$ is an AP-morphism. If we assume φ_1 to be defined only on D, it is not clear whether the assertion of (2.6) holds.

(2.7) Remark. It is fairly easy to show that if $(\mathcal{E}, \mathcal{P})$ is a projection scheme in a reflexive Banach space E, then the pair $(\{E_n'\}, \{P_n'\})$, where $P_n' : E' \to E'$ is dual to P_n and $E_n' = \text{Range}(P_n')$, is a projection scheme in E'. Moreover, $\sup_n \|P_n'\| = \sup_n \|P_n\|$.

(2.8) Example. Let E satisfy the assumptions of (2.7). Consider a closed $D \subset E$ and a morphism $\varphi \in M_m(D, E')$ which determines a strictly monotone map, i.e. for any $x, x' \in D$, $y \in \varphi(x)$, $y' \in \varphi(x')$, we have

$$|(y - y', x - x')| \ge c\|x - x'\|^2.$$

We claim that φ is an AP-morphism (provided in E' we consider the projection scheme given by (2.7)). To see this, observe that

(a) if $y_n \to y$ in E', then $P_n' y_n \to y$ in E';
(b) if $x_n \to x$ weakly in E, then $P_n x - x_n \to 0$ weakly.

To prove (b), take $u \in E'$ and then $(u, P_n x - x_n) = (u, P_n x) - (u, x_n) \to 0$.

Now, let a sequence (x_{n_k}) be bounded, $x_{n_k} \in D_{n_k}$. Take $y_{n_k} \in \varphi(x_{n_k})$ and suppose that $P_{n_k}' y_{n_k} \to y \in E'$. Since E is reflexive, we may assume that actually

$x_{n_k} \to x \in E$ weakly. Since D is closed convex, it is wekly closed; hence $x \in D$. Let $y'_{n_k} \in \varphi(P_{n_k}x)$. By the assumption,

$$c\|x_{n_k} - P_{n_k}x\|^2 \le |(y_{n_k} - y'_{n_k}, x_{n_k} - P_{n_k}x)| = |(P'_{n_k}(y_{n_k} - y'_{n_k}), x_{n_k} - P_{n_k}x)|.$$

Since $P_{n_k}x \to x$ and $y'_{n_k} \in \varphi(P_{n_k}x)$, it follows that, in view of the upper semicontinuity of φ, we may assume $y'_{n_k} \to y' \in \varphi(x)$. Then $P'_{n_k}y'_{n_k} \to y'$. Hence $P_{n_k}x - x_{n_k} \to 0$ in E, therefore $x_{n_k} \to x$. Once again, by the upper semicontinuity of φ, $y \in \varphi(x)$.

The most natural situation when the above examples arise is when one considers a Hilbert space with a countable orthonormal basis. The projection scheme built with the aid of this basis satisfies condition (π_1), the space is reflexive and complete. Moreover, the dual space is isomorphic to itself, hence one may consider strictly monotone maps with values in the domain.

(2.9) Example. If H is a Hilbert space with a countable orthonormal basis, $(\{H_n\}, \{P_n\})$ is the projection scheme constructed by means of this basis, $\varphi_1 \in M_1(H, H)$ is a k-β-contraction, $\varphi_2 \in M_1^K(H, H)$ is a compact morphism and $\varphi_3 \in M_1(H, H)$ determines a strictly monotone map, then $\Phi = \varphi_1 + \varphi_2 + \varphi_3 \in M_1(H, H)$ is an AP-1-morphism provided $k < c$ where c is a constant from the definition of a strictly monotone map.

The above examples show how important is the role played by AP-maps in nonlinear analysis. However, it should be stressed that these examples are valid only under rather strong and restrictive hypotheses concerning the structure of spaces on which the given maps act.

All the examples given above were well-known in the single-valued setting (comp. [92]).

Now, we shall compare the class of AP-morphisms with that of A-morphisms. We intend to show that A-maps can be defined and studied in far more general spaces and under weaker assumptions.

(2.10) Theorem. *Let the assumptions of* (2.3) *concerning* X, $\{X_n\}$, E, $\{E_n\}$, $\{P_n\}$ *and* D *be satisfied. If the scheme* $(\mathcal{E}, \mathcal{P})$ *satisfies condition* (π_r) *and a morphism* $\varphi \in M_m^A((D, \mathcal{D}), (E, \mathcal{E}))$ *determines a proper map, then* φ *is an* AP*-*m*-morphism.*

Proof. Let a sequence (x_{n_k}) with $x_{n_k} \in D_{n_k}$ be bounded. If $y_{n_k} \in \varphi(x_{n_k})$ and $P_{n_k}y_{n_k} \to y \in E$, then $\lim_{k\to\infty} d(y_{n_k}, E_{n_k}) = 0$; hence

$$\|y_{n_k} - P_{n_k}y_{n_k}\| \le (r+1)d(y_{n_k}, E_{n_k}) \to 0$$

and therefore, $y_{n_k} \to y$ as well. The fact that the map determined by φ is proper entails that the sequence (x_{n_k}) has a limit point x. By the upper semicontinuity of φ and the compactness of values of φ, we get $y \in \varphi(x)$. ■

Remark. Observe that an important property of single-valued AP-maps is that their restrictions to bounded closed subsets are proper. In the set-valued setting, this is not true unless we assume that the map under consideration is u.s.c. and l.s.c.

(2.11) Example. We provide an example of an A-map which is not AP (comp. [70]). Let $E = l^2$, $E_n = \operatorname{span}(e_1, \ldots, e_n)$ where $e_k = (\delta_{ki})_{i=1} \in l^2$. Let P_n be the orthogonal projection onto E_n. For simplicity, we shall build a single-valued map. Let $f(e_1) = e_1$, $f(e_2) = e_2$, $f(e_k) = e_{k-1} + e_k$ for $k \geq 3$. Extend f linearly onto E. Then f is not AP. To see this, set $x_n = \sqrt{n^{-1}}(e_1 - e_2 + \ldots + (-1)^n e_n)$. Then $\|x_n\| = 1$, $P_n f(x_n) \to 0$. However, $\|x_n - x_m\| > n^{-1}(n-m)$ for any $n > m$. On the other hand, for any n, $f(E_n) \subset E_n$ and, hence, f is an A-map.

Now, we proceed to constructing the degree theory for AP-morphisms. In the single-valued case, this was done by Browder and Petryshyn [15].

Let X be a normed space with a linear filtration $\{X_n\}_{n=1}^{\infty}$ and let D be an open and bounded subset of X. Let E be a normed space furnished with a projection scheme $(\{E_n\}, \{P_n\})$. Assume that $\dim X_n = \dim E_n > m$ for a.a. n. Take an AP-morphism $\varphi \in M_m(\operatorname{cl} D, E)$ such that $y \notin \varphi(\operatorname{bd} D)$. Observe that, for a.a. n, $P_n y \notin P_n \varphi(\operatorname{bd} D \cap X_n)$. This follows immediately from the definition of AP-maps. Therefore, for sufficiently large n, we may define a number $s_n = \deg(P_n\varphi|D_n, P_n y)$. Following Browder and Petryshyn, we let $\widetilde{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$ (where $\mathbb{Z}$ stands for the integers) and define $D(\varphi, y) \subset \widetilde{\mathbb{Z}}$ by

(i) an integer $s \in D(\varphi, y)$ if there is a subsequence $s_{n_k} \to s$;
(ii) $\pm\infty \in D(\varphi, y)$ if there is a subsequence $s_{n_k} \to \pm\infty$.

However, we prefer another definition. Namely, we put

$$\operatorname{Deg}(\varphi, y) = \nu((s_n)) \in G.$$

This second definition seems to be more aappropriate and carries more information. Consider a set-valued map t which, to any sequence $(s_n) \in \prod_{n=1}^{\infty} \mathbb{Z}$, assigns a subset of $\mathbb{Z}$ such that

(i) $s \in t((s_n))$ if s is a limit point of (s_n);
(ii) $\pm\infty \in t((s_n))$ if (s_n) has a subsequence divergent to $\pm\infty$.

It is easy to see that if $s_n = z_n$ for a.a. n, i.e. $(s_n) - (z_n) \in \bigoplus_{n=1}^{\infty} \mathbb{Z}$, then $t((s_n)) = t((z_n))$. Hence, we have a map $\overline{t} : G \to \widetilde{\mathbb{Z}}$ and then

$$t(\operatorname{Deg}(\varphi, y)) = D(\varphi, y).$$

We will not go into details as concerns the properties of the defined degree Deg (or D). These may be the subject of another large paper. Let us only mention that Deg satisfies all the properties of the degree, while, for D, there are problems with the additivity property.

The constructed degree theory for AP-maps is especially useful in applications (comp. [92]). Let us stress once again that the theory of AP-maps, which is hardly topological, was built and studied mainly from the point of view of broad and extensive applications.

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