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Algebraic theory of fundamental dimension

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CONTENTS

Introduction

By the fundamental dimension of a compactum X (denoted by $\mathrm{Fd}(X)$) we understand the minimum of $\dim Y$, where Y runs through all compacta with $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$. This notion has been introduced by K. Borsuk. In the theory of shape it corresponds to the notion of dimension.

The aim of the present paper is to compile and organize the results of my papers [N_2], [N_4] and [N_5] concerning relationships between fundamental dimension and some algebraic and homological properties.

Analogous questions for the homotopy theory and CW complexes were studied by C. T. C. Wall ([W], p. 63).

The above-mentioned algebraic methods allow us to compute the fundamental dimension of the Cartesian product in some cases. These applications are also presented in this paper. Some of the theorems of this kind are new and have never been published before. A similar programme for ordinary dimension was carried out by P. S. Alexandroff, M. Bockstein, V. G. Boltyanskii and L. Pontryagin (see [Ko_1] and [Ku]).

We assume that the reader is familiar with the theory of shape and knows the notion of a procategory ([B_3], [M-S_1], [M-S_2], [M-S_3], [Mo] and [Ma]). A knowledge of the basic concepts and facts of the theory of retracts, the PL topology and the theory of CW complexes ([B_1], [R-S] and [S]) is also assumed.

Chapter I

Elementary topological characterizations of fundamental dimension

One of the principal results of the theory of fundamental dimension is the formula:

$$\mathrm{Fd}(X) = \mathrm{Min}\{\dim Y\colon \mathrm{Sh}(X) = \mathrm{Sh}(Y)\}.$$

Historically the first proof of this fact was given (unpublished) by W. Holsztyński, who developed an axiomatic approach ([Ho]) to the theory of shape (in particular, the continuity of the shape functor with respect to inverse sequences and their limits) for this purpose. He presented the above-mentioned proof at Professor Borsuk's seminar, December 1968 – January 1969.

In this chapter we prove some topological characterizations of compacta with the fundamental dimension $\leqslant n$, which can be regarded as a generalization of the Holsztyński theorem. We obtain also (as a consequence of these characterizations) the inequality

$$\mathrm{Fd}(X_1 \cup X_2) \leqslant \max\big(\mathrm{Fd}(X_1), \mathrm{Fd}(X_2), \mathrm{Fd}(X_1 \cap X_2)+1\big)$$

and a theorem which states that the fundamental dimension of a compactum X depends only on the fundamental dimensions of its components.

If X is a CW complex, then $X^{(n)}$ denotes the n-dimensional skeleton of X.

If X and Y are polyhedra and $f\colon X \to Y$ is a map such that $f(X^{(n)}) \subset Y^{(n)}$ for $n = 0, 1, 2, \ldots, \dim X$, then f is said to be *cellular*. This terminology agrees with and is motivated by the terminology of the theory of CW complexes.

In this chapter we denote by $\tau_k\colon Q = [0,1]\times[0,1]\times \ldots \to [0,1]^k$ the natural projection given by the formula

$$\tau_k(x_1, x_2, \ldots, x_k, x_{k+1}, \ldots) = (x_1, x_2, \ldots, x_k) \quad \text{for every } (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots) \in Q.$$

1. Characterizations of fundamental dimension. Suppose that W is a finite CW complex and $f\colon X \to W$ is a map. We say (see [N₁]) that $\omega(f) \leqslant n$

iff there exists a homotopy $\varphi\colon X\times[0,1]\to W$ such that

$$\varphi(x,0)=f(x)\quad \text{for every } x\in X \tag{1.1}$$

and

$$\varphi(X\times\{1\})\subset W^{(n)}. \tag{1.2}$$

If (X,x_0), (W,w_0) are pointed CW complexes and $f\colon (X,x_0)\to(W,w_0)$, then the condition $\omega(f)\leqslant n$ implies that there exists a homotopy $\varphi\colon X\times[0,1]\to W$ which satisfies (1.1) and (1.2) and fixes x_0. We can infer this fact from the cellular approximation theorem (see [S], p. 404 and [S], p. 57, Exercise D4).

If X is a compactum and $x_0\in X$ and $\underline{X}=\{X_k,p_k^{k+1}\}$ (or $\underline{(X,x)}=\{(X_k,x_k),p_k^{k+1}\}$) is an inverse sequence of compacta such that $\mathrm{Sh}(\varprojlim \underline{X})=\mathrm{Sh}(X)$ (or $\mathrm{Sh}(\varprojlim \underline{(X,x)})=\mathrm{Sh}(X,x_0)$), then we say that $\underline{X}$ (or $\underline{(X,x)}$) is *associated with* X (or with (X,x_0)) (see [Mo]).

The following theorem characterizes compacta with fundamental dimension $\leqslant n$ (see [N_1]; p. 214 and cf. [D_1]; p. 80):

(1.3) THEOREM. *Let a pointed compactum (X,x_0) be the inverse limit of an inverse sequence $\{(X_k,x_k),p_k^{k+1}\}$ of finite* CW *complexes and let n be a natural number or* 0. *Then the following conditions are equivalent:*

(a) $\mathrm{Fd}(X)\leqslant n$.

(b) *For every k there exists a k' such that $\omega(p_k^{k'})\leqslant n$.*

(c) *$\omega(p_k)\leqslant n$ for every $k=1,2,\ldots$, where $p_k\colon X\to X_k$ is the natural projection.*

(d) *There exists a pointed compactum (Y,y_0) such that* $\dim Y\leqslant n$ *and* $\mathrm{Sh}(X,x_0)=\mathrm{Sh}(Y,y_0)$.

(e) *There exists an inverse sequence $\{Y_k,q_k^{k+1}\}$ of n-dimensional polyhedra and simplicial maps associated with X.*

(f) *$\omega(f)\leqslant n$ for every map $f\colon X\to W$ from X to a finite* CW *complex W.*

Proof. (a) ⇒ (b). There exists an n-dimensional compactum Y such that $\mathrm{Sh}(X)\leqslant\mathrm{Sh}(Y)$. Since $\dim Y\leqslant n$, we infer that there is an inverse sequence $\{Y_k,q_k^{k+1}\}$ of n-dimensional polyhedra associated with Y.

Then there exist maps $(\alpha,f_k)\colon \{X_k,p_k^{k+1}\}=\underline{X}\to\underline{Y}=\{Y_k,q_k^{k+1}\}$ and $(\beta,g_k)\colon\underline{Y}\to\underline{X}$ such that $(\alpha\beta,g_kf_{\beta(k)})\colon\underline{X}\to\underline{X}$ is homotopic with the identity $\underline{\mathrm{id}}_X$ of the system $\underline{X}$.

This means that for every k there is a natural number $k'\geqslant\alpha\beta(k)$ such that $p_k^{k'}\simeq g_kf_{\beta(k)}p_{\alpha\beta(k)}^{k'}$. Obviously we can assume that $g_k(Y_{\beta(k)})\subset X_k^{(n)}$. Therefore $\omega(p_k^{k'})\leqslant n$.

(b) ⇒ (d). Condition (d) implies that there exists a sequence $k_1<k_2<$ of natural numbers such that

$$p_{k_i}^{k_i+1}\colon (X_{k_{i+1}}, x_{k_{i+1}}) = (W_{i+1}, w_{i+1}) \to (W_i, w_i) = (X_{k_i}, x_{k_i})$$

is homotopic (in the pointed sense) with a map $q_i^{i+1}\colon (W_{i+1}, w_{i+1}) \to (W_i, w_i)$ such that $q_i^{i+1}(W_{i+1}) \subset W_i^{(n)}$. It is clear that $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0)$ and $\dim Y \leqslant n$, where $(Y, y_0) = \varprojlim \{(W_i, w_i), q_i^{i+1}\}$.

(d) $\Rightarrow$ (e), (e) $\Rightarrow$ (a), (b) $\Rightarrow$ (c), (f) $\Rightarrow$ (c) and (e) $\Rightarrow$ (f) are obvious.

(c) $\Rightarrow$ (b). Let k be a fixed natural number. Let $q_m^{m+1} = p_{m+k}^{m+k+1}\colon Y_{m+1} = X_{k+m+1} \to Y_m = X_{m+k}$ for every $m = 1, 2, \ldots$ and let $f\colon X \to X_k^{(n)}$ be a map such that if $if \simeq p_k$ then $i\colon X_k^{(n)} \to X_k$ is the inclusion.

Since the inverse sequence $\{Y_m, p_m^{m+1}\}$ is associated with X, we infer that there exist an index m_0 and a map $g\colon Y_{m_0} \to X_k^{(n)}$ such that $gq_{m_0} \simeq f$, where $q_{m_0} = p_{m_0+k}$.

Hence $igp_{m_0+k} \simeq if \simeq p_k$ and $\omega(p_k^{m_0+k}) \leqslant n$. The proof of Theorem (1.3) is finished.

Remark. The equivalence (a) $\Leftrightarrow$ (d) was proved by S. Spież [Sp]. It is a sharpenning of the Holsztyński theorem. This proof of (1.3) is inspired by the proof of the Holsztyński theorem given by A. Trybulec.

D. A. Edwards and R. Geoghegan have proved (see [E-G], Theorem (4.2)) the following

(1.4) THEOREM. *If (X, x_0) is a pointed polyhedron, then there exists a pointed polyhedron (Y, y_0) such that* $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0)$ *and* $\dim Y = \max(3, \mathrm{Fd}(X))$.

The compacta lying in the Hilbert cube Q with the fundamental dimension $\leqslant n$ are characterized in the following way:

(1.5) THEOREM. *Let $X \subset Q$ be a compactum. Then the following conditions are equivalent*:

(a) $\mathrm{Fd}(X) \leqslant n$.

(b) *For every neighborhood U of X in Q there exists a homotopy $\varphi\colon X \times [0,1] \to U$ such that*

$$\varphi(x, 0) = x \ \textit{for}\ x \in X \quad \textit{and} \quad \dim \varphi(X \times \{1\}) \leqslant n.$$

Proof. (a) $\Rightarrow$ (b). We can assume that U is a prism in Q, i.e. there exist a positive integer k and a polyhedron $W \subset [0, 1]^k$ such that $U = \tau_k^{-1}(W)$ (see [B$_1$], p. 105). It is clear that $A = W \times \{0\} \times \{0\} \times \ldots \subset U \subset Q$ is a strong deformation retract of U and the inclusion map of X into U is homotopic in U to some map $f\colon X \to W$ with values belonging to A. Since A is homeomorphic with the polyhedron W, we infer that $\omega(f) \leqslant n$. Hence (a) implies (b).

(b) $\Rightarrow$ (a). It is sufficient to show that $\omega(f) \leqslant n$ for every map $f\colon X \to W$ from X to a polyhedron W. Since $W \in \mathrm{ANR}$, we infer that there are a neighborhood U of X in Q and a map $f\colon U \to W$ such that

$$f(x) = \hat{f}(x) \quad \text{for} \quad x \in X.$$

Let $\varphi\colon X \times [0,1] \to U$ be a homotopy satisfying (b). It is clear that $f \simeq \hat{f}g \simeq \hat{f}|_{g(X)} \circ g$, where $g\colon X \to U$ is defined by the formula

$$g(x) = \varphi(x, 1) \quad \text{for every } x \in X.$$

Since $\dim g(X) \leqslant n$, we infer that there is a homotopy joining $\hat{f}|_{g(X)}$ (in W) with a map $h\colon g(X) \to W$ such that $h(g(X)) \subset W^{(n)}$. This completes the proof.

Using our characterizations, one can easily obtain the following

(1.6) COROLLARY. *Suppose that* $X = \varprojlim \{X_k, p_k^{k+1}\}$, *where* $X_1, X_2, \ldots$ *are compacta and* $\mathrm{Fd}(X_k) \leqslant n$ *for* $k = 1, 2, \ldots$ *Then* $\mathrm{Fd}(X) \leqslant n$.

Proof. Let $f\colon X \to W$ be a map from X to a polyhedron W. Since $\{X_k, p_k^{k+1}\}$ is associated with X, we infer that there is an index k_0 and a map $g\colon X_{k_0} \to X$ such that $f \simeq gp_{k_0}$. Since $\omega(g) \leqslant n$, we conclude that $\omega(f) \leqslant n$. The proof of Corollary (1.6) is completed.

2. The fundamental dimension of components of compacta. The following lemma will be employed:

(2.1) LEMMA. *Let* X *be a compactum lying in the Hilbert cube* Q. *Then* $\mathrm{Fd}(X) \leqslant n$ *if and only if for every neighborhood* U *of* X *there exists a neighborhood* $V \subset U$ *of* X *such that* V *is deformable in* U *to a subset* A *of* U *with* $\dim A \leqslant n$.

Proof. If one can deform X in every neighborhood U of X to a set A with $\dim A \leqslant n$, then $\mathrm{Fd}(X) \leqslant n$ (see Theorem (1.5)).

Let U be a neighborhood of X such that X is deformable in U to a n-dimensional set $A \subset U$ and U is a prism in Q. Then A is deformable in U to a subset of the n-dimensional skeleton B of the base of U and therefore X is deformable in U to a subset of B.

Since $B, U \in \mathrm{ANR}$, we infer that there exists a neighborhood V of X which is deformable in U to B. This completes the proof.

Let us prove the following

(2.2) THEOREM. *A compactum* X *has a fundamental dimension* $\leqslant n$ *iff all its components have a fundamental dimension* $\leqslant n$.

Proof. Suppose that $\mathrm{Fd}(X) \leqslant n$. Then there exists a compactum Y such that $\dim Y \leqslant n$ and $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$. Moreover ([B₃], p. 215), for every component X_μ of X there exist a component Y_μ of Y such that $\mathrm{Sh}(X_\mu) \leqslant \mathrm{Sh}(Y_\mu)$. This implies that all components of X have a fundamental dimension $\leqslant n$.

Suppose that every component of X has a fundamental dimension $\leqslant n$. We can assume that $X \subset Q$.

Consider a neighborhood U of X. Then by Lemma (2.1) for every component X_μ of X there are an open neighborhood $\hat{V}_\mu$ of X_μ such that its boundary is disjoint with X and a homotopy $\varphi_\mu\colon \hat{V}_\mu \times [0,1] \to U$ such that

$$\varphi_\mu(x,0) = x \quad \text{for every } x \in \hat{V}_\mu$$

and

$$\dim \varphi_\mu(\hat{V}_\mu \times \{1\}) \leqslant n.$$

Since X is compact, there is a finite system of indices $\mu_1, \mu_2, \dots, \mu_k$ such that

$$\hat{V} = \hat{V}_{\mu_1} \cup \hat{V}_{\mu_2} \cup \dots \cup \hat{V}_{\mu_k}$$

is a neighborhood of X. Setting

$$V_i = \hat{V}_{\mu_i} \setminus \overline{\bigcup_{j<i} \hat{V}_{\mu_j}} \quad \text{for} \quad i = 1, 2, \dots, k,$$

we get a system of open and disjoint sets $V_1, V_2, \dots, V_k$, such that the set $V' = \bigcup_{i=1}^{k} V_i$ is a neighborhood of X.

Setting

$$\varphi'(x,t) = \varphi_{\mu_i}(x,t) \quad \text{for every } (x,t) \in V_i \times [0,1],$$

we get a homotopy $\varphi'\colon V' \times [0,1] \to U$.

Let V be a closed neighborhood of X such that $V \subset V'$ and let $\varphi = \varphi'|V \times [0,1]$.

It follows that $V \cap V_i$ is a closed subset of Q. Hence $\dim \varphi(V \times \{1\}) \leqslant n$ and $\varphi(x,0) = x$ for every $x \in V$.

Using Theorem (1.5), we infer that $\mathrm{Fd}(X) \leqslant n$ and the proof of Theorem (2.2) is finished.

3. The fundamental dimension of the union of two compacta. Let us prove the following

(3.1) THEOREM. *Let X_1, X_2 be compacta. Then* $\mathrm{Fd}(X_1 \cup X_2) = \max(\mathrm{Fd}(X_1), \mathrm{Fd}(X_2), \mathrm{Fd}(X_1 \cap X_2)+1)$.

Proof. Taking the nerves of open finite coverings $\mathcal{U}_1, \mathcal{U}_2, \dots$ of $X_1 \cup X_2$ such that $\mathcal{U}_{m+1}$ is a refinement of $\mathcal{U}_m$ for $m = 1, 2, \dots$ and the sequence $\{\alpha_m\}$ of maximal diameters of the elements of $\mathcal{U}_m$ converges to zero, one can prove that there exist inverse sequences $\{A_n \cup B_n, p_n^{n+1}\}$, $\{A_n, q_n^{n+1}\}$, $\{B_n, r_n^{n+1}\}$ and $\{C_n, s_n^{n+1}\}$ associated (respectively) with $X_1 \cup X_2$, X_1, X_2 and $X_0 = X_1 \cap X_2$ and satisfying the following conditions:

A_n, B_n and $C_n = A_n \cap B_n$ are subcomplexes of a finite CW complex $A_n \cup B_n$

and

p_n^{n+1}, $q_n^{n+1} = p_n^{n+1}|_{A_{n+1}}$, $r_n^{n+1} = p_n^{n+1}|_{B_{n+1}}$, $s_n^{n+1} = p_n^{n+1}|_{C_{n+1}}$ are cellular maps

for every $n = 1, 2, \dots$

Let $m_i = \mathrm{Fd}(X_i)$ for $i = 0, 1, 2$.

Then for every natural number n there exist indices n_1 and n_2 such that $n_2 > n_1 > n$ and $\omega(s_{n_1}^{n_2}) \leqslant m_0$ and $\omega(q_n^{n_1}) \leqslant m$ and $\omega(r_n^{n_1}) \leqslant m_2$.

From the cellular approximation theorem and the homotopy extension theorem we conclude that there are cellular maps $s\colon C_{n_2} \to C_{n_1}$ and $p\colon A_{n_2} \cup B_{n_2} \to A_{n_1} \cup B_{n_1}$ such that

$$s \simeq s_{n_1}^{n_2} \quad \text{and} \quad s(C_{n_2}) \subset C_{n_1}^{(m_0)}$$

and

$$p \simeq p_{n_1}^{n_2} \quad \text{and} \quad p|_{C_{n_2}} = s.$$

Analogously we infer that there are cellular homotopies $\varphi_1\colon A_{n_1} \times [0, 1] \to A_n$ and $\varphi_2\colon B_{n_1} \times [0, 1] \to B_n$ such that

$$\varphi_1(x, 0) = q_n^{n_1}(x) \text{ for } x \in A_{n_1} \quad \text{and} \quad \varphi_2(x, 0) = r_n^{n_1}(x) \text{ for } x \in B_{n_1}$$

and

$$\varphi_1(A_{n_1} \times \{1\}) \subset A_n^{(m_1)} \quad \text{and} \quad \varphi_2(B_{n_1} \times \{1\}) \subset B_n^{(m_2)}.$$

Setting

$$\varphi_1'(x, t) = \varphi_1(p(x), t) \quad \text{for} \quad (x, t) \in A_{n_2} \times [0, 1]$$

and

$$\varphi_2'(x, t) = \varphi_2(p(x), t) \quad \text{for} \quad (x, t) \in B_{n_2} \times [0, 1],$$

we get cellular homotopies $\varphi_1'\colon A_{n_2} \times [0, 1] \to A_n$ and $\varphi_2'\colon B_{n_2} \times [0, 1] \to B_n$ such that

$$\varphi_1'(C_{n_2} \times [0, 1] \cup A_{n_2} \times \{1\}) \subset A_n^{(r)}$$

and

$$\varphi_2'(C_{n_2} \times [0, 1] \cup B_{n_2} \times \{1\}) \subset B_n^{(r)}$$

where $r = \max(m_1, m_2, m_0 + 1)$.

Then there are (see [S], p. 57, Exercise D4) homotopies $\psi_1\colon A_{n_2} \times [0, 1] \to A_n$ and $\psi_2\colon B_{n_2} \times [0,1] \to B_n$ such that

$$\psi_1(x, t) = p_n^{n_1} p(x) \quad \text{for} \quad (x, t) \in A_{n_2} \times \{0\} \cup C_{n_2} \times [0, 1]$$

and

$$\psi_2(x, t) = p_n^{n_1} p(x) \quad \text{for} \quad (x, t) \in B_{n_2} \times \{0\} \cup C_{n_2} \times [0, 1]$$

and

$$\psi_1(A_{n_2} \times \{1\}) \cup \psi_2(B_{n_2} \times \{1\}) \subset A_n^{(r)} \cup B_n^{(r)} = (A_n \cup B_n)^{(r)}.$$

Setting

$$\psi(x, t) = \begin{cases} \psi_1(x, t) & \text{for} \quad (x, t) \in A_{n_2} \times [0, 1], \\ \psi_2(x, t) & \text{for} \quad (x, t) \in B_{n_2} \times [0, 1], \end{cases}$$

we get a homotopy $\psi\colon (A_{n_2} \cup B_{n_2}) \times [0, 1] \to A_n \cup B_n$ such that

$$\psi((A_n \cup B_n) \times \{1\}) \subset (A_n \cup B_n)^{(r)}.$$

Since p is homotopic with $p_{n_1}^{n_2}$, we infer that $\omega(p_n^{n_2}) = \omega(p_n^{n_1} p) \leqslant r$. The proof is finished.

Using Theorem (3.1) one can prove the following

(3.2) COROLLARY. *Let X, A be compacta and $A \subset X$. Then* $\operatorname{Fd}(X/A) \leqslant \max(\operatorname{Fd}(X), \operatorname{Fd}(A)+1)$.

Proof. Let B be a continuum with a trivial shape such that $B \cap X = A$. Then ([B_3], p. 321) $\operatorname{Sh}(X/A) = \operatorname{Sh}(X/B) = \operatorname{Sh}(X \cup B)$. Using (3.1) for the case where $X_1 = X$ and $X_2 = B$, we obtain $\operatorname{Fd}(X/A) = \operatorname{Fd}(X \cup B) \leqslant \max(\operatorname{Fd}(X), \operatorname{Fd}(A)+1)$. This proves (3.2).

Chapter II

Cohomology groups over local systems and generalized local systems

The first three sections of this chapter deal with the theory of cohomology groups with coefficients in local systems of abelian groups (or, in other words, with coefficients in bundles) for finite cell complexes ([Hi-Wy], [S], [St_2] and [St_3]), but we do not intend to develop this theory systematically here. Our intention is to fix the terminology and to prove some simple auxiliary facts.

The aim of the last section is to generalize the concept of cohomology over local systems. This generalization is closely related to the construction of the cohomology groups given by E. Čech.

We will denote by Z the group of integer numbers.

Let $H^n(X, A; G)$ (or $H_n(X, A; G)$) denote, for every pair (X, A) of compacta and every abelian group G, the n-dimensional Čech cohomology (or homology) group of (X, A) with coefficients in G.

1. Local systems of groups. We say that we have a *local system* $\mathscr{L}$ *of abelian groups* on a space X iff the following conditions are satisfied:

(a) For each point $x \in X$, there is given an abelian group $\mathscr{L}(x)$.

(b) For every path d: $[0, 1] \to X$ from x_0 to x_1 there is an isomorphism $\mathscr{L}(d)$: $\mathscr{L}(x_1) \to \mathscr{L}(x_0)$.

(c) $\mathscr{L}(d_0)$ is the identity automorphism on $\mathscr{L}(x_0)$ for every degenerate path d_0: $[0, 1] \to X, d_0([0, 1]) = \{x_0\}$.

(d) If paths d_1, d_2: $[0, 1] \to X$ are equivalent, then $\mathscr{L}(d_1) = \mathscr{L}(d_2)$.

(e) If d_1, d_2: $[0, 1] \to X$ are paths and $d_1(1) = d_2(0)$, then $\mathscr{L}(d_1 d_2) = \mathscr{L}(d_1) \circ \mathscr{L}(d_2)$.

If $\mathscr{L}(x)$ is a free abelian group or $\mathscr{L}(x)$ is an infinity cyclic group for $x \in X$, then $\mathscr{L}$ is said to be a *local system of free abelian groups* or a *local system of infinity cyclic groups.*

Since the homomorphism $\mathscr{L}(d)$ depends only on the homotopy class of d: $[0, 1] \to X$, we can write $\mathscr{L}([d])$ for the homotopy class $[d]$ of d. In particular, we will write $\mathscr{L}(a)$: $\mathscr{L}(x) \to \mathscr{L}(x)$ when $a \in \pi_1(X, x)$.

(1.1) EXAMPLE. Let $n \geqslant 2$ (or $n \geqslant 3$) and let X be an arcwise connected space (or let (X, A) be a pair of arcwise connected spaces). If we assign to every $x \in X$ (or $x \in A$) the group $\pi_n(X, x)$ (or $\pi_n(X, A, x)$) and to every path d: $[0, 1] \to X$ (or d: $[0, 1] \to A$) the homomorphism h_d: $\pi_n(X, d(1)) \to \pi_n(X, d(0))$ (or h_d: $\pi_n(X, A, d(1)) \to \pi_n(X, A, d(0))$) induced by d, then we obtain a local system of abelian groups $\Pi_n(X)$ (or $\Pi_n(X, A)$) on X (or on A).

These notations will be used in the sequel.

Suppose that $\mathscr{L}$ is a local system of groups on X and $\mathscr{K}$ is a local system of groups on Y. By a *morphism from X and* $\mathscr{L}$ *to Y and* $\mathscr{K}$ we understand a pair $(f, \{f_x\}_{x \in X})$ consisting a map f: $X \to Y$ and a family of homomorphisms f_x: $\mathscr{K}(f(x)) \to \mathscr{L}(x)$, where $x \in X$, such that $f_{x_0} \circ \mathscr{K}(fd) = \mathscr{L}(d) \circ f_{x_1}$ for every path d: $[0, 1] \to X$ from x_0 to x_1.

(1.2) EXAMPLE. Let $\mathscr{K}$ be a local system of groups on Y and let f: $X \to Y$ be a map. If we assign to every $x \in X$ an abelian group $\mathscr{L}(x) = \mathscr{K}(f(x))$ and to every path d: $[0, 1] \to X$ a homomorphism $\mathscr{L}(d) = \mathscr{K}(fd)$: $\mathscr{L}(d(1)) = \mathscr{K}(fd(1)) \to \mathscr{K}(fd(0)) = \mathscr{L}(d(0))$, then we obtain a local system $\mathscr{L}$ of groups on X. We say that $\mathscr{L}$ is induced on X by f and $\mathscr{K}$ One can easily check that the pair $(f, \{\mathrm{id}_{\mathscr{K}(x)}\}_{x \in X})$ is a morphism from $\mathscr{L}$ to $\mathscr{K}$

If $X \subset Y$ and f is an inclusion, we write $\mathscr{L} = \mathscr{K}|_X$.

A local system of groups $\mathscr{L}$ on X is said to be *simple* if the isomorphism $\mathscr{L}(d)$ depends only on $d(0)$ and $d(1)$.

The following proposition holds true:

(1.3) PROPOSITION. *Suppose that* $\mathscr{K}$ *is a local system of groups on* Y *and* f: $X \to Y$ *is a map which induces a trivial homomorphism* $f_\#$: $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, *where* $x_0 \in X$ *and* $y_0 \in Y$. *Then the local system induced on* X *by* $\mathscr{K}$ *and* f *is simple.*

(1.4) EXAMPLE. Let $\mathscr{L}$ be a local system of abelian groups on X. We denote by $\hat{\mathscr{L}}(x)$ the quotient group of $\mathscr{L}(x)$ generated by all $\mathscr{L}(a)(\alpha) - \alpha$, where $\alpha \in \mathscr{L}(x)$ and $a \in \pi_1(X, x)$. One can verify that the isomorphism $\mathscr{L}(d)$: $\mathscr{L}(d(1)) \to \mathscr{L}(d(0))$ induces an isomorphism $\hat{\mathscr{L}}(d)$: $\hat{\mathscr{L}}(d(1)) \to \hat{\mathscr{L}}(d(0))$ for every path d and that $\hat{\mathscr{L}}(d_1) = \hat{\mathscr{L}}(d_2)$ for all paths d_1, d_2: $[0, 1] \to X$ from x_0 to x_1. Then we obtain a simple local system of groups $\hat{\mathscr{L}}$ on X. Let f_x: $\mathscr{L}(x) \to \hat{\mathscr{L}}(x)$ be a canonical projection for every $x \in X$. It is clear that $(\mathrm{id}_X, \{f_x\}_{x \in X})$ is a morphism from $\hat{\mathscr{L}}$ to $\mathscr{L}$

Let $\mathscr{L}, \mathscr{K}, \mathscr{H}$ be local systems of abelian groups on X, Y, Z (respectively), let $(f, \{f_x\}_{x \in X})$ be a morphism from $\mathscr{L}$ to $\mathscr{K}$ and let $(g, \{g_y\}_{y \in Y})$ be a morphism from $\mathscr{K}$ to $\mathscr{H}$. Then $(gf, \{f_x g_{f(x)}\}_{x \in X})$ is a morphism from $\mathscr{L}$ to $\mathscr{H}$ which is called the composition of $(f, \{f_x\}_{x \in X})$ and $(g, \{g_y\}_{y \in Y})$.

We say that two morphisms $(f, \{f_x\}_{x \in X})$ and $(g, \{g_x\}_{x \in X})$, from a local system $\mathscr{L}$ on X to a local system $\mathscr{K}$ on Y are homotopic iff there exists a morphism $(\varphi, \{\varphi_{(x,t)}\}_{(x,t)\in X\times[0,1]})$ from a local system $\bar{\mathscr{L}}$ induced on $X\times[0,1]$ by the Cartesian projection $p_1\colon X\times[0,1]\to X$ and $\mathscr{L}$ such that

$$\varphi(x,0)=f(x) \quad \text{and} \quad \varphi(x,1)=g(x)$$

and

$$f_x=\varphi_{(x,0)}\colon \mathscr{K}(\varphi(x,0))=\mathscr{K}(f(x))\to\bar{\mathscr{L}}(x,0)=\mathscr{L}(x)$$

and

$$g_x=\varphi_{(x,1)}\colon \mathscr{K}(\varphi(x,1))=\mathscr{K}(g(x))\to\bar{\mathscr{L}}(x,1)=\mathscr{L}(x)$$

for every $x\in X$.

(1.5) EXAMPLE. Let $\mathscr{L}$ be a local system of abelian groups induced on X by a map $f\colon X\to Y$ and a local system $\mathscr{K}$ on Y and suppose that $g\colon X\to Y$ is a map and $\varphi\colon X\times[0,1]\to Y$ is a homotopy such that

$$\varphi(x,0)=f(x) \quad \text{and} \quad \varphi(x,1)=g(x) \quad \text{for} \quad x\in X.$$

We denote by $d_{(x,t)}\colon [0,1]\to X$ the path defined by the formula

$$d_{(x,t)}(s)=\varphi(x,st) \quad \text{for every } s\in[0,1].$$

Setting

$$g_x=\mathscr{K}(d_{(x,1)})\colon \mathscr{K}(g(x))\to\mathscr{K}(f(x))=\mathscr{L}(x)$$

and

$$\varphi_{(x,t)}=\mathscr{K}(d_{(x,t)})\colon \mathscr{K}(\varphi(x,t))\to\mathscr{K}(f(x))=\mathscr{L}(x),$$

we obtain morphisms $(g, \{g_x\}_{x\in X})$ and $(\varphi, \{\varphi_{(x,t)}\}_{(x,t)\in X\times[0,1]})$ from local systems $\mathscr{L}$ and $\bar{\mathscr{L}}$ induced on $X\times[0,1]$ by the projection $p_1\colon X\times[0,1]\to X$ to the local system $\mathscr{K}$

In order to prove this fact it is sufficient to prove that paths $c_1, c_2\colon [0,1]\to Y$ defined by the formulas

$$c_1(u)=\begin{cases}\varphi(d_1(0), 2ud_2(0)) & \text{for} \quad 0\leqslant u\leqslant\frac{1}{2},\\ \varphi(d_1(2u-1), d_2(2u-1)) & \text{for} \quad \frac{1}{2}\leqslant u\leqslant 1\end{cases}$$

and

$$c_2(u)=\begin{cases}f(d_1(2u))=\varphi(d_1(2u),0) & \text{for} \quad 0\leqslant u\leqslant\frac{1}{2},\\ \varphi(d_1(1),(2u-1)d_2(1)) & \text{for} \quad \frac{1}{2}\leqslant u\leqslant 1\end{cases}$$

are homotopic for every path $d_1\colon [0,1]\to X$ and every map $d_2\colon [0,1]\to[0,1]$ (we have $d(t)=(d_1(t), d_2(t))\in X\times[0,1]$ for every path $d\colon [0,1]\to X\times[0,1]$, where $d_1\colon [0,1]\to X$ and $d_2\colon [0,1]\to[0,1]$ are continuous functions).

Let $\psi\colon [0,1]\times[0,1]\to Y$ and $b_i\colon [0,1]\to[0,1]\times[0,1]$ for $i=1,2$ be maps given by

$$\psi(s,t)=\varphi(d_1(s), td_2(s)) \quad \text{for} \quad (s,t)\in[0,1]\times[0,1]$$

and

$$b_1(u) = \begin{cases} (0, 2u) & \text{for} \quad 0 \leqslant u \leqslant \frac{1}{2}, \\ (2u-1,1) & \text{for} \quad \frac{1}{2} \leqslant u \leqslant 1 \end{cases}$$

and

$$b_2(u) = \begin{cases} (2u, 0) & \text{for} \quad 0 \leqslant u \leqslant \frac{1}{2}, \\ (1, 2u-1) & \text{for} \quad \frac{1}{2} \leqslant u \leqslant 1. \end{cases}$$

Because $c_1 = \psi b_1$ and $c_2 = \psi b_2$, we infer that c_1 and c_2 are equivalent.

It is known that the relation of the homotopy of morphisms of local systems is reflexive, symmetric and transitive, and that the homotopy class of the composition of two morphisms $(f, \{f_x\}_{x \in X})$ and $(g, \{g_y\}_{y \in Y})$ depends only on their homotopy classes.

Therefore we obtain a category if we consider the homotopy classes of morphisms of local systems as morphisms and local systems as objects.

2. Cohomology with coefficients in local systems. By a *finite n-dimensional cell complex K* we understand an n-dimensional compactum $|K|$ along with finite families $\mathscr{A}(i) = \{|\sigma_j^i|\}_{j=1,2,\ldots,s(i)}$ of i-dimensional subcontinua, where $0 \leqslant i \leqslant n$, satisfying the conditions

(a) $|\sigma_j^i|$ is an i-dimensional cell (i.e. a homeomorph of the closed i-simplex).

(b) $|\sigma_i^p| \cap |K^{(p)}|$ is the boundary $\partial|\sigma_i^p|$ of $|\sigma_i^p|$ (i.e. the subset of $|\sigma_i^p|$ which corresponds to the boundary of the p-simplex) and it is an exact union of $(p-1)$-cells of $\mathscr{A}(p-1)$ called faces of $|\sigma_i^p|$, where $|K^{(q)}| = \bigcup_{i=0}^{q} \bigcup_{j=1}^{s(i)} |\sigma_j^i|$.

(c) $|K^{(n)}| = |K|$.

(d) The interiors $|\sigma_i^p| \backslash \partial|\sigma_i^p|$ and $|\sigma_j^p| \backslash \partial|\sigma_j^p|$ of $|\sigma_i^p|$ and $|\sigma_j^p|$ are disjoint if $i \neq j$ and $p = 0, 1, \ldots, n$.

A subcomplex L of K consists a subspace $|L|$ of $|K|$ and subcolections $\mathscr{B}(i) \subset \mathscr{A}(i)$ such that Conditions (a)-(d) are satisfied if we replace $|K|$ by $|L|$ and $\mathscr{A}(i)$ by $\mathscr{B}(i)$ and n by $\dim |L|$.

(2.1) Example. If K is a finite cell complex, then $K^{(p)}$ together with $\mathscr{A}(i)$ (where $0 \leqslant i \leqslant p$) form a subcomplex $K^{(p)}$ of K.

(2.2) Remark. Each cell complex is a finite CW complex and one easily checks that every simplicial complex is a finite cell complex.

(2.3) Example. If K_1 and K_2 are cell complexes, then $|K_1| \times |K_2|$ and the Cartesian product of cells of K_1 and K_2 form a cell complex $K_1 \times K_2$.

We say that a cell complex K' is a subdivision ([St$_3$], p. 161) of a cell complex K iff $|K'| = |K|$ and each cell of K is the union of the cells of K' which it contains.

(2.4) Remark. It is known ([St$_3$], p. 162) that for every cell complex K there exists a subdivision K' such that K' is a simplicial complex.

When no confusion can arise we shall abbreviate $|K|$ to K and $|K^{(p)}|$ to $K^{(p)}$.

In the same way as for simplicial complexes, one can introduce the notions of an oriented q-cell and an incidence number and construct a chain complex $\{C_q(K;G):\partial_q\}$ and the groups $H_q(K;G)$ and $H^q(K;G)$.

Oriented cells will be denoted by $\sigma, \tau, \ldots$ and non-oriented cells which are carriers of $\sigma, \tau, \ldots$ will be denoted (respectively) by $|\sigma|, |\tau|, \ldots$

The incidence number of an oriented $(q+1)$-cell σ and its q-dimensional face τ will be denoted by $[\sigma;\tau] = \pm 1$.

It is the purpose of this section to introduce cohomology groups with coefficients in local system of abelian groups ([S], p. 282, [St$_2$], [St$_3$], [Hi-Wy]).

Let K be a finite cell complex with a local system $\mathscr{L}$ of abelian groups on K and let us choose a point $x_\sigma \in |\sigma|$ for every q-cell $|\sigma|$.

An abelian group $C^q(K;\mathscr{L}), 0 \leqslant q \leqslant n = \dim K$, is the set of all functions c assigning to each oriented q-cell σ an element $c(\sigma) \in \mathscr{L}(x_\sigma)$ such that $c(-\sigma) = -c(\sigma)$ with addition given by the formula

$$(c_1+c_2)(\sigma) = c_1(\sigma)+c_2(\sigma).$$

We set $C^q(K;\mathscr{L}) = 0$ if $q > n$.

Elements of $C^q(K;\mathscr{L})$ are called *q-cochains*.

For oriented q-cell σ and $g \in \mathscr{L}(x_\sigma)$ we shall denote by $\chi_{\sigma,g}$ a cochain (called an *elementary q-cochain*) defined by the formula

$$\chi_{\sigma,g}(\tau) = \begin{cases} -g & \text{when} \quad \sigma = -\tau, \\ g & \text{when} \quad \sigma = \tau, \\ 0 & \text{when} \quad \sigma \neq \tau, -\tau. \end{cases}$$

Any cochain is the sum of elementary cochains.

Hence, if we assign to every elementary q-cochain $\chi_{\sigma,g}$ a $(q+1)$-cochain $\delta\chi_{\sigma,g}$ defined by the formulas

$\delta\chi_{\sigma,g}(\tau) = [\tau;\sigma]\mathscr{L}(d)(g)$ if σ is a face of oriented $(q+1)$-cell τ and $d\colon [0, 1] \to |\tau|$ is a path from x_τ to x_σ,

and

$\delta\chi_{\sigma,g}(\tau) = 0$ for every $(q+1)$-cell τ such that σ is not a face of τ,

then we obtain a homomorphism $\delta_q\colon C^q(K;\mathscr{L}) \to C^{q+1}(K;\mathscr{L})$.

We set $\delta_q = 0$ if $q \geqslant n$.

Let us suppose that E is a contractible subcomplex of K and $c \in C^q(K;\mathscr{L})$ and $\chi = n_1\tau_1 + \quad + n_k\tau_k \in C_q(K;Z)$, where $n_i \in Z$ and τ_i is an oriented q-cell of E (and K). Clearly, $\mathscr{L}|_E$ is a simple system on E. This means that an element

$$c(\chi)_{x_0} = \sum_{i=1}^{k} \mathscr{L}(d_i)(c(\tau_i)) \in \mathscr{L}(x_0)$$

does not depend on the choice of paths d_i: $[0, 1] \to E$ from x_0 to x_{τ_i}, where $i = 1, 2, \dots, k$.

The element $c(\chi)_{x_0}$ will be called the *index of* c *and* χ in x_0 (cf. [St$_3$], p. 158).

Using this convention, we can write (cf. [St$_3$], p. 158)

(2.5) $$(\delta c)(\sigma) = c(\partial\sigma)_{x_\sigma}.$$

Let $Z^q(K; \mathscr{L}) = \operatorname{Ker} \delta_q$ and $B^{q+1}(K; \mathscr{L}) = \operatorname{Im} \delta_q$ for $q \geqslant 0$ and $B^0(K; \mathscr{L}) = 0$.

The groups $B^q(K; \mathscr{L})$ and $Z^q(K; \mathscr{L})$ are called (respectively) the *group of q-coboundaries* and the *group of q-cocycles.*

Since $\delta_{q+1}\delta_q = 0$, we infer that $Z^q(K; \mathscr{L}) \supset B^q(K; \mathscr{L})$.

We define the *q-dimensional cohomology group of K with coefficients in* $\mathscr{L}$ by the formula

$$H^q(K; \mathscr{L}) = Z^q(K; \mathscr{L})/B^q(K; \mathscr{L}).$$

It is known that $H^n(K; \mathscr{L})$ does not depend on the decomposition of K into cells and the choice of $x_\sigma \in |\sigma|$.

If cocycles c_1 and c_2 represent the same element of $H^q(K; \mathscr{L})$, then we say that c_1 and c_2 are *cohomologous* and we write $c_1 \sim c_2$.

Let K_1 and K_2 be cell complexes and let f: $K_1 \to K_2$ be a continuous function satisfying the following conditions:

(2.6) $$f(K_1^{(p)}) \subset K_2^{(p)} \quad \text{for every } p = 0, 1, 2, \dots$$

(2.7) If $|\sigma|$ is a cell of K_1, then the smallest subcomplex E_σ of K_2 containing $f(|\sigma|)$ is contractible.

Then f is said to be *admissible.*

(2.8) Remark. If f: $K_1 \to K_2$ is a simplicial map from a polyhedron K_1 to a polyhedron K_2, then f is admissible. The Cartesian product of admissible maps is an admissible map. If K' is a subdivision of K, then $\mathrm{id}_{|K|}$: $|K| \to |K'| = |K|$ is admissible.

Let $(f, \{f_x\}_{x\in K})$ be a morphism from a local system $\mathscr{L}_1$ on K_1 to a local system $\mathscr{L}_2$ on K_2 such that f satisfies (2.6) and (2.7) (i.e. f is admissible). Then for every $q = 0, 1, 2, \dots$ there exists ([St$_3$], p. 159) a chain transformation h_f: $\{C_q(K_1; Z)\} \to \{C_q(K_2; Z)\}$ such that h_f carries a vertex into a vertex and $h_f(\sigma) \in C_q(E_\sigma; Z)$ (we can regard $C_q(E_\sigma; Z)$ as a subgroup of $C_q(K_2; Z)$).

Let $c \in C^q(K_2; \mathscr{L}_2)$ and τ be an oriented q-cell of K. Then $g = c(h_f(\tau))_{f(x_\tau)} \in \mathscr{L}_2(f(x_\tau))$ and $f_{x_\tau}(g) = g_\tau \in \mathscr{L}_1(x_\tau)$.

Setting $\bar{f}_q c(\tau) = g_\tau$, we obtain a homomorphism $\bar{f}_q$: $C^q(K_2; \mathscr{L}_2) \to C^q(K_1; \mathscr{L}_1)$.

It is known that $\delta_q \bar{f}_q = \bar{f}_{q+1}\delta_q$.

Therefore f_q induces a homomorphism $f^*\colon H^q(K_2; \mathscr{L}_2) \to H^q(K_1; \mathscr{L}_1)$ which does not depend on the choice of h_f and the decompositions K_1 and K_2 into cells.

Moreover, it is well known that one can extend this definition of f^* onto an arbitrary morphism $(f, \{f_x\}_{x\in K_1})$ of local systems and that for every $n \geqslant 0$ there exists a contravariant functor from the category of local systems of groups on cell complexes and the homotopy classes of morphisms of local systems to the category of abelian groups and homomorphisms which assign to every local system $\mathscr{L}$ on K the group $H^n(K; \mathscr{L})$ and to every homotopy class of morphisms $(f, \{f_x\}_{x\in K})$ from $\mathscr{L}_1$ on K_1 to $\mathscr{L}_2$ on K_2 a homomorphism $f^*\colon H^n(K_2; \mathscr{L}_2) \to H^n(K_1; \mathscr{L}_1)$.

(2.9) Remark. The above-mentioned functor coincides with the functor of the ordinary cohomology in the case where all systems are simple (if $\mathscr{L}$ is a simple system and $x \in K$, then $H^q(K; \mathscr{L}(x)) = H^q(K; \mathscr{L})$ for $q = 0, 1, 2, \ldots$).

The following proposition will be very useful:

(2.10) Proposition. *Suppose that a local system $\mathscr{K}$ is induced on a cell complex K by an admissible map $f\colon K \to L$ and a local system $\mathscr{L}$ on a cell complex L and $\omega(f) \leqslant n$. Then a homomorphism $f_q^*\colon H^q(L; \mathscr{L}) \to H^q(K; \mathscr{K})$ induced by f is trivial for every $n < q$.*

Proof. Using Remark (2.4) and the simplicial-approximation theorem, we find that without loss of generality one can assume that K and L are polyhedra and that there exists a simplicial map $g\colon K \to L$ such that $g \simeq f$ and $g(K) \subset L^{(n)}$. Then g induces (see Example (1.5)) a homomorphism $g_q^*\colon H^q(L; \mathscr{L}) \to H^q(K; \mathscr{K})$ such that $g_q^* = f_q^*$. Since $g(K) \subset L^{(n)}$, we conclude from the definition of g_q^* that $g_q^* = 0$ if $q > n$. The proof is finished.

3. The Künneth formula. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be local systems of groups on cell complexes K_1 and K_2 (respectively). Then we denote by $\mathscr{L}_1 \otimes \mathscr{L}_2$ a local system of groups on $K_1 \times K_2$ defined by the formulas

$$\mathscr{L}_1 \otimes \mathscr{L}_2(x_1, x_2) = \mathscr{L}_1(x_1) \otimes \mathscr{L}_2(x_2)$$

and

$$\mathscr{L}_1 \otimes \mathscr{L}_2(a) = \mathscr{L}_1(p_1 a) \otimes \mathscr{L}_2(p_2 a),$$

where $(x_1, x_2) \in K_1 \times K_2$, $a\colon [0, 1] \to K_1 \times K_2$ is a path and $p_i\colon K_1 \times K_2 \to K_i$ is the Cartesian projection for $i = 1, 2$. If $(f, \{f_x\}_{x\in K_1})$ and $(g, \{g_y\}_{y\in K_2})$ are morphisms from a local system $\mathscr{L}$ on K_1 to a local system $\mathscr{L}'$ on K_1' and from a local system $\mathscr{K}$ on K_2 to a local system $\mathscr{K}'$ on K_2' (respectively), then $(f, \{f_x\}_{x\in K_1}) \otimes (g, \{g_y\}_{y\in K_2}) = (f \times g, \{f_x \otimes g_y\})$ is a morphism from $\mathscr{L} \otimes \mathscr{K}$ to $\mathscr{L}' \otimes \mathscr{K}'$

It is well known that on the category of cell complexes and admissible maps there exists a natural chain equivalence of the functor $\{C_q(K_1 \times K_2; Z)\}$

with the functor $\{C_q(K_1; Z)\} \otimes \{C_q(K_2; Z)\}$. The image of a chain $c_1 \otimes c_2 \in C_q(K_1; Z) \otimes C_p(K_2; Z)$ under this equivalence is denoted by $c_1 \times c_2 \in C_{q+p}(K_1 \times K_2; Z)$. If σ is an oriented p-cell of K_1 and τ is an oriented q-cell of K_2, then $\sigma \times \tau \in C_{p+q}(K_1 \times K_2; Z)$ is a chain concentrated on the cell $|\sigma| \times |\tau|$. Therefore, one can regard $\sigma \times \tau$ as an oriented $(p+q)$-cell of $K_1 \times K_2$ with the carrier $|\sigma| \times |\tau|$.

The following formula will be employed:

$$\partial(\sigma \times \tau) = \partial\sigma \times \tau + (-1)^p \sigma \times \partial\tau, \tag{3.1}$$

where σ is an oriented p-cell of K_1 and τ is an oriented q-cell of K_2.

We must extend these results to cochain complexes $\{C^q(K_1; \mathscr{L}_1)\} \otimes \otimes \{C^p(K_2; \mathscr{L}_2)\}$ and $\{C^q(K_1 \times K_2; \mathscr{L}_1 \otimes \mathscr{L}_2)\}$.

Suppose that $c_1 \in C^p(K_1; \mathscr{L}_1)$ and $c_2 \in C^q(K_2; \mathscr{L}_2)$ and let us denote by $c_1 \times c_2$ a cochain of $C^{p+q}(K_1 \times K_2; \mathscr{L}_1 \otimes \mathscr{L}_2)$ given by the formula

$$c_1 \times c_2(\gamma) = \begin{cases} c_1(\sigma) \otimes c_2(\tau) & \text{if } \gamma = \sigma \times \tau, \text{ where } \sigma \text{ is an oriented } p\text{-cell of } K_1 \text{ and } \tau \text{ is an oriented } q\text{-cell of } K_2, \\ 0 & \text{if } \gamma = \sigma \times \tau, \text{ where } \dim|\sigma| \neq p. \end{cases}$$

Using repeatedly (2.5) and (3.1), we can prove that $\delta(c_1 \times c_2)(\sigma \times \tau) = \delta c_1(\sigma) \otimes c_2(\tau) + (-1)^p c_1(\sigma) \otimes \delta c_2(\tau)$ for every p-cell σ and q-cell τ (cf. [St$_3$], p. 170).

Hence

$$\delta(c_1 \times c_2) = \delta c_1 \times c_2 + (-1)^p c_1 \times \delta c_2,$$

and we can identify the tensor product of the cochains complexes $\{C^q(K_1; \mathscr{L}_1)\}$ and $\{C^q(K_2; \mathscr{L}_2)\}$ with the cochain complex $\{C^q(K_1 \times K_2; \mathscr{L}_1 \otimes \mathscr{L}_2)\}$. This "identification" has a functorial character (with respect to the category of admissible morphisms of local systems and cell complexes).

The Künneth formula for cochain complexes ([S], p. 237, Theorem 2) and the above facts imply the following

(3.2) THEOREM. *If $\mathscr{L}_i$ is a local system of groups on a cell complex K_i for $i = 1, 2$ and $\mathscr{L}_2$ is a system of free abelian groups, then there is a functorial short exact sequence*

$$0 \to \sum_{p+q=n} H^q(K_1; \mathscr{L}_1) \otimes H^p(K_2; \mathscr{L}_2) \to H^n(K_1 \times K_2; \mathscr{L}_1 \otimes \mathscr{L}_2)$$
$$\to \sum_{p+q=n+1} \operatorname{Tor}\left(H^q(K_1; \mathscr{L}_1), H^p(K_2; \mathscr{L}_2)\right) \to 0.$$

4. Generalized local systems. Let X be a continuum and let $\{X_k, p_k^{k+1}\}$ be an inverse sequence of connected finite cell complexes and admissible maps such that

$$\operatorname{Sh}(X) = \operatorname{Sh}(\varprojlim \{X_k, p_k^{k+1}\})$$

and assume that for every $k = 1, 2, \ldots$ on X_k we have a local system of abelian groups $\mathscr{L}_k$. If for every k the local system $\mathscr{L}_{k+1}$ is induced on X_{k+1} by the map $p_k^{k+1}\colon X_{k+1} \to X_k$ and the local system $\mathscr{L}_k$, then the pair $\underline{\mathscr{L}} = (\{X_k, p_k^{k+1}\}, \mathscr{L}_k)$ is called a *generalized local system of abelian groups on* X.

If $\mathscr{L}_1$ is a system of free abelian groups on X_1 (this implies that $\mathscr{L}_k$ is a system of free groups on X_k for $k = 1, 2, \ldots$), then $\underline{\mathscr{L}}$ is called a *generalized local system of free abelian groups on* X.

If $\mathscr{L}$ is a local system on a connected cell complex, then we identify $\mathscr{L}$ with a generalized local system $(\{X, \mathrm{id}_X\}, \mathscr{L})$.

If $\underline{\mathscr{L}} = (\{X_k, p_k^{k+1}\}, \mathscr{L}_k)$ is a generalized local system of abelian groups on a continuum X, the map $p_k^{k+1}\colon X_{k+1} \to X_k$ induces a homomorphism $(p_k^{k+1})^*\colon H^n(X_k; \mathscr{L}_k) \to H^n(X_{k+1}; \mathscr{L}_{k+1})$ and it is easily seen that $\{H^n(X_k; \mathscr{L}_k), (p_k^{k+1})^*\}$ is a direct sequence of abelian groups. Its direct limit will be denoted by $H^n(X; \underline{\mathscr{L}})$ and will be called an *n-dimensional cohomology group of X with coefficients* in $\underline{\mathscr{L}}$.

If $\mathscr{L}_k$ is a simple local system on X_k for almost all k, then $H^n(X; \underline{\mathscr{L}})$ is canonically isomorphic with the ordinary Čech group $H^n(X; \mathscr{L}(x_0))$, where $x_0 \in X$.

Therefore, we can regard the cohomology groups of X with coefficients in generalized local systems as a natural generalization of the Čech cohomology groups.

Let $(\{X_k, p_k^{k+1}\}, \mathscr{L}_k) = \underline{\mathscr{L}}$ and $(\{Y_k, q_k^{k+1}\}, \mathscr{K}_k) = \underline{\mathscr{K}}$ be generalized local systems of abelian groups on continua X and Y (respectively).

Then $(\{X_k \times Y_k, p_k^{k+1} \times q_k^{k+1}\}, \mathscr{L}_k \otimes \mathscr{K}_k) = \underline{\mathscr{L}} \otimes \underline{\mathscr{K}}$ is a generalized local system of groups on $X \times Y$.

If we additionally assume that $\underline{\mathscr{K}}$ is a generalized local system of free abelian groups, then we have a short functorial exact sequence

$$0 \to \sum_{p+q=n} H^p(X_k; \mathscr{L}_k) \otimes H^q(Y_k; \mathscr{K}_k) \to H^{p+q}(X_k \times Y_k; \mathscr{L}_k \otimes \mathscr{K}_k) \to \sum_{p+q=n+1} \operatorname{Tor}\big(H^p(X_k; \mathscr{L}_k), H^q(Y_k; \mathscr{K}_k)\big) \to 0$$

for every $k = 1, 2, \ldots$

Because the tensor product and the torsion product commute with the direct limit, we have the following

(4.1) THEOREM. *Let* $(\{X_k, p_k^{k+1}\}, \mathscr{L}_k) = \underline{\mathscr{L}}$ *and* $(\{Y_k, q_k^{k+1}\}, \mathscr{K}_k) = \underline{\mathscr{K}}$ *be generalized local systems of abelian groups on* X *and* Y *(respectively). If* $\underline{\mathscr{K}}$ *is a generalized local system of free groups, then we obtain the following exact sequence*:

$$0 \to \sum_{p+q=n} H^p(X; \underline{\mathscr{L}}) \otimes H^q(Y; \underline{\mathscr{K}}) \to H^n(X \times Y; \underline{\mathscr{L}} \otimes \underline{\mathscr{K}}) \to \sum_{p+q=n+1} \operatorname{Tor}\big(H^p(X; \underline{\mathscr{L}}), H^q(Y; \underline{\mathscr{K}})\big) \to 0.$$

Chapter III

Homological characterizations of fundamental dimension

This chapter contains the main theorems of our paper, which state (roughly speaking) that Fd (X) is equal to the smallest integer number n such that X is acyclic in the sense of cohomology over generalized local systems of abelian groups on X.

In the case where Fd $(X) > 2$ the proof of this fact is based on the obstruction theory and it is presented in the first four sections.

In fact, we give two independent proofs for the case Fd $(X) = 1$. The first is contained in this chapter. The second proof is longer and more complicated, but at the same time richer, since we investigate additional algebraic properties of $H^1(X; \underline{\mathscr{L}}) \neq 0$. To pick out it the reader should read Lemma (5.7) and Section 2 of the next Chapter, avoiding (5.1) and Section 6.

The problem of finding an algebraic characterization of continua with the fundamental dimension equal to 2 has been solved only for some special cases and it is presented in Sections 7 and 8.

We introduce also some class of continua $\mathscr{F}$ containing all orientable closed manifolds and we prove that Fd $(X \times Y) =$ Fd $(X)+$Fd (Y) if $Y \in \mathscr{F}$ and Fd $(X) \neq 2$ or Fd $(X) = 2$ and X is not approximatively 2-connected.

This yields a partial answer to Borsuk's problem ([B$_2$] and [B$_3$], p. 350) whether Fd $(X \times S^n) =$ Fd $(X)+n$ for $X \neq \emptyset$.

For all CW complexes X, Y and a map $f\colon (X, x_0) \to (Y, y_0)$ ($x_0 \in X$ and $y_0 \in Y$) we will denote by $\tilde{X}, X^{(n)}, X^n, \Sigma(X), \Sigma(f), f_\#$ and $f_*^s (s \geqslant 2)$ (respectively) the universal covering space of X, the n-skeleton of X, the Cartesian power $\underbrace{X \times X \times X \times \ldots \times X}_{n \text{ times}}$, the suspension of X, the suspension of f and the homomorphisms $f_\#\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $f_*^s\colon \pi_s(X, x_0) \to \pi_s(Y, y_0)$ induced by f.

In this and the next chapters Z, Z_p and Q denote (respectively) the group of integer numbers, the cyclic group of finite order p and the group of rational numbers.

1. Deformability of maps and the number $\omega(f)$. Let $Y, Y_0 \in \mathrm{ANR}$ be continua such that $Y \supset Y_0$ and assume that K is a finite polyhedron and X is a compactum.

A map $f: K \to Y$ is said to be *n-normal relative to* Y_0 ([Hu$_2$], p. 107 and [Hu$_3$], p. 202) iff $f(K^{(n-1)}) \subset Y_0$.

We say also that a map $f: X \to Y$ is *n-deformable into* Y_0 ([Hu$_3$], p. 211) iff there exist a $(n+1)$-normal map $\xi: W \to Y$ from a polyhedron W to Y and a map $\alpha: X \to W$ such that $\xi\alpha \simeq f$. In the case where X is a polyhedron n-deformability into Y_0 of a map $g: X \to Y$ is equivalent to the existence of a map $h: X \to Y$ $(n+1)$-normal relative to Y_0, which is homotopic with g.

Let us prove the following

(1.1) PROPOSITION. *Let $\{X_k, p_k^{k+1}\}$ be an inverse sequence of polyhedra associated with a continuum X and let $f: X \to Y$ be a map from X to $Y \in \mathrm{ANR}$ and $Y \supset Y_0 \in \mathrm{ANR}$. Then f is n-deformable into Y_0 iff there exist a natural number k and a map $\eta: X_k \to Y$, $(n+1)$-normal relative to Y_0 and such that $\eta p_k \simeq f$.*

Proof. Suppose that f is n-deformable into Y_0. This means that there exist a polyhedron W and continuous functions $\alpha: X \to W$ and $\xi: W \to Y$ such that $\xi(W^{(n)}) \subset Y_0$ and $\xi\alpha \simeq f$.

Elementary properties of inverse sequences of polyhedra imply that there are a natural number k and a PL map $\beta: X_k \to W$ such that $\beta p_k \simeq \alpha$ and $\beta(X_k^{(n)}) \subset W^{(n)}$.

This implies that $\eta \simeq \xi\beta$ satisfies the required conditions. The proof of Proposition (1.1) is finished.

The following proposition will be very useful:

(1.2) PROPOSITION. *Suppose that X is a movable continuum and $f: X \to Y$ is a map from X to a polyhedron Y. If n is a natural number such that f is m-deformable into $Y^{(n)}$ for every $m = 1, 2, \ldots$, then $\omega(f) \leqslant n$.*

Proof. We can assume that X is the inverse limit of an inverse sequence $\{X_k, p_k^{k+1}\}$ of polyhedra such that

(1.3) For every $k \geqslant 2$ there exists a map $r: X_2 \to X_k$ satisfying $p_1^k r \simeq p_1^2$ and there exists a map $f_1: X \to Y$ satisfying $f_1 p_1 \simeq p$.

Let $\dim X_2 = m$.

The assumptions and Proposition (1.1) imply that there are $k \geqslant 2$ and a continuous function $q: X_k \to X_1$ which is homotopic with p_1^k and satisfies the following condition:

$$f_1 q(X_k^{(m)}) \subset Y^{(n)}.$$

We have

$$f_1 p_1^k r p_2 \simeq f \simeq f_1 q r p_2.$$

We can also assume that the map $r\colon X_2 \to X_k$ (see (1.3)) is a PL map and $r(X_2^{(m)}) = r(X_2) \subset X_k^{(n)}$.

We obtain

$$f_1 qrp_2(X) \subset f_1 q(X_k^{(m)}) \subset Y^{(n)}$$

and

$$\omega(f) \leqslant n.$$

This completes the proof of our proposition.

Using Theorem (1.3) of Chapter I and Proposition (1.1), one can prove that the following theorem holds:

(1.4) PROPOSITION. *Suppose that X is a continuum with* $\operatorname{Fd}(X) < \infty$ *and $f\colon X \to Y$ is a map, where Y is a polyhedron. If n is a natural number such that f is m-deformable into $Y^{(n)}$ for every $m = 1, 2, \ldots$, then $\omega(f) \leqslant n$.*

2. Obstructions to deformability. Suppose that a map $f\colon K \to Y$ from a polyhedron K to a continuum $Y \in \mathrm{ANR}$ is n-normal relative to a subcontinuum $Y_0 \in \mathrm{ANR}$ of Y, where $n \geqslant 3$. Then $\Pi_n(Y, Y_0)$ and a map $g\colon K^{(n-1)} \to Y_0$, which is defined by

$$g(x) = f(x) \quad \text{for} \quad x \in K^{(n-1)},$$

induce a local system $\mathscr{K}$ of abelian groups on $K^{(n-1)}$.

Since $n \geqslant 3$, we see that there exists a local system of abelian groups $\mathscr{L}(f, n)$ on K such that

$$\mathscr{L}(f, n)|_{K^{(n-1)}} = \mathscr{K}$$

For every q-simplex σ of K, we choose a point $x_\sigma \in |\sigma|$ in such a way that x_σ belongs to the boundary $\partial|\sigma|$ of $|\partial|$ for $q \geqslant 1$ (cf. Section 2 of Chapter II).

If we assign to every oriented n-simplex σ of K the unique element (f, σ) of the group $\pi_n(Y, Y_0, f(x_\sigma))$ determined by the partial map

$$f|_{|\sigma|}\colon (\sigma, \partial\sigma, x_\sigma) \to (Y, Y_0, f(x_\sigma)),$$

we obtain an n-cochain $c^n(f)$ of K with coefficients in $\mathscr{L}(f, n)$ (cf. Exercise E, Chapter VI of [Hu$_2$] and [Hu$_3$], p. 203). These notations will be used in the sequel.

The following theorem will be very useful.

(2.1) THEOREM. *Let $n \geqslant 3$ and $Y, Y_0 \in \mathrm{ANR}$ be two continua and $Y \supset Y_0$. For every n-normal map $f\colon K \to Y$ the cochain $c^n(f)$ is a cocycle of K and its cohomology class $[c^n(f)]$ in $H^n(K; \mathscr{L}(f, n))$ is equal to 0 iff there exists a homotopy $\varphi\colon K \times [0, 1] \to Y$ such that*

$$\varphi(x, 0) = f(x) \quad \textit{for} \quad (x, t) \in K \times \{0\} \cup K^{(n-2)} \times [0, 1]$$

and

$$\varphi(K^{(n)} \times \{1\}) \subset Y_0.$$

Proof. The proof of this theorem is a straightforward modification of the proofs of analogous facts for the case where the pair (Y, Y_0) is n-simple (Exercise E-7, Chapter VI of [Hu$_2$], [Hu$_1$] and [Hu$_3$], p. 203).

One can check that the following proposition holds:

(2.2) PROPOSITION. *Suppose that a map $f\colon K \to Y$ from a polyhedron K to a continuum $Y \in$ ANR is n-normal relative to its subcontinuum $Y_0 \in$ ANR and that $g\colon L \to K$ is a simplicial map of a polyhedron L. Then a local system $\mathscr{L}(fg, n)$ is induced by g and $\mathscr{L}(f, n)$ and $g^*([c^n(f)]) = [c^n(fg)] \in H^n(L; \mathscr{L}(fg, n))$ where g^* is induced by g.*

3. Coefficients of cyclicity and $\mathscr{F}$-continua. If X is a non-empty compactum, then we say that a *coefficient of cyclicity of X with respect to an abelian group G* (denoted by $c_G(X)$) is equal to n (where n is an integer number) if $H^m(X; G) = 0$ for all $m > n$ and $H^n(X; G) \neq 0$. Moreover, we set $c_G(\emptyset) = -1$ and $c_G(X) = \infty$ if $X \neq \emptyset$ and for every m there is an $n \geqslant m$ with $H^n(X; G) \neq 0$.

In the sequel the coefficient of cyclicity of X with respect to the group of integer numbers Z is denoted by $c(X)$.

Remark. Let $\mathscr{R}_1$ denote the group of real numbers modulo 1. It is well known ([H-W], p. 137 and p. 124) that for every compactum X the group $H_n(X; \mathscr{R}_1)$ is the character group of $H^n(X; Z)$. This implies that $\max(m;\ H^m(X; Z) \neq 0) = \max(m;\ H_m(X; \mathscr{R}_1) \neq 0) = c(X)$.

Remark. N. Steenrod has proved ([St$_1$], p. 690) that for each abelian group G and every compactum X the group $H_n(X; G)$ is the direct sum of two groups, one determined uniquely by G and $H_n(X; \mathscr{R}_1)$, the other by $H_{n+1}(X; \mathscr{R}_1)$. These groups are trivial if $H_{n+1}(X; \mathscr{R}_1) = H_n(X; \mathscr{R}_1) = 0$. Therefore for every compactum X and each abelian group G and every natural number $m > c(X)$ the group $H_m(X; G)$ is trivial.

Remark. From the universal-coefficient formula for the Čech cohomology ([S], p. 336) we infer that for every compactum X and every $m > c(X)$ the group $H^m(X; G)$ is trivial for all G.

In this section we denote by Q the group of rational numbers and for every prime natural number p we denote by R_p the group of rational numbers which can be represent in the form m/n, where m and n are integer numbers and p does not divide n. Let us also put $Q_p = Q/R_p$ for every prime number p.

If G is an abelian group, then we denote by $\sigma(G)$ the collection of abelian groups defined by the following conditions:

(a) $Q \in \sigma(G)$ iff G contains an element of infinite order.

(b) If p is a prime number, then $Z_p \in \sigma(G)$ iff G contains an element g of order p^k (where k is a natural number) such that g is not divisible by p.

(c) $Q_p \in \sigma(G)$ iff G contains an element of order p.

(d) $R_p \in \sigma(G)$ iff there is an element a of G such that for every integer number n the number p^{n+1} does not divide $p^n a$.

(e) If $H \neq Q, Q_p, Z_p, R_p$, then H does not belong to $\sigma(G)$ (p is the prime number).

In the next chapter we need the following three propositions:

(3.1) PROPOSITION. *For every compactum X and every abelian group G the equality $c_G(X) = \max_{H \in \sigma(G)} \{c_H(X)\}$ holds true.*

(3.2) PROPOSITION. *If X is a compactum and p is a prime number, then*

$$\begin{aligned} c_{Q_p}(X) &\leqslant c_{Z_p}(X) \leqslant c_{Q_p}(X)+1, \\ c_Q(X) &\leqslant c_{R_p}(X), \\ c_{Q_p}(X) &\leqslant \max\left(c_Q(X), c_{R_p}(X)-1\right), \\ c_{R_p}(X) &\leqslant \max\left(c_Q(X), c_{Q_p}(X)+1\right). \end{aligned}$$

(3.3) PROPOSITION. *Let X, Y be compacta and let p be a prime number. Then*

$$\begin{aligned} c_{Z_p}(X\times Y) &= c_{Z_p}(X)+c_{Z_p}(Y), \\ c_Q(X\times Y) &= c_Q(X)+c_Q(Y), \\ c_{Q_p}(X\times Y) &= \max\left(c_{Q_p}(X)+c_{Q_p}(Y), c_{Z_p}(X\times Y)-1\right), \end{aligned}$$

$$c_{R_p}(X\times Y) = \begin{cases} c_{R_p}(X)+c_{R_p}(Y) & \text{if } c_{Q_p}(X) = c_{R_p}(X) \text{ or } c_{Q_p}(Y) = c_{R_p}(Y), \\ \max\left(c_{Q_p}(X\times Y)+1, c_Q(X\times Y)\right) & \text{if } c_{Q_p}(X) < c_{R_p}(X) \\ & \text{and } c_{Q_p}(Y) < c_{R_p}(Y). \end{cases}$$

It is known (cf. [N_2], p. 75) that if we replace in (3.1), (3.2) and (3.3) c_A by $\dim_A$ (where $A = G, H, Q, Z_p, R_p, Q_p$), then we get theorems which hold true ([Ko], p. 219 and p. 231 or [Ku], p. 12 and p. 14 and p. 15). Moreover, studying [Ko] or [Ku], one can observe that the proofs of those theorems contain the proofs of Propositions (3.1), (3.2) and (3.3). Therefore we omit them.

Using the exact Mayer-Vietoris cohomology sequence, one can prove the following

(3.4) PROPOSITION. *If X, Y are compacta and G is an abelian group and $c_G(X\cap Y) < \max\left(c_G(X), c_G(Y)\right)$, then $c_G(X\cup Y) = \max\left(c_G(X), c_G(Y)\right)$.*

A continuum X with $0 \neq \mathrm{Fd}(X) = n < \infty$ belongs to a class $\mathscr{F}$ (in other words X is a *$\mathscr{F}$-continuum*) iff $c_G(X) = n$ for every abelian group $G \neq 0$ (cf. [N_5]).

From the universal-coefficient theorem for homology and cohomology one can conclude that the following proposition holds:

(3.5) Proposition. *If* $X \in \mathrm{ANR}$ *is an n-dimensional continuum and* $H_n(X; Z) \neq 0$, *then* $X \in \mathscr{F}$

In particular $\mathscr{F}$ contains all closed orientable manifolds.

Using the Künneth formula or Proposition (3.4), one can obtain the following propositions:

(3.6) Proposition. *If* $X, Y \in \mathscr{F}$, *then* $X \times Y \in \mathscr{F}$

(3.7) Proposition. *If* $X \in \mathscr{F}$ *and* Y *is a continuum such that* $0 \leqslant \mathrm{Fd}(X \cap Y), \mathrm{Fd}(Y) < \mathrm{Fd}(X)$, *then* $X \cup Y \in \mathscr{F}$

Remark. Proposition (3.1) implies that if X is a continuum with $n = \mathrm{Fd}(X) < \infty$ and the groups $H^n(X; Q), H^n(X; Z_p), H^n(X; Q_p)$ and $H^n(X; R_p)$ are non-trivial for every prime number p, then $X \in \mathscr{F}$ This fact together with the theorem stating ([H-W], p. 137) that for every countable group G and its character group G^* the group $H_n(X; G^*)$ is the character group of $H^n(X; G)$ and with the Pontriagin duality ([P], p. 259) implies that if X is a n-dimensional continuum and $H_n(X; G) \neq 0$ for every $G \neq 0$, then $X \in \mathscr{F}$

If $X \neq \emptyset$ is a continuum, then let us denote by $c[X]$ the maximum of numbers n (finite or infinite) such that there is a generalized local system of coefficients $\underline{\mathscr{L}}$ on X such that $H^n(X; \underline{\mathscr{L}}) \neq 0$. We set also $c[\emptyset] = -1$. The number $c[X]$ will be called a *generalized coefficient of cyclicity* (cf. [N_4], p. 1025).

We will use the following

(3.8) Proposition. *If* X *is a continuum and* $Y \in \mathscr{F}$, *then* $c[X \times Y] \geqslant c[X] + \mathrm{Fd}(Y)$.

Proof. Let $\underline{\mathscr{L}} = (\{X_k, p_k^{k+1}\}, \mathscr{L}_k)$ be a generalized local system of abelian groups on X such that

$$H^m(X; \underline{\mathscr{L}}) \neq 0 \quad \text{where} \quad m = c[X]$$

and let $n = \mathrm{Fd}(Y)$.

The universal coefficients theorem for Čech cohomology and Theorem (4.1) of Chapter II imply that

$$H^{n+m}(X \times Y; \underline{\mathscr{L}} \otimes Z) \approx H^n(X; \underline{\mathscr{L}}) \otimes H^m(Y; Z) \approx H^m(Y; H^n(X; \underline{\mathscr{L}})) \neq 0.$$

Therefore $c[X \times Y] \geqslant m+n$ and the proof of (3.8) is finished.

From Proposition (1.9) of Chapter II and Remark (2.9) we conclude that the following proposition holds:

(3.9) Proposition. *For every approximatively* 1-*connected continuum, we have* $c[X] = c(X)$.

Using Proposition (2.10) of Chapter II and Theorem (1.3) of Chapter I, one can prove the following

(3.10) Proposition. *If* X *is a continuum, then* $c[X] \leqslant \mathrm{Fd}(X)$.

Remark. It is clear that $c(X)$ and $c[X]$ are shape invariants. In Section 8 we shall prove that there exists a sequence of polyhedra X_n such that $c(X_n) = 0$ and $c[X_n] = n$ for every $n = 1, 2, \ldots$

4. Continua with fundamental dimension $\geqslant 3$. Let us prove the following

(4.1) THEOREM. *Let $f: X \to Y$ be a map from a continuum X to a polyhedron Y. If $n = c[X] < \infty$ and $n_0 = \max(2, n)$, then f is m-deformable into $Y^{(n_0)}$ for every $m = 1, 2, \ldots$*

Proof. Let $Y_0 = Y^{(n_0)}$ and $\{X_k; p_k^{k+1}\}$ be an inverse sequence of polyhedra and simplicial maps associated with X. Then there exist a natural number $k(n_0)$ and a cellular map $f_{k(n_0)}: X_{k(n_0)} \to Y$ such that $f_{k(n_0)} p_{k(n_0)} \simeq f$.

Since $f_{k(n_0)}$ is cellular, we have $f_{k(n_0)}(X^{(n_0)}) \subset Y_0$ and f is m-normal relative to Y_0 for every $m \leqslant n_0+1$.

In order to prove that our theorem holds it is sufficient to show that

(4.2) if $m > n_0$ and we have a map $f_{k(m)}: X_{k(m)} \to Y$ which is m-normal relative to Y_0 and $f_{k(m)} p_{k(m)} \simeq f$, then there exist a natural number $k(m+1) \geqslant k(m)$ and a map $f_{k(m+1)}: X_{k(m+1)} \to Y$ $(m+1)$-normal relative to Y_0 such that $f_{k(m+1)} p_{k(m+1)} \simeq f$.

Setting $Y_k = X_{k(m)+k-1}$ and $q_k^{k+1} = p_{k(m)+k-1}^{k(m)+k}: Y_{k+1} = X_{k(m)+k} \to X_{k(m)+k-1} = Y_k$, we obtain an inverse sequence $\{Y_k, q_k^{k+1}\}$ associated with X.

Let $\mathscr{L}_1 = \mathscr{L}(f_{k(m)}, m)$ and let $\mathscr{L}_k$ be a local system of groups induced on Y_k by $\mathscr{L}_1$ and the map $q_1^k: Y_k \to Y_1$.

Then $\underline{\mathscr{L}}. = (\{Y_k, q_k^{k+1}\}, \mathscr{L}_k)$ is a generalized local system of abelian groups on X.

Consider now the element $[c^m(f_1)]$ of $H^m(Y_1; \mathscr{L}(f_1, m)) = H^m(Y_1; \mathscr{L}_1)$. Since $H^m(X; \underline{\mathscr{L}}) = 0$, this element represent the zero of $H^m(X; \underline{\mathscr{L}})$ and there exists a natural number k_0 such that $(q_1^{k_0})^*([c^m(f_{k(m)})]) = 0$.

Since $q_1^{k_0}$ is simplicial and $f_{k(m)}$ is m-normal, we infer that $f_{k(m)} q_1^{k_0}$ is m-normal relative to Y_0. Proposition (2.2) implies that $\mathscr{L}_{k_0} = \mathscr{L}(f_{k(m)} q_1^{k_0}; m)$ and that $(q_1^{k_0})^*([c^m(f_{k(m)})])$ is represented by the cocycle $c^m(f_{k(m)} q_1^{k_0})$ in the group $H^m(Y_{k_0}; \mathscr{L}_{k_0}) = H^m(Y_{k_0}; \mathscr{L}(f_{k(m)} q_1^{k_0}; m))$. From Theorem (2.1) we infer that $f_{k(m)} q_1^{k_0}$ is homotopic with a map $f_{k(m+1)}: Y_{k_0} = X_{k(m)+k_0-1} \to Y$ which is $(m+1)$-normal.

It is clear that $f_{k(m+1)} p_{k(m+1)} \simeq f$.

This completes the proof of (4.2) and our theorem.

It follows from the analysis of the proof of Theorem (4.1) (see Proposition (1.8) and Theorem (1.3) of Chapter I) that the following two theorems hold:

(4.3) THEOREM. *Suppose that X is a continuum with $3 \leqslant n = \mathrm{Fd}(X) < \infty$ and $\{X_k, p_k^{k+1}\}$ is a sequence of polyhedra and simplicial maps associated with X. Then there exist a natural number k_0 and a generalized local*

system of groups $\underline{\mathscr{L}} = (\{Y_k, q_k^{k+1}\}, \mathscr{L}_k)$ *on* Y *such that* $H^n(X; \underline{\mathscr{L}}) \neq 0$ *and* $q_k^{k+1} = p_{k+k_0}^{k+k_0+1}: X_{k+k_0+1} = Y_{k+1} \to Y_k = X_{k+k_0}$ *for every* $k = 1, 2, \ldots$
and

(4.4) THEOREM. *If* X *is a continuum with* $\mathrm{Fd}(X) < \infty$, *then* $c[X] \leqslant \mathrm{Fd}(X) \leqslant \max(2, c[X])$.

Combining Proposition (1.2) with Theorems (1.3) of Chapter I and (4.1), we obtain

(4.5) THEOREM. *If* X *is a movable continuum and* $\mathrm{Fd}(X) = \infty$, *then* $c[X] = \infty$.

Theorem (4.5) and Proposition (3.8) imply that the following theorem holds:

(4.6) THEOREM. *Suppose that* $Y \in \mathscr{F}$ *and* X *is a continuum with* $\mathrm{Fd}(X) > 2$. *Then* $\mathrm{Fd}(X \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$.

A consequence of (4.6) is the corollary:

(4.7) COROLLARY. *Suppose that* X *is a continuum with* $\mathrm{Fd}(X) < \infty$ *and* $Y \in \mathscr{F}$ *Then*

$$\mathrm{Fd}(X \times Y) = \mathrm{Fd}(Y) + \mathrm{Fd}(X \times S^3) - 3$$

if $\mathrm{Fd}(X \times Y) \geqslant 3$ *and*

$$\mathrm{Fd}(X \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$$

if there exists a $Y_0 \in \mathscr{F}$ *such that* $\mathrm{Fd}(X \times Y_0) = \mathrm{Fd}(X) + \mathrm{Fd}(Y_0) \geqslant 3$.

Proof. Let $Y \in \mathscr{F}$ and $\mathrm{Fd}(X \times Y) \geqslant 3$. From (4.6) we conclude that

$$\mathrm{Fd}(Y) + \mathrm{Fd}(X \times S^3) = \mathrm{Fd}((X \times S^3) \times Y) = \mathrm{Fd}((X \times Y) \times S^3) = \mathrm{Fd}(X \times Y) + 3$$

and

$$\mathrm{Fd}(X \times Y) = \mathrm{Fd}(Y) + \mathrm{Fd}(X \times S^3) - 3.$$

This completes the proof of the first part of (4.7).

Let us assume that Y, Y_0 are $\mathscr{F}$-continua such that $\mathrm{Fd}(X) + \mathrm{Fd}(Y_0) = \mathrm{Fd}(X \times Y_0) \geqslant 3$ and $\mathrm{Fd}(X \times Y) < \mathrm{Fd}(X) + \mathrm{Fd}(Y)$. From Theorem (4.6) we infer that

$$\mathrm{Fd}(X \times Y_0 \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y_0) + \mathrm{Fd}(Y).$$

On the other hand, we have

$$\mathrm{Fd}(X \times Y_0 \times Y) \leqslant \mathrm{Fd}(X \times Y) + \mathrm{Fd}(Y_0) < \mathrm{Fd}(X) + \mathrm{Fd}(Y_0) + \mathrm{Fd}(Y).$$

The proof of our corollary is finished.

5. Two algebraic lemmas. If G is a multiplicative group, then the *integral group ring* $Z(G)$ of G is the set of all finite formal sums $\sum n_i g_i$,

$n_i \in Z$ and $g_i \in G$, with addition and multiplication given by

$$\sum n_i g_i + \sum m_i g_i = \sum (n_i + m_i) g_i$$

and

$$(\sum n_i g_i)(\sum m_j g_j) = \sum (n_i m_j) g_i g_j .$$

We will employ the following lemma:

(5.1) LEMMA. *Let G be a non-trivial multiplicative group and $0 \neq z = n_1 g_1 + n_2 g_2 + \quad + n_k g_k \in Z(G)$. Then $(1a - 1e)z \neq 0$ for every element $a \in G$ with the order $\geqslant k$, where e is the unit of G.*

Proof. Without loss of generality we may assume that

(5.2) $n_i \neq 0$ and $g_i \neq g_j$ for all $i, j = 1, 2, \ldots, k$ such that $i \neq j$.

Let us suppose that our lemma does not hold.

This means that there is an $a \in G$ satisfying the following condition:

(5.3) $a^s \neq e$ and $n_1 g_1 + n_2 g_2 + \quad + n_k g_k = n_1 a g_1 + n_2 a g_2 + \quad + n_k a g_k = n_1 a^2 g_1 + n_2 a^2 g_2 + \quad + n_k a^2 g_k = n_1 a^s g_1 + n_2 a^s g_2 + \quad + n_k a^s g_k$
for $s = 1, 2, \ldots, k$.

From (5.2) and (5.3) we infer that for every $s = 1, 2, \ldots, k$ there exists a function $\varkappa_s\colon \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$ such that

(5.4) $$n_i g_i = n_{\varkappa_s(i)} a^s g_{\varkappa_s(i)} \quad \text{and} \quad \varkappa_s(i) \neq \varkappa_s(j)$$

for $i, j = 1, 2, \ldots, k$ and $i \neq j$.

If $\varkappa_s(1) = 1$, then $n_1 g_1 = n_1 a^s g_1$ and $g_1 = a^s g_1$ and $a^s = e$. Therefore

(5.5) $$\varkappa_s(1) \neq 1 \quad \text{for every } s = 1, 2, \ldots, k.$$

Let us observe also that

$$\varkappa_s(i) \neq \varkappa_p(i) \quad \text{for} \quad s, p = 1, 2, \ldots, k \text{ such that } s \neq p.$$

Indeed, from $s > p$ and $\varkappa_s(i) = \varkappa_p(i)$ we conclude that $n_i g_i = n_{\varkappa_s(i)} a^s g_{\varkappa_s(i)} = n_{\varkappa_s(i)} a^p g_{\varkappa_s(i)}$ and $s^{s-p} = e$.

Therefore $1 \in \{\varkappa_1(1), \varkappa_2(1), \ldots, \varkappa_k(1)\}$ in contradiction to (5.4) and (5.5). Thus the proof of Lemma (5.1) is completed.

If (W, w) is a pointed topological space, then $Z(\pi_1(W, w))$ is denoted by $\Lambda(W, w)$.

It is well known that for every connected polyhedron W and every $w \in W$ the group $\pi_k(W, w) = \Pi_k(w)$ is a left $\Lambda(W, w)$-module, where

$$z\alpha = n_1 \Pi_k(a_1)(\alpha) + n_2 \Pi_k(a_2)(\alpha) + \quad + n_l \Pi_k(a_l)(\alpha)$$

for $\alpha \in \Pi_k(w)$ and $z = n_1 a_1 + n_2 a_2 + \quad + n_l a_l \in \Lambda(W, w)$ and $k > 1$.

We recall [L] that if M is a left R-module and if there exists a $B \subset M$ such that for every $m \in M$, we have

(5.6) $m = r_1 b_1 + \quad + r_k b_k,$ where $r_i \in R$ and $b_i \in B$ for $i = 1, 2, \ldots, k$

and such that Presentation (5.6) is unique, then M is said to be a *free R-module* and B is said to be a *basis* for M.

Let $(X, x_0) \underset{\text{top}}{+} (Y, y_0) = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y$ for all pointed compacta (X, x_0) and (Y, y_0).

We shall also use the following

(5.7) LEMMA. *Let $k \geqslant 2$ and $(Y, y_0) = (X, x_0) \underset{\text{top}}{+} (S^k, s_0)$, where (X, x_0) is a pointed connected* CW *complex with $\pi_i(X, x_0) = 0$ for every $i = 2, 3, \ldots, k$. Then $\pi_k(Y, y_0)$ is a free left $\Lambda(Y, y_0)$-module and the basis of $\pi_k(Y, y_0)$ consists of one element $\varepsilon \in \pi_k(Y, y_0)$.*

Proof. Let $p \colon (\tilde{X}, x) \to (X, x_0)$ be a universal covering projection for (X, x_0). It is easy to check that a map $q \colon (\tilde{Y}, y) = \big(X \times \{s_0\} \cup p^{-1}(x_0) \times S^k, (x, s_0)\big) \to (Y, y_0)$ given by the formula

$$q(x) = \begin{cases} (p(y), s_0) & \text{for} \quad x = (y, s_0) \in X \times \{s_0\}, \\ (y, s) & \text{for} \quad x = (y, s_0) \in p^{-1}(x_0) \times S^k \end{cases}$$

is a universal covering projection for (Y, y_0).

Using the fact that $q_*^k \colon \pi_k(Y, y_0) \to \pi_k(Y, y_0)$ is an isomorphism, one can easily verify that for every $g \in \pi_k(Y, y_0)$ there exists an element $n_1 a_1 + \quad + n_l a_l$ of $Z(\pi_1(Y, y_0))$ such that

$$g = (n_1 a_1 + \quad + n_l a_l)\varepsilon$$

where ε is an element of $\pi_k(Y, y_0)$ which is induced by a map $\alpha \colon (S^k, s_0) \to (Y, y_0)$ defined by the formula

$$\alpha(s) = (x_0, s) \quad \text{for every } s \in S^k.$$

It is clear that $z_1 \varepsilon \neq z_2 \varepsilon$ for all $z_1, z_2 \in \Lambda(Y, y_0)$ such that $z_1 \neq z_2$. The proof is finished.

6. Continua with fundamental dimension equal to 1. Let us prove the following

(6.1) THEOREM. *If* $\mathrm{Fd}(X) = 1$ *and* $Y \in \mathscr{F}$, *then* $\mathrm{Fd}(X \times Y) = \mathrm{Fd}(Y) + 1$.

The proof of Theorem (6.1) is based on the following lemma, the proof of which may be left to the reader (cf. [N_5]).

(6.2) LEMMA. *Let $(K, k_0), (W, w_0)$ be finite pointed connected* CW *complexes, $s_0 \in S^2$ and $f \colon (W, w_0) \to \big(K \times S^2, (k_0, s_0)\big)$ be a map. The following conditions are equivalent:*

(a) $\omega(f) \leqslant 2$,

(b) *there exists a homotopy* $\varphi\colon W\times[0,1]\to K\times S^2$ *such that*

$$\varphi(x,t)=f(x) \quad \textit{for every } (x,t)\in W\times\{0\}\cup\{w_0\}\times[0,1]$$

and

$$\varphi(W\times\{1\})\subset K^{(2)}\times\{s_0\}\cup\{k_0\}\times S^2.$$

Proof of Theorem (6.1). It is sufficient to show that $\operatorname{Fd}(X\times S^2)=3$ (see Corollary (4.6)).

For simplicity, we will assume that $(X,x_0)=\varprojlim\{(X_k,x_k),p_k^{k+1}\}$ where (X_k,x_k) is a finite connected polyhedron with $\dim X_k=1$.

Let us suppose that $\operatorname{Fd}(X\times S^2)=2$.

From Theorem (1.3) of Chapter I and Lemma (6.2) we conclude that for every k there exist a $k'>k$ and a homotopy $\varphi\colon (X_{k'}\times S^2)\times[0,1]\to X_k\times S^2$ such that

$$\varphi(y,t)=(p_k^{k'}(x),s) \quad \text{for} \quad (y,t)=((x,s),t)\in(X_{k'}\times S^2)\times\{0\}\cup\{(x_k,s_0)\}\times[0,1]$$

and

$$\varphi(Y_1\times\{1\})\subset X_k\times\{s_0\}\cup\{x_k\}\times S^2=(X_k,x_k)\underset{\text{top}}{+}(S^2,s_0)=(Y_2,y_2)$$

and such that $(p_k^{k'})_\#\ \ \pi_1(X_{k'},x_{k'})\to\pi_1(X_k,x_k)$ is a non-trivial homomorphism, where $(Y_1,y_1)=(X_{k'}\times S^2,(x_{k'},s_0))$.

Let $\varepsilon_1\in\pi_2(Y_1,y_1)$ be a generator of $\pi_2(Y_1,y_1)$ and let ε_2 be a $\Lambda(Y_2,y_2)$-generator of $\pi_2(Y_2,y_2)$ (see Lemma (5.7)).

Setting

$$q(x,s)=\varphi((x,s),1) \quad \text{for} \quad (x,s)\in Y_1,$$

we obtain a map $q\colon (Y_1,y_1)\to(Y_2,y_2)$.

It is clear that the homomorphism $q_\#\colon \pi_1(Y_1,y_1)\to\pi_1(Y_2,y_2)$ is non-trivial.

We have

$$q^2_*(\varepsilon_1)=(n_1b_1+\quad+n_kb_k)\varepsilon_2$$

where $0\neq n_1b_1+\quad+n_kb_k\in\Lambda(Y_2,y_2)$.

Let us denote by c an element of $\pi_1(Y_1,y_1)$ such that $a=q_\#(c)$ is a non-trivial element of $\pi_1(Y_2,y_2)$.

Since $\pi_1(Y_2,y_2)$ is isomorphic with a free group $\pi_1(X_k,x_k)$, we conclude that $q_\#(c^s)=a^s$ is a non-trivial element of $\pi_1(Y_2,y_2)$ for every $s=1,2,\ldots$

Lemma (5.1) implies that

$$(1e-1a)(n_1b_1+\quad+n_kb_k)\neq 0$$

(e is the unit of $\pi_1(Y_2,y_2)$).

It follows that

$$n_1 b_1 + \quad + n_k b_k \neq n_1 ab_1 + \quad + n_k ab_k$$

and

$$q_*^2(\varepsilon_1) = (n_1 b_1 + \quad + n_k b_k)\varepsilon_2 \neq (n_1 ab_1 + \quad + n_k ab_k)\varepsilon_2.$$

On the other hand,

$$\begin{aligned} q_*^2(\varepsilon_1) &= q_*^2(1c\varepsilon_1) = \big(1q_{\#}(c)\big)q_*^2(\varepsilon_1) = 1aq_*^2(\varepsilon_1) \\ &= \big(n_1 q_{\#}(c)b_1 + \quad + n_k q_{\#}(c)b_k\big)\varepsilon_2 \\ &= (n_1 ab_1 + \quad + n_k ab_k)\varepsilon_2. \end{aligned}$$

Thus the proof is finished.

7. Continua with fundamental dimension equal to 2. In this section we denote by $H_n(X)$ the n-dimensional singular homology group of X with integer coefficients and by $(f)_n\colon H_n(X) \to H_n(Y)$ the homomorphism which is induced by $f\colon X \to Y$.

Let $(X, x_0), (Y, y_0)$ be connected pointed CW complexes and let $p\colon (\tilde{X}, a_0) \to (X, x_0)$, $q\colon (\tilde{Y}, b_0) \to (Y, y_0)$ be universal covering projections. Then for every map $f\colon (X, x_0) \to (Y, y_0)$ we denote by $\tilde{f}\colon (\tilde{X}, a_0) \to (\tilde{Y}, b_0)$ the (unique) lifting of $fp\colon (\tilde{X}, a_0) \to (Y, y_0)$.

Let $\mathfrak{W}_0$ be a category whose objects are connected pointed CW complexes and whose morphisms are homotopy classes (in the pointed sense) of maps. It is well known that the tilde $\sim$ induces a functor from $\mathfrak{W}_0$ to $\mathfrak{W}_0$. This functor assigns to every object (W, w_0) of $\mathfrak{W}_0$ its universal covering space $(\tilde{W}, w)$ and to every morphism $[f]$ of $\mathfrak{W}_0$ represented by a map $f\colon (W, w_0) \to (V, v_0)$ the homotopy class $[\tilde{f}]$ of $\tilde{f}\colon (\tilde{W}, w) \to (\tilde{V}, v)$.

As an immediate consequence of this fact we obtain the following

(7.1) Lemma. *If* $\underline{X} = \{X_k, p_k^{k+1}\}$ *and* $\underline{Y} = \{Y_k, q_k^{k+1}\}$ *are sequences of polyhedra and* $\mathrm{Sh}(\varprojlim \underline{X})\ \mathrm{Sh}(\varprojlim \underline{Y})$, *then* $\underline{H} = \{H_n(\tilde{X}_k), \widehat{(\tilde{p}_k^{k+1})_n}\}$ *and* $\{H_n(\tilde{Y}_k), \widehat{(\tilde{q}_k^{k+1})_n}\} = \underline{G}$ *are isomorphic progroups for every* $n = 0, 1, 2, \ldots$

Let us prove the following (see [N5])

(7.2) Theorem. *If* (X, x_0) *is not an approximatively* 2*-connected pointed continuum and if* $\mathrm{Fd}(X) = 2$, *then* $\mathrm{Fd}(X \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$ *for every* $Y \in \mathcal{F}$

Proof. Without loss of generality we may assume that (X, x_0) is the inverse limit of an inverse sequence of 2-dimensional polyhedra $\{(X_k, x_k), p_k^{k+1}\}$.

Let $\gamma_k\colon (\tilde{X}_k, a_k) \to (X_k, x_k)$ be a universal covering projection for every $k = 1, 2, \ldots$

We know that $(X \times S^2, (x_0, s_0)) = \varprojlim \{(X_k \times S^2, (x_k, s_0)), p_k^{k+1} \times \mathrm{id}_{S^2}\}$

and that

$$\gamma_k \times \mathrm{id}_{S^2}\colon (\tilde{X}_k \times S^2, (a_k, s_0)) \to (X_k \times S^2, (x_k, s_0))$$

is a universal covering projection for every $k = 1, 2, \ldots$

It is clear that for every $k = 1, 2, \ldots$ we have

$$(\gamma_k)^2_* (p_k^{k+1})^2_* = (p_k^{k+1})^2 (\gamma_{k+1})^2_* \quad \text{and} \quad \theta_2^k (p_k^{k+1})^2_* = \widehat{(\tilde{p}_k^{k+1})}_2 \circ \theta_2^k$$

where $\theta_2^i\colon \pi_2(X_i, a_i) \to H_2(X_i)$ is the Hurewicz homomorphism.

This means that the pair $\alpha = (\mathrm{id}_N, \theta_2^k)$ is a morphism from a progroup $\{\pi_2(X_k, x_k), (p_k^{k+1})^2_*\}$ to a progroup $H' = \{H_2(\tilde{X}_k), \widehat{(\tilde{p}_k^{k+1})}_2\}$ where N denotes the set of natural numbers.

Since $\theta_2^k\colon \pi_2(\tilde{X}_k, a_k) \to H_2(\tilde{X}_k)$ is an isomorphism, we conclude α is an isomorphism of progroups and H' is not trivial.

The Künneth theorem for singular homology ([S], p. 235) implies that the progroups $\{H_4(X_k \times S^2), \widehat{(p_k^{k+1} \times \mathrm{id}_{S^2})}_4\} = H''$ and H' are isomorphic.

Therefore

(7.3) H'' is not a trivial progroup.

The hypothesis that $\mathrm{Fd}(X \times S^2) \leqslant 3$ implies that $X \times S^2$ has the same shape as the inverse limit of an inverse sequence $\{Y_k, q_k^{k+1}\}$ of 3-dimensional polyhedra. From Lemma (3.4) we conclude that the progroups H'' and $H''' = \{H_4(Y_k), \widehat{(q_k^{k+1})}_4\}$ are isomorphic.

Since $\dim Y_k \leqslant 3$ and $H_4(Y_k) = 0$, we infer that H'' and H''' are trivial progroups, in contradiction to (7.3). Thus the proof of (7.2) is finished.

8. The main results. Let us prove the following

(8.1) THEOREM. *Let X be a continuum with* $\mathrm{Fd}(X) < \infty$. *Then the following conditions are equivalent*

(a) $\mathrm{Fd}(X \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$ *for every* $Y \in \mathscr{F}$,

(b) $\mathrm{Fd}(X \times S^3) = \mathrm{Fd}(X) + 3$,

(c) $\mathrm{Fd}(X) = c[X]$.

Proof. (a) $\Rightarrow$ (b). Obvious.

(b) $\Rightarrow$ (a) (see Corollary (4.7)).

(c) $\Rightarrow$ (b) (see Proposition (3.8)).

(b) $\Rightarrow$ (c) We can assume that $X = \varprojlim \{X_k, p_k^{k+1}\}$, where X_k is a polyhedron and $\dim X_k = \mathrm{Fd}(X) = \dim X = n$.

It follows from Theorem (4.3) that we may additionaly assume that there exists a generalized local system of coefficients $\underline{\mathscr{L}} = (\{X_k \times S^3, p_k^{k+1} \times \mathrm{id}_{S^3}\}, \mathscr{L}_k)$ on $X \times S^3$ such that

(8.2) $$H^{n+3}(X \times S^3; \underline{\mathscr{L}}) \neq 0.$$

Let $s_0 \in S^3$ and $\alpha_k\colon X_k \to X_k \times S^3$ be a map defined by the formula

$$\alpha_k(x) = (x, s_0) \quad \text{for every } x \in X_k.$$

Let us denote by $\mathcal{K}_k$ a local system on X_k induced by α_k and $\mathcal{L}_k$. It is clear that $\underline{\mathcal{K}} = (\{X_k, p_k^{k+1}\}, \mathcal{K}_k)$ is a generalized local system of abelian groups on X.

Since S^3 is simple-connected, we infer that $\mathcal{K} \otimes \mathcal{Z}$ and $\mathcal{L}$ are canonically equivalent, where $\mathcal{Z}$ is a simple local system of infinite cyclic groups on S^3.

Theorem (4.1) of Chapter II and (8.2) imply that

$$H^{n+3}(X \times S^3; \underline{\mathcal{L}}) \approx H^n(X; \underline{\mathcal{K}}) \otimes H^3(S^3; \mathcal{Z}) = H^n(X; \underline{\mathcal{L}}) \otimes Z \approx H^n(X; \underline{\mathcal{L}}) \neq 0.$$

The proof of Theorem (8.1) is finished.

The following theorem is an immediate consequence of Theorems (4.6), (6.1), (7.2) and (8.1).

(8.3) THEOREM. *Let X be a continuum with* $\mathrm{Fd}(X) < \infty$. *Then* $c[X] \leqslant \mathrm{Fd}(X) \leqslant \max(2, c[X])$. *If* $\mathrm{Fd}(X) \neq 2$ *or* $\mathrm{Fd}(X) = 2$ *and X is not approximatively 2-connected, then* $\mathrm{Fd}(X) = c[X]$ *and* $\mathrm{Fd}(X \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$ *for every* $Y \in \mathcal{F}$

(8.4) REMARK. R. Swan has proved ([Sw], see also [Sta]) that if (K, k_0) is a connected CW complex (necessarily finite) such that $\pi_i(K, k_0) = 0 = H^i(K; \mathcal{L})$ for every $i \geqslant 2$ and every local system $\mathcal{L}$ on K, then $\pi_1(K, k_0)$ is a free group. Using this theorem, one can prove that if (X, x_0) is a pointed continuum having the shape of a CW complex, then $\mathrm{Fd}(X) = c[X]$ and $\mathrm{Fd}(X \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$ for $Y \in \mathcal{T}$

(8.5) REMARK. In the above-mentioned theorem the assumption that $\mathrm{Fd}(X) < \infty$ may be replaced by the assumption that X is movable (cf. (4.5)).

If X is approximatively 1-connected, then $c[X] = c(X)$ (see Proposition (3.9)).

If X is a 2-dimensional approximatively 1-connected and 2-connected continuum, then X has a trivial shape. Therefore we get the following

(8.6) THEOREM. *If X is an approximatively 1-connected continuum with* $\mathrm{Fd}(X) < \infty$, *then* $\mathrm{Fd}(X) = c(X)$ *and* $\mathrm{Fd}(X \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$ *for every* $Y \in \mathcal{F}$

and

(8.7) THEOREM. *If X is movable approximatively 1-connected continuum and* $\mathrm{Fd}(X) = \infty$, *then* $c(X) = \infty$.

Remark. Theorem (8.7) and the first assertion of (8.6) were proved in [N_2] and subsequently generalized (Theorems (8.3) and (8.1)) by the author (see $[N_4]$ and $[N_5]$) and independently by J. Dydak (generalization of (8.5) to the case of topological spaces).

(8.8) REMARK. The hypothesis of finite-dimensionality or movability in Theorems (8.3), (4.5), (8.6) and (8.7) cannot be omitted.

Indeed, let X be an acyclic approximatively 1-connected continuum described by Kahn in [Ka]. Then $\mathrm{Fd}(X) = \infty$ and $c(X) = c[X] = 0$.

(8.9) REMARK. For every natural number n there exists a polyhedron X_n such that $\mathrm{Fd}(X_n) = c[X_n] \geqslant n$ and $c(X_n) = 0$. Hence, assertions (8.6) and (8.7) are false if one omits the assumption that X is approximatively 1-connected. The following proof of this fact was communicated to the author by J. Hollingsworth.

Let G denote an open 3-simplex of a triangulation of a Poincaré sphere P (i.e. of a homology 3-sphere P with finite and non-trivial $\pi_1(P)$ and $X = P\backslash G$. It is known that $\tilde{P} = S^3$ ([He]). Therefore $H_{2n}(Y^n; Z) \neq 0$ is a free abelian group for every natural number n, where $Y = \tilde{X}$. If there exists a $k \geqslant 3$ such that $\mathrm{Fd}(X^n) \leqslant k$ for $n = 1, 2, \ldots$, then for every natural number n there exists (see Theorem (1.4) of Chapter I) a polyhedron W_n such that $\mathrm{Sh}(W_n) = \mathrm{Sh}(X^n)$ and $\dim W_n \leqslant k$. Hence $\tilde{W}_n$ has the same homotopy type as Y^n and $\dim \tilde{W}_n \leqslant k$. We obtain a contradiction if $n > k$. Therefore $X_n = X^n$ satisfies the required conditions.

Remark. It was proved by J. Keesling ([K], p. 355) that $\mathrm{Fd}(X) = c(X)$ if X is a compact connected abelian topological group.

Remark. Since the compactum X is approximatively n-connected ($\mathrm{Fd}(X) \leqslant n$) iff each of its components is approximatively n-connected (each of its components has the fundamental dimension $\leqslant n$), we conclude that the assumption of connectivity in (8.6) is not essential, i.e. $\mathrm{Fd}(X) = c(X)$ for every approximatively 1-connected compactum X with $\mathrm{Fd}(X) < \infty$.

We say that a continuum X belongs to $\mathfrak{U}$ iff $\mathrm{Fd}(X) = c[X]$.

Chapter IV

Applications of the homological characterizations of fundamental dimension to some special problems

Homological characterizations of fundamental dimension are useful tools in computing the fundamental dimension of the Cartesian product. In particular, we prove in this chapter that $\mathrm{Fd}(X \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$ if $Y \in \mathfrak{U}$ and Y is a closed PL n-manifold or $\mathrm{Fd}(Y) = 1$ and that there exists a sequence of polyhedra $\{X_n\}_{n=1}^{\infty}$ such that X_1 is the suspension of the projective plane and $\mathrm{Fd}(X_i \times X_j) = \mathrm{Fd}(X_1 \times X_2 \times \ldots) = \mathrm{Fd}(X_i) = \dim X_i = 3$ where $i, j = 1, 2, \ldots$ and $i \neq j$.

Using the main results of the last chapter, we show also that the fundamental dimension of the suspension of a compactum X depends only on the Čech cohomology of X (for the case where $\mathrm{Fd}(X) < \infty$ or X is movable) and that for every natural number $n \geqslant 0$ there are a polyhedron X and its subpolyhedron A such that $\mathrm{Fd}(X) \geqslant n$ and $\dim A + 1 = 2 \geqslant \mathrm{Fd}(X/A)$.

These investigations were inspired by the following problems of Borsuk ([B_2] and [B_3], p. 350):

(1) Is it true that $\mathrm{Fd}(X \times Y) \geqslant \mathrm{Fd}(X) + 1$ for all non-empty compacta X and Y such that $\mathrm{Fd}(Y) \geqslant 1$?

(2) Is it true that $\mathrm{Fd}(X \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$ for all polyhedra X and Y such that $\mathrm{Fd}(X) \neq 0 \neq \mathrm{Fd}(Y)$?

(3) Let Y be compacta and let $f: X \underset{\text{onto}}{\to} Y$ be a map. Is it true that $\mathrm{Fd}(X) \leqslant \mathrm{Fd}(Y) + \mathrm{Fd}(A)$, where A is the closure of the union of all sets $f^{-1}(y)$ such that $y \in Y$ and $f^{-1}(y)$ contains at least two points?

1. The fundamental dimension of the Cartesian product of a closed manifold and a continuum. It is well known ([St_2], p. 620 and [St_3], p. 201) that if M is a closed non-orientable PL n-manifold, then there exists a local system $\mathscr{M}$ of cyclic infinite groups on M such that $H^n(M; \mathscr{M}) \approx Z$.

From Theorem (4.1) of Chapter II we conclude that

$$H^{m+n}(X \times M; \underline{\mathscr{L}} \otimes \mathscr{M}) \approx H^n(X; \underline{\mathscr{L}}) \otimes H^m(M; \mathscr{M}) \neq 0$$

for every generalized local system of coëfficients $(\{X_k, p_k^{k+1}\}, \mathscr{L}_k) = \underline{\mathscr{L}}$ on a continuum X such that

$$H^m(X; \underline{\mathscr{L}}) \neq 0 \quad \text{and} \quad m = c[X].$$

Combining this fact with Theorem (3.8) of Chapter III, we obtain

(1.1) THEOREM. *If M is a closed n-manifold, then $c[X \times M] \geqslant c[X]+n$ for every continuum X*

and

(1.2) THEOREM. $\operatorname{Fd}(X \times M) = \operatorname{Fd}(X)+n$ *for every continuum $X \in \mathfrak{U}$ and every closed n-manifold M.*

(1.3) REMARK. The assumption that M is a closed n-manifold is essential. For every $n \geqslant 2$ there is an S^n-like continuum Y such that $\operatorname{Fd}(\Sigma(P^2) \times Y) < \operatorname{Fd}(Y)+\operatorname{Fd}(\Sigma(P^2)) = n+3$, where $\Sigma(P^2)$ is the suspension of the projective plane P^2.

Indeed, if $n \geqslant 2$ and $Y = \varprojlim \{Y_k, p_k^{k+1}\}$ (where $p_k^{k+1}\colon Y_{k+1} = S^n \to Y_k = S^n$ is any map of degree 2), then every element of $H^n(Y; Z)$ is divisible by 2 ([M-S$_1$], p. 55) and

$$H^n(Y \times \Sigma(P^2); Z) \approx H^n(Y; Z) \otimes H^3(\Sigma(P^2); Z) \approx H^n(Y; Z) \otimes Z_2 = 0$$

and $\operatorname{Fd}(\Sigma(P^2) \times Y) < n+3$ (see Theorem (3.3) of Chapter II).

In Section 3 we will prove that there is a polyhedron W such that $\operatorname{Fd}(W \times \Sigma(P^2)) = 3 = \dim W = \dim \Sigma(P^2)$.

2. The fundamental dimension of the Cartesian product of a curve and a continuum. The main result of this section is the following theorem:

(2.1) THEOREM. *If Y is a continuum, then $c[X \times Y] = c[Y]+1$ for every continuum X with* $\operatorname{Fd}(X) = 1$.

First we shall introduce some special notations, which will be used in the proof of this theorem.

If X is a connected 1-dimensional polyhedron, then $\mathscr{X}$ denotes a local system of free abelian groups on X induced by the inclusion X into Y and $\Pi_2(Y)$, where $x_0 \in X$ and $(Y, y_0) = (X, x_0) \underset{\text{top}}{+} (S^2, s_0)$ (the isomorphism class of $\mathscr{X}$ does not depend on the choice of x_0).

We know (see. Lemma (5.7) of Chapter III) that $\mathscr{X}(x)$ is a free $\Lambda(X, x)$-module for every $x \in X$ and its $\Lambda(X, x)$-basis consists of one element. If ε is a $\Lambda(X, x)$-generator of $\mathscr{X}(x)$, then $\mathscr{X}_\varepsilon = \{\mathscr{X}(a)(\varepsilon)\}_{a \in \Pi_1(X,x)}$. It is clear that $\mathscr{X}(x)$ is freely generated by $\mathscr{X}_\varepsilon$ (in the abelian sense).

Suppose that (W_1, w_1) and $(W_2, w_2) = \bigoplus_{i=1}^{n} (S_i^1, s_i)$ are bouquets of circles. A map $f\colon (W_1, w_1) \to (W_2, w_2)$ is said to be *fine* iff $f^{-1}(w_2)$ is a finite set and for every component A of $W_1 \setminus f^{-1}(w_2)$ there exists an index $i \leqslant n$ such that $f|_A$ is a homeomorphism onto $S_i \setminus \{w_2\}$.

We will also assume that the reader is familiar with Example (1.4) of Chapter II.

The length $l(g)$ of an element $g \neq e$ of a group G (e is the unity of G) freely generated by $A \neq \emptyset$ is the minimum of such numbers n that $g = a_1^{\delta(1)} a_2^{\delta(2)} \ldots a_n^{\delta(n)}$, where $\delta(i) = \pm 1$ and $a_i \in A$ for every $i = 1, 2, \ldots, n$. The length of e is 0.

We shall need the following

(2.2) LEMMA. *Suppose that (X, x_0) is a connected 1-dimensional polyhedron and $\mathcal{L}$ is a local system of free abelian groups on S^1 induced by $\mathcal{X}$ and a map $f\colon (S^1, s_0) \to (X, x_0)$. Let ε be a $\Lambda(X, x_0)$-generator of $\mathcal{X}(x_0)$ and let A be the set of the "$\hat{\ }$" classes in $\mathcal{L}(s_0)$ of elements of $\mathcal{X}_\varepsilon$. Elements $\mathcal{X}(a_1)(\varepsilon)$ and $\mathcal{X}(a_2)(\varepsilon)$ of $\mathcal{X}_\varepsilon$ represent the same element of A iff a_1 and a_2 belong to the same right coset of $\pi_1(X, x_0)$ with respect to its subgroup* $\operatorname{Im} f_\#$

Proof. Let $p\colon \mathcal{X}_\varepsilon \to A$ be the canonical projection and let $\varkappa\colon A \to \mathcal{X}_\varepsilon$ be a right inverse of p. It is clear that A generates $\hat{\mathcal{L}}(s_0)$ and that the functions $p\colon \mathcal{X}_\varepsilon \to A$ and $q = p|_{\varkappa(A)}\colon \varkappa(A) \to A$ induce epimorphisms $p_*\colon \mathcal{X}(x_0) \to \hat{\mathcal{L}}(s_0)$ and $q_*\colon G \to \mathcal{L}(s_0)$, where G is a subgroup of $\hat{\mathcal{L}}(s_0)$ freely generated by $\varkappa(A)$.

Let us observe that for every $\alpha \in \mathcal{X}(x_0)$ we have

$$(2.3) \qquad \alpha \in \operatorname{Ker} p_* \supset \operatorname{Ker} q_* \text{ iff } \alpha = \sum_{i=1}^{k} \big(\mathcal{X}(c_{2i-1})(\varepsilon) - \mathcal{X}(c_{2i})(\varepsilon)\big),$$

where $c_{2i-1} = b_i c_{2i}$ and $b_i \in \operatorname{Im} f_\#$ for every $i = 1, 2, \ldots, k$.

Since $\mathcal{X}(c_{2i-1})(\varepsilon)$ and $\mathcal{X}(c_{2i})(\varepsilon)$ represent the same element of A, we infer that $\operatorname{Ker} q_* = \{e\}$.

This means that q_* is an isomorphism and $\hat{\mathcal{L}}(s_0)$ is freely generated by A.

From (2.3) we deduce that $\mathcal{X}(a_1)(\varepsilon)$ and $\mathcal{X}(a_2)(\varepsilon)$ represent the same element of A iff the following equality holds (in $\Lambda(X, x_0)$):

$$(2.4) \qquad 1a_1 - 1a_2 = \sum_{j=1}^{2k} (-1)^{1+j} c_j, \text{ where } c_{2i-1} = b_i c_{2i} \text{ and } b_i \in \operatorname{Im} f_\# \text{ for every } i = 1, 2, \ldots, k.$$

If (2.4) holds, then there exists a permutation μ of the set $\{1, 2, \ldots, k\}$ such that

$$c_{2\mu(1)-1} = a_1 \text{ and } c_{2\mu(k)} = a_2 \text{ and } c_{2\mu(i)} = c_{2\mu(i+1)-1} \text{ for } i = 1, 2, \ldots, k-1.$$

Since $c_{2\mu(i)-1}, c_{2\mu(i)}$ and $c_{2\mu(i+1)-1}$ represent the same right coset of $\pi_1(X, x_0)$ with respect to $\operatorname{Im} f_\#$, we infer that all c_i represent the same coset. The proof is finished.

The following lemma holds true:

(2.5) LEMMA. *If X is a continuum and* $\operatorname{Fd}(X) = \dim X = 1$, *then there exist a finite bouquet of circles (W, w_0) and an essential map $f\colon X \to W$ such that $g(X) = W$ for every map $g\colon X \to W$ homotopic with f.*

Now we are ready to prove the main theorem.

Proof of Theorem (2.1). Let X_n denote the circle in E^2 consisting of all points (x_1, x_2) such that $x_1^2+(x_2-1/n)^2 = 1/n^2$, where $n = 1, 2, \ldots$

Without loss of generality we can assume that X is the inverse limit of an inverse sequence $\{W_k, p_k^{k+1}\}$ such that

$$p_k^{k+1}\colon (W_{k+1}, (0,0)) = \Big(\bigcup_{i=1}^{n_{k+1}} X_i, (0,0)\Big) \to \Big(\bigcup_{i=1}^{n_k} X_i, (0,0)\Big) = (W_k, (0,0))$$

is a simplicial map with respect to triangulations $\mathscr{T}_k$ of $(W_k, (0,0))$ and $\mathscr{T}_{k+1}$ of $(W_{k+1}, (0,0))$ and that

(2.6) $\quad p_k^{k+1}$ is fine.

Lemma (2.5) and elementary properties of inverse sequences imply that we may additionally assume that

(2.7) $\quad$ for every $k = 1, 2, \ldots$ and for every map $q'\colon X_1 \to W_1$ homotopic with $q = p_1^k|_{X_1}\colon X_1 \to W_1$ we have $q'(X_1) \supset X_1$.

Let $\mathscr{L}_k$ be a local system of free abelian groups on W_k induced by $p_1^k\colon W_k \to W_1 = W$ and $\mathfrak{W}$ on W.

$\underline{\mathscr{L}} = (\{W_k, p_k^{k+1}\}, \mathscr{L}_k)$ is a generalized local system of groups on X. We will study its properties and will prove that

$$H^{n+1}(X \times Y;\ \underline{\mathscr{L}} \otimes \underline{\mathscr{K}}) \neq 0 \tag{2.8}$$

for every generalized local system of coefficients $\underline{\mathscr{K}} = (\{Y_k, q_k^{k+1}\}, \mathscr{K}_k)$ on a continuum Y such that

$$H^n(Y;\ \underline{\mathscr{K}}) \neq 0 \quad \text{and} \quad c[Y] = n. \tag{2.9}$$

In order to prove (2.8) it is enough to show that there exist $\alpha \in H^1(W_1;\ \mathscr{L}_1)$ and $\beta \in H^n(Y_1;\ \mathscr{K}_1)$ such that

$$0 \neq (p_1^k)^*(\alpha) \otimes (q_1^k)^*(\beta) \in H^1(W_k;\ \mathscr{L}_k) \otimes H^n(Y_k;\ \mathscr{K}_k) \quad \text{for every } k = 1, 2, \ldots$$

The counter clockwise orientation on E^2 will be called positive.

Let ε be a $\Lambda(W, (0,0))$-generator of $\mathscr{L}_1(0,0)$ and let σ_0 be a 1-simplex of $\mathscr{T}_1|_{X_1}$ with the first vertex $(0,0) = x_{\sigma_0}$ such that σ_0 induces positive orientation on X_1.

We will denote by a_i a generator of $\pi_1(X_i, (0,0)) \approx Z$ which is induced by the positive orientation of the circle X_i.

One can consider that $\pi_1(W_k,(0,0))$ is freely generated by $a_1, a_2, \dots, a_{n_k}$ for every natural number k.

Suppose that k is a natural number.

From (2.6) and (2.7) we deduce that the set of all 1-simplexes $\mathscr{T}_k|_{X_1}$ which induce positive orientation on X_1 and which are mapped by p_1^k onto $\pm\sigma_0$ is the union of three disjoint sets $\{\sigma_1, \sigma_2, \dots, \sigma_m\} \neq \emptyset$, $\{\sigma_1', \sigma_2', \dots, \sigma_s'\}$ and $\{\sigma_1'', \sigma_2'', \dots, \sigma_s''\}$ satisfying the following conditions:

(2.10) A path $p_1^k d$: $[0,1] \to W_1$ from $y_0 = p_1^k d(0)$ to y_0 represents a non-trivial element of $\pi_1(W_1, y_0)$ for every path d: $[0,1] \to X_1$ from $d(0) \in |\sigma_i|$ to $d(1) \in |\sigma_j|$, where $i, j = 1, 2, \dots, m$ and $i \neq j$.

(2.11) $$p_1^k(\sigma_i') = -p_1^k(\sigma_i'') \quad \text{for every } i = 1, 2, \dots, s.$$

(2.12) For every $i = 1, 2, \dots, s$ there exists a path d: $[0,1] \to X_1$ with $p_1^k(d(0)) = p_1^k(d(1)) = y_0$ from $d(0) \in \sigma_i'$ to $d(1) \in \sigma_i''$ such that the path $p^k d$: $[0,1] \to W_1$ is homotopic with the degenerate path e: $[0,1] \to W_1$, $e([0,1]) = \{y_0\}$.

Every 1-cochain of $C^1(X_1; \mathscr{L})$ or $C^1(X_1; \hat{\mathscr{L}})$ is a cocycle, where $\mathscr{L} = \mathscr{L}_1|_{X_1}$.

If σ' and σ'' are 1-simplexes of $\mathscr{T}_k|_{X_1}$ having a common vertex and if σ' and σ'' induce the same orientation on X_1, then for every path d: $[0,1] \to |\sigma'| \cup |\sigma''|$ from $x_{\sigma''}$ to $x_{\sigma'}$ and for every $z \in \mathscr{L}(x_{\sigma'})$ we have $\chi_{\sigma',z} \sim \chi_{\sigma'',\mathscr{L}(d)(z)}$ in $C^1(X_1; \mathscr{L})$.

Hence, $[\chi_{\sigma',z}] = [\chi_{\sigma'',\mathscr{L}(d)(z)}]$ (in $H^1(X_1; \mathscr{L})$) for all 1-simplexes σ' and σ'' of $\mathscr{T}_k|_{X_1}$ and every path d: $[0,1] \to X_1$ such that σ' and σ'' induce the same orientation on X_1 and $d(0) = x_{\sigma''}$ and $d(1) = x_{\sigma'}$.

One can select $x_{\sigma_i} \in |\sigma_i|$ and choose path d_i: $[0,1] \to X_1$ from x_{σ_1} to x_{σ_i} such that

(2.13) $$p_1^k(x_{\sigma_i}) = x_{\sigma_0} \quad \text{for every } i = 1, 2, \dots, m$$

and

(2.14) $0 = l(g_1) < l(g_i) \neq l(g_j) < l((p_1^k)_\#(a_1))$ for all $i, j = 2, 3, \dots, m$, where $i \neq j$ and $g_i \in \pi_1(W_1, x_{\sigma_0})$ is represented by the path $p_1^k d_i$: $[0,1] \to W_1$.

Conditions (2.10), (2.11), (2.12) and (2.13) imply that

(2.15) $q^*([\chi_{\sigma_0,\varepsilon}]) = \sum_{i=1}^{m} \delta(i)[\chi_{\sigma_i,\varepsilon}] = \sum_{i=1}^{m} \delta(i)[\chi_{\sigma_1,\varepsilon_i}]$ where q^*: $H^1(W_1; \mathscr{L}_1) \to H^1(X_1; \mathscr{L})$ is induced by $q = p_1^k|_{X_1}$: $X_1 \to W_1$ and $\delta(i)\sigma_0 = p_1^k(\sigma_i)$ and $\varepsilon_i = \mathscr{L}(d_i)(\varepsilon) = \mathscr{L}(g_i)(\varepsilon)$ for every $i = 1, 2, \dots, m$.

Condition (2.14) implies that for every $i \geq 2$ the right coset of g_i with respect to $\operatorname{Im} q_\#$ and the right coset of g_1 (i.e. $\operatorname{Im} q_\#$) are different.

Combining (2.15) with Lemma (2.2) we infer that $h([\chi_{\sigma_1,\varepsilon}]) = \alpha_0$ generates a direct summand G of $H^1(X_1;\hat{\mathscr{L}}) = G\otimes H$ and $\alpha_1 = h(\sum_{i=2}^{m}[\chi_{\sigma_1,\varepsilon_i}])\in H$, where $h\colon H^1(X_1;\mathscr{L})\to H^1(X_1;\hat{\mathscr{L}})$ is a homomorphism induced by id_{X_1} and $\hat{}$

Suppose that $\underline{\mathscr{K}} = (\{Y_k, q_k^{k+1}\}, \mathscr{K}_k)$ is a generalized local system of groups on a continuum Y satisfying (2.9).

Let β be an element of $H^n(Y_1;\mathscr{K}_1)$ which represents a non-trivial element of $H^n(Y;\underline{\mathscr{K}})$ and $\alpha = [\chi_{\sigma_0,\varepsilon}]\in H^1(W_1;\mathscr{L}_1)$.

Let us denote by $j_{X_1}\colon X_1\to W_1$ the inclusion.

We have

$$hq^*(\alpha)\otimes(q_1^k)^*(\beta) = hq^*\otimes(q_1^k)^*(\alpha\otimes\beta)$$
$$= (\alpha_0+\alpha_1)\otimes\beta'\in H^1(X_1;\hat{\mathscr{L}})\otimes H^n(W_k;\mathscr{K}_k),$$

where $\beta' = (q_1^k)^*(\beta)$.

Since $\beta' = 0$ and $\alpha\otimes\beta'\neq 0$, we infer that

$$hq^*(\alpha)\otimes(q_1^k)^*(\beta) = h\otimes(\mathrm{id}_{W_k})^*(q^*(\alpha)\otimes\beta')\neq 0$$

and

$$0\neq q^*(\alpha)\otimes(q_1^k)^*(\beta).$$

Therefore

$$q^*(\alpha)\otimes(q_1^k)^*(\beta) = (j_{X_1})^*(p_1^k)^*(\alpha)\otimes(q_1^k)^*(\beta)\neq 0$$

and

$$(p_1^k)^*(\alpha)\otimes(q_1^k)^*(\beta)\neq 0.$$

The proof of Theorem (2.1) is finished.

As an immediate consequence of (2.1) we get the following

(2.16) Corollary. *Suppose that* $Y\in\mathfrak{U}$ *and* $X_1, X_2, \ldots, X_n$ *are continua such that* $\mathrm{Fd}(X_i) = 1$ *for* $i = 1, 2, \ldots, n$. *Then* $\mathrm{Fd}(Y\times X_1\times X_2\times \ldots \times X_n) = \mathrm{Fd}(Y)+n$.

It is known ([C-Ch] and [M-S$_3$]) that there is a curve C, non-movable and acyclic (in the sense of Čech homology and cohomology).

Corollary (2.16) answers Question (6.8) of [N$_2$], namely whether $\mathrm{Fd}(C^n) = n$ for every natural number n.

3. An example of a finite-dimensional continuum with an infinite family of shape factors and the fundamental dimension of the Cartesian product of polyhedra. In this section $H^n(X)$ denotes the n-dimensional Čech cohomology group of X with coefficients in the group of integer numbers.

Let $2 < p_1 < p_2 < p_3 < \ldots$ be a sequence of prime numbers and let Q^n_m denote the matching of an n-dimensional cube and its boundary S by a simplicial map $\alpha\colon S\to S$ with $\deg = p_m$, where $n\geqslant 3$.

It is clear that Q^n_m is a simply connected polyhedron and that

(3.1) $H^n(Q^n_m) \approx Z_{p_m}$ and $H^i(Q^n_m) = 0$ for every natural number $i < n$.

Let

$$X_k = Q^n_{p_1} \times Q^n_{p_2} \times \quad \times Q^n_{p_k}$$

and

$$X = Q^n_{p_1} \otimes Q^n_{p_1} \times \ldots$$

By easy induction (using (3.1) and the Künneth formula) we can show that

$$H^n(X_k) \approx Z_{p_1} \oplus Z_{p_2} \oplus \ldots \oplus Z_{p_k} \quad \text{and} \quad H^i(X_k) = 0 \text{ for } i > n.$$

Since X is homeomorphic with the inverse limit of an inverse sequence $\{X_k, p_k^{k+1}\}$ (where $p_k^{k+1}\colon X_{k+1} \to X_k$ is a projection map from $X_{k+1} = X_k \times \times Q^n_p$ to the first factor), we deduce (from the continuity property for the Čech cohomology) that

(3.2) $$H^i(X) = 0 \text{ for } i > n \quad \text{and} \quad H^k(X) \neq 0.$$

Since a countable product of movable compacta is a movable compactum, (3.2) and Theorem (8.7) of Chapter III imply that X is a movable continuum and $\operatorname{Fd}(X) = n$.

From the Holsztyński theorem we deduce that there exists a movable continuum Y such that $\dim Y = n$ and $\operatorname{Sh}(X) = \operatorname{Sh}(Y)$.

It is easy to check that $\operatorname{Sh}(X_i) \neq \operatorname{Sh}(X_j)$ and $\operatorname{Sh}(X_i)$ is a factor of $\operatorname{Sh}(Y)$ for $i, j = 1, 2, \ldots, i \neq j$.

Since $\operatorname{Sh}(X) \geqslant \operatorname{Sh}(X_j)$ and $\operatorname{Fd}(X) = n$, we infer that $\operatorname{Fd}(X_j) = n$.

Thus we obtain the following ([N_2], p. 71 and [N_3])

(3.3) THEOREM. *For every* $n \geqslant 3$ *there exist a movable n-dimensional continuum* Y *with an infinite family of factors of* $\operatorname{Sh}(Y)$ *and a sequence of polyhedra* $\{Q^n_j\}_{j=2}^{\infty}$ *such that* $\operatorname{Fd}(Q^n_i \times Q^n_j) = \operatorname{Fd}(Q^n_i) = \dim Q^n_i = n$ *for all* $i, j = 1, 2, \ldots, i \neq j$. *The polyhedron* Q^3_1 *has the same homotopy type as the suspension of the projective plane.*

4. The fundamental dimension of the union of two compacta and of the quotient space. It is known (see Remark (8.9) of Chapter III) that for every $n = 1, 2, \ldots$ there exists a polyhedron X such that $\operatorname{Fd}(X) = c[X] \geqslant n$ and $c(X) = 0$. Let A be the cone with the base $X^{(1)}$. It is clear that $X \cup A$ is a simply connected polyhedron. Using Theorem (8.6) and Proposition (3.4) of Chapter III, we obtain the following

(4.1) THEOREM. *For every natural number n there exist connected polyhedra* X, A *and* $Y = X \cup A$ *such that* $\operatorname{Fd}(X) \geqslant n$ *and* $\dim(X \cap A) = \operatorname{Fd}(A)+2 = \dim A = 2 \geqslant \operatorname{Fd}(Y)$.

This theorem implies the following

(4.2) THEOREM. *There are a movable continuum X and a movable compactum A such that* $\mathrm{Fd}(X) = \infty$ *and* $\mathrm{Fd}(A)+1 = \dim(X \cap A) = 1$ *and* $\mathrm{Fd}(X \cup A) \leqslant 2$.

Proof. Let X_n, A_n and $Y_n = X_n \cup A_n$ be such polyhedra that $\mathrm{Fd}(X_n) \geqslant n$

$$\dim(X_n \cap A_n) = \mathrm{Fd}(A_n)+2 = \dim A_n = 2 \geqslant \mathrm{Fd}(Y_n) \quad \text{for} \quad n = 1, 2, \ldots$$

We can assume that $a_n \in X_n \subset Y_n \subset Q$ for $n = 1, 2, \ldots$

Let X'_n, A'_n and $Y'_n = X'_n \cup A'_n \subset Q \times Q \times$ be the sets consisting of all points $\{x_k\}$ such that $x_i = a_i$ for $i \neq n$ and $x_n \in X_n, A_n$ and Y_n (respectively) and

$$X = \bigcup_{n=1}^{\infty} X'_n \quad \text{and} \quad A = \bigcup_{n=1}^{\infty} A'_n.$$

Since $X \cup A$ is homeomorphic with the inverse limit of finite bouquets of Y_n and $\mathrm{Fd}(Y_n) \leqslant 2$, we infer from Theorem (4.5) of Chapter III that $\mathrm{Fd}(X \cup Y) \leqslant 2$.

It is clear that $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(X_n)$ and $\mathrm{Fd}(X) = \infty$.

This completes the proof of Theorem (4.2).

Let n be a natural number and let X, A and $Y = X \cup A$ be polyhedra satisfying the conditions of the thesis of Theorem (4.1). We have $\mathrm{Sh}(X/X \cap A) = \mathrm{Sh}(Y/A) = \mathrm{Sh}(Y)$ and we get

(4.3) COROLLARY. *For every natural number n there exist a polyhedron X and its subpolyhedron A such that* $\mathrm{Fd}(X) \geqslant n$ *and* $\dim A+1 = \mathrm{Fd}(X/A) = 2.$

and

(4.4) COROLLARY. *There are a movable compactum X and a map $f\colon X \to Y$ of X onto a compactum Y such that* $\mathrm{Fd}(X) = \infty$ *and* $\mathrm{Fd}(Y) \leqslant 2$ *and the closure of the union of all sets $f^{-1}(y)$, where $y \in Y$ and $f^{-1}(y)$ contains at least two points, has dimension equal to* 1.

Remark. Corollary (4.4) answers Problem (7.4) of Chapter XII of [B_3].

The following remarkable theorem has been proved by J. Dydak ([D_2]).

(4.5) THEOREM. *Let A, X and Y be compacta such that $A \subset X$ and the shaping $S(i)\colon A \to X$ induced by the inclusion $i\colon A \to X$ is trivial. If $f\colon X \to Y$ is a map of X onto a compactum Y and $f^{-1}(f(x)) = \{x\}$ for every $x \in X \backslash A$, then* $\mathrm{Fd}(Y) \geqslant \mathrm{Fd}(X)$.

5. The fundamental dimension of the suspension of a compactum. Let X be a continuum. Then there is an inverse sequence $\{X_k, p_k^{k+1}\}$, where X_k is a connected polyhedron for every $k = 1, 2, \ldots$ such that X is homeomorphic to $\lim \{X_k, p_k^{k+1}\}$. This implies that $\Sigma(X)$ is homeomorphic to $\varprojlim \{\Sigma(X_k), \Sigma(p_k^{k+1})\}$. It is clear that $\Sigma(X_k)$ is a simply connected polyhedron for every $k = 1, 2, \ldots$

Hence we obtain

(5.1) LEMMA. *If X is a continuum, then $\Sigma(X)$ is an approximatively 1-connected continuum.*

It is known (see Theorem (3.2) of Chapter I) that if X is a compactum and A is a closed subset of X, then

$$\text{(5.2)} \qquad \operatorname{Fd}(X/A) \leqslant \max\big(\operatorname{Fd}(X), \operatorname{Fd}(A)+1\big).$$

From (5.2) we infer (by easy induction) that the following lemma holds.

(5.3) LEMMA. *Let $X \neq \emptyset$ be a compactum and let $A_1, A_2, \ldots, A_n$ be non-empty disjoint closed subsets of X. If Y denotes the hyperspace of the upper semicontinuous decomposition of X into the sets $A_1, A_2, \ldots, A_n$ and the single points of $X \setminus \bigcup_{i=1}^{n} A_i$, then* $\operatorname{Fd}(Y) \leqslant \max\big(\operatorname{Fd}(X), \operatorname{Fd}(A_1)+1, \ldots$ $\ldots, \operatorname{Fd}(A_n)+1\big)$.

Now let us prove the following

(5.4) THEOREM. *For every compactum $X \neq \emptyset$ with a finite fundamental dimension*

$$\text{(5.5)} \qquad \operatorname{Fd}(\Sigma(X)) = \begin{cases} c(X)+1 & \textit{when } c(X) > 0, \\ 0 & \textit{when } c(X) = 0 \textit{ and } X \textit{ is connected}, \\ 1 & \textit{when } c(X) = 0 \textit{ and } X \textit{ is not a continuum.} \end{cases}$$

Proof. It is known that

(5.6) For every compactum X and each abelian group G the groups $H^n(X; G)$ and $H^{n+1}(\Sigma(X); G)$ are isomorphic if $n \geqslant 1$.

Let us suppose that X is a continuum. Since $\Sigma(X)$ is approximatively 1-connected (see (5.1)), we infer (see (5.6)) that

$$\operatorname{Fd}(\Sigma(X)) = c(\Sigma(X)) = c(X)+1 \quad \text{if} \quad c(X) > 0$$

and

$$\operatorname{Fd}(\Sigma(X)) = c(\Sigma(X)) = 0 \quad \text{if} \quad c(X) = 0.$$

Let us suppose that X is not connected.

Consider the hyperspace Y of the upper semicontinuous decomposition of $X \times [-1, 1]$ into the classes of the equivalence relation $\sim$ defined by

$(x, t) \sim (y, s)$ iff $s = t = 1$ and x, y belong to the same component of X or $s = t = -1$ and x, y belong to the same component of X or $(x, t) = (y, s)$.

It is clear that the components of Y are homeomorphic with the suspensions of components of X and that $\Sigma(X)$ is homeomorphic with the hyperspace W of the decomposition of Y into the sets $A_1 = p(X \times \{-1\})$, $A_2 = p(X \times \{1\})$ and the single points of $Y \setminus (A_1 \cup A_2)$, where $p\colon X \times [-1, 1] \to Y$ is the natural projection.

The number $c(Y)$ is equal to the maximum of $c(Y_\mu)$, where $Y_\mu \in \square(Y)$ runs through all components of Y.

Therefore $c(Y) = \mathrm{Fd}(Y)$ (see Theorem (2.2) of Chapter III).

Since $\dim(A_1 \cup A_2) = 0$, we infer (see (5.3)) that

$$c(W) \leqslant \mathrm{Fd}(W) \leqslant \max(\mathrm{Fd}(Y), 1) = \max(c(Y), 1) = \max(c(W), 1) = \max(c(X)+1, 1).$$

Hence $\mathrm{Fd}(X) = c(X)+1$ for every compactum X with $\mathrm{Fd}(X) < \infty$ and $c(X) > 0$.

If $c(X) = 0$ and X is not a continuum, then every component of Y has a trivial shape and W has the same shape as the suspension of 0-dimensional compactum $\square(Y)$. Therefore $c(\Sigma(X)) = c(W) = 1$ and $\mathrm{Fd}(\Sigma(X)) = 1$.

This completes the proof of Theorem (5.4).

K. Borsuk has proved ([B$_3$], p. 156) that the suspension $\Sigma(X)$ of a movable compactum X is movable. This fact, Theorem (4.5) of Chapter III and Lemma (5.1) imply at once the following

(5.7) THEOREM. *Let $X \neq \emptyset$ be a movable continuum. Then*

$$\mathrm{Fd}(\Sigma(X)) = \begin{cases} c(X)+1 & \text{when } c(X) > 0, \\ 0 & \text{when } c(X) = 0. \end{cases}$$

Remark. The assumption of movability in Theorem (5.7) is essential. Let X be the Kahn continuum (see Remark (8.8) of Chapter III). Then $\mathrm{Fd}(\Sigma(X)) = \infty$ and $c(X) = 0$.

We also have the following

(5.8) THEOREM. *There exists a movable continuum X such that* $\mathrm{Fd}(X) = \infty$ *and* $\mathrm{Fd}(\Sigma(X)) = 0$.

Proof. Let X_n be a polyhedron such that $c(X_n) = 0$ and $\mathrm{Fd}(X_n) > n$ for $n = 1, 2, \ldots$ (see Remark (8.9) of Chapter III) and

$$X = X_1 \times X_2 \times \ldots$$

Then $\mathrm{Fd}(X) = \infty$ and $c(X) = 0$. Since X is movable, we conclude that $\mathrm{Fd}(\Sigma(X)) = 0$. The proof of Theorem (5.8) is finished.

6. The fundamental dimension of the Cartesian product of approximatively 1-connected compacta. In this section we adopt the notations of Section 3 of Chapter III and [Ku]. One knows ([Ku], p. 24 and [Ko]) that for every fixed prime number p there is a continuum FQ_{p2} and a simple closed curve $B_p \subset FQ_{p2}$ (denoted in [Ku] by X_∞) such that

$$H^2(FQ_{p2}/B_p; R_p) \neq 0$$

and

$$H^2(FQ_{p2}/B_p; G) = 0 \quad \text{when} \quad G = Q, Z, Z_p, Z_q, R_q, Q_p, Q_q \text{ and } q \text{ is a prime number} \neq p.$$

Let $A_p = \Sigma(FQ_{p2}/B_p)$. From Lemma (5.1) we infer that A_p is an approximatively 1-connected continuum such that

$$c_{R_p}(A_p) = \mathrm{Fd}(A_p) = c(A_p) = 3$$

and

$$c_G(A_p) < 3 \quad \text{when } G = Q, Z, Z_p, Z_q, R_q, Q_p, Q_q \text{ and } p \neq q.$$

Let us prove the following

(6.1) THEOREM. *For every approximatively* 1-*connected continuum* X *with* $\mathrm{Fd}(X) < \infty$ *the following conditions are equivalent*:

(i) $X \in \mathscr{F}$,

(ii) $\mathrm{Fd}(X \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$ *for every approximatively* 1-*connected continuum* $Y \neq \emptyset$.

(iii) $\mathrm{Fd}(X \times A_p) = \mathrm{Fd}(X) + 3$ *for every prime number* p.

Proof. (i) $\Rightarrow$ (ii). Let Y be an approximatively 1-connected continuum and $\mathrm{Fd}(Y) < \infty$. From Theorem (8.6) and (3.1) of Chapter III we infer that $c(X) = \max\{c_{R_p}(X)\}$, $c(Y) \Rightarrow \max\{c_{R_p}(Y)\}$, $c(X \times Y) = \mathrm{Fd}(X \times Y) = \max\{c_{R_p}(X \times Y)\}$. If $c_{Q_p}(X) = c_{R_p}(X) = c(X) = \mathrm{Fd}(X)$ for every prime number p, then $c_{R_p}(X \times Y) = c_{R_p}(X) + c_{R_p}(Y)$ and $\max\{c_{R_p}(X \times Y)\} = \max\{c_{R_p}(X) + c_{R_p}(Y)\} = \max\{\mathrm{Fd}(X) + \mathrm{Fd}(Y)\} = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$.

It is clear that (ii) implies (iii).

(iii) $\Rightarrow$ (ii). From Proposition (3.3) of Chapter III we infer that $c_{R_p}(X \times A_p) = \mathrm{Fd}(X) + 3$ for every $p \neq q$. Therefore $c_{R_p}(X \times A_p) = \mathrm{Fd}(X) + 3$. This implies that $c_{Q_p}(X) = c_{R_p}(X) = \mathrm{Fd}(X) = c(X)$. Propositions (3.1), (3.2) and (3.3) of Chapter III imply that $c_G(X) = c(X) = \mathrm{Fd}(X)$ for every $G \neq 0$.

We also have the following

(6.2) THEOREM. *Let* X *be an approximatively* 1-*connected continuum with* $\mathrm{Fd}(X) < \infty$. *Then*

$$\mathrm{Fd}(X^n) = \begin{cases} n\,\mathrm{Fd}(X) & \text{if there exists a prime number } p \text{ such that } c(X) = \max(c_Q(X), c_{Z_p}(X)), \\ n(\mathrm{Fd}(X) - 1) + 1 & \text{if } c(X) \neq \max(c_Q(X), c_{Z_p}(X)) \text{ for every prime number } p. \end{cases}$$

Proof. By easy induction (using Propositions (3.1), (3.2) and (3.3) of

Chapter III) we obtain

$$\begin{aligned} c_{Z_p}(X^n) &= nc_{Z_p}(X), \\ c_Q(X^n) &= nc_Q(X), \\ c_{Q_p}(X^n) &= \max\big(nc_{Q_p}(X), nc_{Z_p}(X)-1\big), \\ c_{R_p}(X^n) &= \begin{cases} nc_{R_p}(X) & \text{if } c_{Q_p}(X) = c_{R_p}(X), \\ \max\big(c_{Q_p}(X^n)+1, c_Q(X^n)\big) & \text{if } c_{Q_p}(X) < c_{R_p}(X). \end{cases} \end{aligned}$$

If there is a such prime number p that $c(X) = \max\big(c_Q(X), c_{Z_p}(X)\big)$, then $\max\big(c_{Z_p}(X^n), c_Q(X^n)\big) = nc(X) = c(X^n)$ and therefore $\mathrm{Fd}(X^n) = n\,\mathrm{Fd}(X)$.

If $c(X) > \max\big(c_Q(X), c_{Z_p}(X)\big)$ for every prime number p, then $c_{R_p}(X^n) \leqslant n\big(c_{R_p}(X)-1\big)$ when $c_{R_p}(X) < c(X)$ and $c_{R_p}(X^n) = n\big(c_{R_p}(X)-1\big)+1$ when $c_{R_p}(X) = c(X)$. Hence $c(X^n) = n\big(c(X)-1\big)+1$.

The proof of Theorem (6.2) is finished.

7. The fundamental dimension of a subset of manifold. A connected n-manifold M is said to be *regular* ([N_1], p. 219) provided for every continuum $X \subsetneqq M$ there exists a sequence $N_0 = M \supset N_1 \supset N_2 \supset N_3 \supset$ of compact connected submanifolds with boundary of M such that $X = \bigcap_{k=1}^{\infty} N_k$.

M. Brown and B. Cassler have shown (see [T_1], p. 94) that if N is a compact and connected n-manifold with boundary $B \neq \emptyset$, then there is a map $g\colon B \to R$, $\dim R \leqslant n-1$, such that the mapping cylinder C_g of g is homeomorphic to M.

Therefore we have the following

(7.1) Lemma. *Let N be a connected compact n-manifold with boundary $B \neq \emptyset$. Then* $\mathrm{Fd}(N) \leqslant n-1$.

This lemma and Theorem (1.6) of Chapter I imply that the following lemma holds:

(7.2) Lemma. *If M is a regular n-manifold and compactum $X \subsetneqq M$, then* $\mathrm{Fd}(X) \leqslant n-1$.

If M is a connected PL manifold and $X \subsetneqq M$ is a continuum, then for every neighborhood U of X, $U \subset M$, there are a connected polyhedral neighborhood W of X and a regular neighborhood $P \subset U$ of W and we infer that M is a regular manifold. Thus we get the following

(7.3) Proposition. *Let M be a connected* PL *manifold and let $X \subsetneqq M$ be a compactum. Then* $\mathrm{Fd}(X) \leqslant n-1$.

It is known ([T_2], p. 70) that an n-manifold M without boundary is a handlebody if $n \geqslant 6$. Using this theorem, M. Štan'ko has shown (unpublished) that every closed n-manifold is regular for $n \geqslant 6$. The idea of his proof is as follows: For every n-manifold $M, n \geqslant 6$, and every open neighborhood U of a continuum $X \subsetneqq M$, U is a n-manifold and admits a handlebody decomposition. Taking those handles which meets X, we obtain a compact submanifold $N \subset U$ of M. N is a neighborhood of X in M.

The results of Štan'ko implies the following

(7.4) PROPOSITION. *If M is a closed n-manifold and $n \neq 4, 5$, then* $\mathrm{Fd}(X) < n$ *for every compactum $X \subsetneqq M$.*

Let M be a topological n-manifold and let $X \subsetneqq M$ be a compactum such that $\mathrm{Fd}(X) = n$, where $n = 4$ or 5. Then $X \times S^6 \subsetneqq M \times S^6$ and $\mathrm{Fd}(X \times S^6) \leqslant n+5$.

On the other hand, Theorem (8.3) of Chapter III implies that $\mathrm{Fd}(X \times S^6) = n+6$.

We get the following

(7.5) THEOREM. *If M is a closed n-manifold and a compactum X is a proper subset of M, then* $\mathrm{Fd}(X) < n$.

Final remarks and problems

Our knowledge concerning fundamental dimension is incomplete. Many problems are left unsolved in the present paper. In particular we fail (except the cases where X has a polyhedral shape or X is approximatively i-connected and $i = 1$ or $i = 2$) to give sufficient algebraical conditions for a continuum X to have the fundamental dimension equal to 2.

(1) PROBLEM. *Is it true that* $c[X] = 2$ *for every continuum* X *with* $\mathrm{Fd}(X) = 2$?

Some questions concerning the fundamental dimension of the Cartesian product of continua also remain open. They are connected with Problem (1).

Let us prove the following

(2) PROPOSITION. *Let* X *be a continuum with* $\mathrm{Fd}(X) = 2 > c[X]$. *Then the following conditions are satisfied*:

(a) $\mathrm{Fd}(X^n) \leqslant n+1$ *for every natural number* $\geqslant 2$ (*in particular* $\mathrm{Fd}(X\times X) = 2$),

(b) $\mathrm{Fd}(X\times Y) = 2$ *for every continuum* Y *with* $\mathrm{Fd}(Y) = 1$,

(c) $\mathrm{Fd}(X\times M) < n+2$ *for every closed* PL *n-manifold* M,

(d) $\mathrm{Fd}(X\times Y) < \mathrm{Fd}(X)+\mathrm{Fd}(Y)$ *for every* $Y\in\mathscr{F}$

Proof. Let us suppose that $\mathrm{Fd}(X^n) > n$, where $n \geqslant 2$. Then (see Theorem (8.3) of Chapter III)

$$\mathrm{Fd}\big(X^n\times(S^3)^n\big) \geqslant 4n+1.$$

On the other hand (see Theorem (8.1) of Chapter III),

$$\mathrm{Fd}\big(X^n\times(S^3)^n\big) = \mathrm{Fd}\big((X\times S^3)^n\big) \leqslant 4n.$$

Hence (a) is satisfied.

Let $\mathrm{Fd}(X\times Y) = 3$ for Y with $\mathrm{Fd}(Y) = 1$. Then

$$\mathrm{Fd}(X\times S^1) = 2 \quad\text{and}\quad \mathrm{Fd}(X\times S^1\times Y) \leqslant 3.$$

On the other hand,

$$\mathrm{Fd}(X\times Y\times S^1) = 4$$

and (b) must be satisfied.

The proof of (c) is analogous to that of (b).

Condition (d) is a consequence of Theorem (8.1) of Chapter III.

The proof is finished.

Remark. If we restrict ourselves to an approximatively 1-connected continuum X, then $\mathrm{Fd}(X^n)$ is equal to $n\,\mathrm{Fd}(X)$ or $\mathrm{Fd}(X^n) = n(\mathrm{Fd}(X)-1)+1$ (cf. Section 6 of the last chapter). Therefore $\mathrm{Fd}(X^n) = 2n$ or $\mathrm{Fd}(X^n) = n+1$ if X is an approximatively 1-connected continuum and $\mathrm{Fd}(X) = 2$.

Let us formulate the following problem:

(3) PROBLEM. *Is it true that* $\mathrm{Fd}(X^n) > n$ *for every continuum* X *with* $\mathrm{Fd}(X) = 2$?

(4) PROBLEM. *Is it true that* $\mathrm{Fd}(X\times Y) = \mathrm{Fd}(Y)+2$ *if* X *is a continuum with* $\mathrm{Fd}(X) = 2$ *and* $Y \in \mathscr{F}$?

(5) PROBLEM. *Is it true that* $\mathrm{Fd}(X\times Y) = 3$ *if* X *and* Y *are continua and* $\mathrm{Fd}(Y)+1 = \mathrm{Fd}(X) = 2$?

(6) PROBLEM. *Is it true that* $\mathrm{Fd}(X\times M) = n+2$ *for every continuum* X *with* $\mathrm{Fd}(X) = 2$ *and every closed* PL *n-manifold* M?

The affirmative answer to any of these problems would give also the affirmative answer to Problem (1).

Added in proof. Recently S. Spież has solved Problem (1) and has proved that there is a continuum X with $\mathrm{Fd}(X) = 2$ and $c[X] = 1$ (see S. Spież, *An example of a continuum X with* $\mathrm{Fd}(X\times S^1) = \mathrm{Fd}(X) = 2$, to appear in Bull. Acad. Polon. Sci.).

It is also known that if $\mathrm{Fd}(X) < \infty$ and $c[X] = 0$, then $\mathrm{Fd}(X) = 0$ (A. Kadlof and S. Spież, *Remark on the fundamental dimension of Cartesian product of metric compacta*, to appear in Bull. Acad. Polon. Sci.).

References

[B_1] K. Borsuk, *Theory of Retracts*, Warszawa 1967.

[B_2] — *On several problems of the theory of shape*, Studies in Topology (Proceedings of a Conference held at Charlote, North Carolina, March 14-16, 1974), Academic Press, Inc., New York 1975.

[B_3] — *Theory of Shape*, Warszawa 1975.

[C-Ch] J. H. Case and R. E. Chamberlin, *Characterization of treelike compacta*, Pacific J. Math. 10 (1960), pp. 73–84.

[D_1] J. Dydak, *A generalization of cohomotopy groups*, Fund. Math. 90 (1975), pp. 77–98.

[D_2] — *On a paper by Y. Kodama*, Bull. Acad. Polon. Sci. 25 (1977), p. 165.

[E-G] D. A. Edwards and R. Geoghegan, *The stability problem in shape and a Whitehead theorem in pro-homotopy*, Trans. Amer. Math. Soc. 214 (1975), pp. 261–277.

[H-S] D. Handel and J. Segal, *An acyclic continuum with non-movable suspensions*, Bull. Acad. Polon. Sci. 17 (1969), pp. 171–172.

[He] J. Hempel, 3-*Manifolds*, Ann. of Math. Stud. 86, Princeton 1976.

[Hi-Wy] P. J. Hilton and S. Wylie, *Homology Theory*, Cambridge 1960.

[Ho] W. Holsztyński, *An extension and axiomatic characterization of the Borsuk's theory of shape*, Fund. Math. 70 (1971), pp. 157–168.

[Hu_1] Sze-Tsen Hu, *Cohomology and deformation retracts*, Proc. London Math. Soc. (2) 53 (1951), pp. 191–219.

[Hu_2] — *Homotopy Theory*, New York 1959.

[Hu_3] — *Theory of Retracts*, Detroit 1965.

[H-W] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton 1941.

[Ka] D. S. Kahn, *An example in Čech cohomology*, Proc. Amer. Math. Soc. 16 (1965), p. 584.

[Ko_1] Y. Kodama, *Cohomological dimension theory*, Appendix to the book of K. Nagami "Dimension Theory", New York 1970.

[Ko_2] — *On Δ-spaces and fundamental dimension in the sense of Borsuk*, Fund. Math. 89 (1975), pp. 13–22.

[K] J. Keesling, *Shape theory and compact connected abelian topological groups*, Trans. Amer. Math. Soc. 194 (1974), pp. 349–358.

[Ku] V. I. Kuz'minov, *Homological dimension theory* (in Russian), Uspehi Mat. Nauk 23. 5 (143) (1968), pp. 3–49.

[L] J. Lambek, *Lectures on Rings and Modules*, London 1966.

[Ma] S. Mardešić, *On the Whitehead theorem in shape theory I*, Fund. Math. 91 (1976), pp. 51–64.

[$M\text{-}S_1$] — and J. Segal, *Shapes of compacta and* ANR-*systems*, Fund. Math. 72 (1971), pp. 41–59.

[$M\text{-}S_2$] — — *Equivalence of the Borsuk and the* ANR-*system approach to shapes*, Fund. Math. 72 (1971), pp. 61–68.

[$M\text{-}S_3$] — — *Movable compacta and* ANR-*systems*, Bull. Acad. Polon. Sci. 18 (1970), pp. 649–654.

[Mo] K. Morita, *On shapes of topological spaces*, Fund. Math. 86 (1975), pp. 251–259.

[N_1] S. Nowak, *Some properties of fundamental dimension*, Fund. Math. 85 (1974), pp. 211–227.

[N_2] – *On the fundamental dimension of approximatively 1-compacta*, Fund. Math. 89 (1975), pp. 61–79.

[N_3] – *An example of finite dimensional movable continuum with an infinite family of shape factors*, Bull. Acad. Polon. Sci. 24 (1976), pp. 1019–1020.

[N_4] – *On the fundamental dimension of the Cartesian product of two compacta*, Bull. Acad. Polon. Sci. 24 (1976), pp. 1021–1028.

[N_5] – *Some remarks concerning the fundamental dimension of the Cartesian product of two compacta*, Fund. Math. 103 (1979), pp. 31–41.

[R-S] C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise-Linear Topology*, New York 1972.

[P] L. S. Pontryagin, *Continuous Groups* (in Russian), Moscow 1973.

[S] E. Spanier, *Algebraical Topology*, New York 1966.

[Sp] S. Spież, *On characterization of shapes of several compacta*, Bull. Acad. Polon. Sci. 24 (1976), pp. 257–263.

[St_1] N. E. Steenrod, *Universal homology groups*, Amer. J. Math. 58 (1936), pp. 661–701.

[St_2] – *Homology with local coefficients*, Ann. of Math. 44 (1943), pp. 610–627.

[St_3] – *The Topology of Fibre Bundles*, Princeton 1951.

[Sta] J. R. Stallings, *On torsion-free groups with infinitely many ends*, Ann. Math. 88 (1968), pp. 312–334.

[Sw] R. Swan, *Groups of cohomological dimension "one"*, J. of Algebra 12 (1969), pp. 585–601.

[W] C. T. C. Wall, *Finiteness conditions for CW complexes*, Ann. of Math. 81 (1965), pp. 56–69.

[T_1] *Topology of 3-Manifolds*, Proceedings of the University of Georgia Institute 1961, Prentice – Hall, Inc., Englewood Cliffs N. J., 1962.

[T_2] *Topology of Manifolds*, Proceedings of the University of Georgia Institute, Markhan Published Company, Chicago 1970.

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