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THE CONTINUOUS INVERTIBILITY OF FUNCTIONAL OPERATORS IN BANACH SPACES

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1. Introduction. This paper is the survey of results on the one- and two-sided (continuous) invertibility for some classes of functional operators in Hölder, Lebesgue and Orlicz spaces, which were obtained in the theory of singular integral operators with discrete groups of shifts. In particular, we consider the invertibility in these spaces for binomial and polynomial operators generated by shift operators, local theory of invertibility in algebras of functional operators with discrete groups of shifts, the solvability of systems of difference equations with incommensurable differences on the semi-axis and finite intervals, and approximation approach to the problem of invertibility of such operators.

2. Invertibility of binomial functional operators

2.1. Let Γ be a simple closed smooth curve and let α be an orientation-preserving diffeomorphism of Γ onto itself having a finite set Λ of fixed points. We consider the binomial functional operator

$$A = aI - bW, \tag{2.1}$$

where a, b are some functions, I is the identity operator and W is the shift operator: $(W\varphi)(t) = \varphi[\alpha(t)],\ t \in \Gamma$.

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THEOREM 2.1. *Let* $a, b \in C(\Gamma)$. *The operator* $A : L_p(\Gamma) \to L_p(\Gamma)$, $1 \le p \le \infty$, *is invertible if and only if one of the two conditions holds*:

1) $(\forall t \in \Gamma)\ a(t) \neq 0,\ (\forall \tau \in \Lambda)\ |a(\tau)| > |\alpha'(\tau)|^\nu |b(\tau)|$,
2) $(\forall t \in \Gamma)\ b(t) \neq 0,\ (\forall \tau \in \Lambda)\ |a(\tau)| < |\alpha'(\tau)|^\nu |b(\tau)|$,

where $\nu = -1/p$. *The respective inverse operators have the form*

$$1)\ A^{-1} = \sum_{n=0}^{\infty} (a^{-1} bW)^n a^{-1} I, \qquad 2)\ A^{-1} = -W^{-1} \sum_{n=0}^{\infty} (b^{-1} aW^{-1})^n b^{-1} I.$$

Theorem 2.1 was actually proved in [37]. It remains valid for $A : C(\Gamma) \to C(\Gamma)$ with $\nu = 0$ (see [38]).

For Hölder spaces an invertibility criterion for the operator A is qualitatively different from Theorem 2.1.

Let $H_\mu^0(\Gamma, Y) = \{\varphi \in H_\mu(\Gamma) : \varphi(t) = 0, t \in Y\}$ for $Y \subset \Lambda$.

THEOREM 2.2. *Let* $a, b \in H_\mu(\Gamma)$, $0 < \mu \le 1$. *Then*

1) *the operator* $A : H_\mu^0(\Gamma, \Lambda) \to H_\mu^0(\Gamma, \Lambda)$ *is invertible if and only if one of the two conditions of Theorem* 2.1 *for* $\nu = \mu$ *is satisfied*,

2) *the operator* $A : H_\mu(\Gamma) \to H_\mu(\Gamma)$ *is invertible if and only if the operator* $A : H_\mu^0(\Gamma, \Lambda) \to H_\mu^0(\Gamma, \Lambda)$ *is invertible and*

$$(\forall \tau \in \Lambda) \quad a(\tau) \neq b(\tau). \tag{2.2}$$

This result was obtained in [10]. The additional condition (2.2) is connected with decomposition of $H_\mu(\Gamma)$ into the direct sum of $H_\mu^0(\Gamma, \Lambda)$ and a finite-dimensional subspace generated by traces of Hölder functions on Λ and also with the triangular representation of A in this direct sum of subspaces. In this connection, the spectrum of the weighted shift operator bW in the space $H_\mu(\Gamma)$ in contrast to the spaces $L_p(\Gamma), C(\Gamma), H_\mu^0(\Gamma, \Lambda)$, as a rule, is not invariant with respect to rotations around zero.

Criteria for the invertibility of the operator A in the Banach spaces $X = L_p(\Gamma), C(\Gamma), H_\mu^0(\Gamma, \Lambda)$ have an alternative nature. We can write them in the following form, well expressing their essence.

THEOREM 2.3. *Let* $X = L_p(\Gamma), C(\Gamma), H_\mu^0(\Gamma, \Lambda)$. *Then the operator* $A : X \to X$ *is invertible if and only if one of the two conditions is fulfilled*:

1) *the operator* aI *is invertible in* X *and* $r(a^{-1} bW) < 1$,
2) *the operator* bI *is invertible in* X *and* $r(b^{-1} aW^{-1}) < 1$,

where $r(B)$ *is the spectral radius of an operator* B.

2.2. Results of Section 2.1 are extended to Orlicz spaces [4] and generalized Hölder spaces [21]. Invertibility criteria for A in these spaces are characterized by more complicated dependence on the shift derivative α' at fixed points $\tau \in \Lambda$.

Let mutually complementary N-functions $M(x)$ and $N(x) = \max\{v|x| - M(v) : v \geq 0\}$ satisfy the Δ_2-condition:

$$\limsup_{x\to+\infty}[M(2x)/M(x)] < \infty, \qquad \limsup_{x\to+\infty}[N(2x)/N(x)] < \infty.$$

Then the corresponding Orlicz space $L_M(\Gamma)$ is reflexive and vice versa [36]. Let

$$K_e(c) = \lim_{n\to\infty}\left[\sup_{x>1}\frac{M^{-1}(x)}{M^{-1}(c^n x)}\right]^{1/n}, \qquad K_i(c) = \lim_{n\to\infty}\left[\inf_{x>1}\frac{M^{-1}(x)}{M^{-1}(c^n x)}\right]^{1/n}$$

if $c \geq 1$, and $K_e(c) = [K_i(c^{-1})]^{-1}$, $K_i(c) = [K_e(c^{-1})]^{-1}$ if $0 < c < 1$, where $M^{-1}(\cdot)$ is the inverse function for $M(\cdot)$. It is obvious that $K_e(c) \geq K_i(c)$ for $c > 0$ and moreover $K_e(c) = K_i(c) = c^{-1/p}$ in the special case of $L_p(\Gamma), 1 < p < \infty$.

Theorem 2.4. *The operator A with coefficients $a, b \in C(\Gamma)$ is invertible in the reflexive Orlicz space $L_M(\Gamma)$ if and only if one of the two conditions holds*:

1) $(\forall t \in \Gamma)$ $a(t) \neq 0$, $(\forall \tau \in \Lambda)$ $|a(\tau)| > K_e(|\alpha'(\tau)|)|b(\tau)|$;
2) $(\forall t \in \Gamma)$ $b(t) \neq 0$, $(\forall \tau \in \Lambda)$ $|a(\tau)| < K_i(|\alpha'(\tau)|)|b(\tau)|$.

The quantities $K_e(c)$ and $K_i(c)$ are closely connected with interpolated characteristics ν_0, ν_1 (named Boyd's indices) of the rearrangement-invariant space $L_M(\Gamma)$. According to [6],

$$\nu_0 = -\lim_{c\to+0}\frac{\ln h(c)}{\ln c} = -\sup_{0<c<1}\frac{\ln h(c)}{\ln c}, \qquad \nu_1 = -\lim_{c\to+\infty}\frac{\ln h(c)}{\ln c} = -\inf_{c>1}\frac{\ln h(c)}{\ln c},$$

where $h(c) = \limsup_{x\to+\infty}[M^{-1}(x)/M^{-1}(cx)]$ for $c > 0$. Moreover, $0 < \nu_1 \leq \nu_0 < 1$. Then [4]

$$K_e(c) = \max\{c^{-\nu_0}, c^{-\nu_1}\}, \qquad K_i(c) = \min\{c^{-\nu_0}, c^{-\nu_1}\}.$$

Let Φ be the set of moduli of continuity $\omega \not\equiv 0$ on $[0, d]$, where d is the diameter of Γ. We consider the generalized Hölder space $H_\omega(\Gamma)$ and its subspace $H^0_\omega(\Gamma, \Lambda) = \{\varphi \in H_\omega(\Gamma) : \varphi(\tau) = 0,\ \tau \in \Lambda\}$, $\omega \in \Phi$. By analogy with Orlicz spaces, we introduce the quantities

$$K_e(c) = \lim_{n\to\infty}\left[\sup_{0<\delta<d}\frac{\omega(c^n\delta)}{\omega(\delta)}\right]^{1/n}, \qquad K_i(c) = \lim_{n\to\infty}\left[\inf_{0<\delta<d}\frac{\omega(c^n\delta)}{\omega(\delta)}\right]^{1/n}$$

if $0 < c < 1$, and $K_e(c) = [K_i(c^{-1})]^{-1}$, $K_i(c) = [K_e(c^{-1})]^{-1}$ if $c \geq 1$. It is clear that $K_e(c) = K_i(c) = c^\mu$ in the case of $\omega(\delta) = \delta^\mu$ $(0 < \mu \leq 1)$, while in general case, $K_e(c) \geq K_i(c)$.

Theorem 2.5. *The operator A with coefficients $a, b \in H_\omega(\Gamma)$ is invertible in the space $H_\omega(\Gamma), \omega \in \Phi$, if and only if A is invertible in $H^0_\omega(\Gamma, \Lambda)$ and condition* (2.2) *holds. The operator A is invertible in the space $H^0_\omega(\Gamma, \Lambda)$ if and only if one of the two conditions of Theorem* 2.4 *holds.*

2.3. Let now an orientation-preserving diffeomorphism $\alpha : \Gamma \to \Gamma$ have an arbitrary non-empty set Λ of fixed points. In particular, Λ can be the Cantor set and have positive measure. Set $u_\pm(t) = \lim_{n\to\pm\infty} u[\alpha_n(t)]$ for $t \in \Gamma$, where

$\alpha_n(t) = \alpha[\alpha_{n-1}(t)]$ for $n \in \mathbb{Z}$ and $\alpha_0(t) = t$. Let $\partial\Lambda$ be the boundary of Λ, $\Psi = \Gamma\backslash\Lambda$ and for the operator (2.1) let

$$\sigma_A(t) = \begin{cases} a(t) - b(t), & t \in \Gamma_1 = \Gamma\backslash\Psi, \\ a(t), & t \in \Gamma_2 = \{\tau \in \Psi : |a_\pm(\tau)| > K_e(|\alpha'_\pm(\tau)|)|b_\pm(\tau)|\}, \\ -b(t), & t \in \Gamma_3 = \{\tau \in \Psi : |a_\pm(\tau)| < K_i(|\alpha'_\pm(\tau)|)|b_\pm(\tau)|\}, \\ 0, & t \in \Gamma\backslash(\Gamma_1 \cup \Gamma_2 \cup \Gamma_3). \end{cases} \tag{2.3}$$

THEOREM 2.6. *If the diffeomorphism $\alpha : \Gamma \to \Gamma$ preserves orientation and the set Λ of fixed points of α is non-empty, then the operator A is invertible in the space $L_M(\Gamma), H^0_\omega(\Gamma, \partial\Lambda)$ if and only if $\sigma_A(t) \neq 0$ for each $t \in \Gamma$. The operator A is invertible in $H_\omega(\Gamma)$ if and only if A is invertible in $H^0_\omega(\Gamma, \partial\Lambda)$ and $a(t) \neq b(t)$ for all $t \in \Xi$, where Ξ is the set of isolated points in Λ.*

Theorem 2.6 for the spaces $X = L_p(\Gamma), H_\mu(\Gamma), L_M(\Gamma)$ was proved, respectively, in [15, 27, 28, 29, 30, 4]. The case of $X = H_\omega(\Gamma)$ was considered by the author (see [21]).

The general case of the non-empty set Λ of periodic points is reduced to the previous case with the help of the following lemmas.

LEMMA 2.1. *The operator A is invertible in the space $X = L_M(\Gamma), H^0_\omega(\Gamma, \partial\Lambda)$ if and only if the operator $A_m = a_m I - b_m W^m$ is invertible in X, where $f_m(t) = f(t)f[\alpha(t)]\dots f[\alpha_{m-1}(t)]$ and m is the (same) multiplicity of periodic points for α orientation-preserving and $m = 2$ for α orientation-reversing.*

LEMMA 2.2. *If the shift α changes orientation on Γ and fixed points of α are not isolated in Λ, then the operator A is invertible in the space $H_\omega(\Gamma)$ if and only if the operator $A_m = a_m I - b_m W^m$ is invertible in the space $H^0_\omega(\Gamma, \partial\Lambda)$ and*

$$(\forall t \in \Xi\backslash Z) \quad a_2(t) \neq b_2(t), \quad (\forall t \in \Xi \cap Z) \quad a(t) \neq b(t), \tag{2.4}$$

where Ξ is the set of isolated points in Λ and Z is the set of fixed points of the orientation-reversing shift α. In other cases the operators A and A_m are invertible in $H_\omega(\Gamma)$ only simultaneously.

COROLLARY 2.1. *The spectrum of the weighted shift operator $T = bW$ with coefficient $b \in C(\Gamma)$ in the space $L_M(\Gamma)$ has the form*

$$\begin{aligned} \sigma_0(T) = \Big(\bigcup_{t\in\Gamma_1} \{z : \ z^m = b_m(t)\}\Big) \cup \Big(\bigcup_{\gamma\subset\Gamma\backslash\Lambda} \sigma(T, \gamma)\Big) \\ \cup \Big(\bigcup_{\tau\in Y'} \{z : |b_m(\tau)| K_i(|\alpha'_m(\tau)|) \leq |z|^m \leq |b_m(\tau)| K_e(|\alpha'_m(\tau)|)\}\Big), \end{aligned} \tag{2.5}$$

where Y' is the derivative set for $Y = \partial\Lambda$ and

$$\sigma(T,\gamma) = \begin{cases} \{z : \min_{\tau\in\Lambda\cap\overline{\gamma}} |b_m(\tau)|K_i(|\alpha'_m(\tau)|) \le |z|^m \le \max_{\tau\in\Lambda\cap\overline{\gamma}} |b_m(\tau)|K_e(|\alpha'_m(\tau)|)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } \min\{|b(t)| : t \in \overline{\gamma}\} > 0, \\ \{z : |z|^m \le \max_{\tau\in\Lambda\cap\overline{\gamma}} |b_m(\tau)|K_e(|\alpha'_m(\tau)|)\} \quad \textit{if } \min\{|b(t)| : t\in\overline{\gamma}\} = 0; \end{cases}$$

for connected components $\gamma \subset \Gamma\backslash\Lambda$.

COROLLARY 2.2. *The spectrum of the weighted shift operator* $T = bW$ *with coefficient* $b \in H_\omega(\Gamma)$ *in the space* $H^0_\omega(\Gamma, \partial\Lambda)$ *and in the space* $H_\omega(\Gamma)$ *has the form* $\sigma_0(T)$ *and* $\sigma_0(T) \cup \sigma_1(T)$, *respectively*, *where* $\sigma_0(T)$ *is defined by* (2.5), *and*

$$\sigma_1(T) = \begin{cases} \bigcup_{t\in\Xi}\{z : z^m = b_m(t)\} \quad \textit{if } \alpha \textit{ preserves orientation}; \\ (\bigcup_{t\in\Xi\backslash Z}\{z : z^2 = b_2(t)\}) \cup \{b(t) : t \in \Xi\cap Z\} \\ \qquad\qquad\qquad\qquad \textit{if } \alpha \textit{ changes orientation.} \end{cases}$$

2.4. In this subsection we present criteria for one-sided invertibility of the operator A in reflexive Orlicz spaces and generalized Hölder spaces.

Let a diffeomorphism $\alpha : \Gamma \to \Gamma$ preserve or change the orientation on Γ and suppose the set Λ of periodic points of α is non-empty. In this case we set (cf. (2.3)):

$$\eta_e(t) = |a_m(t)| - K_e(|\alpha'_m(t)|)|b_m(t)|, \quad \eta_i(t) = |a_m(t)| - K_i(|\alpha'_m(t)|)|b_m(t)|,$$
$$\eta_e^\pm(t) = \lim_{n\to\pm\infty} \eta_e[\alpha_n(t)], \quad \eta_i^\pm(t) = \lim_{n\to\pm\infty}\eta_i[\alpha_n(t)] \quad \text{for } t\in\Gamma,$$

$$\Psi = \overline{\Gamma\backslash\Lambda}, \quad \Gamma_1 = \Gamma\backslash\Psi, \quad \Gamma_2 = \{t\in\Psi : \eta_e^\pm(t) > 0\}, \quad \Gamma_3 = \{t\in\Psi : \eta_i^\pm(t) < 0\},$$
$$\Gamma_4 = \{t\in\Psi : \eta_i^+(t) < 0 < \eta_e^-(t)\}, \quad \Gamma_5 = \{t\in\Psi : \eta_i^-(t) < 0 < \eta_e^+(t)\}.$$

Let $\sigma_A(t)$ be equal to $a_m(t) - b_m(t), a_m(t), -b_m(t)$ and 0 for $t \in \Gamma_1, \Gamma_2, \Gamma_3$ and $\Gamma\backslash(\Gamma_1\cup\Gamma_2\cup\Gamma_3)$, respectively.

THEOREM 2.7. *If a diffeomorphism* α *preserves or changes orientation on* Γ *and the set* Λ *of periodic points of* α *is non-empty*, *then the operator* A *with coefficients* $a, b \in C(\Gamma)$ *is right invertible* (*left invertible*) *in the reflexive Orlicz space* $L_M(\Gamma)$ *if and only if*:

1) $\sigma_A(t) \ne 0$ *for every* $t \in \Gamma\backslash\Gamma_4$ (*respectively*, *for every* $t\in\Gamma\backslash\Gamma_5$) *and*

2) $(\forall t\in\Gamma_4)$ $(\exists k_0\in\mathbb{Z})$ $a[\alpha_k(t)] \ne 0$ *for* $k < k_0$ *and* $b[\alpha_k(t)] \ne 0$ *for* $k \ge k_0$ (*respectively*, $(\forall t\in\Gamma_5)$ $(\exists k_0\in\mathbb{Z})$ $a[\alpha_k(t)]\ne 0$ *for* $k > k_0$ *and* $b[\alpha_k(t)] \ne 0$ *for* $k < k_0$).

This result was established in [4]. The case of $M(x) = p^{-1}|x|^p$ (i.e. the space $L_p(\Gamma)$ for $p \in (1,\infty)$) is considered in [43, 44]. Theorem 2.7 is also valid for the space $H^0_\omega(\Gamma,\partial\Lambda)$, if $a, b\in H_\omega(\Gamma)$ (see [27, 28, 29, 30] in the case of $\omega(\delta) = \delta^\mu$, the general case of $\omega\in\Phi$ is considered by analogy with the spaces $L_M(\Gamma)$ and $H^0_\mu(\Gamma,\partial\Lambda)$ by the author).

COROLLARY 2.3. *If the set* Λ *of periodic points of a diffeomorphism* $\alpha : \Gamma\to\Gamma$ *is finite*, *then the intersection of the right spectrum and the left spectrum of the* *shift operator W in the space $L_M(\Gamma)$ or $H^0_\omega(\Gamma,\Lambda)$ is the union of annuli*

$$\mathbb{K}_\tau = \{z : K_i(|\alpha'_m(\tau)|) \le |z|^m \le K_e(|\alpha'_m(\tau)|)\}, \qquad \tau \in \Lambda. \tag{2.6}$$

Consequently, the passage from the spaces $L_p(\Gamma)$ (respectively, $H^0_\mu(\Gamma,\Lambda)$) to the more general spaces $L_M(\Gamma)$ (H^0_ω (Γ,Λ)) in the case of the finite set Λ brings a new quality: the intersection of the right spectrum and the left spectrum of W can have non-zero two-dimensional Lebesgue measure.

In contrast to the case of $L_M(\Gamma)$ or $H^0_\omega(\Gamma,\partial\Lambda)$, the criteria for the right invertibility and left invertibility of A in the space $H_\omega(\Gamma)$ are essentially different.

Theorem 2.8. *Under the assumptions of Theorem* 2.7, *the operator A with coefficients $a, b \in H_\omega(\Gamma)$ is right invertible in the space $H_\omega(\Gamma)$, $\omega \in \Phi$, if and only if the operator A is right invertible in the space $H^0_\omega(\Gamma,\partial\Lambda)$ and either $a_m(t) \neq b_m(t)$ for every $t \in \Xi$ in the case of α orientation-preserving or condition* (2.4) *is fulfilled for α orientation-reversing.*

Lemma 2.3. *If either $\partial\Lambda = \emptyset$ or $\partial\Lambda = \{\tau, \alpha(\tau), \dots, \alpha_{m-1}(\tau)\}$, then the left* (*respectively*, *right*) *invertibility of the operator A in the space $H_\omega(\Gamma)$ is equivalent to its two-sided invertibility in $H_\omega(\Gamma)$.*

In other cases, the left invertibility criterion for the operator A in $H_\omega(\Gamma)$ is much more complicated. Let $Y_0 = (\partial\Lambda \backslash \Xi) \cup \Xi_0$, where

$$\Xi_0 = \begin{cases} \{t \in \Xi : a_m(t) \neq b_m(t)\} & \text{if } \alpha \text{ preserves orientation,} \\ \{t \in \Xi \backslash Z : a_2(t) \neq b_2(t)\} \cup \{t \in \Xi \cap Z : a(t) \neq b(t)\} & \\ \qquad\qquad\qquad\qquad\qquad \text{if } \alpha \text{ changes orientation.} & \end{cases}$$

Lemma 2.4. *If $\partial\Lambda \neq \emptyset$, then the operator A is left invertible in the space $H_\omega(\Gamma)$ if and only if it is left invertible in the space $H^0_\omega(\Gamma,\partial\Lambda)$ and its kernel in $H^0_\omega(\Gamma, Y_0)$ is trivial.*

It remains to find a criterion for triviality of $\operatorname{Ker} A$ in $H^0_\omega(\Gamma, Y_0)$ under the condition for the strict left invertibility of A in $H^0_\omega(\Gamma,\partial\Lambda)$. For simplicity, we present it only in the case of Γ consisting of one open arc γ with endpoints $\tau_\pm$ and $\Lambda = \{\tau_\pm\}$. Then $m = 1$. Here τ_+ is the attracting point and τ_- is the repelling point for the shift α, i.e. $\tau_\pm = \lim_{n\to\pm\infty} \alpha_n(t)$ for any point $t \in \gamma\backslash\Lambda$.

Let M_1 denote the class of arcs γ such that

$$\inf\{|a[\alpha_k(t)]b[\alpha_k(t)]| : \ k \in \mathbb{Z}\} > 0$$

for all t belonging to a non-empty set $N \subset \gamma\backslash\{\tau_\pm\}$ and let

$$\inf\{|a[\alpha_k(t)]| : \ k \in \mathbb{Z}\} = \inf\{|b[\alpha_k(t)]| : \ k \in \mathbb{Z}\} = 0$$

for other points $t \in \gamma\backslash\{\tau_\pm\}$. In other cases, we put γ into the class M_2. If $a(\tau_\pm) = b(\tau_\pm)$ and $K_e(|\alpha'(\tau_+)|) < 0 < K_i(|\alpha'(\tau_-)|)$, then for each $t \in N$ the product

$$d(t) = \prod_{k=-\infty}^{+\infty} (b[\alpha_k(t)])^{-1} a[\alpha_k(t)]$$

converges. For $f \in H_\omega(\gamma)$ we set

$$\nu(f) = \sup_{t\in\Gamma\setminus\{\tau_\pm\}} \inf_{k\in\mathbb{Z}} |f[\alpha_k(t)]|.$$

THEOREM 2.9. *If the operator A is strictly left invertible in the space $H^0_\omega(\gamma, \partial\Lambda)$, then its kernel in the space $H^0_\omega(\gamma, Y_0)$ is trivial if and only if*:

1) *in the case $Y_0 = \emptyset$ the function $d(t) \not\equiv const$ on N if $\gamma \in M_1$, and $\nu(a)\nu(b) \neq 0$ if $\gamma \in M_2$*;
2) *$\nu(a) \neq 0$ in the case $Y_0 = \{\tau_+\}$*;
3) *$\nu(b) \neq 0$ in the case $Y_0 = \{\tau_-\}$*;
4) *$Y_0 = \{\tau_\pm\}$.*

In the case of a general contour Γ and a non-empty set Λ, the criterion for triviality of $\operatorname{Ker} A$ in $H^0_\omega(\Gamma, Y_0)$ is formulated and proved by analogy with [27, 28, 29, 30]. We remark that, in view of continuous junction of solutions of the equation $A\varphi = 0$ at points $\tau \in \partial\Lambda$, the kernel of the operator A in $H^0_\omega(\Gamma, Y_0)$ can be trivial also in the case of violation of conditions 1)–4) of Theorem 2.9 for some connected components $\gamma \subset \Gamma\setminus\Lambda$.

3. Invertibility of polynomial functional operators

3.1. Let B^n denote the space of n-dimensional vectors with elements in the Banach space B and let Γ and α satisfy the conditions of Subsection 2.1. In the reflexive Orlicz space $L^n_M(\Gamma), n \geq 1$, we consider a polynomial functional operator

$$A = \sum_{j=0}^{m} a_j W^j, \tag{3.1}$$

with matrix-valued functions $a_j \in C^{n\times n}(\Gamma)$ and $(W\varphi)(t) = \varphi[\alpha(t)], t \in \Gamma$. We suppose that at least one of the coefficients a_0 or a_m is nonsingular everywhere on Γ.

LEMMA 3.1. *The operator A of the form* (3.1) *is invertible in the space $L^n_M(\Gamma)$ if and only if the operator*

$$B = b_0 I - b_1 W \tag{3.2}$$

is invertible in the space $L^{nm}_M(\Gamma)$, where

$$b_0 = \begin{pmatrix} a_0 & 0 \\ 0 & I_{n(m-1)} \end{pmatrix}, \quad b_1 = \begin{pmatrix} -c & -a_m \\ I_{n(m-1)} & 0 \end{pmatrix}, \quad c = (a_1, a_2, \ldots, a_{m-1})$$

and I_k is the $k \times k$ identity matrix.

If the coefficient a_0 (or a_m) is nonsingular, then b_0 (respectively, b_1) is also nonsingular. Now we present an invertibility criterion for the operator (3.2) in the space $L^n_M(\Gamma)$. First, we introduce several notations and definitions following [16, 19].

Let $\mathbb{T}$ be the unit circle, let Δ_+ and Δ_- be the open interior and exterior of the unit disk and let $\sigma(Q)$ denote the spectrum of a matrix Q. Set

$$B(\tau, z) = \det(b_0(\tau) - b_1(\tau)z), \qquad z \in \mathbb{K}_\tau,\ \tau \in \Lambda,$$

where the annulus $\mathbb{K}_\tau$ is defined by (2.6) for $m = 1$, and

$$\begin{aligned}\kappa(\tau, B) &= \frac{1}{2\pi}\{\arg B(\tau, K_e(|\alpha'(\tau)|)z)\}_{z\in\mathbb{T}} \\ &= \frac{1}{2\pi}\{\arg B(\tau, K_i(|\alpha'(\tau)|)z)\}_{z\in\mathbb{T}}\end{aligned}$$

if $B(\tau, z) \neq 0$ for each $z \in \mathbb{K}_\tau, \tau \in \Lambda$ (here $\{\cdot\}$ denotes the increment of the function in braces). If all numbers $\kappa(\tau, B), \tau \in \Lambda$, coincide, then their common value is denoted by κ.

Lemma 3.2. *If the operator* (3.2) *is invertible in the space* $L^n_M(\Gamma)$, *then*

$$(\forall z \in \mathbb{K}_\tau, \tau \in \Lambda) \quad B(\tau, z) \neq 0, \quad \kappa(\tau, B) = const \stackrel{\text{def}}{=} \kappa. \tag{3.3}$$

Definition 3.1. A matrix $Q(t)$ will be called a *matrix of α-normal form on* $\gamma \subset \Gamma$ if

$$Q(t) = \operatorname{diag}\{Q_1(t), Q_2(t)\}, \quad (\forall t \in \gamma) \quad \det Q_2(t) \neq 0,$$
$$(\forall \tau \in \gamma \cap \Lambda) \quad \sigma(Q_1(\tau)K_e(|\alpha'(\tau)|)) \subset \Delta_+, \quad \sigma(Q_2(\tau)K_i(|\alpha'(\tau)|)) \subset \Delta_-.$$

Definition 3.2. A continuous matrix $d(t)$ is *α-reducible to α-normal form on* γ if there exists a continuous nonsingular matrix $v(t)$ such that

$$v^{-1}(t)d(t)v[\alpha(t)] = Q(t),$$

where $Q(t)$ is a matrix of α-normal form on γ.

Definition 3.3. If a vector $\varphi \in C(\Gamma^0)$ is a solution of the equation

$$b_0(t)\varphi(t) = b_1(t)K_i(|\alpha'(\tau)|)\varphi[\alpha(t)]$$

such that

$$\|\varphi[\alpha_{-j}(t)]\|_{\mathbb{C}^n} \to +\infty, \qquad \|\varphi[\alpha_j(t)]\|_{\mathbb{C}^n} \to 0 \tag{3.4}$$

as $j \to +\infty$ for each $t \in \Gamma^0$, where $\Gamma^0 = \Gamma\backslash\Lambda$, then we call φ an *α-solution corresponding to the operator* B. If a vector $\varphi \in C(\Gamma^0)$ is a solution of the equation

$$b_0(t)\varphi(t) = b_1(t)K_e(|\alpha'(\tau)|)\varphi[\alpha(t)]$$

such that conditions (3.4) are fulfilled as $j \to -\infty$ for each $t \in \Gamma^0$, then we call φ an *α_{-1}-solution corresponding to the operator* B.

Let $\mathfrak{N}^+(B)$ (respectively, $\mathfrak{N}^-(B)$) denote the collection of all α-solutions (α_{-1}-solutions) corresponding to B, supplemented by the zero vector. If $\mathfrak{N}^+ \subset \mathfrak{N}^+(B)$ is a linear space (over the field of complex numbers), then $\dim \mathfrak{N}^+$ denotes its dimension, defined as the maximal number of α-solutions in $\mathfrak{N}^+$ which are linearly independent at each point $t \in \Gamma^0$. Similarly, $\dim \mathfrak{N}^-$ denotes the maximal number of α_{-1}-solutions in the linear space $\mathfrak{N}^- \subset \mathfrak{N}^-(B)$ which are linearly independent at each point $t \in \Gamma^0$. Let $\mathfrak{M}^+(B)$ denote the set of $n \times \kappa$ matrices whose columns are α-solutions in $\mathfrak{N}^+(B)$, and let $\mathfrak{M}^-(B)$ denote the set of $n \times (n-\kappa)$ matrices whose columns are α_{-1}-solutions in $\mathfrak{N}^-(B)$.

DEFINITION 3.4. A matrix $\Phi \in C^{n\times r}(\Gamma^0)$ is said to be *normalizable on* Γ if there exists a pointwise nonsingular matrix $M \in C^{r\times r}(\Gamma^0)$ such that the matrix ΦM^{-1} can be continuously extended to the whole contour Γ.

If the matrix $b_0(t)$ (respectively, $b_1(t)$) is nonsingular on Γ, then set

$$d_+(t) = b_0^{-1}(t)b_1(t) \quad (d_-(t) = b_1^{-1}[\alpha_{-1}(t)]b_0[\alpha_{-1}(t)]) \quad \text{for } t \in \Gamma.$$

THEOREM 3.1. *If* $\det b_0(t) \neq 0$ *everywhere on* Γ, *then the following conditions are equivalent*:

1) *The operator* B *is invertible in the space* $L^n_M(\Gamma)$.

2) *Conditions* (3.3) *are fulfilled and* $\mathfrak{N}^+(B)$ *contains a linear space* $\mathfrak{N}^+$ *of dimension* κ.

3) *Conditions* (3.3) *are valid*, *and either* $\kappa = 0$, *or* $\kappa > 0$ *and then there exists a matrix* $\Phi(t) \in \mathfrak{M}^+(B)$ *which is normalizable on* Γ *and a normalizing matrix* $M(t)$ *such that*

$$\lim_{t\to\tau} \Phi(t)M^{-1}(t) = \omega(\tau)\begin{pmatrix} O_{(n-\kappa)\times\kappa} \\ I_\kappa \end{pmatrix}, \qquad \tau \in \Lambda,$$

where $\omega(\tau)$ *denotes a matrix which reduces* $d_+(\tau)$ *to the Jordan normal form*: $\omega^{-1}(\tau)d_+(\tau)\omega(\tau) = Q(\tau)$, *and* $Q(\tau)$ *is a matrix of* α*-normal form on* Λ.

4) *The matrix* $d_+(t)$ *is* α*-reducible on* Γ *to* α*-normal form.*

5) *The operator* $B_\rho = b_0 I - b_1\rho W$ *is invertible in* $L^n_M(\Gamma)$ *for any function* $\rho \in C(\Gamma)$ *such that* $|\rho(\tau)| = 1$ *on* Λ *and* $\rho(t) \neq 0$ *everywhere on* Γ.

The next assertion is dual to Theorem 3.1.

THEOREM 3.2. *If* $\det b_1(t) \neq 0$ *everywhere on* Γ, *then the following conditions are equivalent*:

1) *The operator* B *is invertible in the space* $L^n_M(\Gamma)$.

2) *Conditions* (3.3) *are fulfilled and* $\mathfrak{N}^-(B)$ *contains a linear space* $\mathfrak{N}^-$ *of dimension* $n-\kappa$.

3) *Conditions* (3.3) *are valid*, *and either* $\kappa = n$, *or* $\kappa < n$ *and then there exists a matrix* $\Phi(t) \in \mathfrak{M}^-(B)$ *which is normalizable on* Γ *and a normalizing matrix* $M(t)$ *such that*

$$\lim_{t\to\tau} \Phi(t)M^{-1}(t) = \omega(\tau)\begin{pmatrix} O_{\kappa\times(n-\kappa)} \\ I_{n-\kappa} \end{pmatrix}, \qquad \tau \in \Lambda,$$

where $\omega(\tau)$ *denotes a matrix which reduces* $d_-(\tau)$ *to the Jordan normal form*: $\omega^{-1}(\tau)d_-(\tau)\omega(\tau) = Q(\tau)$, *and* $Q(\tau)$ *is a matrix of* α_{-1}*-normal form on* Λ.

4) *The matrix* $d_-(t)$ *is* α_{-1}*-reducible on* Γ *to* α_{-1}*-normal form.*

5) *Condition* 5) *of Theorem* 3.1.

Corollary 3.1. *If* $\det[b_0(t)b_1(t)] \neq 0$ *for all* $t \in \Gamma$, *then the operator* B *is invertible in the space* $L^n_M(\Gamma)$ *if and only if* d_+W (*or*, *equivalently*, d_-W^{-1}) *is a hyperbolic operator*, *i.e. its spectrum in* $L^n_M(\Gamma)$ *does not intersect the unit circle.*

Corollary 3.2. *If* $b_0, b_1 \in C^{n\times n}(\Gamma)$ *and* $\det[b_0(t)b_1(t)] \neq 0$ *for every* $t \in \Gamma$, *then the operator* (3.2) *is invertible in* $L^n_M(\Gamma)$ *if and only if*

$$(\forall z \in \mathbb{K}_\tau,\ \tau \in \Lambda) \qquad \det[b_0(\tau) - b_1(\tau)z] \neq 0$$

and there exist linear subspaces $\mathfrak{N}^+ \subset \mathfrak{N}^+(B)$ *and* $\mathfrak{N}^- \subset \mathfrak{N}^-(B)$ *such that* $\dim \mathfrak{N}^+ + \dim \mathfrak{N}^- = n$.

Equivalent conditions of Theorems 3.1 and 3.2 in different aspects characterize the invertibility of the operator B. So, conditions 2) (and Corollary 3.2) are connected with the exponential dichotomy of solutions of a system of difference equations. Conditions 3) characterize the asymptotics of solutions of the corresponding system of homogeneous functional equations and they are connected with classical results of Poincaré and Perron on the asymptotic properties of solutions of homogeneous difference equations (see [7], Chapter 5). Condition 4) asserts the existence of a special factorization with a shift for matrix-valued function d_+ (or d_-) and, consequently, for the operator $D^+ = I - d_+W$ (or $D^- = I - d_-W^{-1}$). Condition 5) affirms, in particular, that the spectrum of the operator d_+W (and d_-W^{-1}) is invariant with respect to rotations around zero (cf. [42]).

For the spaces $L^n_p(\Gamma)$ these results were proved in [16, 18, 19]. The general case of reflexive Orlicz spaces is considered by the author. Conceptually close results for the Lebesgue spaces connected with the theory of linear extensions of dynamical systems were obtained in [1, 2, 3]. Further development of such approach on the basis of the multiplicative ergodic theorem is contained in [40, 41].

3.2. The investigation of the operator (3.1) in the case of the finite set Λ of periodic points is reduced to the previous case with the help of the following lemma (see [19]).

Here it is more convenient to replace (3.1) by the operator

$$A = \sum_{j=mr}^{m\nu} a_j W^j, \tag{3.5}$$

where r and ν are integers with $\nu \geq r$, and m is the multiplicity of periodic points of the shift α.

Lemma 3.3. *The operator* (3.5) *is invertible in* $L^n_M(\Gamma)$ *if and only if the operator*

$$H_A = \sum_{j=r}^{\nu} b_j W^{mj} \tag{3.6}$$

is invertible in $L_M^{nm}(\Gamma)$, *where*

$$b_r(t)=\begin{pmatrix} a_{mr}(t) & a_{mr+1}(t) & \dots & a_{mr+m-1}(t)\\ O_{n\times n} & a_{mr}[\alpha(t)] & \dots & a_{mr+m-2}[\alpha(t)]\\ \vdots & \vdots & \ddots & \vdots\\ O_{n\times n} & O_{n\times n} & \dots & a_{mr}[\alpha_{m-1}(t)]\end{pmatrix},$$

$$b_\nu(t)=\begin{pmatrix} a_{m\nu}(t) & O_{n\times n} & \dots & O_{n\times n}\\ a_{m\nu-1}[\alpha(t)] & a_{m\nu}[\alpha(t)] & \dots & O_{n\times n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m\nu-m+1}[\alpha_{m-1}(t)] & a_{m\nu-m+2}[\alpha_{m-1}(t)] & \dots & a_{m\nu}[\alpha_{m-1}(t)]\end{pmatrix},$$

$$b_j(t)=\begin{pmatrix} a_{mj}(t) & a_{mj+1}(t) & \dots & a_{mj+m-1}(t)\\ a_{mj-1}[\alpha(t)] & a_{mj}[\alpha(t)] & \dots & a_{mj+m-2}[\alpha(t)]\\ \vdots & \vdots & \ddots & \vdots\\ a_{mj-m+1}[\alpha_{m-1}(t)] & a_{mj-m+2}[\alpha_{m-1}(t)] & \dots & a_{mj}[\alpha_{m-1}(t)]\end{pmatrix},$$

$j=r+1,r+2,\dots,\nu-1$ $(\nu>r)$.

4. Algebras of functional operators

4.1. Let Γ and α satisfy the conditions of Subsection 2.1 and let $\mathcal{L}(X)$ denote the algebra of all bounded linear operators acting in a Banach space X. In the space $L_p(\Gamma), 1\le p\le\infty$, we consider the Banach algebra $\mathfrak{A}$ generated by polynomial functional operators A of the form

$$A=\sum a_kU^k, \tag{4.1}$$

where the coefficients a_k are continuous functions on Γ and U is the isometric shift operator: $(U\varphi)(t)=|\alpha'(t)|^{1/p}\varphi[\alpha(t)],\ t\in\Gamma$. To each operator (4.1) there corresponds an operator-valued function defined by the infinite finite-diagonal matrix

$$\mathcal{A}(t)=(a_{j-i}[\alpha_i(t)])_{i,j=-\infty}^{+\infty},\qquad t\in\Gamma\backslash\Lambda,$$

and

$$\|A\|_{\mathcal{L}(L_p(\Gamma))}=\sup_{t\in\Gamma\backslash\Lambda}\|\mathcal{A}(t)I\|_{\mathcal{L}(l_p)}, \tag{4.2}$$

$$\|A\|_0\overset{\text{def}}{=}\inf_{\|f\|=1}\|Af\|=\inf_{t\in\Gamma\backslash\Lambda}\|\mathcal{A}(t)I\|_0. \tag{4.3}$$

Hence the mappings $A\to\mathcal{A}(t)I,\ t\in\Gamma\backslash\Lambda$, are extended by continuity to homomorphisms of $\mathfrak{A}$ into $\mathcal{L}(l_p)$ and, moreover, the equalities (4.2)–(4.3) are valid for all $A\in\mathfrak{A}$.

Lemma 4.1. *An operator* $A\in\mathfrak{A}$ *is invertible in the space* $L_p(\Gamma), 1\le p\le\infty$, *if and only if for all* $t\in\Gamma\backslash\Lambda$ *the operators* $\mathcal{A}(t)I$ *are invertible in the space* l_p.

Every operator $\mathcal{A}(t)I$ $(t \in \Gamma\backslash\Lambda)$ is a compact perturbation of a discrete Wiener-Hopf operator pair in l_p $(1 \le p \le \infty)$ [45, 46]. Therefore, with the use of passage in the case $p = \infty$ to a preconjugate operator, we can define the continuous functions $\mathcal{A}_\tau$ $(\tau \in \Lambda)$ on the unit circle $\mathbb{T}$:

$$\mathcal{A}_\tau(z) = \sum a_k(\tau) z^k, \qquad z \in \mathbb{T},$$

where $\|\mathcal{A}_\tau\|_{C(\mathbb{T})} \le \|A\|_{\mathcal{L}(L_p(\Gamma))}$ for all $\tau \in \Lambda$ and $1 \le p \le \infty$. The mappings $A \to \mathcal{A}_\tau$ are homomorphisms of $\mathfrak{A}$ into $C(\mathbb{T})$.

Theorem 4.1. *An operator $A \in \mathfrak{A}$ is invertible in the space $L_p(\Gamma), 1 \le p \le \infty$, if and only if*:

1) $\mathcal{A}_\tau(z) \ne 0$ *for all* $z \in \mathbb{T}$ *and* $\tau \in \Lambda$,
2) *the numbers* $\kappa_\tau(A) = \frac{1}{2\pi}\{\arg \mathcal{A}_\tau(z)\}_{z\in\mathbb{T}}$ *coincide for all* $\tau \in \Lambda$,
3) *for all* $t \in \Gamma\backslash\Lambda$ *there exists a positive integer* m_0 *such that for all* $m \ge m_0$,

$$\Omega_{A,m}(t) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{\det(a_{\kappa+j-i}[\alpha_i(t)])_{i,j=-n}^{n}}{\det(a_{\kappa+j-i}[\alpha_i(t)])_{i,j=-n,\dots,-m,m,\dots,n}} \ne 0, \qquad z \in \mathbb{T},$$

where κ *is the common value of* $\kappa_\tau(A)$.

Theorem 4.1 for $1 < p < \infty$ was proved in [17, 20]. Its conditions 1)–2) follow from the Noether property of the discrete Wiener-Hopf operator pair corresponding to the operator $\mathcal{A}(t)I \in \mathcal{L}(l_p)$, while condition 3) is connected with the projection method of solution of a Wiener-Hopf equation (for $1 \le p < \infty$).

The generalizations of these results for the case of $L_p^n(\Gamma), n \ge 1$, and piecewise continuous coefficients are contained in [17, 20] (see also [45, 46, 47]).

4.2. Let us consider the one-sided invertibility of operators $A \in \mathfrak{A}$ in $L_p(\Gamma)$, $1 \le p \le \infty$.

Lemma 4.2. *An operator $A \in \mathfrak{A}$ is right* (*left*) *invertible in the space $L_p(\Gamma)$ if and only if the operators $\mathcal{A}(t)I$ is right* (*left*) *invertible in the space l_p for all $t \in \Gamma\backslash\Lambda$.*

Lemma 4.2 was proved in [47]. From this lemma and $n(d)$-normality criterion for a Wiener-Hopf operator pair ([8], p. 291) corresponding to the operator $\mathcal{A}(t)I$ we obtain the following result.

Lemma 4.3. *If an operator $A \in \mathfrak{A}$ is right* (*left*) *invertible in the space $L_p(\Gamma)$, then $\mathcal{A}_\tau(z) \ne 0$ for all $z \in \mathbb{T}$ and $\tau \in \Lambda$.*

Fix $t \in \Gamma\backslash\Lambda$ and $\tau_\pm = \lim_{n\to\pm\infty} \alpha_n(t) \in \Lambda$. If $\mathcal{A}_{\tau_\pm}(z) \ne 0$ for each $z \in \mathbb{T}$, then set

$$\kappa_\pm = \frac{1}{2\pi}\{\arg \mathcal{A}_{\tau_\pm}(z)\}_{z\in\mathbb{T}}.$$

Let

$$\mathcal{A}_n^{ij}(t) = \Pi_n^i \mathcal{A}(t) \tilde{\Pi}_n^j \qquad (i, j = 1, 2),$$

where $\Pi_n^2 = I - \Pi_n^1$, $\tilde{\Pi}_n^2 = I - \tilde{\Pi}_n^1$, Π_n^1 and $\tilde{\Pi}_n^1$ are the projections in l_p which preserve all components of vectors $\{\varphi_k\}_{-\infty}^{+\infty} \in l_p$ with indices $|k| \le n$ and $-n + \kappa_- \le k \le n + \kappa_+$, respectively, and annihilate the others.

LEMMA 4.4. *If $\mathcal{A}_{\tau_\pm}(z) \neq 0$ for each $z \in \mathbb{T}$, then for all sufficiently large n the operators $\mathcal{A}_n^{22}(t) : \tilde{\Pi}_n^2 l_p \to \Pi_n^2 l_p$ are invertible.*

Consequently, under the conditions of Lemma 4.4 the right (left) invertibility of $\mathcal{A}(t)I$ in l_p is equivalent to the right (left) invertibility of the rectangular $(2n+1) \times (2n+1+\kappa)$ matrices

$$\mathcal{K}_n[\mathcal{A}(t)] = \mathcal{A}_n^{11}(t) - \mathcal{A}_n^{12}(t)[\mathcal{A}_n^{22}(t)]^{-1}\mathcal{A}_n^{21}(t),$$

where $\kappa = \kappa_+ - \kappa_-$. Finally, the criterion for one-sided invertibility of $A \in \mathfrak{A}$ has the following form.

THEOREM 4.2. *An operator $A \in \mathfrak{A}$ is right (left) invertible in the space $L_p(\Gamma)$ if and only if*

1) *$\mathcal{A}_\tau(z) \neq 0$ for all $z \in \mathbb{T}$ and $\tau \in \Lambda$;*
2) *for every $t \in \Gamma \backslash \Lambda$ there exists a positive integer n_0 such that for all $n > n_0$*

$$\operatorname{rank} \mathcal{K}_n[\mathcal{A}(t)] = 2n+1 \qquad (\textit{respectively},\ 2n+1+\kappa_+ - \kappa_-),$$

where $\kappa_\pm$ correspond to $\tau_\pm = \lim_{n\to\pm\infty} \alpha_n(t)$.

Theorem 4.2 was proved in [23, 24].

4.3. In this subsection we study the invertibility of functional operators in reflexive Orlicz spaces $L_M(\Gamma)$ and generalized Hölder spaces $H_\omega(\Gamma)$. We consider the operators

$$A = \sum_{k=-\infty}^{+\infty} a_k W^k, \tag{4.4}$$

where $a_k \in C(\Gamma)$ or $H_\omega(\Gamma)$, respectively, $(W\varphi)(t) = \varphi[\alpha(t)]$ for $t \in \Gamma$ and $\sum \|a_k W^k\| < \infty$. We assume also that the diffeomorphism $\alpha : \Gamma \to \Gamma$ preserves orientation and has a finite set Λ of fixed points. Let

$$\mathbb{K}_\tau = \{z : K_i(|\alpha'(\tau)|) \le |z|^m \le K_e(|\alpha'(\tau)|)\}, \qquad \tau \in \Lambda,$$

where $K_i(c)$ and $K_e(c)$ are defined for $L_M(\Gamma)$ and $H_\omega(\Gamma)$ in Subsection 2.2, and

$$\mathcal{A}_\tau(z) = \sum a_k(\tau) z^k, \qquad z \in \mathbb{K}_\tau,\ \tau \in \Lambda.$$

THEOREM 4.3. *The operator* (4.4) *is invertible in the space $L_M(\Gamma)$ if and only if*

1) *$\mathcal{A}_\tau(z) \neq 0$ for all $z \in \mathbb{K}_\tau$ and $\tau \in \Lambda$,*
2) *the numbers $\kappa_\tau(A) = \frac{1}{2\pi}\{\arg \mathcal{A}_\tau(K_i(|\alpha'(\tau)|)z)\}_{z\in\mathbb{T}}$ coincide for all $\tau \in \Lambda$,*
3) *condition* 3) *of Theorem* 4.1 *is fulfilled.*

From Theorems 4.1 and 4.3 we obtain the next important corollary.

THEOREM 4.4. *The operator* (4.4) *is invertible in the space* $L_M(\Gamma)$ *if and only if it is invertible in every space* $L_p(\Gamma)$, *where* $1/p \in [\nu_1, \nu_0]$ *and* ν_0 (ν_1) *is the upper* (*lower*) *Boyd index of the reflexive Orlicz space* $L_M(\Gamma)$.

The same results are valid for generalized Hölder spaces.

THEOREM 4.5. 1) *The operator* (4.4) *is invertible in the space* $H_\omega(\Gamma)$ *if and only if it is invertible in the space* $H^0_\omega(\Gamma, \Lambda)$ *and* $\sum a_n(\tau) \neq 0$ *for all* $\tau \in \Lambda$.

2) *The operator* (4.4) *is invertible in the space* $H^0_\omega(\Gamma, \Lambda)$ *if and only if conditions* 1)–3) *of Theorem* 4.3 *are fulfilled.*

For generalized Hölder spaces $H_\omega(\Gamma)$ we can define the following analogues of Boyd's indices (cf. [48]):

$$\mu_0 = \lim_{c\to+0} \frac{\ln h(c)}{\ln c} = \sup_{0<c<1} \frac{\ln h(c)}{\ln c}, \qquad \mu_1 = \lim_{c\to+\infty} \frac{\ln h(c)}{\ln c} = \inf_{c>1} \frac{\ln h(c)}{\ln c},$$

where $h(c) = \limsup_{x\to+0} \frac{\omega(cx)}{\omega(x)}$ for $c > 0$ and $0 \le \mu_0 \le \mu_1 \le 1$.

THEOREM 4.6. *The operator* (4.4) *is invertible in the space* $H_\omega(\Gamma)$ (*or* $H^0_\omega(\Gamma, \Lambda)$) *if and only if it is invertible in every space* $H_\mu(\Gamma)$ (*respectively*, $H^0_\mu(\Gamma, \Lambda)$), *where* $\mu \in [\mu_0, \mu_1]$.

To each point $t \in \Gamma\backslash\Lambda$ we assign the Banach algebra $h_\omega(t) = H_\omega(\overline{G_t})$, where $\overline{G_t}$ is the closure of the orbit $G_t = \{\alpha_k(t) : k \in \mathbb{Z}\}$. Furthermore, we define the Banach space

$$h^0_\omega(t) = \{\varphi \in h_\omega(t) : \varphi(\lim_{n\to\pm\infty} \alpha_n(t)) = 0\}.$$

In the space $h^0_\omega(t)$ we consider the discrete operator A_t given by

$$(A_t\varphi)(x) = \sum a_k(x)\varphi[\alpha_k(x)], \qquad x \in \overline{G_t}.$$

THEOREM 4.7. *The operator* (4.4) *is invertible in the space* $H^0_\omega(\Gamma, \Lambda)$ *if and only if for every* $t \in \Gamma\backslash\Lambda$ *the operator* A_t *is invertible in the space* $h^0_\omega(t)$.

For binomial functional operators we can prove the following stronger result (see [27, 28] in the case $\omega(\delta) = \delta^\mu$).

THEOREM 4.8. *The operator* $A = aI - bW$ *with coefficients* $a, b \in H_\omega(\Gamma)$ *is right* (*left*) *invertible in the space* $H^0_\omega(\Gamma, \Lambda)$ *if and only if for every* $t \in \Gamma\backslash\Lambda$ *the operator* A_t *defined by*

$$(A_t\varphi)(x) = a(x)\varphi(x) - b(x)\varphi[\alpha(x)], \qquad x \in \overline{G_t},$$

is right (*left*) *invertible in the space* $h^0_\omega(t)$.

THEOREM 4.9. *If* $t \in \Gamma\backslash\Lambda$, *then the operator* A_t *is invertible in the space* $h^0_\omega(t)$ *if and only if*

1) $\mathcal{A}_\tau(z) \neq 0$ *for all* $z \in \mathbb{K}_\tau$ *and* $\tau = \tau_\pm \stackrel{\text{def}}{=} \lim_{n\to\pm\infty} \alpha_n(t)$,
2) $\kappa_{\tau_+}(A) = \kappa_{\tau_-}(A)$,
3) $\Omega_{A,m}(t) \neq 0$ *for all sufficiently large* $m \in \mathbb{N}$.

The results of Subsection 4.3 were proved by the author. Theorem 4.5 was announced in [21].

4.4. Let $\Gamma = \mathbb{T}$ and $\alpha(t) = e^{i\gamma}t$ for $t \in \Gamma$, where $\gamma/2\pi$ is an irrational number. Then α does not have periodic points, and the closure of each orbit G_t coincides with Γ. In the space $L_p(\Gamma), 1 \le p \le \infty$, we consider the Banach algebra $\mathfrak{A}$ generated by polynomial functional operators (4.1) with piecewise continuous coefficients a_k and the shift operator $U : (U\varphi)(t) = \varphi(e^{i\gamma}t),\ t \in \Gamma$. To each operator $A \in \mathfrak{A}$ there correspond two operator-valued functions

$$\mathcal{A}^{\pm}(t) = (a_{j-i}[\alpha_i(t \pm 0)])_{i,j=-\infty}^{+\infty}, \qquad t \in \Gamma,$$

with values in the algebra $\mathcal{L}(l_p)$.

LEMMA 4.5. *For every $A \in \mathfrak{A}$ and $t \in \Gamma$*

$$\|A\| = \|\mathcal{A}^{\pm}(t)\|, \qquad \|A\|_0 = \|\mathcal{A}^{\pm}(t)\|_0.$$

THEOREM 4.10. *An operator $A \in \mathfrak{A}$ is invertible in $L_p(\Gamma), 1 \le p \le \infty$, if and only if one of the operators $\mathcal{A}^{\pm}(t)$ (equivalently, all the operators $\mathcal{A}^{\pm}(t)$) is (are) invertible in l_p.*

The local-trajectory theory for the invertibility of functional operators with piecewise continuous coefficients in the space $L_p(\Gamma)$ was constructed by the author also for much more complicated groups of shifts.

DEFINITION 4.1. A discrete group is called *subexponential* [2] if for any finite subset F of it, $\lim_{n\to\infty} |F^n|^{1/n} = 1$, where $|F^n|$ is the number of different words of length n constructed from elements of F.

In particular, the following groups are subexponential: finite groups, commutative groups, and groups with a polynomial growth of the number of words or with a growth of the number of words that is between polynomial and exponential growth.

DEFINITION 4.2. A discrete group is called *amenable* [9] if on the space $l_\infty(G)$ of all bounded complex-valued functions at G with supremum-norm there exists an invariant mean, i.e. a state (positive linear functional with unit norm) m satisfying the condition

$$m(f) = m({}_hf) = m(f_h)$$

for all $h \in G$ and $f \in l_\infty(G)$, where

$$({}_hf)(g) = f(h^{-1}g), \quad (f_h)(g) = f(gh), \quad g \in G.$$

All subexponential (or solvable) groups are amenable. As there exist solvable groups with an exponential growth of the number of words, the class of amenable groups is strictly wider than the class of subexponential groups.

DEFINITION 4.3. A discrete group G of diffeomorphisms $g : \Gamma \to \Gamma$ *acts topologically freely on* Γ (cf. [2]) if for any finite subset $F \subset G$ and any arc $\gamma \subset \Gamma$ there exists a point $t \in \gamma$ with mutually different images $g(t)$ $(g \in F)$.

Let G be a discrete group of diffeomorphisms of Γ onto itself. We consider the Banach subalgebra $\mathfrak{A} \subset \mathcal{L}(L_p(\Gamma)), 1 \le p \le \infty$, generated by operators

$$A = \sum_{g \in F} a_g U_g \tag{4.5}$$

with piecewise continuous coefficients a_g and isometric shift operators U_g : $(U_g\varphi)(t) = |g'(t)|^{1/p}\varphi[g(t)]$, $t \in \Gamma$ (here F runs through finite subsets of G). To each operator (4.5) and each point $t \in \Gamma$ there correspond the discrete operators $A_t^\pm \in \mathcal{L}(l_p(G)), 1 \le p \le \infty$, defined by

$$(A_t^\pm f)(h) = \sum a_g[h(t \pm 0)]f(hg), \quad h \in G,$$

where $(hg)(\cdot) = g[h(\cdot)]$.

LEMMA 4.6. *If G acts topologically freely on Γ, then the mappings $A \to A_t^\pm$ $(t \in \Gamma)$ can be extended to homomorphisms of $\mathfrak{A}$ into $\mathcal{L}(l_p(G)), 1 \le p \le \infty$, and*

$$(\forall A \in \mathfrak{A})\ (\forall t \in \Gamma) \quad \|A_t^\pm\| \le \|A\|, \quad \|A\|_0 \le \|A_t^\pm\|_0.$$

COROLLARY 4.1. *Under the condition of Lemma 4.6, the invertibility of $A \in \mathfrak{A}$ in $L_p(\Gamma)$ implies the invertibility in $l_p(G)$ of all operators $A_t^\pm$ $(t \in \Gamma)$.*

THEOREM 4.11. *If G is an amenable group of diffeomorphisms $g : \Gamma \to \Gamma$ and G acts topologically freely on Γ, then an operator $A \in \mathfrak{A}$ is invertible (right invertible, left invertible) in the space $L_2(\Gamma)$ if and only if all operators $A_t^\pm$ $(t \in \Gamma)$ are invertible (right invertible, left invertible) in the space $l_2(G)$.*

This invertibility criterion was proved by the author (cf. [21]) on the basis of the local-trajectory method [11, 14] of studying the invertibility in C*-algebras of operators with discrete groups of shifts. This method is closely connected with the study of crossed products of C*-algebras and groups of their automorphisms. A corresponding result for subexponential groups of shifts was proved in [2].

4.5. For other $p \in [1, \infty]$ we have a weaker result.

THEOREM 4.12. *If G is a subexponential group of diffeomorphisms acting topologically freely on Γ, then for the invertibility of an operator $A \in \mathfrak{A}$ in the space $L_p(\Gamma), p \in [1, \infty]\backslash\{2\}$, it is necessary, and under the additional condition*

$$(\forall A \in \mathfrak{A}) \quad \|A\| = \sup_{t \in \Gamma} \max\{\|A_t^+\|, \|A_t^-\|\}, \tag{4.6}$$

it is also sufficient, that the operators $A_t^\pm$ be invertible in the space $l_p(G)$ for all $t \in \Gamma$ and

$$\sup_{t \in \Gamma} \max\{\|(A_t^+)^{-1}\|, \|(A_t^-)^{-1}\|\} < \infty. \tag{4.7}$$

We remark that for the invertibility of the operator (4.5) condition (4.6) is not necessary. Theorem 4.11 (with the additional condition (4.7)), Theorem 4.12 and their generalizations for the case of violation of the topologically free action of G on Γ were obtained by the author in [12, 21, 22, 25].

5. Systems of difference equations with incommensurable differences

5.1. Let $\Gamma = \mathbb{R}_+$ or $\Gamma = (0, \lambda), 0 < \lambda < \infty$. In the space $L_p^n(\Gamma)$, $1 \le p \le \infty$, we consider the functional operator

$$A = \chi_\Gamma \sum a_h U_h, \tag{5.1}$$

where χ_Γ is the characteristic function of Γ, a_h are constant complex matrices and $(U_h\varphi)(t) = \varphi(t+h)$, $t \in \Gamma$ (here h runs through an at most countable subset of $\mathbb{R}$, and we imbed $L_p^n(\Gamma)$ into $L_p^n(\mathbb{R})$ with the help of the zero extension on $\mathbb{R}\backslash\Gamma$). We suppose that $\sum \|a_h\| < \infty$. Then the almost periodic matrix-valued function (briefly MVF)

$$\mathcal{A}(x) = \sum a_h e^{ihx} \quad (x \in \mathbb{R}) \tag{5.2}$$

belongs to the class AP_W of absolutely convergent Fourier series. Obviously, AP_W is the Banach algebra with norm $\|\mathcal{A}\|_W = \sum \|a_h\|$.

Since the MVF $\mathcal{A} \in AP_W$, either the operator (5.1) is invertible in all spaces $L_p^n(\Gamma)$, $1 \le p \le \infty$, or it is invertible in no space $L_p^n(\Gamma)$ (see [39]).

The invertibility criteria for A on a semi-axis and a finite interval are different, but in both cases they are closely connected with factorizability of some almost periodic MVF's.

Let $AP_W^\pm$ be the subalgebra of AP_W consisting of MVF's with spectrum in $\mathbb{R}_\pm$.

Definition 5.1. An $n \times n$ MVF G defined on $\mathbb{R}$ is called *AP_W-factorizable* if it is represented in the form $G = G^+\Lambda G^-$, where $\Lambda(x) = \operatorname{diag}\{e^{i\lambda_1 x}, \dots, e^{i\lambda_n x}\}$, all $\lambda_j \in \mathbb{R}$ and MVF's $G^\pm$, $(G^\pm)^{-1} \in AP_W^\pm$.

Such and similar factorizations of almost periodic MVF's and their connections with the Fredholm theory for singular integral operators with oscillating coefficients and convolution type operators with oscillating presymbols were investigated e.g. in [13, 21, 26, 31, 32, 33, 34, 35, 50, 51, 52]. In particular, according to [33, 35, 51] the n-tuple $\{\lambda_j\}$ is uniquely determined for every AP_W-factorizable MVF G. The numbers λ_j are called the *partial P-indices* of the MVF G.

Theorem 5.1. *If $\Gamma = \mathbb{R}_+$, then the operator* (5.1) *is invertible in the space $L_p^n(\Gamma), 1 \le p \le \infty$, if and only if the MVF $\mathcal{A}(x)$ is AP_W-factorizable with $\lambda_j = 0$ $(j = 1, \dots, n)$.*

Theorem 5.2. *If $\Gamma = \mathbb{R}_+$, then the operator* (5.1) *is right* (*left*) *invertible in $L_2^n(\Gamma)$ if and only if the MVF*

$$\begin{pmatrix} \mathcal{A}(x) & 0 \\ I_n + \mathcal{A}^*(x)\mathcal{A}(x) & \mathcal{A}^*(x) \end{pmatrix} \quad \left(\textit{respectively}, \begin{pmatrix} \mathcal{A}^*(x) & 0 \\ I_n + \mathcal{A}(x)\mathcal{A}^*(x) & \mathcal{A}(x) \end{pmatrix}\right)$$

is AP_W-factorizable with $\lambda_j = 0$ $(j = 1, \dots, 2n)$.

For other $p \in [1, \infty]$ we have only sufficient conditions for one-sided invertibility of A in $L_p^n(\mathbb{R}_+)$.

LEMMA 5.1. *If an MVF $\mathcal{A}$ is AP_W-factorizable and all $\lambda_j \geq 0$ (respectively, $\lambda_j \leq 0$), then the operator* (5.1) *is right (left) invertible in the space $L_p^n(\mathbb{R}_+)$.*

Note that AP_W-factorization of $\mathcal{A}$ does not always follow from the one-sided invertibility of A in $L_p^n(\mathbb{R}_+)$.

THEOREM 5.3. *If $\Gamma = (0, \lambda)$ and $0 < \lambda < \infty$, then the operator* (5.1) *is invertible in the space $L_p^n(\Gamma), 1 \leq p \leq \infty$, if and only if the MVF*

$$\mathcal{B}(x) = \begin{pmatrix} e^{i\lambda x} I_n & 0 \\ \mathcal{A}(x) & e^{-i\lambda x} I_n \end{pmatrix} \tag{5.3}$$

is AP_W-factorizable with $\lambda_j = 0$ $(j = 1, \ldots, 2n)$.

In the case $n = 1$ Theorem 5.1 is contained in [8]. This result for $n > 1$ and Theorems 5.2–5.3 were obtained by the author (see [26, 21, 13]). Generalizations of these results to uniform almost periodic presymbols $\mathcal{A}$ in the case $p = 2$ can be found in [13]. They require a generalization of the concept of factorizability for MVF's $\mathcal{A} \in AP$ (in particular, factors $\mathcal{A}^{\pm}$ and their inverses belong to the space $B_2^{\pm}$ of Besicovitch almost periodic MVF's with spectrum in $\mathbb{R}_{\pm}$).

5.2. A number of effective criteria and algorithms for AP_W-factorization of an MVF $G \in AP_W$ are contained in [31, 32, 33, 34, 35, 50, 52, 5, 21]. They allow one to construct effectively the two-sided and one-sided inverse operators for the operators (5.1). The above mentioned algorithms are closely connected with the theory of continued fractions (both numerical and functional) and the theory of matrix bundles. We present two corresponding examples (see [33, 35]).

THEOREM 5.4. *If $\mathcal{A}(x) = Ce^{i\mu x} - De^{i\sigma x}$ and $-\lambda < \sigma < \mu < \lambda$, then the MVF* (5.3) *is AP_W-factorizable with almost periodic polynomials, and its partial P-indices λ_j are equal to zero if and only if one of the following conditions holds*: 1) $\mu = 0, \det C \neq 0$; 2) $\sigma = 0, \det D \neq 0$; 3) $\mu\sigma < 0$, *the number $\lambda/(\mu - \sigma)$ is an integer and* $\det CD \neq 0$.

In the case of almost periodic trinomial $\mathcal{A}(x)$ the MVF (5.3) is not always AP_W-factorizable.

THEOREM 5.5. *Let $\Gamma = (0, \lambda)$, $\mathcal{A}(x) = C_{-1}e^{-i\nu x} - C_0 + C_1 e^{i\alpha x}$, $\det(C_{-1}C_0C_1) \neq 0$ and suppose the matrices $a_{\pm} = C_0^{-1}C_{\pm 1}$ commute. Let $\alpha, \nu > 0$, $\alpha + \nu = \lambda$ and let the number $\beta = \nu/\alpha$ be irrational. Then the operator* (5.1) *is invertible in the space $L_p^n(\Gamma)$ (equivalently, the MVF* (5.3) *is AP_W-factorizable with zero P-indices) if and only if $\sigma(a_+^{\beta} a_-) \cap \mathbb{T} = \emptyset$.*

The proof of Theorem 5.5 is essentially based on the decomposition of the irrational number β in a numerical continued fraction.

5.3. Finally, we present an approximate approach to the problem of invertibility of operators (5.1) which was developed by P. M. Tishin and the author.

Fix a MVF $\mathcal{A} \in AP_W$. It is represented in the form (5.2), where h runs through an Abelian group $H = H(\mathcal{A})$ generated by an at most countable set of real numbers γ_j which are linearly independent over the ring $\mathbb{Z}$. We consider arbitrary numerical sequences $\gamma_{j,k} \to \gamma_j$ as $k \to \infty$. Let H_k be Abelian groups generated by $\{\gamma_{j,k}\}$ and let $\nu_k : H \to H_k$ be corresponding homomorphisms. To the MVF $\mathcal{A}$ there correspond the sequence of MVF's

$$\mathcal{A}_k(x) = \sum_{h \in H} a_h e^{i\nu_k(h)x}, \qquad k = 1, 2, \dots$$

We point out that the corresponding sequence of operators A_k converges to the operator (5.1) in $\mathcal{L}(L_p^n(\Gamma))$ only strongly.

THEOREM 5.6. *For an MVF $\mathcal{A} \in AP_W$ the following conditions are equivalent*:

1) *the MVF $\mathcal{A}$ is AP_W-factorizable with zero P-indices*,

2) *for all sufficiently large k the MVF's $\mathcal{A}_k$ are AP_W-factorizable with zero P-indices* (*i.e.* $\mathcal{A}_k = \mathcal{A}_k^+ \mathcal{A}_k^-$) *and the norms $\|(\mathcal{A}_k^{\pm})^{-1}\|_W$ are uniformly bounded.*

COROLLARY 5.1. *If either $\Gamma = \mathbb{R}_+$ or $\Gamma = (0, \lambda)$, then the operator* (5.1) *is invertible in the space $L_p^n(\Gamma)$, $1 \le p \le \infty$, if and only if for all sufficiently large k the operators A_k are invertible in $L_p^n(\Gamma)$ and the norms of their inverse operators are uniformly bounded.*

The last result is similar to [49], but the method of its proof is quite different.

With the help of Theorem 5.6 and Corollary 5.1 we can obtain the following

THEOREM 5.7. *Let $\Gamma = (0, \lambda)$ and*

$$\mathcal{A}(x) = a_0 + \sum_{j=1}^{N} a_j (e^{-ij\mu x} + e^{i(\lambda - j\mu)x}),$$

where $\lambda > N\mu > 0$ and λ/μ is an irrational number. Then the operator (5.1) *is invertible in the space $L_p(\Gamma)$, $1 \le p \le \infty$, if and only if*

$$a_0 + \sum_{j=1}^{N} a_j t^{-j} \ne 0 \quad \textit{for every} \ \ t \in \mathbb{T}.$$

Theorem 5.6 can also be applied for the proof of Theorem 5.5.

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