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NGUYEN VAN THU

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WROCŁAWSKA DRUKARNIA NAUKOWA

## CONTENTS

## Introduction

In the present paper([1]) we study the limit laws arising from affine modification of certain multi-parameter normed sums of independent Banach space valued random variables. We describe these limit laws in terms of their multi-dimensional decomposability algebraic structures and obtain the Urbanik representation theorems for the characteristic functionals.

The classical limit problem of characterizing of limit laws of normed sums of real-valued random variables was proposed by A. Ya. Khinchin in 1936 and solved by P. Lévy in [6] (p. 195) (see also M. Loéve [7], p. 319). The attempt to extend the theory to the multi-dimensional linear space case developed by H. Shape [13] has resulted in several recent papers of K. Urbanik (see [16], [17]). Namely, he introduced a concept of decomposability semigroup $D(\mu)$ associated with a probability measure $\mu$ on a Banach space and characterized all full Lévy's measures $\mu$ in terms of $D(\mu)$. Moreover, applying the extreme points method he obtained a representation for the characteristic functionals of Lévy's measures on $X$.

The purpose of this paper is to generalize the Lévy–Khinchin–Urbanik problem to the case where the summands are indexed by a countable lattice and take values in a Banach space. The technique developed in [17] by K. Urbanik will be widely exploited.

The Author would like to express his sincere gratitude to Professor K. Urbanik for many helpful discussions.

## 1. Notation and preliminaries

This paper is concerned with probability measures defined on Borel subsets of a real separable Banach space $X$ with the norm $\| \ \|$ and the topological dual space $X^*$. For a probability measure $\mu$ on $X$ its characteristic functional is defined by the formula

$$\hat{\mu}(y) = \int_X \exp i\langle y, x\rangle \, \mu(dx) \qquad (y \in X^*)$$

([1]) This paper was written during the author's stay at the University of Wroclaw (Poland) in the academic year 1976/77.

where $\langle\, ,\,\rangle$ denotes the dual pairing between $X$ and $X^*$. A sequence $\{\mu_j\}$ of probability measures on $X$ is said to *converge to a probability measure* $\mu$ if for every bounded continuous real-valued function $f$ on $X$

$$\int_X f\,d\mu_j \to \int_X f\,d\mu.$$

A probability measure $\mu$ on $X$ is called *full* if its support is not contained in any proper hyperplane of $X$. Further, by $\delta_x$, $x \in X$, we shall denote the unit mass at $x$.

Let $B(X)$ denote the algebra of all continuous linear operators on $X$ with the norm topology. The unit and zero elements of $B(X)$ will be denoted by $I$ and $0$ respectively. An element $P$ of $B(X)$ is called a *projector* if $P^2 = P$. Given a subset $F$ of $B(X)$ let $\mathrm{Sem}(F)$ denote the closed multiplicative semigroup of operators spanned by $F$.

The concept of decomposability semigroup $D(\mu)$ of linear operators associated with a probability measure $\mu$ on $X$ was introduced in [16] and [17] by K. Urbanik. Namely, $D(\mu)$ consists of all operators $A$ from $B(X)$ for which the equality

$$\mu = A\mu * \mu_A \tag{1.1}$$

holds for a certain probability measure $\mu_A$ on $X$. Here $*$ denotes the convolution of measures and $A\mu$ denotes a probability measure defined by the formula

$$A\mu(E) = \mu(A^{-1}(E))$$

for all Borel subsets $E$ of $X$. Since $\widehat{A\mu}(y) = \hat{\mu}(A^* y)$, $y \in X^*$, we can write (1.1) in the form

$$\hat{\mu}(y) = \hat{\mu}(A^* y)\hat{\mu}_A(y) \qquad (y \in X^*). \tag{1.2}$$

In the sequel we shall need a generalization of the concept of decomposability semigroups. Let $d$ be a fixed positive integer and $A_1, A_2, \ldots, A_d$ be some operators from $B(X)$. Then a probability measure $\mu$ on $X$ is said to be *$\langle A_1, A_2, \ldots, A_d\rangle$-decomposable* if there exist probability measures $\mu_{A_1}, \mu_{A_1,A_2}, \ldots, \mu_{A_1,A_2,\ldots,A_d}$ such that $\mu = A_1\mu * \mu_{A_1}$, $\mu_{A_1} = A_2\mu_{A_1} * \mu_{A_1,A_2}, \ldots$ $\ldots, \mu_{A_1,A_2,\ldots,A_{d-1}} = A_d\mu_{A_1,A_2,\ldots,A_{d-1}} * \mu_{A_1,A_2,\ldots,A_d}$. It is evident that if $\hat{\mu}(y) \neq 0$ for every $y \in X^*$ and $\mu$ is $\langle A_1, A_2, \ldots, A_d\rangle$-decomposable then the measures $\mu_{A_1}, \mu_{A_1,A_2}, \ldots$ are uniquely determined. Further, let $A_1, A_2, \ldots$ be an infinite sequence of operators from $B(X)$. Then a probability measure $\mu$ on $X$ is said to be *$\langle A_1, A_2, \ldots\rangle$-decomposable* if there exist probability measures $\mu_{A_1}, \mu_{A_1,A_2}$, such that $\mu = A_1\mu * \mu_{A_1}$, $\mu_{A_1} = A_1\mu_{A_1} * \mu_{A_1,A_2}, \ldots$ Let us introduce the notation:

$D^d(\mu) = \{\langle A_1, A_2, \ldots, A_d\rangle;\ A_j \in B(X),\ j = 1, 2, \ldots, d$ and $\mu$ is $\langle A_1, A_2, \ldots$ $\ldots, A_d\rangle$-decomposable$\}$ and

$D^{\infty}(\mu) = \{\langle A_1, A_2, \ldots\rangle: A_j \in B(X), j = 1, 2, \ldots$ and $\mu$ is $\langle A_1, A_2, \ldots$ decomposable$\}$.

It is clear that for every probability measure $\mu$ on $X$ the sets $D^d(\mu)$, $d = 1, 2, \ldots, \infty$, are non-empty and closed under the product weak* operator topology. In the sequel every set $D^d(\mu)$, $d = 1, 2, \ldots$, will be called a *d-dimensional decomposability algebraic structure* associated with a probability measure $\mu$ on $X$. For $d = 1$ $D^d(\mu) = D(\mu)$ is a semigroup under multiplication of operators. One may expect that $D^d(\mu)$, $d \geqslant 2$, should be a semigroup under multiplication of corresponding coordinates. Unfortunately this fails to be true if the measure $\mu$ is not concentrated at a single point of $X$. The loss of semigroup properties of $D^d(\mu)$ for the case $d \geqslant 2$ seems to be the main dificulty to use the method developed in [17] by K. Urbanik.

Let $F$ be a subset of $B(X)$. We say that a probability measure $\mu$ on $X$ is *d-times*, $d = 1, 2, \ldots$, (resp. *completely*) *F-decomposable* if the Cartesian product $F \times F \times \quad \times F$ ($d$-times) (resp. $F \times F \times \ldots$) is contained in $D^d(\mu)$ (resp. $D^{\infty}(\mu)$). For the further convenience if $\mu$ is completely $F$-decomposable we shall write that $\mu$ is $\infty$-times $F$-decomposable. In particular, for $F = \{cI: 0 < c < 1\}$ the concept of multiply $F$-decomposable probability measures coincides with the concept of multiply self-decomposable probability measures introduced in [8].

Now we shall establish some simple properties of the multi-dimensional decomposability algebraic structures associated with probability measures on $X$.

PROPOSITION 1.1. *Let $F_1, F_2, \ldots$ be a sequence of subsets of $B(X)$ with the property that for any $i \neq j$, $A \in F_i$ and $B \in F_j$ we have $AB = BA$. Suppose that the Cartesian product $F_1 \times F_2 \times \quad \times F_d$, $d = 1, 2,$ (resp. $F_1 \times F_2 \times \ldots$) is contained in $D^d(\mu)$ (resp. $D^{\infty}(\mu)$). Then*

$$\mathrm{Sem}(F_1) \times \mathrm{Sem}(F_2) \times \quad \times \mathrm{Sem}(F_d) \subset D^d(\mu)$$

(*resp.* $\mathrm{Sem}(F_1) \times \mathrm{Sem}(F_2) \times \quad \subset D^{\infty}(\mu)$).

Proof. It suffices to prove the proposition for the case $d < \infty$. Further, since $D^d(\mu)$ is closed in the norm product topology of $B(X) \times B(X) \times$ $\times B(X)$ ($d$ times) and by a simple induction it suffices to prove that if $\langle A_1^{(i)}, A_2, \ldots, A_d\rangle$, $i = 1, 2$, belong to $F_1 \times F_2 \times \quad \times F_d$ then the element $\langle A_1^{(1)} A_1^{(2)}, A_2, \ldots, A_d\rangle$ belongs to $D^d(\mu)$.

Accordingly, let $\mu_{A_1^{(i)}}, \mu_{A_1^{(i)}, A_2}, \ldots, \mu_{A_1^{(i)}, A_2, \ldots, A_d}$ $(i = 1, 2)$ be such probability measures that $\mu = A_1^{(i)} \mu * \mu_{A_1^{(i)}}$,

$$\mu_{A_1^{(i)}} = A_2 \mu_{A_1^{(i)}} * \mu_{A_1^{(i)}, A_2}, \ldots, \mu_{A_1^{(i)}, A_2, \ldots, A_{d-1}}$$
$$= A_d \mu_{A_1^{(i)}, A_2, \ldots, A_{d-1}} * \mu_{A_1^{(i)}, A_2, \ldots, A_d} \quad (i = 1, 2).$$

Then we have the equation

$$A_1\mu = A_1^{(2)}A_1^{(1)}\mu * A_1^{(2)}\mu_{A_1^{(1)}}$$

and consequently,

$$\mu = A_1^{(1)}A_1^{(2)}\mu * \mu_{A_1^{(1)}A_1^{(2)}}$$

where the measure $\mu_{A_1^{(1)}A_1^{(2)}}$ is defined as $\mu_{A_1^{(2)}} * A_1^{(2)}\mu_{A_1^{(1)}}$. Moreover, we have

$$\begin{aligned}\mu_{A_1^{(1)}A_1^{(2)}} &= A_2\,\mu_{A_1^{(2)}} * \mu_{A_1^{(2)},A_2} * A_1^{(2)}A_2\,\mu_{A_1^{(1)}} * A_1^{(2)}\mu_{A_1^{(1)},A_2}\\ &= A_2\,\mu_{A_1^{(1)}A_1^{(2)}} * \mu_{A_1^{(1)}A_1^{(2)},A_2}\end{aligned}$$

where the measure $\mu_{A_1^{(1)}A_2^{(2)},A_2}$ is defined as $\mu_{A_1^{(2)},A_2} * A_1^{(2)}\mu_{A_1^{(1)},A_2}$. This means that $\langle A_1^{(1)}A_1^{(2)}, A_2\rangle$ belongs to $D^2(\mu)$. Proceeding successively, by induction, it follows that $\langle A_1^{(1)}A_1^{(2)}, A_2, \ldots, A_d\rangle \in D^d(\mu)$. The proposition is thus proved.

PROPOSITION 1.2. *Let $P_1, P_2, \ldots, P_r$ be some projectors in $B(X)$ with the property that $P_iP_j = P_jP_i = 0$ for all indexes $i \neq j$. Given a number $v = 1, 2, \ldots, d$ let $A_1, A_2, \ldots, A_v^{(i)}, \ldots, A_d$ $(i = 1, 2, \ldots, r)$ be such operators that for every $i = 1, 2, \ldots, r$ $A_1, A_2, \ldots, A_v^{(i)}, \ldots, A_d$ commute one another and for each choice of $1 \leqslant j_1 < j_2 < \quad < j_s \leqslant d$ we have $\langle A_{j_1}, A_{j_2}, \ldots, A_{j_s}\rangle \in D^s(\mu)$, where by $A_v$ we denote an arbitrary operator from the set $\{P_j, A_v^{(j)}: j = 1, 2, \ldots, r\}$. Put $B_j = A_j$ for $j = 1, 2, \ldots, d$ and $j \neq v$ and put $B_v = \sum_{i=1}^{r} P_iA_v^{(i)}$. Then for any $1 \leqslant j_1 < j_2 < \quad < j_s \leqslant d$ we have $\langle B_{j_1}, B_{j_2}, \ldots, B_{j_s}\rangle \in D^s(\mu)$.*

Proof. Let $j_1 < j_2 < \quad < j_s$ be a fixed subsequence of the sequence $\{1, 2, \ldots, d\}$. Without loss of generality we may assume that $\{j_1, j_2, \ldots, j_s\} = \{1, 2, \ldots, d\}$ and $v = 1$. Put $B = \sum_{i=1}^{r} P_iA_1^{(i)}$. Since $A_1^{(i)} \in D(\mu)$ we have the decompositions

$$\mu = A_1^{(i)}\mu * \mu_{A_1^{(i)}} \quad (i = 1, 2, \ldots, r).$$

Moreover by the proof of Proposition 1.2 [17] we have

$$\mu = B\mu * \mu_B$$

where the measure $\mu_B$ is defined by the formula

$$\mu_B = \mathop{*}_{i=1}^{r} P_i\,\mu_{A_1^{(i)}} * \Big(I - \sum_{i=1}^{r} P_i\Big)\mu \tag{1.3}$$

where the symbol $\mathop{*}\limits_{i=1}^{r}$ denotes the convolution of relevant measures. Further, we have the decompositions

$$\mu_{A_1^{(i)}} = A_2\,\mu_{A_1^{(i)}} * \mu_{A_1^{(i)},A_2} \qquad (i = 1, 2, \ldots, r),$$

and

$$\mu = A_2\,\mu * \mu_{A_2}.$$

Consequently, by virtue of (1.3) it follows that

$$\mu_B = A_2\,\mu_B * \mu_{B,A_2}$$

where the measure $\mu_{B,A_2}$ is defined by the formula

$$\mu_{B,A_2} = \mathop{*}_{i=1} P_i\,\mu_{A_1^{(i)},A_2} * \Big(I - \sum_{i=1}^{r} P_i\Big)\mu_{A_2}.$$

Finally, by induction, it follows that $\langle B, A_2, \ldots, A_d\rangle$ belongs to $D^d(\mu)$ which completes the proof of the proposition.

## 2. Statement of the problem

Let $N^d$, $d = 1, 2, \ldots$, denote the lattice of all $d$-vectors with natural components and with the natural ordering $\leqslant$. For $n = (n^1, n^2, \ldots, n^d) \in N^d$ we shall write $n \to \infty$ whenever $n^1, n^2, \ldots, n^d \to \infty$ simultaneously.

In the sequel, we shall use the letters $n$, $m$, $k$, $h$ to denote the vectors of $N^d$ and use the letters $i, j, r, v, u, d, t, s$ to denote real or natural numbers.

We say that a collection of probability measures $\mu_{n,k}$ ($n, k \in N^d$, $k \leqslant k_n$, $k_n \in N^d$, $k_n \to \infty$ whenever $n \to \infty$ and $d = 1, 2, \ldots$) on a Banach space $X$ is *uniformly infintesimal* if for every subsequence $\{i_1, i_2, \ldots, i_s\}$ of the sequence $\{1, 2, \ldots, d\}$ such that $i_1 < i_2 < \quad < i_s$ and for every neighbourhood $U$ of 0 in $X$

$$\lim_{n\to\infty} \min_{\substack{1 \leqslant k^{i_r} \leqslant k_n^{i_r} \\ r = 1, 2, \ldots, s}} \mathop{*}_{\substack{1 \leqslant k^j \leqslant k_n^j \\ j \in \{1, 2, \ldots, d\} \setminus \{i_1, i_2, \ldots, i_s\}}} \mu_{n,k}(U) = 1$$

where $k = (k^1, k^2, \ldots, k^d)$ and $k_n = (k_n^1, k_n^2, \ldots, k_n^d)$.

It is evident that the collection $\{\mu_{n,k}\}$, $n, k \in N^d$, is uniformly infinitesimal if and only if for every subsequence $\{i_1, i_2, \ldots, i_s\}$ of $\{1, 2, \ldots, d\}$ with $i_1 < i_2 < \quad < i_s$ and for every choice of $h_n^{i_r}$, $1 \leqslant h_n^{i_r} \leqslant k_n^{i_r}$, $n \in N^d$ and $r = 1, 2, \ldots, s$,

$$\mathop{*}_{\substack{1 \leqslant k^j \leqslant k_n^j \\ j \in \{1, 2, \ldots, d\} \setminus \{i_1, i_2, \ldots, i_s\} \\ k_r^i = h_n^{i_r} \\ r = 1, 2, \ldots, s}} \mu_{n,k} \to \delta_0.$$

Moreover, we have the following proposition:

PROPOSITION 2.1. *For every $d = 1, 2, \ldots$ the class of all probability measures $\mu$ on $X$ for which there exists a uniformly infinitesimal collection $\{\mu_{n,k}\}$, $n$, $k \in N^d$ and $k \leqslant k_n$, such that $\mathop{*}\limits_{k \leqslant k_n} \mu_{n,k} \to \mu$ as $n \to \infty$ coincides with the class of all infinitely divisible probability measures on $X$.*

Proof. Suppose that $\{\mu_{n,k}\}$, $n$, $k \in N^d$, be a uniformly infinitesimal collection of probability measures on $X$ such that

$$\mathop{*}_{k \leqslant k_n} \mu_{n,k} \to \mu \quad \text{as } n \to \infty.$$

We shall prove that $\mu$ is infinitely divisible.

Put, for $t$, $s = 1, 2, \ldots$

$$\nu_{t,s} = \mathop{*}_{\substack{k \leqslant (s_t, s_t, \ldots, s_t) \\ k^1 = s}} \mu_{\underset{(d \text{ times})}{(t,t,\ldots,t)},k}$$

where $s_t = k^1_{(t,t,\ldots,t)}$. By the uniform infinitesimality condition of $\{\mu_{n,k}\}$ it follows that the triangular array $\{\nu_{t,s}\}$ $(t, s = 1, 2, \ldots, s \leqslant s_t)$ is uniformly infinitesimal too. Moreover,

$$\mathop{*}_{s \leqslant s_t} \nu_{t,s} = \mathop{*}_{k \leqslant (s_t, s_t, \ldots, s_t)} \mu_{(t,t,\ldots,t),k} \to \mu \quad \text{as } t \to \infty.$$

Consequently, $\mu$ is infinitely divisible.

Conversely, given an infinitely divisible probability measure $\mu$ on $X$ define a collection $\{\mu_{n,k}\}$ $(n, k \in N^d)$ by

$$\mu_{n,k} = \mu^{\frac{1}{n^1 n^2 \ldots n^d}}$$

whenever $n = (n^1, n^2, \ldots, n^d)$ and $k \in N^d$. It hints at the collection $\{\mu_{n,k}\}$ $(n, k \in N^d;\ k \leqslant n)$ is uniformly infinitesimal and moreover,

$$\mathop{*}_{k \leqslant n} \mu_{n,k} = \mu \quad \text{for every } n \in N^d.$$

Thus the proposition is proved.

In terms of random variables, the problem we study can be formulated as follows:

PROBLEM I. Suppose that $\{\xi_n\}$, $n \in N^d$, $d = 1, 2, \ldots$, is a random field of $X$-valued random variables with distributions $\{\mu_n\}$, $\{x_n\}$, $n \in N^d$, is a vector field in $X$ and $A_1$, $A_2$, is a sequence of operators from $B(X)$ such that

(I.1) $A_1, A_2,$ are invertible and commute one another,

(I.2) $\operatorname{Sem}(\{A_r A_s^{-1}: s = 1, 2, \ldots, r;\ r = 1, 2, \ldots\})$ is compact (in the norm topology of $B(X)$), (I.3) the probability measures $\{A_n \mu_k\}$, where $k$, $n \in N^d$; $k \leqslant n$ and $A_n = A_{n^1} A_{n^2} \ldots A_{n^d}$ if $n = (n^1, n^2, \ldots, n^d)$, form a uniformly infinitesimal collection and the distribution of

$$A_n \sum_{k \leqslant n} \xi_k + x_n$$

converges to a probability measure $\mu$ as $n \to \infty$; What can be said about the limit measure $\mu$?

PROBLEM II. Suppose that $A_1, A_2, \ldots$ is a sequence of operators from $B(X)$ with the properties (I.1) and (I.2) and $\mu$ is a probability measure on $X$ such that for every $d = 1, 2, \ldots$ there exist a random field $\{\xi_n\}$, $n \in N^d$, of independent $X$-valued random variables with distributions $\{\mu_n\}$ and a vector field $\{x_n\}$, $n \in N^d$, in $X$ such that (I.3) holds. What can be said about the limit measure $\mu$?

Let us denote by $N_d(X)$, $d = 1, 2, \ldots$, the set of all limit measures in the Problem I and by $N_\infty(X)$ the set of all limit measures in the Problem II. Our further aim is to give a description of full measures belonging to $N_d(X)$, $d = 1, 2, \ldots, \infty$. In the case $d = 1$ K. Urbanik [17] solved the Problem I without the assumption that the operators commute one another. It is interesting how to solve the Problem I for $d \geqslant 2$ omitting the extra condition that $A_1, A_2, \ldots$ commute one another.

We note that for full measures in $N_d(X)$, $d = 1, 2, \ldots, \infty$, on finite-dimensional spaces the compactness condition (I.2) can be omitted ([16], Proposition 3.3). The same is true for non-denegerate measures on a Banach space $X$ when $A_1, A_2, \ldots$ are multiples of I. In this case, the limit measures in the Problems I and II are multiply self-decomposable ones. We refer the reader to [5] and [8] for an account of multiply self-decomposable probability measures on Banach spaces.

## 3. Norming sequences

We say that a sequence $A_1, A_2, \ldots$ of operators from $B(X)$ with the properties (I.1) and (I.2) is a *norming sequence corresponding to a probability measure $\mu$ in $N_d(X)$*, $d = 1, 2, \ldots$, if there exist sequences $\{\mu_n\}$, $n \in N^d$, and $\{x_n\}$, $n \in N^d$, of probability measures on $X$ with the property (I.3) and elements of $X$ respectively, such that

$$A_n \mathop{*}_{k \leqslant n} \mu_k * \delta_{x_n}$$

converges to $\mu$ as $n \to \infty$. Here $A_n = A_{n^1} A_{n^2} \ldots A_{n^d}$ if $n = (n^1, n^2, \ldots, n^d) \in N^d$. Further, a sequence $A_1, A_2, \ldots$ of operators from $B(X)$ with the properties (I.1) and (I.2) is a norming sequence corresponding to a measure $\mu$ in $N_\infty(X)$

if it is a norming sequence corresponding to the measure $\mu$ treated as an element of $N_d(X)$ for every $d = 1, 2, \ldots$.

PROPOSITION 3.1. *For every norming sequence $\{A_j\}$, $j = 1, 2, \ldots$ corresponding to a full measure $\mu$ from $N_d(X)$, $d = 1, 2, \ldots, \infty$, we have $A_j \to 0$.*

Proof. It suffices to prove the proposition for the case $d < \infty$. Suppose that $A_n \nu_n * \delta_{x_n} \to \mu$ where $\mu$ is full, $\nu_n = \underset{k \leqslant n}{*} \mu_n$, $n \in N^d$, and $A_n = A_{n^1} A_{n^2} \ldots A_{n^d}$ if $n = (n^1, n^2, \ldots, n^d) \in N^d$. By the condition (I.2) Sem $(\{A_j : j = 1, 2, \ldots\})$ is compact. Let $A$ be an arbitrary cluster point of the sequence $\{A_j\}$ and $A_{j_r} \to A$ as $j_r \to \infty$. Since for each $n$, $n \leqslant (j_r, j_r, \ldots, j_r)$,

$$A_{(j_r, j_r, \ldots, j_r)} \nu_{(j_r, j_r, \ldots, j_r)} * \delta_{x_{(j_r, j_r, \ldots, j_r)}} = A_{j_r}^d \nu_n * A_{j_r}^d \underset{\substack{k \leqslant (j_r, j_r, \ldots, j_r) \\ \text{and } \exists\, i = 1, 2, \ldots, d \\ \text{such that } n^i < k^i}}{*} \mu_k * \delta_{x_{(j_r, j_r, \ldots, j_r)}}$$

and

$$A_{j_r}^d \mu_k \to \delta_0 \quad \text{for every } k \in N^d \text{ when } j_r \to \infty,$$

we have the equation

$$\mu = A^d \nu_n * \mu \quad (n \in N^d). \tag{3.1}$$

Further, by the condition (I.2), Sem $(\{AA_{j_r}^{-1} : r = 1, 2, \ldots\})$ is compact. Let $B$ be a cluster point of the sequence $\{AA_{j_r}^{-1}\}$. Passing if necessary, to a subsequence we may assume without loss of generality that $AA_{j_r}^{-1} \to B$. Consequently,

$$A = BA. \tag{3.2}$$

By (3.1) we have the equation

$$\mu = A^d A_{j_r}^{-d} (A_{j_r}^d \nu_{(j_r, j_r, \ldots, j_r)} * \delta_{x_{(j_r, j_r, \ldots, j_r)}}) * \mu * \delta_{u_{(j_r, j_r, \ldots, j_r)}} \tag{3.3}$$

where $u_{(j_r, j_r, \ldots, j_r)} = -A_{x_{(j_r, j_r, \ldots, j_r)}}$. Since the sequence $\{\delta_{x_{(j_r, j_r, \ldots, j_r)}}\}$ is conditionally compact ([12], Chapter III, Theorem 2.1), we may assume without loss of generality that $\delta_{u_{(j_r, j_r, \ldots, j_r)}} \to \delta_u$. Then (3.3) implies

$$\mu = B^d \mu * \mu * \delta_u.$$

Consequently, $|\widehat{B^d \mu}(y)| = 1$ for every $y \in X^*$. Thus $B^d \mu = \delta_x$ for a certain $x \in X$ ([3], Proposition 2.3). But this is possible for a full measure $\mu$ if and only if $B^d = 0$ and $x = 0$. Now by (3.2) we get

$$A = B^d A = 0$$

and hence $A_j \to 0$ which completes the proof of the proposition.

LEMMA 3.1. *Let $\{A_j\}$, $j = 1, 2, \ldots$, be a norming sequence corresponding to a measure $\mu \in N_d(X)$, $d = 1, 2, \ldots$ (resp. $d = \infty$). Let $F$ denote the set of all cluster points of sequences $A_{j_r} A_{i_r}^{-1}$ with $i_r \leqslant j_r$, $r = 1, 2, \ldots$ and $i_r \to \infty$.*

*Then we have*

$$F \times F \times \underset{(d \text{ times})}{\ldots} \times F \subset D^d(\mu)$$

(*resp.* $F \times F \times \quad \subset D^\infty(\mu)$).

Proof. It suffices to prove the lemma for the case $d < \infty$. Let $n_r, m_r \in N^d$, $r = 1, 2, \ldots$, such that $n_r \leqslant m_r$ and $n_r \to \infty$. Let $C_i$, $i = 1, 2, \ldots, d$, be a cluster point of the sequence $\{A_{m_r^i} A_{n_r^i}^{-1}\}$, $i = 1, 2, \ldots, d$. Suppose that $A_n \nu_n * \delta_{x_n} \to \mu$, where $\nu_n = \underset{k \leqslant n}{*} \mu_k$, $A_n = A_{n^1} A_{n^2} \ldots A_{n^d}$ whenever $n = (n^1, n^2, \ldots, n^d)$. Then

(3.4)

$$A_{m_r} \nu_{m_r} * \delta_{x_{m_r}} = A_{m_r^1} A_{n_r^1}^{-1} (A_{n_r^1} A_{m_r^2} \ldots A_{m_r^d} \nu_{(n_r^1, m_r^2, \ldots, m_r^d)} * \delta_{x_{(n_r^1, m_r^2, \ldots, m_r^d)}}) * \omega_r^{(1)}$$

where $\omega_r^{(1)} = A_{m_r} \underset{\substack{k \leqslant m_r \\ k^1 > n_r^1}}{*} \mu_k * \delta_{x_{m_r}^{(1)}}$ for some points $x_{m_r}^{(1)} \in X$. Further, we have

$$(3.5) \quad \omega_r^{(1)} = A_{m_r^2} A_{n_r^2}^{-1} (A_{m_r^1} A_{n_r^2} A_{m_r^3} \ldots A_{m_r^d} \underset{\substack{n_r^1 < k^1 \\ k^2 \leqslant n_r^2 \\ k^3 \leqslant m_r^3 \\ \vdots \\ k^d \leqslant m_r^d}}{*} \mu_k * \delta_{x^{(1)}_{(m_r^1, n_r^2, m_r^3, \ldots, m_r^d)}}) * \omega_r^{(2)}$$

where $\omega_r^{(2)} = A_{m_r} \underset{\substack{n_r^1 < k^1 \\ n_r^2 < k^2 \\ k^3 \leqslant m_r^3 \\ \vdots \\ k^d \leqslant m_r^d}}{*} \mu_k * \delta_{x_{m_r}^{(2)}}$ for some $x_{m_r}^{(2)} \in X$. Proceeding successively, we get a probability measure $\omega_r^{(d)}$ such that

(3.6)

$$\omega_r^{(d-1)} = A_{m_r^d} A_{n_r^d}^{-1} (A_{m_r^1} A_{m_r^2} \quad A_{m_r^{d-1}} A_{n_r^d} \underset{\substack{n_r^1 < k^1 \\ n_r^2 < k^2 \\ \vdots \\ n_r^{d-1} < k^{d-1} \\ k^d \leqslant m_r^d}}{*} \mu_k * \delta_{x_{(m_r^1, m_r^2, \ldots, m_r^{d-1}, n_r^d)}}) * \omega_r^{(d)}.$$

For the simplicity of the notation we may assume that the sequences $\{\omega_r^{(j)}\}_{r=1,2,\ldots}$ ($j = 1, 2, \ldots, d$) being conditionally compact ([12], Chapter III, Theorem 2.1) converge to some probability measures $\omega^{(j)}$ ($j = 1, 2, \ldots, d$), respectively and for every $i = 1, 2, \ldots, d$

$$C_i = \lim_{r \to \infty} A_{m_r^i} A_{n_r^i}^{-1}.$$

Letting $r \to \infty$ we get, by virtue of (3.4), (3.5) and (3.6), the following equations:

$$\mu = C_1 \mu * \omega^{(1)}, \ \omega^{(1)} = C_2 \omega^{(1)} * \omega^{(2)}, \ \ldots, \ \omega^{(d-1)} = C_d \omega^{(d-1)} * \omega^{(d)}$$

which show that $\langle C_1, C_2, \ldots, C_d \rangle \in D^d(\mu)$. The lemma is thus proved.

Given a probability measure $\mu$ on $X$ by $A(\mu)$ we shall denote the subset of $D(\mu)$ consisting of all operators $A$ with the property that $\mu = A\mu * \delta_x$ for a certain $x \in X$. It is clear that $A(\mu)$ is a closed subsemigroup of $D(\mu)$ and $I \in A(\mu)$.

LEMMA 3.2. *For every norming sequence* $\{A_j\}$ $(j = 1, 2, \ldots)$ *corresponding to a full measure* $\mu$ *in* $N_d(X)$ $(d = 1, 2, \ldots, \infty)$

$$A(\mu) \cap \mathrm{Sem}(F) \tag{3.7}$$

*where F is the set defined in Lemma* 3.1, *is a compact group containing all cluster points of the sequence* $\{A_{j+1} A_j^{-1}\}$ $(j = 1, 2, \ldots)$.

Proof. It suffices to prove the lemma for the case $d < \infty$. The compactness of the set (3.7) is clear. Suppose that $A$ is a cluster point of the sequence $\{A_{j+1} A_j^{-1}\}$ and $A_{j_r+1} A_{j_r}^{-1} \to A$. From the equation

$$A_{(n^1+1, n^2, \ldots, n^d)} \nu_{(n^1+1, n^2, \ldots, n^d)} * \delta_{x_{(n^1+1, n^2, \ldots, n^d)}} = A_{n^1+1} A_{n^1}^{-1} (A_n \nu_n * \delta_{x_n}) * A_{(n^1+1, n^2, \ldots, n^d)} \mathop{*}_{\substack{k^1 = n^1+1 \\ k^2 \le n^2 \\ \vdots \\ k^d \le n^d}} \mu_k * \delta_{u_n}$$

where $\nu_n = \mathop{*}_{k \le n} \mu_k (n \in N^d)$ and $u_n$ are some points of $X$, we get, by virtue of (I.3), $\mu = A\mu * \delta_x$. Thus $A \in A(\mu)$ and consequently, $A$ belongs to the set (3.7).

Suppose now that $B$ is an element of the set (3.7). Consider the monothetic compact semigroup Sem $(\{B\})$. By virtue of Numakura Theorem ([11], Theorem 3.1.1) the cluster points of the sequence $\{B^j\}$ form a group $G$. Moreover, $G$ is the minimal ideal of Sem $(\{B\})$ and Sem $(\{B\})$ contains exactly one idempotent $P$, namely the unit of $G$. Hence it follows that $G$ contains an element $C$ with the property that

$$BC = CB = P.$$

Of course $P$ and $C$ belong to the set (3.7). Thus $\mu = P\mu * \delta_x$ for a certain point $x \in X$. Since $\mu$ is full and $P$ is an idempotent, the last formula yields $PX = X$. Thus $P = I$ and consequently, $C = B^{-1}$ which completes the proof of the lemma.

PROPOSITION 3.2. *To every full measure* $\mu \in N_d(X)$ $(d = 1, 2, \ldots, \infty)$ *there corresponds a norming sequence* $\{B_j\}$, $j = 1, 2, \ldots$, *with the property that*

$$B_{j+1} B_j^{-1} \to I. \tag{3.8}$$

Proof. It suffices to prove the proposition for the case $d < \infty$. Let $A_1, A_2,$ be an arbitrary norming sequence corresponding to a full measure $\mu \in N_d(X)$. Put

$$G = A(\mu) \cap \mathrm{Sem}\cdot(F)$$

where the set $F$ is defined in Lemma 3.1. By Lemma 3.2 $G$ is a compact group containing all cluster points of the sequence $\{A_{j+1} A_j^{-1}\}$. Consequently, we can choose a sequence $\{C_j\}$ $(j = 1, 2, \ldots)$ of elements of $G$ with the property

$$C_j^{-1} - A_{j+1} A_j^{-1} \to 0 \quad \text{as } j \to \infty. \tag{3.9}$$

Putting $B_1 = A_1$ and $B_j = C_1 C_2 \ldots C_{j-1} A_j$ $(j = 2, 3, \ldots)$ we infer that $B_1, B_2, \ldots$ are invertible and moreover, $\mathrm{Sem}(\{B_j B_r^{-1}: r = 1, 2, \ldots, j; j = 1, 2, \ldots\})$ being a closed subsemigroup of $\mathrm{Sem}(\{A_j A_r^{-1}: r = 1, 2, \ldots, j; j = 1, 2, \ldots\})$ is compact. Further, by assumption, $A_n \mu_{j_n} \to \delta_0$ for every choice of $\{j_n\}$, $j_n \leqslant n$, $n \in N^d$ and $A_n = A_{n^1} A_{n^2} \ldots A_{n^d}$ if $n = (n^1, n^2, \ldots, n^d)$. Since the sequence $\{C_1 C_2 \quad C_j\}$, $j = 1, 2, \ldots$, is conditionally compact the last relation yields $B_n \mu_{j_n} \to \delta_0$, where $B_n = B_{n^1} B_{n^2} \ldots B_{n^d}$ if $n = (n^1, n^2, \ldots, n^d) \in N^d$. Thus condition (I.3) is fulfilled. Moreover, the conditional compactness of the sequence $\{C_1 C_2 \quad C_j\}$, $j = 1, 2, \ldots$, implies the conditional compactness of the sequence $\{B_n \nu_n * \delta_{u_n}\}$, where for $n = (n^1, n^2, \ldots, n^d)$ with $n^j \geqslant 2$ $(j = 1, 2, \ldots, d)$ $u_n = \prod_{j=1}^{d} (C_1 C_2 \quad C_{n^j - 1}) x_n$ and $\nu_n = \mathop{*}_{k \leqslant n} \mu_k$. From the relation $A_n \nu_n * \delta_{x_n} \to \mu$ it follows that each cluster point of $\{B_n \nu_n * \delta_{u_n}\}$ is of the form $C\mu$, where $C$ is a cluster point of the sequence $\{\prod_{j=1}^{d} (C_1 C_2 \quad C_{n^j - 1})\}$, $n = (n^1, n^2, \ldots, n^d) \in N^d$ with $n^j \geqslant 2$, $j = 1, 2, \ldots, d$. But $C \in G$ and, consequently, $\mu = C\mu * \delta_{v_C}$ for a certain $v_C \in X$. Hence it follows that we can choose elements $v_n$, $n \in N^d$, in $X$ in such a way that

$$B_n \nu_n * \delta_{v_n} \to \mu \quad \text{as } n = (n^1, n^2, \ldots, n^d) \to \infty.$$

Thus $B_1, B_2, \ldots$ is a norming sequence corresponding to $\mu$.

To prove the condition (3.8) we observe that the norms of elements of the compact set $G$ are bounded in common, say by a constant $b$. Thus

$$\begin{aligned} \|B_{j+1} B_j^{-1} - I\| &= \|C_1 C_2 \quad C_j (A_{j+1} A_j^{-1} - C_j) C_{j-1}^{-1} C_{j-2}^{-1} \ldots C_1^{-1}\| \\ &\leqslant b^2 \|A_{j+1} A_j^{-1} - C_j^{-1}\| \end{aligned}$$

which, by (3.9), implies (3.8). The proposition is thus proved.

## 4. A characterization of full measures belonging to $N_d(X)$

Let $\mu$ be a full probability measure in $N_d(X)$, $d = 1, 2, \ldots, \infty$. By Proposition 3.2 we choose a norming sequence $\{A_j\}$, $j = 1, 2, \ldots$, corresponding to $\mu$ with the property $A_{j+1}A_j^{-1} \to I$. We fix this norming sequence for the remainder of this section. Define the set $F$ as in Lemma 3.1 and put $S = \mathrm{Sem}(F)$. Let $P$ be a projector belonging to $S$ and

$$S_P = \{A \in S\colon AP = PA = A\}.$$

It is clear that $S_P$ is a compact subsemigroup of $S$. Further, by $G_P$ we denote the subset of $S_p$ consisting of those operators $A$ for which

$$P\mu = A\mu * \delta_x \quad \text{for a certain } x \in X.$$

LEMMA 4.1. *$G_P$ is a compact group with the unit $P$.*

Proof. It is easy to check that $G_P$ is a closed subsemigroup of $S_P$ which implies the compactness of $G_P$. By the definition of $G_P$ the projector $P$ is the unit of $G_P$. Let $A \in G_P$. Then the monothetic semigroup $\mathrm{Sem}(A)$ is compact and, by Numakura Theorem ([11], Theorem 3.1.1), contains a projector $Q$ and an operator $B$ with the property

$$AB = BA = Q. \tag{4.1}$$

Of course $PQ = QP = Q$ and $P\mu = Q\mu * \delta_x$ for a certain $x \in X$. Since $\mu$ is full the last formula yields $PX = QX$. Consequently, $P = Q$ and, in view of (4.1), $G_P$ is a group.

LEMMA 4.2. *If $A \in S_P$ and $P \in \mathrm{Sem}(A)$, then $A \in G_P$.*

Proof. Let $A^{k_r} \to P$ for a subsequence $\{k_r\}$ of $\{1, 2, \ldots\}$. Of course, without loss of generality, we may assume that $k_r \geqslant 2$ and the sequence $\{A^{k_r-1}\}$ is convergent to an operator $B$. Then we have $AB = P$ and for some probability measures $\nu$ and $\lambda$

$$\mu = A\mu * \nu, \tag{4.2}$$

$$\mu = B\mu * \lambda, \tag{4.3}$$

because $A, B \in D(\mu)$. From (4.3) we get $A\mu = P\mu * A\lambda$. Hence and from (4.2) we obtain the equation $\mu = P\mu * A\lambda * \nu$. Consequently, $P\mu = P\mu * A\lambda * P\nu$ or in terms of the characteristic functionals

$$\widehat{P\mu}(y) = \widehat{P\mu}(y)\,\widehat{A\lambda}(y)\,\widehat{P\nu}(y) \qquad (y \in X^*).$$

Thus $|\widehat{P\mu}(y)| = 1$ in a neighbourhood of 0 in $X^*$ which implies $P\nu = \delta_x$ for a certain $x \in X$ ([3], Proposition 2.3). Now taking into account (4.2) we have $P\mu = A\mu * \delta_x$ which completes the proof of the lemma.

LEMMA 4.3. *For every non-zero projector $P$ belonging to $S$ the semigroup $S_P$ contains a one-parameter semigroup $P\exp tV$ ($t \geq 0$, $V \in B(X)$ with $PV = VP = V$). Moreover, $S_P$ contains a projector $Q$ with the properties $P \neq Q$ and*

$$\lim_{t\to\infty} (P-Q)\exp tV = 0.$$

Proof. By Lemma 4.1 the group $G_P$ is compact. Put

$$a_{t,u} = \min\{\|P - A_u A_t^{-1} H\|: H \in G_P\}.$$

Obviously,

$$a_{u,u} = 0 \quad (u = 1, 2, \ldots) \tag{4.4}$$

and by Proposition 3.1,

$$\lim_{u\to\infty} a_{t,u} = \|P\| \geq 1 \quad (t = 1, 2, \ldots). \tag{4.5}$$

Since the semigroup $\operatorname{Sem}(\{A_r A_s^{-1}: s = 1, 2, \ldots, r;\ r = 1, 2, \ldots\})$ is compact, all its elements have the norm bounded in common by a constant $b$. Consequently, for $t \leq u$

$$\begin{aligned} a_{t,u+1} &\leq \min\{\|P - A_u A_t^{-1} H\| + \|(A_{u+1} A_u^{-1} - I) A_u A_t^{-1} H\|: H \in G_P\} \\ &\leq a_{t,u} + b\,\|A_{u+1} A_u^{-1} - I\| \end{aligned}$$

and

$$\begin{aligned} a_{t,u} &\leq \min\{\|P - A_{u+1} A_t^{-1} H\| + \|(A_{u+1} A_u^{-1} - I) A_u A_t^{-1} H\|: H \in G_P\} \\ &\leq a_{t,u+1} + b\,\|A_{u+1} A_u^{-1} - I\| \end{aligned}$$

which imply that

$$\lim_{u\to\infty} \max_{1 \leq t \leq u} |a_{t,u+1} - a_{t,u}| = 0. \tag{4.6}$$

Given a number $c$ satisfying the condition $0 < c < 1$ we can find, by virtue of (4.4) and (4.5), an index $u_t \geq t$ such that $a_{t,u_t} < c$ and $a_{t,u_t+1} \geq c$ ($t = 1, 2, \ldots$). From (4.6) it follows that $a_{t,u_t} \to c$. Further, by the conditional compactness of the sequence $\{A_{u_t} A_t^{-1}\}$ and the compactness of $G_P$ we can choose a cluster point $A_c$ of $\{A_{u_t} A_t^{-1}\}$ and $D_c \in G_P$ such that

$$\|P - D_c A_c\| = c = \min\{\|P - A_c H\|: H \in G_P\}.$$

By Lemma 3.1 $A_c \in S$. Consequently, setting $B_c = D_c A_c$ we have $B_c \in S_P$ and

$$\|P - B_c\| = c = \min\{\|P - B_c H\|: H \in G_P\} \tag{4.7}$$

which yields

$$B_c \notin G_P. \tag{4.8}$$

Put

$$b_{t,c} = \min\{\|P - B_c^t H\|:\ H \in G_P\} \qquad (t = 1, 2, \ldots).$$

By (4.7) we have

(4.9) $$b_{1,c} = c.$$

Consider the semigroup Sem$(\{B_c\})$. By Numakura Theorem ([11], Theorem 3.1.1) it contains a projector $P_c$. Of course

(4.10) $$\limsup_{t \to \infty} b_{t,c} \geq \min\{\|P - P_c H\|:\ H \in G_P\}.$$

Since $P_c \in S_P$, $P - P_c$ is also a projector and, by Lemma 4.2, $P_c \neq P$. Thus

(4.11) $$\|P - P_c\| \geq 1.$$

Put

$$a = \inf\{\|P - P_c H\|:\ H \in G_P,\ 0 < c < 1\}.$$

We shall show that $a > 0$. Contrary to this let us assume that $a = 0$. Then by the compactness of $S_P$ and $G_P$, we can find an element $D$ of $G_P$ and a cluster point $R$ of $\{P_c:\ 0 < c < 1\}$ with the property $P = DR$. Since $R$ is also a projector and $R \in S_P$, we have $R = PR = DR = P$. Consequently, $P$ is a cluster point of $\{P_c:\ 0 < c < 1\}$ which contradicts (4.11). Thus $a > 0$ and, by (4.10),

(4.12) $$\limsup_{t \to \infty} b_{t,c} \geq a > 0 \quad \text{for every } c\ (0 < c < 1).$$

Further, taking into account that all elements of the compact semigroup $S$ have norm bounded by a constant $b$, we have, in view of (4.7),

$$b_{t+1,c} \leq \min\{\|P - B_c^t H\| + \|(B_c^t - B_c^{t+1})H\|:\ H \in G_P\} \leq b_{t,c} + bc.$$

and

$$b_{t,c} \leq \min\{\|P - B_c^{t+1} H\| + \|(B_c^{t+1} - B_c^t)H\|:\ H \in G_P\} \leq b_{t+1,c} + bc$$

which imply that

(4.13) $$\lim_{c \to 0} \sup_{t = 1, 2, \ldots} |b_{t+1,c} - b_t| = 0.$$

Let $c_t \to 0$. Given a number $d$ satisfying the condition $0 < d < a$, we can find, by virtue of (4.9) and (4.12), an integer $t_u$ such that $b_{t_u, c_u} < d$ and $b_{t_u+1, c_u} \geq d$. From (4.13) it follows that $b_{t_u, c_u} \to d$. The sequence $\{B_{c_u}^{t_u}\}$ is conditionally compact. Let $E_d$ be its cluster point. Then

(4.14) $$\min\{\|P - E_d H\|:\ H \in G_P\} = d \quad (0 < d < a)$$

and, consequently,

(4.15) $$E_d \notin G_P \quad (0 < d < a).$$

The set $\{E_d\colon 0 < d < a\}$ is also conditionally compact. Let $E_0$ be its cluster point when $d \to 0$. Then by (4.14) and the compactness of $G_P$, $P = H_0 E_0$ for a certain $H_0$ of the group $G_P$. Since $E_0 \in S_P$, this implies that $E_0 \in G_P$. Consequently, by Numakura Theorem ([11], Theorem 3.1.1), there exists a positive integer $q$ such that

$$\|P - E_0^q\| < \tfrac{1}{4}.$$

Taking a positive number $d_0$ with the property

$$\|E_0^q - E_{d_0}^q\| < \tfrac{1}{4}$$

we put

$$W = E_{d_0}^q. \tag{4.16}$$

Then

$$\|P - W\| < \tfrac{1}{2} \tag{4.17}$$

and, by the definition of the operators $E_d$,

$$B_{c_i}^{r_i} \to W \tag{4.18}$$

where $r_i \in \{1, 2, \ldots\}$ and $r_i \to \infty$. From (4.7) and (4.17) it follows that the operators $B_{c_i}$ and $W$ can be represented in an exponential form

$$B_{c_i} = P \exp U_i, \qquad W = P \exp V \tag{4.19}$$

where $U_i$, $V \in B(X)$, $PV = VP = V$, $PU_i = U_i V = U_i$ $(i = 1, 2)$,

$$WV = VW \tag{4.20}$$

and, by (4.18),

$$r_i U_i \to V \tag{4.21}$$

Let $t$ be an arbitrary positive real number. Then, by (4.19) and (4.21),

$$B_{c_i}^{[r_i t]} \to P \exp tV,$$

where the square brackets denote the integral part. Since $B_{c_i} \in S_P$ we infer that the one-parameter semigroup $\{P \exp tV\}$ $(t \geqslant 0)$ is contained in $S_P$. Consider the semigroup $\mathrm{Sem}(\{W\})$. By the Numakura Theorem ([11]. Theorem 3.1.1) it contains a projector $Q$. By (4.16) $Q \in \mathrm{Sem}(\{E_{d_0}\})$. By (4.15) and Lemma 4.2 we have the inequality $P \neq Q$. Obviously, $Q \in S_P$ and the set $\{(P-Q)\exp tV\colon t \geqslant 0\}$ is conditionally compact. Let $H$ be its cluster point when $t \to \infty$. Then for a sequence $\{t_r\}$ tending to $\infty$ we have

$$\lim_{r\to\infty} (P-Q)\exp t_r V = H. \tag{4.22}$$

Passing to a subsequence if necessary we may assume without loss of generality that both sequences $\{P \exp [t_r] V\}$ and $\{P \exp (t_r - [t_r]) V\}$ are convergent to $H_1$ and $H_2$ respectively. By (4.19) $H_1$ is a cluster point of the sequence $\{W^r\}$. Consequently, $QH_1 = H_1 Q = H_1$. Thus $(P-Q)H_1 = 0$, because $H_1 \in S_P$. Furthermore, by (4.22), $H = (P-Q)H_1 H_2$ which implies $H = 0$. Thus we have proved that

$$\lim_{t\to\infty} (P-Q)\exp tV = 0$$

which completes the proof of the lemma.

The following theorem gives a characterization of full measures belonging to $N_d(X)$ $(d = 1, 2, \ldots, \infty)$ in terms of their multi-dimentional decomposability algebraic structures.

THEOREM 4.1. *A full probability measure* $\mu$ *on* $X$ *belongs to the set* $N_d(X)$ $(d = 1, 2, \ldots)$ *(resp.* $d = \infty$*) if and only if there exists a one-parameter semigroup* $T_t := \exp tV$ $(t \geqslant 0)$ *with* $V \in B(X)$ *and* $\lim_{t\to\infty} T_t = 0$ *such that* $\mu$ *is d-times (resp. completely)* $\{T_t\}_{t\geqslant 0}$*-decomposable.*

Proof. It suffices to prove the theorem for the case $d < \infty$.

The necessity. Suppose that $\mu$ is a full measure from the set $N_d(X)$ $(d = 1, 2, \ldots)$. By Proposition 3.2 we choose a norming sequence $\{A_j\}$, $j = 1, 2, \ldots$, corresponding to $\mu$ with the property that $A_{j+1}A_j^{-1} \to I$. By Lemma 3.2 $I \in S$. By consecutive application of Lemma 4.3 we get a system of projectors $P_0 = I, P_1, \ldots, P_r$ and a system of operators $V_1, V_2, \ldots, V_r$ with the following properties: $S_{P_j}$ contains the one-parameter semigroup $P_j \exp tV_{j+1}$ $(t \geqslant 0)$, $P_j V_{j+1} = V_{j+1} P_j = V_{j+1}$, $P_{j+1} \in S_{P_j}$, $P_{j+1} V_{j+1} = V_{j+1} P_{j+1}$, $P_j \neq P_{j+1}$ and $\lim_{t\to\infty} (P_j - P_{j+1}) \exp tV_{j+1} = 0$ $(j = 0, 1, 2, \ldots, r-1)$. Moreover, by the compactness of $S$ we may assume that $P_r = 0$. Further, the condition $P_j \in S_{P_{j-1}}$ yields $P_j P_{j-1} = P_{j-1} P_j = P_j$ $(j = 1, 2, \ldots, r)$. Put $Q_j = P_{j-1} - P_j = P_j(I - P_{j-1})$ and $U = \mathrm{Sem}(\{S, I-P_j\colon j = 1, 2, \ldots, r\})$. By Proposition 1.3 we have the inclusion

$$U \times U \times \underset{(d \text{ times})}{\ldots} \times U \in D^d(\mu). \tag{4.23}$$

It is clear that $Q_j \in U$ $(j = 1, 2, \ldots, r)$, $\sum_{j=1}^{r} Q_j = I$, $Q_j V_j = V_j Q_j$ and the one-parameter semigroup $Q_j \exp tV_j$ $(t \geqslant 0)$ is contained in $U$. Moreover,

$$\lim_{t\to\infty} \sum_{j=1}^{r} Q_j \exp tV_j = 0.$$

Now by (4.23) and by Proposition 1.2 we infer that for any $t_1, t_2, \ldots, t_d \geqslant 0$ $\langle \sum_{j=1}^{r} Q_j \exp t_1 V_j, \sum_{j=1}^{r} Q_j \exp t_2 V_j, \quad \sum_{j=1}^{d} Q_j \exp t_d V_j \rangle \in D^d(\mu)$.

Setting $V = \sum_{j+1}^{r} Q_j V_j$ we have $\exp tV = \sum_{j=1}^{r} Q_j \exp tV_j$. Then the one-parameter semigroup $T_t = \exp tV$ satisfies the condition $\lim_{t\to\infty} T_t = 0$ and, moreover, the measure $\mu$ is $d$-times $\{T_t\}$-decomposable.

The sufficiency. Let $T_t = \exp tV$ $(t \geq 0)$ be a one-parameter semigroup such that $\lim_{t\to\infty} T_t = 0$ and $\mu$ is $d$-times $\{T_t\}$-decomposable. Setting, for $j = 1, 2,$ $B_j = \exp \frac{1}{j} V$, we have the formula

$$\mu = B_{n^1} \mu * \lambda_{n^1} \quad (n^1 = 1, 2, \ldots). \tag{4.24}$$

Put

$$A_j = \exp \sum_{i=1}^{j} \frac{1}{i} V \quad (j = 1, 2, \ldots)$$

and

$$\mu_{n^1} = \begin{cases} A_1^{-1} \mu & \text{for} \quad n^1 = 1, \\ A_{n^1}^{-1} \lambda_{n^1} & \text{for} \quad n^1 = 2, 3, \ldots \end{cases} \tag{4.25}$$

Since for any $n = (n^1, n^2, \ldots, n^d) \in N^d$ $\langle B_{n^1}, B_{n^2}, \ldots, B_{n^d} \rangle \in D^d(\mu)$ and by (4.24) we have $B_{n^2} \in D(\lambda_{n^1})$. Consequently, by (4.25), $B_{n^2} \in D(\mu_{n^1})$ and hence there exists a probability measure $\lambda_{n^1,n^2}$ such that

$$\mu_{n^1} = B_{n^2} \mu_{n^1} * \lambda_{n^1,n^2}. \tag{4.26}$$

Put, for $n^1, n^2 = 1, 2, \ldots,$

$$\mu_{n^1,n^2} = \begin{cases} A_1^{-1} \lambda_{n^1} & \text{for} \quad n^2 = 1, \\ A_{n^2}^{-1} \lambda_{n^1,n^2} & \text{for} \quad n^2 > 1. \end{cases} \tag{4.27}$$

By the same reason as above we infer that for every $n^3 = 1, 2, \ldots$ $B_{n^3} \in D(\mu_{n^1,n^2})$. Proceeding successively, we get 2 systems of probability measures $\mu_{n^1}, \mu_{n^1,n^2}, \ldots, \mu_{n^1,n^2,\ldots,n^d}$ and $\lambda_{n^1}, \lambda_{n^1,n^2}, \ldots, \lambda_{n^1,n^2,\ldots,n^d}$ $(n^1, n^2, \ldots$ $\ldots, n^d = 1, 2, \ldots)$ with the properties that for every $r = 1, 2, \ldots, d-1$

$$\mu_{n^1,n^2,\ldots,n^r} = B_{n^{r+1}} \mu_{n^1,n^2,\ldots,n^r} * \lambda_{n^1,n^2,\ldots,n^{r+1}} \tag{4.28}$$

and

$$\mu_{n^1,n^2,\ldots,n^{r+1}} = \begin{cases} A_1^{-1} \lambda_{n^1,n^2,\ldots,n^r} & \text{for} \quad n^{r+1} = 1, \\ A_{n^{r+1}}^{-1} \lambda_{n^1,n^2,\ldots,n^{r+1}} & \text{for} \quad n^{r+1} > 1. \end{cases} \tag{4.29}$$

Now, by (4.25), (4.28) and (4.29), we have

$$\mu = A_{n^1} A_{n^2} \ldots A_{n^d} \mathop{*}_{k \leq n} \mu_k \quad (n = (n^1, n^2, \ldots, n^d)).$$

It remains to prove that the collection of probability measures $\{A_n \mu_k\}$ $(n, k \in N^d, k \leq n$ and $A_n = A_{n^1} A_{n^2} \ldots A_{n^d}$ whenever $n = (n^1, n^2, \ldots, n^d) \in N^d)$ is uniformly infinitesimal. We shall prove this by induction.

Let $i_1 < i_2 < \quad < i_s$ be a subsequence of $\{1, 2, \ldots, d\}$. Then by virtue of (4.25), (4.28) and (4.29) we have

$$\mathop{*}_{\substack{1 \leqslant k^j \leqslant n^j \\ j \in \{1,2,\ldots,d\} \setminus \{i_1, i_2, \ldots, i_s\}}} A_n \mu_k = A_{n^{i_1}} A_{n^{i_2}} \ldots A_{n^{i_s}} \mu_{k^{i_1}, k^{i_2}, \ldots, k^{i_s}} \tag{4.30}$$

whenever $n = (n^1, n^2, \ldots, n^d)$ and $k = (k^1, k^2, \ldots, k^d)$.

For $s = 1$ it is known (see the proof of Theorem 4.1 [17]) that $A_{n^{i_1}} \mu_{k^{i_1}} \to \delta_0$ for each choice of $k^{i_1} \leqslant n^{i_1}$. Suppose that for any choice of $k^{i_1} \leqslant n^{i_1}, k^{i_2} \leqslant n^{i_2}, \ldots, k^{i_r} \leqslant n^{i_r}$ $(r = 1, 2, \ldots, s-1)$ we have

$$A_{n^{i_1}} A_{n^{i_2}} \ldots A_{n^{i_r}} \mu_{k^{i_1}, k^{i_2}, \ldots, k^{i_r}} \to \delta_0 .$$

From (4.28) and (4.29) we get the equations

$$A_{n^{i_1}} A_{n^{i_2}} \ldots A_{n^{i_{r+1}}} \mu_{k^{i_1}, k^{i_2}, \ldots, k^{i_{r+1}}} = A_{n^{i_1}} A_{n^{i_2}} \ldots A_{n^{i_r}} (A_{n^{i_{r+1}}} A_{k^{i_{r+1}}}^{-1}) \lambda_{k^{i_1}, k^{i_2}, \ldots, k^{i_{r+1}}} \qquad (k^{i_{r+1}} > 1)$$

and

$$A_{n^{i_1}} A_{n^{i_2}} \ldots A_{n^{i_r}} \mu_{k^{i_1}, k^{i_2}, \ldots, k^{i_r}} = B_{k^{i_{r+1}}} A_{n^{i_1}} A_{n^{i_2}} \quad A_{n^{i_r}} \mu_{k^{i_1}, k^{i_2}, \ldots, k^{i_r}} * A_{n^{i_1}} A_{n^{i_2}} \ldots A_{n^{i_r}} \lambda_{k^{i_1}, k^{i_2}, \ldots, k^{i_{r+1}}} .$$

Consequently, by the induction assumption and by the fact that the sequence $\{A_{n^{i_{r+1}}} A_{k^{i_{r+1}}}^{-1}\}$ is conditionally compact $A_{n^{i_1}} A_{n^{i_2}} \quad A_{n^{i_{r+1}}} \mu_{k^{i_1}, k^{i_2}, \ldots, k^{i_{r+1}}} \to \delta_0$ for each choice of $k^{i_1} \leqslant n^{i_1}$, $k^{i_2} \leqslant n^{i_2}, \ldots, k^{i_{r+1}} \leqslant n^{i_{r+1}}$ Thus the condition (I.3) is also fulfilled which completes the proof of the theorem.

## 5. A characterization of multiply $\{T_t\}$-decomposable probability measures on $X$

It is well-known ([14], [15], [2]) that every infinitely divisible probability measure $\mu$ on $X$ has a unique representation

$$\mu = \varrho * \tilde{e}(M) \tag{5.1}$$

where $\varrho$ is a symmetric Gaussian measure and $\tilde{e}(M)$ is a generalized Poisson measure on $X$. In terms of the characteristic functional we have the formulas

$$\hat{\varrho}(y) = \exp\{-\tfrac{1}{2}\langle y, Ry\rangle\} \qquad (y \in X^*) \tag{5.2}$$

$R$ being a covariance operator i.e. a compact operator from $X^*$ into $X$ such that $\langle y_1, Ry_2\rangle = \langle y_2, Ry_1\rangle$ (symmetry) and $\langle y, Ry\rangle \geqslant 0$ (non-negativity)

([4], [18]) and

$$\hat{\tilde{e}}(M)(y) = \exp\{i\langle y, x_0\rangle + \int_X K(x, y)\, M(dx)\} \tag{5.3}$$

for a certain $x_0 \in X$. The kernel $K$ is defined by the formula

$$K(x, y) = \exp i\langle y, x\rangle - 1 - i\langle y, x\rangle\, 1_W(x)$$

where $1_W$ denotes the indicator of a compact subset $W$ of $X$. Furthermore, the measure $M$ being a generalized Poisson exponent has a finite mass outside every neighbourhood of 0 in $X$.

Let $R(X)$ denote the set of all covariance operators of symmetric Gaussian measures on $X$ and $M(X)$ denote the set of all generalized Poisson exponents on $X$. Recall that if $R_1$ is a symmetric non-negative operator from $X^*$ into $X$ and $R_2 - R_1$ is non-negative for a certain operator $R_2 \in R(X)$, then also $R_1 \in R(X)$ ([18], p. 151). Moreover, $M(X)$ is a cone, i.e. if $M \in M(X)$ and $M \geqslant N \geqslant 0$, then $N$, $M - N \in M(X)$.

Given an operator $V \in B(X)$ with the property that $T_t := \exp tV \to 0$ as $t \to \infty$ we shall denote by $L_d(X, V)$ $(d = 1, 2, \ldots, \infty)$ the set of all $d$-times $\{T_t\}$-decomposable probability measures on $X$. In particular, $L_\infty(X, V)$ denotes the set of all completely $\{T_t\}$-decomposable probability measures on $X$. It is evident that

$$L_d(X, V) \subseteq L_{d+1}(X, V) \subseteq L_\infty(X, V) \quad \text{for every } d = 1, 2, \ldots$$

LEMMA 5.1. *Suppose that $\mu = \varrho * \tilde{e}(M)$ where $\varrho$ is a symmetric Gaussian measure with the covariance operator $R$ and $M \in M(X)$. If $\langle A_1, A_2, \ldots, A_d\rangle \in D^d(\mu)$ and*

$$\mu = A_1 \mu * \mu_{A_1},$$

$$\mu_{A_1, A_2, \ldots, A_j} = A_{j+1} \mu_{A_1, A_2, \ldots, A_j} * \mu_{A_1, A_2, \ldots, A_{j+1}} \quad (j = 1, 2, \ldots, d-1)$$

*where the measures $\mu_{A_1}, \mu_{A_1, A_2}, \ldots, \mu_{A_1, A_2, \ldots, A_d}$ are infinitely divisible, then $\langle A_1, A_2, \ldots, A_d\rangle \in D^d(\varrho)$ and $\langle A_1, A_2, \ldots, A_d\rangle \in D^d(\tilde{e}(M))$. Moreover, if $\mu_{A_1, A_2, \ldots, A_j} = \varrho_j * \tilde{e}(M_j)$ $(j = 1, 2, \ldots, d)$ is the Tortrat representation of the measure $\mu_{A_1, A_2, \ldots, A_j}$, then the covariance operator $R_j$ of $\varrho_j$ is given by the formula*

$$R_j = R + \sum_{r=1}^{j} (-1)^r \sum_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant j} A_{i_r} A_{i_{r-1}} \ldots A_{i_1} R A_{i_1}^* A_{i_2}^* \ldots A_{i_r}^* \tag{5.4}$$

*and the generalized Poisson exponent $M_j$ is given by the formula*

$$M_j = M + \sum_{r=1}^{j} (-1)^r \sum_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant j} A_{i_r} A_{i_{r-1}} \ldots A_{i_1} M. \tag{5.5}$$

Proof (by induction). If $d = 1$ the lemma reduces to Urbanik's lemma ([17], Lemma 5.1). Suppose that our lemma is true for $d \geq 1$. Further, let $\langle A_1, A_2, \ldots, A_{d+1}\rangle \in D^{d+1}(\mu)$ and $\mu_{A_1}, \mu_{A_1,A_2}, \ldots, \mu_{A_1,A_2,\ldots,A_{d+1}}$ are infinitely divisible. We shall prove that for every $j = 1, 2, \ldots, d+1$ equations (5.4) and (5.5) hold.

By the induction assumption for every $j = 1, 2, \ldots, d$ $R_j$ and $M_j$ are given by (5.4) and (5.5) respectively. On the other hand,

$$\mu_{A_1,A_2,\ldots,A_d} = A_{d+1}\varrho_d * \varrho_{d+1} * \tilde{e}(A_{d+1}M_d + M_{d+1}).$$

Consequently, by the uniqueness of the Tortrat representation, $\varrho_d = A_{d+1}\varrho_d * \varrho_{d+1}$ and $M_d = A_{d+1}M_d + M_{d+1}$ which together with (5.4) and (5.5) (for $j = d$) imply that the equations (5.4) and (5.5) hold also for $j = d+1$. The lemma is thus proved.

THEOREM 5.1. *Let $V \in B(X)$ and $T_t := \exp tV \to 0$ as $t \to \infty$. Then a probability measure $\mu$ on $X$ is d-times $\{T_t\}$-decomposable ($d = 1, 2, \ldots, \infty$) if and only if $\mu = \varrho * \tilde{e}(M)$, where $\varrho$ is a symmetric Gaussian measure with the covariance operator $R$ and $M \in M(X)$ such that for every $j = 1, 2, \ldots, d$,*

$$(5.6) \qquad (-1)^j (V^j R + V^{j-1} R V^* + \quad + V R V^{*(j-1)} + R V^{*j}) \geq 0$$

*and*

(5.7)

$$M + \sum_{r=1}^{j} (-1)^r \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq j} T_{t_{i_1}+t_{i_2}+\ldots+t_{i_r}} M \geq 0 \quad \textit{for all } t_1, t_2, \ldots, t_j \geq 0$$

Proof. It is enough to prove the theorem for $d < \infty$. Let $\mu \in L_d(X, V)$ and $t_1, t_2, \ldots, t_d \geq 0$. Then there exist probability measures $\mu_{t_1}, \mu_{t_1,t_2}, \ldots$ $\ldots, \mu_{t_1,t_2,\ldots,t_d}$ such that

$$\mu = T_{t_1}\mu * \mu_{t_1},$$

$$\mu_{t_1,t_2,\ldots,t_j} = T_{t_{j+1}}\mu_{t_1,t_2,\ldots,t_j} * \mu_{t_1,t_2,\ldots,t_{j+1}} \qquad (j = 1, 2, \ldots, d-1).$$

Since $L_d(X, V) \subseteq L_1(X, V)$ ($d = 1, 2, \ldots, \infty$) and by Corollary 4.2 [17] the probability measures $\mu, \mu_{t_1}, \ldots, \mu_{t_1,t_2,\ldots,t_d}$ are infinitely divisible. Let

$$\mu = \varrho * \tilde{e}(M) \quad \text{and} \quad \mu_{t_1,t_2,\ldots,t_j} = \varrho_{t_1,t_2,\ldots,t_j} * \tilde{e}(M_{t_1,t_2,\ldots,t_j})$$

be their Tortrat representations, where $\varrho, \varrho_{t_1,t_2,\ldots,t_j}$ ($j = 1, 2, \ldots, d$) are some symmetric Gaussian measures on $X$ with the covariance operators $R_j$, $R_{t_1,t_2,\ldots,t_j}$ respectively and $M, M_{t_1,t_2,\ldots,t_j} \in M(X)$. By virtue of Lemma 5.1

we get the equations

(5.8)

$$R_{t_1,t_2,\ldots,t_j} = R + \sum_{r=1}^{j} (-1)^r \sum_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant j} T_{t_{i_1}+t_{i_2}+\ldots+t_{i_r}} R T^*_{t_{i_1}+t_{i_2}+\ldots+t_{i_r}}$$

and

$$M_{t_1,t_2,\ldots,t_j} = M + \sum_{r=1}^{j} (-1)^r \sum_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant j} T_{t_{i_1}+t_{i_2}+\ldots+t_{i_r}} M. \tag{5.9}$$

Hence it follows that the relation (5.7) holds. Further, by equation (5.8) and taking into account the expansion in a neighbourhood of 0

$$R_{t_1,t_2,\ldots,t_j} = (-1)^j t_1 t_2 \ldots t_j (V^j R + V^{j-1} R V^* + \quad + V R V^{*(j-1)} + R V^{*j}) + \\ + o(t_1, t_2, \ldots, t_j)$$

it follows that (5.6) holds.

Conversely, suppose that (5.6) and (5.7) hold for every $j = 1, 2, \ldots, d$. Given $t_1, t_2, \ldots, t_d \geqslant 0$ define $R_{t_1,t_2,\ldots,t_j}$ and $M_{t_1,t_2,\ldots,t_j}$ according to the formulas (5.8) and (5.9). By an easy induction it follows that $M_{t_1,t_2,\ldots,t_j} \in M(X)$ $(j = 1, 2, \ldots, d)$ and

$$\tilde{e}(M) = T_{t_1} \tilde{e}(M) * \tilde{e}(M_{t_1}),$$

$$\tilde{e}(M_{t_1,t_2,\ldots,t_j}) = T_{t_{j+1}} \tilde{e}(M_{t_1,t_2,\ldots,t_j}) * \tilde{e}(M_{t_1,t_2,\ldots,t_{j+1}}) \quad (j = 1, 2, \ldots, d-1).$$

Consequently, $\langle T_{t_1}, T_{t_2}, \ldots, T_{t_d} \rangle \in D^d(\tilde{e}(M))$. It remains to prove that $\langle T_{t_1}, T_{t_2}, \ldots, T_{t_d} \rangle \in D^d(\varrho)$. Accordingly, it suffices to prove that $R_{t_1,t_2,\ldots,t_j} \geqslant 0$ for every $j = 1, 2, \ldots, d$ and for all $t_1, t_2, \ldots, t_d \geqslant 0$.

For $j = 1$ we have $R_{t_1} \geqslant 0$ ([17], Theorem 5.1). Suppose that $R_{t_1,t_2,\ldots,t_j} \geqslant 0$ $(t_1, t_2, \ldots, t_j \geqslant 0)$. By the definition of $R_{t_1,t_2,\ldots,t_j}$ we have the formula

$$R_{t_1,t_2,\ldots,t_{j+1}} = R_{t_1,t_2,\ldots,t_j} - T_{t_{j+1}} R_{t_1,t_2,\ldots,t_j} T^*_{t_{j+1}}$$

which, by Theorem 5.1 [17], implies that $R_{t_1,t_2,\ldots,t_{j+1}} \geqslant 0$. Thus by induction the theorem is proved.

Let $\{T_t\}$ be the same as in Theorem 5.1. For every $d = 1, 2, \ldots$ let $R_d(X, V)$ denote a subset of $R(X)$ consisting of all covariance operators $R$ of the form

$$R = \int_0^\infty t^{d-1} T_t Q T_t^* \, dt \tag{5.10}$$

for a certain symmetric non-negative operator $Q$ from $X^*$ into $X$. It is evident that

$$R_{d+1}(X, V) \subseteq R_d(X, V) \quad (d = 1, 2, \ldots).$$

Further, we put

$$R_\infty(X, V) = \bigcap_{d=1}^{\infty} R_d(X, V). \tag{5.11}$$

THEOREM 5.2. *The class of all covariance operators of d-times $\{T_t\}$-decomposable Gaussian measures on $X$ coincides with $R_d(X, V)$ $(d = 1, 2, \ldots, \infty)$.*

Proof. Let $d < \infty$ and $R$ be a covariance operator of $d$-times $\{T_t\}$-decomposable Gaussian measure $\varrho$ on $X$. Given $t_1, t_2, \ldots, t_d \geq 0$ define the covariance operators $R_{t_1, t_2, \ldots, t_j}$ by the formula (5.8). For every $y \in X^*$ define the functions $f_y(t_1, t_2, \ldots, t_j)$ $(j = 1, 2, \ldots, d)$ by

$$f_y(t_1, t_2, \ldots, t_j) = \langle y, R_{t_1, t_2, \ldots, t_j} y\rangle.$$

It is evident that

$$\frac{\partial^j}{\partial t_1, \partial t_2, \ldots, \partial t_j} f_y(t_1, t_2, \ldots, t_j)$$
$$= \langle y, T_{t_1+t_2+\ldots+t_j}\{(-1)^j(V^j R + V^{j-1} RV^* + \ldots + VRV^{*(j-1)} + RV^{*j})\} \times$$
$$\times T_{t_1+t_2+\ldots+t_j} y\rangle.$$

Consequently,

$$f_y(t_1, t_2, \ldots, t_j) = \int_0^{t_1}\int_0^{t_2} \ldots \int_0^{t_j} \langle y, T_{s_1+s_2+\ldots+s_j} Q_j T^*_{s_1+s_2+\ldots+s_j} y\rangle \, ds_1 \, ds_2 \ldots ds_j$$

where by $Q_j$ we denote the operator

$$(-1)^j(V^j R + V^{j-1} RV^* + \ldots + VRV^{*(j-1)} + RV^{*j}).$$

Letting $t_1, t_2, \ldots, t_j \to \infty$ and taking into account the fact that $T_t \to 0$ as $t \to \infty$ we get the equation

$$\langle y, Ry\rangle = \int_0^{\infty}\int_0^{\infty} \ldots \int_0^{\infty} \langle y, T_{s_1+s_2+\ldots+s_j} Q_j T^*_{s_1+s_2+\ldots+s_j} y\rangle \, ds_1 \, ds_2 \ldots ds_j$$

which, by a simple changing the variables, implies that

$$\langle y, Ry\rangle = \frac{1}{(j-1)!} \langle y, \int_0^{\infty} t^{j-1} T_t Q_j T_t^* \, dt \, y\rangle.$$

Putting $j = d$ and $Q = \frac{1}{(d-1)!} Q_d$ into this formula we have the equation

$$R = \int_0^{\infty} t^{d-1} T_t Q T_t^* \, dt.$$

It is clear, by Theorem 5.1, that $Q$ is a symmetric non-negative operator from $X^*$ into $X$. Thus we conclude that $R \in R_d(X, V)$.

Conversely, let $R \in R_d(X, V)$. Then there exists a symmetric non-negative operator $Q$ from $X^*$ into $X$ such that the equation (5.10) holds. Let us write (5.10) in an equivalent form

$$(5.12) \qquad R = \int_0^\infty \int_0^\infty \dots \int_0^\infty T_{s_1+s_2+\dots+s_j} Q' T^*_{s_1+s_2+\dots+s_j} ds_1 ds_2 \dots ds_j,$$

where $Q' = (d-1)!\,Q$. Then for any $j = 1, 2, \dots, d$ and $t_1, t_2, \dots, t_j \geqslant 0$ we get the formula

$$R_{t_1,t_2,\dots,t_j} := R + \sum_{r=1}^{j} (-1)^r \sum_{1 \leqslant i_1 < i_2 < \dots < i_r \leqslant j} T_{t_{i_1}+t_{i_2}+\dots+t_{i_r}} R T^*_{t_{i_1}+t_{i_2}+\dots+t_{i_r}}$$

$$= \int_0^{t_1}\int_0^{t_2} \dots \int_0^{t_j} \underbrace{\int_0^\infty \dots \int_0^\infty}_{(d-j)\ \text{times}} T_{s_1+s_2+\dots+s_d} Q' T^*_{s_1+s_2+\dots+s_d} ds_1 ds_2 \dots ds_d,$$

which, by the assumption that $Q$ is symmetric and non-negative implies $R_{t_1,t_2,\dots,t_j} \geqslant 0$. But, by the proof of Theorem 5.1, the last inequality is equivalent to the condition (5.6). Hence and by Theorem 5.1 $R$ is a covariance operator of a $d$-times $\{T_t\}$-decomposable Gaussian measure on $X$. Thus the theorem is proved for the case $d < \infty$. The case $d = \infty$ is quite clear which completes the proof of the theorem.

## 6. A reduction of the problem

In [17] K. Urbanik has introduced a concept of weight functions on a real separable Banach space $X$. Roughly speaking, a weight function on $X$ is every real-valued function $\Phi$ on $X$ such that

(a) $\Phi(0) = 0$ and $\Phi(x) > 0$ for all $x \neq 0$,

(b) $\Phi(x)$ converges to a positive limit as $\|x\| \to \infty$,

(c) $\Phi(x) \leqslant \alpha \|x\|^2$ for a certain positive constant $\alpha$ and for all $x \in X$,

(d) $\int_X \Phi(x) M(dx) < \infty$ for every $M \in M(X)$,

(e) if $M_j \in M(X)$, $\tilde{e}(M_j) \to \mu$ and $\int_X \Phi(x) M_j(dx) \to 0$, then $\mu = \delta_x$ for a certain $x \in X$.

It is known ([17], Proposition 5.2) that for every real separable Banach space $X$ there exists a weight function on $X$.

Given a subset $E$ of $X$ we put

$$\tau(E) = \{T_t x\colon x \in E, -\infty < t < \infty\},$$

where $\{T_t\}$ is a semigroup as described in Theorem 5.1.

Lemma 6.1 (cf. [17], Lemma 5.4). *For every $M \in M(X)$ there exists a sequence $\{E_j\}$ of compact subset of $X$ such that $0 \notin E_j$, $\tau(E_i) \cap \tau(E_j) = \emptyset$*

if $i \neq j$ $(i, j = 1, 2, \ldots)$ *and* $M = \sum_{j=1}^{\infty} M_j$ *where* $M_j$ *is the restriction of* $M$ *to* $\tau(E_j)$.

We note that if $M \in M_d(X, V)$ $(d = 1, 2, \ldots, \infty)$ then for every $\{T_t\}$-invariant subset $U$ of $X$, i.e. such a subset that $\tau(U) = U$, the restriction of $M$ to $U$, denoted by $M|_U$ belongs to $M_d(X, V)$ too. Consequently, from Lemma 6.1 we get the following corollary:

COROLLARY 6.1. *Let* $M \in M_d(X, V)$ $(d = 1, 2, \ldots, \infty)$. *Then there exists a decomposition* $M = \sum_{j=1}^{\infty} M_j$, *where* $M_j \in M_d(X, V)$ $(j = 1, 2, \ldots)$ $M_j$ *are concentrated on disjont sets* $\tau(E_j)$, $0 \notin E_j$, *and* $E_j$ *are compact.*

This corollary reduces our problem of examing measures $M \in M_d(X, V)$ $(d = 1, 2, \ldots, \infty)$ to the case of measures concentrated on $\tau(E)$ where $E$ is compact and $0 \notin E$. We denote this class of measures by $L_d(E, V)$ $(d = 1, 2, \ldots, \infty)$. Following K. Urbanik [17] we shall find a suitable compactification of $\tau(E)$ and determine the extreme points of a certain convex set formed by probability measures on this compactification.

Accordingly, let $[-\infty, \infty]$ be the usual compactification of the real line and let $E$ be a compact subset of $X$ such that $0 \notin E$. Then $E \times [-\infty, \infty]$ endowed with the product topology becomes a compact space. We define an equivalence relation in $E \times [-\infty, \infty]$ as follows: $(x_1, t_1) \sim (x_2, t_2)$ where $x_1, x_2 \in E$ and $t_1, t_2 \in [-\infty, \infty]$, if and only if there exists a real number $s$ such that $T_s x_1 = x_2$ and $t_2 = t_1 - s$. It is known [17] that $\sim$ is continuous. Hence the quotient space $E \times [-\infty, \infty]/\sim$ denoted by $\bar{\tau}(E)$ is compact. The element of $\bar{\tau}(E)$, i.e. the coset containing $(x, t)$ will be denoted by $[x, t]$. Each element of $\tau(E)$ is of the form $T_t x$, where $x \in E$ and $t$ is a real number. In general this representation is not unique. But $T_{t_1} x_1 = T_{t_2} x_2$ if and only if $(x_1, t_1) \sim (x_2, t_2)$. Thus the mapping $T_t x \to [x, t]$ is an embedding of $\tau(E)$ into a dense subset of $\bar{\tau}(E)$. In other words, $\bar{\tau}(E)$ is a compactification of $\tau(E)$. In the sequel we shall identify the elements $T_t x$ of $\tau(E)$ and the corresponding elements $[x, t]$ of $\bar{\tau}(E)$. Further, we extend the functions $T_s$ $(-\infty < s < \infty)$ and $\| \ \|$ from $\tau(E)$ onto $\bar{\tau}(E)$ by continuity, i.e. we put $T_s[x, -\infty] = [x, -\infty]$, $T_s[x, \infty] = [x, \infty]$, $\|[x, -\infty]\| = \infty$, $\|[x, \infty]\| = 0$ for all $x \in E$. Then we have the formula

$$T_s[x, t] = [x, t+s].$$

Let $\Phi$ be a weight function on $X$. By Lemma 5.3 [17] and the condition (b) $\Phi$ is bounded from below on every set $\{x\colon \|x\| \geqslant r\} \cap \tau(E)$ with $r > 0$. Further, $\Phi$ can be extended to $\bar{\tau}(E)$ by assuming $\Phi([x, \infty]) = 0$ and $\Phi([x, -\infty]) = \lim_{\|z\| \to \infty} \Phi(z)$. Let $N$ be a finite Borel measure on $\bar{\tau}(E)$. Put

$$M_N(U) = \int_U \frac{N(du)}{\Phi(u)} \tag{6.1}$$

for every subset $U$ of $\bar{\tau}(E)$ with the property $\inf\{\|u\|: u \in U\} > 0$. This formula defines a $\sigma$-finite measure $M_N$ on the set $\{u \in \bar{\tau}(E): \|u\| > 0\}$. Let $H_d(E, V)$ $(d = 1, 2, \ldots)$ denote the class of all finite measures $N$ on $\bar{\tau}(E)$ for which the corresponding measures $M_N$ fulfil the condition

$$M_N + \sum_{r=1}^{j} (-1)^r \sum_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant j} T_{t_{i_1} + t_{i_2} + \ldots + t_{i_r}} M_N \geqslant 0 \tag{6.2}$$

for every $j = 1, 2, \ldots, d$ and for all $t_1, t_2, \ldots, t_d \geqslant 0$. Moreover, we put

$$H_\infty(E, V) = \bigcap_{d=1}^{\infty} H_d(E, V).$$

It is easy to check that the sets $H_d(E, V)$ $(d = 1, 2, \ldots, \infty)$ are closed and convex. Let us consider the measures $M$ from $L_d(E, V)$ as measures on $\bar{\tau}(E)$. Set

$$N^M(U) = \int_U \Phi(u) M(du) \tag{6.3}$$

for every Borel subset $U$ of $\bar{\tau}(E)$. It is evident that $M \in L_d(E, V)$ if and only if $N^M \in H_d(E, V)$ $(d = 1, 2, \ldots, \infty)$. By $I_d(E, V)$ we shall denote the subset of $H_d(E, V)$ consisting of probability measures. Clearly, $I_d(E, V)$ $(d = 1, 2, \ldots, \infty)$ is convex and compact. Further, for every Borel subset $E_1$ of $E$ the sets $\tau(E_1)$, $\{[x, -\infty]: x \in E_1\}$ and $\{[x, \infty]: x \in E_1\}$ are $\{T_t\}$-invariant. Hence if $N \in H_d(E, V)$, the restriction of $N$ to any of these sets is again in $H_d(E, V)$. This implies that every extreme point of $I_d(E, V)$ $(d = 1, 2, \ldots, \infty)$ must be concentrated on orbits of elements of $\bar{\tau}(E)$, i.e. on one of the following sets: $\tau(\{x\})$, $\{[x, -\infty]\}$ and $\{[x, \infty]\}$ where $x \in E$. Obviously, all measures $\delta_z$, $z \in \bar{\tau}(E) \setminus \tau(E)$ are extreme points of $I_d(E, V)$. Then the problem of examing the measures $M$ from $L_d(E, V)$ is reduced to finding extreme points of sets $I_d(E, V)$ $(d = 1, 2, \ldots, \infty)$ concentrated on $\tau(\{x\})$, where $x \in E$.

## 7. Multiply monotone functions

A real-valued function $g$ defined on the real line is called *d-times monotone* $(d = 1, 2, \ldots)$ if the following conditions are satisfied:

(i) $g$ is left-continuous and $\lim_{t \to \infty} g(t) = 0$,

(ii) for any $t_1, t_2, \ldots, t_d > 0$ and $a, b \in (-\infty, \infty)$ with $a < b$ we have the inequality

$$\Delta_{t_1, t_2, \ldots, t_d}\, g(b) \geqslant \Delta_{t_1, t_2, \ldots, t_d}\, g(a)$$

where $\Delta_{t_1,t_2,\ldots,t_d}$ is a difference operator defined on the real functions $g$ inductively as follows:

$$\Delta_{t_1,t_2,\ldots,t_d} g(s) = \begin{cases} \Delta_{t_1,t_2,\ldots,t_{d-1}} g(s) - g(s-t_d) & \text{if } d \geqslant 2, \\ g(s) - g(s-t_1) & \text{if } d = 1 \end{cases}$$

($t_1, t_2, \ldots, t_d > 0$ and $-\infty < s < \infty$).

Further, if for every $d = 1, 2, \ldots$ a function $g$ is $d$-times monotone, then it is called *completely monotone.*

PROPOSITION 7.1. *Let $g$ a $d$-times monotone function on the real line ($d = 1, 2, \ldots$). Then there exists a unique non-negative left-continuous monotone non-decreasing function $q$ such that for every $t \in (-\infty, \infty)$*

$$g(t) = \int_{-\infty}^{t} \int_{-\infty}^{u_{d-1}} \int_{-\infty}^{u_{d-2}} \ldots \int_{-\infty}^{u_1} q(u)\, du\, du_1\, du_2 \ldots du_{d-1}. \tag{7.1}$$

Proof. It is evident that every $d$-times monotone function $g$ on the real line is convex non-negative and monotone non-decreasing. Consequently, there exists a unique non-negative left-continuous monotone non-decreasing function $g_1$ such that for every $t \in (-\infty, \infty)$

$$g(t) = \int_{-\infty}^{t} g_1(s)\, ds.$$

It is easy to check that if $d > 1$ then the function $g_1$ is $(d-1)$-times monotone. Hence and by an easy induction it follows that there exists a unique non-negative left-continuous monotone non-decreasing function $q$ such that (7.1) holds. The proposition is thus proved.

PROPOSITION 7.2. *Let $g$ be a completely monotone function on the real line. Then there exists a unique completely monotone function $q$ such that for every $t \in (-\infty, \infty)$*

$$g(t) = \int_{-\infty}^{t} q(u)\, du. \tag{7.2}$$

Proof. By the definition of completely monotone functions it follows that the function $g$ is convex non-negative monotone non-decreasing. Consequently, there exists a unique non-negative left-continuous monotone non-decreasing function $q$ such that for every $t \in (-\infty, \infty)$ equation (7.2) holds. It is clear that

$$\lim_{t \to -\infty} q(t) = 0$$

and, moreover, for any $d = 1, 2, \ldots, t_1, t_2, \ldots, t_d > 0$ and $a, b \in (-\infty, \infty)$ with $a < b$

$$\int_a^b \underset{t_1, t_2, \ldots, t_d}{\Delta} q(s)\, ds = \underset{t_1, t_2, \ldots, t_d}{\Delta} g(b) - \underset{t_1, t_2, \ldots, t_d}{\Delta} g(a) \geqslant 0.$$

Consequently, $\underset{t_1, t_2, \ldots, t_d}{\Delta} q(s) \geqslant 0$ for every $s \in (-\infty, \infty)$ which shows that the function $q$ is completely monotone. Thus the proposition is proved.

## 8. The Urbanik representation for $d$-times $\{T_t\}$-decomposable ($d = 1, 2, \ldots$) probability measures on $X$

Consider a compact subset $E$ of $X$ such that $0 \notin E$ and an arbitrary probability measure $N$ concentrated on $\tau(\{x\})$ ($x \in E$). Setting

$$g_N(b) = M_N(\{[x, t]:\ t < b\}) \tag{8.1}$$

we infer, by (6.2), that $N \in I_d(E, V)$ if and only if for any $t_1, t_2, \ldots, t_d > 0$ and $a < b$

$$\begin{aligned} &\underset{t_1, t_2, \ldots, t_d}{\Delta} g_N(b) - \underset{t_1, t_2, \ldots, t_d}{\Delta} g_N(a) \\ &= M_N(U) + \sum_{r=1}^{d} (-1)^r \sum_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant d} T_{t_{i_1} + t_{i_2} + \ldots + t_{i_r}} M_N(U) \geqslant 0 \end{aligned} \tag{8.2}$$

where $U = \{[x, t]:\ a \leqslant t < b\}$. In other words, $N \in I_d(E, V)$ if and only if the function $g_N$ defined by the formula (8.1) is $d$-times monotone. Further, by Proposition 7.1, there exists a unique non-negative left-continuous monotone non-decreasing function $q_N$ such that for every $t \in (-\infty, \infty)$

$$g_N(t) = \int_{-\infty}^{t} \int_{-\infty}^{u_{d-1}} \int_{-\infty}^{u_{d-2}} \ldots \int_{-\infty}^{u_1} q_N(u)\, du\, du_1\, du_2 \ldots du_{d-1}. \tag{8.3}$$

which, by virtue of (6.1) and (8.1), implies that

$$N(\{[x, t]:\ a \leqslant t < b\}) = \int_a^b \Phi([x, t])\, g_N^*(t)\, dt \tag{8.4}$$

where the function $g_N^*$ is defined by the formula

$$g_N^*(t) = \begin{cases} \int_{-\infty}^{t} \int_{-\infty}^{u_{d-2}} \ldots \int_{-\infty}^{u_1} q_N(u)\, du\, du_1\, du_2 \ldots du_{d-2} & \text{whenever } d \geqslant 2, \\ q_N(t) & \text{whenever } d = 1. \end{cases} \tag{8.5}$$

Consequently, we have

$$\int_{-\infty}^{\infty} \Phi([x,t])\, g_N^*(t)\, dt = 1. \tag{8.6}$$

Conversely, every non-negative monotone non-decreasing left-continuous function $q_N$ with the property (8.6) determines, by (8.4) and (8.5), a probability measure $N$ concentrated on $\tau(\{x\})$ for which the corresponding function $g_N$ is $d$-times monotone. Consequently, $N \in I_d(E, V)$. Hence we conclude that a measure $N \in I_d(E, V)$ is an extreme point of $I_d(E, V)$ if and only if the corresponding function $q_N$ cannot be decomposed into a non-trivial convex combination of two functions $q_{N_1}$ and $q_{N_2}$ ($N_1, N_2 \in I_d(E, V)$). But this is possible only in the case $q_N(t) = 0$ if $t \leqslant t_0$ and $q_N(t) = c$ if $t > t_0$ for some constants $t_0$ and $c$. By (8.4), (8.5), and by some computation we get the formula

$$N(\{[x,t]:\ a \leqslant t < b\}) = c(d-1)! \int_a^b \Phi([x,t])\, \{(t-t_0)_+\}^{d-1}\, dt \tag{8.7}$$

where for a real number $\lambda$ we write $\lambda_+ = \max(\lambda, 0)$. The constant $c$ is determined by (8.6) and (8.7). Namely,

$$c^{-1} = (d-1)! \int_{-\infty}^{\infty} \Phi([x,t])\, \{(t-t_0)_+\}^{d-1}\, dt. \tag{8.8}$$

We remark, by condition (c) and by Lemma 5.2 [17], that the last integral is finite for every $d = 1, 2, \ldots$ Thus we have proved that every extreme point of $I_d(E, V)$ concentrated on $\tau(\{x\})$ is of the form (8.7), where the constant $c$ is given by (8.8).

Conversely, let $N$ be a probability measure on $\bar{\tau}(E)$ defined by the formula (8.7) where $x \in E$ and the constant $c$ is given by (8.8). Then the corresponding measure $M_N$ is of the form

$$M_N(\{[x,t]:\ a \leqslant t < b\}) = c(d-1)! \int_a^b \{(t-t_0)_+\}^{d-1}\, dt.$$

It is easy to check that the measure $M_N$ defined by the last formula satisfies the condition (6.2) and hence it belongs to the set $I_d(E, V)$. Moreover, $M_N$ is an extreme point of $I_d(E, V)$.

For every $z \in \tau(E)$, $z = T_s x$ and $x \in E$, we put

$$N_z^{(d)}(U) = C_d(z) \int_0^{\infty} 1_U(T_t z)\, \Phi(T_t z)\, t^{d-1}\, dt \tag{8.9}$$

where $1_U$ denotes the indicator of a subset $U$ of $\bar{\tau}(E)$ and

$$C_d^{-1}(z) = (d-1)! \int_0^{\infty} \Phi(T_t z)\, t^{d-1}\, dt. \tag{8.10}$$

By virtue of (8.7) and (8.8) it follows that $N_z^{(d)}$ are extreme points of $I_d(E, V)$. We extend the definition of $N_z^{(d)}$ to $z \in \bar{\tau}(E)\setminus\tau(E)$ by assuming $N_z^{(d)} = \delta_z$. In this case we have also $N_z^{(d)} \in I_d(E, V)$. It hints at, the mapping $z \to N_z^{(d)}$ from $\bar{\tau}(E)$ into $I_d(E, V)$ is one-to-one and continuous. Consequently, it is a homeomorphism between $\bar{\tau}(E)$ and the set $e(I_d(E, V))$ of all extreme points of $I_d(E, V)$. Thus we have proved the following lemma:

LEMMA 8.1. *The set $\{N_z^{(d)}: z \in \bar{\tau}(E)\}$ is identical with the set $e(I_d(E, V))$ of all extreme points of $I_d(E, V)$ and the mapping $z \to N_z^{(d)}$ is a homeomorphism between $\bar{\tau}(E)$ and $e(I_d(E, V))$.*

Once the extreme points of $I_d(E, V)$ are found we can apply a well-known Krein–Milman–Choquet Theorem ([14], Chapter 3). Since each element of $H_d(E, V)$ is of the form $cN_1$ where $N_1 \in I_d(E, V)$ and $c \geqslant 0$ we then get the following proposition:

PROPOSITION 8.1. *A measure $N$ belongs to $H_d(E, V)$ $(d = 1, 2, \ldots)$ if and only if there exists a finite Borel measure $m$ on $\bar{\tau}(E)$ such that*

$$\int_{\bar{\tau}(E)} f(x)N(dx) = \int_{\bar{\tau}(E)} \int_{\bar{\tau}(E)} f(u)N_z^{(d)}(du)m(dz)$$

*for every continuous function $f$ on $\bar{\tau}(E)$. If $N$ is concentrated on $\tau(E)$ then $m$ does the same.*

From this proposition and by (6.1) and (8.9) we get, after some computation, the following corollary:

COROLLARY 8.1. *Let $M$ be a measure from $M(X)$ concentrated on $\tau(E)$. Then $M \in L_d(E, V)$ $(d = 1, 2, \ldots)$ if and only if there exists a finite Borel measure $m$ on $\tau(E)$ such that*

$$\int_{\tau(E)} f(x)M(dx) = \int_{\tau(E)} C_d(z) \int_0^\infty f(T_t z)t^{d-1}\,dt\,m(dz)$$

*for every $M$-integrable function $f$ on $\tau(E)$. The function $C_d(z)$ is defined by the formula* (8.10).

We now turn to the consideration of arbitrary measures $M \in M(X)$ corresponding to $d$-times $\{T_t\}$-decomposable probability measures on $X$. By Lemma 6.1 there exists a decomposition $M = \sum_{j=1}^{\infty} M_j$, where $M_j \in M(X)$ are restrictions of $M$ to disjoint sets $\tau(E_j)$, $0 \notin E_j$ and $E_j$ are compact. Then we have $M_j \in L_d(E_j, V)$ $(j = 1, 2, \ldots)$. Let $m_j$ denote a finite measure on $\tau(E_j)$ corresponding to $M_j$ in the representation given by Corollary 8.1. Then

$$\int_X f(x)M(dx) = \sum_{j=1}^{\infty} \int_{\tau(E_j)} C_d(z) \int_0^\infty f(T_t z)t^{d-1}\,dt\,m_j(dz)$$

for every $M$-integrable function $f$. Substituting $f = \Phi$ into this formula, we get the equation

$$\int_X \Phi(x) M(dx) = \sum_{j=1}^{\infty} m_j(\tau(E_j)).$$

Consequently, setting $m = \sum_{j=1}^{\infty} m_j$, we get a finite measure on $X$ satisfying the equation

$$\int_X f(x) M(dx) = \int_X C_d(z) \int_0^{\infty} f(T_t z) t^{d-1} dt m(dz) \tag{8.11}$$

for every $M$-integrable function $f$ on $X$. Moreover, $m(\{0\}) = 0$.

Putting, for any $x \in X$ and $y \in X^*$,

$$K_{\Phi,V}^{(d)}(x, y) = C_d(x) \int_0^{\infty} K(T_t x, y) t^{d-1} dt \tag{8.12}$$

where the kernel $K$ is given by the formula (5.3), we get the formula

$$\int_X K(x, y) M(dx) = \int_X K_{\Phi,V}^{(d)}(x, y) m(dx) \qquad (y \in X^*)$$

which together with (5.1), (5.2), (5.3) and (5.6) yields the following theorem:

THEOREM 8.1. *Let $\Phi$ be a weight function on $X$, $V \in B(X)$ and $T_t := \exp tV \to 0$ as $t \to \infty$. Then a probability measure $\mu$ on $X$ is d-times $\{T_t\}$-decomposable $(d = 1, 2, \ldots)$ if and only if there exist a finite measure $m$ on $X$ vanishing at 0, an element $x_0 \in X$ and an operator $R \in R_d(X, V)$ such that*

$$\hat{\mu}(y) = \exp\left(i\langle y, x_0\rangle - \tfrac{1}{2}\langle y, Ry\rangle + \int_X K_{\Phi,V}^{(d)}(x, y) m(dx)\right) \tag{8.13}$$

*for every $y \in X^*$. The kernel $K_{\Phi,V}^{(d)}$ is defined by the formula* (8.12).

Combining Theorems 4.1 and 8.1 we get the following solution of the Problem I which is a generalization of the Urbanik representation theorem for full Levy's measures on Banach spaces ([17], Theorem 5.3).

THEOREM 8.2. *Let $\Phi$ be a weight function on $X$. A full probability measure $\mu$ on $X$ belongs to $N_d(X)$ $(d = 1, 2, \ldots)$ if and only if there exists a one-parameter semigroup $T_t := \exp tV$ $(t \geqslant 0)$ with $V \in B(X)$ and $\lim_{t\to\infty} T_t = 0$, an element $x_0 \in X$, an operator $R \in R_d(X, V)$ and a finite measure $m$ on $X$ vanishing at 0 such that the equation* (8.13) *holds.*

## 9. The Urbanik representation for completely $\{T_t\}$-decomposable probability measures on X

Consider a compact subset $E$ of $X$ such that $0 \notin E$ and an arbitrary probability measure $N$ concentrated on $\tau(\{x\})$ where $x \in E$. Define a function $g_N$ by virtue of the formula (8.1). From (6.2) and (8.2) it follows that

$N \in I_\infty(E, V)$ if and only if $g_N$ is completely monotone. Further, by Proposition 7.2 there exists a unique completely monotone function $p_N$ such that for every $t \in (-\infty, \infty)$

$$g_N(t) = \int_{-\infty}^{t} p_N(u)\,du \tag{9.1}$$

which together with (6.3) and (8.1) implies the formula

$$N\{[x, t]:\ a \leqslant t < b\} = \int_a^b \Phi([x, t])\,P_N(t)\,dt. \tag{9.2}$$

Consequently, we have

$$\int_{-\infty}^{\infty} \Phi([x, t])\,p_N(t)\,dt = 1. \tag{9.3}$$

Conversely, every completely monotone function $p_N$ on the real line with the property (9.3) determines, according to the formula (9.2), a probability measure $N$ concentrated on $\tau(\{x\})$. Moreover, we have $N \in I_\infty(E, V)$. Hence we conclude that a measure $N \in I_\infty(E, V)$ concentrated on $\tau(\{x\})$ is an extreme point of $I_\infty(E, V)$ if and only if the corresponding function $p_N$ cannot be decomposed into a non-trivial convex combination of two functions $p_{N_1}$ and $p_{N_2}$ $(N_1, N_2 \in I_\infty(E, V))$. Given $t > 0$ and a function $p$ with such a property define two auxiliary functions $p_1$ and $p_2$ as follows:

$$p_1(u) = \frac{p(u)+p(u-t)}{1+c} \quad \text{and} \quad p_2(u) = \frac{p(u)+p(u-t)}{1+c} \quad (-\infty < u < \infty),$$

where $c = \int_{-\infty}^{\infty} \Phi([x, u])\,p(u-t)\,du$. It is evident that for sufficiently large $t$ we have $0 < c < 1$ and then the functions $p_1$ and $p_2$ are both completely monotone. Moreover, they are normalized by the condition (9.3) and for every $u \in (-\infty, \infty)$

$$p(u) = \tfrac{1}{2}(1+c)\,p_1(u) + \tfrac{1}{2}(1-c)\,p_2(u).$$

Consequently, for every $u \in (-\infty, \infty)$ and sufficiently large $t > 0$

$$p(u-t) = p(u) \int_{-\infty}^{\infty} \Phi([x, s])\,p(s-t)\,ds$$

which, by a simple reason, implies that the function $p$ is of the form

$$p(u) = Ce^{su} \quad (-\infty < u < \infty) \tag{9.4}$$

where $C$, $s$ are some positive constants.

Given a subset $U$ of $X$ and an operator $V \in B(X)$ such that $T_t := \exp tV \to 0$ as $t \to \infty$ define a congruence relation in $U$ as follows:

$x_1 \varrho x_2$, where $x_1, x_2 \in U$, if and only if there exists a number $t \in (-\infty, \infty)$ such that $T_t x_1 = x_2$. It is evident that the relation $\varrho$ is continuous. Let $U/\varrho$ denote the quotient space. Then for every $U \subset X$ we have $U/\varrho = \tau(U)/\varrho$.

Suppose that $\Phi$ is a weight function on $X$. Put

$$\sigma_{\Phi,V}(U) = \{([x], s) \in U/\varrho \times R_+ : \int_{-\infty}^{\infty} \Phi(T_t x) e^{st} dt < \infty\} \tag{9.5}$$

where $U \subset X \setminus \{0\}$, $[x]$ is an equivalence class of $U/\varrho$ and $R_+$ is the positive half-line.

By Lemma 5.2 [17] and the condition (c) it follows that for every non-void $U \subset X\setminus\{0\}$ the set $\sigma_{\Phi,V}(U)$ is non-void. Further, for every sequence $U_1, U_2, \ldots$ of subsets of $X\setminus\{0\}$ such that the sets $\tau(U_j)$ $(j = 1, 2, \ldots)$ are disjoint, the sets $\sigma_{\Phi,V}(U_j)$ are disjoint too and

$$\sigma_{\Phi,V}\Big(\bigcup_{j=1}^{\infty} U_j\Big) = \bigcup_{j=1}^{\infty} \sigma_{\Phi,V}(U_j). \tag{9.6}$$

Given an element $([z], s) \in \sigma_{\Phi,V}(E)$, where $E$ is a compact subset of $X\setminus\{0\}$ and $[z]$ is an equivalence class of $E/\varrho$ with $z \in E$ we put

$$N_{[z],s}(Q) = C_{\Phi,V}(z, s) \int_{-\infty}^{\infty} 1_Q(T_t z) \Phi(T_t z) e^{st} dt \tag{9.7}$$

where $1_Q$ denotes the indicator of a subset $Q$ of $\bar{\tau}(E)$ and

$$C_{\Phi,V}^{-1}(z, s) = \int_{-\infty}^{\infty} \Phi(T_t z) e^{st} dt. \tag{9.8}$$

It is easy to check that the right-hand side of (9.7) does not depend on any choice of the representing element $z$ of $[z]$.

Since for every extreme point $N$ of $I_\infty(E, V)$ the corresponding function $P_N$ is of the form (9.4) and normalized by the condition (9.3) the set $\{N_{[z],s}: ([z], s) \in \sigma_{\Phi,V}(\tau(E))\}$ contains all extreme points of $I_\infty(E, V)$ concentrated on $\tau(E)$. Our further aim is to prove that every measure $N_{[z],s}(([z], s) \in \sigma_{\Phi,V}(\tau(E)))$ defined by the formula (9.7) is an extreme point of $I_\infty(E, V)$.

Accordingly, from (9.7) it follows that the measure $N_{[z],s}$ is concentrated on $\tau(\{z\})$ and the corresponding measure $M_{N_{[z],s}}$ is of the form

$$M_{N_{[z],s}}(\{T_t z: a \leq t < b\}) = C_{\Phi,V}(z, s) \int_a^b e^{st} dt$$

$(-\infty < a < b < \infty)$. Consequently, $M_{N_{[z],s}} \in L_\infty(E, V)$ and hence the measure $N_{[z],s}$ defined by the formula (9.7) is an extreme point of the set $I_\infty(E, V)$. It is easily seen that the mapping $([z], s) \to N_{[z],s}$ from $\sigma_{\Phi,V}(\tau(E))$ into $I_\infty(E, V)$ is one-to-one and continuous. Thus we have proved the following lemma:

LEMMA 9.1. *The set* $\{N_{[z],s}\colon ([z], s) \in \sigma_{\Phi,V}(\tau(E))\}$ *is identical with the set of all extreme points of the set* $I_\infty(E, V)$ *concentrated on* $\tau(E)$ *and the mapping* $([z], s) \to N_{[z],s}$ *is a homeomorphism between them.*

Denoting by $e(I_\infty(E, V))$ the set of all extreme points of $I_\infty(E, V)$ and taking into account the fact that each element of $H_\infty(E, V)$ is of the form $cN_1$, where $N_1 \in I_\infty(E, V)$ and $c \geqslant 0$, we then get the following proposition:

PROPOSITION 9.1. *A measure* $N$ *belongs to the set* $H_\infty(E, V)$ *if and only if there exists a finite Borel measure* $m$ *on* $e(I_\infty(E, V))$ *such that*

$$\int_{\bar{\tau}(E)} f(x)\, N(dx) = \int_{e(I_\infty(E,V))} \int_{\bar{\tau}(E)} f(u)\, \pi(du)\, m(d\pi)$$

*for every continuous function* $f$ *on* $\bar{\tau}(E)$. *If* $N$ *is concentrated on* $\tau(E)$ *then* $m$ *is concentrated on the subset of* $e(I_\infty(E, V))$ *consisting of probability measures concentrated on* $\tau(E)$.

Combining (6.1), (9.7), Lemma 9.1 and Proposition 9.1 we get the following corollary:

COROLLARY 9.1. *Let* $M$ *be a measure from* $M(X)$ *and concentrated on* $\tau(E)$. *Then* $M$ *belongs to the set* $L_\infty(E, V)$ *if and only if there exists a finite Borel measure* $m$ *on the set* $\sigma_{\Phi,V}(\tau(E))$ *such that*

$$\int_{\tau(E)} f(x)\, M(dx) = \int_{\sigma_{\Phi,V}(\tau(E))} C_{\Phi,V}(z, u) \int_{-\infty}^{\infty} f(T_t z)\, e^{ut}\, dt\, m(d([z], u))$$

*for every* $M$*-integrable function* $f$ *on* $\tau(E)$. *The function* $C_{\Phi,V}(z, u)$ *is defined by the formula* (9.8).

Consider an arbitrary measure $M \in M(X)$ corresponding to a completely $\{T_t\}$-decomposable probability measure on $X$. By Corollary 6.1 there exists a decomposition $M = \sum_{j=1}^{\infty} M_j$, where $M_j \in M_\infty(X, V)$ $(j = 1, 2, \ldots)$, $M_j$ are concentrated on disjoint sets $\tau(E_j)$, $0 \notin E_j$ and $E_j$ are compact. Let $m_j$ denote a finite Borel measure on $\sigma_{\Phi,V}(\tau(E_j))$ corresponding to $M_j$ in the representation given by Corollary 9.1. Then, for every $M$-integrable function $f$ on $X$

$$\int_X f(x)\, M(dx) = \sum_{j=1}^{\infty} \int_{\sigma_{\Phi,V}(\tau(E_j))} C_{\Phi,V}(z, u) \int_{-\infty}^{\infty} f(T_t z)\, e^{ut}\, dt\, m(d([z], u)).$$

Substituting $f = \Phi$ into this formula we get the equation

$$\int_X f(x)\, M(dx) = \sum_{j=1}^{\infty} m_j\big(\sigma_{\Phi,V}(\tau(E_j))\big) < \infty.$$

Consequently, setting $m = \sum_{j=1}^{\infty} m_j$ and taking into account the fact that the sets $\sigma_{\Phi,V}(\tau(E_j))$ $(j = 1, 2, \ldots)$ are disjoint, $\sum_{j=1}^{\infty} \sigma_{\Phi,V}(\tau(E_j)) = \sigma_{\Phi,V}(X \backslash \{0\})$ we get a finite Borel measure $m$ on $\sigma_{\Phi,V}(X\setminus\{0\})$ satisfying the equation

$$(9.9)\qquad \int_X f(x)M(dx) = \int_{\sigma_{\Phi,V}(X\setminus\{0\})} C_{\Phi,V}(z,u)\int_{-\infty}^{\infty} f(T_t z)e^{ut}\,dt\,m(d([z],u))$$

which, by virtue of (9.8), can be written in the form

(9.10)

$$\int_X f(x)M(dx) = \int_{\sigma_{\Phi,V}(X\setminus\{0\})} \int_{-\infty}^{\infty} f(T_t z)e^{ut}\,dt\Big[\int_{-\infty}^{\infty} \Phi(T_t z)e^{ut}\,dt\Big]^{-1} m(d([z],u)).$$

Hence and by (5.1), (5.2), (5.3), (5.6) we get the following theorem:

THEOREM 9.1. *Let $\Phi$ be a weight function on $X$ and $V$ an operator from $B(X)$ such that $T_t := \exp tV \to 0$ as $t\to\infty$. A probability measure $\mu$ on $X$ is completely $\{T_t\}$-decomposable if and only if there exists a finite Borel measure $m$ on $\sigma_{\Phi,V}(X\setminus\{0\})$, a covariance operator $R\in R_\infty(X,V)$ and an element $x_0\in X$ such that for every $y\in X^*$*

$$(9.11)\qquad \hat{\mu}(y) = \exp\{i\langle y, x_0\rangle - \tfrac{1}{2}\langle y, Ry\rangle +$$
$$+\int_{\sigma_{\Phi,V}(X\setminus\{0\})}\int_{-\infty}^{\infty} K(T_t z, y)e^{ut}\,dt\Big[\int_{-\infty}^{\infty}\Phi(T_t z)e^{ut}\,dt\Big]^{-1} m(d([z],u))$$

*where the set $\sigma_{\Phi,V}(X\setminus\{0\})$ is defined by the formula* (9.5) *and the kernel $K$ is given by* (5.3). *The integrand over $\sigma_{\Phi,V}(X\setminus\{0\})$ does not depend on any choice of the representing elements $z$ of the equivalence classes $[z]$ of $X\setminus\{0\}/\varrho$.*

Combining Theorems 4.1 and 9.1 we get the following solution of Problem II:

THEOREM 9.2. *Let $\Phi$ be a weight function on $X$. A full probability measure $\mu$ on $X$ belongs to the set $N_\infty(X)$ if and only if there exist an operator $V\in B(X)$ with $\lim_{t\to\infty}\exp tV = 0$, an element $x_0\in X$, an operator $R\in R_\infty(X,V)$ and a finite Borel measure $m$ on $\sigma_{\Phi,V}(X\setminus\{0\})$ such that the characteristic functional of $\mu$ is given by the formula* (9.11).

# References


[1] G. Choquet, *Le théorem de représentation intégrale dans les ensemble convexes compact*, Ann. Inst. Fourier 10 (1960), pp. 333–344.

[2] E. Dettweiler, *Grenzwertsätze für Wahrscheinlichkeitsmasse auf B-adriklanschen Räumen*, Thesis, Eberhard-Karls Universität zu Tübingen, 1974.

[3] K. Ito and M. Nisio, *On the convergence of sums of independent Banach space valued random variables*, Osaka J. of Math. 5 (1968), pp. 35–48.

[4] N. S. Jain and G. Kallianpur, *Norm convergent expansions for Gaussian processes in Banach spaces*, Proc. Amer. Math. Soc. 25 (1970), pp. 890–895.

[5] A. Kumar and B. M. Schreiber, *Self-decomposable probability measures on Banach spaces*, Studia Math. 53 (1975), pp. 55–71.

[6] P. Lévy, *Théorie de l'addition des variables aléatoires*, Paris 1954.

[7] M. Loéve, *Probability Theory*, New York 1950.

[8] Nguyen Van Thu, *Multiply self-decomposable probability measures on Banach spaces*, Studia Math. 66 (1980), pp. 161–175.

[9] – *Banach space valued Brownian motions I*, Acta Math. Vietnam 3 (1978), pp. 35–40.

[10] – *Banach space valued Brownian motions II: A stochastic integral of operator-valued functions*, ibid. 3 (1978), 44–46.

[11] A. B. Paalman-de Miranda, *Topological Semigroup*, Amsterdam 1964.

[12] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, New York–London 1967.

[13] M. Sharpe, *Operator-stable probability distributions on vector groups*, Trans. Amer. Math. Soc. 136 (1969), pp. 51–65.

[14] A. Tortrat, *Structure des lois indéfiniment divisibles dans un espace vectorial topologique (separe) X*, Symposium on Probability Methods in Analysis, Lecture Notes in Mathematics 31, Berlin–Heidelberg–New York 1967, pp. 299–328.

[15] – *Sur la structure des lois indéfiniment divisibles dans les espaces vectoriels*, Z. Warhscheinlichkeitstheorie verw. Geb. 11 (1969), pp. 311–326.

[16] K. Urbanik, *Levy's probability measures on Euclidean spaces*, Studia Math. 44 (1972), pp. 119–148.

[17] – *Levy's probability measures on Banach spaces*, ibid. 63 (1978), pp. 284–308.

[18] N. N. Vahaniya, *Probability distributions on linear spaces* (in Russian), Tbilisi 1971.

[19] R. E. Williamson, *Multiply monotone functions and their Laplace transforms*, Duke Math. J. 23 (1956), pp. 189–207.