PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 19
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1987

SOME LEAKAGE PROBLEMS FOR IDEAL INCOMPRESSIBLE FLUID MOTION IN DOMAINS WITH EDGES

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1. Introduction

In this paper we consider two cases of motions of an ideal incompressible fluid in a bounded domain $\Omega \subset \boldsymbol{R}^3$. In the first case the tangent components of the vorticity are prescribed in an inflow, and the normal component of the velocity on the whole boundary [7, 9]. In the second case the velocity vector in an inflow and the pressure in an outflow are given [8, 9]. The motions are considered in a domain with edges on its boundary. Therefore, elliptic boundary value problems theory in domains with edges on the boundary [5, 6, 11] implies some restrictions on the smoothness of the fluid motion near the edges.

Let us assume for simplicity that the boundary of Ω consists of three parts S_0, S_1, S_2 such that S_0 is between S_1 and S_2. Moreover, between S_0 and S_i, $i = 1, 2$, there is a dihedral angle. Therefore our domain is similar to a cylinder; however, all results of this paper are valid for more general domains too. The physical picture is such that the fluid enters the domain through S_1, leaves it through S_2 and S_0 is impermeable for it.

The first motion is described as follows:

$$v_t + v \cdot \nabla v + \nabla p = f, \tag{1.1}$$

$$\operatorname{div} v = 0, \tag{1.2}$$

$$v|_{t=0} = a, \quad \operatorname{div} a = 0, \tag{1.3}$$

$$v_n|_{\partial\Omega} = b, \quad b|_{S_1} < 0, \quad b|_{S_0} = 0, \quad b|_{S_2} \geqslant 0, \tag{1.4}$$

$$\omega|_{S_1} = \eta \in TS_1, \tag{1.5}$$

where v is the velocity vector, p the pressure, $\omega = \operatorname{rot} v$ the vorticity and f

the external force. Moreover, $v_n = v \cdot \bar{n}$, where $\bar{n}$ is the outward unit normal vector to the boundary, $\partial\Omega = S_0 \cup S_1 \cup S_2$ and TS_1 is the tangent space to S_1.

From (1.2)–(1.5) the following compatibility conditions have to be assumed:

$$\int_{\partial\Omega} b(s, t)\,ds = 0, \tag{1.6}$$

$$b|_{t=0} = a \cdot \bar{n}|_{\partial\Omega}, \tag{1.7}$$

$$\eta \cdot \bar{\tau}_\mu = \operatorname{rot} a \cdot \bar{\tau}_\mu|_{S_1}, \quad \mu = 1, 2, \tag{1.8}$$

where $\bar{\tau}_\mu \in TS_1$, $\mu = 1, 2$.

We know from [7, 9] that to prove the existence of solutions of the problem (1.1)–(1.5) one has to replace it by the following system of two problems:

$$\text{(A)} \qquad \begin{aligned} &\omega_t + v \cdot \nabla\omega - \omega \cdot \nabla v = \operatorname{rot} f \equiv F, \\ &\omega|_{t=0} = \omega_0 = \operatorname{rot} a, \quad \omega|_{S_1} = \eta, \end{aligned}$$

where v is treated as a given function, and

$$\text{(B)} \qquad \operatorname{rot} v = \omega, \quad \operatorname{div} v = 0, \quad v_n|_{\partial\Omega} = b,$$

where ω is given.

For the second motion the equations (1.1)–(1.3) are satisfied and we assume the following boundary conditions:

$$v|_{S_1} = \chi, \quad v_n|_{S_0} = 0, \tag{1.9}$$

$$p|_{S_2} = \varkappa. \tag{1.10}$$

Moreover, in this case the following compatibility conditions are necessary:

$$v|_{t=0} = a|_{S_1}, \quad a_n|_{S_0} = 0. \tag{1.11}$$

To prove the existence of solutions of this problem we need to replace it by the system of two problems [8, 9]:

$$\text{(C)} \qquad \begin{aligned} &v_t + v \cdot \nabla v = -\nabla p + f, \\ &v|_{t=0} = a(x), \quad \operatorname{div} a = 0, \\ &v|_{S_1} = \chi, \quad v_n|_{S_0} = 0, \end{aligned}$$

where p is treated as a given function, and

$$\Delta p = \operatorname{div} f - v^k_{x^i} v^i_{x^k} \equiv h(v, f),$$

$$\frac{1}{H_n} \frac{\partial p}{\partial n}\Big|_{S_1} = f_n|_{S_1} - \chi_{n,t} + \sum_{\mu=1}^{2} (\chi_n \chi_{\mu,\tau_\mu} + \chi_n \chi_\mu \operatorname{div} \bar{\tau}_\mu

$$
\text{(D)}\quad
\begin{aligned}
&+\chi_n \chi_\mu \bar{\tau}_\mu \cdot \bar{n}_{,n} - \chi_\mu \chi_{n,\tau_\mu}) + \sum_{\mu,\nu=1}^{2} \chi_\mu \chi_\nu \bar{\tau}_\nu \cdot \bar{n}_{,\tau_\mu} \\
&+\chi_n^2 \operatorname{div} \bar{n} \equiv g_1(\chi, \bar{\tau}, \bar{n}), \\
\frac{1}{H_n} \frac{\partial p}{\partial n}\bigg|_{S_0} &= -v^k v \cdot \bar{n}_{,x^k} + f_n|_{S_0} \equiv g_0(v, f), \\
&p|_{S_2} = \varkappa(x', t), \qquad x' \in S_2,
\end{aligned}
$$

where v is a given function and $\chi_\mu = \chi \cdot \bar{\tau}_\mu$. The other quantities are described in Section 2.

The paper is organized as follows. In Section 2 weighted Sobolev spaces are introduced and solvability theorems for the elliptic problems (B), (D) in domains with edges and in those spaces are formulated. In Sections 3 and 5 the existence of solutions of the evolution problem (A) and (C) in suitable weighted Sobolev spaces is shown by the method of characteristics. To prove the existence of solutions of (A, B) or (C, D) we use a method of successive approximations (see for example Section 5). Then it is shown in Sections 4 and 5 that the presence of an inflow implies that the weighted Sobolev spaces cannot be used. Indeed, the method of construction of solutions of the problems (A), (C) in the form (3.10) and (5.4), respectively, implies that the vorticity and velocity on S_1 are estimated by expressions in which there appear the last term in (3.12) and the first two terms in (5.7) (terms with g_1 and g_2), respectively. But if we use the method of successive approximations then these terms imply that η, χ and f have to be small in suitable norms. To avoid these restrictions (the only restriction we admit is a restriction on time, local solutions) we have to get an a priori estimate for solutions of the problems (A) and (C) by a direct method (differentiating the equations $(A)_1$, $(C)_1$, multiplying by the same derivatives of unknown functions and then integrating over Ω, and so on), as was done in [7, 8, 9]. But it is shown in [9] that weighted Sobolev spaces cannot be used in the case of an inflow also. However, weighted Sobolev spaces are very convenient to consider boundary value problems in domains with edges because less restrictions on the maximal magnitude of angles in the domains must be imposed. In view of this fact we start our considerations in weighted Sobolev spaces but the presence of an inflow forces us to use ordinary Sobolev spaces (see Theorems 4.1 and 5.1). We met with a similar situation in [14], where weighted Hölder spaces could not be used because an inflow occurred. In papers [12, 13] the existence of solutions of the problems (A, B) and (C, D) without inflow and outflow ($\partial\Omega = S_0$) is proved in weighted Sobolev and Hölder spaces, respectively. The restrictions obtained there on the magnitude of the maximal angle in the considered domains are less than those calculated in Remarks 4.1 and 5.1.

Finally, in Sections 4 and 5 the existence of solutions of the problems (A, B) and (C, D) is proved in Sobolev spaces (see Theorems 4.1 and respectively 5.1).

In [4] similar considerations to those in Lemmas 3.4 and 5.3 are done for the equation of continuity (equation for the density of a compressible fluid).

2. Notation and auxiliary results

We assume that S_ν, $\nu = 0, 1, 2$, are manifolds of class C^l such that $S_i \cap S_0 = L_i$, $i = 1, 2$, are edges of the boundary and $S_1 \cap S_2 = \emptyset$. Therefore there are dihedral angles between the tangent spaces $T_x S_i$ and $T_x S_0$ at each point $x \in L_i$, $i = 1, 2$. Let $\vartheta_i(x)$ be the magnitude of the dihedral angle at a point $x \in L_i$, $i = 1, 2$. Let

$$\vartheta_0 = \max_{i=1,2} \max_{x \in L_i} \vartheta_i(x).$$

In a neighbourhood of S_1 we introduce a curvilinear system of orthonormal coordinates $\tau_1(x)$, $\tau_2(x)$, $n(x)$ such that S_1 is determined by $n(x) = 0$ and $\tau_1(x)$, $\tau_2(x)$, $x \in S_1$, are tangent coordinates on S_1. Moreover, we denote by $\bar{\tau}_1(x)$, $\bar{\tau}_2(x)$, $\bar{n}(x)$ the orthonormal basis corresponding to this coordinate system such that for $x \in S_1$, $\bar{\tau}_1(x)$, $\bar{\tau}_2(x)$ are vectors tangent to S_1 and $\bar{n}(x)$ is the outward unit normal vector to S_1. Finally, we introduce the Lamé coefficients [2] determined by:

$$\frac{\partial}{\partial \tau_i} = H_i \bar{\tau}_i \cdot \nabla, \qquad i = 1, 2, \qquad \frac{\partial}{\partial n} = H_n \bar{n} \cdot \nabla$$

and we write

$$\alpha_{,n} = \frac{\partial}{\partial n} \alpha, \qquad \alpha_{,\tau} = \frac{\partial}{\partial \tau} \alpha$$

and so on.

In this paper the summation convention is used. Now we introduce weighted Sobolev spaces $W^l_{p,\mu}(\Omega)$, $0 \leqslant l$ integer, $1 \leqslant p$ real and $0 < \mu$ real, with the norm

$$\|u\|_{l,p,\mu,\Omega} \equiv \|u\|_{W^l_{p,\mu}(\Omega)} = \Big(\sum_{|\alpha| \leqslant l} \int_\Omega |D^\alpha u(x)|^p \varrho^{p\mu}(x)\, dx \Big)^{1/p},$$

where $\varrho(x)$ is the distance from x to the nearest edge. Moreover, we denote the space of traces of functions from $W^l_{p,\mu}(\Omega)$ by $W^{l-1/p}_{p,\mu}(\partial\Omega)$. Finally, we denote by $L^k_{p,\mu}(\Omega)$ the space with the norm

$$\|u\|_{L^k_{p,\mu}(\Omega)} = \Big(\sum_{|\alpha| = k} \int_\Omega |D^\alpha u(x)|^p \varrho^{p\mu}(x)\, dx \Big)^{1/p}.$$

Hence $L^0_{p,\mu}(\Omega) = W^0_{p,\mu}(\Omega) = L_{p,\mu}(\Omega)$. For functions in $L^k_{p,\mu}(\Omega)$ the Hardy inequality

$$\|u\|_{L_{p,\mu-k}(\Omega)} \leqslant c\|u\|_{L^k_{p,\mu}(\Omega)}, \quad p(\mu-k) > -2, \tag{2.1}$$

is valid.

From [10] we recall

THEOREM 2.1. *Let* $u \in W^l_{p,\mu}(\Omega)$, $\Omega \subset \boldsymbol{R}^3$, *$l$, k natural, $p, q > 1$, $\mu, \nu > 0$ real and*

$$l-k-3\left(\frac{1}{p}-\frac{1}{q}\right)-(\mu-\nu) \geqslant 0. \tag{2.2}$$

Then $u \in L^k_{q,\nu}(\Omega)$ *and*

$$\|u\|_{L^k_{q,\nu}(\Omega)} \leqslant c\|u\|_{W^l_{p,\mu}(\Omega)}. \tag{2.3}$$

The problems (B) and (C) are elliptic. In papers [5, 6, 11] the existence, uniqueness and regularity properties of solutions of those problems are shown. We recall these results.

THEOREM 2.2 [5, 6, 11]. *Let* $\omega \in W^k_{p,\mu}(\Omega)$, $b \in W^{k+1-1/p}_{p,\mu}(\partial\Omega)$, $S_\nu \in C^{k+2}$, $\nu = 1, 2$, *$k > 0$ natural, $p > 1$, $\mu > 0$ real and*

$$\frac{\pi}{\vartheta_0} > k+3-\mu-\frac{2}{p} > 0, \quad \vartheta_0 < \pi. \tag{2.4}$$

Then the problem (B) *has a unique solution* $v \in W^{k+1}_{p,\mu}(\Omega)$ *and the estimate*

$$\|v\|_{W^{k+1}_{p,\mu}(\Omega)} \leqslant c(\|\omega\|_{W^k_{p,\mu}(\Omega)}+\|b\|_{W^{k+1-1/p}_{p,\mu}(\partial\Omega)}) \tag{2.5}$$

is valid. For dihedral angles equal to π/n, where $n > 1$ is a natural number, the condition (2.4) *can be omitted.*

THEOREM 2.3 [5, 6]. *Let* $h \in W^k_{p,\mu}(\Omega)$, $g_i \in W^{k+1-1/p}_{p,\mu}(S_i)$, $i = 0, 1$, $\varkappa \in W^{k+2-1/p}_{p,\mu}(S_2)$, $S_\nu \in C^{k+2}$, $\nu = 0, 1, 2$, *$k > 0$ natural, $p > 1$, $\mu > 0$ real and*

$$\frac{\pi}{2\vartheta_0} > k+2-\mu-\frac{2}{p} > 0. \tag{2.6}$$

Then the problem (D) *has a solution* $p \in W^{k+2}_{p,\mu}(\Omega)$ *such that*

$$\|p\|_{W^{k+2}_{p,\mu}(\Omega)} \leqslant c\Big(\|h\|_{W^k_{p,\mu}(\Omega)}+\sum_{i=0}^{1}\|g_i\|_{W^{k+1-1/p}_{p,\mu}(S_i)} + \|\varkappa\|_{W^{k+2-1/p}_{p,\mu}(S_2)}+\|p\|_{L_2(\Omega)}\Big). \tag{2.7}$$

If we assume that

$$\int_\Omega p\,dx = 0, \tag{2.8}$$

the above solution will be unique and the last term in (2.7) can be omitted.

For dihedral angles equal to π/n, $n > 1$ a natural number, the condition (2.6) can be omitted but some compatibility conditions on the right-hand side functions must be assumed [13].

LEMMA 2.1. *Let T be given and $v \in C(0, T; W^2_{p,\mu}(\Omega))$. Then there exists a domain Ω_T and a function $\tilde{v} \in C(0, T; W^2_{p,\mu}(\Omega_T))$ which is an extension of the function v such that $\tilde{v}_n|_{\partial\Omega_T} = 0$ and*

$$\|\tilde{v}\|_{C(0,T;W^2_{p,\mu}(\Omega_T))} \leqslant c\|v\|_{C(0,T;W^2_{p,\mu}(\Omega))} \leqslant c\|\tilde{v}\|_{C(0,T;W^2_{p,\mu}(\Omega_T))}. \tag{2.9}$$

Proof. Use the Calderón extension [1, Ch. 3, § 9] such that $\tilde{v}_n|_{\partial\Omega_T} = 0$.

3. Existence of solutions of the problem (A)

To prove the existence of solutions of the problems (A) and (C) we introduce the characteristic curves

$$\begin{aligned} &\frac{d}{ds} y(x, t; s) = v(y(x, t; s), s), \\ &y(x, t; t) = x. \end{aligned} \tag{3.1}$$

Let $v \in C(0, T; W^2_{p,\mu}(\Omega))$, $1 - 3/p - \mu > 0$. Then Theorem 2.1 implies that $v(x, t)$ is Lipschitz continuous with respect to x, so there exists a unique solution of (3.1) which is C^1 with respect to the parameter s. There are two kinds of these curves:

(a) $y(x, t; s) \in \Omega \quad \forall s \in [0, t]$,

(b) there exists a moment $t_*(x, t) \geqslant 0$ such that $y(x, t; t_*(x, t)) \in S_1$.

Now we recall some properties of solutions of (3.1). Let $v \in C(0, T; C^2(\Omega))$. Then from (3.1) we have

$$\frac{d}{ds} y^i_{x^k}(x, t; s) = v^i_{y^l}(y(x, t; s), s) y^l_{x^k}(x, t; s), \tag{3.2}$$

$$\frac{d}{ds} y^i_{x^k x^l} = v^i_{y^r y^p} y^r_{x^k} y^p_{x^l} + v^i_{y^r} y^r_{x^k x^l}. \tag{3.3}$$

Let $J = \det\|y^i_{x^j}\|$. Then

$$J = \exp\int_t^\tau \operatorname{div}_y x(y(x, t; \tau), \tau) d\tau.$$

Let $A^k_j = \partial J/\partial y^k_{x^j}$. Then

$$\frac{d}{ds} A^k_j = -\frac{\partial v^s}{\partial x^j} A^k_s. \tag{3.4}$$

Moreover, we introduce for $t \in [0, T]$

$$(3.5)\qquad \begin{aligned} &\frac{d}{ds}\tilde{y}(x, t; s) = \tilde{v}(\tilde{y}(x, t; s), s),\\ &\tilde{y}(x, t; t) = x, \end{aligned}$$

where $\tilde{v}$ is determined by Lemma 2.1. Hence $\Omega_T \ni x \to \tilde{y}(x, t; s) \in \Omega_T$ is a diffeomorphism.

LEMMA 3.1. *Let $v \in C(0, T; W^2_{p,\mu}(\Omega))$ and $1-3/p-\mu > 0$, where $1 < p$, $0 < \mu$ are real. Then the following estimate is valid:*

$$(3.6)\qquad |y_x(x, t; s)| \leqslant c \exp[ct \sup_t \|v\|_{W^2_{p,\mu}(\Omega)}], \qquad \forall s \in [t'(x, t); t], t \in [0, T],$$

where $t'(x, t) = 0$ and $t'(x, t) = t_*(x, t)$ for the curves satisfying (a) *and* (b), *respectively. Moreover, we have*

$$(3.7)\qquad \Big(\int_\Omega |y_{xx}(x, t; s)|^p \varrho^{p\mu}(x)\, dx\Big)^{1/p} \leqslant ct \sup_t \|v\|_{W^2_{p,\mu}(\Omega)} \cdot \exp[ct \sup_t \|v\|_{W^2_{p,\mu}(\Omega)}], \qquad \forall s \in [t'(x, t), t], t \in [0, T].$$

Proof. Let $v \in C(0, T; C^2(\Omega))$. Integrating (3.2) and using Theorem 2.1 we get (3.6). Integrating (3.3), using Lemma 2.1 and (3.6) we obtain

$$\Big(\int_\Omega |y_{xx}(x, t; \tau)|^p \varrho^{p\mu}(x)\, dx\Big)^{1/p} \leqslant c \exp[ct \sup_t \|v\|_{W^2_{p,\mu}(\Omega)}] \cdot \int_0^t \Big(\int_{\Omega_T} |\tilde{v}_{\tilde{y}\tilde{y}}(\tilde{y}(x, t; s), s)|^p \varrho^{p\mu}(x)\, dx\Big)^{1/p} ds.$$

Then using the fact that $\Omega_T \ni x \to \tilde{y}(x, t; s) \in \Omega_T$ is a diffeomorphism we get (3.7). We conclude the proof with a density argument.

By a *generalized solution* of the problem (A) we mean a solution of the following integral identity:

$$(3.8)\qquad -\int_{\Omega^T} \omega\varphi_t + \int_\Omega \omega_0 \varphi(x, 0) + \int_{S_1^T} v_n \eta\varphi - \int_{\Omega^T} [\omega v \cdot \nabla\varphi + \omega \cdot \nabla v\varphi] = \int_{\Omega^T} F\varphi$$
$$\forall \varphi \in C^1(\bar{\Omega}^T),\ \Omega^T = \Omega \times [0, T],\ \varphi|_{t=T} = 0,\ \varphi|_{S_2} = 0.$$

First we shall show uniqueness.

LEMMA 3.2. *Let $\nabla v \in L_\infty(\Omega^T)$, $\omega_0 \in L_2(\Omega)$, $\eta \in C(0, T; L_2(S_1))$. Then the problem* (A) *has a unique solution in $C(0, T; L_2(\Omega))$.*

Proof. Let ω_1, ω_2 be two solutions of the problem (A). Let $\eta = \omega_1 - \omega_2$. Therefore the problem (A) implies

$$\frac{d}{dt}\|\eta\|^2_{L_2(\Omega)} \leqslant c \sup_\Omega |\nabla v|\, \|\eta\|^2_{L_2(\Omega)}, \qquad \eta|_{t=0} = 0.$$

This concludes the proof.

From the definition of $t_*(x, t)$ we have

$$(\partial_t + v\cdot\nabla)t_*(x, t) = 0. \tag{3.9}$$

Therefore our domain Ω^T is divided by the curves $t_*(x, t) = 0$ starting from the points $t = 0$, $x \in S^1$ into two subdomains: Ω^T_+ $(t_*(x, t) > 0)$ and Ω^T_- $(t_*(x, t) < 0)$, where Ω^T_- is filled up by the curves of the family (a) and Ω^T_+ by the curves of the family (b).

LEMMA 3.3. *Let $v \in C(0, T; C^1(\Omega))$, $\omega_0 \in C(\Omega)$, $\eta \in C(S_1^T)$, $F \in C(\Omega^T)$. Then there exists a solution $\omega \in C(\Omega^T_\pm)$ of the problem* (A) *such that*

$$\omega^k(x, t) = A_j^k\big(y_x(x, t; t'(x, t))\big)\omega^j\big(y(x, t; t'(x, t)), t'(x, t)\big) + \int_{t'(x,t)}^{t} A_j^k(y_x(x, t; s))F^j\big(y(x, t; s), s\big)ds, \tag{3.10}$$

where $t'(x, t) = 0$ for curves of (a) *and $t'(x, t) = t_*(x, t)$ for curves of* (b).

Proof. Using the curves (3.1) the problem (A) can be written in the form

$$\frac{d\omega}{ds}(y(x, t; s), s) - \omega\big(y(x, t; s), s\big)\cdot\nabla_y v\big(y(x, t; s), s\big) = F\big(y(x, t; s), s\big),$$

$$\omega\big(y(x, t; 0), 0\big) = \omega_0\big(y(x, t; 0)\big), \tag{3.11}$$

$$\omega\big(y(x, t; t_*(x, t)), t_*(x, t)\big) = \eta\big(y(x, t; t_*(x, t)), t_*(x, t)\big).$$

Then after integrating [3, 9] we get (3.10). This concludes the proof.

LEMMA 3.4. *Suppose that*

1. $v \in C\big(0, T; W^2_{p,\mu}(\Omega)\big)$, $1 - 3/p - \mu > 0$, $p > 1$, $\mu \geqslant 0$ *are real.*
2. $\omega_0 \in W^1_{p,\mu}(\Omega)$, $\eta \in C\big(0, T; W^1_{p,\mu}(S_1)\big)$, $\eta_t \in C\big(0, T; L_{p,\mu}(\Omega)\big)$, $F \in C\big(0, T; W^1_{p,\mu}(\Omega)\big)$.
3. $-v\cdot\bar{n}|_{S_1} \geqslant a_0 > 0$.
4. $S_1 \in C^2$.
5. *The compatibility condition* (1.8) *is satisfied.*

Then the solution determined by Lemma 3.3 *belongs to $C\big(0, T; W^1_{p,\mu}(\Omega)\big)$ and*

$$\begin{aligned}\sup_t \|\omega\|_{W^1_{p,\mu}(\Omega)} \leqslant c\exp\big[ct \sup_t \|v\|_{W^2_{p,\mu}(\Omega)}\big]\big[(t \sup_t \|v\|_{W^2_{p,\mu}(\Omega)})\\ \cdot(\|\omega_0\|_{W^1_{p,\mu}(\Omega)} + \sup_t \|\eta\|_{W^1_{p,\mu}(S_1)} + t\sup_t \|F\|_{W^1_{p,\mu}(\Omega)})\\ + t\sup_t \|F\|_{W^1_{p,\mu}(\Omega)} + \|\omega_0\|_{W^1_{p,\mu}(\Omega)} + \sup_t \|\eta\|_{W^1_{p,\mu}(S_1)}\\ + \sup_t \|\eta_t\|_{L_{p,\mu}(S_1)} + \sup_t \|F\|_{W^1_{p,\mu}(\Omega)} + \sup_t \|v\|_{W^2_{p,\mu}(\Omega)} \sup_t \|\eta\|_{W^1_{p,\mu}(S_1)}\big].\end{aligned} \tag{3.12}$$

Proof. Let $\omega \in C\big(0, T; C^1(\Omega)\big)$. First we calculate the x-derivative of ω.

For $(x, t)\in\Omega_+^T \cup \Omega_-^T$ from (3.10) we have

$$(3.13)\qquad \omega^k_{,x^r}(x, t) = A^k_{j,y^l_{x^i}} y^l_{x^i x^r} \omega^j + A^k_j \omega^j_{,y^l} y^l_{x^r} + \int_{t'(x,t)}^{t} \big(A^k_{j,y^\sigma_{x^i}} y^\sigma_{x^i x^r} F^j$$
$$+ A^k_j F^j_{,y^\sigma} y^\sigma_{x^r}\big)\, ds + \bigg[\frac{d}{d\tau} A^k_j\big(y_x(x, t; \tau)\big)\omega^j\big(y(x, t; \tau), \tau\big)$$
$$+ A^k_j \frac{d}{d\tau}\omega^j\big(y(x, t; \tau), \tau\big) - A^k_j\big(y_x(x, t; \tau)\big) F^j\big(y(x, t; \tau), \tau\big)\bigg]\bigg|_{\tau = t'(x,t)} t'_{x^r}.$$

From (3.4) and (3.11) it follows that the coefficient of t'_{x^r} vanishes. The main difficulty in proving the estimate (3.12) is to estimate the second term on the right-hand side of (3.13). To do this we have to use the curvilinear coordinates introduced in Section 2 and to calculate $\omega_{,n}|_{S_1}$ for $(\mathrm{A})_1$. The $L_{p,\mu}(\Omega)$ norm of this term is estimated by the last four terms in (3.12). The other terms can be estimated simply by using Lemma 3.1.

Now we show that $\omega\in C(0, T; W^1_{p,\mu}(\Omega))$. To do this we choose sequences $\{\overset{k}{\omega}_0\}$ in $C^1(\Omega)$, $\{\overset{k}{\eta}\}$ in $C^1(S_1^T)$, $\{\overset{k}{F}\}$ in $C(0, T; C^1(\Omega))$, $\{\overset{k}{v}\}$ in $C(0, T; C^2(\Omega))$, where $k = 1, 2, \ldots$, which converge to ω_0, η, F, v in the norms of assumptions 1, 2, respectively. Then from (3.10) and (3.13) it follows that $\overset{k}{\omega}$ which corresponds to $\overset{k}{v}$, $\overset{k}{\omega}_0$, $\overset{k}{\eta}$, $\overset{k}{F}$ belongs to $C(0, T; C^1(\Omega))$ if the compatibility condition (1.8) is satisfied and additionally the following compatibility conditions are assumed:

$$(3.14)\qquad \overset{k}{\eta}_t + \overset{k}{v}_\mu \overset{k}{\eta}_{,\tau_\mu} + \overset{k}{b}\,\overset{k}{\omega}_{0,n} - \overset{k}{\eta}\cdot\nabla\overset{k}{v} = \overset{k}{F}, \qquad t = 0,\ x\in S_1,\ k \text{ natural}.$$

From (3.12) it follows that the sequence $\{\overset{k}{\omega}\}$ is bounded in $L_\infty(0, T; W^1_{p,\mu}(\Omega))$ and converges strongly in $L_\infty(0, T; L_{p,\mu}(\Omega))$. Therefore it has a limit ω in $L_\infty(0, T; L_{p,\mu}(\Omega))$ and from the weak star compactness of $L_\infty(0, T; W^1_{p,\mu}(\Omega))$ it follows that the limit ω belongs to $L_\infty(0, T; W^1_{p,\mu}(\Omega))$.

Observe that for the limits of the sequences $\{\overset{k}{\omega}_0\}$, $\{\overset{k}{\eta}\}$, $\{\overset{k}{F}\}$, $\{\overset{k}{v}\}$, the condition (3.14) is not satisfied.

Now we have to show that ω is a weak solution of the problem (A). To do this we put $\overset{k}{\omega}$ into the integral identity (3.8) and letting $k\to\infty$ we deduce that ω satisfies (3.8). This concludes the proof.

4. Existence of solutions of the problem (A, B)

To prove the existence of solutions of the problem (A, B) we first have to get an a priori estimate. From Theorem 2.2 and Lemma 3.4 it follows that the desired estimate can be obtained if

$$(4.1)\qquad \sup_t \|\eta\|_{W^1_{p,\mu}(S_1)}$$

is sufficiently small, since the term

$$\sup_t \|v\|_{W^2_{p,\mu}(\Omega)} \sup_t \|\eta\|_{W^1_{p,\mu}(S_1)}$$

appears on the right-hand side of (3.12). We wish to underline that this term appears because an inflow is considered. The above method of obtaining the a priori estimate (3.12) is used because we want to get an a priori estimate of solutions of the problem (A, B) in weighted Sobolev spaces. For domains with edges, these spaces are more convenient than ordinary Sobolev spaces because less restriction on the maximal magnitude of angles is needed. On the other hand we want to obtain an a priori estimate for the problem (A, B) under a restriction on time only, so the above considerations imply that it has to be done in the same way as in [7, 9]. But as was shown in [9] also the method used in [7, 9] implies that the presence of an inflow excludes the possibility of getting an a priori estimate for solutions to the problem (A, B) in weighted Sobolev spaces. Therefore, repeating the considerations from [7, 9] and using Lemma 3.4 (for existence only) and Theorem 2.2 for $\mu = 0$ we get

THEOREM 4.1. *Suppose that*

1. $\pi/\vartheta_0 > 4 - 2/p$, $p > 3$,
2. $-b_n|_{S_1} \geqslant a_0 > 0$,
3. $\omega_0 \in W^1_p(\Omega)$, $b \in C(0, t; W^{2-1/p}_p(\partial\Omega))$, $\eta \in C(0, t; W^1_p(S_1))$, $F \in C(0, t; W^1_p(\Omega))$,
4. $S_\nu \in C^3$, $\nu = 0, 1, 2$,

and $t \leqslant T$, *where* T *is sufficiently small.*

Then there exists a unique solution of the problem (A, B) *such that* $v \in C(0, T; W^2_p(\Omega))$, $\omega \in C(0, T; W^1_p(\Omega))$.

Remark 4.1. From assumption 1 of Theorem 4.1 it follows that

$$\vartheta_0 < \tfrac{3}{10}\pi. \tag{4.2}$$

5. Existence of solutions of the problem (C, D)

To prove the existence of solutions of the problem (C, D) we shall use the following method of successive approximations:

$$\overset{m+1}{v}_t + \overset{m}{v} \cdot \nabla \overset{m+1}{v} = f - \nabla \overset{m}{p} \equiv \overset{m}{G},$$

$$\overset{m+1}{v}|_{t=0} = a, \quad \overset{m+1}{v}|_{S_1} = \chi, \quad \overset{m+1}{v}_n|_{S_0} = 0, \tag{5.1}$$

and

$$\Delta \overset{m}{p} = \operatorname{div} f - \overset{m}{v}_x \overset{m}{v}_x,$$
(5.2)
$$\frac{\partial \overset{m}{p}}{\partial n}\bigg|_{S_1} = g_1, \quad \frac{\partial \overset{m}{p}}{\partial n}\bigg|_{S_0} = (-\overset{m}{v}^k \overset{m}{v} \cdot \bar{n}_{x_k} + f_n)|_{S_0}, \quad \overset{m}{p}|_{S_2} = \varkappa,$$

where $m = 1, 2, \ldots$, and $\overset{0}{v} = a$.

To consider the linearized problem (C) we write for simplicity $v = \overset{m}{v}$, $u = \overset{m+1}{v}$, $G = \overset{m}{G}$. By a weak solution of the problem (5.1) we mean a solution of the following integral identity:

$$(5.3) \qquad -\int_{\Omega^T} u(\varphi_t + v\nabla\varphi) + \int_{\Omega} a\varphi(x, 0) + \int_{S_1^T} u_n \chi\varphi = \int_{\Omega^T} G\varphi$$

$$\forall \varphi \in C^1(\bar{\Omega}^T) \text{ such that } \varphi|_{t=T} = 0, \ \varphi|_{S_2} = 0.$$

Now we shall show uniqueness.

LEMMA 5.1. *Let $a \in L_2(\Omega)$, $G \in C(0, T; L_2(\Omega))$. Then the problem* (5.1) *has a unique solution in $C(0, T; L_2(\Omega))$.*

Proof. Let u_1, u_2 be two solutions of (5.1) and let $\eta = u_1 - u_2$. Then (5.1) implies that

$$\frac{1}{2}\frac{d}{dt}\|\eta\|_{L_2(\Omega)}^2 \leqslant \|G\|_{L_2(\Omega)} \|\eta\|_{L_2(\Omega)}^2$$

and $\eta|_{t=0} = 0$. This concludes the proof.

LEMMA 5.2. *Let $v \in C(0, T; C^1(\Omega))$, $a \in C(\Omega)$, $\chi \in C(S_1^T)$, $G \in C(\Omega^T)$. Then there exists a solution of the problem* (5.1) *such that $u \in C(\Omega_\pm^T)$ and*

$$(5.4) \qquad u(x, t) = u\big(y(x, t; t'(x, t)), t'(x, t)\big) + \int_{t'(x,t)}^{t} G\big(y(x, t; s), s\big)\, ds,$$

where $t'(x, t)$ is defined in Lemma 3.3.

Proof. Using the curves (3.1) the problem (5.1) can be written in the form

$$\frac{du}{ds}\big(y(x, t; s), s\big) = G\big(y(x, t; s), s\big),$$
(5.5)
$$u\big(y(x, t; 0), 0\big) = a\big(y(x, t; 0)\big),$$
$$u\big(y(x, t; t_*(x, t)), t_*(x, t)\big) = \chi\big(y(x, t; t_*(x, t)), t_*(x, t)\big).$$

The above expressions easily imply (5.4). This ends the proof.

LEMMA 5.3. *Suppose that*

1. $1 - 3/p - \mu > 0$, $p > 1$, $\mu \geqslant 0$ *are real,*
2. $a \in W_{p,\mu}^2(\Omega)$,

3. $v \in C(0, T; W^2_{p,\mu}(\Omega))$,
4. $v \cdot \bar{n}|_{S_1} \geq a_0 > 0$,
5. $S_1 \in C^3$,
6. $\chi \in C(0, T; W^2_{p,\mu}(S_1))$, $\chi_t \in C(0, T; W^1_{p,\mu}(S_1))$, $\chi_{tt} \in C(0, T; L_{p,\mu}(S_1))$,
7. $\overset{m}{\bar{p}}, \overset{m-1}{\bar{p}} \in C(0, T; W^3_{p,\mu}(\Omega))$, $\overset{m}{\bar{p}}_t \in C(0, T; W^2_{p,\mu}(\Omega))$,
8. $G \in C(0, T; W^2_{p,\mu}(\Omega))$, $G_t \in C(0, T; W^1_{p,\mu}(\Omega))$,

and g_1, g_2 are positive increasing functions with respect to their arguments (see (5.11)).

Then the solution determined by Lemma 5.2 *is such that*

$$u \in C(0, T; W^2_{p,\mu}(\Omega)), \quad u_t \in C(0, T; W^1_{p,\mu}(\Omega)), \tag{5.6}$$

and the following estimate is valid:

$$\begin{aligned} \|u\|_{W^2_{p,\mu}(\Omega)} + \|u_t\|_{W^1_{p,\mu}(\Omega)} &\leq c \exp[ct \sup_t \|v\|_{W^2_{p,\mu}(\Omega)}][g_1 + g_2 \cdot \\ &\cdot (\sup_t \|\overset{m}{\bar{p}}\|_{W^3_{p,\mu}(\Omega)} + \sup_t \|\overset{m}{\bar{p}}_t\|_{W^2_{p,\mu}(\Omega)} + \sup_t \|\overset{m}{\bar{p}}\|_{W^3_{p,\mu}(\Omega)} \sup_t \|\overset{m-1}{\bar{p}}\|_{W^3_{p,\mu}(\Omega)}) \\ &+ t(\sup_t \|\overset{m}{G}\|_{W^2_{p,\mu}(\Omega)} + \sup_t \|\overset{m}{G}_t\|_{W^1_{p,\mu}(\Omega)})(1 + t \sup_t \|v\|_{W^2_{p,\mu}(\Omega)})], \end{aligned} \tag{5.7}$$

where $t \in [0, T]$.

Proof. First we shall obtain (5.7) for sufficiently smooth data functions. Let

$$\begin{gathered} v \in C(0, T; C^2(\Omega)), \quad a \in C^2(\Omega), \\ \chi \in C(0, T; C^2(S_1)), \quad \chi_t \in C(0, T; C^1(S_1)), \quad \chi_{tt} \in C(0, T; C(S_1)), \\ \overset{m}{\bar{p}}, \overset{m-1}{\bar{p}} \in C(0, T; C^3(\Omega)), \quad \overset{m}{\bar{p}}_t \in C(0, T; C^2(\Omega)), \\ G \in C(0, T; C^2(\Omega)), \quad G_t \in C(0, T; C^1(\Omega)). \end{gathered} \tag{5.8}$$

Then from (5.4) we have

$$\begin{aligned} u_{x^r} = u_{y^i}(y(x, t; t'(x, t)), t'(x, t)) y^i_{x^r}(x, t; t'(x, t)) + \Big[\frac{\partial v}{\partial \tau}(y(x, t; \tau), \tau) \\ + \frac{dy^i}{d\tau}(x, t; \tau) v_{y^i}(y(x, t; \tau), \tau) - G(y(x, t; \tau), \tau)\Big]\Big|_{\tau = t'(x,t)} t'_{x^r}(x, t) \\ + \int_{t'(x,t)}^{t} G_{y^i}(y(x, t; s), s) y^i_{x^r}(x, t; s) ds, \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} u_{x^r x^q} = u_{y^i y^j}(y(x, t; t'(x, t)), t'(x, t)) y^i_{x^r}(x, t; t'(x, t)) \\ \cdot y^j_{x^q}(x, t; t'(x, t)) + u_{y^i}(y(x, t; t'(x, t)), t'(x, t)) y^i_{x^r x^q}(x, t; t'(x, t)) \end{aligned} \tag{5.10}$$

$$+\left[t'_{x^r}\frac{\partial}{\partial x^q}+t'_{x^q}\frac{\partial}{\partial x^r}+t'_{x^r}t'_{x^q}\frac{\partial}{\partial \tau}+t'_{x^r x^q}\right]$$

$$\cdot\left[\frac{dv}{d\tau}(y(x,t;\tau),\tau)-G(y(x,t;\tau),\tau)\right]\bigg|_{\tau=t'(x,t)}$$

$$+\int_{t'(x,t)}^{t}\left[G_{y^i y^j}(y(x,t;s),s)\,y^i_{x^r}y^j_{x^q}\right.$$

$$\left.+G_{y^i}(y(x,t;s),s)\,y^i_{x^r x^q}\right]ds.$$

From (5.5) it follows that the coefficients of the derivatives of $t'(x, t)$ vanish.

To obtain the estimate (5.7), the most difficult point is to estimate the first terms on the right-hand sides of (5.4), (5.9), (5.10). To estimate these terms on curves of the family (b) we have to use the curvilinear coordinates introduced in Section 2. Then they lead to the following boundary integral:

$$I = \int_{S_1} v_n\left[|u_{,\tau}|^p+|u_{,\tau\tau}|^p+|u_{,n\tau}|^p+|u_{,nn}|^p+|u_{,n}|^p\right]\varrho^{p\mu}(x')\,dx'.$$

Calculating the normal derivative $u_{,n} = \overset{m+1}{v}_{,n}$ from (5.1), repeating the consideration in [8, 9], and using Theorem 2.1 and the properties of traces of functions in $W^l_{p,\mu}(\Omega)$ [10], we can estimate the boundary term I by

(5.11) $$I \leqslant g_1\Big(\sup_t\|\chi\|_{W^2_{p,\mu}(S_1)},\ \sup_t\|\chi_t\|_{W^1_{p,\mu}(S_1)},\ \sup_t\|\chi_{tt}\|_{L_{p,\mu}(S_1)},$$

$$\sup_t\|f\|_{W^1_{p,\mu}(S_1)},\ \sup_t\|f_t\|_{L_{p,\mu}(S_1)}\Big)+g_2\Big(\sup_t\|\chi\|_{W^2_{p,\mu}(S_1)},$$

$$\sup_t\|\chi_t\|_{W^1_{p,\mu}(S_1)},\ \sup_t\|f\|_{W^1_{p,\mu}(S_1)}\Big)\Big(\sup_t\|\overset{m}{p}\|_{W^3_{p,\mu}(\Omega)}+\sup_t\|\overset{m}{p}_t\|_{W^2_{p,\mu}(\Omega)}$$

$$+\sup_t\|\overset{m}{p}\|_{W^3_{p,\mu}(\Omega)}\ \sup_t\|\overset{m-1}{p}\|_{W^3_{p,\mu}(\Omega)}\Big).$$

Finally, using (5.11), from (5.4), (5.9), (5.10), Lemma 2.1 and Lemma 3.1 we obtain (5.7).

From (5.6) it follows that we look for classical solutions of (5.1) (all expressions in (5.1) are Hölder continuous with exponent $\alpha = 1-3/p$). To show that u satisfies (5.1) we choose sequences $\{v^k\}$, $\{a^k\}$, $\{\overset{m}{p}{}^k\}$, $\{\overset{m-1}{p}{}^k\}$, $\{\chi^k\}$, $\{G^k\}$ in the spaces determined by (5.8) which converge in the norms from the assumptions of the lemma to the functions v, a, χ, $\overset{m}{p}$, $\overset{m-1}{p}$, G. From (5.8) it follows that the functions v^k, a^k, χ^k, $\overset{m}{p}{}^k$, $\overset{m-1}{p}{}^k$, G^k must satisfy the compatibility condition (1.11) and

(5.12) $$\chi_t+\chi_\mu\chi_{,\tau_\mu}+\chi_n a_{,n} = G \quad \text{for } x\in S_1,\ t=0,$$

and

(5.13) The first derivatives of $(5.1)_1$ with respect to t, τ_μ, n, $\mu = 1, 2$, calculated for $x\in S_1$ and $t = 0$ are equal to zero.

To obtain (5.13) the unknown derivatives of u must be calculated from $(5.1)_1$.

Knowing that u^k, v^k, a^k, χ^k, $\overset{m}{\bar{p}}{}^k$, $\overset{m-1}{\bar{p}}{}^k$, G^k satisfy (5.1) and the inequality (5.7) and letting $k \to \infty$ we deduce that (5.1) as well as (5.7) are satisfied for the limit functions. Observe that the condition (5.13) is not satisfied in the limit. This ends the proof.

Now we want to prove the existence of solutions of the problem (C, D). From Theorem 2.3 and Lemma 5.3 it follows that it is not possible to obtain an a priori estimate without some restriction on data functions χ and f (see Section 5.2 in [9], where the form of g_2 is described). Another way of getting the a priori estimate (and without the restriction on data functions) is presented in [8, 9], where it is also shown that only nonweighted spaces can be used, so $\mu = 0$. These facts are connected with estimating the boundary term I, which appears in the case of an inflow only.

Without repeating the considerations from [8, 9] we can prove

THEOREM 5.1. *Suppose that*

1. $\pi/(2\vartheta_0) > 3-2/p$, $p > 3$,
2. $a \in W_p^2(\Omega)$,
3. $\chi \in C(0, T; W_p^2(\Omega))$, $\chi_t \in C(0, T; W_p^1(\Omega))$, $\chi_{tt} \in C(0, T; L_p(\Omega))$,
4. $-v \cdot \bar{n}|_{S_1} \geqslant a_0 > 0$,
5. $f \in C(0, T; W_{p,\mu}^2(\Omega))$, $f_t \in C(0, T; W_{p,\mu}^1(\Omega))$,
6. $S_\nu \in C^3$, $\nu = 0, 1, 2$,

and T is sufficiently small. Then for all $t \leqslant T$ there exists a unique solution of the problem (C, D) *such that*

$$v \in (0, T; W_p^2(\Omega)), \quad v_t \in C(0, T; W_p^1(\Omega)), \quad p \in C(0, T; W_p^3(\Omega)),$$
$$p_t \in (0, T; W_p^2(\Omega)).$$

Remark 5.1. Assumption 1 of Theorem 5.1 implies that

$$\vartheta_0 < \tfrac{3}{14}\pi. \tag{5.14}$$

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