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PROJECTIVE REPRESENTATIONS OF S_n: RING STRUCTURE AND INDUCTIVE FORMULAE

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Let $\tilde{S}_n$, $n \geqslant 0$, be either of the two sequences of essential double covers for the symmetric groups. There is an epimorphism $\tilde{S}_n \xrightarrow{\pi} S_n$ with kernel $\{1, z\}$. For $n \geqslant 4$, the central involution z is in the commutator subgroup of $\tilde{S}_n$; and the complex projective representations of S_n (up to equivalence) are in 1-1 correspondence with those linear representations of $\tilde{S}_n$ in which z acts as -1. The same is true for the alternating group A_n, letting $\tilde{A}_n = \pi^{-1} A_n$, except that A_6 and A_7 have additional families of projective representations. These facts, and many more, including a complete determination of the characters in terms of certain symmetric functions, the Q-functions, were discovered by Schur [Schur 1911].

There is a graded ring $C = \bigoplus_{n=0}^{\infty} C_n$ in which, if $n \geqslant 4$, C_n is the free abelian group generated by the irreducible projective representations above. Both S_n and A_n are involved. The ring structure was introduced in [Hoffman–Humphreys 1985], with detailed proofs in [Hoffman–Humphreys 1986]. This initial treatment had two drawbacks, whose sum was greater than its parts. The first one was removed [Hoffman–Humphreys 1987] by the introduction of the base ring

$$L := \mathbf{Z}[\lambda]/(\lambda^3 - 2\lambda),$$

to replace $\mathbf{Z}$. Let $\varrho := \lambda^2 - 1$. There is a $\mathbf{Z}/2$-grading on L given by requiring λ to be in $L^{(1)}$. Each C_n has a $\mathbf{Z}/2$-grading, $C_n = C_n^{(0)} \oplus C_n^{(1)}$, where, for $n \geqslant 4$, and for $i = 0$ (respectively $i = 1$), $C_n^{(i)}$ is generated by the irreducible projective representations of A_n (respectively, S_n). In fact, C_n is a free $\mathbf{Z}/2$-graded L-module, where λ acts by inducing and restricting between A_n and S_n. An L-basis is found by choosing exactly one from each pair $\{x, \varrho \cdot x\}$, where

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x ranges over irreducibles for which $x \neq \varrho \cdot x$. The introduction of L both allowed a simpler description of C, and provided a Hopf algebra structure for C. The description is:

THEOREM A. *There are irreducible $c_n \in C_n^{(n+1)}$ such that C is the L-algebra generated by $\{c_1, c_2, \ldots\}$ subject only to relations*

(COMM) $$c_i c_j = \varrho^{i+j+1} c_j c_i,$$

(SQ) $$c_n^2 = (-1)^{n+1} \lambda \left[c_{2n} + \lambda \sum_{i=1}^{n-1} (-1)^i c_{2n-i} c_i\right].$$

Consequently, an L-basis for C (not the basis of irreducibles) is $\{c_{i_1} c_{i_2} \ldots : i_1 > i_2 > \ldots\}$.

Note that the general form of (COMM) is

$$xy = \varrho^{mn+ij} yx \quad \text{for} \quad x \in C_m^{(i)},\ y \in C_n^{(j)}.$$

The second drawback was that the proof of this theorem, which involved some Hopf algebra theory (and worse at the prime 2, before the use of L), did not identify the irreducibles. For this it was still necessary to follow the intricacies of Schur's original treatment.

In the first section below it is shown how this second drawback may be overcome by use of an inductive formula for the irreducibles. This simplifies earlier work considerably, and avoids Hopf algebra theory.

Recently there have appeared two interesting updated expositions of Schur's theory, both of which introduce a ring structure. (See also a forthcoming book by John Humphreys and the author). In [Józefiak 1989], a major feature is the use of the theory of $\mathbf{Z}/2$-graded algebras, known also as superalgebras. In terms of the above ring C, the ring structure in Józefiak's exposition is $C^{(0)}/(1-\varrho)\cdot C^{(0)}$, the multiplication being defined using one of the four formulae for multiplying in C, namely the $\mathbf{Z}/2$-graded tensor product. This type of product for modules apparently first appeared in [Atiyah–Bott–Shapiro 1964].

The other modernized treatment is in the introductory sections of [Stembridge 1987]. His operation gives an associative product on the sum over n of the Grothendieck groups of those negative representations V of $\tilde{S}_n$ for which $\varrho \cdot V \cong V$. This group is $\lambda \cdot C^{(0)} \subset C^{(1)}$, and the multiplication, say $\circ$, in terms of that in C, may be given by

$$(\lambda \cdot x) \circ (\lambda \cdot y) = \lambda \cdot (xy).$$

(See also [H-H 1986, 3.2].) Using this multiplication, the group isomorphism $C^{(0)}/(1-\varrho)C^{(0)} \xrightarrow{\lambda \cdot} \lambda \cdot C^{(0)}$ becomes a ring isomorphism. This ring is also isomorphic to the algebra of Q-functions, at least after tensoring with $\mathbf{Q}[\sqrt{2}]$. It contains slightly less information than C itself does, and certainly has more complicated ring structure than C, although easily determinable from that of C. For example, due to arithmetical problems, it lacks a basis consisting of monomials in a set of ring generators. However, it is useful for applications of this representation theory to Q-function theory. Most applications in the opposite direction are more easily done by using the operators below, rather than Q-functions.

1. The Bernstein-type operators

Define an L-bilinear symmetric positive definite inner product,

$$\langle\ ,\ \rangle\colon C \times C \to L,$$

by requiring that $\langle x, x\rangle = 1$ if x is an irreducible with $\varrho \cdot x \neq x$, and that $\langle x, y\rangle = 0$ if x and y irreducible for which x, y, $\varrho \cdot x$, and $\varrho \cdot y$ are all distinct. Positivity is with respect to the canonical basis $\{1, \varrho, \lambda\}$ for L.

Now, for each $u \in C$, define $u^*\colon C \to C$ by $\langle u^*(v), w\rangle = \langle v, uw\rangle$.

Finally, for each $n \geqslant 1$, define $A_n\colon C \to C$ by

$$A_n(x) = c_n x + \lambda \sum_{i>0} (-1)^i c_{n+i} c_i^*(x).$$

See [Zelevinsky 1981; p. 69] for the motivating analogue re linear representations of S_n, due to Bernstein.

THEOREM B [Hoffman 1989]. *Define*

$$a_{n_1, n_2, \ldots} := A_{n_1} A_{n_2} \ldots (1).$$

Then:

(i) *the* **Z**-*basis for* C *of irreducibles is*

$$\{\theta \cdot a_{n_1, n_2, \ldots}\colon\ \theta = 1,\ \varrho \text{ or } \lambda;\ n_1 > n_2 > \ldots > 0\};$$

(ii) *the sets* $\{a_\alpha\colon \alpha \in \mathscr{D}\}$ *and* $\{c_\alpha\colon \alpha \in \mathscr{D}\}$ *are related by unitriangular matrices over* L.

Above, $\mathscr{D}$ is the set $\{(n_1, n_2, \ldots)\colon n_1 > n_2 > \ldots\}$ of strict integer partitions, and

$$c_{n_1, n_2, \ldots} := c_{n_1} c_{n_2} \ldots .$$

Part (ii) is the L-version of a fact due to Schur for Q-functions. The proof proceeds by showing $\langle a_\alpha, a_\beta\rangle = \delta_{\alpha\beta}$, the main calculation being

$$c_k^*(a_{n_1, n_2 \ldots}) = \begin{cases} a_{n_2, n_3, \ldots} & \text{if } k = n_1; \\ 0 & \text{if } k > n_1. \end{cases}$$

The only other routes from the definition of $\tilde{S}_n$ to a description of its irreducible negative representations (in the Grothendieck group; construction of the modules is another matter) seem to be several orders of magnitude lengthier.

Once one has defined the basic irreducibles c_n (say, as Clifford modules) and verified (COMM) and (SQ), Theorem A is equivalent to the fact that $\{c_\alpha : \alpha \in \mathscr{D}\}$ is an L-basis for C. But this is immediate from part (ii) of Theorem B, since $\{a_\alpha : \alpha \in \mathscr{D}\}$ is an L-basis. This argument is the bypass of Hopf algebra theory mentioned above.

2. The formula for characters

For brevity we shall avoid discussing $\tilde{A}_n$. Letting χ_h denote the character of a (possibly virtual) representation h, define, for $\alpha \in \mathscr{D}_n$

$$\langle \alpha \rangle = \begin{cases} \chi_{a_\alpha} & \text{if } \alpha \text{ is odd;} \\ \chi_{\lambda \cdot a_\alpha} & \text{if } \alpha \text{ is even.} \end{cases}$$

Oddness of partitions is in the sense of cycle types, that is, the parity of the number of even parts. For each cycle type ω, the number of conjugacy classes in $\tilde{S}_n$ which project to ω is either one or two. If one, then $\langle \alpha \rangle (g) = 0$ for g in that class, since $g \sim zg$. When ω consists entirely of odd parts, the number is two. The number $\langle \alpha \rangle (g)$ therefore has different signs for g in the two classes of $\tilde{S}_n$ which project to a cycle type in $\mathscr{P}_n^{\text{odd}}$. The aim below is to give a new proof of an inductive method for calculating this number. The method is analogous to the Murnaghan–Nakayama rule for linear representations of S_n. It is the only calculation which is needed, as is clear from a theorem of Schur which says that $\langle \alpha \rangle (g)$ is zero for all α whenever the cycle type of $\pi(g)$ has at least one even part, with the one exception of $\langle \alpha \rangle (g)$ when α is odd and g projects to the cycle type α itself. In that case the value is given rather simply [J. 5.8]. A slight elaboration of the theory below will yield this result as well as all character information on $\tilde{A}_r$. This includes the character values of the basic (Clifford module) irreducibles c_r. Only two direct trace calculations need be made: the value of $\langle r \rangle$ on the r-cycle, and the corresponding value of basic irreducible character on $\tilde{A}_r$ when r is odd. We shall however confine ourselves to the more interesting case of $\langle \alpha \rangle (g)$ where g projects to a cycle type in $\mathscr{P}^{\text{odd}}$.

For this, we need the following elements of C:

$$p_{2k+1} := (1 + k\lambda^2) c_{2k+1} + \lambda \sum_{i=1}^{k} (-1)^i (2k - 2i + 1) c_{2k-i+1} c_i;$$

$$p_{2k} := (1 - \varrho) \cdot c_{2k}.$$

These are primitive elements, satisfying $\langle p_{2k+1}, c_{2k+1} \rangle = 1$, introduced in [H-H 1987]. Choose $s_{2k+1} \in \tilde{S}_{2k+1}$ so that it projects to a $(2k+1)$-cycle and so that $\chi_{\lambda c_{2k+1}}(s_{2k+1}) = 1$. Then, for all $h \in C_{2k+1}^{(1)}$,

$$\chi_h(s_{2k+1}) = \langle h, \lambda p_{2k+1} \rangle_{\varrho = 0}.$$

This follows by noting that both sides are group homomorphisms as functions of h, and by checking that they agree on those $h = \theta \cdot c_\alpha$ which are in $C^{(1)}_{2k+1}$, where $\theta = 1$, ϱ or λ and $\alpha \in \mathscr{D}_{2k+1}$. When $\alpha = (2k+1)$, both sides give 1, and for all other α, we get zero. Note that $\langle\ ,\ \rangle_{\varrho=0}$ is the usual $\mathbf{Z}$-valued inner product for representations of S_n. This is because it is $\mathbf{Z}$-bilinear and has the irreducibles as an orthonormal basis.

We shall use the fact from [H-H 1987] that, defining $p_{a_1,a_2,\ldots} := p_{a_1} p_{a_2} \cdots$, the set $\{p_\alpha\}$, as α ranges over *all* partitions of n, generates $C_n \otimes_{\mathbf{Z}} \mathbf{Q}$ as an $L \otimes_{\mathbf{Z}} \mathbf{Q}$-module. It is not a basis, nor are any of its subsets. If $\alpha \notin \mathscr{D} \cup \mathscr{P}^{\text{odd}}$, we have $p_\alpha = 0$. It now follows that the coproduct $\Delta\colon C \to C \otimes C$ can be written in the form

$$\Delta y = \sum_\alpha y_\alpha \otimes p_\alpha$$

for certain (not always unique) elements $y_\alpha \in C \otimes_{\mathbf{Z}} \mathbf{Q}$ which depend on y.

LEMMA C. *Let r be an odd positive integer. Let $y \in C^{(1)}_{n+r}$, and let $g \in \tilde{S}_n$ so that $(g, s_r) \in \tilde{S}_n \hat{\curlyvee} \tilde{S}_r \subset \tilde{S}_{n+r}$. Assume that g is even. Then*

$$\chi_y(g, s_r) = \chi_{p_r^*(y)}(g).$$

This is the basic reduction formula. It shows how to "remove an r-cycle", at the expense of having to calculate $p_r^*(y)$, which will usually be a *virtual* representation, not an actual one. For the inductive calculation of the irreducible characters we shall therefore need to learn how to write $p_r^*(a_\alpha)$ as a linear combination of irreducibles.

Proof of Lemma C. We have

$$\chi_y(g, s_r) = \chi_{\sum\limits_{\alpha \vdash r} y_\alpha \otimes p_\alpha}(g, s_r) = \sum_{\alpha \vdash r} \chi_{y_\alpha \otimes p_\alpha}(g, s_r).$$

We have identified the set of virtual negative representations of $\tilde{S}_n \hat{\curlyvee} \tilde{S}_r$ with the sumodule spanned by the image of $(C^{(0)}_n \times C^{(1)}_r) \cup (C^{(1)}_n \times C^{(0)}_r)$ in $C_n \otimes_L C_r$; and we identified the restriction map to $\tilde{S}_n \hat{\curlyvee} \tilde{S}_r$ with the component $C_{n+r} \to C_n \otimes_L C_r$ of the coproduct on C. Using that g is even, and using the definition of the isomorphism between $C_n \otimes_L C_r$ and the group of graded and ungraded negative representations of $\tilde{S}_n \hat{\curlyvee} \tilde{S}_r$, one gets the formula

$$\chi_{y_\alpha \otimes p_\alpha}(g, s_r) = \begin{cases} \chi_{y_\alpha}(g)\, \chi_{\lambda p_\alpha}(s_r) & \text{if } \alpha \text{ is even;} \\ \chi_{\lambda y_\alpha}(g)\, \chi_{p_\alpha}(s_r) & \text{if } \alpha \text{ is odd.} \end{cases}$$

Thus

$$\chi_y(g, s_r) = \chi_{y_r}(g)\, \chi_{\lambda p_r}(s_r), \tag{1}$$

since, for $\alpha \neq (r)$, the second factor in each of the earlier formula is zero. This is because p_α is induced from a "Young subgroup" $\tilde{S}_{a_1} \hat{\curlyvee} \tilde{S}_{a_2} \hat{\curlyvee} \ldots$ of $\tilde{S}_r$, where s_r is an "r-cycle". Alternatively, $\langle p_\alpha, p_r \rangle = 0$, as we now see.

Now if u and v both have positive $\mathbf{Z}$-grading which add to r, we have

$$\langle p_r, uv\rangle = \langle\langle \Delta p_r, u\otimes v\rangle\rangle = \langle\langle 0, u\otimes v\rangle\rangle = 0.$$

In particular, $\langle p_r, p_\alpha\rangle = 0$ if $\alpha \neq (r)$, and

$$\langle p_r, p_r\rangle = \left\langle\left(1+\frac{r-1}{2}\lambda^2\right)c_r + \text{decomposables},\ p_r\right\rangle$$
$$= \left(1+\frac{r-1}{2}\lambda^2\right)\langle c_r, p_r\rangle = 1+\frac{r-1}{2}\lambda^2.$$

Thus $\chi_{\lambda p_r}(s_r) = \langle \lambda p_r, \lambda p_r\rangle_{\varrho=0} = \left[\lambda^2\left(1+\frac{r-1}{2}\lambda^2\right)\right]_{\varrho=0} = r$. Going back to (1), we see that

$$\chi_y(g, s_r) = r\chi_{y_r}(g). \tag{2}$$

But, for all z,

$$\langle p_r^*(y), z\rangle = \langle y, zp_r\rangle = \langle\langle \Delta y, z\otimes p_r\rangle\rangle$$
$$= \langle\langle \sum_\alpha y_\alpha \otimes p_\alpha, z\otimes p_r\rangle\rangle = \sum_\alpha \langle y_\alpha, z\rangle\langle p_\alpha, p_r\rangle$$
$$= \langle y_r, z\rangle\langle p_r, p_r\rangle = \left(1+\frac{r-1}{2}\lambda^2\right)\langle y_r, z\rangle.$$

Therefore,

$$p_r^*(y) = \left(1+\frac{r-1}{2}\lambda^2\right)y_r = \frac{r+1}{2}y_r + \frac{r-1}{2}\varrho\cdot y_r.$$

Since g is even, $\chi_{\varrho V}(g) = \chi_V(g)$ for any V. Thus

$$\chi_{p_r^*(y)}(g) = \frac{r+1}{2}\chi_{y_r}(g) + \frac{r-1}{2}\chi_{\varrho\cdot y_r}(g) = r\chi_{y_r}(g).$$

Comparing this with (2) yields the result.

The remaining problem of determining $p_r^*(a_\alpha)$ in terms of irreducibles can be readily disposed of by manipulations with the operators. The combined answer below in Theorem F is a formula which was discovered (perhaps empirically) many years ago [Morris 1965], and later occurred in [Humphreys 1986] and [Morris 1988]. It is a rather ungainly formula, making its discovery all the more admirable. Our method of proof makes clear the stages at which the complexities arise.

LEMMA D. *For all $n\in\mathbf{Z}$ and odd $r \geqslant 1$, we have*

$$p_r^* A_n - \varrho^n A_n p_r^* = \begin{cases} \lambda A_{n-r} & \text{if } n \neq 0 \text{ or } r; \\ A_0 & \text{if } n = r; \\ \lambda^2 A_{-r} & \text{if } n = 0, \end{cases}$$

where, to define A_n *for* $n \leqslant 0$, *we take*

$$A_0(x) := x + \lambda^2 \sum_{i>0} (-1)^i c_i^*(x) c_i,$$

and

$$A_{-k}(x) := (-1)^k c_k^*(x) + \lambda \sum_{i>k} (-1)^i c_i^*(x) c_{i-k} \quad \textit{for } k > 0.$$

Proof. The primitivity of p_r is equivalent to $p_r^*(uv) = (p_r^* u) v + \varrho^{\deg u} u (p_r^* v)$, from which the identities follow easily.

COROLLARY. *For positive odd* r *and for* $k_1 > k_2 > \ldots > k_s > 0$, *we have*

$$p_r^*(a_{k_1,\ldots,k_s}) = \sum_{\{i:k_i \neq r\}} \lambda a_{k_1,\ldots,k_i - r,\ldots,k_s} + \sum_{\{i:k_i = r\}} \varrho^{k_1+\ldots+k_{i-1}} a_{k_1,\ldots,k_i - r,\ldots,k_s}.$$

Here we have extended, to any sequence $n_1, n_2, \ldots$, *the definition:* $a_{n_1,n_2,\ldots} := A_{n_1} A_{n_2} \ldots (1)$. *The second summation is over a singleton or empty set, and the subscript* $k_i - r$ *is zero, but should not be omitted.*

The proof is immediate by induction on s.

LEMMA E.
(i) $A_0 A_0 = \mathrm{Id}$.
(ii) $A_n A_0 + \varrho A_0 A_n = 0$, *if* $n \neq 0$.
(iii) $A_{-n} A_n + A_n A_{-n} = (-1)^n \mathrm{Id}$, *if* $n \neq 0$.
(iv) $A_k A_l + \varrho^{k+l} A_l A_k = 0$ *if all of* k, l *and* $k+l$ *are nonzero.*
(v) $A_k A_k = 0$ *if* $k \neq 0$.

Sketch proof. All of these can be proved by straightforward manipulation with the definitions of the operators, using (SQ) several times. It is less laborious to only do this for (iv) with $k > 0$, separately for $l > 0$ and $l < 0$. Then all the other identities are quickly deduced using the following inductive principle: If an L-linear operator V: $C \to C$ satisfies:

(a) $V(1) = 0$;

and

(b) $V(x) = 0$ implies $V A_k(x) = 0$ for all $k > 0$;

then V is zero. This is clear from Theorem B(i). Note also that (v) is an immediate consequence of (iv).

COROLLARY. *Since* $A_0(1) = 1$ *and* $A_n(1) = 0$ *for* $n < 0$, *the element in the summations of the previous corollary is determined as either zero or* $\pm$ *(an irreducible), according to the following five cases. We take* $k_{s+1} = 0$ *in the second case* (I). *"Omit* k_v*" is denoted* $\hat{k}_v$. *The element* $a_{k_1,\ldots,k_i - r,\ldots,k_s}$ *is:*

	0, *if* $k_i > r$, *and* $\exists u$ *with* $k_i - r = k_u$;
(I)	$(-1)^{j-i} \varrho^N a_{k_1, \ldots, \hat{k}_i, \ldots, k_j, k_i - r, k_{j+1}, \ldots, k_s}$, *if* $k_i > r$, *and* $\exists j$ *with* $k_j > k_i - r > k_{j+1}$;
(II)	$(-1)^{s-i} a_{k_1, \ldots, \hat{k}_i, \ldots, k_s}$, *if* $k_i = r$;
(III)	$(-1)^{j-i} \varrho^M a_{k_1, \ldots, \hat{k}_i, \ldots, \hat{k}_j, \ldots, k_s}$, *if* $k_i < r$, *and* $\exists j > i$ *with* $k_j + k_i = r$;
	0, *if* $k_i < r$, *and* $\lvert k_i - r \rvert \neq k_j \forall j > i$.

In (I), $N = (j-i)(k_i - r) + k_{i+1} + \ldots + k_j$;
in (III), $M = (j-i-1)(k_i - r) + k_{i+1} + \ldots + k_{j-1}$.

Proof. In each case, move $k_i - r$ past other subscripts to the right, using (iv) or (ii). If $k_i - r > 0$, use (iv). Either move it until reaching an equal entry, giving 0 by (v); or until reaching a smaller entry, giving formula (I). If $k_i - r = 0$, use (ii). Move it to the right hand end and delete it, giving formula (II). If $k_i - r < 0$, use (iv). Either move it till reaching an entry $r - k_i$, then apply (iii), giving formula (III) after applying the next argument to the other term; or move it to the right hand end, giving 0.

Combining the two corollaries with the basic method, Lemma C, we obtain the theorem referred to earlier.

THEOREM F. *Let g be an even element of $\tilde{S}_n$ and let r be an odd integer. For each $\alpha = (k_1, \ldots, k_s) \in \mathcal{D}_{n+r}$, define*

$$\operatorname{par}(\alpha) := \begin{cases} 0 & \text{if } \alpha \text{ is even} \\ 1 & \text{if } \alpha \text{ is odd} \end{cases} := \text{the reduction } \operatorname{mod} 2$$

of the number of i's for which k_i is even. With summation over the sets (I), (II), (III) *of the previous corollary, we have*

$$\begin{aligned} \langle\alpha\rangle(g, s_r) = & \sum_{i \in (\mathrm{I})} (-1)^{j-i} 2^{1-\operatorname{par}\alpha} \langle k_1, \ldots, \hat{k}_i, \ldots, k_j, k_i - r, k_{j+1}, \ldots, k_s\rangle(g) \\ & + \sum_{i \in (\mathrm{II})} (-1)^{s-i} \langle k_1, \ldots, \hat{k}_i, \ldots, k_s\rangle(g) \\ & + \sum_{i \in (\mathrm{III})} (-1)^{j-i+k_i} 2^{1-\operatorname{par}\alpha} \langle k_1, \ldots, \hat{k}_i, \ldots, \hat{k}_j, \ldots, k_s\rangle(g). \end{aligned}$$

The partitions occurring in this formula are sometimes described using shifted Young diagrams and various paraphernalia such as hooks, skew hooks, bars, leg lengths, coat hangers, etc. ...

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