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Hyperspace retractions for curves

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CONTENTS

Abstract

We study retractions from the hyperspace of all nonempty closed subsets of a given continuum onto the continuum (which is naturally embedded in the hyperspace). Some necessary and some sufficient conditions for the existence of such a retraction are found if the continuum is a curve. It is shown that the existence of such a retraction for a curve implies that the curve is a uniformly arcwise connected dendroid, and that a universal smooth dendroid admits such a retraction. The existence of this retraction for a given dendroid implies that the dendroid admits a mean. An example of a (nonplanable) smooth dendroid that admits no mean is constructed. Some related results are obtained and open problems are stated. The results answer several questions asked in the literature.

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1. Introduction

Let X be a metric continuum. We denote by 2^X (respectively, $C(X)$) the hyperspace of all nonempty closed subsets (respectively, subcontinua) of X, equipped with the Hausdorff metric. Sam B. Nadler, Jr. asks in [60, (3.1), p. 193] the following question.

1.1. Problem (Nadler). When is X a continuous image of 2^X or of $C(X)$?

In the same paper [60] he gives some necessary and some sufficient conditions for existence of a mapping from 2^X or $C(X)$ onto X. The existence of such a mapping implies that X is weakly chainable (in the sense of [51] or—equivalently—X is a continuous image of a pseudo-arc) and that X is the union of two proper subcontinua each of which is arcwise connected [60, Theorem 3.2, p. 193]. In case when X is chainable (circle-like) such a mapping exists if and only if X is an arc (a simple closed curve, respectively) [60, 3.3, p. 193]. A space is said to be *g-contractible* [6] provided that there exists a surjective mapping from the space onto itself which is homotopic to a constant mapping. Nadler shows that if X is g-contractible, then there is a mapping from 2^X onto X [60, Theorem 3.4, p. 193].

Since the hyperspace $F_1(X)$ of singletons of X is a subspace of 2^X and is homeomorphic to X, the continuum X can be considered as naturally embedded in $C(X)$. Thus, if we identify X with $F_1(X)$, we have

$$X \subset C(X) \subset 2^X. \tag{1.2}$$

So, the following is a particular case of Problem 1.1 (see [59, p. 413] and also [62, (6.2), p. 270]).

1.3. Problem (Nadler). What are necessary and sufficient conditions in order that a continuum X be a retract of 2^X or $C(X)$?

Let $\mathcal{H}$ be 2^X or $C(X)$. By a *selection* for $\mathcal{H}$ we mean a mapping $s : \mathcal{H} \to X$ such that $s(A) \in A$ for each $A \in \mathcal{H}$. Then a selection for $\mathcal{H}$ is a retraction of $\mathcal{H}$ onto X. Therefore Problem 1.3 is related to the existence of a selection for the corresponding hyperspace (see [63], [54], [55] and an expository paper [16], where a number of references are given).

There are a number of results that are related to mappings from, onto or between some hyperspaces (which are subspaces of 2^X), in particular to hyperspace retractions. For example, $C(X)$ is always a continuous image of 2^X [60, Theorem 3.6, p. 194], not necessarily being its retract, [32]. Many results concern local connectedness of X or of a hyperspace at some of its points (see [26], [29], [30], [35], [61] for example). In the late thirties M. Wojdysławski proved that locally connected continua have contractible hyperspaces [68] and that $C(X)$ is an absolute retract if and only if X is locally connected

[69] (compare also [41, Theorem 4.4, p. 28]). A characterization of dendrites in terms of continuity of the function $I : 2^X \to C(X)$ that assigns to a closed subset A of a hereditarily unicoherent continuum X the continuum $I(A)$ irreducible with respect to containing A is given as Theorem 1 of [31, p. 3]. For characterizations of smooth dendroids in terms of continuity of functions related to hyperspaces see [31, Theorem 8, p. 7]; compare also [52, Theorem 1, p. 112].

Conditions mentioned in Problem 1.3 are known in a very particular case when X is locally connected. Namely, we have the following important result (see [59, p. 413] and [62, Theorem (6.4), p. 270]).

1.4. THEOREM (Nadler). *A locally connected continuum X is a retract of 2^X if and only if X is an absolute retract.*

As a consequence we get a corollary.

1.5. COROLLARY (Nadler). *A locally connected curve X is a retract of 2^X if and only if X is a dendrite.*

For arbitrary continua, not necessarily locally connected, the situation is much more complicated and it is not likely to be clarified soon. However, there are partial results and examples which describe various situations. Hyperspace retractions of half-line compactifications are studied and many very interesting results are obtained by D. W. Curtis in [24]. Some interrelations between several conditions concerning hyperspace retractions, as well as suitable examples, are presented by S. B. Nadler, Jr. in his book [62]. It is known that if a one-dimensional continuum X is a retract of either 2^X or $C(X)$, then it is a dendroid, i.e., it is arcwise connected and hereditarily unicoherent [34, p. 122]. J. T. Goodykoontz, Jr. shows in [32] (in [33], respectively) an example of a nonlocally connected continuum X which is a smooth dendroid such that $C(X)$ is (is not, respectively) a retract of 2^X. A. Illanes in [38] constructs an example of a continuum X which is a retract of $C(X)$ but not of 2^X, and which does not admit any mean. Recall that a *mean* on a space X is a mapping $\mu : X \times X \to X$ such that $\mu(x, y) = \mu(y, x)$ and $\mu(x, x) = x$ for every $x, y \in X$ (in other words, it is a symmetric, idempotent, continuous binary operation on X). In [39] two examples of dendroids are constructed relating to the existence of a selection and of a retraction on $C(X)$.

In [34] a complete discussion is given of the existence of retractions, deformation retractions and strong deformation retractions between $F_1(X)$, $C(X)$ and 2^X; conclusions are collected in Table I of [34, p. 130]. We erase some question marks from that table and answer some other questions from [34].

The main subject is related to Problem 1.3 of Nadler in its part concerning 2^X rather than $C(X)$. In the light of Theorem 1.4 we consider nonlocally connected continua. Recall that there are continua X in all dimensions such that X is a deformation retract of 2^X [34, Proposition 2.10, p. 126]. We discuss the above question under the additional assumption that $\dim X = 1$ (i.e., that the continuum X is a curve). In other words, we are interested in the following problem.

1.6. PROBLEM. Characterize curves X such that X is a retract of 2^X.

The paper consists of five chapters. After an introduction and preliminaries, hyperspace retractions are studied in Chapter 3. We start by showing that if a curve X admits such a retraction from either 2^X or $C(X)$, then X is a uniformly arcwise connected dendroid (Theorems 3.1 and 3.3). This is an extension of a statement in [34]. We also give some conditions for the inverse limit X of finite dendrites to admit a retraction from 2^X onto X (Theorem 3.19). We conclude that the Mohler–Nikiel universal smooth dendroid admits the considered retraction (Theorem 3.21).

The fourth chapter is devoted to selections for the hyperspace $C(X)$. A sufficient condition is given in Corollary 4.3 for the restriction of a retraction of Theorem 3.3 to $C(X)$ to be a selection. Extra conditions are formulated that imply smoothness of the dendroid X. In Example 4.4 a noncontractible (thus nonsmooth) dendroid is constructed admitting a retraction from 2^X onto X whose restriction to $C(X)$ is a selection.

Chapter 5 is devoted to means. It is shown that for each continuum X the existence of a retraction from 2^X onto X implies the existence of a mean on X (Propositions 5.11 and 5.16). It is known that if a fan X (i.e., a dendroid having just one ramification point) is smooth, then X is a deformation retract of 2^X [34, Theorem 2.9, p. 125]. A fan X that admits no mean is presented in [5, Example 3.7, p. 42]; so there is no retraction from 2^X onto X. One of the most important results of Chapter 5 is Example 5.52 which shows a smooth dendroid admitting no mean. The example answers in the negative several questions concerning means asked in the literature (see Remarks 5.54). These results can be considered as steps towards finding a characterization of continua (of curves) which admit a mean. This problem was the starting point of the present investigations.

2. Preliminaries

All spaces considered are assumed to be metric. We denote by $\mathbb{N}$ the set of positive integers, and by $\mathbb{R}$ the space of reals. A *continuum* means a compact connected space, and a *mapping* means a continuous function. If $Y \subset X$, then Y is a *retract* of X means that there exists a mapping $r : X \to Y$ (called a *retraction*) such that the restriction $r|Y$ is the identity on Y. Given a continuum X we let 2^X denote the hyperspace of all nonempty closed subsets of X with the Hausdorff metric (equivalently: with the Vietoris topology; see e.g. [62, (0.1), p. 1, and (0.12), p. 10]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X, and for each $n \in \mathbb{N}$ we put

$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ elements}\}.$$

In particular, $F_1(X)$ is the hyperspace of singletons, i.e.,

$$F_1(X) = \{A \in 2^X : A \text{ is a singleton}\}.$$

Note that $F_1(X)$ is homeomorphic to X. We identify X and $F_1(X)$, and we consider X as a subspace of 2^X under the natural embedding. Similarly, we think of $C(X)$ as a subspace of 2^X. Hence, in particular, we shall consider a retraction $r : 2^X \to X$ rather than a retraction $r : 2^X \to F_1(X)$, although the former notation is perhaps less formal.

A *curve* means a one-dimensional continuum. A continuum is said to be *unicoherent* if the intersection of any two subcontinua whose union is the whole continuum is connected. A property of a continuum X is *hereditary* provided that the whole space X has the property, as well as every subcontinuum of X also has this property. Thus, in particular, a continuum is said to be *hereditarily unicoherent* provided that each its subcontinuum is unicoherent. If S is an arbitrary set in a continuum X, we denote by $I(S)$ a continuum in X containing S whose no proper subcontinuum contains S, i.e., $I(S)$ means a continuum in X which is *irreducible about the set* $S \subset X$. It is known (see [10, Theorem T1, p. 187]) that in hereditarily unicoherent continua X the continuum $I(S)$ is unique (equal to the intersection of all subcontinua containing S); moreover, the above uniqueness characterizes hereditarily unicoherent continua (see [56, Theorem 1.1, p. 179]). Therefore, for hereditarily unicoherent continua, the assignment I described above can be considered as a function $I : 2^X \to C(X)$. For its application to characterizations of some curves see [31, Theorems 1 and 8, pp. 3 and 7].

A space is said to be *uniquely arcwise connected* provided that for any two points there is exactly one arc joining these points. A *dendrite* means a locally connected continuum that contains no simple closed curve. A *dendroid* means an arcwise connected and hereditarily unicoherent continuum. It follows that the concept of a dendrite coincides with that of a locally connected dendroid. An *end point* of a dendroid X is defined as a point p of X which is an end point of each arc containing p. By a *ramification point* of a dendroid X we understand a point which is the centre of a simple triod contained in X. A dendroid having at most one ramification point v is called a *fan*, and v is called its *top*. The cone over the closure of the harmonic sequence of points is called the *harmonic fan*. The cone over the Cantor middle-thirds set is called the *Cantor fan*. A dendrite is said to be *finite* provided that the set of all its end points is finite.

A continuum X is said to be *uniformly pathwise connected* provided that it is a continuous image of the Cantor fan. The original definition of this concept, given in [45, p. 316], is more complicated, but it agrees with the one above by Theorem 3.5 of [45, p. 322]. A space X is said to be *uniformly arcwise connected* provided that it is arcwise connected and that for each $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that every arc in X contains k points that cut it into subarcs of diameters less than ε. By Theorem 3.5 in [45, p. 322], each uniformly arcwise connected continuum is uniformly pathwise connected (but not conversely) and it is easy to see that for uniquely arcwise connected continua these two notions coincide (compare [45, p. 316]). In particular, this holds for dendroids.

A dendroid X is said to be *smooth at a point* $v \in X$ provided that for each sequence of points $a_n \in X$ which converges to a point $a \in X$ the sequence of arcs $va_n \subset X$ converges to the arc va. A dendroid X is said to be *smooth* provided it is smooth at some point $v \in X$. Then the point v is called an *initial point* of X. It is known that every smooth dendroid is uniformly arcwise connected (see [20, Corollary 16, p. 318]).

The following observation is a consequence of the definition.

2.1. OBSERVATION. *If a sequence* $\{a_n\}$ *in a smooth dendroid* X *with an initial point* v *converges to* $a \neq v$, *then there is a sequence of homeomorphisms* $h_n : [0,1] \to va_n$ *which converges* (*uniformly*) *to a homeomorphism* $h : [0,1] \to va$.

A continuum X is said to have the *property of Kelley* provided that for each point x of X, for each sequence of points x_n converging to x, and for each subcontinuum K of X containing the point x, there exists a sequence of subcontinua K_n containing the points x_n such that K is its limit (see [41, Property (3.2), p. 26]).

We have the following result [25, Corollary 5, p. 730].

2.2. THEOREM (Czuba). *Each dendroid having the property of Kelley is smooth.*

The opposite implication is not true: consider a harmonic fan with the limit segment prolonged beyond the limit point of end points.

A subset A of a space X is said to be *planable* if it is embeddable in the plane, i.e., if there is a homeomorphism $\phi : A \to \phi(A) \subset \mathbb{R}^2$.

A mapping $f : X \to Y$ from a topological space X onto a topological space Y is called:

• *monotone* provided that the inverse image of each subcontinuum of Y is a subcontinuum of X;

• *monotone relative to a point* $p \in X$ provided that for each subcontinuum Q of Y such that $f(p) \in Q$ the inverse image $f^{-1}(Q)$ is connected;

• *confluent* provided that for each subcontinuum Q of Y each component of the inverse image $f^{-1}(Q)$ is mapped onto Q under f.

A mapping is monotone if and only if it is monotone relative to each point of its domain [53, Theorem 2.1, p. 720], and open as well as monotone mappings between continua are confluent [11, V and VI, p. 214].

We use the symbols Li, Ls and Lim to denote the lower limit, the upper limit and the limit of a sequence of subsets of a metric space, according to the notation given in Kuratowski's monograph [46, §29, pp. 335–340]. Further, we write $f = \lim f_n$ to denote that a sequence of mappings f_n converges uniformly to the limit f.

3. Hyperspace retractions

It is observed in [34, p. 122] that if a one-dimensional continuum X is a retract of either 2^X or $C(X)$, then it is a dendroid. The only argument used is Theorem (6.9) of [62, p. 272], which leads to Vietoris homology theory or to other homology theories (see [62, Theorems (1.172)–(1.180), pp. 176–179]). We present here another (and more complete) argument, to obtain a stronger result. Both these arguments are based on the possibility to represent the hyperspace 2^X or $C(X)$ as the intersection of a decreasing sequence of Hilbert cubes [62, Theorem (1.171), p. 175].

3.1. THEOREM. *Let X be a curve. If*

(3.2) *there exists a retraction from $C(X)$ onto X,*

then X is a uniformly arcwise connected dendroid.

P r o o f. For every continuum X the hyperspace $C(X)$ has trivial shape (see [62, Corollary (1.182), p. 180]). Trivial shape is preserved under retraction (simply by definition), and therefore (3.2) implies that X has trivial shape. Since every curve having trivial shape is treelike [44, Theorem 2.1(B), p. 237], it follows that X is treelike. Every treelike continuum is hereditarily unicoherent (see [9, Theorem 1, p. 74] and [47, §57, II, Theorem 2, p. 437] and treelikeness is a hereditary property; compare [14, (2.1) and (4.3), pp. 144 and 147]), so we infer that X is hereditarily unicoherent. Further, for each continuum X the hyperspace $C(X)$ is a continuous image of the Cantor fan ([41, Theorem 2.7, p. 25]; compare [62, Theorem (1.33), p. 81]), i.e., it is uniformly pathwise connected. Therefore so is X, uniform pathwise connectedness being a continuous invariant. Thus the conclusion follows by the definition of a dendroid.

3.3. THEOREM. *Let X be a curve. If*

$$\text{(3.4)} \qquad \textit{there exists a retraction } r : 2^X \to X,$$

then X is a uniformly arcwise connected dendroid.

Proof. It follows from (1.2) that (3.4) implies (3.2). Indeed, the restriction $r|C(X)$ is a retraction from $C(X)$ onto X. Thus the conclusion follows from Theorem 3.1.

3.5. REMARK. The converses to Theorems 3.1 and 3.3 are not true. This can be shown by one example. Recall the following construction. Let X_1 be the geometric cone in the plane with vertex $v_1 = (-1, 0)$ and the base consisting of the points $(1, 0)$ and $(1, 2^{-n})$ for $n \in \mathbb{N}$. Similarly, let X_2 be the geometric cone in the plane with vertex $v_2 = (1, 0)$ and the base consisting of the points $(-1, 0)$ and $(-1, -2^{-n})$ for $n \in \mathbb{N}$. Thus X_1 and X_2 are harmonic fans having the limit segment v_1v_2 in common. Put

$$X = X_1 \cup X_2.$$

Thus X is a uniformly arcwise connected dendroid. It is known [63, Theorem 2, p. 372] that $C(X)$ does not admit a selection. The proof of this property given in [62, (5.10), p. 258], uses the fact that $C(X)$ is contractible. However, the argument for the contractibility is not correct (as observed in [34, Example 2.13, p. 128]; see also [25]) because it is based on a false statement that X, being obviously nonsmooth, has the property of Kelley, which contradicts Theorem 2.2. The proof of contractibility of $C(X)$ has been repaired by J. T. Goodykoontz, Jr. in [34, Example 2.13, p. 128]. It follows that 2^X is also contractible (in fact both these properties are equivalent, [41, Lemma 3.1, p. 25]). Since contractibility is preserved under retractions [47, §54, V, Theorem 3, p. 371] and since X is not contractible (because it is of type N between v_1 and v_2, see [64, Theorem 2.1, p. 838], it follows that there is no retraction from either 2^X or $C(X)$ onto X.

3.6. REMARK. If the converse to Theorem 3.3 is under consideration only (so, if we pay no attention to existence of a retraction from $C(X)$ onto X), then counterexamples can be even more specialized. Apart from the dendroid X of Remark 3.5, other uniformly arcwise connected dendroids X for which there is no retraction from 2^X onto X are known. An example of a plane fan with no such retraction is constructed in [5, Example 3.7, p. 42]. A smooth dendroid X with no retraction from 2^X onto X is constructed in Example 5.52. It obviously is a retract of $C(X)$, since each smooth dendroid X is a retract (even a strong deformation retract) of $C(X)$ (see [34, Proposition 2.14, p. 129]).

Now we shall prove a partial converse to Theorem 3.3. We need some auxiliary definitions and propositions.

Let X be a continuum. A retraction $r : 2^X \to X$ is said to be *associative* provided that

$$r(A \cup B) = r(\{r(A)\} \cup B) \quad \text{for every } A, B \in 2^X. \tag{3.7}$$

Let X be hereditarily unicoherent. A retraction $r : 2^X \to X$ is said to be *internal* provided that

$$r(A) \in I(A) \quad \text{for each } A \in 2^X. \tag{3.8}$$

The next proposition says that this property is hereditary.

3.9. Proposition. *Let Y be a hereditarily unicoherent continuum. If a retraction $r : 2^Y \to Y$ is internal, then for every subcontinuum X of Y the restriction of r to 2^X is also an internal retraction of 2^X onto X.*

Proof. For an arbitrary subcontinuum X of a hereditarily unicoherent continuum Y the condition $A \subset X$ implies $I(A) \subset X$ for each set A in Y. Hence, if A is nonempty and closed, then $A \subset X$ implies $r(A) \in I(A) \subset X$ by (3.8), and the conclusion holds by the definition of a retraction.

3.10. Corollary. *Let Y be a hereditarily unicoherent continuum. If a retraction $r : 2^Y \to Y$ is internal and associative, then for every subcontinuum X of Y the restriction of r to 2^X is also an internal and associative retraction of 2^X onto X.*

For the proof of the following result see [34, Theorem 2.9, p. 125].

3.11. Theorem (Goodykoontz). *Every smooth fan X is a deformation retract of 2^X.*

3.12. Remarks. (a) In [5, Example 3.7, p. 42], a (nonsmooth) plane fan X is constructed such that there is no retraction from 2^X onto X. Thus smoothness is an essential assumption in Theorem 3.11.

(b) In Example 5.52 a (nonplanable) smooth dendroid X will be constructed for which there is no retraction from 2^X onto X. This shows that Goodykoontz's Theorem 3.11 cannot be generalized to all smooth dendroids (i.e., that the property of X having just one ramification point is indispensable).

In connection with Remark 3.12(b) we have the following question which is related to Theorem 3.11.

3.13. Question. For what smooth dendroids X does there exist a *deformation* retraction from 2^X onto X?

Our next goal is to show that a universal smooth dendroid Y admits an associative retraction from the hyperspace 2^Y onto Y. Since the dendroid is defined using inverse limits, we need some auxiliary results on inverse limits of dendrites with open bonding mappings.

3.14. Lemma. *Let a dendrite X contain a point p and a subdendrite Y such that $p \in Y$ and $X \setminus Y$ is connected. Let $f : X \to Y$ be an open retraction which is monotone relative to p, such that* card $f^{-1}(y) \le 2$ *for each point $y \in Y$, and let $\lambda : X \to [0, \infty)$ be such that*

$\lambda(p) = 0$, and, for each $x \in X$, $\lambda(f(x)) = \lambda(x)$ and the restriction $\lambda|px : px \to [0, \lambda(x)]$ is a homeomorphism.

If $r_Y : 2^Y \to Y$ is an associative retraction such that

$$\lambda(r_Y(B)) = \min\{\lambda(y) : y \in B\} \quad \textit{for each } B \in 2^Y,$$

then there exists an associative retraction $r_X : 2^X \to X$ satisfying

$$\lambda(r_X(A)) = \min\{\lambda(x) : x \in A\} \quad \textit{for each } A \in 2^X, \tag{3.15}$$

and $r_X|2^Y = r_Y$.

Proof. Let $A \in 2^X$. The set $f^{-1}(r_Y(f(A)))$ consists of at most two points. If it consists of two points, then exactly one of them is in Y. Define $r_X(A)$ as a point satisfying the following two conditions:

$$r_X(A) \in f^{-1}(r_Y(f(A))) \tag{3.16}$$

and

$$r_X(A) \in Y \quad \text{if and only if} \quad A \cap Y \neq \emptyset. \tag{3.17}$$

Observe that r_X is well-defined, its restriction to 2^Y is r_Y, and condition (3.15) holds by the definition of r_X. We shall prove continuity of r_X.

By assumption the mapping f can be considered as the combination of two homeomorphisms: the identity on Y and $f|(X \setminus Y)$ on $X \setminus Y$. Note also that $\text{bd}(X \setminus Y)$ is a one-point set, say $\{b\}$, and we have

$$\lambda(x) \geq \lambda(b) \quad \text{for each } x \in X \setminus Y. \tag{3.18}$$

For each $n \in \mathbb{N}$ take $A_n \in 2^X$ and assume that the sequence $\{A_n\}$ has A_0 as its limit. If either $A_0 \subset X \setminus Y$ or $A_n \cap Y \neq \emptyset$ for each $n \in \mathbb{N}$, then the condition $\lim r_X(A_n) = r_X(A_0)$ is a consequence of the definitions. So, assume $A_n \subset X \setminus Y$ and $A_0 \cap Y \neq \emptyset$. Then $b \in A_0$ and, according to the convergence, we have

$$\lim \min\{\lambda(x) : x \in A_n\} \leq \lambda(b).$$

On the other hand, (3.18) implies that $\lambda(x) \geq \lambda(b)$ for each $x \in A_n$, whence we get $\lim \min\{\lambda(x) : x \in A_n\} = \lambda(b)$. Therefore

$$\lim \lambda(r_X(A_n)) = \lambda(b) = \lambda(r_X(A_0)).$$

Since b is the only point of $f^{-1}(f(b))$, we conclude that $\lim r_X(A_n) = b = r_X(A_0)$, so r_X is continuous.

Now we shall prove that it is associative. We have to show (3.7). Let $A, B \in 2^X$. The definition of r_X and associativity of r_Y imply that

$$\begin{aligned} r_Y(f(A \cup B)) &= r_Y(f(A) \cup f(B)) = r_Y(\{r_Y(f(A))\} \cup f(B)) \\ &= r_Y(f(\{r_X(A)\} \cup B)) = f(r_X(\{r_X(A)\} \cup B)). \end{aligned}$$

Further,

$$\begin{aligned} r_X(A \cup B) \in Y &\Leftrightarrow (A \cup B) \cap Y \neq \emptyset \Leftrightarrow A \cap Y \neq \emptyset \text{ or } B \cap Y \neq \emptyset \\ &\Leftrightarrow r_X(A) \in Y \text{ or } B \cap Y \neq \emptyset \\ &\Leftrightarrow (\{r_X(A)\} \cup B) \cap Y \neq \emptyset \Leftrightarrow r_X(\{r_X(A)\} \cup B) \in Y. \end{aligned}$$

Since r_X was defined using (3.16) and (3.17), and since we showed appropriate associativity properties of both of them, the proof of associativity of r_X is complete.

3.19. THEOREM. *Let* $\{X_n, f_n : n \in \mathbb{N}\}$ *be an inverse sequence such that* X_1 *is a straight line segment with an end point* p *and, for each* $n \in \mathbb{N}$, *the following conditions are satisfied*:

- X_n *is a finite dendrite*;
- $X_n \subset X_{n+1}$;
- *the difference* $X_{n+1} \setminus X_n$ *is connected*;
- *the bonding mapping* $f_n : X_{n+1} \to X_n$ *is an open retraction which is also monotone relative to* p.

Then $X = \varprojlim\{X_n, f_n : n \in \mathbb{N}\}$ *is a smooth dendroid having the property of Kelley*, *and there exists an associative retraction* $r : 2^X \to X$.

Proof. Since the bonding mappings f_n are retractions, we can consider the continua X_n as subsets of X (and then X is known to be homeomorphic with the closure of their union). Thus $p \in X$. Since the properties of being a dendroid and being smooth are preserved under the inverse limit operation if the bonding mappings are monotone relative to points forming a thread (see [18, Corollary 3, p. 145, and Theorem 1, p. 144]) we conclude that the continuum X is a dendroid which is smooth at p. Further, since each dendrite X_n as a locally connected continuum has the property of Kelley, and since the property of Kelley is preserved under the inverse limit operation if the bonding mappings are confluent (see [23, Theorem 2, p. 190]), thus in particular if they are open, it follows that X has this property, too (it is enough to show that X is a dendroid with the property of Kelley; then its smoothness follows by Theorem 2.2).

Let $\pi_1 : X \to X_1$ be the natural projection. Define $\lambda : X \to [0, \infty)$ taking as $\lambda(x)$ the distance from p to $\pi_1(x)$ in X_1 for each $x \in X$. To begin the inductive procedure observe that an associative retraction $r_1 : 2^{X_1} \to X_1$ which agrees with the mapping λ, i.e., for which the condition

$$\lambda(r_1(A)) = \min\{\lambda(x) : x \in A\} \quad \text{for each } A \in 2^{X_1} \tag{3.15}$$

holds, can be defined taking $r_1(A)$ as the point of A nearest to the end point p.

Assume that for some $n \in \mathbb{N}$ an associative retraction $r_n : 2^{X_n} \to X_n$ is defined. Because of finiteness of the dendrite X_{n+1} we can define intermediate dendrites

$$X_n = X_{n,0} \subset X_{n,1} \subset \ldots \subset X_{n,k} \subset X_{n,k+1} = X_{n+1}$$

and open retractions

$$f_{n,i} : X_{n,i+1} \to X_{n,i} \quad \text{for} \quad i \in \{0, \ldots, k\}$$

such that

1. the mappings $f_{n,i}$ are monotone relative to p;
2. card $f_{n,i}^{-1}(x) \le 2$ for each $x \in X_{n,i}$;
3. $f_n = f_{n,0} \circ f_{n,1} \circ \ldots \circ f_{n,k}$.

Since $X_{n,i+1} \subset X$, the mapping $\lambda | X_{n,i+1} : X_{n,i+1} \to [0,\infty)$ is defined. One can check that the assumptions of Lemma 3.14 are satisfied with $X_{n,i+1}$ in place of X and $X_{n,i}$ in place of Y, as well as $f_{n,i}$ in place of f, and $\lambda | X_{n,i+1}$ in place of λ. Thus, for each $i \in \{0,\ldots,k\}$ we can define an associative retraction $r_{n,i} : 2^{X_{n,i}} \to X_{n,i}$ by the inductive procedure starting with $r_{n,0} = r_n$. By Lemma 3.14 we then have

$$r_{n,i+1} | 2^{X_{n,i}} = r_{n,i}. \tag{3.20}$$

Putting $r_{n+1} = r_{n,k+1}$ we can see by (3.20) that $r_{n+1} | 2^{X_n} = r_n$. The limit mapping $r = \varprojlim r_n$ is the required associative retraction.

Let $\mathcal{K}$ be a class of continua. A continuum $Y \in \mathcal{K}$ is said to be *universal* in $\mathcal{K}$ provided that every member of $\mathcal{K}$ can be embedded in Y.

By Theorem 3.19 there is a universal smooth dendroid Y admitting a retraction from 2^Y onto Y. It is the one constructed by Mohler and Nikiel in [58], and it will be called the *Mohler–Nikiel universal smooth dendroid*. We recall its construction.

The dendroid under consideration is defined as the inverse limit of an inverse sequence of finite dendrites Y_n, where $n \in \{0,1,2,\ldots\}$, with bonding mappings $g_n : Y_{n+1} \to Y_n$ which are finite compositions of mappings satisfying the assumptions of Theorem 3.19. Let $Y_0 = [0,1]$ and let $p = 0 \in Y_0$. We define $\lambda_0 : Y_0 \to [0,1]$ as the identity. Let $d_1, d_2, d_3, \ldots$ be a sequence of rationals in $[0,1)$ such that every rational in $[0,1)$ appears infinitely many times in the sequence. Assume that for some $n \in \{0,1,2,\ldots\}$ we have defined a finite dendrite Y_n, a mapping $\lambda_n : Y_n \to [0,1]$ such that $\lambda_n | Y_k = \lambda_k$ for each $k \in \{0,1,\ldots,n\}$, and open retractions $g_k : Y_{k+1} \to Y_k$ for $k \in \{0,1,\ldots,n-1\}$. To define Y_{n+1} consider two copies $Y_n \times \{0\}$ and $Y_n \times \{1\}$ of Y_n. Now Y_{n+1} is obtained from the free union of the two copies of Y_n by identification of each two points $(y,0)$ and $(y,1)$ if and only if $\lambda_n(y) \le d_n$. Define $g_n : Y_{n+1} \to Y_n$ as the natural projection reidentifying the doubled portion of Y_n. Finally put $\lambda_{n+1}(y,0) = \lambda_{n+1}(y,1) = \lambda_n(y)$ for each $y \in Y$. It is shown in [58] that $Y = \varprojlim\{Y_n, g_n : n \in \mathbb{N}\}$ is a universal smooth dendroid. Because the difference $Y_{n+1} \setminus Y_n$ has finitely many components, the mapping $g_n : Y_{n+1} \to Y_n$ can be described as the composition of a finite number of intermediate mappings each of which corresponds to the natural projection of one of the components of the difference $Y_{n+1} \setminus Y_n$ onto Y_n.

Applying Theorem 3.19 we get the following result.

3.21. Theorem. *The Mohler–Nikiel universal smooth dendroid Y has the property of Kelley, and there exists an associative retraction $r : 2^Y \to Y$.*

3.22. Remark. There is an example (viz. Example 5.52) of a (nonplanable) smooth dendroid X for which there is no retraction from 2^X onto X. By universality of the Mohler–Nikiel universal smooth dendroid Y the dendroid X can be considered as a subspace of Y. Therefore it follows from Proposition 3.9 that the retraction r of Theorem 3.21 is not internal. In other words, the Mohler–Nikiel universal smooth dendroid admits an associative retraction, while it does not admit any internal one.

3.23. Question. Let a dendroid X have the property of Kelley. Does there exist a retraction $r : 2^X \to X$?

4. Applications to selections

Let a continuum X and a family $\mathcal{F} \subset 2^X$ be given. By a *continuous selection* (briefly a *selection*) on $\mathcal{F}$ we mean a mapping $s : \mathcal{F} \to X$ such that $s(A) \in A$ for each $A \in \mathcal{F}$. Since $F_1(X)$ and X are homeomorphic, if $F_1(X) \subset \mathcal{F}$, a selection on $\mathcal{F}$ may be viewed as a special kind of retraction from $\mathcal{F}$ onto X. A selection is said to be *rigid* provided that

$$\text{(4.1)} \qquad \text{if} \quad A, B \in \mathcal{F} \text{ and } s(B) \in A \subset B, \quad \text{then} \quad s(A) = s(B).$$

Kuratowski, Nadler and Young proved in [48] that for every continuum X a selection on the family 2^X exists if and only if X is an arc. If one seeks a continuous selection on the family $C(X)$ of all subcontinua of X, then such a simple characterization of continua which admit a selection (they are said to be *selectible*) is not known and seems to be a rather hard problem. A very important step was made by Nadler and Ward, who proved that each selectible continuum is a dendroid (see [63, Lemma 3, p. 370]), and that a locally connected continuum is selectible if and only if it is a dendrite [63, Corollary, p. 371]. Since a selection on $C(X)$ is a retraction, the above result of Nadler and Ward can be sharpened by Theorem 3.1 to the statement that if a continuum is selectible, then it is a uniformly arcwise connected dendroid [13, Proposition 2, p. 110].

Consider now the restriction of the retraction $r : 2^X \to X$ to the family $C(X) \subset 2^X$ of all subcontinua of X. Note that if $A \in C(X)$, then $I(A) = A$, whence condition (3.8) implies $r(A) \in A$, i.e., $r|C(X)$ is a selection on $C(X)$.

Assume that the retraction $r : 2^X \to X$ satisfies one more condition:

$$\text{(4.2)} \qquad \text{if} \quad A, B \in 2^X \text{ and } r(B) \in A \subset B, \quad \text{then} \quad r(A) = r(B).$$

Observe that condition (4.2) implies (4.1), and hence it assures that this selection is rigid. Ward proved [67, Theorem 2, p. 1043] that a continuum X admits a rigid selection on $C(X)$ if and only if X is a smooth dendroid. Thus Theorem 3.3 implies the following corollary.

4.3. Corollary. *Let X be a curve. If there exists an internal retraction $r : 2^X \to X$, then $r|C(X)$ is a selection on $C(X)$, and X is a selectible dendroid.*

Moreover, if the retraction r satisfies the implication

$$\text{(4.2)} \qquad \textit{if} \quad A, B \in 2^X \textit{ and } r(B) \in A \subset B, \quad \textit{then} \quad r(A) = r(B),$$

then the selection $r|C(X)$ is rigid, and the dendroid X is smooth.

Recall that the existence of a retraction from $C(X)$ onto X does not suffice for the existence of a selection from $C(X)$ onto X. Namely A. Illanes has constructed in [39, Section 4, p. 70] an example of a dendroid X_2 which is a retract of $C(X_2)$, while $C(X_2)$ does not admit any selection. Thus, by Corollary 4.3, there is no internal retraction from 2^{X_2} onto X_2. It will be shown in Theorem 5.58 that there is a retraction from 2^{X_2} onto X_2. The dendroid X_2 coincides with the dendroid D described by T. Maćkowiak in [55, Example, p. 321]. Its various properties are collected in Theorem 5.78.

In the same paper [39] A. Illanes constructs a selectible dendroid X_1 which is a retract, but not a deformation retract, of its hyperspace $C(X_1)$. This permits replacing two question marks in row (7) of Table I of [34, p. 130] by "no". It will be shown in the next chapter that some other questions from this table also have negative answers.

Looking for a characterization of curves for which a retraction (3.4) exists (compare Problem 1.6) one can ask if Theorem 3.3 can be sharpened to get smoothness of the dendroid in question, or, in other words, if condition (4.2), which is responsible for smoothness of the dendroid, can be deduced from conditions (3.4) and (3.8). We now show, by constructing a proper example, that this is not the case.

Every contractible curve is a uniformly arcwise connected dendroid (a result which corresponds to our Theorem 3.3, see [12, Propositions 1 and 4, p. 73]; see also [21, Theorem 3, p. 94]), and each smooth dendroid is contractible (see [20, Corollary 12, p. 311]; cf. [21, Corollary, p. 93] and [57, Theorem 1.16, p. 371]).

4.4. Example. *There exists a noncontractible dendroid X with an internal retraction $r : 2^X \to X$ which does not satisfy condition* (4.2).

Proof. In the Cartesian rectangular coordinate system in the plane $\mathbb{R}^2$ put $v^+ = (0,1)$, $v^- = (0,-1)$, and

$$H^+ = \{(0,0)\} \cup \{(1/n,0) : n \in \{1,3,5,\ \ldots\}\},$$
$$H^- = \{(0,0)\} \cup \{(1/n,0) : n \in \{2,4,6,\ \ldots\}\}.$$

Let X^+ denote the cone over H^+ with vertex v^+, let X^- have the corresponding meaning (with H^- and v^-), and let

$$X = X^+ \cup X^-.$$

Thus X is a plane dendroid having only two ramification points v^+ and v^- by construction, and it is known to be noncontractible and selectible (see [13, Proposition 3, p. 110]).

We distinguish maximal straight line segments in X, i.e., segments in X joining either the two ramification points of X or a ramification point to an end point of X, and we label them with elements of the closure $\mathbb{H}$ of the harmonic sequence

$$\mathbb{H} = \{0\} \cup \{1/n : n \in \mathbb{N}\} = \{0, 1, 1/2, 1/3,\ \ldots\}$$

as follows. S_0 denotes v^-v^+. For each $h \in \mathbb{H} \setminus \{0\}$, let S_h denote a maximal straight line segment in X that ends at $(h,0)$, i.e., S_h joins either v^- or v^+ to $(h,0) = (1/n,0)$. Therefore

$$X = \bigcup \{S_h : h \in \mathbb{H}\}$$

is another description of X.

To define the required retraction we introduce two auxiliary mappings. The first of them, $\pi_0 : X \to [-1,1]$, is defined by $\pi_0(x,y) = y$ for each point $(x,y) \in X$. It corresponds to the projection of X onto the straight line segment v^-v^+. The domain of the other one, f, is the triangle

$$T = \{(x,y) \in \mathbb{R}^2 : x \in [-1,1] \text{ and } x \le y \le 1\},$$

and $f : T \to [-1, 1]$ is defined for each $(x, y) \in T$ by

$$f(x,y) = \begin{cases} x & \text{for } x \in [-1,0] \text{ and } x \le y \le 0, \\ x + y & \text{for } x \in [-1,0] \text{ and } y \in [0,1], \\ y & \text{for } x \in [0,1] \text{ and } x \le y \le 1. \end{cases}$$

Now we are ready to define $r : 2^X \to X$. We put

$$r(\{v^-\}) = v^- \quad \text{and} \quad r(\{v^+\}) = v^+,$$

and for $A \in 2^X \setminus \{\{v^-\}, \{v^+\}\}$ we describe $r(A) = (x, y)$ by indicating a segment S_k to which the point $r(A)$ belongs and by determining its coordinate y. We put

$$k = \inf\{h \in \mathbb{H} : I(A) \cap (S_h \setminus \{v^-, v^+\}) \neq \emptyset\}.$$

In this way the segment S_k to which $r(A)$ belongs is fixed. We localize $r(A) = (x, y)$ in S_k putting

$$y = f(\min \pi_0(A), \max \pi_0(A)). \tag{4.5}$$

Thus $r : 2^X \to X$ is defined. Observe that for each point p of X we have $r(\{p\}) = p$, i.e., r is a retraction.

Now we prove continuity of r. Take $A \in 2^X$ and a sequence $\{A_m \in 2^X : m \in \mathbb{N}\}$ such that $A = \operatorname{Lim} A_m$. If $A = \{v^-\}$ or $A = \{v^+\}$, continuity of r at A follows directly from the definition of r. So let $A \in 2^X \setminus \{\{v^-\}, \{v^+\}\}$. We define two auxiliary sets. Let

$$Y^- = X^- \setminus (S_0 \setminus \{v^-\}) = \bigcup\{S_h : h \in \{1/2, 1/4, 1/6, \ldots\}\},$$

and similarly

$$Y^+ = X^+ \setminus (S_0 \setminus \{v^+\}) = \bigcup\{S_h : h \in \{1, 1/3, 1/5, \ldots\}\}.$$

Thus Y^- and Y^+ are connected and dense, but not closed, subsets of X^- and X^+ respectively.

We examine four cases.

Case 1. $k > 0$, $\min \pi_0(A) \neq -1$, $\max \pi_0(A) \neq 1$. Then $I(A) \subset Y^+$ or $I(A) \subset Y^-$, and $I(A)$ is a finite fan, an arc or a point. Similarly, the sets A_m have the same properties for almost all $m \in \mathbb{N}$, and we have $k = \min\{h \in \mathbb{H} : I(A_m) \cap (S_h \setminus \{v^-, v^+\}) \neq \emptyset\}$. Thus continuity of r follows from continuity of f.

Case 2. $A \subset X^+$ and $\max \pi_0(A) = 1$, or $A \subset X^-$ and $\min \pi_0(A) = -1$. By symmetry we fix our attention on the first possibility. Then we have $\min \pi_0(A) \geq 0$. Therefore if $r(A) = (x, y)$, then $y = 1$ by (4.5) and the definition of f. Thus $r(A) = v^+$. It follows that if A_m's converge to A, then

$$\lim \min \pi_0(A_m) \geq 0 \quad \text{and} \quad \lim \max \pi_0(A_m) = 1,$$

which implies, again by continuity of f, that the sequence $r(A_m)$ converges to v^+.

Case 3. $A \cap S_0 = \emptyset$ and $k = 0$. Then

$$A \cap (Y^- \setminus \{v^-\}) \neq \emptyset \neq A \cap (Y^+ \setminus \{v^+\}),$$

and the same two inequalities hold for almost all sets A_m, i.e., $A_m \cap (Y^- \setminus \{v^-\}) \neq \emptyset \neq A_m \cap (Y^+ \setminus \{v^+\})$ for all but finitely many indices $m \in \mathbb{N}$. Thus by the definition of r we have $r(A_m) \in S_0$ for almost all $m \in \mathbb{N}$, and therefore continuity of r at A follows from continuity of the second coordinate y of $r(A)$, which in turn is a consequence of continuity of π_0 and of f by (4.5).

CASE 4. $A \cap S_0 \neq \emptyset$. Then obviously $k = 0$. Take a point $p_0 \in A \cap S_0$. Choose a sequence of points $p_m \in A_m$ with $p_0 = \lim p_m$ and let an index k_m be determined by $r(A_m) \in S_{k_m}$. If $p_m \in S_{j_m}$ for some $j_m \in \mathbb{H}$, then we obviously have $k_m \leq j_m$. Therefore, according to the definition of r, we have $\mathrm{Ls}\{r(A_m)\} \subset S_0$, and so continuity of r at A follows from that of π_0 and of f as previously.

Since Cases 1 through 4 cover all possibilities, continuity of r is shown.

Observe that implication (3.8) easily follows from the definition of r. Finally, it is immediate to see that the dendroid X is nonsmooth. Therefore the retraction r does not satisfy (4.2) by Corollary 4.3.

4.6. REMARK. Since the dendroid X of Example 4.4 satisfies assumptions (3.4) and (3.8) of Corollary 4.3, it follows that $r|C(X)$ is a selection on $C(X)$. In Proposition 3 of [13, pp. 110 and 111], a selection on $C(X)$ is explicitly defined.

5. Applications to means

Given a Hausdorff space X, a *mean* μ on X is defined as a (continuous) mapping $\mu : X \times X \to X$ such that for each $x, y \in X$ we have

$$\mu(x, x) = x; \tag{5.1}$$

$$\mu(x, y) = \mu(y, x). \tag{5.2}$$

More generally, by an *n-mean* on X we understand a mapping $\mu_n : X^n \to X$ which is the identity on the diagonal and which is symmetric in the sense that it takes the same values for each permutation of the variables. Further, by a *strong n-mean* we understand a mapping $\mu_n : X^n \to X$ which is the identity on the diagonal of the product, and which, for any two elements

$$(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in X^n$$

such that $\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_n\}$, satisfies the equality

$$\mu_n(a_1, \ldots, a_n) = \mu_n(b_1, \ldots, b_n).$$

Every strong n-mean is an n-mean, and any 2-mean is strong. The 3-mean $\mu_3 : [0, 1]^3 \to [0, 1]$ defined by $\mu_3(x, y, z) = (x + y + z)/3$ is not strong.

For a continuum X and $n \in \mathbb{N}$, the existence of a strong n-mean

$$\mu_n : X^n \to X$$

is equivalent to the existence of a retraction

$$r_n : F_n(X) \to X,$$

where the two concepts are related to each other by the equality

$$\mu_n(a_1, \ldots, a_n) = r_n(\{a_1, \ldots, a_n\}). \tag{5.3}$$

A mean is said to be *associative* provided that

$$\mu(x, \mu(y, z)) = \mu(\mu(x, y), z) \quad \text{for every } x, y, z \in X. \tag{5.4}$$

A space X equipped with an associative mean μ is a *semilattice*, i.e., μ can be considered as an idempotent, commutative and associative operation. In particular, when X is a continuum, the semilattice (X,μ) is compact and connected. The reader is referred to Chapter 6 of [28] for more detailed information on compact topological semilattices.

A semilattice (X,μ) defines a partial order $\le$ by

$$x \le y \quad \text{if and only if} \quad \mu(x,y) = y. \tag{5.5}$$

The *supremum* of a nonempty set $A \subset X$ is defined as a point $z = \sup A$ such that

$$\text{for each } x \in A \text{ we have } x \le z, \tag{5.6}$$

and

$$\text{for each } t \in X, \text{if } x \le t \text{ for each } x \in A, \text{ then } z \le t. \tag{5.7}$$

Then

$$\sup\{x,y\} = \mu(x,y) \quad \text{for every } x,y \in X. \tag{5.8}$$

Therefore a supremum of a finite set does exist, and it is continuous as a function from X^n onto X. Note that this supremum is a strong n-mean. Thus we have the following result.

5.9. Proposition. *If a continuum X admits an associative mean μ, then for each $n \in \mathbb{N}$ it admits a strong n-mean $\mu_n : X^n \to X$ with*

$$\mu_n(a_1,\dots,a_n) = \sup\{a_1,\dots,a_n\}. \tag{5.10}$$

The existence of a mean on a continuum is related to the existence of a retraction. One implication is considered in the next proposition. Another will be discussed later (see Proposition 5.25).

5.11. Proposition. *Let X be a continuum. If*

$$\textit{there exists a retraction } r : 2^X \to X, \tag{3.4}$$

then, for each $n \in \mathbb{N}$, there exists a strong n-mean

$$\mu_n : X \times \ldots \times X \to X.$$

Further, for $n = 2$, if the retraction r is associative, then the 2-mean is associative, too, and

$$r(A) = \sup A \quad \textit{for each } A \in 2^X. \tag{5.12}$$

P r o o f. It is enough to define

$$\mu_n(a_1,\dots,a_n) = r(\{a_1,\dots,a_n\}). \tag{5.13}$$

If $n = 2$ and r is associative, then by (3.7) we have

$$r(\{x,y,z\}) = r(\{x\} \cup \{r(\{y,z\})\}) = r(\{r(\{x,y\})\} \cup \{z\}),$$

whence (5.4) follows by (5.13).

To show (5.12) note that, by (3.7), for each $x \in A$ we have

$$r(A) = r(A \cup \{x\}) = r(\{r(A)\} \cup \{x\}) = \mu(r(A),x),$$

whence $x \le \mu(r(A), x) = r(A)$ by (5.5), so (5.6) holds with $z = r(A)$. Take $t \in X$ such that

(5.14) $$x \le t \quad \text{for each } x \in A.$$

Consider a sequence of finite sets $A_n \subset A$ with $A_n \in F_n(X)$ and $A = \operatorname{Lim} A_n$. Thus for each $n \in \mathbb{N}$ we have $r|F_n(X) = r_n$, whence

$$r(A_n) = r_n(A_n) = \mu_n(A_n) = \sup A_n$$

by (5.3) and (5.10). It follows from (5.14) that for each $n \in \mathbb{N}$ we have $r(A_n) \le t$, whence by continuity of r we infer $r(A) \le t$, i.e., (5.7) holds with $z = r(A)$.

A mean $\mu : X \times X \to X$ on a dendroid X is said to be *internal* provided that

(5.15) $$\mu(a_1, a_2) \in a_1 a_2 \quad \text{for every } a_1, a_2 \in X.$$

The next proposition is related to the previous. Its proof is left to the reader.

5.16. PROPOSITION. *Let X be a dendroid. If*

(3.4) *there exists a retraction* $r : 2^X \to X$,

then, *for each $n \in \mathbb{N}$*, *the mapping*

$$\mu_n : X \times \ldots \times X \to X$$

defined by

(5.13) $$\mu_n(a_1, \ldots, a_n) = r(\{a_1, \ldots, a_n\})$$

is a strong n-mean on X. Moreover, *if the retraction r is internal* (*associative*), *then the mean $\mu_2 = \mu$ is also internal* (*associative*, *respectively*).

Recall that a continuum X is said to be *arc-smooth at a point $p \in X$* provided that there exists a mapping $\alpha : X \to C(X)$ satisfying the following conditions:

(5.17) $\alpha(p) = \{p\}$;
(5.18) $\alpha(x)$ is an arc from p to x, for each $x \in X \setminus \{p\}$;
(5.19) if $x \in \alpha(y)$, then $\alpha(x) \subset \alpha(y)$.

A continuum X is said to be *arc-smooth* provided that it is arc-smooth at some point p. The reader is referred to [27] for more detailed information.

As a consequence of Theorems 5.11(4) and 5.12 of Chapter 6 of [28, p. 300], we have the following theorem.

5.20. THEOREM. *If a continuum admits an associative mean*, *then it is arc-smooth.*

Using a partial order structure on the hereditarily unicoherent continua (see [42, Theorem, p. 680]) M. Bell and S. Watson have observed in the proof of Example 4.8 of [5, p. 45] that a dendroid which admits an associative mean is smooth. A much more general result can be shown, which is a consequence of Theorem 5.20.

5.21. THEOREM. *Let a continuum X be either one-dimensional or hereditarily unicoherent. If X admits an associative mean $\mu : X \times X \to X$*, *then X is a smooth dendroid.*

Proof. By Theorem 5.20 the continuum X is arc-smooth. Assume it is one-dimensional. Since each arc-smooth one-dimensional continuum is a smooth dendroid (see [27, Theorem II-4-B, p. 559]), the conclusion follows. Assume that X is hereditarily unicoherent. Being arc-smooth it is arcwise connected, so it is a dendroid by the definition. Thus it is one-dimensional (see e.g. [10, T24, p. 197]), and the previous case applies.

A topological semilattice X is said to *have small semilattices at a point* $x \in X$ provided that the point x has a basis of neighbourhoods which are subsemilattices of X. A semilattice is said to *have small semilattices* provided that it has small semilattices at every point. Equivalently, a topological space X with an associative mean $\mu : X \times X \to X$ has small semilattices if for each point $x \in X$ and for each open neighbourhood U of x there exists a set V such that $x \in \operatorname{int} V \subset V \subset U$ and that $\mu(V \times V) = V$. An example of a one-dimensional metric compact semilattice which has no basis of subsemilattices is described in [28, Section 4 of Chapter 6, pp. 293–296]. See in particular Theorem 4.5, p. 296, and Notes, p. 297, of [28].

5.22. Proposition [28, Chapter 6, Exercise 2.9(2), p. 279]. *If a continuum* X *admits an associative mean*, *then the supremum* $\sup A$ *exists for each* $A \in 2^X$.

5.23. Proposition [28, Chapter 6, Exercise 3.20, p. 289]. *Let* X *be a compact topological semilattice. The function* $r : 2^X \to X$ *defined by*

$$r(A) = \sup A \quad \text{for each } A \in 2^X \tag{5.24}$$

is continuous if and only if X *has small semilattices.*

Note that $r : 2^X \to X$ defined by (5.24) is a retraction. Thus, as an immediate consequence of Proposition 5.23 we have the following partial converse to Proposition 5.11.

5.25. Proposition. *Let a continuum* X *admit an associative mean* $\mu : X \times X \to X$. *If*

$$\text{the semilattice } (X, \mu) \text{ has small semilattices}, \tag{5.26}$$

then there exists a retraction $r : 2^X \to X$, *namely the one defined by* (5.24).

It is tempting to get a more general result by deleting condition (5.26) from the assumptions of Proposition 5.25 in the following sense. Given a continuum X admitting an associative mean, we are looking for μ' such that the induced partial order determines a semilattice having small semilattices. Then applying Proposition 5.25 to μ' we get a retraction from 2^X onto X. Thus the following question is of some interest.

5.27. Question. Let a continuum X admit an associative mean. Does there exist an associative mean $\mu' : X \times X \to X$ such that the semilattice (X, μ') has small semilattices?

A natural question that comes with the definition of a mean is: which spaces, especially metric continua, admit such means? For more than half a century this question has been answered for a very small class of spaces. So, the following problems can be considered as main ones, constituting a research program.

5.28. Problems. Characterize topological spaces (in particular metric continua) that admit a mean (or an associative mean).

Means on $[0, 1]$, even in a more general setting, were studied by A. N. Kolmogoroff who described a structural form of these mappings in [43]. An extensive study of this topic was the subject of the habilitation thesis of G. Aumann, and was developed in some of his papers [1], [2] and [3] and in papers of other authors. In particular, Aumann showed [3, Theorems 1 and 2, pp. 211 and 212] that the circle S^1 and, more generally, the k-dimensional sphere S^k, where $k \geq 1$, does not admit any mean, while every dendrite does [3, Theorem 7, p. 214]. An outline of a quite different proof that S^1 does not admit any mean is given in [62, (0.71.1), p. 50]. These fundamental results were generalized later in various ways.

K. Borsuk showed in [7, p. 184] that if a locally connected continuum is not unicoherent, then it contains a simple closed curve (i.e., a homeomorphic image of the circle S^1) as its retract. Since each retract of a space admitting a mean also admits a mean [66, Lemma, p. 85] and since no simple closed curve admits a mean [3, Theorem 1, p. 211], we get the following result (see [66, Theorem, p. 85]).

5.29. Proposition (Sigmon). *If a locally connected continuum admits a mean, then it is unicoherent.*

Since every locally connected unicoherent curve is a dendrite (see [47, §57, III, Corollary 8, p. 442]), and since each dendrite admits a mean [3, Theorem 7, p. 214], a corollary follows [66, p. 85].

5.30. Proposition (Sigmon). *A one-dimensional locally connected continuum admits a mean if and only if it is a dendrite.*

The above result, as well as Nadler's one (see Corollary 1.5) can be supplemented as follows:

5.31. Theorem. *Let a continuum X be locally connected. Then the following conditions are equivalent*:

(5.32) *X is an absolute retract*;
(5.33) *there exists a retraction $r : 2^X \to X$.*

Moreover, if X is one-dimensional, then each of them is equivalent to any of the following:

(5.34) *X is a dendrite*;
(5.35) *there exists an associative retraction $r : 2^X \to X$*;
(5.36) *there exists an associative mean $\mu : X \times X \to X$*;
(5.37) *there exists a mean $\mu : X \times X \to X$.*

Proof. Assume (5.32). Since X can be embedded into 2^X, the retraction r exists by the definition of an absolute retract. Assume (5.33). Since X is locally connected, 2^X is an absolute retract [62, Theorem (1.96), p. 136], and thus X is a retract of an absolute retract. So, the first part of the result is shown.

Now assume (5.34). Then the function $I : 2^X \to C(X)$ that assigns to a set $A \in 2^X$ the continuum $I(A)$ irreducible with respect to containing A is continuous by Theorem 1 of [31, p. 3]. Further, if we fix a point $p \in X$, then X can be partially ordered by the relation $\leq_p$ defined by $x \leq_p y$ provided that $x \in py$; it is well-known that this partial order is continuous and it has p as the least element compare e.g. [42, Theorem, p. 680]). Since X is a smooth dendroid with p as an initial point (see [20, Corollary 4, p. 298]), the function min : $C(X) \to X$ that assigns to a continuum K its minimal point with respect to $\leq_p$, i.e., such a point q of K that $pq \cap K = \{q\}$ is a continuous selection (see the proof of Theorem 2 in [67, p. 1043]). Putting $r = \min \circ I$ we get a retraction $r : 2^X \to X$ (see Corollary 1.5) for which associativity condition (3.7) is satisfied by its definition. Thus (5.35) follows.

Further, (5.35) implies (5.36) by Proposition 5.11, and the implication from (5.36) to (5.37) is trivial. Next, (5.37) implies (5.34) by Corollary 5.30. Finally, (5.35) trivially implies (5.33), and since each one-dimensional absolute retract is a dendrite (see [8, Corollary 13.5, p. 138]), conditions (5.32) and $\dim X = 1$ imply (5.34). The proof is complete.

5.38. Question. Assume that the continuum X is locally connected. Does (5.37) imply (5.36)? Does (5.36) imply (5.32)?

Most of the results about means are related to locally connected spaces. Little is known on ones that are not locally connected. P. Bacon [4] has shown that the $\sin(1/x)$-curve admits no mean. M. Bell and S. Watson [5] obtained criteria for the existence and for the non-existence of means on continua. Their non-existence criterion [5, Theorem 3.5, p. 42] has been generalized by K. Kawamura and E. D. Tymchatyn in [40] as follows. Let a continuum X contain an arc-like subcontinuum A with opposite end points a and b of A. A sequence $\{A_n : n \in \mathbb{N}\}$ of subcontinua of X is called a *folding sequence with respect to the point* a provided that it satisfies the following conditions: for each $n \in \mathbb{N}$ there are two subcontinua P_n and Q_n of A_n such that $A_n = P_n \cup Q_n$ and

$$\operatorname{Lim}(P_n \cap Q_n) = \{a\} \quad \text{and} \quad \operatorname{Lim} P_n = \operatorname{Lim} Q_n = A.$$

5.39. Theorem (Kawamura and Tymchatyn, [40, Theorem 2.2, p. 99]). *Let a hereditarily unicoherent continuum X contain an arc-like subcontinuum A with opposite end points a and b of A. If there exist folding sequences $\{A_n\}$ and $\{B_n\}$ with respect to a and b respectively, then X admits no mean.*

The concept of a folding sequence is a generalization of Oversteegen's concept of type N [64, p. 837] which in turn generalizes Graham's concept of a zigzag [36, p. 78], and is related to Maćkowiak's notion of a bend set [54, p. 548]. These concepts were exploited to obtain some criteria for noncontractibility and nonselectibility of dendroids (see [36, Theorem 2.1, p. 81]; [49, Theorem 2, p. 416]; [50, Corollary 6, p. 126]; [64, Theorem 2.1, p. 838]; [65, Theorem 3.4, p. 393]). Namely, replacing in the assumptions of Theorem 5.39 the continua A, A_n and B_n in a dendroid X by arcs, we get just the concept of a dendroid X *of type N between points a and b*, which means that there is an arc $A = ab \subset X$ and two sequences of arcs A_n and B_n in X with end points a_n, a'_n and b_n, b'_n respectively, and two sequences of points $a''_n \in B_n \setminus \{b_n, b'_n\}$ and $b''_n \in A_n \setminus \{a_n, a'_n\}$ such that

$$A = \operatorname{Lim} A_n = \operatorname{Lim} B_n, \quad a = \lim a_n = \lim a'_n = \lim a''_n, \quad b = \lim b_n = \lim b'_n = \lim b''_n.$$

Thus Theorem 5.39 leads to the following corollary.

5.40. Corollary. *If a dendroid is of type N between some two points, then it admits no mean.*

A. Illanes in [38] constructs a continuum X which is a retract of $C(X)$ but not of 2^X. To show that there is no retraction from 2^X onto X he proves that X does not admit any mean. This result shows that three question marks in column (4) of Table I of [34, p. 130], have to be replaced by "no". It is also a negative answer to the question on equivalence of (6.28.1) and (6.28.2) in [62, p. 290].

Our aim now is to present some results concerning means on some special continua, mainly curves, which are not locally connected in general. In particular, we construct in Example 5.52 a smooth dendroid having the same properties as the above mentioned continuum X (for which $\dim X > 1$) of Illanes [38].

As a consequence of Theorem 3.11 and Proposition 5.16 we have the following corollary.

5.41. Corollary. *Every smooth fan admits an associative and internal mean.*

5.42. Remarks. (a) The result of Corollary 5.41 has independently been obtained by M. Bell and S. Watson in [5, Proposition 4.2, p. 43].

(b) Recall that there exists a contractible and selectible fan that admits a mean while it admits neither an internal nor an associative one (see [5, Example 4.8, p. 45]; for a picture see [19, Fig. 7, p. 69]).

5.43. Remark. In the light of Proposition 5.16 one can say that, roughly speaking, for dendroids X the existence of a retraction $r : 2^X \to X$ is stronger than the existence of a mean $\mu : X \times X \to X$. However, the authors do not know to what extent it is stronger, or even whether it is essentially stronger. More precisely, we have the following question.

5.44. Question. Does there exist a dendroid X which admits a mean $\mu : X \times X \to X$ and for which there is no retraction $r : 2^X \to X$?

The next proposition is an analog of Proposition 3.9 for retractions. Its proof runs in the same way, so it is left to the reader.

5.45. Proposition. *If a dendroid Y admits an internal mean $\mu : Y \times Y \to Y$, then for every subdendroid X of Y the restriction $\mu|X \times X$ is an internal mean on X.*

5.46. Corollary. *If a dendroid Y admits an internal and associative mean $\mu : Y \times Y \to Y$, then for every subdendroid X of Y the restriction $\mu|X \times X$ is an internal and associative mean on X.*

The existence of a mean on a curve X does not imply the existence of a retraction from 2^X to X nor arcwise connectedness of X. As the authors have been informed by Murray Bell, the following example is due to John M. Franks.

5.47. Example (J. M. Franks). *The dyadic solenoid admits a mean.*

Proof. Let S^1 be the unit circle, and let $f : S^1 \to S^1$ be defined by $f(z) = z^2$. Put $X_n = S^1$ and $f_n = f$ for each $n \in \mathbb{N}$. Then $X = \varprojlim\{X_n, f_n : n \in \mathbb{N}\}$ is the dyadic solenoid. Define the shift mapping $h : X \to X$ by $h(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. Then $\mu(x, y) = h(x \cdot y)$, where the dot denotes the usual complex multiplication of the coordinates, is a mean on X.

The next questions, which are related to Theorem 3.3, concern the existence of a mean on a dendroid.

5.48. Question. Let a dendroid X admit a mean. Must X be uniformly arcwise connected?

5.49. Question. Let a dendroid X have the property of Kelley. Does there exist a mean $\mu : X \times X \to X$?

5.50. Problem. Characterize dendroids admitting a mean.

5.51. Remarks. (a) Using an argument that is very close to Corollary 5.40, M. Bell and S. Watson showed in [5, Example 3.7, p. 42] that the planar fan X of Example 1.2 of [64, p. 838] (which is of type N between some two points; see e.g. [21, p. 95], for its picture) admits no mean. It follows from Proposition 5.16 that there is no retraction from 2^X onto X. Furthermore, Kawamura and Tymchatyn's result (Theorem 5.39), and—consequently—Corollary 5.40, present such dendroids (or fans) in profusion.

(b) Smoothness is a necessary assumption in Corollary 5.41. This can be seen by the above mentioned example of a nonsmooth uniformly arcwise connected plane fan X that admits no mean [5, Example 3.7, p. 42].

(c) The same example shows by Proposition 5.16 that condition (3.4) cannot be satisfied for this X, i.e., that there is no retraction r from 2^X onto X. Therefore the converse to Theorem 3.3 is not true.

5.52. Example. *There exists a smooth dendroid admitting no mean.*

Proof. If p and q are points in the Euclidean 3-space $\mathbb{R}^3$, we denote by pq the straight line segment joining p and q, and we use $z : \mathbb{R}^3 \to \mathbb{R}$ for the third coordinate of points.

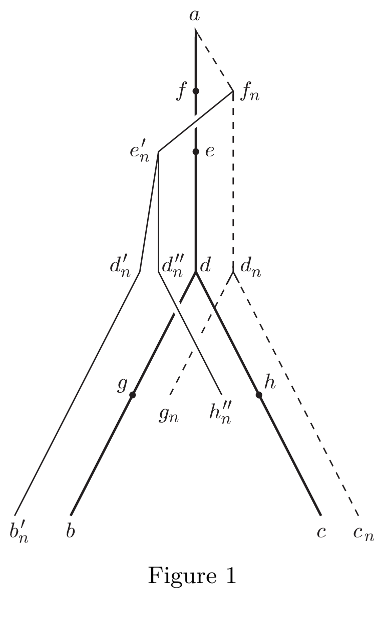

Figure 1

Take in the 3-space a simple triod T which is the union of three straight line segments ad, bd, cd pairwise disjoint except for d. The triod T can be situated in the space so that

$$z(b) = z(c) = 0, \quad z(d) = 1/2 \quad \text{and} \quad z(a) = 1$$

(see Figure 1). Let e, g and h stand for the midpoints of the segments ad, bd and cd respectively. Further, let f be the midpoint of the segment ae. For each $n \in \mathbb{N}$ choose points c_n, g_n and d_n such that

$$z(c_n) = 0, \quad z(g_n) = 1/4 \quad \text{and} \quad z(d_n) = 1/2,$$
$$c = \lim c_n, \quad g = \lim g_n \quad \text{and} \quad d = \lim d_n.$$

Similarly, for each $n \in \mathbb{N}$ define points b_n', h_n'', d_n', d_n'' and e_n' such that

$$z(b_n') = 0, \quad z(h_n'') = 1/4, \quad z(d_n') = z(d_n'') = 1/2, \quad z(e_n') = 3/4,$$
$$\lim b_n' = b, \quad \lim h_n'' = h, \quad \lim d_n' = \lim d_n'' = d, \quad \lim e_n' = e.$$

For each $n \in \mathbb{N}$ consider the triods

$$T_n = ad_n \cup g_n d_n \cup c_n d_n.$$

Define f_n as the point in ad_n with $z(f_n) = 7/8$ and consider the triods

$$T_n' = (b_n' d_n' \cup d_n' e_n') \cup (h_n'' d_n'' \cup d_n'' e_n') \cup e_n' f_n$$

having the points e_n' as their centres. All these points can be chosen in such a way that the union

$$D = T \cup \bigcup \{T_n \cup T_n' : n \in \mathbb{N}\}$$

is a smooth dendroid.

In the set $D \times \{0, 1\}$ we identify the points $(x, 0)$ and $(y, 1)$ if and only if

1. $x \in ad$ and $x = y$, or
2. $x \in bd$ and $y \in cd$ and $z(x) = z(y)$, or
3. $x \in cd$ and $y \in bd$ and $z(x) = z(y)$.

The resulting space Y is again a smooth dendroid. To simplify notation, let us put $x = (x, 0)$ for $x \in D$, so that D is considered as contained in Y. Note that there is a natural autohomeomorphism α of Y such that

$$\alpha|ad \quad \text{is the identity}, \quad \alpha(b) = c \quad \text{and} \quad \alpha(c) = b.$$

Consider now a sequence of embeddings $\beta_i : Y \to \mathbb{R}^3$ with $i \in \mathbb{N}$ such that

(i) $\beta_i(Y) \cap \beta_j(Y) = \emptyset$ if $i \neq j$;
(ii) $z(\beta_i(y)) = z(y)$ for each point $y \in Y$;
(iii) $\lim \beta_i(y) = (0, 0, z(y))$ for each point $y \in Y$;
(iv) $\beta_i(a) = (1/i, 0, 1)$ for each $i \in \mathbb{N}$.

Define

$$X = \{(u, 0, 1) : u \in [0, 1]\} \cup \{(0, 0, v) : v \in [0, 1]\} \cup \bigcup \{\beta_i(Y) : i \in \mathbb{N}\}.$$

Observe that X is a dendroid which is smooth at the point $(0, 0, 1)$. We will show that X admits no mean. Assume on the contrary that there is a mean $\mu : X \times X \to X$. Since

$\mu((0,0,v),(0,0,v)) = (0,0,v)$ for $v \in [0,1]$, there is an index $k \in \mathbb{N}$ such that if p and q are points of Y with $z(p) = z(q)$, we have

$$z(\mu(\beta_k(p),\beta_k(q))) \in (z(p) - 1/16, z(p) + 1/16). \tag{5.53}$$

We will say that two points p and q of X are *near* provided that if A denotes the arc joining p and q in X, then $\operatorname{diam} z(A) < 1/16$.

Now we prove several claims that will lead to a contradiction.

Claim 1. *$\mu(\beta_k(b),\beta_k(c))$ is near $\beta_k(b)$ or near $\beta_k(c)$.*

To show the claim we denote, for each $t \in [0,1/2]$, by $b(t)$ and $c(t)$ the points in the segments $\beta_k(bd)$ and $\beta_k(cd)$ respectively, satisfying $z(b(t)) = z(c(t)) = t$. Put

$$M = \{\mu(b(t),c(t)) : t \in [0,1/2]\}.$$

Thus M is a continuum containing the point $\beta_k(d)$ (for $t = 1/2$) and such that $z(M) \subset [0,1/2+1/16)$, therefore $M \subset \beta_k(T)$. Note that for $t = 0$ we have $\mu(\beta_k(b),\beta_k(c)) \in M$, and by (5.53) we see that

$$z(\mu(\beta_k(b),\beta_k(c))) \in [0,1/16],$$

and so the conclusion follows.

Taking into account the autohomeomorphism $\alpha : Y \to Y$ we can fix one of the two possibilities mentioned in Claim 1. Thus we can assume that the following claim holds.

Claim 2. *$\mu(\beta_k(b),\beta_k(c))$ is near $\beta_k(b)$.*

Claim 3. *$\mu(\beta_k(b'_n),\beta_k(c_n))$ is near $\beta_k(b'_n)$ for sufficiently large $n \in \mathbb{N}$.*

Really, arguing as for Claim 1, we see that

$$\mu(\beta_k(b'_n),\beta_k(c_n)) \in \beta_k(T_n \cup T'_n),$$

and by continuity of μ the conclusion follows from Claim 2.

Claim 4. *If $p \in \beta_k(T_n)$, $q \in \beta_k(T'_n)$ and $z(p) = z(q) \in [0,3/4]$, then $\mu(p,q) \in \beta_k(T'_n)$ for sufficiently large $n \in \mathbb{N}$.*

Indeed, we can find, for each $t \in [0,1]$, points $p(t)$ and $q(t)$ such that $p(0) = \beta_k(c_n)$, $q(0) = \beta_k(b'_n)$, $p(1) = p$, $q(1) = q$, and $z(p(t)) = z(q(t))$. Denoting $M = \{\mu(p(t),q(t)) : t \in [0,1]\}$ we see that the point $\mu(p(0),q(0)) \in M$ is near $\beta_k(b'_n)$ according to Claim 3, and that $z(M) \subset [0,3/4+1/16]$. Thus $M \subset \beta_k(T'_n)$, and taking $t = 1$ the conclusion follows.

For each $n \in \mathbb{N}$ denote by h_n the midpoint of the segment $c_n d_n$. As a consequence of Claim 4 and of continuity of μ we have the next claim.

Claim 5. *$\mu(\beta_k(h_n),\beta_k(h''_n))$ is near $\beta_k(h''_n)$ for sufficiently large $n \in \mathbb{N}$.*

Arguing as previously, using Claim 5 we get the next results.

Claim 6. *$\mu(\beta_k(d_n),\beta_k(d''_n))$ is near $\beta_k(d''_n)$ for sufficiently large $n \in \mathbb{N}$.*

Claim 7. *$\mu(\beta_k(g_n),\beta_k(h''_n))$ is near $\beta_k(h''_n)$ for sufficiently large $n \in \mathbb{N}$.*

Taking the limit for $n \to \infty$ we have the following.

Claim 8. *$\mu(\beta_k(g),\beta_k(h))$ is near $\beta_k(h)$.*

Again arguing in the same way as in the proof of Claim 4 we infer from Claim 8 the next claim.

CLAIM 9. *$\mu(\beta_k(b), \beta_k(c))$ is near $\beta_k(c)$.*

Now Claim 9 contradicts Claim 2. The proof is complete.

5.54. REMARKS. (a) Since a dendroid X is smooth if and only if there exists a rigid selection on $C(X)$ [67, Theorem 2, p. 1043] which is obviously a retraction from $C(X)$ onto X, and since nonexistence of a mean on X implies nonexistence of a retraction from 2^X onto X by Proposition 5.16, the dendroid X constructed in Example 5.52 has all the properties of the continuum described in [38]. In particular, it answers in the negative the three questions in column (4) of Table 1 of [34, p. 130], even for a much narrower class of continua.

(b) The same example answers in the negative the following questions: Question 2.16 of [34, p. 129] whether any smooth dendroid X is a retract of 2^X; Problem 4.1 of [5, p. 42] whether any smooth dendroid admits a mean; Problem 4.3 of [5, p. 43] whether any selectible dendroid has a mean (since each smooth dendroid is selectible, [67, Theorem 2, p. 1043]) and whether any contractible dendroid has a mean (since each smooth dendroid is (hereditarily) contractible, [20, Corollary 12, p. 311] and [22, Proposition 14, p. 235]).

(c) Since the Mohler–Nikiel universal smooth dendroid admits a mean (even an associative one) by Theorem 3.21 and Proposition 5.16, while the dendroid of Example 5.52 does not, it follows that having a mean is not a hereditary property even for a class as narrow as that of smooth dendroids.

Example 5.52 and Proposition 5.45 imply the following result.

5.55. PROPOSITION. *No universal smooth dendroid admits an internal mean.*

In connection with Example 5.52 the following problems are of some interest.

5.56. PROBLEM. Characterize smooth dendroids admitting a mean.

5.57. PROBLEM. Characterize smooth dendroids X admitting a retraction from 2^X onto X.

Now we shall discuss some properties of a special dendroid, viz. the dendroid defined and investigated by T. Maćkowiak in [55, Example, p. 321]. The dendroid coincides with the dendroid X_2 studied by A. Illanes in [39, Section 4, p. 70], who proved that there exists a retraction from $C(X_2)$ onto X_2. M. Bell and S. Watson proved in [5, Example 4.10, p. 47] that it admits a mean. Maćkowiak showed in [55, (4), p. 323] that it is contractible. Since contractibility of any continuum X implies its g-contractibility, which in turn implies that X is a continuous image of 2^X [60, Theorem 3.4, p. 193], we infer that there is a mapping from 2^{X_2} onto X_2. However, a much stronger result can be shown which is a common generalization of the Illanes and Bell–Watson results mentioned above, namely that there exists a retraction from 2^{X_2} onto X_2. To show that, we start with recalling the definition of the example, following Sections 4 and 5 of [39], which is much simpler than the one in [55].

In the Cartesian coordinates in the 3-space $\mathbb{R}^3$ let $p = (1,0,0)$ and $T = (\{0\}\times[-1,1]\times\{0\}) \cup ([0,1]\times\{(0,0)\})$. For $n \in \mathbb{N}$ put $p_n = (0,(-1)^n, 1/n)$. Let $\phi : [1,\infty) \to \mathbb{R}^3$ be the function which maps linearly the intervals of the form $[n, n+1]$ onto the straight line segment $p_n p_{n+1}$ sending n to p_n. For each $n \in \mathbb{N}$ put

$$L_n = \phi([6n+1, 6n+3]) \cup \phi(6n+3/2)p$$

and

$$M_n = \phi([6n+9/2, 6n+6]) \cup \phi(6n+9/2)p,$$

where $\phi(6n+3/2)p$ and $\phi(6n+9/2)p$ are both understood as straight line segments. Define

$$X_2 = T \cup \Big(\bigcup\{L_n : n \in \mathbb{N}\}\Big) \cup \Big(\bigcup\{M_n : n \in \mathbb{N}\}\Big).$$

Then X_2 is a dendroid. We will refer to X_2 as to the *Illanes–Maćkowiak dendroid* (see Figure 2).

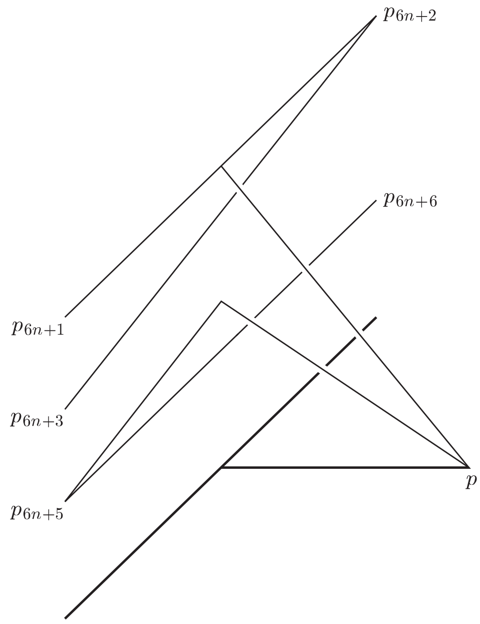

Figure 2

Further, for each $n \in \mathbb{N}$ consider the following sets:

$$A_n = \phi([6n+2, 6n+3]) \subset L_n,$$
$$B_n = \phi([6n+1, 6n+2]) \cup \phi(6n+3/2)p \subset L_n,$$
$$C_n = \phi([6n+5, 6n+6]) \subset M_n,$$
$$D_n = \phi([6n+9/2, 6n+5]) \cup \phi(6n+9/2)p \subset M_n.$$

Thus we have $L_n = A_n \cup B_n$ and $M_n = C_n \cup D_n$, whence it follows that X_2 can be redefined as

$$X_2 = T \cup \Big(\bigcup\{(A_n \cup B_n) \cup (C_n \cup D_n) : n \in \mathbb{N}\}\Big).$$

Note that $T = \operatorname{Lim} L_n = \operatorname{Lim} M_n$.

5.58. THEOREM. *For the Illanes–Maćkowiak dendroid X_2 there exists a retraction $r : 2^{X_2} \to X_2$.*

Proof. We define the projections

$$\pi : \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\}, \quad \pi_1 : \mathbb{R}^3 \to \mathbb{R}, \quad \text{and} \quad \pi_2 : \mathbb{R}^3 \to \mathbb{R}$$

by the formulas

$$\pi(x, y, z) = (x, y, 0), \quad \pi_1(x, y, z) = x, \quad \pi_2(x, y, z) = y.$$

For $A \in 2^{X_2}$ we put $s_1(A) = \min \pi_1(A)$, $s_2(A) = \min \pi_2(A)$, $t_1(A) = \max \pi_1(A)$, $t_2(A) = \max \pi_2(A)$. We will write s_1, s_2, t_1 or t_2 if A is understood. Thus $s_1 \le t_1$ and $s_2 \le t_2$.

We define a retraction for the limit triod T first. Namely for $A \in 2^T$ satisfying $s_1 = 0$ we consider the following six cases:

(5.59) if $t_1 \in [0, 1/4]$ and $s_2 \le t_2 \le 0$, then $r_0(A) = (0, s_2, 0)$;
(5.60) if $t_1 \in [0, 1/4]$ and $s_2 \le 0$ and $t_2 \ge 0$, then $r_0(A) = (0, (1-4t_1)(1-s_2)t_2 + s_2, 0)$;
(5.61) if $t_1 \in [0, 1/4]$ and $t_2 \ge s_2 \ge 0$, then $r_0(A) = (0, 4t_1(s_2 - t_2) + t_2, 0)$;
(5.62) if $t_1 \in [1/4, 1/2]$, then $r_0(A) = (0, (1 - 4t_1)(1 + s_2) + s_2, 0)$;
(5.63) if $t_1 \in [1/2, 3/4]$, then $r_0(A) = (0, 4t_1 - 3, 0)$;
(5.64) if $t_1 \in [3/4, 1]$, then $r_0(A) = (4t_1 - 3, 0, 0)$.

Thus for all subsets $A \in 2^T$ with $s_1 = 0$ the point $r_0(A)$ is determined. Note the following:

(5.65) if $s_1 = 0$ and $t_1 = 1$, then $r_0(A) = p = (1, 0, 0)$;
(5.66) if $s_1 = 0$ and A is a singleton $\{q\}$, then $r_0(A) = q$.

Recall that a space Y is called an *absolute extensor for spaces of a class* $\mathcal{K}$ provided that for every space $X \in \mathcal{K}$ and for every closed subspace A of X, every mapping $f : A \to Y$ can be continuously extended over X. It is known that a compact metric space Y is an absolute extensor for metric spaces (or—equivalently—for compact metric spaces) if and only if Y is an absolute retract [37, Chapter 3, Theorems 3.1 and 3.2, pp. 83 and 84]. Thus T is an absolute extensor. Consider the following three closed subsets of 2^T:

$$\mathcal{S} = \{A \in 2^T : s_1(A) = 0\}, \quad \mathcal{T} = \{A \in 2^T : t_1(A) = 1\}, \quad \mathcal{F} = F_1(T),$$

and note that for $A \in \mathcal{S}$ the point $r_0(A)$ has already been defined by conditions (5.59)–(5.64). We further define:

(5.67) if $A \in \mathcal{T}$, then $r_0(A) = p = (1, 0, 0)$;
(5.68) if $A = \{q\} \in \mathcal{F}$, then $r_0(A) = q$.

Thus r_0 is defined on the union $\mathcal{S} \cup \mathcal{T} \cup \mathcal{F}$. Since T is an absolute extensor, there exists an extension of r_0 from $\mathcal{S} \cup \mathcal{T} \cup \mathcal{F}$ over 2^T (which is still denoted by r_0), i.e., a mapping $r_0 : 2^T \to T$ which satisfies conditions (5.59)–(5.64) and (5.67)–(5.68). Thus $r_0 : 2^T \to T$ is a retraction.

Now we define a retraction $r : 2^{X_2} \to X_2$. Let $A \in 2^{X_2}$, and put $r(A) = v$. Then

(5.69) if $A \in 2^T$, then $v = r_0(A)$;
(5.70) $\pi(v) = r_0(\pi(A))$;

(5.71) if $A \subset A_n$, or $A \subset B_n$, or $A \subset C_n$, or $A \subset D_n$, then $v \in A_n$, or $v \in B_n$, or $v \in C_n$, or $v \in D_n$, respectively;

(5.72) if $A \subset A_n \cup B_n$ and $A \cap A_n \neq \emptyset \neq A \cap B_n$, then $v \in B_n$;

(5.73) if $A \subset C_n \cup D_n$ and $A \cap C_n \neq \emptyset \neq A \cap D_n$, and
(a) if $t_1 \in [0, 1/2]$, then $v \in C_n$;
(b) if $t_1 \in [1/2, 1]$, then $v \in D_n$;

(5.74) for all other cases we put $v \in T$.

Therefore $r : 2^{X_2} \to X_2$ just defined is a retraction. The proof is complete.

5.75. Corollary (A. Illanes, [39, Section 5, p. 70]). *For the Illanes–Maćkowiak dendroid X_2 there exists a retraction from $C(X_2)$ onto X_2.*

As a consequence of Theorem 5.58 and Proposition 5.11 we obtain the following result.

5.76. Corollary (M. Bell and S. Watson, [5, Example 4.10, p. 47]). *The Illanes–Maćkowiak dendroid X_2 admits a mean.*

5.77. Remark. The retraction $r : 2^{X_2} \to X_2$ of Theorem 5.58 is a modification of the retraction from $C(X_2)$ onto X_2 defined by A. Illanes in [39, Section 5, pp. 70–71]. However, it is not an extension of the Illanes retraction. And what is more, it can be shown that the Illanes retraction from $C(X_2)$ onto X_2 cannot be extended to any retraction from 2^{X_2} onto X_2.

Properties of the dendroid X_2 are summarized below.

5.78. Theorem. *The Illanes–Maćkowiak dendroid X_2 has the following properties*:

(5.79) *X_2 is contractible*;

(5.80) *X_2 is not hereditarily contractible*;

(5.81) *the hyperspaces 2^{X_2} and $C(X_2)$ are both contractible*;

(5.82) *if Y is an open image of X_2, then the hyperspaces 2^Y and $C(Y)$ are both contractible*;

(5.83) *X_2 is a retract of $C(X_2)$*;

(5.84) *there is no selection on $C(X_2)$*;

(5.85) *X_2 admits a mean*;

(5.86) *X_2 admits no internal mean*;

(5.87) *X_2 admits no associative mean*;

(5.88) *there is a retraction from 2^{X_2} onto X_2*;

(5.89) *there is no internal retraction from 2^{X_2} onto X_2*;

(5.90) *there is no associative retraction from 2^{X_2} onto X_2*;

(5.91) *X_2 is not planable.*

P r o o f. (5.79) was shown in [55, (4), p. 323]. (5.80) is proved in [15, Observation 8, p. 29]. (5.81) follows from (5.79) and Corollary (16.8) of [62, p. 537]. It implies (5.82) by [62, Theorem (16.39), p. 559]. (5.83) and (5.84) are proved in Sections 5 and 6 of [39, p. 70]; for (5.83) see our Corollary 5.75; for (5.84) see also [55, (3), p. 322]. (5.85) is shown in Example 4.10 of [5, p. 47]. Since X_2 contains a homeomorphic image of a fan that admits no internal mean, viz. the one of [5, Example 4.8, p. 45] (mentioned here in

Remark 5.42(b)), (5.86) is a consequence of Proposition 5.45. Since X_2 is not smooth, (5.87) follows from Theorem 5.21. (5.88) is just our Theorem 5.58. (5.89) is a consequence of (5.86) and Proposition 5.16; and (5.90) follows from (5.87) again by Proposition 5.16. Finally, for (5.91) see [15, Observation 4, p. 28].

5.92. Remark. For other structural properties of the dendroid X_2 see [17, Remark 2.28, p. 569].

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