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The Freudenthal compactification

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CONTENTS

1. Introduction

This paper is intended to give an elementary development of the Freudenthal compactification φX of a rim-compact space X. It is not intended to be comprehensive, although most of the results of Morita [M_2] and Isbell [I], as well as the seminal work of Freudenthal [F_3], are covered in this paper. Many authors have used the existence and some of the properties of the Freudenthal compactification, however they did not have a complete source to cite. Freudenthal's development of φX was pointed toward the case when X is a separable metric space. Morita removed the second countable limitation of Freudenthal's results, however his derivation of φX is dependent upon "the theory of simple extensions of a space with respect to a uniformity" [M_1]. Isbell's derivation of φX, via proximities, is a section of his book on uniform spaces [I] and does not include some of the cardinality results found in Morita's papers. Skljarenko also developed the Freudenthal compactification of a rim-compact space via proximities [Sk_4]; his results appear to be directed at the study of the perfectness of the Freudenthal compactification. Zippin [Z] developed a compactification for the so-called semipeanian spaces, i.e. locally connected, connected, rim-compact, separable complete metric spaces, and he obtained a locally connected metric compactification for such spaces.

The development of the Freudenthal compactification given herein is similar to that of Freudenthal, in that a base $\mathscr{B}$ for X consisting of open subsets with compact boundaries satisfying certain properties is employed. However, instead of using maximal linked systems in $\mathscr{B}$, a technique employing ultra $\mathscr{B}$-filters is considered. The elementary properties of φX such as the weight of φX or the perfectness of φX or the zero-dimensional embeddedness of $\varphi X - X$ are developed herein. Some of the results given here, e.g. Section 6, appear to be new.

Notation and Preliminaries. Throughout this paper all spaces will be Hausdorff spaces. If $A \subset X$, $\operatorname{cl}_X A$ (or simply $\operatorname{cl} A$ when no confusion is likely to result) will denote the closure of A in X and $\operatorname{Fr}_X A$ (or simply $\operatorname{Fr} A$) will denote the frontier of A in X, i.e. $\operatorname{Fr}_X A = \operatorname{cl}_X(X - A) \cap \operatorname{cl}_X A$. Following Morita [$M_2$], if A is an open (closed) subset of X and $\operatorname{Fr}_X A$ is compact, we will say that A is a *γ-open* (*γ-closed*) subset of X.

An *extension of* X is a space Y containing X such that $\text{cl}_Y X = Y$; the set $Y - X$ is the *remainder of* X *in* Y. A *compactification* αX of X is a compact Hausdorff extension of X. If αX and γX are compactifications of X, $\alpha X \geqslant \gamma X$ means that there exists a continuous surjection $h: \alpha X \to \gamma X$ such that $h|_X$ is the identity. It is well known that if $\alpha X \geqslant \gamma X$ and $\gamma X \geqslant \alpha X$, then there exists a homeomorphism $h: \alpha X \to \gamma X$ where $h|_X$ is the identity map; in this case we identify αX and γX and write $\alpha X = \gamma X$.

A Hausdorff space X is *rim-compact* if X has a base consisting of γ-open subsets of X. (Some authors have used the terms (locally) peripherally (bi)compact or semi-(bi)compact instead of rim-compact.)

Definitions not given herein may be found in most general topology texts, e.g. in [D], [E₁], [K], or [Wi].

2. $\mathscr{B}$-filters and $\mathscr{B}$-compactifications

In this and subsequent sections, $\mathscr{B}$ will denote a collection of open subsets of X which is a base for the topology on X. The following four properties of such a $\mathscr{B}$ will be used to develop the extension $X(\mathscr{B})$ of X.

b1. If $A \in \mathscr{B}$ and $B \in \mathscr{B}$, then $A \cap B \in \mathscr{B}$.

b2. If $A \in \mathscr{B}$, then $X - \text{cl}_X A \in \mathscr{B}$.

b3. If $A \in \mathscr{B}$, then $\text{Fr}_X A$ is compact.

b4. If U and V are γ-open subsets of X and $\text{cl}_X U \cap \text{cl}_X V = \emptyset$, then there exists $A \in \mathscr{B}$ such that $U \subset A \subset X - \text{cl}_X V$.

Recall that a collection $\mathscr{F} \subset \mathscr{B}$ is a *$\mathscr{B}$-filter* provided that the members of $\mathscr{F}$ are non-empty, if $C \in \mathscr{B}$ and C contains a member of $\mathscr{F}$ then $C \in \mathscr{F}$; and whenever $A, B \in \mathscr{F}$, then $A \cap B \in \mathscr{F}$. An *ultra $\mathscr{B}$-filter* is a maximal $\mathscr{B}$-filter (cf. [Wi], p. 83).

LEMMA 2.1. *Suppose that $\mathscr{B}$ has property* b1 *and that $\mathscr{F}$ is a ·$\mathscr{B}$-filter. Then*

(i) *$\mathscr{F}$ is contained in an ultra $\mathscr{B}$-filter;*

(ii) *$\mathscr{F}$ is an ultra $\mathscr{B}$-filter if and only if $\mathscr{F}$ contains every member of $\mathscr{B}$ that meets each member of $\mathscr{F}$;*

(iii) *if $\mathscr{F}$ is an ultra $\mathscr{B}$-filter with $\{A_1, ..., A_n\} \subset \mathscr{B}$ and if $A = A_1 \cup ... \cup A_n$ is either dense in X or $A \in \mathscr{F}$, then some A_i belongs to $\mathscr{F}$ for $1 \leqslant i \leqslant n$.*

Proof. Part (i) employs a Zorn's Lemma argument and part (ii) is straightforward. To see part (iii), suppose to the contrary that no A_i belongs to $\mathscr{F}$. Then by part (ii), there exists $\{B_1, ..., B_n\} \subset \mathscr{F}$ such that $A_i \cap B_i = \emptyset$ for all $1 \leqslant i \leqslant n$. Then $A \cap (B_1 \cap ... \cap B_n) = (A_1 \cap B_1) \cup ... \cup (A_n \cap B_n) = \emptyset$. But this is impossible if $A \in \mathscr{F}$ or if A is dense in X. Hence for some i, $A_i \in \mathscr{F}$. ∎

LEMMA 2.2. *Suppose that $\mathscr{B}$ has properties* b1 *and* b2 *and that $\mathscr{F}$ is a $\mathscr{B}$-filter. Then $\mathscr{F}$ is an ultra $\mathscr{B}$-filter if and only if whenever $B \in \mathscr{B}$, either $B \in \mathscr{F}$ or $X - \mathrm{cl}_X B \in \mathscr{F}$.*

Proof. The necessity follows from Lemma 2.1 since $A = B \cup (X - \mathrm{cl}_X B)$ is dense in X and both B and $X - \mathrm{cl}_X B$ belong to $\mathscr{B}$. For the sufficiency, suppose that there exists a $\mathscr{B}$-filter $\mathscr{D}$ having $\mathscr{F}$ as a proper subset. Let $B \in \mathscr{D} - \mathscr{F}$. Then by the hypothesis, $X - \mathrm{cl}_X B \in \mathscr{F}$ and so $X - \mathrm{cl}_X B \in \mathscr{D}$. But then $\mathscr{D}$ must contain $B \cap (X - \mathrm{cl}_X B)$ and this is impossible. Therefore $\mathscr{F}$ is an ultra $\mathscr{B}$-filter. ∎

Definition of the extension $X(\mathscr{B})$. Let $F(\mathscr{B})$ denote the set of all free ultra $\mathscr{B}$-filters ($\mathscr{F}$ is free if $\mathrm{ad}_X \mathscr{F} = \bigcap \{\mathrm{cl}_X F: F \in \mathscr{F}\}$ is empty). For each $A \in \mathscr{B}$, let A^* denote the subset of $X \cup F(\mathscr{B})$ defined by

$$A^* = A \cup \{\mathscr{F} \in F(\mathscr{B}): A \in \mathscr{F}\},$$

and let $\mathscr{B}^*$ be the set $\{A^*: A \in \mathscr{B}\}$.

LEMMA 2.3. *Let $\mathscr{B}$ have property* b1. *If $A, B \in \mathscr{B}$, then $A^* \cap B^* = (A \cap B)^*$.*

Proof. Clearly $X \cap A^* \cap B^* = X \cap (A \cap B)^*$. Let $\mathscr{F} \in F(\mathscr{B}) \cap A^* \cap B^*$. Then $A \in \mathscr{F}$ and $B \in \mathscr{F}$, so that $A \cap B \in \mathscr{F}$. Of course the latter implies that $\mathscr{F} \in (A \cap B)^*$. If $\mathscr{F} \in F(\mathscr{B}) \cap (A \cap B)^*$, then $A \cap B \in \mathscr{F}$, and so $A \in \mathscr{F}$ and $B \in \mathscr{F}$. Thus $\mathscr{F} \in A^* \cap B^*$. ∎

Let $\mathscr{B}$ satisfy b1, so, by Lemma 2.3, $\mathscr{B}^*$ is a base for a topology on the set $X \cup F(\mathscr{B})$; and let $X(\mathscr{B})$ denote this set, $X \cup F(\mathscr{B})$, with this topology generated by $\mathscr{B}^*$. We use $\mathrm{cl}^* S$ ($\mathrm{Fr}^* S$) to denote the closure (frontier) of a subset S of $X(\mathscr{B})$. Note that $X(\mathscr{B})$ contains X as a densely embedded subset.

If αX is a compactification of X, then the remainder $\alpha X - X$ is said to be *zero-dimensionally embedded in* αX provided that each point of $\alpha X - X$ has a neighborhood base whose frontier in αX misses $\alpha X - X$.

THEOREM 2.4. *Suppose $\mathscr{B}$ has properties* b1 *and* b2. *Then the space $X(\mathscr{B})$ is a Hausdorff H-closed extension of X, and $X(\mathscr{B}) - X$ is zero-dimensionally embedded in $X(\mathscr{B})$.*

Proof. It is clear that X is a dense subset of $X(\mathscr{B})$. To see that $X(\mathscr{B})$ is Hausdorff, let $p, q \in X(\mathscr{B})$, $p \neq q$. Since X is a Hausdorff space, we may suppose that $q \in F(\mathscr{B})$. Suppose first that $p \in X$. Since q is a free ultra $\mathscr{B}$-filter on X, there exists $A \in q$ so that $p \notin \mathrm{cl}_X A$. Then A^* and $(X - \mathrm{cl}_X A)^*$ are disjoint open subsets of q and p respectively. Suppose now that $p \in F(\mathscr{B})$. Then since $p \neq q$, there exist $B \in p$ and $C \in q$ so that $B \notin q$ and $C \notin p$. Then $X - \mathrm{cl}_X B \in q$ and $X - \mathrm{cl}_X C \in p$, and so $B^* \cap (X - \mathrm{cl}_X C)^*$ and $C^* \cap (X - \mathrm{cl}_X B)^*$ are disjoint open subsets of $X(\mathscr{B})$ containing p and q respectively; so that $X(\mathscr{B})$ is a Hausdorff space.

Recall that a Hausdorff space Y is *H-closed* if every open cover of Y

contains a finite collection whose union is dense in Y (cf. [Wi], p. 126). Let $\mathscr{C}$ be a subfamily of $\mathscr{B}$ such that $\mathscr{C}^* = \{B^*: B \in \mathscr{C}\}$ covers $X(\mathscr{B})$. Suppose by way of contradiction, that for every finite subfamily $\mathscr{C}_0$ of $\mathscr{C}$, $X(\mathscr{B}) \neq \text{cl}^*(\bigcup \{B^*: B \in \mathscr{C}_0\})$. Then $\bigcap \{X - \text{cl}_X B: B \in \mathscr{C}_0\} \neq \emptyset$. Hence the collection of complements (in X) of the closures of finite unions of members of $\mathscr{C}^*$ is contained in a free ultra $\mathscr{B}$-filter $\mathscr{F}$. It follows that if $\mathscr{F} \in B^*$ where $B \in \mathscr{C}$, then $\mathscr{F}$ is also in $(X - \text{cl}_X B)^*$. But $B^* \cap (X - \text{cl}_X B)^* = \emptyset$, and this contradiction implies that $X(\mathscr{B})$ is H-closed.

It remains to see that $X(\mathscr{B}) - X$ is zero-dimensionally embedded in $X(\mathscr{B})$. We assert that if $A \in \mathscr{B}$, then $\text{Fr}^* A^*$ is a subset of X. For suppose that $\mathscr{F} \in F(\mathscr{B}) \cap \text{cl}^* A^*$, and that $B \in \mathscr{F}$. Then $\mathscr{F} \in B^*$ and so, since B^* is open in $X(\mathscr{B})$, $A^* \cap B^* \neq \emptyset$. But $A^* \cap B^* = (A \cap B)^*$ and so $A \cap B \neq \emptyset$. Then A has a non-empty intersection with every member of $\mathscr{F}$ and thus $A \in \mathscr{F}$. But this means that $\mathscr{F} \in A^*$, so that $F(\mathscr{B}) \cap \text{cl}^* A^*$ is a subset of A^*; and therefore $\text{Fr}^* A^* = (\text{cl}^* A^*) - A^*$ is a subset of X as required. ∎

THEOREM 2.5. *Suppose $\mathscr{B}$ has properties* b1 *and* b2. *Then $X(\mathscr{B})$ is compact if and only if $\mathscr{B}$ has property* b3.

Proof. Suppose that $X(\mathscr{B})$ is compact and let $A \in \mathscr{B}$. Suppose further that $\text{Fr}_X A$ is not compact. Then since $\mathscr{B}$ is a base for the topology of X, $\mathscr{B}$ contains a subfamily $\mathscr{B}_0$ whose union covers $\text{Fr}_X A$. Then $\{A^*, (X - \text{cl}_X A)^*\} \cup \{B^*: B \in \mathscr{B}_0\}$ covers $X(\mathscr{B})$, but this cover does not contain a finite subcover of $X(\mathscr{B})$. Of course this contradicts the compactness of $X(\mathscr{B})$ and so $\text{Fr}_X A$ is compact.

On the other hand, suppose that $\mathscr{B}$ has property b3, i.e. suppose that every member of $\mathscr{B}$ has a compact frontier in X. To see that $X(\mathscr{B})$ is compact, let $\{A^*: A \in \mathscr{B}_0\}$ be an open cover of $X(\mathscr{B})$ by basic open subsets of $X(\mathscr{B})$. By Theorem 2.4, there exist $A_1, \ldots, A_n \in \mathscr{B}_0$ such that $X(\mathscr{B}) = \text{cl}^*(A_1^* \cup \ldots \cup A_n^*)$. Then $A = A_1 \cup \ldots \cup A_n$ is dense in X. Also $\text{Fr}_X A \subset \text{Fr}_X A_1 \cup \ldots \cup \text{Fr}_X A_n$ and so $\text{Fr}_X A$ is a compact subset of X. Then there exist a finite collection $\{A_{n+1}, \ldots, A_{n+m}\}$ in $\mathscr{B}_0$ so that $\text{Fr}_X A \subset A_{n+1} \cup \ldots \cup A_{n+m}$. Then $X = A_1 \cup \ldots \cup A_{n+m}$ and by Lemma 2.1, every $\mathscr{F}$ is in some A_i^* for $1 \leq i \leq n+m$. Thus $X(\mathscr{B}) = A_1^* \cup \ldots \cup A_{n+m}^*$ and $X(\mathscr{B})$ is compact. ∎

A compactification αX of X will be called *rim-perfect* provided that whenever U and V are γ-open subsets of X and $\text{cl}_X U \cap \text{cl}_X V = \emptyset$, then $\text{cl}_{\alpha X} U \cap \text{cl}_{\alpha X} V = \emptyset$.

THEOREM 2.6. *If $\mathscr{B}$ has properties* b1, b2 *and* b4, *then $X(\mathscr{B})$ is an H-closed, rim-perfect extension of X.*

Proof. In light of Theorem 2.4, we need only prove that $X(\mathscr{B})$ is rim-perfect. Let U and V be γ-open subsets of X with $\text{cl}_X U \cap \text{cl}_X V = \emptyset$. Suppose $\mathscr{F} \in F(\mathscr{B})$ and $\mathscr{F} \in \text{cl}^* U \cap \text{cl}^* V$. By property b4, there exist $A \in \mathscr{B}$ such that

$U \subset A \subset X - \mathrm{cl}_X V$. Now by Lemma 2.3, either $A \in \mathscr{F}$ or $B = X - \mathrm{cl}_X V \in \mathscr{F}$. If $A \in \mathscr{F}$, then $\mathscr{F} \in A^*$ and A^* is an open subset of $X(\mathscr{B})$ that misses $\mathrm{cl}^* V$. Similarly if $B \in \mathscr{F}$, then B^* is an open subset of $X(\mathscr{B})$ containing $\mathscr{F}$ and missing $\mathrm{cl}^* U$. Of course, this is impossible. Thus $\mathrm{cl}^* U \cap \mathrm{cl}^* V = \emptyset$ and $X(\mathscr{B})$ is rim-perfect. ∎

EXAMPLE 2.7. We obtain Urysohn's well-known example of a non-compact minimal Hausdorff space Z as a space $X(\mathscr{B})$ where $\mathscr{B}$ satisfies b1, b2 and b4.

Let $\{c_i : i \in N\}$, $\{a_{ij} : (i, j) \in N \times N\}$, $\{b_{ij} : (i, j) \in N \times N\}$, $\{\alpha\}$ and $\{\beta\}$ be pairwise disjoint collections of (distinct) points, let $X = \{c_i : i \in N\} \cup \{a_{ij} : (i, j) \in N \times N\} \cup \{b_{ij} : (i, j) \in N \times N\}$ and let $Z = X \cup \{\alpha\} \cup \{\beta\}$. We define a topology on Z by declaring each $\{a_{ij}\}$ and $\{b_{ij}\}$ to be open; a basic open set containing c_i is of the form $C(i, j) = \{c_i : i \in N\} \cup \{a_{ik} : (i, k) \in N \times N, k \geqslant j\} \cup \{b_{ik} : (i, k) \in N \times N, k \geqslant j\}$; a basic open set about α is of the form $A(j) = \{\alpha\} \cup \{a_{ik} : (i, k) \in N \times N, i \geqslant j\}$ and a basic open set about β is of the form $B(j) = \{\beta\} \cup \{b_{ik} : (i, k) \in N \times N, i \geqslant j\}$. Then if $\mathscr{B}$ denotes the smallest base for X satisfying b1 and b2 containing $\mathscr{B}_1 = (\bigcup \{C(i, j) : (i, j) \in N \times N\}) \cup (\bigcup \{A(j) : j \in N\} - \{\alpha\}) \cup (\bigcup \{B(j) : j \in N\} - \{\beta\})$, $\mathscr{B}$ satisfies b1, b2 and b4. Furthermore $X(\mathscr{B})$ is a rim-perfect, H-closed extension of X homeomorphic to Z.

The next theorem is an immediate consequence of Theorems 2.5 and 2.6.

THEOREM 2.8. If $\mathscr{B}$ satisfies b1, b2, b3 and b4, then $X(\mathscr{B})$ is a rim-perfect compactification of X such that $X(\mathscr{B}) - X$ is zero-dimensionally embedded in $X(\mathscr{B})$.

The uniqueness of this kind of compactification is established by the following sequence of theorems.

THEOREM 2.9. Let αX be a compactification of X. Then $\alpha X - X$ is zero-dimensionally embedded in αX if and only if $\alpha X = X(\mathscr{B})$ where $\mathscr{B}$ is a base for X satisfying b1, b2, and b3.

Proof. The sufficiency follows from Theorems 2.4 and 2.5. Suppose now that αX is a compactification of X and $\alpha X - X$ is zero-dimensionally embedded in αX. Let $\mathscr{B}_\alpha$ be the base for αX consisting of all open subsets U of αX such that $\mathrm{Fr}_{\alpha X} U \subset X$. Let $\mathscr{B} = \{U \cap X : U \in \mathscr{B}_\alpha\}$.

It is clear that $\mathscr{B}$ satisfies b1 and b2 since $\mathscr{B}_\alpha$ satisfies b1 and b2. Furthermore, $\mathrm{Fr}_X(U \cap X)$ is a closed subset of the compact set $\mathrm{Fr}_{\alpha X} U$ and so $\mathscr{B}$ satisfies b3. Thus $X(\mathscr{B})$ is a compactification of X and $X(\mathscr{B}) - X$ is zero-dimensionally embedded in $X(\mathscr{B})$.

We define $f : X(\mathscr{B}) \to \alpha X$ by $f(p) = p$ if $p \in X$ and $f(p) = \bigcap \{\mathrm{cl}_{\alpha X} U : U \cap X \in p\}$ if $p \in F(\mathscr{B})$. In the latter case, p is a filter-base in αX and $f(p)$ converges to $f(p)$ in αX. It follows from the Hausdorff property for αX that f is well-defined. In order to see that f is continuous, we assert that if $U \in \mathscr{B}_\alpha$ and $A = U \cap X$, then $f^{-1}(U) = A^*$. First it is clear that $f^{-1}(U \cap X)$

$= X \cap f^{-1}(U)$ so suppose that $p \in F(\mathscr{B})$ and $p \in f^{-1}(U)$. Then $f(p) \in U \subset \mathrm{cl}_{\alpha X} U$ and so $U \cap X = A \in p$. Thus $p \in A^*$. Now suppose that $\mathscr{F} \in A^*$ and $\mathscr{F} \in F(\mathscr{B})$. Then $A \in \mathscr{F}$ so that $f(\mathscr{F}) \in \mathrm{cl}_{\alpha X} A \subset \mathrm{cl}_{\alpha X} U$. Hence $\mathscr{F} \in f^{-1}(U)$. Thus we have shown that f is a continuous map and $f|_X$ is the identity.

Since f is an extension of the identity on X, f is a surjection. Finally, we note that f is one-to-one since $\mathscr{B}_\alpha$ is a base for αX. Hence f is a homeomorphism and $\alpha X = X(\mathscr{B})$. ∎

Some of the above propositions, at least in an equivalent form, have appeared elsewhere, e.g. a variation of Theorem 2.4 may be found in Flachsmeyer's paper ([Fl₁], Theorem 1) and Theorem 2.5 is Theorem 7 (ibid.). Theorem 2.9 is essentially Lemma 2.10 of [Sk₄].

THEOREM 2.10. *Let αX and γX be compactifications of X. If γX is rim-perfect and $\alpha X - X$ is zero-dimensionally embedded in αX, then $\gamma X \geqslant \alpha X$.*

Proof. By Theorem 2.8, there exists a base $\mathscr{B}$ for X satisfying b1, b2 and b3 such that $\alpha X = X(\mathscr{B})$. For $A \in \mathscr{B}$, let A_γ be the largest open subset of γX whose intersection with X is A, i.e. let $A_\gamma = X - (\gamma X - \mathrm{cl}_{\gamma X} B)$. We wish to define a map $f: \gamma X \to X(\mathscr{B})$. To that end let A and B be members of B such that $\mathrm{cl}_X A \cap \mathrm{cl}_X B = \emptyset$. Then, since γX is rim-perfect, $\mathrm{cl}_{\gamma X} A \cap \mathrm{cl}_{\gamma X} B = \emptyset$; so if $p \in \gamma X - X$, $p \in \mathrm{cl}_{\gamma X} A$ or $p \notin \mathrm{cl}_{\gamma X} B$. Hence, if $p \in \gamma X - X$, the family $\{B \in \mathscr{B}: p \in \mathscr{B}_\gamma\}$ is non-empty. Furthermore, since γX is a compact Hausdorff space, this family is a free ultra $\mathscr{B}$-filter. Thus we define $f: \gamma X \to X(\mathscr{B})$ by: $f(x) = x$ if $x \in X$ and $f(p) = \{B \in \mathscr{B}: p \in \mathscr{B}_\gamma\}$ if $p \in \gamma X - X$. By the above, f is well-defined. It remains to show that f is continuous. We first assert that if $A \in \mathscr{B}$, $f(A_\gamma) \subset A^*$. If $x \in X$ and $x \in A_\gamma$, $f(x) = x \in A^*$. Suppose then that $p \in \gamma X - X$. Then $A \in \{B \in \mathscr{B}: p \in A_\gamma\}$ and so $A \in f(p)$ and $f(p) \in A^*$. Hence, $f(A_\gamma) \subset A^*$ if $A \in \mathscr{B}$. Now, in order to see that f is continuous, we need only consider the case when $q \in \gamma X - X$ and $f(q) \in B^*$ where $B \in \mathscr{B}$. Let $A_1 \in \mathscr{B}$ be such that $f(p) \in A_1^* \subset \mathrm{cl}^* A_1^* \subset B^*$. Then $A_1 \subset f(p)$ and so $X - \mathrm{cl}_X A_1 \notin f(p)$. This means that there exists $C \in \mathscr{B}$ such that $p \in C_\gamma$ and $C \cap X - \mathrm{cl}_X A_1 = \emptyset$. Thus $C \subset \mathrm{cl}_X A_1 \subset \mathrm{cl}^* A_1^* \subset B_*$, so that $C \subset B$ and $C^* \subset B^*$. Then $p \in f(C_\gamma) \subset C^* \subset B^*$, and hence f is continuous. ∎

THEOREM 2.11. *Let X be a rim-compact Hausdorff space. Then there exists a base $\mathscr{B}$ for X satisfying b1, b2, b3 and b4; and so there exists a unique compactification $X(\mathscr{B})$ which is rim-perfect and has $X(\mathscr{B}) - X$ zero-dimensionally embedded in $X(\mathscr{B})$.*

Proof. Let $\mathscr{B} = \{U: U$ is γ-open in $X\}$. It is routine to show that $\mathscr{B}$ has properties b1, b2, b3 and b4. The uniqueness of $X(\mathscr{B})$ follows from Theorem 2.9. ∎

Definition of the Freudenthal compactification. For any rim-compact Hausdorff space X, φX will denote the topologically unique rim-perfect

compactification of X with $\varphi X - X$ zero-dimensionally embedded in φX. This compactification is called the *Freudenthal compactification* of X. The Freudenthal compactification has been denoted by several symbols; e.g. FX in [I], γX in [M$_2$], φX in [H], μX in [ES], and X^* by Freudenthal in [F$_3$].

Relations to other constructions. A base $\mathcal{B}$ of open sets of X is called a π-*compact base* if it satisfies the following conditions: (i) if A, $B \in \mathcal{B}$, then $A \cap B \in \mathcal{B}$ and $A \cup B \in \mathcal{B}$; (ii) if $A \in \mathcal{B}$, then $X - \mathrm{cl}_X A \in \mathcal{B}$; (iii) if $A \in \mathcal{B}$, then $\mathrm{Fr}_X A$ is compact. If $\mathcal{B}$ is a π-compact base for X, then there is a *proximity* $\delta_{\mathcal{B}}$ defined on X by the condition: $S \delta_{\mathcal{B}} T$ if and only if there exists $A \in \mathcal{B}$ such that $\mathrm{cl}_X S \subset A$ and $\mathrm{cl}_X T \subset X - \mathrm{cl}_X A$. It is known that every π-compact base yields a π-*compactification* $v_{\mathcal{B}} X$ such that $v_{\mathcal{B}} X - X$ is zero-dimensionally embedded in $v_{\mathcal{B}} X$ (cf. [Sk$_4$], p. 228).

A π-compact base $\mathcal{B}$ is *full* if $\mathcal{B}$ contains all the sets of the form $U \cap X$ where U is open in $v_{\mathcal{B}} X$ and $\mathrm{Fr}_{v_{\mathcal{B}}X} U \subset X$. There is a one-to-one correspondence between full π-compact bases $\mathcal{B}$ and π-compactifications αX of a rim-compact space X such that $\alpha X - X$ is zero-dimensionally embedded in αX ([Sk], Thm. 6). Thus, if $\mathcal{B}$ is any base for X satisfying b1, b2 and b3, we define $\mathcal{B}$ to be a *full $\mathcal{B}$-base* if $\mathcal{B}$ contains all the sets of the form $U \cap X$ where U is open in $X(\mathcal{B})$ and $\mathrm{Fr}_{X(\mathcal{B})} U \subset X$. The obvious one-to-one correspondence between full π-compact bases and full $\mathcal{B}$-bases yields the following theorem.

THEOREM 2.12. *Let X be rim-compact. There is a one-to-one correspondence between full $\mathcal{B}$-bases and compactifications αX of X such that $\alpha X - X$ is zero-dimensionally embedded in αX.*

Skljarenko observed that if $\mathcal{B}$ denotes the π-compact base consisting of all γ-open subsets of a rim-compact space, then $v_{\mathcal{B}} X$ is the maximal π-compactification of X and $v_{\mathcal{B}} X$ is the minimal perfect compactification of X ([Sk$_4$], p. 231). Consequently, for such a base $\mathcal{B}$ of a rim-compact space X, $\varphi X = v_{\mathcal{B}}(X)$ (cf. (2.10)).

J. R. Isbell defined a proximity relation δ on a rim-compact space X by the following: $S \delta T$ unless the closures of S and T are separated by some compact subset of X. He also showed that δ determines a precompact uniformity μ on X, and thus (X, δ) has a completion $\beta(\mu X)$. It then follows that $\beta(\mu X)$ is a compactification of X ([I], p. 111). Isbell defines $\beta(\mu X)$ to be the Freudenthal compactification of X. In Theorems 30 and 39 of [I], he shows that $\beta(\mu X)$ is a perfect compactification of X and $\beta(\mu X) - X$ is zero-dimensionally embedded in $\beta(\mu X)$. Thus $\beta(\mu X) = \varphi X$ by Theorem 2.12.

K. Morita obtained a compactification γX of a rim-compact space X as the completion γX of the uniformity on X consisting of all finite γ-open covers of X, and he proved that γX is the maximal compactification of X with $\gamma X - X$ zero-dimensionally embedded in X ([M$_2$], Thm. 1). In Theorem 2 of [M$_2$], Morita proves that $\gamma X - X$ is rim-perfect; thus $\gamma X = \varphi X$.

H. Freudenthal gave a method for obtaining a compactification R^* of a rim-compact space R such that $R^* - R$ is zero-dimensionally embedded in R^*. Furthermore, when R is a second countable, rim-compact space, he showed that there is a maximal compactification R^* of R such that $\dim(R^* - R) = 0$. When X is second countable, $\dim(R^* - R) = 0$ is equivalent to $R^* - R$ is zero-dimensionally embedded in R^* ([I], Lemma 34, p. 113); thus, in this case, $R^* = \varphi R$.

Freudenthal's construction was observed to be applicable to other (not necessarily rim-compact) spaces by Fan and Gottesman [FG]. They observed that Freudenthal's construction employing maximal binding families of open sets could be used to construct other compactifications αX of X where $\alpha X - X$ is not necessarily zero-dimensionally embedded in αX.

In [Fr], O. Frink generalized Wallman's method of constructing a compactification of a space X to obtain a compactification of a Tychonoff space X associated with a certain type of normal base for the closed sets of X. Frink called this type of compactification a Wallman compactification of X and, after showing that βX is a Wallman compactification, he asked whether every compactification was a Wallman compactification. In [Na] Njastad showed that the Freudenthal compactification of a rim-compact space is a Wallman compactification. Baayen and van Mill [BvM] showed that the Freudenthal compactification of a locally compact, metrizable space is a regular Wallman compactification; i.e. it is a Wallman compactification of each of its dense subspaces.

3. The weight of φX

If X is a space, $w(X)$ will denote the *weight* of X, i.e. $w(X)$ is the least cardinality for a base for X. We will use $|S|$ to denote the cardinality of a set S.

LEMMA 3.1. *If $\mathscr{B}$ is any base for X, there exists $\mathscr{B}_0 \subset \mathscr{B}$ such that $|\mathscr{B}_0| = w(X)$.*

Proof. Let $\mathscr{U}$ be any base for X such that $|\mathscr{U}| = w(X)$. We may suppose that $|\mathscr{U}| \geqslant \aleph_0$. Let $P = \{(U, V) \in \mathscr{U} \times \mathscr{U}:$ there exists $B \in \mathscr{B}$ with $U \subset B \subset V\}$. Then $|P| \leqslant |\mathscr{U}|$. For each $p = (U, V) \in P$, let $B_p \in \mathscr{B}$ be such that $U \subset B_p \subset V$ and let $\mathscr{B}_0 = \{B_p: p \in P\}$. Then $|\mathscr{B}_0| = w(X)$. It remains to note that $\mathscr{B}_0$ is a base for X. ∎

LEMMA 3.2. *Let $\mathscr{B}$ be a base for X satisfying* b1, b2 *and* b3 *so that* $X(\mathscr{B})$ *is a compactification of X. Let $\mathscr{B}_0$ be a subset of $\mathscr{B}$ such that $\mathscr{B}_0$ satisfies* b1 *and* b2 *and such that $\mathscr{B}_0^* = \{A^*: A \in \mathscr{B}_0\}$ is also a base for $X(\mathscr{B})$. Then if $X(\mathscr{B})$ is rim-perfect, $\mathscr{B}_0$ contains every clopen subset of X.*

Proof. Let U be a clopen subset of X. Let $V = X - U$, $U' = X(\mathscr{B})$

$-\mathrm{cl}^* V$ and $V' = X(\mathscr{B}) - \mathrm{cl}^* U$. Since U and V are disjoint clopen subsets of X and since $X(\mathscr{B})$ is rim-perfect, $\mathrm{cl}^* U \cap \mathrm{cl}^* V = \emptyset$. Also, since $X = U \cup V$, $X(\mathscr{B}) = \mathrm{cl}^* U \cup \mathrm{cl}^* V$ and so $\mathrm{cl}^* U = U'$ and $\mathrm{cl}^* V = V'$, and each of U' and V' is clopen in $X(\mathscr{B})$.

Now, since $\mathscr{B}_0^*$ is a base for $X(\mathscr{B})$ and U' is compact, there exists a finite subfamily $\{A_1, \ldots, A_n\}$ of $\mathscr{B}_0$ such that $U' = A_1^* \cup \ldots \cup A_n^* = \mathrm{cl}^* A_1^* \cup \ldots \cup \mathrm{cl}^* A_n^*$. Then $U = A_1 \cup \ldots \cup A_n = \mathrm{cl}_X A_1 \cup \ldots \cup \mathrm{cl}_X A_n$; and since $\mathscr{B}_0$ satisfies b1 and b2, $V = (X - \mathrm{cl}_X A_1) \cap \ldots \cap (X - \mathrm{cl}_X A_n)$ belongs to $\mathscr{B}_0$, as does $U = X - \mathrm{cl}_X V = X - V$. ∎

LEMMA 3.3. *Let X be a rim-compact Hausdorff space and let $\mathcal{O}$ be the set of all clopen subsets of X. Then $w(\varphi X) \geqslant w(X) \cdot |\mathcal{O}|$.*

Proof. Let $\mathscr{B}$ be the set of all γ-open subsets of X so that $\mathscr{B}$ satisfies b1, b2, b3 and b4, and so that $\varphi X = X(\mathscr{B})$. By Lemma 3.1, $\mathscr{B}$ contains a subset $\mathscr{B}_0$ such that $w(\varphi X) = |\mathscr{B}_0^*|$ (where $\mathscr{B}_0^* = \{A^*: A \in \mathscr{B}_0\}$). We may assume that $\mathscr{B}_0$ satisfies b1 and b2 since any such base may be augmented so as to satisfy b1 and b2 without increasing the cardinality of $\mathscr{B}_0$. By Lemma 3.2, $\mathcal{O} \subset \mathscr{B}_0$ so $|\mathscr{B}_0^*| = |\mathscr{B}_0| \geqslant |\mathcal{O}|$. Since $\mathscr{B}_0$ is a base for X, $|\mathscr{B}_0| \geqslant w(X)$ as required. ∎

LEMMA 3.4. *Let X be a rim-compact Hausdorff space and let $\mathscr{B}'$ be a base for X satisfying b3. Then X has a base $\mathscr{B}$ containing $\mathscr{B}'$ such that $\mathscr{B}$ satisfies b1, b2 and b3 and $|\mathscr{B}'| = |\mathscr{B}|$.*

Proof. Let $\mathscr{B}_1$ be the set of all intersections of finite subfamilies of $\mathscr{B}'$. Then $|\mathscr{B}_1| = |\mathscr{B}'|$, $\mathscr{B}' \subset \mathscr{B}_1$ and $\mathscr{B}_1$ satisfies b1 and b3. Let $\mathscr{B}_2 = \mathscr{B}_1 \cup \{X - \mathrm{cl}_X A: A \in \mathscr{B}_1\} \cup \{X - \mathrm{cl}_X(X - \mathrm{cl}_X A): A \in \mathscr{B}_1\}$. Then $|\mathscr{B}_2| = |\mathscr{B}_1| = |\mathscr{B}'|$ and $\mathscr{B}' \subset \mathscr{B}_1 \subset \mathscr{B}_2$ and $\mathscr{B}_2$ satisfies b2 and b3. Let $\mathscr{B}_3$ be the set of all intersections of finite subfamilies of $\mathscr{B}_2$ and let $\mathscr{B}_4 = \mathscr{B}_3 \cup \{X - \mathrm{cl}_X A: A \in \mathscr{B}_3\} \cup \{X - \mathrm{cl}_X(X - \mathrm{cl}_X A): A \in \mathscr{B}_3\}$. Continue in this fashion and define an increasing sequence of bases $\mathscr{B}' \subset \mathscr{B}_1 \subset \mathscr{B}_2 \subset \ldots \subset \mathscr{B}_i \subset \mathscr{B}_{i+1} \subset \ldots$, so that for each $i \in N$, $|\mathscr{B}_i| = |\mathscr{B}'|$, $\mathscr{B}_i$ satisfies b3, $\mathscr{B}_{2i}$ satisfies b2 and $\mathscr{B}_{2i-1}$ satisfies b1. Finally $\mathscr{B} = \bigcup\{\mathscr{B}_i: i \in N\}$ is the desired base for X.

THEOREM 3.5. *Every rim-compact Hausdorff space X has a compactification αX whose weight is the same as the weight of X and such that $\alpha X - X$ is zero-dimensionally embedded in αX.*

Proof. By Lemma 3.4, there exists a base $\mathscr{B}$ for X such that $|\mathscr{B}| = w(X)$ and $\mathscr{B}$ satisfies b1, b2 and b3. Then $X(\mathscr{B})$ is a compactification of X and $w(X(\mathscr{B})) = |\mathscr{B}| = w(X)$. Also, $X(\mathscr{B}) - X$ is zero-dimensionally embedded in $X(\mathscr{B})$. ∎

Freudenthal ([F_3], Satz 6.3) and Morita ([M_2], Theorem 9) obtained Theorem 3.5 in the case when X is second countable and has only countably many clopen subsets. Skljarenko ([Sk_4], Lemma 14) removed the second countable limitation of Freudenthal's result. Isbell obtained Freudenthal's theorem as a corollary to the following result.

THEOREM 3.6 ([I], p. 114). *Let X be a space such that every compact set has a countable neighborhood base. The following are equivalent*:

(a) *X is rim-compact.*

(b) *X has a compactification αX with $\dim(\alpha X - X) = 0$.*

(c) *X has a compactification $\overset{\bullet}{\alpha} X$ with $\operatorname{ind}(\alpha X - X) = 0$.*

The equivalence of (a) and (b) in Theorem 3.6 was originally shown by Skljarenko in [Sk₁]. Engelking [E₂] later showed that such a compactification αX can be found which preserves the weight of X.

J. Smirnov has given an example of a space X where X is not rim-compact, however $\operatorname{ind}(\alpha X - X) = 0$ ([Sm₁]). Also B. Diamond has investigated 0-*spaces*, i.e. spaces X such that X has a compactification αX with $\operatorname{ind}(\alpha X - X) = 0$ ([Di₁], [Di₂], [Di₃], [Di₄]).

Isbell notes that it is unknown whether every rim-compact space X has a compactification αX with $\dim(\alpha X - X) = 0$ ([I], p. 114). Of course, every such space has a compactification γX with $\gamma X - X$ zero-dimensionally embedded in X and thus every such space has a compactification αX with $\operatorname{ind}(\alpha X - X) = 0$. The reader is referred to Isbell's book or to Diamond's papers for further discussion on the dimension and connectedness properties of the remainder of a compactification.

It is known (e.g. [I], p. 118) that there exists a rim-compact space X that does not have a compactification αX which preserves the dimension of X and at the same time introduces a zero-dimensional remainder.

We now use Theorem 3.5 and obtain a proposition due to L. Zippen ([Z], Thm. 1).

COROLLARY 3.7. *Let X be a rim-compact topologically complete, separable metric space. Then X has a metric compactification αX such that $\alpha X - X$ is a countable set.*

Proof. By Theorem 3.5, there exists a metric compactification Y of X and $Y - X$ is zero-dimensionally embedded in Y. Since X is topologically complete, $Y - X$ is an F_σ-set in Y. Then $Y - X$ is the union of countably many pairwise disjoint compact sets $\{C_i: i \in N\}$, where $\{C_i: i \in N\}$ is a null-sequence in Y. Thus the decomposition $\{C_i: i \in N\} \cup \{x: x \in X\}$ is upper semi-continuous ([Wh], p. 133), and the decomposition space αX is a separable metric space. Clearly $\alpha X - X$ is a countable set. ∎

The proof of the following result is due to Morita ([M₂], Theorem 11).

LEMMA 3.8. *Let X be a rim-compact Hausdorff space and let $\mathcal{O}$ be the set of all clopen subsets of X. Then X has a base $\mathcal{B}$ which satisfies* b3 *and* b4 *and such that $|\mathcal{B}| \leqslant w(X) \cdot |\mathcal{O}|$.*

Proof. By Lemma 3.1, there exists a base $\mathcal{B}_0$ such that $\mathcal{B}_0$ has property b3, $\mathcal{B}_0$ is closed under finite unions and $|\mathcal{B}_0| = w(X)$. For each $B \in \mathcal{B}_0$, let $\mathcal{D}_B = \{C: C$ is clopen in $X - B\}$. Let A be a fixed member of $\mathcal{B}_0$. We wish to show that $|\mathcal{D}_A - \mathcal{O}| \leqslant w(X) \cdot |\mathcal{O}|$. For each $C \in \mathcal{D}_A - \mathcal{O}$, $C \cap \operatorname{Fr}_X A$

is a clopen subset of the compact set $\operatorname{Fr}_X A$. Thus, $\operatorname{Fr}_X A$ has at most $w(X)$ clopen sets (in its relative topology). We define an equivalence relation on the set $\mathscr{D}_A - \mathcal{O}$ by: for C, C' in $\mathscr{D}_A - \mathcal{O}$, C is equivalent to C' if and only if $C \cap \operatorname{Fr}_X A = C' \cap \operatorname{Fr}_X A$. This relation partitions $\mathscr{D}_A - \mathcal{O}$ into at most $w(X)$ equivalence classes. Now let C be a fixed member of $\mathscr{D}_A - \mathcal{O}$ and let E_C be the equivalence class of $\mathscr{D}_A - \mathcal{O}$ which contains C. We assert that $|E_C| \leqslant |\mathcal{O}|$. For if $C' \in E_C$, then $C \cap \operatorname{Fr}_X A = C' \cap \operatorname{Fr}_X A$, and so both $C - C'$ and $C' - C$ are in $\mathcal{O}$. Also $C' = (C \cap C') \cup (C' - C) = (C - (C - C')) \cup (C' - C)$, so that every member of E_C can be obtained from C by using only finitely members of $\mathcal{O}$. Thus, $|E_C| \leqslant |\mathcal{O}|$. This implies that $|\mathscr{D}_A - \mathcal{O}| \leqslant w(X) \cdot |\mathcal{O}|$, as we asserted. It follows from this assertion that $|\mathscr{D}_A| \leqslant w(X) \cdot |\mathcal{O}|$ for each $A \in \mathscr{B}_0$.

Let $\mathscr{B}_1 = \{B \cup C : B \in \mathscr{B}_0 \text{ and } C \in \mathscr{D}_B\}$. Now the member $B \cup C$ of $\mathscr{B}_1$ is open in X and $\operatorname{Fr}_X(B \cup C) \subset \operatorname{Fr}_X B_1$, and so $\mathscr{B}_1$ satisfies b3. Let $\mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_1$; clearly $\mathscr{B}$ satisfies b3 and $\mathscr{B}$ is a base for X. In order to see that $\mathscr{B}$ satisfies b4, let U and V be γ-open subsets of X such that $\operatorname{cl}_X U \cap \operatorname{cl}_X V = \emptyset$. Then, since $\mathscr{B}_0$ is closed under finite unions, there exists $A \in \mathscr{B}$, such that $\operatorname{Fr}_X U \subset A \subset X - \operatorname{cl}_X V$. Let $B = A \cup [(X - A) \cap U]$. Then $U \subset B \subset X - \operatorname{cl}_X V$. Furthermore $(X - A) \cap U$ is clopen in $X - A$ so that $B \in \mathscr{B}_1 \subset \mathscr{B}$. Thus $\mathscr{B}$ has property b4. ∎

THEOREM 3.9. *If X is a rim-compact Hausdorff space and $\mathcal{O}$ is the set of all clopen subsets of X, then $w(\varphi X) = w(X) \cdot |\mathcal{O}|$.*

Proof. By Lemma 3.3, $w(\varphi X) \geqslant w(X) \cdot |\mathcal{O}|$. By Lemma 3.8, there exists a base $\mathscr{B}'$ for X satisfying b3 and b4 and $|\mathscr{B}'| \leqslant w(X) \cdot |\mathcal{O}|$. Now, by Lemma 3.4, there exists a base $\mathscr{B}$ for X containing $\mathscr{B}'$ such that $\mathscr{B}$ satisfies b1, b2 and b3 and $|\mathscr{B}'| = |\mathscr{B}|$. Since $\mathscr{B}'$ has property b4, so also does $\mathscr{B}$. Thus, by Theorem 2.11, $\varphi X = X(\mathscr{B})$. Therefore, $w(\varphi X) \leqslant |\mathscr{B}^*| = |\mathscr{B}| = |\mathscr{B}'| \leqslant w(X) \cdot |\mathcal{O}|$ and so $w(\varphi X) = w(X) \cdot |\mathcal{O}|$. ∎

COROLLARY 3.10. *Let X be a rim-compact, connected space. Then φX is a metric space if and only if X is a separable metric space.*

Proof. This is an immediate consequence of Theorem 3.9. ∎

4. Other properties of φX

A compactification αX of X is said to be *perfect* provided that whenever $p \in \alpha X - X$ and W is an open subset of αX containing p, then $W \cap X$ is not the union of disjoint, relatively open sets U and V such that $p \in \operatorname{cl}_{\alpha X} U \cap \operatorname{cl}_{\alpha X} V$. Recall that αX is rim-perfect provided that whenever U and V are γ-open subsets of X and $\operatorname{cl}_X U \cap \operatorname{cl}_X V = \emptyset$, then $\operatorname{cl}_{\alpha X} U \cap \operatorname{cl}_{\alpha X} V = \emptyset$. It is easy to verify that every perfect compactification is rim-perfect.

EXAMPLE 4.1. There exists a compactification of the plane, R^2, which is

rim-perfect but not perfect. Let $X = \{\varrho e^{i\theta} \in C: 1 < \varrho < 2$ and $0 < \theta < 2\pi\}$ and let Y be the closure of X in C. Note that X is homeomorphic to R^2 and Y is a closed annulus in C. Let $U = \{\varrho e^{i\theta} \in X: 0 < \theta < \pi/2\}$ and $V = \{\varrho e^{i\theta} \in X: -\pi/2 < \theta < 0\}$. Then U and V are open subsets of X whose closures in X are disjoint. However, $\mathrm{cl}_Y U \cap \mathrm{cl}_Y V \neq \emptyset$ and so Y is not perfect. Now if S and T are γ-open subsets of X and $\mathrm{cl}_X S \cap \mathrm{cl}_X T = \emptyset$, at least one of $\mathrm{cl}_X S$ or $\mathrm{cl}_X T$ is compact. Thus $\mathrm{cl}_Y S \cap \mathrm{cl}_Y T = \emptyset$ and Y is rim-perfect. ∎

Te compactification in the above example is not the Freudenthal compactification, since the remainder is not zero-dimensional. The Freudenthal compactification of R^2 is in fact the one-point-compactification S^2 of R^2 (cf. (4.8)). It follows from the next theorem that no such compactification as that in Example 4.1 can be found which has a zero-dimensionally embedded remainder.

THEOREM 4.2. *Let X be rim-compact and let αX be a compactification of X where $\alpha X - X$ is zero-dimensionally embedded in αX. Then αX is rim-perfect if and only if αX is perfect.*

Proof. We need only show that if αX is a rim-perfect compactification of X, then αX is perfect. To that end let $p \in \alpha X - X$ and suppose that W is an open subset of αX containing p such that $W \cap X$ is the union of disjoint open sets U_1 and U_2 and $p \in \mathrm{cl}_{\alpha X} U_1 \cap \mathrm{cl}_{\alpha X} U_2$. Since αX has a zero-dimensionally embedded remainder, there exists an open set S of Y containing p so that $\mathrm{cl}_{\alpha X} S \subset W$ and $\mathrm{Fr}_{\alpha X} S \subset X$. Thus $V_1 = S \cap U_1$ and $V_2 = S \cap U_2$ are each γ-open subsets of X. Furthermore $\mathrm{cl}_X V_1 \cap \mathrm{cl}_X V_2 \subset U_1 \cap U_2$, $\mathrm{cl}_X V_1 \subset X - V_2$ and $\mathrm{cl}_X V_2 \subset X - V_1$. Thus $\mathrm{cl}_X V_1 \cap \mathrm{cl}_X V_2 \subset [(U_1 \cup U_2) \cap (X - U_1) \cap (X - U_2)] = \emptyset$. By the rim-perfectness of αX, $\mathrm{cl}_{\alpha X} V_1 \cap \mathrm{cl}_{\alpha X} V_2 = \emptyset$. Of course this is a contradiction, so that αX is perfect. ∎

THEOREM 4.3. *Let X be rim-compact. Then φX is the smallest perfect compactification of X, and φX is the largest compactification such that $\varphi X - X$ is zero-dimensionally embedded in φX.*

Proof. Let γX be any perfect compactification of X. Now γX is rim-perfect, and, by Theorem 2.10, $\gamma X \geqslant \varphi X$ as required. On the other hand, suppose that αX is a compactification of X and $\alpha X - X$ is zero-dimensionally embedded in αX. Then, again by Theorem 2.10, $\varphi X \geqslant \alpha X$. ∎

The perfectness of φX implies the following maximality condition.

THEOREM 4.4. *Let X be rim-compact. Then φX is the largest compactification of X such that $\varphi X - X$ is totally disconnected.*

Proof. This follows from Corollary 1 of [R]. ∎

In (3.3) of [Mc], it was shown that φX is the largest compactification of X with $\mathrm{ind}(\varphi X - X) = 0$. Skljarenko proved that φX is the largest compactification such that $\varphi X - X$ is punctiform, i.e. compact components of $\varphi X - X$ are singletons.

Perfect compactifications have been studied extensively (e.g. [I], p. 114; [S₁], [S₂] and [S₃]) and several equivalent conditions for the perfectness of a compactification of a (not necessarily rim-compact) space are known. Among these results are the following:

THEOREM 4.5. *Let* αX *be a compactification of a completely regular space. The following are equivalent*:

(a) αX *is a perfect compactification of* X.

(b) *If a closed set* C *of* X *separates* A *and* B *in* X, *then the closure of* C *in* αX *separates* A *and* B *in* αX.

(c) *The extension of the map of* αX *into the Stone–Čech compactification of* X *is monotone.*

(d) *For any pair of disjoint open sets* U *and* V *of* X, $\mathrm{O}\langle U \cup V \rangle = \mathrm{O}\langle U \rangle \cup \mathrm{O}\langle V \rangle$, *where, if* P *is open in* X, $\mathrm{O}\langle P \rangle$ *is the set* $\alpha X - \mathrm{cl}_{\alpha X}(X - P)$.

The proofs of these equivalences may be found in [I] and [Sk₂].

THEOREM 4.6. ([No₁]). *Let* X *be a locally compact space and let* $\mathscr{S}$ $= \{\alpha X : \alpha X$ *is a compactification of* X *and* $\alpha X - X$ *is a finite set*$\}$. *Then* φX *is the least upper bound of* $\mathscr{S}$.

Proof. The set of all compactifications of X is a complete upper semi-lattice ([C], p. 13) and so $\alpha X = \mathrm{lub}\ \mathscr{S}$ exists. Moreover, by Theorem 2.11, φX is greater than every member of $\mathscr{S}$; thus $\varphi X \geqslant \alpha X$. We will show that $\varphi X = \alpha X$ by showing that αX is rim-perfect. Suppose to the contrary that there exist γ-open subsets U_1 and U_2 of X such that $\mathrm{cl}_X U_1 \cap \mathrm{cl}_X U_2 = \emptyset$ and $\mathrm{cl}_{\alpha X} U_1 \cap \mathrm{cl}_{\alpha X} U_2 \neq \emptyset$. Let $U_3 = X - \mathrm{cl}_X(U_1 \cup U_2)$ and let $Y = X \cup \{p_1, p_2, p_3\}$ where $\{p_1, p_2, p_3\}$ are distinct points not in X. We define a base for a topology on Y as follows. Let $V \subset Y$. Then V is a basic open set if either (i) $V \subset X$ and V is open in X, or (ii) there exists $i \in \{1, 2, 3\}$ such that $p_i \in V$, $V \cap X \subset U_i$ and $(\mathrm{cl}_X U_i) - V$ is compact. Then Y with this topology is a three-point compactification of X, and so $Y \in \mathscr{S}$. Thus there exists a continuous surjection $h: \alpha X \to Y$ where $h|_X$ is the identity. This means that $\mathrm{cl}_{\alpha X} h^{-1}(U_1) \cap \mathrm{cl}_{\alpha X} h^{-1}(U_2) = \emptyset$ and this is a contradiction. Thus αX is rim-perfect. By Theorem 2.10, $\alpha X \geqslant \varphi X$, so that $\varphi X = \alpha X$. ∎

In the following theorem, X_∞ denotes the one-point-compactification of locally compact Hausdorff space X.

THEOREM 4.7. *Let* X *be a non-compact connected, locally connected and locally compact space. The following are equivalent*:

(a) $\varphi X = X_\infty$.

(b) *If* αX *is any compactification of* X, $\alpha X - X$ *is connected.*
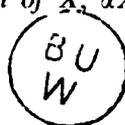
(c) *If K is any compact subset of X, $X - K$ has exactly one component whose closure in X is not compact.*

Proof. (a) implies (b). Suppose $\varphi X = X_\infty$ and suppose that αX is a compactification of X and that $\alpha X - X$ is the union of disjoint closed sets H and K. Let γX be the decomposition of αX whose non-degenerate elements are $\{H\}$ and $\{K\}$. Then γX is a compactification of X and $\gamma X - X$ is a doubleton. By Theorem 2.10, there is a continuous map $f\colon \varphi X \to \gamma X$. But this means $\varphi X \neq X_\infty$ since $\varphi X - X$ must then contain at least two members. Thus $\alpha X - X$ is connected.

(b) implies (c). Suppose that K is a compact subset of X and $X - K$ has at least two components whose closures in X are not compact. It then follows from the connectedness, local connectedness and local compactness of X that there exists an open set V of X containing K such that $\mathrm{cl}_X V$ is compact and $X\text{-}\mathrm{cl}_X V$ has only finitely many components $Q_1, \ldots, Q_n$ with $\mathrm{cl}_X Q_i$ not compact. Furthermore, by our supposition, $n \geqslant 2$, and so $H = \mathrm{cl}_X Q_1$ and $K = \mathrm{cl}_X Q_2 \cup \ldots \cup \mathrm{cl}_X Q_n$ are non-empty and non-compact. We will now define a compactification γX of X such that $\gamma X - X$ is a doubleton $\{p, q\}$ and $H \cup \{p\}$ and $K \cup \{q\}$ are each compact. Let $p, q \notin X$, $p \neq q$ and define a set $U \subset \gamma X = X \cup \{p, q\}$ to be a basic open set if (i) U is an open subset of X; or (ii) $p \in U$ and U is an open subset of Q_1 and $H - U$ is compact; or (iii) $q \in U$ and U is an open subset of $Q_2 \cup \ldots \cup Q_n$ and $K - U$ is compact. (Essentially, we have merely one-point-compactified each of the sets H and K.) Then γX is a compactification of X and $\gamma X - X$ is not connected. Thus (b) implies (c).

(c) implies (a). Suppose $\varphi X \neq X_\infty$, i.e. suppose $\varphi X - X$ is non-degenerate. Since $\varphi X - X$ is closed, we may suppose that $\varphi X - X$ is the union of non-empty disjoint compact sets L and M, and so we may choose open sets U and V of φX containing L and M such that $\mathrm{cl}^* U \cap \mathrm{cl}^* V = \emptyset$. Let $K = \varphi X - (U \cup V)$; then K is compact. Also since X is connected, each of U and V contains a component of $X - K$ whose closure is not compact. Since $K = \varphi X - (U \cup V)$, K is not compact. Thus (c) implies (a). ∎

COROLLARY 4.8. *Let X and Y each be non-compact, locally compact, locally connected, connected spaces and let $Z = X \times Y$. Then $\varphi Z = Z_\infty$.*

Proof. We will show that if K is a compact subset of Z, then $Z - K$ has exactly one component with a non-compact closure in Z. Since each projection, $\pi_X\colon Z \to X$ and $\pi_Y\colon Z \to Y$ is continuous, each set $\pi_X(K)$ and $\pi_Y(K)$ is compact and there exist open and connected sets U of X and V of Y containing $\pi_X(K)$ and $\pi_Y(K)$ respectively such that $\mathrm{cl}_X U$ and $\mathrm{cl}_Y V$ are compact. Then every component of $Z - K$ whose closure is not compact must meet $\big(X \times (Y - \mathrm{cl}_Y V)\big) \cup \big((X - \mathrm{cl}_X U) \times Y\big)$. Since this latter set is connected, there is only one such component. By Theorem 4.7, $\varphi Z = Z_\infty$. ∎

K. Nowiński [No$_1$] has shown that if X and Y are connected, non-compact, locally compact, metacompact spaces, $\varphi(X \times Y) = (X \times Y)_\infty$.

COROLLARY 4.9. *Let $n > 1$ be an integer. Then $\varphi(R^n) = S^n$.*

THEOREM 4.10. *Let X be a connected, rim-compact space. Then φX is locally connected if and only if X is locally connected.*

Proof. Suppose that X is locally connected. Then $\varphi X - X$ is zero-dimensionally embedded in φX; in particular, $\varphi X - X$ does not contain a non-degenerate continuum. By Propositions 2.8 and 3.4 of [GMc], φX is locally connected.

On the other hand, suppose that φX is locally connected. Let $\mathscr{D}$ be a base for the topology of φX consisting of open connected subsets of φX. Then if $U \in \mathscr{D}$, $U \cap X$ is connected by Theorem 4.2. Hence X is locally connected. ∎

COROLLARY 4.11. *Let X be rim-compact. Then φX is locally connected if and only if X is locally connected and has only finitely many components.*

Proof. Suppose that φX is locally connected. Then by the above theorem, X is locally connected. Thus if Q is a component of X, Q is clopen in X and so the closure, cl* Q, of Q in φX is an open subset of φX. Since a compact, locally connected space can only have finitely many components, X can have only finitely many components.

On the other hand, suppose that X is locally connected and $Q_1, \ldots, Q_n$ are the (finitely many) components of X. Then each Q_i is rim-compact, connected and locally connected and so φQ_i is locally connected. Then since φX is homeomorphic to the disjoint union of the φQ_i's, φX is locally connected. ∎

J. de Groot and R. H. McDowell [GMc] proved that a locally connected, rim-compact space X has a locally connected compactification αX if and only if X has at most finitely many compact components. Their proof employed the fact that any non-compact component Q of such a space has a locally connected Freudenthal compactification φX (e.g. Theorem 4.10). The compactification αX was obtained as a certain quotient of the union of the φQ's.

THEOREM 4.12. *Let X be a connected, rim-compact space. Then φX is a locally connected metric space if and only if X is a locally connected, separable, metric space.*

Proof. By Theorem 3.8, φX is a metric space if and only if X is a separable metric space, and by Theorem 4.10, φX is locally connected if and only if X is locally connected. ∎

A metric r for a space X is a *Property S metric* for X if for any $\varepsilon > 0$, X is the union of finitely many connected sets of r-diameter less than ε ([W], p.

20). A compact locally connected, connected metric space is called a *Peano continuum*.

COROLLARY 4.13. *Let X be a rim-compact connected, locally connected, separable metric space. Then φX is a Peano continuum, and if d is any metric for φX, $r = d|_X$ is a Property S metric for X.*

Proof. By Theorem 4.12, φX is a Peano continuum. Let d' be any metric for φX and let $r = d|_X$. To see that r is a Property S metric for X, let $\varepsilon > 0$. Since φX is locally connected, there exists a finite collection $\{V_1, \ldots, V_n\}$ of open, connected sets of φX such that $\{V_1, \ldots, V_n\}$ covers φX and the d-diameter of each V_i is less than ε. Then since φX is perfect, $U_i = V_i \cap X$ is connected for each i, $1 \leqslant i \leqslant n$ (cf. (4.2)). Thus $\{U_1, \ldots, U_n\}$ is a cover of X by connected sets of r-diameter less than ε. ■

EXAMPLE 4.14. Here αX is a locally connected metric compactification of a locally connected, connected, locally compact metric space X; and if d is any metric for αX, $d|_X$ is not a Property S metric for X.

Let $Y_0 = \{(x, y) \in R^2 : x \geqslant 0$ and $y \in \{0, 1\}\}$ and for $k \in N$, let $Y_k = \{(x, y) \in R^2 : 2k-1 \leqslant x \leqslant 2k$ and $0 \leqslant y \leqslant 1\}$ and let $L_k = \{(x, y) \in R^2 : 2k \leqslant x < 2k+1$ and for some $j \in N$, $1 \leqslant j < 2^k$, $y = j2^{-k}\}$. Let αX be the closure of $Y = Y_0 \cup (\bigcup \{Y_n : n \in N\}) \cup (\bigcup \{L_n : n \in N\})$ in $R_\infty \times R_\infty$; $\alpha X = Y \cup \{(x, y) : x = \infty$ and $0 \leqslant y \leqslant 1\}$.

Let $Z_R = \{(x, y) \in R^2 : x = 2^{2k+1}$ and for some $j \in N$, $1 < j < k$, $y = j2^{-k}\}$. Now $X = Y - \bigcup \{Z_n : n \in N\}$ is connected and locally connected, and αX is a locally connected metric compactification of X. Moreover, if d is any metric for αX and $p = (\infty, 2^{-1})$ and V is any connected open subset of αX containing p but missing $Y_0 \cup \{(\infty, 0), (\infty, 1)\}$, then $V \cap X$ has infinitely many components. Thus $d|_X$ is not a Property S metric for X.

It follows from perfectness of φX that if γX is any locally connected compactification of a rim-compact space such that $\gamma X - X$ is zero-dimensionally embedded in γX and whenever V is an open connected set in γX, $V \cap X$ is also connected, then $\gamma X = \varphi X$ (cf. Theorem 38 of [I], p. 114). Thus we may obtain φX as follows. The compactification φX of X adds $2^n - 1$ points to each set L_n and adds one point q in place of the set $\{\infty\} \times R$ (the reader may envision each set L_n sticking up from R^2 and compactified by the addition of $2^n - 1$ points). Of course the diameter of the L_i's in φX goes to zero as i approaches ∞. ■

EXAMPLE 4.15. In this example X is a non-locally compact, locally connected, rim-compact subset of R^2, $\varphi X = \mathrm{cl}_R X$ and φX is a Peano continuum, i.e. φX is a locally connected, connected metric space.

Let $Y = \{(x, y) \in R^2 : x^2 \leqslant 1$ and $y^2 \leqslant 1\}$, let $Z = \{(x, y) \in Y : x \in Q$ and $y \in Q\}$ and let $X = Y - Z$. Then each of X and Y are locally connected, connected metric spaces and Y is a compactification of X. Furthermore, since $Y - X = Z$ is countable, $Y - X$ is zero-dimensionally embedded in Y and

X is rim-compact. It is well known that R^2 cannot be separated by any countable set (cf. Thm. IV 4. of [HW]), and so if V is any connected and open subset of Y, $V \cap X$ is connected. Thus $\varphi X = Y$. ∎

The following corollary to a result of Nowiński yields many examples of the Freudenthal compactification of spaces.

THEOREM 4.16. ([No₁]). *Let* M *be a compact manifold of dimension greater than* 1, *and let* Z *be a closed zero-dimensional subset of* M. *Then* $\varphi(M - Z) = M$.

A connected space X is said to be *unicoherent* (*γ-unicoherent*) provided that whenever H and K are closed (*γ-closed*) subsets of X and $X = H \cup K$, then $H \cap K$ is also connected.

THEOREM 4.17 ([CD]). *Let* X *be a locally connected, connected, rim-compact Hausdorff space. Then:*

(i) X *is γ-unicoherent if and only if* φX *is unicoherent.*

(ii) *If* X *is also a locally compact separable metric space, the following are equivalent:*

(a) φX *is unicoherent.*

(b) X *has a uncoherent compactification* αX *and there exists a monotone map* h: $\alpha X \to \varphi X$ *such that* $h|_X$ *is the identity.*

One can also show that φX is the smallest unicoherent compactification of a connected rim-compact, locally connected, γ-unicoherent space. Proofs of the above results depend upon the perfectness of φX.

Several authors, most notably B. J. Ball and R. B. Sher, have utilized the Freudenthal compactification in their study of shape or homotopy properties (see for example [B₁], [B₂], [BS], [Sh₁], [Sh₂], or [Sh₃]).

K. Morita characterized the Freudenthal compactification φX in the case where $\varphi X - X$ has exactly m points and X is a non-compact, locally compact, locally connected, connected metric space as follows. Let $m \geqslant 1$ and Q_m be the union of m closed segments (a_i, a_0), $i = 1, 2, \ldots, m$, each having only one point a_0 in common, and let $P_m = Q_m - \{a_1, \ldots, a_m\}$. Let X be a metric space. Then there exists a closed continuous surjection f: $P_m \to X$ if and only if X is a non-compact, locally compact, locally connected, connected separable metric space and $\varphi X - X$ consists of at most m points. In this case there exists a continuous extension φf: $Q_m \to \varphi X$.

5. Extensions of maps and subspaces

The first of two topics in this section is concerned with extensions of maps to $\mathscr{B}$-compactifications.

LEMMA 5.1. *Let* f: $Y \to X$ *be a continuous map and let* $\mathscr{B}'$ *and* $\mathscr{B}$ *be bases for* Y *and* X *respectively, satisfying:*

(a) $\mathscr{B}'$ *has property* **b1** (*so that* $Y(\mathscr{B}')$ *is an extension of* Y);

(b) $\mathscr{B}$ *has properties* **b1, b2** *and* **b3**;

(c) *for each* $A \in \mathscr{B}$, $f^{-1}(A) \in \mathscr{B}'$.

Then f *has a* (*unique*) *continuous extension* $\hat{f}: Y(\mathscr{B}') \to X(\mathscr{B})$.

Proof. Let $\mathscr{F}$ be a free ultra $\mathscr{B}'$-filter, i.e. let $\mathscr{F} \in Y(\mathscr{B}') - Y$ and define $\mathscr{F}_X = \{B \in \mathscr{B}: f^{-1}(B) \in \mathscr{F}\}$. We note that $\mathscr{F}_X$ is non-empty; for if $A_1^*, \ldots, A_n^*$ is a finite open cover of the compact space $X(\mathscr{B})$, then $X = A_1 \cup \ldots \cup A_n$ and so $Y = f^{-1}(A_1) \cup \ldots \cup f^{-1}(A_n)$. Now each set $f^{-1}(A_i)$ belongs to $\mathscr{B}'$, so that by Lemma 2.1, some such set $f^{-1}(A_j)$ belongs to $\mathscr{F}$. Of course this means that $A_j \in \mathscr{F}_X$. We now assert that the adherence, $\mathrm{ad}_{X(\mathscr{B})} \mathscr{F}_X$, of $\mathscr{F}_X$ is a singleton member of X. To see this, suppose that $x, y \in \mathrm{ad}_{X(\mathscr{B})} \mathscr{F} = \bigcap \{\mathrm{cl}^* B: B \in \mathscr{F}_X\}$ and $x \neq y$. Since $\mathscr{B}^*$ is base for the regular space $X(B)$ there exist $C, D \in \mathscr{B}$ such that C^* and D^* have disjoint closures in $X(\mathscr{B})$, and $x \in C^*$ and $y \in D^*$. Now $C_1 = X - \mathrm{cl}_X D$ and $D_1 = X - \mathrm{cl}_X C$ belong to $\mathscr{B}$, and so by Lemma 2.1, either $f^{-1}(C_1)$ or $f^{-1}(D_1)$ belongs to $\mathscr{F}$ and consequently, either C_1 or D_1 belongs to $\mathscr{F}_X$. In the former case y is not a member of the adherence of $\mathscr{F}_X$, and in the latter case x is not a member of the adherence of $\mathscr{F}_X$. Thus $\mathrm{ad}_{X(\mathscr{B})} \mathscr{F}_X$ is a singleton; call the point $y_{\mathscr{F}}$.

We define $\hat{f}: Y(\mathscr{B}') \to X(\mathscr{B})$ as follows: if $y \in Y$, let $\hat{f}(y) = f(y)$; if $\mathscr{F} \in F(\mathscr{B}')$, let $\hat{f}(\mathscr{F}) = y_{\mathscr{F}}$. We now assert that if $A \in \mathscr{B}$, $\hat{f}(f^{-1}(A)^*) \subset \mathrm{cl}^* A$. Thus we need to show that if $\mathscr{F} \in F(\mathscr{B}') \cap f^{-1}(A)^*$, then $\hat{f}(\mathscr{F}) \in \mathrm{cl}^* A$. Now $\mathscr{F} \in F(\mathscr{B}') \cap f^{-1}(A)^*$ means that $f^{-1}(A) \in \mathscr{F}$ and so $A \in \mathscr{F}_X$. But $\hat{f}(\mathscr{F}) = y_{\mathscr{F}} \in \mathrm{ad}_{X(\mathscr{B})} \mathscr{F}_X$, so that $\hat{f}(\mathscr{F}) \in \mathrm{cl}^* A$ as required.

It now follows from the regularity of $X(\mathscr{B})$ and the fact that $\hat{f}(f^{-1}(A)^*) \subset \mathrm{cl}^* A$ for $A \in \mathscr{B}$, that $\hat{f}$ is continuous. Since $X(\mathscr{B})$ is a Hausdorff space, $\hat{f}: Y(\mathscr{B}') \to X(\mathscr{B})$ is unique. ∎

THEOREM 5.2. *Let* $f: Y \to X$ *be a continuous map where* X *and* Y *are rim-compact. If* $f^{-1}(G)$ *is* γ-*open in* Y *for each* γ-*open subset* G *of* X, *then there exists a continuous extension* $\varphi f: \varphi Y \to \varphi X$.

Proof. Let $\mathscr{B}'$ and $\mathscr{B}$ be the set of all γ-open subsets of Y and X respectively. Then $\varphi Y = Y(\mathscr{B}')$ and $\varphi X = X(\mathscr{B})$, and the theorem follows from Lemma 5.1. ∎

COROLLARY 5.3 ([M₅]). *Let* Y *and* X *be rim-compact and let* $f: Y \to X$ *be a closed continuous map with compact boundaries of point inverses. Then there exists a continuous extension* $\varphi f: \varphi Y \to \varphi X$.

Proof. By Lemma 3 of [M₄], if A is a γ-closed subset of X, $\mathrm{Fr}\, f^{-1}(A)$ is compact. Thus if G is a γ-open subset of X, $\mathrm{Fr}\, f^{-1}(G)$ is a subset of the compact set $\mathrm{Fr}\, f^{-1}(\mathrm{Fr}\, G) \cup \mathrm{Fr}\, f^{-1}(Y-G)$ and hence $f^{-1}(G)$ is γ-open in Y. ∎

COROLLARY 5.4 ([M₃]). *Let* $f: Y \to X$ *be a closed continuous map between rim-compact metric spaces. Then there exists a continuous extension* $\varphi f: \varphi Y \to \varphi X$.

Proof. By Theorem 1 of [St], point inverses of f have compact boundaries. The corollary now follows from Corollary 5.3. ∎

COROLLARY 5.5 ([ES]). *Let* $f: Y \to X$ *be a perfect map between rim-compact spaces. Then there exists a continuous extension* $\varphi f: \varphi Y \to \varphi X$.

Proof. This is an immediate consequence of Corollary 5.3. ∎

The following propositions are included, without proofs, for completeness.

THEOREM 5.6 ([No$_1$]). *Let* X *be a rim-compact, metacompact space, let* Y *be a compact space, and let* $f: X \to Y$ *be a continuous closed map. Then there exists a continuous extension* $\varphi f: \varphi X \to Y$.

THEOREM 5.7 ([No$_1$]). *Let* $f: X \to Y$ *be a continuous closed mapping between locally compact, metacompact spaces. Then there exists a continuous extension* $\varphi f: \varphi X \to \varphi Y$.

K. Morita proved that every continuous closed map f of a locally compact, paracompact space X onto a locally compact space Y had a paracompact range, and thus there exists a continuous extension $\varphi f: \varphi X \to \varphi Y$ ([M$_4$]). Nowiński showed by example that Theorem 5.7 was an essential generalization of Morita's result. We include here an example of a closed map $f: X \to Y$ where Y is compact, X is not metacompact, and f does not admit an extension $\varphi f: \varphi X \to \varphi Y$. Of course, point inverses of Y do not have compact boundaries.

EXAMPLE 5.8 ([No$_1$]). Let Ω_0 be the space of all countable ordinals, let $Y = [0, 1]$ and let $X = \Omega_0 \times Y$. Let $f: X \to Y$ be the projection. Then X is locally compact, the map f is closed, and point inverses are nowhere dense in X and they are non-compact countably compact. Now X is not metacompact since the closed subset $\Omega_0 \times \{0\}$ of X is not metacompact. Also X_∞ is perfect and $X_\infty - X$ is zero-dimensionally embedded in X_∞. Thus $\varphi X = X_\infty$. Clearly f does not admit a continuous extension to X_∞.

THEOREM 5.9 ([No$_1$]). *Let* X *be a locally compact, metacompact space and let* αX *be a compactification of* X. *If it is not true that* $\alpha X \geqslant \varphi X$, *then there exists a compact space* Y *and a continuous closed surjection* $f: X \to Y$ *that does not admit a continuous extension to* αX.

The second topic in this section is concerned with $\mathscr{B}$-compactifications of subspaces of a space.

THEOREM 5.10. *Let* Y *be a* γ-closed subset of X and let $\mathscr{B}$ be a base for X satisfying b1 and b2. Let $\mathscr{B}' = \{A \cap Y: A \in \mathscr{B}\}$. *Then* $\mathscr{B}'$ *satisfies* b1, *and if* $\mathscr{B}$ *satisfies* b3 (*and* b4), $\mathscr{B}'$ *also satisfies* b3 (*and* b4). *Furthermore* $Y(\mathscr{B}')$ *is homeomorphic to* $\mathrm{cl}_{X(\mathscr{B})} Y$.

Proof. Clearly, $\mathscr{B}'$ satisfies b1 so that $Y(\mathscr{B}')$ is an extension of Y. We first define a function f on $Y(\mathscr{B}')$ as follows: $f(p) = p$, if $p \in Y$, and $f(p) = \{A \in \mathscr{B}: A \cap Y \in p\}$ if $p \in F(\mathscr{B}')$. We assert that f is a well-defined function

into $X(\mathscr{B})$. Clearly if $p \in Y$, $f(p) \in X \subset X(\mathscr{B})$. Suppose that $p \in F(\mathscr{B}')$ and that A, $B \in f(p)$. Then $(A \cap Y) \cap (B \cap Y) \in p$ and so $(A \cap B) \cap Y \in p$, and thus $A \cap B \in f(p)$. Now suppose $C \in \mathscr{B}$ and $A \in f(p)$ and $A \subset C$. Then $A \cap Y \subset C \cap Y$, so that $C \in f(p)$ and $f(p)$ is a $\mathscr{B}$-filter. We show that $f(p)$ is a free $\mathscr{B}$-filter by showing that if $x \in X$, then x is not an adherent point of $f(p)$. Since Y is γ-closed, there exists $A \in \mathscr{B}$ so that $x \in A$ and $(\text{cl}_X A) \cap Y = \emptyset$. But $(X - \text{cl}_X A) \cap Y = Y$ and Y belongs to p, so that $X - \text{cl}_X A \in f(p)$. Hence $f(p)$ is free.

Finally we assert that $f(p)$ is an ultra $\mathscr{B}$-filter. Since $\mathscr{B}$ satisfies b1 and b2, we may employ Lemma 2.2 and show that if $f(p)$ contains either A or $X - \text{cl}_X A$ whenever $A \in \mathscr{B}$, then $f(p)$ is an ultra $\mathscr{B}$-filter. Suppose then that $A \in \mathscr{B}$ and $A \notin f(p)$, i.e. $A \cap Y \notin p$. Then by Lemma 2.1, the $\mathscr{B}$-filter $f(p)$ contains a set of the form $B \cap Y$, $B \in \mathscr{B}$, where $(A \cap Y) \cap (B \cap Y) = \emptyset$. Then $Y \cap \text{cl}_X(A \cap B) = \text{Fr}_X Y \cap \text{cl}_X(A \cap B)$ is compact. Since p is free, for each $y \in Y \cap \text{cl}_X(A \cap B)$, there exist A_y, $B_y \in \mathscr{B}$ so that $y \in A_y$, $B_y \cap Y \in p$ and $A_y \cap B_y = \emptyset$. From the compactness, there exist $y_1, \ldots, y_n \in Y \cap \text{cl}_X(A \cap B)$ with $Y \cap \text{cl}_X(A \cap B) \subset A_{y_1} \cup \ldots \cup A_{y_n}$. Now $B \cap B_{y_1} \cap \ldots \cap B_{y_n} \cap Y \subset (B \cap Y) - (A_{y_1} \cup \ldots \cup A_{y_n}) \subset (B \cap Y) - (Y \cap \text{cl}_X(A \cap B)) = (X - \text{cl}_X A) \cap B \cap Y$. Therefore $(X - \text{cl}_X A) \cap B_{y_1} \cap \ldots \cap B_{y_n} \cap B = B_{y_1} \cap \ldots \cap B_{y_n} \cap B$ belongs to p. But this means $(\text{cl}_X A) \cap Y \in p$ and so $X - \text{cl}_X A \in p$. Thus $f(p)$ is an ultra $\mathscr{B}$-filter.

We will now show that for each $B \in \mathscr{B}$, $f^{-1}(B^*) = (B \cap Y)^*$. Let $p \in (B \cap Y)^* - Y$ so that $B \cap Y \in p$. Then $B \in f(p)$, so that $f(p) \in B^*$. Of course this means $p \in f^{-1}(B^*)$. On the other hand, let $p \in f^{-1}(B^*) - Y$. Then $B \in f(p)$ and so $B \cap Y \in p$. Thus $p \in (B \cap Y)^*$ as required. Hence, if $B \in \mathscr{B}$, $f^{-1}(B^*) = (B \cap Y)^*$ and $f: Y(\mathscr{B}') \to X(\mathscr{B})$ is continuous. (Note that $f(Y(\mathscr{B}')) = \text{cl}^* Y$ and so f is a continuous map of $Y(\mathscr{B}')$ onto $\text{cl}^* Y$.)

We will now define a map g of $\text{cl}^* Y$ onto $Y(\mathscr{B}')$ so that g is continuous and $g = f^{-1}$ and $f = g^{-1}$. Define $g: \text{cl}^* Y \to Y(\mathscr{B}')$ as follows: $g(q) = q$ if $q \in Y$; and $g(q) = \{B \cap Y: B \in q\}$, if $q \in \text{cl}^* Y - Y$. By arguments similar to those above, we can show that $g(q)$ is a free ultra $\mathscr{B}'$-filter whenever $q \in \text{cl}^* Y - Y$, i.e. whenever q is a free ultra $\mathscr{B}$-filter in $\text{cl}^* Y$. We now argue that for each $B \in \mathscr{B}$, $g^{-1}((B \cap Y)^*) \overset{\bullet}{=} B^* \cap \text{cl}^* Y$. Let $q \in \text{cl}^* Y - Y$. Then $g(q) \in (B \cap Y)^*$ and so $B \cap Y \in g(q)$. Thus $B \in q$ and $q \in B^* \cap \text{cl}^* Y$. Now let $q \in B^* \cap (\text{cl}^* Y - Y)$. Then $B \in q$ and $B \cap Y \in g(q)$, so that $g(q) \in (B \cap Y)^*$. Hence $g^{-1}((B \cap Y)^*) = B^* \cap \text{cl}^* Y$.

This means g is continuous. Clearly, $g = f^{-1}$ and $f = g^{-1}$, by definition. Hence f is a homeomorphism of $Y(\mathscr{B}')$ onto $\text{cl}^* Y$.

If $\mathscr{B}$ also satisfies b1, b2 and b3, $\mathscr{B}'$ also satisfies b3 since Y is γ-closed. Suppose then that $\mathscr{B}$ satisfies b1, b2, b3, and b4; we wish to show that $\mathscr{B}'$ also satisfies b4. To that end, let S and T be open subsets of Y so that $\text{Fr}_Y S$ and $\text{Fr}_Y T$ are compact and $\text{cl}_Y S \cap \text{cl}_Y T = \emptyset$. Then $\text{Fr}_Y S = \text{Fr}_Y S \cup (\text{cl}_X S \cap \text{Fr}_X Y)$ which is compact. Similarly, $\text{Fr}_X T$ is compact.

Since $\mathscr{B}$ satisfies b3, there exist γ-open sets U and V in X so that $\mathrm{cl}_X S \subset U \subset \mathrm{cl}_X U \subset V \subset X - \mathrm{cl}_X T$. Then by b4, there exists $B \in \mathscr{B}$ with $U \subset B \subset V$. But then $S \subset U \cap Y \subset B \cap Y \subset V \cap Y \subset Y - \mathrm{cl}_X T$. Since $B \cap Y \in \mathscr{B}'$, $\mathscr{B}'$ satisfies b4. ∎

COROLLARY 5.11. *Let X be rim-compact and let Y be a γ-closed subset of X. Then φY is homeomorphic to $\mathrm{cl}_{\varphi X} Y$.*

Proof. Let $\mathscr{B}$ be the set of all γ-open subsets of X and $\mathscr{B}' = \{B \cap Y \colon B \in \mathscr{B}\}$. By Theorem 3.4, $\mathrm{cl}_{\varphi X} Y$ is homeomorphic to $Y(\mathscr{B}')$ and so $Y(\mathscr{B}')$ is a compactification of Y. By the proof of Theorem 2.6, $Y(\mathscr{B}')$ is rim-perfect. Similarly, the proof of Theorem 2.4 yields that $Y(\mathscr{B}') - Y$ is zero-dimensionally embedded in $Y(\mathscr{B}')$. Thus, by Theorem 2.9 and Theorem 2.10, $Y(\mathscr{B}')$ is the unique rim-perfect compactification of Y with a zero-dimensionally embedded remainder, i.e. $Y(\mathscr{B}') = \varphi Y$. ∎

Note that in Theorem 5.10 and Corollary 5.11 we did not show that $\mathscr{B}'$ satisfied b2. In fact, such a $\mathscr{B}'$ will not satisfy b2; it differs from b2 by the sets in $\mathrm{Fr}_X Y$. We could define a $\mathscr{B}'$ satisfying b2 by using Lemma 3.4.

6. Subordinate subsets of $C^*(X)$

For any $f \in C^*(X)$, I_f denotes the smallest closed interval in $\mathbf{R}$ that contains $f(X)$. If $F \subset C^*(X)$, P_F denotes the product of the spaces $\{I_f \colon f \in F\}$ and $e_F \colon X \to P_F$ the *evaluation map* of F defined by $(e_F(x))_f = f(x)$ for each $f \in F$. If e_F is an embedding, we denote the closure of $e_F(X)$ in P_F by $e_F X$. It is clear that every $f \in F$ has a continuous extension to $e_F X$.

Let $F \subset C^*(X)$ and let $\mathscr{B}$ be a base for X. We say that F *is subordinate to $\mathscr{B}$* provided whenever $f \in F$ and $s, t \in I_f$ and $s < t$, there exist $A, B \in \mathscr{B}$ such that:

$$f^{-1}((-\infty, s)) \subset A \subset f^{-1}((-\infty, t))$$

and

$$f^{-1}((t, \infty)) \subset B \subset f^{-1}((s, \infty)).$$

We say that F *separates $\mathscr{B}$* provided that whenever $C, D \in \mathscr{B}$ and $\mathrm{cl}_X C \cap \mathrm{cl}_X D = \emptyset$, then there exists $f \in F$ such that $\mathrm{cl}_{\mathbf{R}} f(C) \cap \mathrm{cl}_{\mathbf{R}} f(D) = \emptyset$.

LEMMA 6.1. *If $\mathscr{B}$ satisfies b1 and b2 and $F \subset C^*(X)$ is subordinate to $\mathscr{B}$, then every $f \in F$ admits a continuous extension $f' \in C^*(X(\mathscr{B}))$, i.e. $f'|_X = f$.*

Proof. Let $f \in F$. We define $f' \colon X(\mathscr{B}) \to I_f$ as follows: Let $x \in X(\mathscr{B})$; $f'(x) = f(x)$, whenever $x \in X$, and $f'(x) = \bigcap \{[p, q] \subset I_f \colon f^{-1}([p, q])$ contains a member of $x\}$, whenever $x \notin X$.

Suppose that $\mathscr{F} \in F(\mathscr{B})$. We wish to show that $f'(\mathscr{F})$ is well-defined. To

that end, let $\mathscr{C} = \{[p, q] \subset I_f : f^{-1}([p, q])$ contains a member of $\mathscr{F}\}$. Since $f^{-1}(I_f)$ contains X, $\mathscr{C} \neq \emptyset$. Since $\mathscr{F}$ is a $\mathscr{B}$-filter, $\mathscr{C}$ has f.i.p. and $\bigcap \mathscr{C} \neq \emptyset$. To see that $\bigcap \mathscr{C}$ is a singleton, suppose that p, $q \in \bigcap \mathscr{C}$ and $p > q$. Let r, s, t, $w \in I_f$ such that $q < r < s < t < w < p$. Then, since F is subordinate to $\mathscr{B}$, there exist A, $B \in \mathscr{B}$, such that $f^{-1}((-\infty, t)) \subset A \subset f^{-1}((-\infty, w))$ and $f^{-1}((s, \infty)) \subset B \subset f^{-1}((r, \infty))$. Then, since $\mathscr{F}$ is a ultra $\mathscr{B}$-filter and $X = A \cup B$, either $A \in \mathscr{F}$ or $B \in \mathscr{F}$. If $A \in \mathscr{F}$, then $f^{-1}([w, p])$ misses A and so $p \notin \bigcap \mathscr{C}$. Similarly, if $B \in \mathscr{F}$, $f^{-1}([q, r])$ misses B, and so $q \notin \bigcap \mathscr{C}$. This implies that $p = q$ and $f'(\mathscr{F}) = \bigcap \mathscr{C}$ is a singleton.

It remains to see that f' is continuous. But again this follows readily, for if $s < t$ and A, $B \in \mathscr{B}$ and $f^{-1}((-\infty, s)) \subset A \subset f^{-1}((-\infty, t))$ and $f^{-1}((t, \infty)) \subset B \subset f^{-1}((s, \infty))$, $f'(A^*) \subset (-\infty, t]$, $f'(B^*) \subset [s, \infty)$ and $f'((A \cap B)^*) \subset [s, t]$. Thus if $x \in X(\mathscr{B})$ and $q < f'(x) < p$, then there exists $C \in \mathscr{B}$ such that $x \in C^*$ and $f(C^*) \subset [q, p]$. ∎

LEMMA 6.2. *If $\mathscr{B}$ satisfies* b1, b2 *and* b3 *and if $\mathscr{F}$ and $\mathscr{D}$ are distinct free ultra $\mathscr{B}$-filters, then there exist $A \in \mathscr{F}$ and $B \in \mathscr{F}$ such that* $\mathrm{cl}_X A \cap \mathrm{cl}_X B = \emptyset$.

Proof. Let $A \in \mathscr{F}$ and $C \in \mathscr{D}$ be such that $A \cap C = \emptyset$. Since $\mathscr{D}$ is free, for each $x \in \mathrm{cl}_X A \cap \mathrm{cl}_X C$, there exists $C_x \in \mathscr{B}$ such that $x \in C_x$ and $C_x \notin \mathscr{D}$. If $K = \mathrm{cl}_X A \cap \mathrm{cl}_X C$, then K is compact since $K \subset \mathrm{Fr}_X A$. Thus there exist $x_1, \ldots, x_n \in K$ such that $K \subset B_{x_1} \cup \ldots \cup B_{x_n}$. Now we have $\mathrm{cl}_X [C \cap (X - \mathrm{cl}_X B_{x_1}) \cap \ldots \cap (X - \mathrm{cl}_X B_{x_n})] \subset \mathrm{cl}_X C \cap \mathrm{cl}_X (X - \mathrm{cl}_X B_{x_1}) \cap \ldots \cap \mathrm{cl}_X (X - \mathrm{cl}_X B_{x_n}) \subset X - \mathrm{cl}_X A$. Therefore if $B = C \cap (X - \mathrm{cl}_X B_{x_1}) \cap \ldots \cap (X - \mathrm{cl}_X B_{x_n})$, then $B \in \mathscr{D}$ and $\mathrm{cl}_X A \cap \mathrm{cl}_X B = \emptyset$ as required. ∎

THEOREM 6.3. *If $\mathscr{B}$ satisfies* b1, b2 *and* b3 *and if F is subordinate to $\mathscr{B}$ and F separates $\mathscr{B}$, then $e_F : X \to P_F$ is an embedding and $e_F X = X(\mathscr{B})$.*

Proof. It is well known that since F separates points and closed sets in X, then $e_F : X \to P$ is an embedding and the closure of $e_F(X)$ in P_F, $e_F X$, is a compactification of X (cf. [C], p. 12). By Lemma 6.1, every member of F has a continuous extension to $X(\mathscr{B})$; by Theorem 2.3 of [BY$_2$], it remains to show that points of $X(\mathscr{B})$ can be separated by continuous extensions of members of F. To that end, let x, $y \in X(\mathscr{B})$ where $x \neq y$. Since $\mathscr{B}$ is a base for X and F separates $\mathscr{B}$, we may assume that $y \notin X$, say $y = \mathscr{F} \in F(\mathscr{B})$. First of all, if $x \in X$, the free-ness of $\mathscr{F}$ and the base qualities of $\mathscr{B}$, yield that there exist A, $B \in \mathscr{B}$ such that $x \in A$, $B \in \mathscr{F}$ and $\mathrm{cl}_X A \cap \mathrm{cl}_X B = \emptyset$. In this case, since F is subordinate to $\mathscr{B}$, there exists $f \in F$ such that $f(A)$ and $f(B)$ have disjoint closures in I_f. Of course this means that $f(x) \neq f(y)$. On the other hand, if $x \notin X$, say $x = \mathscr{D} \in F(\mathscr{B})$, by Lemma 4.2 there exist H, $G \in \mathscr{B}$ such that $H \in \mathscr{F}$, $G \in \mathscr{D}$ and $\mathrm{cl}_X H \cap \mathrm{cl}_X G = \emptyset$. Then there exists $g \in F$ such that $g(H)$ and $g(G)$ have disjoint closures in I_g. Thus in either case, there exists a continuous extension of a member of F that separates x and y. ∎

LEMMA 6.4. *Suppose that $\mathscr{B}$ satisfies* b1, b2 *and* b3, *and that $f \in C^*(X(\mathscr{B}))$ and $g = f|_X$. Then $\{g\}$ is subordinate to $\mathscr{B}$.*

Proof. Let $s, t \in I_g$ and suppose that $s < t$. Then since $f^{-1}((-\infty, s])$ and $f^{-1}([t, \infty))$ are compact subsets of $X(\mathscr{B})$ and $\mathscr{B}^*$ is a base for $X(\mathscr{B})$, there exist $A_1, \ldots, A_n, B_1, \ldots, B_m$ in $\mathscr{B}$ such that

$$f^{-1}((-\infty, s]) \subset A_1^* \cup \ldots \cup A_n^*,$$

$$f^{-1}([t, \infty)) \subset B_1^* \cup \ldots \cup B_m^*,$$

and

$$\mathrm{cl}^*(A_1^* \cup \ldots \cup A_n^*) \cap \mathrm{cl}^*(B_1^* \cup \ldots \cup B_m^*) = \emptyset.$$

Now, $A = X - \mathrm{cl}_X(\bigcap \{X - \mathrm{cl}_X A_i: 1 \leqslant i \leqslant n\})$ and $B = X - \mathrm{cl}_X(\bigcap \{X - \mathrm{cl}_X B_i: 1 \leqslant i \leqslant m\})$ each belong to $\mathscr{B}$. Also, $\mathrm{cl}_X A \cap \mathrm{cl}_X B = \emptyset$, $g^{-1}((-\infty, s)) \subset A$ and $g^{-1}((t, \infty)) \subset B$. Thus $g^{-1}((-\infty, s)) \subset A \subset g^{-1}((-\infty, t))$ and $g^{-1}((t, \infty)) \subset B \subset g^{-1}((s, \infty))$, and so $\{g\}$ is subordinate to $\mathscr{B}$. ∎

THEOREM 6.5. *Suppose that $\mathscr{B}$ satisfies* b1, b2 *and* b3. *Then* $\{f \in C^*(X): f$ *has a continuous extension to* $X(\mathscr{B})\}$ *is precisely the set* $\{f \in C^*(X): \{f\}$ *is subordinate to* $\mathscr{B}\}$.

Proof. This is an immediate consequence of Lemmas 6.1 and 6.4. ∎

For each $f \in C^*(X)$, let $B(f) = \{t \in I_f: f^{-1}(t)$ contains a compact set K that separates X into disjoint open sets M and N where $f(M) \subset (-\infty, t]$ and $f(N) \subset [t, \infty)\}$. For any space X, let $\Gamma(X) = \{f \in C^*(X): B(f)$ is dense in $I_f\}$.

LEMMA 6.6 ([D]). *Let $\mathscr{B}$ be a base for X satisfying* b1, b2, *and* b3, *let* $A \in \mathscr{B}$ *and let W be an open subset of X containing* $\mathrm{cl}_X A$. *Then there exists* $f \in \Gamma(X)$ *such that*:
  (i) $f(\mathrm{cl}_X A) = \{0\}$ *and* $f(X - W) = \{1\}$, *and*
  (ii) $\{f\}$ *is subordinate to* $\mathscr{B}$.

Proof. We employ the usual Urysohn construction. For each rational number t, we define a γ-open subset R_t of X as follows.

For $t < 0$, let $R_t = \emptyset$. For $t > 1$, let $R_t = X$. Enumerate the rationals in $[0, 1]$ so that $t_1 = 1$ and $t_2 = 0$. Since $\mathrm{Fr}\, A$ is compact, there exists $R_1 \in \mathscr{B}$ such that $\mathrm{cl}_X A \subset R_1 \subset \mathrm{cl}_X R_1 \subset W$. Similarly, there is a γ-open set $R_2 \in \mathscr{B}$ so that $\mathrm{cl}_X A \subset R_2 \subset \mathrm{cl}_X R_2 \subset R_1$. We repeat the above process, so that by induction, we obtain a collection $\{R_t: t \in Q \cap [0, 1]\}$ such that for any pair t and s in $Q \cap [0, 1]$, R_t and R_s belong to $\mathscr{B}$, and if $s < t$, $\mathrm{cl}_X R_s \subset R_t$. Now, as usual, we define $f: X \to [0, 1]$ by $f(x) = \inf \{t: x \in R_t\}$. It is well known that f is continuous and $f(\mathrm{cl}_X A) = \{0\}$ and $f(X - W) = \{1\}$.

It remains to see that $f \in \Gamma(X)$ and $\{f\}$ is subordinate to $\mathscr{B}$. First of all it follows that for each $t \in Q$, $0 < t < 1$, $\mathrm{Fr}_X R_t \in f^{-1}(t)$. Since the former set is γ-open, $f \in \Gamma(X)$. Similarly, if $s, t \in [0, 1]$ with $s < t$, there exist rationals p and q such that $s < p < q < t$. Now $f^{-1}((-\infty, s)) \subset$

$\subset R_p \subset \operatorname{cl}_X R_p \subset f^{-1}((-\infty, p]) \subset f^{-1}((-\infty, t))$,   and $R_p \in \mathscr{B}$.   Likewise,
$f^{-1}((t, \infty)) \subset f^{-1}([q, \infty)) \subset (X - \operatorname{cl}_X R_p) \subset f^{-1}((s, \infty))$. Then, since $R_p \in \mathscr{B}$,
$X - \operatorname{cl}_X R_p$ belongs to $\mathscr{B}$. ∎

THEOREM 6.7. *Let X be rim-compact and let $\mathscr{B} = \{U: U$ is γ-open in $X\}$.
Then*

(a) *$\Gamma(X)$ separates $\mathscr{B}$ and $\Gamma(X)$ is subordinate to $\mathscr{B}$;*

(b) *$\varphi X = e_{\Gamma(X)} X$;*

(c) *φX is the smallest compactification of X such that every member of
$\Gamma(X)$ has a continuous extension to φX;*

(d) *φX is the largest compactification of X such that points of $\varphi X - X$ are
separated by continuous extensions of members of $\Gamma(X)$;*

(e) *φX is the unique compactification of X such that every member of
$\Gamma(X)$ can be extended continuously to φX, and points of $\varphi X - X$ are separated
by continuous extensions of members of $\Gamma(X)$.*

Proof. Property (a) follows from Lemma 6.6, and (b) follows (a) and
Theorem 6.3; (c), (d) and (e) follow from the principal results of $[\text{BY}_1]$. ∎

THEOREM 6.8 (cf. Corollary 5.11). *Let X be a rim-compact space, let Y be
a γ-closed subset of X and let Z be the closure of Y in φX. Then $\varphi Y = Z$.*

Proof. Let $F = \Gamma(Y)$. By Lemma 4.6, F separates points of $Z - Y$,
and so by Theorem 2.1 of $[\text{BY}_1]$, $Z = e_F Y$. It follows from Theorem 4.7
that $\varphi Y = e_F Y$. ∎

THEOREM 6.9 (cf. (6.7) of [GJ]). *Let X be a rim-compact space which is
dense in space Y. The following are equivalent:*

(a) *$X \subset Y \subset \varphi X$.*

(b) *Y is rim-compact and $\varphi X = \varphi Y$.*

Proof. (a) implies (b). Suppose $X \subset Y \subset \varphi X$. Then since $\varphi X - Y$ is
zero-dimensionally embedded in φX, Y is rim-compact, so φY exists. Clearly
φX is a compactification of Y. Moreover, if $f \in \Gamma(X)$ and φf is the
continuous extension of f to φX, $\varphi f|_Y$ belongs to $\Gamma(Y)$. Since points of $\varphi X
- Y$ can be separated by such an extension to φX, by Theorem 6.7, φX must
equal φY.

(b) implies (a). Suppose that Y is rim-compact and $\varphi X = \varphi Y$. Recall that
$X \subset \operatorname{cl}_Y X = Y$. Then $X \subset Y \subset \varphi Y$, and since $\varphi Y = \varphi X$, $X \subset Y \subset \varphi X$ as
required. ∎

Remark 6.10. Let $\mathscr{B}$ be the set of all γ-open subsets of X and let
$\mathscr{B}' = \{A^* \cap Y: A \in \mathscr{B}\}$. One can show that $\mathscr{B}'$ satisfies b1, b2, b3 and b4,
and that $Y(\mathscr{B}') = \varphi Y = \varphi X$.

COROLLARY 6.11. *Let X be a rim-compact space, let $p \in \varphi X - X$ and let
$Y = \varphi X - \{p\}$. Then Y is locally compact, and $Y_\infty = \varphi Y = \varphi X$.*

Proof. By Theorem 6.9, $\varphi Y = \varphi X$. However, $\varphi X - Y$ is a singleton,

and by the uniqueness of the one-point-compactification of a locally compact space, $\varphi X = Y_\infty$. ∎

THEOREM 6.12. *Let* D *be any infinite discrete space. Then* $\varphi D = \beta D$.

Proof. Points of $\beta D - D$ can be separated by a continuous extension of a member of $\Gamma(D)$. By Theorem 6.8, $\beta D = \varphi D$. ∎

Suppose S is a subring of $C^*(X)$ containing the constant functions, and αX is a compactification of X. According to [H], S will be said to *determine* αX if every $f \in S$ has an extension $\alpha f \in C(\alpha X)$ and $S^\alpha = \{\alpha f : f \in S\}$ separates points of αX. In [BY$_1$], an arbitrary subset F of $C^*(X)$ is said to determine αX provided αX is the smallest compactification such that every member of F has a continuous extension to αX. It follows from Theorem 2.1 of [BY$_1$] that the two notions are coincident when F is a subring of $C^*(X)$. We shall denote by $C^*(X)$ the subring of $C^*(X)$ consisting of all $f \in C^*(X)$ such that for every maximal ideal M of $C^*(X)$ there exists a real number r such that $(f - r) \in M$. Let $C_{K,F}(X)$ denote the set of all $f \in C^*(X)$, such that for some compact set C in X, $f(X - C)$ is a finite set.

Henriksen [H] observed that if X is rim-compact and realcompact, then every $f \in C^*(X)$ has a continuous extension φf to φX. He also proved that if X is realcompact and $C^*(X)$ determines a compactification αX of X, then X is rim-compact and $\alpha X = \varphi X$. There are realcompact, rim-compact spaces X for which $C^*(X)$ does not determine a compactification of X [H]. In [Do], Dominguez proved that $C_{K,F}(X)$ determines φX if and only if X is rim-compact and X has a base of open sets whose boundaries have compact neighborhoods.

Nowiński [No$_1$] showed that if X is a locally compact, metacompact space, then $\varphi X = M_X$ where M_X is the set of all proper maximal ideals of the smallest closed subring of $C^*(X)$ that contains the set of all closed maps in $C^*(X)$ with the topology on M_X generated by the set $\{J \in M_X : f \notin J\}$ as f ranges over M_X.

A *dendritic space* is a connected space in which every pair of points can be separated by a third point. Proizvolov [Pr] showed that every rim-compact dendritic space X has a unique dendritic compactification αX; and K. Allen, in his thesis, showed that $\alpha X = \varphi X$, i.e. the Freudenthal compactification of a dendritic rim-compact space X is the unique dendritic compactification of X. Furthermore if M denotes the set of all continuous maps into $[0, 1]$ with connected point inverses on a rim-compact dendritic space, then

(i) every $f \in M$ has a unique continuous monotone extension φf to φX, and

(ii) for every pair of distinct points p and q in $\mathrm{cl}^*(\varphi X - X)$, there exists $f \in M$ such that $\varphi f(p) \neq \varphi f(q)$.

Moreover, φX is the smallest compactification of X satisfying (i) and φX is the largest compactification of X satisfying (ii) ([A]).

7. Quasi-component spaces

Let $\mathcal{O}(Z)$ denote the collection of all clopen subsets of Z. For $z \in Z$, the intersection of all members of $\mathcal{O}(Z)$ containing z is the quasi-component of Z containing z. The collection of all such quasi-components of Z is a partition of Z; let QZ denote this partition, and let $q: Z \to QZ$ be the natural projection of Z into QZ. We topologize QZ as follows: if $z \in Z$, a basic open set of QZ about $q(z)$ is of the form $q(W)$ where $z \in W \in \mathcal{O}(Z)$. We call QZ with this topology, the *quasi-component space of Z*.

We will prove that φQX is homeomorphic to $Q\varphi X$ whenever X is a rim-compact Hausdorff space. To that end, we need some notation. Let $q: X \to QX$ be the map of X onto the quasi-component space described above and let $p: \varphi X \to Q\varphi X$ be the natural map of φX onto its quasi-component space $Q\varphi X$. Recall that φX is the $\mathcal{B}$-compactification $X(\mathcal{B})$ of X where $\mathcal{B}$ is the collection of all γ-open subsets of X. Now $\mathcal{B}' = \{q(W): W \in \mathcal{O}(X)\}$ is a base for the open sets of QX. Moreover, $\mathcal{O}(X) \subset \mathcal{B}$ (since the frontier of any member of $\mathcal{O}(X)$ is empty and therefore compact), so that if $A \in \mathcal{B}'$, $q^{-1}(A) \in \mathcal{B}$. Thus by Theorem 5.2, there exists a continuous surjection $\hat{q}: \varphi X \to \varphi QX$ where $\varphi X = X(\mathcal{B})$ and $\varphi QX = QX(\mathcal{B}')$.

In order to label a diagram we will use $e: X \to \varphi X$ and $d: QX \to \varphi QX$ to denote the identity homeomorphisms on X and QX, respectively, into the Freudenthal compactifications φX and φQX respectively. We will use $p: \varphi X \to Q\varphi X$ to denote the natural map of φX onto its quasi-component space $Q\varphi X$. Thus we have the following:

$$\begin{array}{ccc} X & \xrightarrow{e} \varphi X & \xrightarrow{p} Q\varphi X \\ \downarrow q & \downarrow \hat{q} & \\ QX & \xrightarrow{d} \varphi QX & \end{array}$$

Let $y \in \varphi X$, let t be the quasi-component of φX containing y and let V be a basic open subset of φX containing t. Recall that this means that $V = p(S)$ where S is a clopen subset of φX containing y and $t = p(y)$ is the intersection of all such clopen subsets of φX. We define a function $h: Q\varphi X \to \varphi QX$ by $h(t) = (\hat{q} \circ p^{-1})(t)$.

THEOREM 7.1. $h: Q\varphi X \to \varphi QX$ is a homeomorphism.

Proof. We wish to show that h is a continuous bijection. To see that h is continuous let $A \in \mathcal{B}'$ so that A^* is a basic open set in φQX. Then $A = q(W)$ for some clopen subset W of X. Now $W = q^{-1}(A) \in \mathcal{B}$ and $W^* = \hat{q}^{-1}(A^*)$. Also, W^* is clopen in φX and so $p(W^*)$ is open in $Q\varphi X$. Hence $h^{-1}(A^*) = (p \circ \hat{q}^{-1})(A^*) = p(W^*)$ is open in $Q\varphi X$ and h is continuous.

To see that h is one-to-one, let $t_1, t_2 \in Q\varphi X$, $t_1 \neq t_2$. Let V_1 and V_2 be disjoint clopen subsets of $Q\varphi X$ containing t_1 and t_2 respectively, where $V_i = p(S_i)$, $i = 1, 2$ and S_1 and S_2 are clopen subsets of φX. Now $\hat{q}(S_1) \cap \hat{q}(S_2)$

$= S_1^* \cap S_2^*$ (* in $\mathcal{B}'$) and $S_1^* \cap S_2^* = (S_1 \cap S_2)^* = \emptyset$. Thus, $h(t_1) \in h(V_1) = \hat{q}(S_1)$ and $h(t_2) \in h(V_2) = \hat{q}(S_2)$ are distinct, and h is one-to-one.

Finally, to see that h is a surjection, let $v \in \varphi QX$. Now if $v \in QX$, say $v = \bigcap \{O_\alpha: \alpha \in \Gamma\}$ where each $O_\alpha \in \mathcal{O}(X)$, then each O_α is clopen in X; and $u = \bigcap \{O_\alpha^*: \alpha \in \Gamma\}$ is a non-empty subset of φX such that $u \subset \hat{q}^{-1}(v)$. Consequently, any member of $p(u)$ maps onto v under h. On the other hand, if $v \in \varphi QX - QX$, then $v \in F(\mathcal{B}')$, where $\mathcal{B}' = \{q(W): W \in \mathcal{O}(X)\}$. Say v is a free $\mathcal{B}'$-filter $\mathcal{F}'$. Then $\{W: W \in \mathcal{O}(X)$ and $q(W) \in \mathcal{F}'\}$ is a free $\mathcal{B}$-filter on X, and thus it contains a free ultra $\mathcal{B}$-filter $\mathcal{F}$ so that $\hat{q}(\mathcal{F}) = \mathcal{F}'$. Then $h(p(\mathcal{F})) = \mathcal{F}' = v$, and so h is a surjection. This completes the proof. ∎

LEMMA 7.2. *Let X be a rim-compact Hausdorff space so that $\mathcal{B}$, the collection of all γ-open subsets of X, satisfies* b1, b2, b3 *and* b4 *and* $X(\mathcal{B}) = \varphi X$. *Let $q: X \to QX$ and $p: \varphi X \to Q\varphi X$ be the natural maps of X and φX, respectively, onto their quasi-component spaces QX and $Q\varphi X$, and let $e: X \to \varphi X$ be the identity map. Then $f: QX \to Q\varphi X$ defined by $f(x) = (p \circ e \circ q^{-1})(x)$ is a homeomorphism of QX into $Q\varphi X$.*

Proof. Let S be a basic open subset of QX, i.e. let $S = q(W)$ where W is a clopen subset of X. Then $W \in \mathcal{B}$ and $(X - W) \in \mathcal{B}$. Furthermore, W^* and $(X - W)^*$ are clopen in $Q\varphi X$. This implies that f is a one-to-one, continuous, open map of QX onto $f(QX)$. Thus f is a homeomorphism. ∎

LEMMA 7.3. *If φX is a metric space, then QX is a compact metric space.*

Proof. Since φX is a compact metric space, $Q\varphi X$ is a compact metric space ($Q\varphi X$ is the continuous Hausdorff image of a compact metric space, so $Q\varphi X$ is a compact metric space). Let $f: QX \to Q\varphi X$ be the embedding of the previous lemma and let $h: Q\varphi X \to \varphi QX$ be the homeomorphism of Theorem 7.1. We assert that $h \circ f$ is a surjection. For suppose $t \in \varphi QX$ and $t \notin (h \circ f)(\varphi QX)$. Then, since φQX is a metric space, there exists a sequence $\{t_n: n \in N\}$ in φQX and $t_n \to t$. Also there exists a sequence $\{U_n: n \in N\}$ of pairwise disjoint open subsets of φQX such that for each $n \in N$, $t_n \in U_n$ and the closure of U_n in φQX misses both t and the union of the closures of $U_1, \ldots, U_{n-1}$, and the diameter of U_n is less than $1/2^n$. It then follows that each U_n contains a set of the form $d(A_n) = A_n^*$ where A_n is a clopen subset of QX. Then $U = \bigcup \{A_{2n+1}: n \in N\}$ and $V = \bigcup \{A_{2n}: n \in N\}$ are open subsets of QX and U and V have disjoint closures in QX. By the perfectness of φQX, U and V have disjoint closures in φQX. Of course this is impossible since t is common to each closure. Thus QX is homeomorphic to φQX and QX is a compact metric space. ∎

LEMMA 7.4. *Let X be a separable metric space and let $f: X \to Y$ be a continuous surjection of X onto a compact Hausdorff space Y. Then Y is a metric space.*

Proof. Let $\mathcal{B}$ be a countable base for X, let $\mathcal{C} = \{f(B): B \in \mathcal{B}\}$ and let $p = \{(C, D) \in \mathcal{C} \times \mathcal{C}:$ there exists disjoint open sets U and V in Y such that

$C \subset U$ and $D \subset V\}$. For each $p = (C, D)$, let U_p and V_p be disjoint open sets in Y with $C \subset U_p$ and $D \subset V_p$. Let $\mathscr{C}'$ be the topology on Y having $\{U_p : p \in P\} \cup \{V_p : p \in P\}$ as a subbase, and let Z denote the set Y with this topology. It is straightforward to see that the identity map of Y onto Z is a continuous bijection and that Z is a Hausdorff space. Thus Y and Z are homeomorphic. However Z has a countable base, so that Z is a second countable compact Hausdorff space, and is thus a metric space. ∎

LEMMA 7.5. *Let $\mathcal{O}$ be the collection of clopen subsets of X, where X is a separable metric space. If QX is compact, then QX is metrizable and $w(QX)$ $= |\mathcal{O}| \leqslant \aleph_0$.*

Proof. By Lemma 7.4, QX is metrizable. Thus $w(QX) = \aleph_0$, since QX is compact. Let $\mathscr{U}$ be a countable base for QX and let W be clopen in X. Then $q(W)$ and $q(X - W)$ are disjoint clopen subsets of QX, and so $q(W)$ is a finite union of elements of $\mathscr{U}$. Thus, every member of $\mathcal{O}$ is a finite union of members of $\mathscr{U}$, and so $|\mathcal{O}| = |\mathscr{U}| \leqslant \aleph_0$. Clearly $w(QX) = |\mathcal{O}|$. ∎

THEOREM 7.6. *Let X be rim-compact. The following are equivalent*:

(i) *φX is a metric space*;

(ii) *X is a separable metric space and QX is compact*;

(iii) *X is a separable space and φQX is a metric space*.

Proof. (i) implies (ii). Suppose φX is a metric space. Then X is a separable metric space (since φX is also separable). By Lemma 7.3, QX is a compact space.

(ii) implies (iii). By Lemma 7.5, the weight of QX is countable, and so QX is a compact Hausdorff space of countable weight and thus is a metric space. Since QX is compact, $\varphi QX = QX$, so that φQX is a metric space.

(iii) implies (i). By Theorem 7.1, φQX and $Q\varphi X$ are homeomorphic and so $Q\varphi X$ is second countable. This means $Q\varphi X$ has a countable base of the form $\{p(W) : W$ is clopen in $\varphi X\}$, so that by Lemma 2.15, there exists a countable base $\mathscr{B} = \{A_i^* : A_i$ is γ-open in $X, i \in N\}$ for φX. Now by Lemma 2.16, $\mathscr{B}$ must contain A_i^* whenever A_i is clopen in X. Thus there are at most countably many clopen subsets in X. Finally, by Theorem 3.8, $w(\varphi X) = w(X) \cdot \aleph_0 = \aleph_0$ and thus φX is a second countable compact Hausdorff space. ∎

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