

POLSKA AKADEMIA NAUK • INSTYTUT MATEMATYCZNY

S. 133  
[24]

# ROZPRAWY MATEMATYCZNE

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XXIV

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*On two-to-one continuous functions*

WARSZAWA 1961  
PAŃSTWOWE WYDAWNICTWO NAUKOWE

ś. 7133

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PRINTED IN POLAND

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## INTRODUCTION

The purpose of this paper is to make a general investigation of a special class of continuous functions  $f$ , namely the so-called *k-to-one functions*, i. e. assuming every value in exactly  $k$  points. The general assumption concerning the space of arguments  $\mathcal{X}$  of  $f$  is that of local compactness and separability. We also assume, in order to exclude the triviality, that  $f(\mathcal{X})$  is a subspace of an  $\mathcal{L}^*$ -space (or a topological Hausdorff space)  $\mathcal{Y}$ . The most interesting case is that of  $k = 2$ , i. e. of the two-to-one functions. In this case there is a very interesting problem of the existence or non-existence of such functions. This problem was considered in papers [4], [5], [6], [7], [11], and [14]. The most important result, due to Civin [4], asserts that there do not exist two-to-one continuous functions on  $n$ -cells for  $n \leq 3$ . The most interesting problem which remains open is that of the existence or non-existence of the two-to-one functions on  $n$ -cells for  $n > 3$ . A similar problem for  $k \neq 2$  does not exist (see [7] and [12]).

The investigation of the two-to-one continuous functions is in a natural manner equivalent to the investigation of some involution on  $\mathcal{X}$ , i. e. the involution  $\varphi$ , where  $\varphi(x)$  is the element of  $f^{-1}f(x)$  which is different from  $x$ . This involution is, in general, discontinuous, but it is in some meaning semicontinuous. In paper [4] Civin has shown that the investigation of  $\varphi$  on compact manifolds (or, if  $f$  is closed, on locally compact manifolds) is equivalent to the investigation of some continuous involution.

In this paper the two-to-one functions are considered in a more general situation. We do not assume that  $\mathcal{X}$  is a manifold and, if  $\mathcal{X}$  is non-compact, that  $f$  is closed. The most general problem is that of the behaviour of  $\varphi$  on the neighbourhoods and the so called pseudo-neighbourhoods of points being Euclidean or, in some meaning, pseudo-Euclidean. According to the main theorem of this paper (Theorem 1) the investigation of  $\varphi$  on the closure of neighbourhoods (pseudoneighbourhoods) of Euclidean (pseudo-Euclidean) points is reduced to the investigation of some continuous involution. Using this theorem it is possible to investigate the two-to-one continuous functions on some non-

-manifolds. The results of Civin and other authors are obvious consequences of this theorem and its corollaries.

The method used in this paper is that of multi-valued upper semi-continuous functions. It is known that such functions are of the I-st Baire class (see [10]) and that the same is true for the partial functions. Of course, they are Baire functions with respect to the topology induced by the Hausdorff metric in the space of compact subsets of the topological space in question. This method allows to get proofs of theorems in more natural manner than in above mentioned papers. Although the spaces in question are metrizable, we do not use explicit metric notions.

In the paper [4] of Civin the problem is raised whether there exist or not the two-to-one continuous functions on Euclidean spaces  $E^n$  for  $n > 1$  (if  $n = 1$ , they do not exist, see [4] and Chapter V of this paper). We give a class of examples of such functions of  $E^n$  for  $n \geq 2$ . This is in contrast to  $n$ -cells.

In another example we show that there exist two-to-one continuous functions on the closures of some simply connected plane domains whose boundaries are irreducible cuts of the plane. This is also in contrast to the results of Roberts and Civin concerning 2-cells.

There are investigated problems concerning two-to-one continuous functions (not necessarily closed) on locally compact manifolds. The involution associated to the involution  $\varphi$ , which is in the compact case continuous, is in this case generally discontinuous. Theorem 3 of Chapter V gives some information about the structure of the set of discontinuity points of this involution. The complete characterization of this set of points remains an open problem.

We give only two examples of applications of the general results of this paper to the spaces whose local structure is in some sense "bad"; namely, we consider the irreducible continuum  $\mathcal{N}_r$  and the indecomposable continuum  $\mathcal{B}_0$ . All points of both continua are pseudo-Euclidean. There is no method for the investigation of the two-to-one continuous functions on spaces whose local structure is "worse" than that of  $\mathcal{N}_r$  and  $\mathcal{B}_0$ .

We already mentioned that if  $\mathcal{X}$  is a compact manifold, the investigation of the two-to-one continuous functions is reduced to the investigation of some continuous involution, i. e. to the examination of fixed point sets of this involution. It is a very difficult problem if the dimension is greater than 3. Recently, Bing [2] and other authors showed some examples of continuous involutions on  $I^n$  (for  $n = 4$  and some other integers greater than 4) whose fixed points sets are not  $n$ -cells and, in some cases, even non-manifolds.

The author wishes to thank Professor Bronisław Knaster for many helpful discussions in connection with the problems considered in this paper.

The notations, such as  $Ls$ ,  $Li$ ,  $Lim$ ,  $Fr$ , and so on, are as in [10]. The meaning of “iff” is “if and only if”. The paper is divided into five chapters, chapters into paragraphs. Assertion III (4.3), for example, is in the fourth paragraph of the third chapter (if this assertion is referred to in Chapter III, then the III is omitted).

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## I. GENERAL PROPERTIES OF $k$ -TO-ONE FUNCTIONS ON LOCALLY COMPACT SPACES

Let  $\mathcal{X}$  be a separable, locally compact space.  $\mathcal{X}$  may be considered as a subspace of a compact space  $\mathcal{X}^*$  without one point ([10], II, p. 50). If we denote this point by  $p$ , then  $\mathcal{X} = \mathcal{X}^* - p$ . Let us consider continuous functions  $f: \mathcal{X} \rightarrow \mathcal{Y}$ , where  $\mathcal{Y}$  is an  $\mathcal{L}^*$ -space ([10], I, p. 83). We shall call such functions *k-to-one* if, for every  $y \in \mathcal{Y}$ ,  $f^{-1}(y)$  consists of exactly  $k$  points.

**1. Multi-valued functions  $\Phi$  and  $\Psi$ .** Consider a multi-valued function  $\Phi: \mathcal{X} \rightarrow \mathcal{X}$  given by

$$(1.1) \quad \Phi(x) = f^{-1}f(x).$$

If  $f$  is  $k$ -to-one, then

$$(1.2) \quad \Phi(x) = x^1 \cup x^2 \cup \dots \cup x^k, \quad \text{where } x^1 = x, \quad x^i \neq x^j \text{ if } i \neq j.$$

We note first that

$$(1.3) \quad \Phi(x^i) = \Phi(x^j), \quad i, j = 1, 2, \dots, k,$$

and

$$(1.4) \quad \Phi(u) \cap \Phi(v) \neq \emptyset \text{ implies } \Phi(u) = \Phi(v).$$

We prove that

(1.5) *If  $f: \mathcal{X} \rightarrow \mathcal{Y}$ , where  $\mathcal{Y}$  is an  $\mathcal{L}^*$ -space, is continuous and  $F = f^{-1}f$ , then for every  $x \in \mathcal{X}$  we have  $\text{Ls}_{\xi \rightarrow x} F(\xi) \subset F(x) \cup p$ .*

In fact, let  $\eta \in F(\xi)$  and  $\lim \eta = v$ . Let  $v \neq p$ . Then we have  $F(\xi) = F(\eta)$  and therefore  $f(\xi) = f(\eta)$ . Since  $f$  is continuous, hence <sup>(1)</sup>  $f(x) = f(v)$ , which implies  $v \in F(x)$ .

From the definition of  $\Phi$  it follows that (1.5) is true for  $\Phi$ .

Consider another multi-valued function  $\Psi: \mathcal{X}^* \rightarrow \mathcal{X}^*$  defined by

$$(1.6) \quad \Psi(x) = \Phi(x) \cup p \text{ if } x \neq p,$$

$$(1.7) \quad \Psi(p) = p \cup \text{Ls}_{\xi \rightarrow p} \Phi(\xi).$$

Note that

(1.8)  *$\Psi$  is upper semicontinuous, i. e. for every  $x$  there is  $\text{Ls}_{\xi \rightarrow x} \Psi(\xi) \subset \Psi(x)$ .*

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<sup>(1)</sup>  $\lim \eta = v$  implies  $\lim f(\eta) = f(v)$  as  $\mathcal{Y}$  is an  $\mathcal{L}^*$ -space ([10], I, p. 85). In particular, this implication is true if  $\mathcal{Y}$  is a Hausdorff space ([10], I, p. 90).

It is an immediate consequence of (1.5) and (1.6) if  $X \neq p$ , and of (1.7) if  $x = p$ .

The multi-valued function  $F: \mathcal{X} \rightarrow \mathcal{X}'$  is said to be *continuous at*  $x \in \mathcal{X}$  if there exists topological limit  $\lim_{\xi \rightarrow x} F(\xi)$  and if  $\lim_{\xi \rightarrow x} F(\xi) = F(x)$ .

It is well known ([10], II, p. 38) that if  $\mathcal{X}$  and  $\mathcal{X}'$  are compact metric spaces,  $F$  is upper semicontinuous and  $F(x)$  are compact subsets of  $\mathcal{X}'$ , then  $F$  is of the I-st Baire class. Hence, by (1.8),  $\Psi$  is of the I-st Baire class. Denote by  $\mathcal{X}_F$  the set of points of continuity of  $F$  on  $\mathcal{X}$ . It is clear that

$$(1.9) \quad \mathcal{X}_\phi = \mathcal{X}_\Psi.$$

From well known theorems of Baire functions and from (1.8) it follows that

$$(1.10) \quad \mathcal{X}_\Psi \text{ is a } G_\delta\text{-set, which is dense in } \mathcal{X}.$$

We shall prove further more that (see [14])

$$(1.11) \quad \mathcal{X}_\Psi \text{ is open in } \mathcal{X}.$$

**2. The proof of (1.11).** Let  $x \neq p$ . Denote  $p$  by  $x^{k+1}$ . Then  $\Psi(x) = x^1 \cup x^2 \cup \dots \cup x^{k+1}$ . Let  $V = V_1 \cup V_2 \cup \dots \cup V_{k+1}$  be an open neighbourhood of  $\Psi(x)$  such that

$$(2.1) \quad x^i \in V_i,$$

$$(2.2) \quad \bar{V}_i \cap \bar{V}_j = \emptyset \text{ if } i \neq j.$$

Let  $\Psi$  be continuous at  $x$ . Then there exists an open neighbourhood  $U$  of  $x$  such that

$$(2.3) \quad \bar{U} \subset V_1,$$

$$(2.4) \quad \Psi(\xi) \subset V \text{ if } \xi \in \bar{U}$$

and

$$(2.5) \quad \Psi(\xi) \cap \bar{V}_i \neq \emptyset, \text{ if } \xi \in \bar{U} \text{ and } i = 1, 2, \dots, k+1.$$

From (2.1)-(2.5) it follows that

$$(2.6) \quad \Psi(\xi) \cap \bar{V}_i \text{ is a single point, if } \xi \in \bar{U} \text{ and } i = 1, 2, \dots, k+1.$$

Hence, the functions  $\varphi_i(\xi) = \Psi(\xi) \cap \bar{V}_i$  are well defined on  $\bar{U}$ . In particular, we have  $\varphi_{k+1}(\xi) = p$  for all  $\xi$  in  $\bar{U}$ . We prove that

$$(2.7) \quad \text{If } i \neq k+1, \text{ then } \varphi_i \text{ are homeomorphisms.}$$

1.  $\varphi_i$  are one-to-one. Suppose that there exist points  $\xi' \neq \xi''$  of  $\bar{U}$  such that  $\varphi_i(\xi') = \varphi_i(\xi'')$ . Then  $\xi'' \in \Psi(\xi') \cap \bar{U} \subset \Psi(\xi) \cap V_1$ , contrary to (2.6).

2.  $\varphi_i$  are continuous. Let  $\xi \in \bar{U}$  and  $\xi = \lim \xi'$ , where  $\xi' \in \bar{U}$ . Note that, by (2.6),  $\lim_{\xi' \rightarrow \xi} \varphi_i(\xi')$  exists. We have  $\lim_{\xi' \rightarrow \xi} \varphi_i(\xi') = \varphi_i(\xi)$  because both these points belong to the set  $\bar{V}_i \cap \Psi(\xi)$  which is a single point.

Hence, the function  $\Psi$  may be written in the form

$$(2.8) \quad \Psi(\xi) = p \cup \bigcup_{i=1}^k \varphi_i(\xi),$$

where  $\varphi_i$  are homeomorphisms with the following property

$$(2.9) \quad \varphi_i(\bar{U}) \cap \varphi_j(\bar{U}) = 0 \text{ if } i \neq j,$$

which follows directly from (2.6). Note that in general  $\varphi_i(U)$  need not be open in  $\mathcal{X}$ .

We prove now (1.11). It remains to prove that  $\Psi$  is lower semicontinuous at every  $\xi \in \bar{U}$ , i. e. if  $\eta \in \Psi(\xi)$ ,  $\lim \xi' = \xi$ , and  $W$  is an open neighbourhood of  $\eta$ , then there exists a neighbourhood  $H$  of  $\xi$  such that  $W \cap \Psi(\xi') \neq 0$  for  $\xi' \in H$ . According to (2.8) there exists an  $i$  such that  $\eta = \varphi_i(\xi)$ . Let  $W$  be an open neighbourhood of  $\eta$  contained in  $V_i$ . By (2.7),  $\varphi_i$  are continuous. Therefore there exists a neighbourhood  $H$  of  $\xi$  such that  $\xi' \in H$  implies  $\varphi_i(\xi') \in W$ . Hence, by (2.8),  $\Psi(\xi') \cap W \neq 0$ .

**3. Some conditions for  $\Psi$  to be discontinuous at  $x \neq p$ .** The following criterion of discontinuity of  $\Psi$  at  $x \neq p$  is obvious:

(3.1)  *$\Psi$  is discontinuous at  $x \neq p$  iff there exist sequences  $\{\xi_n\}$ ,  $\{u_n\}$  and  $\{v_n\}$  such that  $x = \lim \xi_n$ ,  $u_n$  and  $v_n$  belong to  $\Psi(\xi_n)$ ,  $u_n \neq v_n$  and  $\lim u_n = \lim v_n$ .*

The following assertion is a necessary condition for the discontinuity of  $\Psi$  at  $x \neq p$ . The condition is stronger than that of (3.1).

(3.2) *If  $x \neq p$  is a point of discontinuity of  $\Psi$ , then there exist sequences  $\{\xi_n\}$ ,  $\{u_n\}$  and  $\{v_n\}$  such that  $\lim \xi_n = x$ ,  $\xi_n \in \mathcal{X}_\Psi$ ,  $u_n$  and  $v_n$  belong to  $\Psi(\xi_n)$ ,  $u_n \neq v_n$  and  $\lim u_n = \lim v_n$ .*

By (3.1) there exist sequences  $\{\xi'_n\}$ ,  $\{v'_n\}$  and  $\{u'_n\}$  having all the properties required in (3.2) except  $\xi'_n \in \mathcal{X}_\Psi$ . Since  $\lim u'_n = \lim v'_n$ , hence there exists an integer  $j$  such that  $u'_n$  and  $v'_n$  belong to  $V_j$ , where  $V_j$  has the same meaning as in (2.1) and (2.2). Therefore there exists an  $i$ ,  $i \neq j$ , such that  $\Psi(\xi'_n) \cap V_i = 0$  for sufficiently large  $n$ . Semicontinuity of  $\Psi$  implies that for every such  $n$  there exist neighbourhoods  $U_n \subset U$  of  $\xi'_n$  such that

$$(1) \quad \Psi(U_n) \cap V_i = 0.$$

$$(2) \quad \text{Lim } U_n = x.$$

By (1.10),  $U_n \cap \mathcal{X}_\Psi \neq 0$ . Choose  $\xi_n \in U_n \cap \mathcal{X}_\Psi$ . The sequence  $\{\xi_n\}$  is the first of sequences required in (3.2);  $\lim \xi_n = x$  follows from (2). From (1) it follows that  $\Psi(\xi_n) \cap V_i = 0$ . Then, since  $\Psi(\xi_n)$  consists of  $k+1$  points, there exists  $m$ ,  $m \neq i$ , such that  $V_m \cap \Psi(\xi_n)$  consists of at least two points. Denote these points by  $u_n$  and  $v_n$ . Since  $\lim u_n = \lim v_n = x_m$ ,  $\{u_n\}$  and  $\{v_n\}$  are the remaining sequences required in assertion (3.2).

**4. Some properties of  $f^{-1}f$  on  $\mathcal{X}$ .** A multi-valued function  $F: \mathcal{X} \rightarrow \mathcal{X}$  is said to be *bounded at  $x \notin \mathcal{X}$*  if  $p \in \text{Ls } F(\xi)$ . It is said to be *bounded on  $\mathcal{X}$*  if it is bounded at every  $x \notin \mathcal{X}$ . In particular,  $\Phi$  is bounded at  $x \in \mathcal{X}$  iff  $x \notin \Psi(p)$ , and  $\Phi$  is bounded on  $\mathcal{X}$  iff  $\Psi(p) = p$ . Note (without proof, which is standard) that

$$(4.1) \quad F = f^{-1}f \text{ is bounded on } \mathcal{X} \text{ iff } f \text{ is closed.}$$

A multi-valued function  $F: \mathcal{X} \rightarrow \mathcal{X}$  is said to be *weakly bounded at  $x \in \mathcal{X}$*  if for every sequence  $\{\xi_n\}$  such that  $\lim_{\xi \rightarrow x} \xi_n = x$  and  $\xi_n \in \mathcal{X}_F$  there is  $p \notin \text{Ls } F(\xi_n)$ . It is said to be *weakly bounded on  $\mathcal{X}$*  if it is weakly bounded at every  $x \in \mathcal{X}$ .

**5. Partial functions.** Let  $X^*$  be a compact subset of  $\mathcal{X}^*$ . Consider the partial function  $\Psi|X^*$ . It is upper semicontinuous and the values of  $\Psi|X^*$  are compact subsets (1.6) and (1.7) of  $\mathcal{X}^*$ . Hence  $\Psi|X^*$  is of the I-st Baire class and therefore the theorems (1.9)-(1.11), (3.1) and (3.2) remain true if we substitute  $\Psi|X^*$  for  $\mathcal{X}^*$ ,  $X_{\Psi|X}$  for  $\mathcal{X}_{\Psi}$ , etc. For convenience we shall write  $X_F$  instead of  $X_{F|X}$ .

We now prove some simple properties of partial functions of  $\Phi$ .

(5.1) *If  $X \subset \mathcal{X} - \Psi(p)$  is compact, then  $\Phi|X$  is upper semicontinuous and the values of  $\Phi|X$  are compact subsets of some compact space  $X' \subset \mathcal{X}$ .*

To prove this, let  $x \in X$ . We have, by hypothesis,  $p \notin \text{Ls } \Phi(\xi)$ . Hence, by (1.5),  $\text{Ls } \Phi(\xi) \subset \Phi(x)$ . The required compact subset  $X'$  of  $\mathcal{X}$  is  $\bigcup_{\xi \in X} \overline{\Phi(\xi)}$ .

By (5.1), we conclude, as before, that if  $X \subset \mathcal{X} - \Psi(p)$  is compact, then  $\Phi|X$  is of the I-st Baire class. Therefore theorems (1.9)-(1.11), (3.1) and (3.2) remain true if we substitute  $\Phi|X$  for  $\Psi$ ,  $X_{\Phi}$  for  $\mathcal{X}_{\Psi}$  etc., where  $X$  satisfies the hypotheses of (5.1).

We now consider a special case, namely that of compact  $\mathcal{X}$ . In this case  $\mathcal{X}^*$  may be taken as  $\mathcal{X}$  with an isolated point  $p$  added to it. We have then  $\Psi(p) = p$ , hence the hypotheses of (5.1) are satisfied. Therefore theorems (1.9)-(1.11), (3.1) and (3.2) remain true if we substitute  $X$  for  $\mathcal{X}^*$ ,  $\Phi|X$  for  $\Psi$  and we make further obvious simplifications, where  $X$  is as in (5.1).

**6. A converse problem.** Let us consider a decomposition of  $\mathcal{X}$  into disjoint compact subsets

$$(6.1) \quad \mathcal{X} = \bigcup_{y \in Y} X_y.$$

Consider the question: under what conditions is it possible to introduce a Hausdorff ( $T_2$ ) topology in  $\mathcal{Y}$  and to define a continuous function

$f: \mathcal{X} \rightarrow \mathcal{Y}$  such that  $f(x) = y$  iff  $x \in X_y$ ? If  $\mathcal{X}$  is compact, then, by the well known theorem of Alexandroff, this is possible iff the decomposition (6.1) is upper semicontinuous ([10], II, p. 42). Without this condition it is possible to introduce in  $\mathcal{Y}$  only a  $T_1$  topology and to define such a function  $f$  ([1], p. 63, Satz II). We shall introduce a Hausdorff topology in  $\mathcal{Y}$  and define  $f: \mathcal{X} \rightarrow \mathcal{Y}$ ,  $f$  having the above described properties, for locally compact  $\mathcal{X}$ , under the assumption that the decomposition (6.1) has the following property:

(6.2) *If  $X_y \cap \text{Li} X_{y_k} \neq 0$ , then  $\text{Ls} X_{y_k} \subset X_y \cup p$ , for every sequence  $\{y_k\}$ , where  $y_k \in \mathcal{Y}$ .*

The topology in  $\mathcal{Y}$  is defined as follows. Neighbourhoods of  $y$  in  $\mathcal{Y}$  are all sets of the form  $W = \{y' \in \mathcal{Y} : X_{y'} \subset U \cup V, U \cap X_{y'} \neq 0\}$ , where  $U$  is an arbitrary open subset of  $\mathcal{X}$ , containing  $X_y$ , and  $V$  is an arbitrary open neighbourhood of  $p$  in  $\mathcal{X}^*$ . All the axioms of Hausdorff spaces are satisfied (axioms A-D of [10], I, p. 33). We shall verify axiom D only (i. e. referring to the separation of two points by disjoint open neighbourhoods). Let  $y_1 \neq y_2$  be some arbitrary points of  $\mathcal{Y}$ . We have  $X_{y_1} \cap X_{y_2} = 0$ . Because  $\mathcal{X}$  is metric and therefore normal, there exist open subsets  $U_1$  and  $U_2$  of  $\mathcal{X}$  such that  $X_{y_1} \subset U_1$ ,  $X_{y_2} \subset U_2$  and  $U_1 \cap U_2 = 0$ , and open neighbourhoods  $V_1$  and  $V_2$  of  $p$  in  $\mathcal{X}^*$  such that  $V_i \cap U_j = 0$ , where  $i, j = 1$  and  $2$ . Hence the neighbourhoods  $W_1$  and  $W_2$  of  $y_1$  and  $y_2$ , defined by  $U_1, V_1$  and  $U_2, V_2$  respectively, are disjoint.

The convergence in  $\mathcal{Y}$  defined by this topology has the following property which is an immediate consequence of (6.2):

(6.3)  $\lim_{k \rightarrow \infty} y_k = y$  iff  $X_y \cap \text{Li} X_{y_k} \neq 0$  and  $\text{Ls} X_{y_k} \subset X_y \cup p$ .

We prove now that

(6.4)  *$\mathcal{Y}$  is compact iff  $\text{Ls} X_{y_k} \neq p$  for every sequence  $\{y_k\}$ , where  $y_k \in \mathcal{Y}$ .*

**Sufficiency.** Let  $x \in \text{Ls} X_{y_k} - p$ . Then there exists a sequence  $\{x_{k_n}\}$  converging to  $x$  such that  $x_{k_n} \in X_{y_{k_n}}$ . Hence,  $x \in \text{Li} X_{y_{k_n}}$ . By (6.2),  $\text{Ls} X_{y_{k_n}} \subset X_y \cup p$ , where  $y$  is such that  $x \in X_y$ . Hence, by (6.3),  $\lim y_{k_n} = y$ .

**Necessity.** Suppose, on the contrary, that there exists a sequence  $\{y_k\}$  such that  $y_k \in \mathcal{Y}$  and  $\text{Ls} X_{y_k} = p$ . Then for every  $y \in \mathcal{Y}$  there exists an open neighbourhood  $U$  of  $X_y$  in  $\mathcal{X}$  such that  $U \cap X_{y_k} = 0$  for  $k$  sufficiently large. Hence,  $y$  cannot be a limit point of  $\{y_k\}$ . But  $\mathcal{Y}$  is assumed to be compact and we have a contradiction.

(6.5) *The function  $f: \mathcal{X} \rightarrow \mathcal{Y}$  defined by  $\{f(x) = y$  iff  $x \in X_y\}$  is continuous.*

To prove this, let  $x = \lim x_k$ ,  $x \in X_y$ ,  $x_k \in X_{y_k}$ . Then  $X_y \cap \text{Li } X_{y_k} \neq \emptyset$ . From (6.2) it follows that  $\text{Ls } X_{y_k} \subset X_y \cup p$  and hence, by (6.3),  $\lim y_k = y$ . From the definition of  $f$  we have  $y = f(x)$  and  $y_k = f(x_k)$ , hence  $\lim f(x_k) = f(x)$ .

Corollaries. Let us consider a  $k$ -valued function  $\Phi: \mathcal{X} \rightarrow \mathcal{X}$  defined by (1.2), having properties (1.3), (1.4), and such that  $\text{Ls } \Phi(\xi) \subset \Phi(x) \cup p$  for every  $x \in \mathcal{X}$ . This function induces a decomposition of  $\mathcal{X}$  into disjoint compact subsets having properties (6.1) and (6.2). We shall verify (6.2). Let  $\Phi(x) \cap \text{Li } \Phi(x_k) \neq \emptyset$ . Then there exists a point  $x^j \in \Phi(x)$  and a sequence  $\{v_k\}$ ,  $v_k \in \Phi(x_k)$ , such that  $\lim_{k \rightarrow \infty} v_k = x^j$ . We have, by (1.4),  $\Phi(x_k) = \Phi(v_k)$  and  $\Phi(x) = \Phi(x^j)$ . According to the last assumption concerning  $\Phi$  we have  $\text{Ls } \Phi(v_k) \subset \Phi(x^j) \cup p$ , i. e.

$$\text{Ls } \Phi(x_k) \subset \Phi(x) \cup p.$$

$k \rightarrow \infty$

We have then

(6.6) *If  $\Phi: \mathcal{X} \rightarrow \mathcal{X}$  is a  $k$ -valued function given by (1.2), having properties (1.3), (1.4), and such that  $\text{Ls } \Phi(\xi) \subset \Phi(x) \cup p$  for every  $x \in \mathcal{X}$ , then there exists a topological Hausdorff space  $\mathcal{Y}$  and a continuous ( $k$ -to-one) function  $f: \mathcal{X} \rightarrow \mathcal{Y}$  such that  $f^{-1}f(x) = \Phi(x)$  for every  $x \in \mathcal{X}$ . Furthermore,  $\mathcal{Y}$  is compact iff  $\text{Ls } \Phi(\xi) \neq p$ .*

$\xi \rightarrow p$



II. GENERAL PROPERTIES OF TWO-TO-ONE FUNCTIONS  
ON LOCALLY COMPACT SPACES

**1. Semicontinuous involution  $\varphi$ .** If  $k = 2$ , then, by I (1.2),  $\Phi(x) = x \cup x^2$ . Let us denote  $x^2$  by  $\varphi(x)$ . We have

$$(1.1) \quad \Phi(x) = x \cup \varphi(x)$$

and  $\Phi\varphi(x) = \Phi(x)$ . Hence  $\varphi\varphi(x) = x$ . Because  $\varphi(x) \neq x$  for every  $x \in \mathcal{X}$  ( $f$  is two-to-one),  $\varphi$  is an involution without fixed points on  $\mathcal{X}$ . It may be discontinuous, but it is *semicontinuous*, i. e.

$$(1.2) \quad \text{Ls}_{\xi \rightarrow x} \varphi(\xi) \subset x \cup \varphi(x) \cup p,$$

which is another form of I (1.5).

Denote by  $\mathcal{X}_\varphi$  the set of points of continuity of  $\varphi$ . It is clear that

$$(1.3) \quad \mathcal{X}_\varphi = \mathcal{X}_\Phi = \mathcal{X}_\Psi.$$

Hence, from I (1.10) and I (1.11), it follows that

$$(1.4) \quad \mathcal{X}_\varphi \text{ is open and dense in } \mathcal{X}.$$

We give now some other properties of  $\varphi$  being obvious translations of that of  $\Phi$  or  $\Psi$ .

(1.5) *If  $x \in \mathcal{X}_\varphi$ , then there exists an open neighbourhood  $U$  of  $x$  contained in  $\mathcal{X}_\varphi$  such that  $\varphi|_{\bar{U}}$  is a homeomorphism and  $\varphi(\bar{U}) \cap \bar{U} = \emptyset$ .*

(This follows from I (2.7) and I (2.9)).

(1.6)  *$x \in \mathcal{X} - \mathcal{X}_\varphi$  iff there exists a sequence  $\{\xi_n\}$  such that  $\lim \xi_n = x$  and  $\lim \varphi(\xi_n)$  is  $x$  or  $p$ .*

(This follows from I (3.1)).

(1.7) *If  $x \in \mathcal{X} - \mathcal{X}_\varphi$ , then there exists a sequence  $\{\xi_n\}$  such that  $\lim \xi_n = x$ ,  $\xi_n \in \mathcal{X}_\varphi$  and  $\lim \varphi(\xi_n)$  is  $x$  or  $p$ .*

(This follows from I (3.2)).

As in I.4,  $\varphi$  is said to be *bounded at  $x \in \mathcal{X}$*  if  $p \notin \text{Ls}_{\xi \rightarrow x} \varphi(\xi)$ , and it is said to be *bounded on  $\mathcal{X}$*  if it is bounded at every  $x \in \mathcal{X}$ . Similarly,  $\varphi$  is said to be *weakly bounded at  $x \in \mathcal{X}$*  if  $p \notin \text{Ls}_{\substack{\xi \rightarrow x, \\ \xi \in \mathcal{X}_\varphi}} \varphi(\xi)$ , and it is said to be *weakly bounded on  $\mathcal{X}$*  if it is weakly bounded at every  $x \in \mathcal{X}$ .

According to I.5, all the assertions (1.4)-(1.7) remain true if we substitute  $\varphi|_{\mathcal{X}}$  for  $\varphi$ ,  $X^* = X \cup p$  for  $\mathcal{X}^*$ , and  $X_{\varphi|_{\mathcal{X}}} = X_\varphi$  for  $\mathcal{X}_\varphi$ .

Finally, from I.6 it follows that

(1.8) *If  $\varphi$  is a semicontinuous involution on  $\mathcal{X}$ , then there exists a topological Hausdorff space  $\mathcal{Y}$  and a continuous function  $f: \mathcal{X} \rightarrow \mathcal{Y}$  such that  $f^{-1}f(x) = x \cup \varphi(x)$  for every  $x \in \mathcal{X}$ . Furthermore,  $\mathcal{Y}$  is compact iff  $\text{Ls } \varphi(\xi) \neq p$  and  $f$  is two-to-one iff  $\varphi$  is without fixed points.*

**2. Classification of points of discontinuity of  $\varphi$ .** For every  $x \in \mathcal{X}$  consider

$$\Psi^*(x) = \text{Ls } \varphi(\xi).$$

Semicontinuity of  $\Psi$  implies  $\Psi^*(x) \subset \Psi(x)$ . Note that if  $x \in \mathcal{X}_\varphi$ , then  $\Psi^*(x) = \varphi(x)$  and if  $\varphi$  is discontinuous at  $x$ , i. e.  $x \in \mathcal{X} - \mathcal{X}_\varphi$ , then, by (1.7),  $x$  or  $p$  is in  $\Psi^*(x)$ . Note also that  $\varphi$  is weakly bounded at  $x$  iff  $p \notin \Psi^*(x)$ . There are three kinds of discontinuities of  $\varphi$  according to several forms of  $\Psi^*(x)$ .

We say that  $x \in \mathcal{X}$  is a *removable point of discontinuity* of  $\varphi$  ( $x \in \text{RD}(\varphi)$ ) or  $x$  is an *RD-point* of  $\varphi$  if  $\Psi^*(x) = x$ . We say that  $x \in \mathcal{X}$  is a *weakly essential point of discontinuity* of  $\varphi$  ( $x \in \text{WED}(\varphi)$ ) or  $x$  is a *WED-point* of  $\varphi$  if  $\Psi^*(x) = x \cup \varphi(x)$ . Finally, we say that  $x \in \mathcal{X}$  is a *strongly essential point of discontinuity* of  $\varphi$  ( $x \in \text{SED}(\varphi)$ ) or  $x$  is a *SED-point* of  $\varphi$  if  $p \in \Psi^*(x)$ . Note that  $\varphi$  is weakly bounded at the RD-points and WED-points, but is not weakly bounded at the SED-points.

This classification holds for partial functions of  $\varphi$ , too. Let  $X \subset \mathcal{X}$  and  $x \in X$ . Then, for example,  $x$  may be a *WED-point* of  $\varphi|X$  ( $x \in \text{WED}(\varphi|X)$ ), i. e.  $\Psi_X^*(x) = x \cup \varphi(x)$ , where  $\Psi_X^*(x) = \text{Ls } \varphi(\xi)$ .

**3. WED-points of  $\varphi|X$ , where  $X$  is a closed subset of  $\mathcal{X}$ .** We shall prove here a simple but important lemma. Let  $X$  be a closed subset of  $\mathcal{X}$ . As before,  $X^* = X \cup p$ . If  $A \subset X$ , then  $\bar{A}$  denotes the closure of  $A$  with respect to  $X^*$ , i. e. with respect to the whole  $\mathcal{X}^*$ .

(3.1) *If  $x \in \text{WED}(\varphi|X)$ , then there exists an open neighbourhood  $U$  of  $x$  in  $X$  such that  $U \cap \text{SED}(\varphi|X) = \emptyset$ , and such that, for every open neighbourhood  $V \subset U$  of  $x$  in  $X$ ,  $\varphi(\overline{V \cap X_\varphi})$  is disconnected between  $x$  and  $\varphi(x)$ , i. e.*

$$\overline{\varphi(V \cap X_\varphi)} = P_V \cup Q_V,$$

where

- (i)  $P_V$  and  $Q_V$  are open-closed subsets of  $\overline{\varphi(V \cap X_\varphi)}$ ,
- (ii)  $P_V \neq \emptyset \neq Q_V$ ,
- (iii)  $P_V \cap Q_V = \emptyset$ ,
- (iv)  $x \in P_V, \varphi(x) \in Q_V$ .

To prove (3.1), consider a sequence  $\{V_n\}$  of open neighbourhoods of  $x$  in  $X$  without SED-points of  $\varphi|X$  and such that  $\bigcap_{n=1}^{\infty} V_n = x$ . Note that if  $n$  is sufficiently large, then  $\overline{\varphi(V_n \cap X_\varphi)}$  is disconnected between  $x$  and  $\varphi(x)$ . In fact, connectedness of  $\overline{\varphi(V_n \cap X_\varphi)}$  between  $x$  and  $\varphi(x)$  for all  $n$  implies connectedness of  $\overline{\text{Ls}\varphi(V_n \cap X_\varphi)}$  between these points. That is impossible because, according to (1.2),  $\overline{\text{Ls}\varphi(V_n \cap X_\varphi)} = \text{Ls}\varphi(V_n \cap X_\varphi) = x \cup \varphi(x)$ . Therefore, it may be assumed that  $U$  is one of  $V_n$ , and (3.1) is proved.

Let  $V \subset U$  be an arbitrary neighbourhood of  $x$  in  $X$  such that

$$(3.2) \quad \overline{V} \cap Q_V = 0.$$

Consider an open (in  $X$ ) set  $N_V \subset V$  defined by

$$(3.3) \quad N_V = \varphi^{-1}[Q_V \cap \varphi(V \cap X_\varphi)].$$

Note that  $N_V \neq 0$ . That is a consequence of (iv) and  $x \in WED(\varphi|X)$ . Furthermore

$$(3.4) \quad W \cap N_V \neq 0 \text{ for every open neighbourhood } W \text{ of } x \text{ in } X.$$

Consider another open (in  $X$ ) set  $M_V$  defined by

$$(3.5) \quad M_V = V - \overline{N}_V.$$

Note that  $M_V \neq 0$ . That is also a consequence of (iv) and  $x \in WED(\varphi|X)$ . Furthermore <sup>(2)</sup>

$$(3.6) \quad \varphi(M_V \cap X_\varphi) \subset P_V,$$

$$(3.7) \quad \text{Fr}(M_V) \subset \overline{N}_V \cap V,$$

and hence, by definition (3.3) of  $N_V$ , we obtain

$$(3.8) \quad \text{Fr}(M_V) \cap V \cap X_\varphi = 0.$$

We shall prove now that

(3.9) *If  $\xi \in \text{Fr}(M_V) \cap V$ , then for every sequence  $\{\xi_n\}$  such that  $\xi_n \in M_V$  and  $\lim \xi_n = \xi$  we have  $\lim \varphi(\xi_n) = \xi$ . Because  $x \in \text{Fr}(M_V) \cap V$ , the same is true for  $x$ .*

Suppose, on the contrary, that  $\lim \varphi(\xi_n) = \varphi(\xi)$ . Then, by (3.6), we have  $\varphi(\xi) \in P_V$ .

By (3.7) and  $\xi \in \text{Fr}(M_V)$ , there exists a sequence  $\{\xi'_n\}$  with  $\lim \xi'_n = \xi$  and  $\xi'_n \in N_V$ . By definition (3.3) of  $N_V$ , we have  $\lim \varphi(\xi'_n) \in Q_V$ . From (3.2) it follows that  $\xi \neq \lim \varphi(\xi'_n) \neq p$ . Hence  $\lim \varphi(\xi'_n) = \varphi(\xi)$  and therefore  $\varphi(\xi) \in Q_V$ . This is in contradiction to (iii) and  $\varphi(\xi) \in P_V$ .

**4. The function  $\tilde{\varphi}$  on  $V$ .** Let  $x$ ,  $U$  and  $V$  be as in (3.1). Consider a function  $\tilde{\varphi}: V \rightarrow \mathcal{X}$  given by

<sup>(2)</sup> If  $A \subset X$ , then  $\text{Fr}(A)$  denotes the boundary of  $A$  with respect to  $X$ , i. e.  $\text{Fr}(A) = \overline{A \cap X} - A$ .

$$(4.1) \quad \tilde{\varphi}(\xi) = \begin{cases} \varphi(\xi) & \text{if } \xi \in M_V \cap X_\varphi, \\ \xi & \text{if } \xi \in V - (M_V \cap X_\varphi). \end{cases}$$

The function  $\tilde{\varphi}$  may be discontinuous. But

(4.2) *If the partial function  $\varphi|_{M_V}$  has no WED-points, then  $\tilde{\varphi}$  is continuous on  $V$ .*

It is clear that  $\tilde{\varphi}$  is continuous at  $\xi \in M_V \cap X_\varphi$  and  $\xi \in N_V$ . If  $\xi \in \text{Fr}(M_V \cap V)$ , then  $\tilde{\varphi}(\xi) = \xi$  and continuity is a consequence of (3.9) and (4.1). Let then  $\xi \in M_V - X_\varphi$ . Since  $U \cap \text{SED}(\varphi|X) = \emptyset$ ,  $\xi \in \text{RD}(\varphi|X)$ . Therefore  $\tilde{\varphi}(\xi) = \xi$  and  $\lim \varphi(\xi_n) = \xi$  for every sequence  $\{\xi_n\}$  which converges to  $\xi$ . Thus (4.2) is proved.

The following special lemma will be used in the next chapter:

(4.3) *If  $\varphi|_{M_V}$  has no WED-points and  $\xi' \neq \xi''$  are points of  $V$  and (i) there exists a sequence  $\{\xi'_n\}$  such that  $\xi'_n \in X_\varphi$ ,  $\varphi(\xi'_n) \in X_\varphi$  and*

$$\lim \xi'_n = \xi,$$

*then  $\tilde{\varphi}(\xi') \neq \tilde{\varphi}(\xi'')$ .*

This inequality is obvious, even without assumption (i), if  $\xi'$  and  $\xi''$  both belong, or not, to  $X_\varphi \cap M_V$ .

It is sufficient to prove (4.3) for  $\xi' \in X_\varphi \cap M_V$  and  $\xi'' \in V - (X_\varphi \cap M_V)$ . We have then, by (4.1),  $\tilde{\varphi}(\xi') = \varphi(\xi')$  and  $\tilde{\varphi}(\xi'') = \xi''$ .

Suppose, on the contrary, that  $\tilde{\varphi}(\xi') = \tilde{\varphi}(\xi'')$ . Then  $\tilde{\varphi}(\xi') = \xi''$  and, by continuity of  $\tilde{\varphi}$  at  $\xi' \in X_\varphi$ , it follows that

$$(*) \quad \lim \tilde{\varphi}(\xi'_n) = \tilde{\varphi}(\xi') = \xi''.$$

Consider two cases.

1.  $\tilde{\varphi}(\xi'_n) \in X_\varphi \cap M_V$ . From (\*) and  $\xi'' \in V - (X_\varphi \cap M_V)$  it follows that  $\xi'' \in \overline{M_V} \cap V$ . If  $\xi'' \in \text{Fr}(M_V)$ , then, by (3.9) and (\*),  $\lim \xi'_n = \lim \tilde{\varphi}(\xi'_n) = \xi''$ .

Because  $\lim \xi'_n = \xi' \neq \xi''$ , we have a contradiction. If  $\xi'' \in M_V$ , then  $\xi'' \in \text{RD}(\varphi|X)$ . Hence, by (\*),  $\lim \xi'_n = \lim \tilde{\varphi}(\xi'_n) = \xi'' \neq \xi'$ . A contradiction.

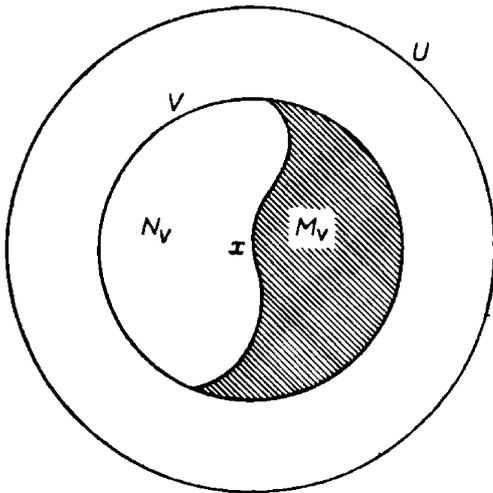


Fig. 1

2.  $\tilde{\varphi}(\xi'_n) \in X_\varphi - M_V$ . From  $\tilde{\varphi}(\xi'_n) \in X_\varphi$  and (3.8) it follows that  $\tilde{\varphi}(\xi'_n) \in N_V$ . Hence, by definition of  $N_V$ , we have  $\tilde{\varphi}(\xi'_n) = \xi'_n \in Q_V$ . On the other hand, by hypothesis, we have  $\lim \xi'_n = \xi' \in V$ . Hence, for sufficiently large  $n$ ,  $\xi'_n \in V$ , which is in contradiction to (3.2).

Thus (4.3) is proved.

**5. The function  $\varphi$  considered on  $\overline{\text{WED}(\varphi|X)}$ .** Let  $x$ ,  $X$ ,  $U$  and  $V$  be as in (3.1). We prove here an important lemma.

(5.1) *If  $x$  is a WED-point of  $\varphi|X$  and if for every open neighbourhood  $V \subset U$  of  $x$  in  $X$  the function  $\varphi|M_V$  has WED-points, then the function  $\varphi|\overline{\text{WED}(\varphi|X)}$  is discontinuous at  $x$ .*

To prove this, let  $V_0 \subset U$  be an open neighbourhood of  $x$  in  $X$  for which (3.2) holds. From hypotheses it follows that there exists a sequence  $\{V_n\}$  of open neighbourhoods of  $x$  in  $X$ , where  $V_n \subset V_0$ , such that  $\bigcap_{n=1}^{\infty} V_n = x$  and such that every function  $\varphi|M_{V_n}$  has WED-points. Choose a WED-point  $\xi_n$  of  $\varphi|M_{V_n}$ . Since  $V_n$  are open in  $X$ ,  $\xi_n$  are WED-points of  $\varphi|X$ , too. We have  $\lim \xi_n = x$ . Then from (3.9) it follows that  $\lim \varphi(\xi_n) = x$ . By the discontinuity criterion I (3.1) applied to  $\varphi|\text{WED}(\varphi|X)$ , we find that  $x$  is a point of discontinuity of  $\varphi|\text{WED}(\varphi|X)$  and hence of  $\varphi|\overline{\text{WED}(\varphi|X)}$ .

### III. BEHAVIOUR OF SEMICONTINUOUS INVOLUTION AT EUCLIDEAN POINTS

**1. Definitions.** A point  $x \in \mathcal{X}$  is said to be a *Euclidean point* ( $x \in E(\mathcal{X})$  or  $x$  is an *E-point* of  $\mathcal{X}$ ) if there exists a neighbourhood  $H$  of  $x$  in  $\mathcal{X}$ , whose closure is a Euclidean solid sphere (of dimension  $n \geq 1$ ). It is said to be an *interior Euclidean point* ( $x \in IE(\mathcal{X})$  or  $x$  is an *IE-point* of  $\mathcal{X}$ ) if  $x$  lies in the geometrical interior of this solid sphere, and a *boundary Euclidean point* ( $x \in BE(\mathcal{X})$  or  $x$  is a *BE-point* of  $\mathcal{X}$ ) if  $x$  lies on its surface. All the IE-points of  $\mathcal{X}$  have the so called *spherical neighbourhoods*, i. e. such neighbourhoods whose closures are solid spheres consisting of IE-points of  $\mathcal{X}$  only. All the BE-points of  $\mathcal{X}$  have the so called *hemispherical neighbourhoods*, i. e. such neighbourhoods the closures of which are solid spheres whose surfaces consist of two hemispheres, one of which consists of BE-points of  $\mathcal{X}$  only and contains  $x$ ; the other consists of IE-points of  $\mathcal{X}$  only.

The homeomorphic image of a spherical (hemispherical) neighbourhood  $H$  of  $x$  in  $\mathcal{X}$  is not, in general, a spherical (hemispherical) neighbourhood of the image of  $x$ . However,

(1.1) Let  $\bar{H} \subset G$ , where  $G$  and  $H$  are spherical (hemispherical) neighbourhoods of  $x$  in  $\mathcal{X}$  and let  $g : G \rightarrow \mathcal{X}$  be a homeomorphism such that  $g(\bar{H}) \subset G$ . Then  $g(H)$  is a spherical (hemispherical) neighbourhood of  $g(x)$  in  $\mathcal{X}$ .

The proof is standard and will be omitted.

If  $U$  is a neighbourhood of  $x$  in  $\mathcal{X}$ , then we shall call the component of  $U$  which contains  $x$  a *pseudo-neighbourhood* of  $x$  in  $\mathcal{X}$ . We shall call a point  $x \in \mathcal{X}$  *pseudo-Euclidean* ( $x \in PE(\mathcal{X})$  or  $x$  is a *PE-point* of  $\mathcal{X}$ ) if there exists a pseudo-neighbourhood of  $x$  in  $\mathcal{X}$  whose closure is a Euclidean solid sphere. A PE-point  $x$  of  $\mathcal{X}$  may be an *interior* ( $x \in IPE(\mathcal{X})$  or  $x$  is an *IPE-point* of  $\mathcal{X}$ ) or a *boundary point* ( $x \in BPE(\mathcal{X})$  or  $x$  is a *BPE-point* of  $\mathcal{X}$ ). We omit the details of full definitions.

**2. The function  $\varphi|_X$ , where  $X$  is the closure of a spherical (hemispherical) pseudo-neighbourhood of a PE-point.** Let  $x$  be a PE-point of  $\mathcal{X}$ . Let  $X$  denote the closure of a spherical or hemispherical pseudo-neighbourhood of  $x$  in  $\mathcal{X}$  according to  $x$  being a IPE-point or a BPE-point of  $\mathcal{X}$ .



Assume that  $x$  is a WED-point of  $\varphi|X$ .

Let  $U, V, P_V, Q_V, M_V$ , and  $N_V$  have the same meaning as in II.3. We assume, in addition, that  $U$  and  $V$  are spherical or hemispherical neighbourhoods of  $x$  in  $X$  according to  $x$  being an IPE-point or a BPE-point of  $X$ .

We prove now another very important lemma which is a specialization of the results of II.4 for the case considered here. Namely, under assumptions as above, we prove that

(2.1) *The function  $\varphi|M_V$  has WED-points.*

Suppose, on the contrary, that there exists an open neighbourhood  $V^* \subset U$  of  $x$  in  $X$  such that  $\varphi|M_{V^*}$  is without WED-points. Then the function  $\tilde{\varphi}$  defined by II (4.1) for  $V = V^*$  is continuous on  $V^*$  and  $x$  is a fixed point of  $\tilde{\varphi}$  (because  $x \in X - X_\varphi$ ). Consider an open, spherical, or hemispherical, neighbourhood  $W \subset V^*$  of  $x$  in  $X$  such that

$$(1) \quad \tilde{\varphi}(\overline{W}) \subset V^*.$$

We set  $M_W = W \cap M_V$ . It is clear, by hypothesis, that  $\varphi|M_W$  has no WED-points. We prove that

$$(2) \quad \tilde{\varphi} \text{ is one-to-one on } W.$$

Let  $\xi' \neq \xi''$  be points of  $W$ . The inequality  $\tilde{\varphi}(\xi') \neq \tilde{\varphi}(\xi'')$  is obvious if both  $\xi'$  and  $\xi''$  lie in  $M_W \cap X_\varphi$  or not. It remains to consider the case  $\xi' \in M_W \cap X_\varphi$  and  $\xi'' \in W - (M_W \cap X_\varphi)$ .

Let  $\{W_n\}$  be a sequence of spherical (hemispherical) neighbourhoods of  $\xi'$  such that

$$(i) \quad \bigcap_{n=1}^{\infty} \overline{W}_n = \xi',$$

$$(ii) \quad \overline{W}_n \subset M_W \cap X_\varphi, \quad n = 1, 2, \dots,$$

and such that  $\varphi|\overline{W}_n$  are homeomorphisms. Such a sequence exists by II(2.7). From (1), (ii) and definition of  $\tilde{\varphi}$  it follows that  $\tilde{\varphi}(\overline{W}_n) = \varphi(\overline{W}_n) \subset V^*$ , where  $\varphi(\overline{W}_n)$  are Euclidean solid spheres of the same dimension. Because  $X_\varphi$  is dense in  $X$ , we have, for all  $n$ ,  $\varphi(\overline{W}_n) \cap X \neq \emptyset$ . Choose  $\xi'_n \in \varphi(\overline{W}_n)$  and set  $\xi'_n = \varphi(\xi''_n)$ . We have  $\xi'_n \in \overline{W}_n$ . Hence, by (i) and (ii),  $\xi'_n \in X_\varphi$  and  $\lim \xi'_n = \xi'$ . We see that all the hypotheses of II (4.3) are satisfied. Therefore  $\tilde{\varphi}(\xi') \neq \tilde{\varphi}(\xi'')$  and (2) is proved.

From (2) it follows that

$$(3) \quad \tilde{\varphi}(M_W \cap X_\varphi) \cap (W - (M_W \cap X_\varphi)) = \emptyset.$$

In fact, if  $\tilde{\varphi}(\xi') = \xi''$ ,  $\xi' \in M_W \cap X_\varphi$  and  $\xi'' \in W - (M_W \cap X_\varphi)$ , then  $\xi' \neq \xi''$ . We have also, by II (4.1),  $\tilde{\varphi}(\xi'') = \xi''$ . Therefore  $\tilde{\varphi}(\xi') = \tilde{\varphi}(\xi'')$ , contrary to (2).

Let now  $U$  be an open neighbourhood (spherical or hemispherical) of  $x$  in  $X$  such that

$$(4) \quad \bar{U}' \subset W.$$

Then

$$(5) \quad \tilde{\varphi} \text{ is a homeomorphism on } U'.$$

Let  $V'$  be an open spherical (hemispherical) neighbourhood of  $x$  in  $X$  such that

$$(6) \quad V' \subset U',$$

$$(7) \quad \tilde{\varphi}(V') \subset U'.$$

Such a neighbourhood exists because  $x$  is a fixed point of  $\tilde{\varphi}$ .

All the hypotheses of (1.1) are satisfied if we substitute  $U'$  for  $G$ ,  $V'$  for  $H$ , and  $\tilde{\varphi}$  for  $g$ . Therefore  $\tilde{\varphi}(V')$  is an open spherical (hemispherical) neighbourhood of  $x$  in  $X$ . Hence

$$(8) \quad V' \cap \tilde{\varphi}(V') \text{ is an open neighbourhood of } x \text{ in } X.$$

Let  $\Delta$  be a component of  $x$  in  $V' \cap \tilde{\varphi}(V')$ . It is an open neighbourhood of  $x$  in  $X$ . By (5),  $\tilde{\varphi}$  is a homeomorphism on  $V' \cap \tilde{\varphi}(V')$ . Hence, because  $\tilde{\varphi}(x) = x$ , we have  $\varphi(\Delta) = \Delta$  and therefore

$$(9) \quad \tilde{\varphi}(\bar{\Delta}) = \bar{\Delta}.$$

We prove that

$$(10) \quad \tilde{\varphi} \text{ is an involution on } \bar{\Delta}.$$

To prove this, consider two cases.

1. Let  $\xi \in (M_W \cap X_\varphi) \cap \bar{\Delta}$ . Then, by (3) and (9), we have  $\tilde{\varphi}(\xi) \in (M_W \cap X_\varphi) \cap \bar{\Delta}$ . According to definition II (4.1) of  $\tilde{\varphi}$  (for  $V = V^*$ ) and the definition of  $M_W$  we obtain  $\tilde{\varphi}\tilde{\varphi}(\xi) = \tilde{\varphi}\varphi(\xi) = \varphi\varphi(\xi) = \xi$ .

2. Let  $\xi \in W - (M_W \cap X_\varphi)$ . Then from definition II (4.1) of  $\tilde{\varphi}$  it follows immediately that  $\tilde{\varphi}\tilde{\varphi}(\xi) = \xi$ .

According to II (4.1) the set of fixed points of  $\tilde{\varphi}|_{\bar{\Delta}}$  contains an open set  $N_V \cap \bar{\Delta}$ , which is non-empty in virtue of II (3.4).

On the other hand, all the points of  $\Delta$  are Euclidean with respect to  $X$  and therefore  $\bar{\Delta}$  is the closure of a domain in the Euclidean manifold  $U'$ . According to the well known theorem of Newman ([10], p. 1), the set of fixed points of continuous involution on the closure of a domain in Euclidean manifold does not contain interior points. We have a contradiction. Thus (2.1) is proved.

**3. The main theorem.** All the theorems which will be proved in Chapters III and IV are consequences of the following one:

**THEOREM 1.** *A PE-point of  $\mathcal{X}$  cannot be a WED-point of  $\varphi|X$ , where  $X$  is a spherical or hemispherical pseudo-neighbourhood of  $x$  in  $\mathcal{X}$  according to whether  $x$  is an IPE-point or a BPE-point of  $\mathcal{X}$ .*

**Proof.** Suppose, on the contrary, that  $x$  is a PE-point of  $\mathcal{X}$  and a WED-point of  $\varphi|X$  simultaneously, where  $X$  is as above. Let also  $U$  and  $V$  have the same meaning as formerly. By lemma (2.1), the function  $\varphi|M_V$  has WED-points, which are also WED-points of  $\varphi|V$ , because  $M_V$  is open in  $V$ . Since all these points lie in  $V$ , they are PE-points of  $\mathcal{X}$  and  $X$  is their spherical (hemispherical) neighbourhood in  $\mathcal{X}$ .

Let  $\xi$  be one of these points. By (2.1), for every open neighbourhood  $V'$  of  $\xi$  in  $X$ , the function  $\varphi|M_{V'}$  has WED-points, which are also WED-points of  $\varphi|V$ , because  $V'$ , and hence  $M_{V'}$ , are open in  $V$ . Therefore we conclude that the set of WED-points of  $\varphi|V$  is dense in itself. By II (5.1),  $\overline{\varphi|WED(\varphi|V)}$  is discontinuous at every point of  $WED(\varphi|V)$ , and therefore (see discontinuity criterion I (3.1)) at every point of  $WED(\varphi|V)$ . But we know that  $\varphi$  is of the I-st Baire class and therefore if  $A$  is a closed subset of  $X$ , then  $\varphi|A$  cannot be discontinuous at every point of  $A$ ; thus a contradiction occurs.

#### 4. Immediate consequences of Theorem 1. We prove

**THEOREM 2.** *Let  $R \subset \mathcal{X}$  be a manifold having the following property:*

- (i) *if  $x \in R$ , then there exists a pseudoneighbourhood of  $x$  in  $\mathcal{X}$ , which is a neighbourhood of  $x$  in  $R$  simultaneously<sup>(3)</sup>.*

*Thus, if  $\varphi$  is weakly bounded on  $R$ , then the function  $\tilde{\varphi} : R \rightarrow \mathcal{X}$  given by*

$$(ii) \quad \tilde{\varphi}(\xi) = \begin{cases} \varphi(\xi) & \text{if } \xi \in R, \\ \xi & \text{if } \xi \in R - R_\varphi \end{cases}$$

*is continuous and one-to-one.*

*Furthermore, if  $\tilde{\varphi}(R) \subset R$ , then  $\tilde{\varphi}(R) = R$ ,*

$$(iii) \quad \tilde{\varphi}(R_\varphi) = \varphi(R_\varphi) = R_\varphi,$$

$$(iv) \quad \tilde{\varphi}(R - R_\varphi) = R - R_\varphi,$$

*and  $\tilde{\varphi}$  is an involution on  $R$ , and it cannot be the identity on open subsets of  $R$ .*

**Proof.** Because  $\varphi$  is weakly bounded on  $R$ , i.e.  $\varphi|R$  is weakly bounded, the function  $\varphi|R$  has no SED-points. Also  $\varphi|R$  has no WED-points. If it had, every such point  $\xi$  would be a WED-point of  $\varphi|W$ , where  $W$  is an arbitrary open spherical (hemispherical) neighbourhood of  $\xi$  in  $R$ . Assume, according to (i), that  $W$  is a pseudoneighbourhood of  $\xi$  in  $\mathcal{X}$ . Denoting by  $X$  the closure of  $W$ , we find that  $\xi$  is WED-point of  $\varphi|X$ , contrary to Theorem 1.

<sup>(3)</sup> Note that if  $R$  is a subarc of a dendrite  $\mathcal{X}$ , and  $R$  contains a point of ramification of  $\mathcal{X}$ , then  $R$  has not the property (i).

Hence,  $R$  consists only of continuity points and RD-points of  $\varphi|R$ , i. e.  $R - R_\varphi = \text{RD}(\varphi|R)$ . By the definition of  $\tilde{\varphi}$  and that of RD-points, we conclude that  $\tilde{\varphi}$  is continuous on  $R$ .

We prove now that  $\tilde{\varphi}$  is one-to-one.

Let  $\xi' \neq \xi''$  be points of  $R$ . The inequality  $\tilde{\varphi}(\xi') \neq \tilde{\varphi}(\xi'')$  is obvious if both  $\xi'$  and  $\xi''$  belong to  $R_\varphi$  or not.

It remains to consider the case:  $\xi' \in R_\varphi$  and  $\xi'' \in R - R_\varphi$ . Suppose, on the contrary, that  $\tilde{\varphi}(\xi') = \tilde{\varphi}(\xi'')$ . Then there is

$$(1) \quad \tilde{\varphi}(\xi') = \varphi(\xi') = \xi''.$$

Let  $\{W_n\}$  be a sequence of spherical (hemispherical) neighbourhoods of  $\xi'$  in  $R$  such that

$$(2) \quad \bigcap_{n=1}^{\infty} W_n = \xi',$$

$$(3) \quad \overline{W}_n \subset R_\varphi,$$

and such that  $\varphi|\overline{W}_n$  are homeomorphisms. Such a sequence exists by II (2.7). By (1) and (3) we have

$$(4) \quad \bigcap_{n=1}^{\infty} \tilde{\varphi}(\overline{W}_n) = \bigcap_{n=1}^{\infty} \varphi(\overline{W}_n) = \xi''.$$

According to (i) we assume that  $\tilde{\varphi}(\overline{W}_n) = \varphi(\overline{W}_n) \subset R$  for all  $n$ . Since, by II (1.4),  $R_\varphi$  is dense in  $R$ , and  $\tilde{\varphi}(\overline{W}_n) = \varphi(\overline{W}_n)$  are Euclidean solid spheres of the same dimension as the manifold  $R$ , then  $R_\varphi \cap \tilde{\varphi}(\overline{W}_n) \neq \emptyset$  for all  $n$ . Consider a sequence  $\{\xi'_n\}$ , where  $\xi'_n \in R_\varphi \cap \tilde{\varphi}(\overline{W}_n)$ . By (4) we have

$$(5) \quad \lim \xi'_n = \xi''.$$

Set  $\xi'_n = \varphi(\xi''_n) = \tilde{\varphi}(\xi''_n)$ . Then we have  $\xi'_n \in \overline{W}_n$ . From (2) it follows that  $\lim \xi'_n = \lim \varphi(\xi''_n) = \xi'$ . On the other hand, the continuity of  $\tilde{\varphi}$  and (5) imply  $\lim \varphi(\xi''_n) = \lim \tilde{\varphi}(\xi''_n) = \tilde{\varphi}(\xi'') = \xi''$  (because  $\xi'' \in R - R_\varphi$ ). A contradiction occurs.

To prove the second part of the theorem, note first that from the definition of  $\tilde{\varphi}$  it follows that  $\tilde{\varphi}(R_\varphi) = \varphi(R_\varphi)$  and that  $\tilde{\varphi}$  is the identity on  $R - R_\varphi$ . Hence (iv).

Then the inclusion  $\tilde{\varphi}(R) \subset R$  implies the inclusion  $\tilde{\varphi}(R_\varphi) \subset R_\varphi$ . Hence, if  $\xi \in R_\varphi$ , then  $\tilde{\varphi}(\xi) \in R_\varphi$ . Then, by the definition of  $\tilde{\varphi}$ , we obtain  $\tilde{\varphi}\varphi(\xi) = \varphi\varphi(\xi) = \xi$ , i. e.  $\xi \in \tilde{\varphi}(R_\varphi)$ . We have then  $R_\varphi \subset \tilde{\varphi}(R_\varphi)$ . Thus (iii) is proved.

From (iii) and (iv) it follows immediately  $\tilde{\varphi}(R) = R$ .

Finally we prove that  $\tilde{\varphi}$  is an involution on  $R$ . Consider two cases. If  $\xi \in R - R_\varphi$ , then the equality  $\tilde{\varphi}\tilde{\varphi}(\xi) = \xi$  is an immediate consequence of (ii). If  $\xi \in R_\varphi$ , this equality is a consequence of (ii) and (iii).

Because  $\tilde{\varphi}(\xi) = \varphi(\xi) \neq \xi$  on  $R_\varphi$ , and  $R_\varphi$  is, according to II (1.4), dense in  $R$ ,  $\tilde{\varphi}$  cannot be the identity on open subsets of  $R$ . This completes the proof of the theorem.

In particular, if  $\mathcal{X}$  is a manifold, Theorem 2 goes into the following one:

(4.1) *If  $\mathcal{X}$  is a manifold and  $\varphi$  is weakly bounded on  $\mathcal{X}$ , then the function  $\tilde{\varphi}$ , given by (ii) for  $R = \mathcal{X}$ , is a continuous involution on  $\mathcal{X}$ , which cannot be the identity on the open subsets of  $\mathcal{X}$ . Moreover,  $\tilde{\varphi}(\mathcal{X}_\varphi) = \varphi(\mathcal{X}_\varphi) = \mathcal{X}_\varphi$  and  $\tilde{\varphi}(\mathcal{X} - \mathcal{X}_\varphi) = \mathcal{X} - \mathcal{X}_\varphi$ .*

If  $\mathcal{X}$  is a manifold, then we denote by  $S$  the set of fixed points of  $\tilde{\varphi}$  on  $\mathcal{X}$ . According to the definition (ii) of  $\tilde{\varphi}$  we have

$$(4.2) \quad S = \mathcal{X} - \mathcal{X}_\varphi.$$

We have also  $\varphi(\mathcal{X}_\varphi) = \mathcal{X}_\varphi$ . Therefore, because  $\varphi(\mathcal{X}) = \mathcal{X}$ , we have  $\varphi(S) = S$ .

From the last equality it follows that

$$(4.3) \quad f|_S \text{ is two-to-one.}$$


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#### IV. SOME APPLICATIONS TO COMPACT SPACES

**1. Civin's theorems.** If  $\mathcal{X}$  is a compact manifold, then  $\varphi$  is bounded and therefore  $\varphi$  is weakly bounded on  $\mathcal{X}$ . A special case of III (4.1) is the following theorem of Civin ([4], p. 54, Theorem III; let us recall that  $\varphi$  is bounded iff  $f$  is closed):

(1.1) *If  $\mathcal{X}$  is a compact manifold (with or without boundary), then the function  $\tilde{\varphi}$  given by III.4 (ii) for  $R = \mathcal{X}$  is a continuous involution on  $\mathcal{X}$  which cannot be the identity on the open subsets of  $\mathcal{X}$ . Moreover,  $\tilde{\varphi}(\mathcal{X}_\varphi) = \varphi(\mathcal{X}_\varphi) = \mathcal{X}_\varphi$  and  $\varphi(\mathcal{X} - \mathcal{X}_\varphi) = \mathcal{X} - \mathcal{X}_\varphi$ .*

We give now the proof of the well known theorem of Civin of the non-existence of two-to-one continuous functions on closed Euclidean  $n$ -cells for  $n \leq 3$ :

(1.2) *Two-to-one continuous functions do not exist on closed  $n$ -cells for  $n \leq 3$ .*

We proceed by induction. Such functions do not exist on 0-cell, i. e. on one-point set. Assume that they do not exist on  $k$ -cells, where  $k \leq n-1$ ,  $n \leq 3$ . Suppose, on the contrary, that  $\mathcal{X}$  is closed  $n$ -cell and that there exists  $f: \mathcal{X} \rightarrow \mathcal{Y}$ , where  $\mathcal{Y}$  is an  $\mathcal{L}^*$ -space and  $f$  is continuous and two-to-one. Let  $\varphi$  be a semicontinuous involution associated with  $f$ , and let  $\tilde{\varphi}$  be a continuous involution on  $\mathcal{X}$  as in (1.1). Let  $S$  be the set of fixed points of  $\tilde{\varphi}$ . By III(4.2),  $f|S$  is two-to-one. According to the well known theorems of Kérékjartó ([8], p. 224-226), for  $n \leq 2$ , and Smith [15], for  $n = 3$ ,  $S$  is a closed  $k$ -cell with  $k \leq n-1$ , contrary to assumption.

The theorem of Civin asserts that there do not exist two-to-one continuous functions on  $n$ -cells,  $n \leq 3$ , if their values lie in some  $\mathcal{L}^*$ -space. But there exist such functions if  $\mathcal{Y}$  is only a  $T_1$ -space. That is a consequence of a theorem of Alexandroff-Hopf ([1], p. 63, Satz II); which asserts that for every decomposition of  $T_1$ -space  $\mathcal{X}$  into disjoint closed subsets  $X_y$ , where  $y \in \mathcal{Y}$ , there exists a  $T_1$ -topology in  $\mathcal{Y}$  and a continuous function  $f: \mathcal{X} \rightarrow \mathcal{Y}$  defined by  $\{(x \in X_y) \equiv (f(x) = y)\}$ . Then the two-to-one continuous functions of  $I^n$  into a  $T_1$ -space are defined by an arbitrary decomposition of  $I^n$  into disjoint two-point subsets.

The problem of the existence or non-existence of the two-to-one continuous functions on  $n$ -cells for  $n \geq 4$  is open.

**2. Two-to-one continuous functions on graphs.** We prove now a theorem of non-existence of the two-to-one continuous functions on graphs whose Euler-Poincaré characteristic is odd. This theorem was first proved by Harrold [6] and Gilbert [11]. In our proof Theorem 2 will be used.

A graph is a finite sum of closed intervals, which are disjoint or have their ends in common, and of isolated points. The Euler-Poincaré characteristic is the difference between the numbers of vertices (i. e. ends of intervals and isolated points) and the number of intervals. It is denoted, as usual, by  $\chi = w - p$ . Note that  $\chi$  does not change with subdivisions.

(2.1) *Two-to-one continuous functions do not exist on graphs, when their Euler-Poincaré characteristic is odd.*

Let  $\mathcal{X} = \Gamma$  be a graph and let  $f$  be a two-to-one continuous function on  $\Gamma$ . We prove that the existence of a function  $f$  implies that  $\chi(\Gamma)$  is even.

Let  $\varphi$  be the semicontinuous involution on  $\Gamma$  associated to  $f$ . Let  $(\gamma)$  be a given division of  $\Gamma$  into intervals. Denote by  $R$  the sum of interiors of intervals of  $(\gamma)$ . We see that  $R$  is an open 1-dimensional manifold in  $\Gamma$  and hence all the hypotheses of the first part of Theorem 2 are satisfied for  $R$ . Then the function  $\tilde{\varphi}$ , given by III.4 (ii), is continuous and one-to-one on  $R$ .

Let  $(\gamma')$  be a subdivision of  $(\gamma)$  by adding new vertices  $\varphi^{-1}(a)$ , where  $a$  are vertices of  $(\gamma)$ . Consider the set  $R'$  consisting of the interiors of intervals of  $(\gamma')$ .  $R'$  is an open manifold contained in  $R$ . We see that Condition III.4 (i) holds for  $R'$ . Consider the function  $\tilde{\varphi}': R' \rightarrow \Gamma$  given by III.4 (ii) for  $R = R'$ . It is a partial function of  $\tilde{\varphi}$ . It will be written for convenience  $\tilde{\varphi}$  instead of  $\tilde{\varphi}'$ . From the definition of  $R'$  it follows that

$$(1) \quad \tilde{\varphi}(R') \subset R.$$

We prove even more, namely

$$(2) \quad \tilde{\varphi}(R') \subset R'.$$

To prove this suppose, on the contrary, that there exists a point  $a' \in R'$  such that  $\tilde{\varphi}(a') = b$  is a vertex of  $(\gamma')$ , i. e.  $b \in \Gamma - R'$ . From (1) it follows that  $b \in R - R'$ . It is clear that  $a' \neq b$ . Hence, by III.4 (ii),  $b = \varphi(a')$ . Since  $b \in R - R'$ ,  $b$  cannot be a vertex of  $(\gamma)$ . Hence there exists a point  $a \in \Gamma - R$  such that  $b = \varphi^{-1}(a)$ , i. e.  $\tilde{\varphi}(b) = a$ . Since  $b \in R$  and  $a \in \Gamma - R$ , we have  $a \neq b$ . Therefore, by III.4 (ii),  $a = \varphi(b)$ . We have then  $a = a'$  because  $b = \varphi(a')$ . Then the point  $a$  would be a vertex of  $(\gamma)$  and a point of  $R'$ . A contradiction occurs.

According to (2) the hypotheses of the second part of Theorem 2 are satisfied. Therefore  $\tilde{\varphi}$  is a continuous involution on  $R'$  such that

$$(3) \quad \tilde{\varphi}(R'_\varphi) = \varphi(R'_\varphi) = R'_\varphi.$$

The interiors of intervals of  $(\gamma')$  are components of  $R'$ . The continuous involution  $\tilde{\varphi}$  maps these intervals onto intervals. If  $\tilde{\varphi}$  maps an interval onto itself, it has only one fixed point.

Consider a subdivision  $(\gamma'')$  of  $(\gamma')$  by adding all the fixed points as new vertices. Denote the sum of the new intervals by  $R''$ . We have  $R'' \subset R'$ ,  $\tilde{\varphi}(R'') = R''$  and  $R''_\varphi = R''$ . Hence, by (3), we obtain

$$(4) \quad \tilde{\varphi}(R'') = \varphi(R'') = R''$$

and therefore

$$(5) \quad \varphi(\Gamma - R'') = \Gamma - R.$$

$\Gamma - R''$  is the set of all vertices of  $(\gamma'')$ . By (5), the number of points in  $\Gamma - R''$  is even. Thus, by (4), the number of components of  $R''$ , i. e. the number of intervals in  $(\gamma'')$  is even. Hence also the Euler-Poincaré characteristic of  $\Gamma$  in subdivision  $(\gamma'')$  is even. Thus (2.1) is proved.

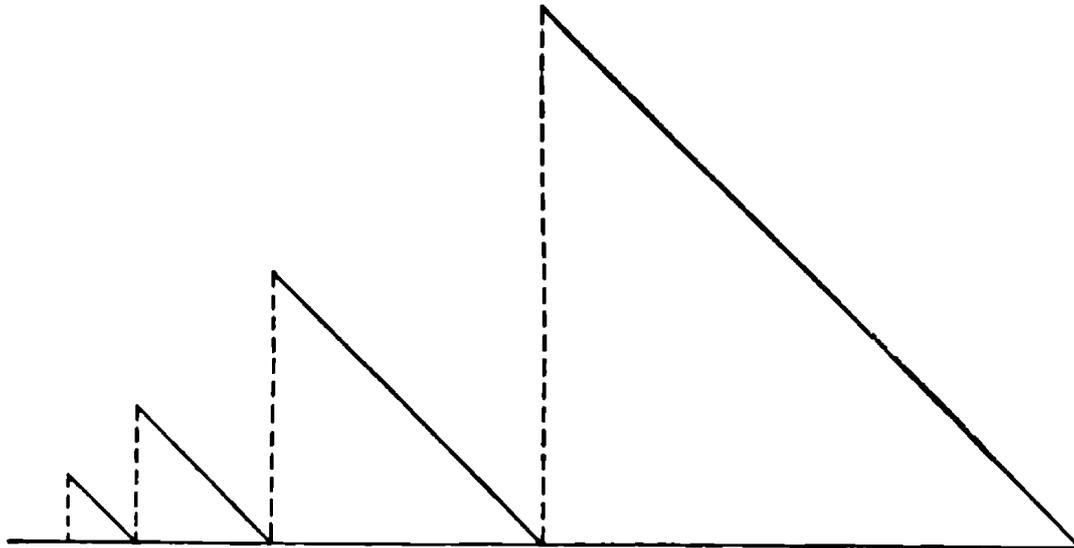


Fig. 2

**Remarks.** The theorem of Harrold-Gilbert holds, in particular, for finite (i. e. having a finite number of points of ramification) dendrites, because their Euler-Poincaré characteristic is even. However (see also paper [3]),

(2.2) *There can exist two-to-one continuous functions on infinite dendrites.*

**Example.** Consider a dendrite  $D$  consisting of the closed interval  $0 \leq x \leq 1$  of  $x$ -axis in the  $xy$ -plane and of straight-line segments  $x + y = 1/2^n$ , where  $1/2^{n+1} \leq x \leq 1/2^n$  and  $n = 1, 2, \dots$  (see fig. 2).

Let  $r$  denote the orthogonal projection of the dendrite  $D$  on  $x$ -axis. Consider the decomposition of  $D$  into pairs  $\{(x, y), (r(x), y)\}$  and the pair  $\{(0, 0), (1, 0)\}$ , where  $y > 0$ . This decomposition is upper semi-continuous and defines a two-to-one continuous function on  $D$ , according to the theorem of Alexandroff.

However, the following problem remains open: does there exist two-to-one continuous functions on 2-dimensional polyhedra whose Euler-Poincaré characteristic is odd.

**3. Two-to-one functions on the closures of plane domains.** We shall not consider the problem in all generality. Only an example will be given showing the difference in behaviour of the two-to-one continuous functions and associated semicontinuous involutions on 2-cells and on closures of plane domains having the same properties with respect to the fundamental group as 2-cells. The example which follows is in contrast to Roberts theorem of [14].

**Example.** Consider the dendrite  $D$  of (2.2). Let  $L_n$ ,  $n = 1, 2, \dots$ , be a polygonal line joining the points  $(0, 3/2^{n+2})$  and  $(1, 1/2^{n+2})$ , having no point in common with  $D$ , and lying on the straight lines:  $y = 1/2^{n+2}$ ,  $y = 1/2^{k+1} + 1/2^{n+2}$ ,  $x + y = 1/2^k \pm 1/2^{n+2}$ , where  $k = 1, 2, \dots, n$ ,  $x + y = 1/2^n + 1/2^{n+2} = 5/2^{n+2}$ ; hence,  $L_n$  approximates, from the upper half-plane, dendrite  $D$  and segment  $\frac{1}{2} \leq x \leq 1, y = 0$  (see fig. 3).

Let  $A_n$  be closed  $1/2^{n+5}$ -neighbourhood of  $L_n$ . It is easy to verify that  $A_n \cap A_m = \emptyset$  if  $m \neq n$ ,  $A_n \cap D = \emptyset$  for all  $n$  in question, and that  $D = \text{Lim } A_n$ . Let

$$B_n = h_n(A_n),$$

where  $h_n$  is a mapping  $(x, y) \rightarrow (x', y')$  defined by  $x' = x$  and  $y' = -a_n y$ . Let  $a_n > 0$  be so chosen that  $a_1 = 1$  and that  $\text{dist}(B_n, X) = \varrho(B_{n-1}, X)$  for  $n = 2, 3, \dots$ , where  $X$  denotes the segment  $\langle 0, 1 \rangle$  of  $x$ -axis. Hence  $B_n \cap B_m = \emptyset$ , if  $m \neq n$ , and  $\text{Lim } B_n = X$ . Let, finally,  $Q$  be the rectangle  $1/2 \leq x \leq 1, -1/2 \leq y \leq 1/2$ .

The required continuum is  $K = Q \cup \bigcup_{n=1}^{\infty} (A_n \cup B_n)$ . It is the closure of the domain  $G = \text{Int } K$ . The boundary of  $K$  is an irreducible cut of the plane. There exist two-to-one continuous functions on  $K$ . In fact, consider the following involution  $\lambda$  on  $K$ :

$$\lambda = \begin{cases} h_n & \text{on } A_n, \\ h_n^{-1} & \text{on } B_n, \end{cases}$$

$$\lambda((x, \frac{1}{2})) = (x, -\frac{1}{2}) \quad \text{if } 1/2 \leq x \leq 1.$$

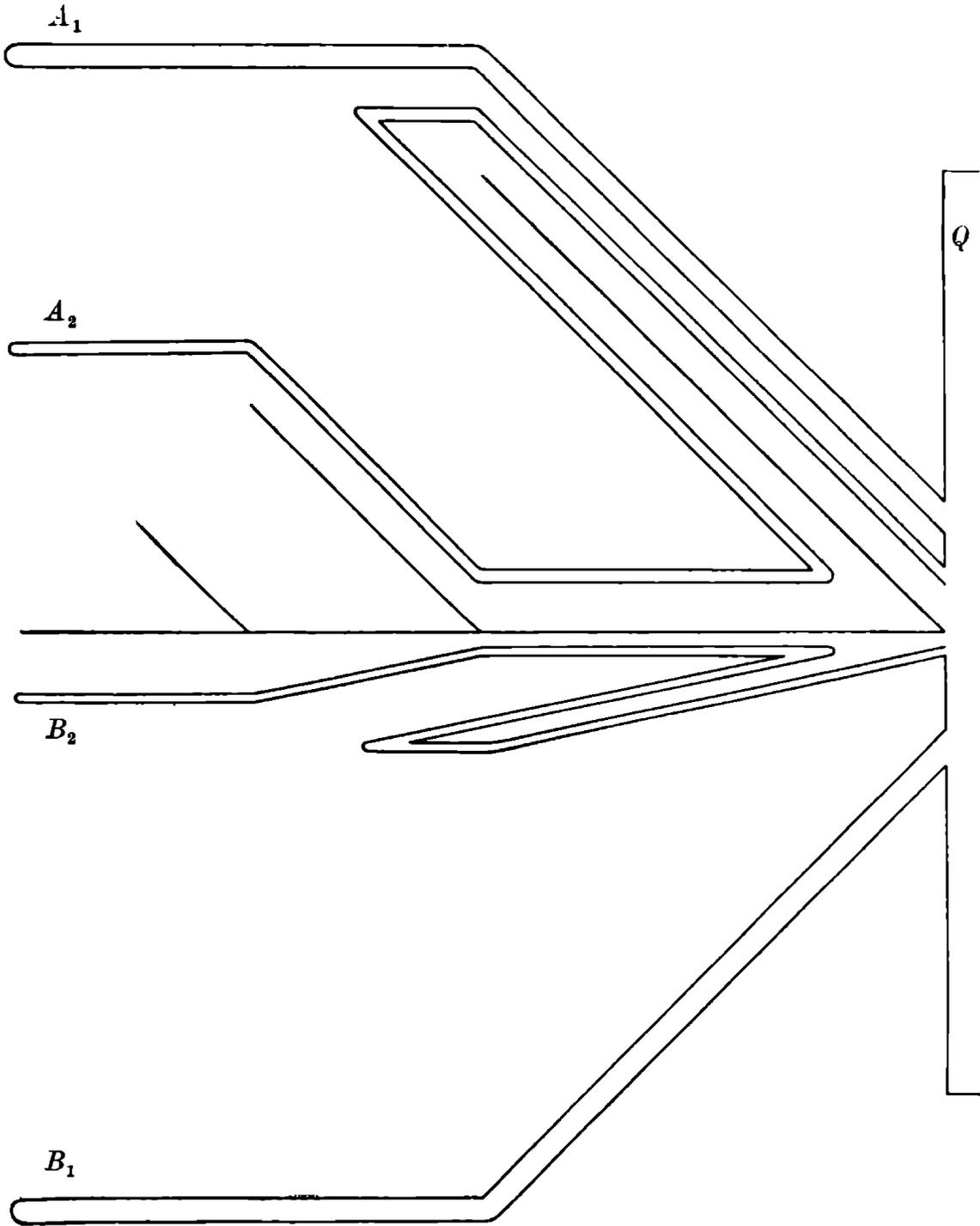


Fig. 3

On  $Q - \bigcup_{n=1}^{\infty} (A_n \cup B_n)$ ,  $\lambda$  is linear with respect to  $y$  and identity with respect to  $x$ .

Then the following semicontinuous decomposition into disjoint pairs of points defines (according to the theorem of Alexandroff) a contin-

uous two-to-one function on  $K$ . This decomposition consists of the following pairs:

$$\begin{aligned} &\{a, \lambda(a)\} \quad \text{for } a \in K - D - X, \\ &\{a, r(a)\} \quad \text{for } a \in D - X, \\ &\{(3/4 - x, 0), (3/4 + x, 0)\} \quad \text{for } 0 < x \leq 1/4, \\ &\{(0, 0), (0, 3/4)\}. \end{aligned}$$

#### 4. Two-to-one functions on some class of irreducible continua.

Consider a compact space  $\mathcal{X}$  with the following properties:

1. every point  $x$  of  $\mathcal{X}$  is PE-point;
2. there exists a semicontinuous decomposition  $\mathcal{X} = \bigcup_{z \in Z} X_z$  into disjoint arcs, where  $Z$  is a compact space;
3.  $X_z$  are components of arc-connectedness of  $\mathcal{X}$ , i. e. the sums of all arcs containing a given point.

Some irreducible continua (the case  $Z = \langle 0, 1 \rangle$ ) have these properties, for example, a continuum of Knaster considered below.

From 1-3 it follows that

(4.1) *Every  $X_z$  is a manifold (of course, of dimension 1) containing with every point one of its neighbourhoods (in  $X_z$ ) which is simultaneously a pseudo-neighbourhood of this point in  $\mathcal{X}$ .*

To prove this consider two cases.

1. Let  $x \in X_z \cap \text{IPE}(\mathcal{X})$ . Then a spherical pseudo-neighbourhood  $H$  of  $x$  in  $\mathcal{X}$  exists. By 3, it is an open interval of  $X_z$ . Thus  $H$ , as an open interval of  $X_z$ , is a neighbourhood of  $x$  in  $X_z$ .

2. Let  $x \in X_z \cap \text{BPE}(\mathcal{X})$ . Then a hemispherical pseudoneighbourhood  $H$  of  $x$  in  $\mathcal{X}$  exists. It is an interval with only one end  $x$  and is contained in  $X_z$ , according to 3. Since  $X_z$  is an arc,  $H$  is a neighbourhood of  $x$  in  $X_z$ .

Let  $f$  be a two-to-one continuous function on  $\mathcal{X}$  and  $\varphi$  the semicontinuous involution associated with  $f$ . We prove now some properties of  $\varphi$  which will be used for defining some involution  $\chi$  on  $Z$  associated with  $\varphi$ .

(4.2) *The function  $\tilde{\varphi}_z$  given by III.4 (ii) for  $R = X_z$  is a homeomorphism on  $X_z$ .*

(This follows directly from (4.1) and Theorem 2).

(4.3) *The function  $\tilde{\varphi} = \bigcup_{z \in Z} \tilde{\varphi}_z$  is a continuous involution on  $\mathcal{X}$  which cannot be the identity on arcs, such that if  $z \in Z$ , then there exists  $z' \in Z$  such that  $\tilde{\varphi}(X_z) = X_{z'}$ ; in other words,  $\tilde{\varphi}$  maps elements of decomposition onto elements of decomposition.*

In fact, the function  $\tilde{\varphi}_z = \tilde{\varphi}|_{X_z}$  is, according to (4.2), a homeomorphism. Then  $\tilde{\varphi}(X_z)$  is an arc. We have, by 3, the inclusion

$$(1) \quad \tilde{\varphi}(X_z) \subset X_{z'} \quad \text{for some } z' \in Z.$$

We prove also that

$$(2) \quad \tilde{\varphi}(X_{z'}) \subset X_z.$$

Let  $L$  be an arc contained in  $(X_z)_\varphi$ . Such an arc exists according to II (1.4). Then, by the definition of  $\tilde{\varphi}$ , there is  $\tilde{\varphi}(\xi) = \varphi(\xi)$  if  $\xi \in L$ . Therefore  $\tilde{\varphi}(L) = \varphi(L)$  is an arc contained in the arc  $X_z$ . Because  $(X_{z'})_\varphi$  is dense in  $X_{z'}$ , we have

$$(3) \quad (X_{z'})_\varphi \cap \varphi(L) \neq \emptyset.$$

Choose a point  $\xi' \in (X_{z'})_\varphi \cap \varphi(L)$ . Then  $\xi' = \varphi(\xi)$  for some  $\xi \in L$ . According to definition III. 4 (ii) for  $R = X_{z'}$ , we have  $\tilde{\varphi}(\xi') = \tilde{\varphi}\varphi(\xi) = \varphi\varphi(\xi) = \xi$ . From (3) it follows that  $\tilde{\varphi}(X_{z'}) \cap X_z \neq \emptyset$ . Since, by (4.2),  $\tilde{\varphi}(X_{z'})$  is an arc, hence (2).

To prove that  $\tilde{\varphi}$  is an involution, consider  $R = X_z \cup X_{z'}$ . By (4.1), (1) and (2) all the hypotheses of Theorem 2 hold for such an  $R$ . Hence  $\tilde{\varphi}$  is a continuous involution on  $R$  and it cannot be the identity on open subsets of  $R$ . Because  $X_z$  and  $X_{z'}$  are components of  $R$ ,  $\tilde{\varphi}(X_z) = X_{z'}$ .

Thus (4.3) is proved.

We prove now that

(4.4) *If  $z \in Z$ , then the inequality  $\tilde{\varphi}(X_z) \cap X_{z'} \neq \emptyset$  holds for only one  $z' \neq z$  of  $Z$ .*

In fact, according to (4.3) there exists  $z' \in Z$ , such that  $\tilde{\varphi}(X_z) = X_{z'}$ . If  $z' \neq z$ , then  $\varphi(X_z) = X_{z'}$ , and therefore there does not exist another  $z'$  with such a property. On the other hand, if  $z' = z$ , then  $\tilde{\varphi}(X_z) = X_z$  and  $\tilde{\varphi}$ , as a continuous involution on the arc  $X_z$ , has, according to (4.3), only one fixed point which we denote by  $a_z$ . Since  $\tilde{\varphi} = \varphi$  on  $X_z - a_z$ , there exists only one  $z' \neq z$  such that  $\varphi(a_z) \in X_{z'}$ . Therefore there exists only one  $z' \neq z$  such that  $\tilde{\varphi}(X_z) \cap X_{z'} \neq \emptyset$ . Thus (4.4) is proved.

We define now a new function  $\chi: Z \rightarrow Z$  by

$$(4.5) \quad \{z' = \chi(z)\} \equiv \{z' \neq z \text{ and } \varphi(X_z) \cap X_{z'} \neq \emptyset\}.$$

By (4.4),  $\chi$  is well defined. We shall show that

(4.6)  *$\chi$  is a semicontinuous involution on  $Z$  without fixed points.*

To prove this, let  $\chi(z) = z'$ . Then  $\varphi(X_z) \cap X_{z'} \neq \emptyset$  and there exists  $\xi \in X_z$  such that  $\eta = \varphi(\xi) \in X_{z'}$ . Since  $\varphi(\eta) = \xi \in X_{z'}$ ,  $\varphi(X_{z'}) \cap X_z \neq \emptyset$ , and therefore  $\chi(z') = z$ .

From the definition of  $\chi$  it follows that  $\chi$  is without fixed points.

It remains to prove that  $\chi$  is semicontinuous, i. e. that if  $\{z_n\}$  is a sequence of points of  $Z$  such that  $\lim z_n = z$  and  $\lim \chi(z_n) = a$ , then

$$(1) \quad a \neq z \text{ implies } a = \chi(z).$$

From the upper semicontinuity of the decomposition 2,  $\lim z_n = z$ , and  $\lim \chi(z_n) = a$  it follows that

$$(2) \quad \text{Ls } X_{z_n} \subset X_z,$$

$$(3) \quad \text{Ls } X_{\chi(z_n)} \subset X_a.$$

From (4.5) it follows that

$$(4) \quad \varphi(X_{z_n}) \cap X_{\chi(z_n)} \neq \emptyset \text{ for every } n.$$

Then, by (2), there exists a point  $x \in X_z$  and a sequence  $\{x_n\}$  converging to  $x$  such that  $x_n \in X_{z_n}$  and  $\varphi(x_n) \in X_{\chi(z_n)}$ . Therefore, by (3), we have

$$(5) \quad \lim \varphi(x_n) \in X_a.$$

Suppose that  $a \neq z$ . Then  $X_z \cap X_a = \emptyset$ . From (5) and from  $x \in X_z$  it follows that  $\lim \varphi(x_n) \neq x$ . Therefore the semicontinuity of  $\varphi$  implies  $\lim \varphi(x_n) = \varphi(x)$ ; hence, by (5),  $\varphi(x) \in X_a$ , i. e.  $\varphi(X_z) \cap X_a \neq \emptyset$ . Then, by the definition of  $\chi$ , we have  $a = \chi(z)$ . Thus the semicontinuity of  $\chi$  and (4.6) is proved.

An application. Let us consider the irreducible continuum  $\mathcal{N}_V$  defined by Knaster ([9], p. 570). The hyperspace  $Z$  for  $\mathcal{N}_V$  is the closed interval  $\langle 0, 1 \rangle$ . All the elements  $X_z$  of the decomposition of  $\mathcal{N}_V$  are arcs. More precisely, they are closed intervals or polygonal lines consisting of two segments (like the letter  $V$ ). The hyperspace of the former is a dense  $G_\delta$ -set in  $Z = \langle 0, 1 \rangle$  and the hyperspace of the latter is a countable subset of  $Z$ . All the points of  $\mathcal{N}_V$  are PE-points. Elements of the decomposition,  $X_z$ , are the components of arc-connectedness of  $\mathcal{N}_V$ . Hence  $\mathcal{N}_V$  has the properties 1-3.

We prove that

(4.7) *Two-to-one continuous functions do not exist on  $\mathcal{N}_V$ .*

Suppose, on the contrary, that such a function exists. Then, by (4.6), there exists a semicontinuous involution  $\chi$  on  $Z = \langle 0, 1 \rangle$  without fixed points, i. e. a two-to-one continuous function on  $\langle 0, 1 \rangle$ . This is impossible according to the theorem of Harrold.

The following problem remains open:

Do there exist two-to-one continuous functions on irreducible continua, if there do not exist two-to-one continuous functions on elements of decomposition of that continuum?

Irreducibility is essential because the dendrite  $D$  of (2.2) has an upper semicontinuous decomposition into arcs and single points and there exist two-to-one continuous functions on  $D$ .

**5. The non-existence of two-to-one continuous functions on the indecomposable continuum  $\mathcal{B}_0$ <sup>(4)</sup>.** All the points of  $\mathcal{B}_0$  are PE-

<sup>(4)</sup> It is possible to investigate in a general manner two-to-one continuous functions on compacta all of whose points are PE-points and for which III (i) holds. The Knaster-Janiszewski continuum  $\mathcal{B}_0$  (see for description [10], II, p. 143) belongs to this class.

points of dimension 1 and all except one are IPE-points. Denote by  $a$  the BPE-point of  $\mathcal{B}_0$ . Note that for IPE-points  $x \in \mathcal{B}_0$  the following two properties are equivalent: 1. being an open arc containing  $x$ , 2. being a spherical pseudo-neighbourhood of  $x$ . Also for the point  $a$  similar properties are equivalent, i. e. 1'. being an arc without an end and containing  $a$ , 2'. being a hemispherical pseudoneighbourhood of  $a$ .

We shall prove, using these properties of  $\mathcal{B}_0$ , that

(5.1) *There do not exist two-to-one continuous functions on  $\mathcal{B}_0$ .*

In short, such functions do not exist on  $\mathcal{B}_0$  because  $\mathcal{B}_0$  contains only one BPE-point. The proof is as follows.

Suppose, on the contrary, that there exists a two-to-one continuous function  $f$  on  $\mathcal{B}_0$ . Let  $\varphi$  be associated with  $f$  semicontinuous involution.

Consider for every  $x \in \mathcal{B}_0$  a spherical (hemispherical) pseudo-neighbourhood  $R_x$  of  $x$  in  $\mathcal{B}_0$ . Consider  $\tilde{\varphi}_{R_x}$  defined by III. 4 (ii) for  $R = R_x$ . We prove that

(1) *The value of  $\tilde{\varphi}_{R_x}$  at  $x$  does not depend on the choice of pseudo-neighbourhood  $R_x$ .*

In fact, let  $R'_x$  and  $R''_x$  be two such pseudo-neighbourhoods. Let  $V \subset R'_x \cap R''_x$  be an open subset of both  $R'_x$  and  $R''_x$ . Note that a point  $\eta$  belongs to  $(R'_x)_\varphi$  iff it belongs to  $(R''_x)_\varphi$ . For if  $\eta \in (R'_x)_\varphi$ , then  $\eta \in V_\varphi$ , because  $V$  is open in  $R'_x$ . On the other hand,  $V$  is open in  $R''_x$ , and therefore  $\eta \in V_\varphi$  implies  $\eta \in (R''_x)_\varphi$ . This remark implies, according to the definition of  $\tilde{\varphi}$ , the assertion (1).

According to (1) the function  $\tilde{\varphi}_{R_x}(x)$  is well defined on the whole continuum  $\mathcal{B}_0$ . We denote this function by  $\tilde{\varphi}$ . It may be discontinuous. But (the proof is standard)

(2) *If  $L \subset \mathcal{B}_0$  is an arc, then  $\tilde{\varphi}|L$  is an homeomorphism.*

We shall prove now that

(3)  $\tilde{\varphi}$  *is one-to-one on  $\mathcal{B}_0$ .*

In fact, let  $x' \neq x''$  and  $R = R_{x'} \cup R_{x''}$ , where  $R_{x'}$  and  $R_{x''}$  are as in (2). According to the properties of  $\mathcal{B}_0$ , which were given at the beginning of this paragraph, all the hypotheses of the first part of Theorem 2 are satisfied for such an  $R$ . We have then  $\tilde{\varphi}_R(x') \neq \tilde{\varphi}_R(x'')$  and, by the definition of  $\tilde{\varphi}$  on  $\mathcal{B}_0$ ,  $\tilde{\varphi}(x') \neq \tilde{\varphi}(x'')$ . This proves (3).

Let now  $B$  denote the set of all points  $x \in \mathcal{B}_0$  for which  $\tilde{\varphi}(x) \neq x$  holds. It is clear that for every spherical (hemispherical) pseudo-neighbourhood  $R$  of  $x$  in  $\mathcal{B}_0$  there is

(4)  $R \cap B = R_\varphi.$

We shall prove that

(5)  $\tilde{\varphi}(B) = \varphi(B) \subset B.$

The equality is obvious. Suppose that the inclusion is not true. Then for some  $x \in B$  we have  $\tilde{\varphi}(x) \notin B$  and therefore  $\tilde{\varphi}(x) \neq x$ .  $\tilde{\varphi}(x) \notin B$  implies, by the definition of  $B$ , the equality  $\tilde{\varphi}\tilde{\varphi}(x) = \tilde{\varphi}(x)$  which is in contradiction to  $\tilde{\varphi}(x) \neq x$  and to (3).

In order to finish the proof of (5.1), consider two cases according to whether the BPE-point  $a$  belongs or not to  $B$ .

1. Let  $a \in \mathcal{B}_0 - B$ . Let  $R$  be a hemispherical neighbourhood of  $a$ . By the assumption  $\tilde{\varphi}(a) = a$ , and by (2),  $\tilde{\varphi}(R)$  is an arc with only one end  $a$ . Then according to the properties of  $\mathcal{B}_0$ , which were given at the beginning of this paragraph,  $R \cup \tilde{\varphi}(R)$  is an arc with only one end  $a$ . Therefore, the inclusion

$$\tilde{\varphi}\tilde{\varphi}(R) \subset R$$

is true. Indeed, if  $x \in R - R_\varphi$ , then  $\tilde{\varphi}\tilde{\varphi}(x) = x$ ; hence  $\tilde{\varphi}\tilde{\varphi}(x) \in R$ . If  $x \in R_\varphi$ , then, by (4) and (5),  $\tilde{\varphi}(x) \in B$  and hence  $\tilde{\varphi}\tilde{\varphi}(x) = \varphi\varphi(x) = x \in R_\varphi \in R$ .

The last inclusion implies  $\tilde{\varphi}(R \cup \tilde{\varphi}(R)) \subset R \cup \tilde{\varphi}(R)$ . Therefore, by the second part of theorem 2,  $\tilde{\varphi}$  is a continuous involution on  $R \cup \tilde{\varphi}(R)$  and it cannot be the identity on open subsets of  $R \cup \tilde{\varphi}(R)$ . It is well known that such an involution does not exist on an arc without an end. A contradiction occurs.

2. Let  $a \in B$ . In this case we have  $\tilde{\varphi}(a) = \varphi(a) \neq a$ ; hence  $\tilde{\varphi}(a) \in \text{IPE}(\mathcal{B}_0)$ . Let  $L$  be an arc whose interior contains  $\tilde{\varphi}(a)$ . By (2),  $\tilde{\varphi}(L)$  is also an arc whose interior contains  $\tilde{\varphi}\tilde{\varphi}(a)$ . According to (4) and  $a \in B$  we conclude that  $a$  would be an interior point of the arc  $\tilde{\varphi}(L)$  contained in  $\mathcal{B}_0$ , which is impossible.

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## V. TWO-TO-ONE CONTINUOUS FUNCTIONS ON LOCALLY COMPACT MANIFOLDS

According to theorem 1, a semicontinuous involution  $\varphi$  induced by a two-to-one continuous function has no WED-point if  $\mathcal{X}$  is a manifold. However, if  $\mathcal{X}$  is only locally compact manifold as, for example, the Euclidean space, then there are examples of two-to-one continuous functions with SED-points. It will be shown (problem raised in [4]) that there exist two-to-one continuous functions on Euclidean spaces  $E^n$  with  $n \geq 2$ , which is in contrast to Civin's theorem concerning closed cells.

**1. The function  $\tilde{\varphi}$  on  $\mathcal{X} - \text{SED}(\varphi)$ .** Let us now consider the function  $\tilde{\varphi} : \mathcal{X} - \text{SED}(\varphi) \rightarrow \mathcal{X}$  defined by

$$(1.1) \quad \tilde{\varphi}(\xi) = \begin{cases} \varphi(\xi) & \text{if } \xi \in \mathcal{X}_\varphi, \\ \xi & \text{if } \xi \in \mathcal{X} - \text{SED}(\varphi) - \mathcal{X}_\varphi. \end{cases}$$

Since  $\text{SED}(\varphi)$  is closed in  $\mathcal{X}$ ,  $\mathcal{X} - \text{SED}(\varphi)$  is a manifold. Therefore, by the Theorem 2,

$$(1.2) \quad \tilde{\varphi} \text{ is continuous and one-to-one.}$$

We shall prove that

(1.3) *If  $x \in \mathcal{X} - \text{SED}(\varphi)$  is an IE-point of  $\mathcal{X}$ , then  $\tilde{\varphi}(x) \in \mathcal{X} - \text{SED}(\varphi)$ ; furthermore, if  $x \in \mathcal{X}_\varphi$ , then  $\tilde{\varphi}(x) = \varphi(x) \in \mathcal{X}_\varphi$ .*

In the proof of (1.3) we shall consider only the case  $x \in \mathcal{X}_\varphi$ , because  $x \in \mathcal{X} - \mathcal{X}_\varphi$  implies  $\tilde{\varphi}(x) = x$ . Let then  $\{W_n\}$  be a sequence of spherical neighbourhoods of  $x$  such that

$$(1) \quad \overline{W}_n \subset \mathcal{X}_\varphi, \quad n = 1, 2, \dots,$$

$$(2) \quad \bigcap_{n=1}^{\infty} \overline{W}_n = x$$

and such that all  $\varphi|_{\overline{W}_n}$  are homeomorphisms. Such a sequence exists by II(1.4). All  $\varphi(\overline{W}_n) = \tilde{\varphi}(\overline{W}_n)$  are solid spheres of the same dimension as  $\mathcal{X}$ . They contain  $\tilde{\varphi}(x)$  in their interiors. Therefore they are spherical neighbourhoods of  $\tilde{\varphi}(x)$  in  $\mathcal{X}$ . From the continuity of  $\tilde{\varphi}$ , (1) and (2) it follows that

$$(3) \quad \bigcap_{n=1}^{\infty} \tilde{\varphi}(\overline{W}_n) = \bigcap_{n=1}^{\infty} \varphi(\overline{W}_n) = \tilde{\varphi}(x).$$

Suppose, on the contrary, that  $\tilde{\varphi}(x) \in \text{SED}(\varphi)$ . Then, according to formula (3), there exists a sequence  $\{x_n\}$  converging to  $\tilde{\varphi}(x)$  such that  $x_n \in \mathcal{X}_\varphi \cap \tilde{\varphi}(\overline{W}_n)$  for all  $n$  and such that

$$(4) \quad \lim \tilde{\varphi}(x_n) = p.$$

From  $x_n \in \tilde{\varphi}(\overline{W}_n) = \varphi(\overline{W}_n)$  it follows that  $\varphi(x_n) \in \overline{W}_n$ . Hence, by (1),  $\lim \varphi(x_n) = x$ , which is in contradiction to (4).

To prove the second part of (1.3), note that formula (1) implies  $\tilde{\varphi}(\mathcal{X} - \text{SED}(\varphi) - \mathcal{X}_\varphi) = \mathcal{X} - \text{SED}(\varphi) - \mathcal{X}_\varphi$ . Because  $\tilde{\varphi}$  is one-to-one  $\tilde{\varphi}(x) \in \mathcal{X} - \text{SED}(\varphi) - \mathcal{X}_\varphi$  implies  $x \in \mathcal{X} - \text{SED}(\varphi) - \mathcal{X}_\varphi$ , which is in contradiction to the assumption.

Note that

(1.4) *If  $h$  is a homeomorphism on  $0 \leq t \leq 1$  such that  $h(t) \in \mathcal{X} - \text{SED}(\varphi)$  for  $t \neq 0$ , then there exists  $\lim_{t \rightarrow 0} \tilde{\varphi}h(t)$ .*

It is an immediate consequence of the semicontinuity of  $\varphi$ , more precisely, of the inclusion,  $\text{Ls}_{t \rightarrow 0} \tilde{\varphi}h(t) \subset \Psi h(0)$ .

The following properties of  $\tilde{\varphi}$  on manifolds without boundary (i. e. consisting of IE-points only) are obvious implications of (1.3) and of Theorem 2:

$$(1.5) \quad \tilde{\varphi}(\mathcal{X} - \text{SED}(\varphi)) \subset \mathcal{X} - \text{SED}(\varphi);$$

(1.6)  *$\tilde{\varphi}$  is a continuous involution on  $\mathcal{X} - \text{SED}(\varphi)$ , and it cannot be the identity on open subsets of  $\mathcal{X} - \text{SED}(\varphi)$ ; moreover,  $\tilde{\varphi}(\mathcal{X}_\varphi) = \varphi(\mathcal{X}_\varphi) = \mathcal{X}_\varphi$  and  $\tilde{\varphi}(\mathcal{X} - \mathcal{X}_\varphi) = \mathcal{X} - \mathcal{X}_\varphi$ .*

**2. The strong accessibility of points of  $\text{SED}(\varphi)$ .** The theorem which follows is the most important one in the investigation of the structure of  $\text{SED}(\varphi)$ . We start with some definitions. Every homeomorphic image of the closed interval  $0 \leq t \leq 1$ , given by a homeomorphism  $h$  such that  $h(0) = x$  and  $h(t) \in \mathcal{X} - \text{SED}(\varphi)$  for  $t \neq 0$ , is called a *path to the point  $x$* . A point  $x \in \text{SED}(\varphi)$  is said to be *strongly accessible* from  $\mathcal{X} - \text{SED}(\varphi)$  if there exists a path to  $x$  such that

$$\lim_{t \rightarrow 0} \tilde{\varphi}h(t) = p.$$

Note that

(2.1) *If  $x$  is an isolated point in  $\text{SED}(\varphi)$ , then  $x$  is strongly accessible from  $\mathcal{X} - \text{SED}(\varphi)$ .*

**THEOREM 3.** *If  $\mathcal{X}$  is a manifold,  $x$  is a SED-point of  $\varphi$ , and  $U$  is an open neighbourhood of  $x$  in  $\mathcal{X}$ , then there exists a SED-point of  $\varphi$  in  $U$ , which is strongly accessible from  $\mathcal{X} - \text{SED}(\varphi)$ .*

**Proof.** Suppose, on the contrary, that there exists a spherical (hemispherical) neighbourhood  $U$  of  $x$  in  $\mathcal{X}$  such that no point of  $U \cap \text{SED}(\varphi)$  is strongly accessible. Then, by (2.1), we conclude that

$$(1) \quad U \cap \text{SED}(\varphi) \text{ is dense in itself.}$$

Let us consider a point  $\xi$  of  $U \cap \text{SED}(\varphi)$  and a sequence  $\{\xi_n\}$  which converges to  $\xi$ , such that  $\xi_n \in \mathcal{X}'_\varphi$ ,

$$(2) \quad \lim \tilde{\varphi}(\xi_n) = \lim \varphi(\xi_n) = p,$$

and such that there exists a sequence  $\{L_n\}$  of arcs  $L_n = \widehat{\xi_n \xi_{n+1}} \subset U - \xi$  having the following properties:

$$(3) \quad L_n \cap L_{n+1} = \xi_{n+1},$$

$$(4) \quad \text{Lim } L_n = \xi.$$

By (3), the sum  $L = \bigcup_{n=1}^{\infty} L_n$  is an arc. By assumption,  $\xi$  is not strongly accessible, hence the arcs  $L$  and  $L_n$  cannot be paths to  $\xi$ . Therefore, by (2), the inequality  $L_n \cap \text{SED}(\varphi) \neq 0$  holds for infinitely many  $n$ .

Let  $\xi'_n \in L_n \cap \text{SED}(\varphi)$  be the first of SED-points lying on the arc  $L_n = \widehat{\xi_n \xi_{n+1}}$ . Denote by  $L'_n$  the subarc  $\widehat{\xi_n \xi'_n}$  of  $L_n$ . According to (4) we have

$$(5) \quad \text{Lim } L'_n = \xi,$$

$$(6) \quad \lim \xi'_n = \xi.$$

Consider  $L'_n$  as a path to  $\xi'_n$  defined by a homeomorphism  $h_n$ . It is clear that

$$(7) \quad \begin{cases} h_n(t) \subset L'_n, & 0 \leq t \leq 1, \\ h_n(0) = \xi'_n, \\ h_n(1) = \xi_n. \end{cases}$$

By assumption, there do not exist strongly accessible points in  $U$ . Hence  $\lim_{t \rightarrow 0} \tilde{\varphi} h_n(t) \neq p$ . Therefore, by (7),

$$(8) \quad \lim_{t \rightarrow 0} \tilde{\varphi} h_n(t) = a_n, \quad \text{where } a_n \text{ is } \xi'_n \text{ or } \varphi(\xi'_n).$$

Since  $\tilde{\varphi}$  is continuous on  $L'_n - \xi'_n$  (see the definition of  $\xi'_n$ ), hence all the sets  $\overline{\tilde{\varphi}(L'_n - \xi'_n)}$  and, therefore, the sets  $\text{Ls } \tilde{\varphi}(L'_n - \xi'_n)$  are continua.

We see that (7) and (8) imply  $a_n \in \overline{\tilde{\varphi}(L'_n - \xi'_n)}$ . We have then

$$(9) \quad \lim a_n \in \text{Ls } \overline{\tilde{\varphi}(L'_n - \xi'_n)} = \text{Ls } \tilde{\varphi}(L'_n - \xi'_n).$$

We return to the function  $\Psi$  which is upper semicontinuous. By (5), we have the following inclusions:

$$\text{Ls } \tilde{\varphi}(L'_n - \xi'_n) \subset \text{Ls } \Psi(L'_n - \xi'_n) \subset \Psi(\xi).$$

From these inclusions it follows that the continuum  $\text{Ls } \tilde{\varphi}(L'_n - \xi'_n)$  consists of only one point. By (2), this point is  $p$ . Then from (9) it follows that  $\lim a_n = p$ . According to (5),  $\lim \xi'_n = \xi \neq p$ , hence, by (9),  $a_n = \varphi(\xi'_n)$  for sufficiently large  $n$ .

In other words, we have shown that for every  $\xi \in U \cap \text{SED}(\varphi)$  there exists a sequence  $\{\xi'_n\}$  of points of  $U \cap \text{SED}(\varphi)$  which converges to  $\xi$ , and such that  $\lim \varphi(\xi'_n) = p$ .

Then, according to the discontinuity criterion for  $\varphi$  (see II(1.6)), every point of  $U \cap \text{SED}(\varphi)$  is the point of discontinuity of the partial function  $\varphi|_{U \cap \text{SED}(\varphi)}$  and hence, by II(1.3), of  $\Psi|_{U \cap \text{SED}(\varphi)}$ . Since, by I(1.11), the points of discontinuity of  $\Psi$  form a closed set of points, every point of  $U \cap \text{SED}(\varphi)$  is the point of discontinuity of  $\Psi|_{U \cap \text{SED}(\varphi)}$ . We have a contradiction because  $\Psi$  is of the I-st Baire class.

Thus the theorem is proved.

**3. The one-dimensional case.** The following theorem, which is given without proof in [4], follows in an elementary way from Theorem 3:

(3.1) *There do not exist two-to-one continuous functions on the straight line.*

To prove this, suppose that such a function  $f$  exists. Let  $\varphi$  be a semi-continuous involution induced by  $f$  and let  $\tilde{\varphi}$  be the continuous involution on  $E^1 - \text{SED}(\varphi)$  defined by (1.1), where  $E^1$  is the straight line.

There exist at most two points in  $\text{SED}(\varphi)$ . Indeed, if  $x_1$  and  $x_2$  belong to  $\text{SED}(\varphi)$  and  $x_1 \neq x_2$ , then, according to Theorem 3, there exist disjoint neighbourhoods of  $x_1$  and  $x_2$  and two points, in these neighborhoods, which are strongly accessible from  $E^1 - \text{SED}(\varphi)$  by disjoint paths. The  $\tilde{\varphi}$ -images of these paths are paths to the point  $p = \infty$ . These paths to  $p$  are disjoint because  $\tilde{\varphi}$  is an involution. There are only two disjoint paths to  $p = \infty$  on  $E^1$ . Hence  $\text{SED}(\varphi)$  consists of at most two points.

Consider three cases.

1.  $\text{SED}(\varphi) = x_1 \cup x_2$ , where  $x_1 \neq x_2$ . Let us denote by  $X_0, X_1$ , and  $X_2$  the components of  $E^1 - x_1 - x_2$ . By (1.6),  $\tilde{\varphi}$  is a continuous involution on  $X_0 \cup X_1 \cup X_2$ , and it cannot be the identity on intervals. Hence

the fixed points set  $S$  of  $\tilde{\varphi}$  consists of one or three points. According to III (4.1) we have  $\varphi(x_1 \cup x_2 \cup S) = x_1 \cup x_2 \cup S$ . This is impossible because  $x_1 \cup x_2 \cup S$  consists of an odd number of points.

2.  $\text{SED}(\varphi) = x_1$ . Let us denote by  $X_1$  and  $X_2$  the components of  $E^1 - x_1$ . By (1.6),  $\tilde{\varphi}$  is a continuous involution on  $X_1 \cup X_2$ , and it cannot be the identity on intervals. The fixed points set  $S$  of  $\tilde{\varphi}$  on  $X_1 \cup X_2$  is empty or consists of two points. Then, as in 1, the number of points in  $x_1 \cup S$  is odd.

3.  $\text{SED}(\varphi) = 0$ . Now  $\tilde{\varphi}$  is a continuous involution on  $E^1$ , not being the identity. There is only one fixed point of  $\tilde{\varphi}$ . If we denote this point by  $s$ , then we have, as before,  $\varphi(s) = s$ , which is impossible.

Thus (3.1) is proved.

As a corollary we obtain

(3.2) *If  $f: E^n \rightarrow \mathcal{Y}$  is a two-to-one continuous function, where  $n \leq 2$  and  $\mathcal{Y}$  is an  $\mathcal{L}^*$ -space, then there exist SED-points.*

Suppose, on the contrary, that there exists a function  $f: E^n \rightarrow \mathcal{Y}$ ,  $n \leq 2$ , where  $\mathcal{Y}$  is an  $\mathcal{L}^*$ -space, such that the associated semicontinuous involution  $\varphi$  is without SED-points. Then  $\varphi$  is weakly bounded on  $E^n$ , and the hypotheses of Theorem 2 are satisfied. According to Theorem 2, the involution  $\tilde{\varphi}$  given by III.4 (ii) is continuous on  $E^n$ . If  $S$  denotes the set of fixed points of  $\tilde{\varphi}$ , then, by III(4.3),  $f|S$  is two-to-one. According to the well known theorem of Kérékjartó ([8], p. 224-226), if  $n \leq 2$ , then  $S$  is a point or the straight line. Hence we have a contradiction; in the second case with regard to (3.1).

Note that the assumption  $n \leq 2$  in (3.2) is essential. Examples will be given in § 5.

Now we are going to investigate the two-to-one continuous functions on the half-straight line. We prove that if  $\mathcal{X}$  is the half-straight line, then there exist three kinds of such functions.

Note first that in this case

(3.3)  $\text{SED}(\varphi) \neq 0$ .

For if  $\text{SED}(\varphi) = 0$ , then, according to Theorem 2, the involution  $\tilde{\varphi}$  given by III.4 (ii) would be continuous and it would not be the identity. It is well known that there does not exist such an involution on the half-straight line.

Assume that in the considered case the compact space  $\mathcal{X}^* = \mathcal{X} \cup p$  is a closed interval, whose ends are  $p$  and  $q$ . There is only one path to  $p$  on  $\mathcal{X}^*$ .

Applying the argument used in the proof of (3.1) we conclude that  $\text{SED}(\varphi)$  consists of at most one point. Hence, by (3.3),  $\text{SED}(\varphi)$  consists of exactly one point, which we denote by  $x_0$ .

Let  $X_1 = (q, x_0)$  and  $X_2 = (x_0, p)$  be open intervals.  $X_1$  may be empty if  $q = x_0$ . According to (1.2), the function  $\tilde{\varphi}$  given by (1.1) on  $X_1 \cup X_2$  is continuous and one-to-one. Then  $\tilde{\varphi}(X_i)$  are intervals and

$$(3.4) \quad \tilde{\varphi}(X_i) \subset X_{j(i)},$$

where  $i = 1$  or  $2$  and also  $j(i) = 1$  or  $2$ . We prove also that

$$(3.5) \quad \tilde{\varphi}(X_i) = X_{j(i)}.$$

We prove (3.5) by showing that if  $a_i$  is the end of  $X_1$ , then

$$(3.6) \quad \lim_{\xi \rightarrow a_i} \tilde{\varphi}(\xi) = a_{j(i)},$$

where  $a_{j(i)}$  is one of the ends of  $X_{j(i)}$ .

Consider three cases according as  $a_i = p$ ,  $a_i = q$  or  $a_i = x_0$ . The first one is very simple, because  $\text{SED}(\varphi) = x_0$  implies  $\lim_{\xi \rightarrow p} \tilde{\varphi}(\xi) = x_0$ .

Let then  $a_i$  be  $q$  or  $x_0$ . As  $\mathcal{X}_\varphi$  is dense in  $\mathcal{X}$ , then there exists a sequence  $\{\xi_n\}$ , where  $\xi_n \in \mathcal{X}_\varphi \cap \text{IE}(\varphi)$ , such that  $\lim \xi_n = a_i$ . We have, by (1.3),  $\tilde{\varphi}(\xi_n) = \varphi(\xi_n)$  and therefore  $\tilde{\varphi}\tilde{\varphi}(\xi_n) = \varphi\varphi(\xi_n) = \xi_n$ . Let us denote by  $b_i$  the limit  $\lim \tilde{\varphi}(\xi_n) = \lim \varphi(\xi_n)$ . Suppose, on the contrary, that  $b_i$  is not an end of  $X_{j(i)}$ . Then  $\tilde{\varphi}$  is continuous at  $b_i$  and, therefore,  $\tilde{\varphi}(b_i) = \lim \tilde{\varphi}\tilde{\varphi}(\xi_n) = \lim \xi_n = a_i$ .

We have a contradiction. Namely, if  $a_i = q$ , then there exists an open interval containing  $b_i$ , which is mapped homeomorphically by  $\tilde{\varphi}$  on an open interval containing  $q$  with  $\tilde{\varphi}(b_i) = q$ ; this is impossible, because  $q$  is the end of  $\mathcal{X}$ . If  $a_i = x_0$ , then  $\tilde{\varphi}(b_i) = a_i = x_0$ ; contrary to (1.3), as  $b_i \in [\mathcal{X} - \text{SED}(\varphi)] \cap \text{IE}(\mathcal{X})$ .

Thus (3.6) and, therefore, (3.5) is proved. Furthermore, because  $\tilde{\varphi}$  is one-to-one on  $X_1 \cup X_2$ , we conclude that

$$(3.7) \quad i_1 \neq i_2 \text{ implies } j(i_1) \neq j(i_2).$$

The final result, being a consequence of (3.5) and (3.7), is as follows:

(3.8) *There exist exactly three kinds of semicontinuous involutions without fixed points (i. e., by II (1.8), three kinds of two-to-one continuous functions) on the half-straight line, which are described by the following relations:*

- (i)  $x_0 \neq q, \quad \tilde{\varphi}(X_1) = X_1,$
- (ii)  $x_0 \neq q, \quad \tilde{\varphi}(X_1) = X_2,$
- (iii)  $x_0 = q.$

Let the half-straight line  $\mathcal{X}$  be represented by the interval  $0 \leq x < 1$ . Then the involution  $\varphi_1$ , given by

$$\varphi_1(x) = \begin{cases} 1/2 - x & \text{if } 0 \leq x \leq 1/2 \text{ and } x \neq 1/4, \\ 1 - x & \text{if } 1/2 < x < 1 \text{ and } x \neq 3/4, \\ 3/4 & \text{if } x = 1/4, \end{cases}$$

is of the first kind. The involution

$$\varphi_2(x) = x + 1/2 \pmod{1}$$

is of the second kind. As an example of involution of the third kind consider the following:

$$\varphi_3(x) = \begin{cases} 1 - x & \text{if } 0 < x < 1 \text{ and } x \neq 1/2, \\ 1/2 & \text{if } x = 0. \end{cases}$$

It is easy to see that the space  $\mathcal{Y}$ , which is according to II (1.8) a Hausdorff space, is in all cases (i)-(iii) the circle.

**4. The existence of two-to-one continuous functions on Euclidean spaces  $E^n$  for  $n \geq 2$ .** This problem was raised by Civin [4]. Note that there is no problem if the space of values,  $\mathcal{Y}$ , is not an  $\mathcal{L}^*$ -space but only a  $T_1$ -space (see I. 6).

Consider  $E^n$  as  $S^n - p$ , where  $S^n$  is the  $n$ -sphere and  $p \in S^n$ . Denote by  $x^-$  the antipodal point of  $x$ , and by  $X^-$  the antipodal set of  $X$ . Consider a continuum  $C \subset S^n$  such that

- (i)  $S^n - C$  is homeomorphic to  $E^n$ ,
- (ii) there exists two-to-one continuous function on  $C$ ,
- (iii)  $C^- \subset S^n - C$ .

According to (i), assume that there exists a continuous function  $h: S^n \rightarrow S^n$  such that

- (iv)  $h|_{S^n - C}$  is a homeomorphism and  $h(S^n - C) = S^n - p$ ,
- (v)  $h(C) = p$ .

According to (ii), there exists a semicontinuous involution without fixed points on  $C$ . Denote this involution by  $\chi$ . Denote also by  $\chi^-$  the similar involution on  $C^-$  given by

$$\chi^-(x) = (\chi(x^-))^-.$$

Consider the function  $\lambda: S^n - C \rightarrow S^n - C$  given by

$$\lambda(x) = \begin{cases} \chi^-(x) & \text{if } x \in C^-, \\ x^- & \text{if } x \in S^n - C - C^-. \end{cases}$$

By definition

(4.1)  $\lambda$  is an involution without fixed points on  $S^n - C$ .

Note that

(4.2) If  $x \in C^-$ , then  $\lim_{\substack{\xi \rightarrow x \\ \xi \notin C^-}} \lambda(\xi)$  exists and is equal to  $x^-$ .

In fact, by definition,  $\xi \notin C$  implies  $\lambda(\xi) = \xi^-$ . Hence (4.2) follows from the continuity of the antipodism.

We prove now that

(4.3)  $\text{Ls}_{\substack{\xi \rightarrow x \\ \xi \notin C}} \lambda(\xi) \subset x \cup \lambda(x) \cup C$  for every  $x \in S^n - C$ .

In fact, if  $x \in C^-$  and  $\xi \in C^-$ , then  $\text{Ls}_{\xi \rightarrow x} \lambda(\xi) \subset x \cup \lambda(x)$ , as  $\lambda|_{C^-} = \chi$  and  $\chi$  is semicontinuous. If  $x \in C^-$  and  $\xi \notin C^-$ , then, by (4.2),  $\text{Ls}_{\xi \rightarrow x} \lambda(\xi) = x^- \in C$ . Finally, let  $x \notin C^-$  and  $\xi \notin C^-$ . Then  $\lambda(\xi) = \xi^-$  and  $\lambda(x) = x^-$ ; therefore  $\text{Ls}_{\substack{\xi \rightarrow x \\ \xi \notin C^-}} \lambda(\xi) = x^- = \lambda(x)$ . Thus (4.3) is proved.

**THEOREM 4.** *There exist two-to-one continuous functions  $f: E^n \rightarrow \mathcal{Y}$ , where  $n \geq 2$  and  $\mathcal{Y}$  is a compact Hausdorff space.*

**Proof.** Let  $C$ ,  $h$ , and  $\lambda$  have the same meaning as before. Consider the function  $\varphi: S^n - p \rightarrow S^n - p$  given by  $\varphi = h\lambda h^{-1}$ .  $\varphi$  is an involution on  $S^n - p$ , because  $\varphi\varphi(x) = h\lambda h^{-1}h\lambda h^{-1}(x) = x$  for  $x \in S^n - p$ . It is without fixed points because  $\lambda$  is without fixed points and  $h$  and  $h^{-1}$  are homeomorphisms.

We prove now that  $\varphi$  is semicontinuous. In fact, from (4.3) it follows that  $\text{Ls}_{\xi \rightarrow x} h\lambda h^{-1}(\xi) \subset h^{-1}(x) \cup \lambda h^{-1}(x) \cup C$ . Hence  $\text{Ls}_{\xi \rightarrow x} \varphi(\xi) = \text{Ls}_{\xi \rightarrow x} h\lambda h^{-1}(\xi) \subset h h^{-1}(x) \cup h\lambda h^{-1}(x) \cup h(C) = x \cup \varphi(x) \cup p$ , in virtue of (v) and the continuity of  $h$ . According to II (1.8), there exists a continuous two-to-one function  $f: S^n - p \rightarrow \mathcal{Y}$ , where  $\mathcal{Y}$  is a Hausdorff space.

We prove that  $\mathcal{Y}$  is compact. According to II (1.8), it is sufficient to show that  $\text{Ls}_{\xi \rightarrow p} \varphi(\xi) \neq p$ . Consider a sequence  $\{\xi_k\}$  converging to  $p$ .

We have  $\lim_{k \rightarrow \infty} h^{-1}(\xi_k) = x \in C$  if the limit exists. We have also, for sufficiently large  $k$ ,  $h^{-1}(\xi_k) = (h^{-1}(\xi_k))^-$ ; hence  $\lim_{k \rightarrow \infty} \lambda h^{-1}(\xi_k) = x^- \in C^-$ . Therefore  $\lim_{k \rightarrow \infty} \varphi(\xi_k) = \lim_{k \rightarrow \infty} h\lambda h^{-1}(\xi_k) = h(x^-) \in h(C^-)$ . Since, by (v),  $p \notin h(C^-)$ , we conclude that  $\text{Ls}_{\xi \rightarrow p} \varphi(\xi) \neq p$ .

In order to finish the proof of the theorem, recall that there exist continua  $C$  having the properties (i)-(iii). This has been shown in (2.2) and § 3 of Chapter IV (the dendrite  $D$  and continuum  $K$ ).

**Remark.** We prove, in addition, that in the above construction of  $\varphi$  the set of SED-points of  $\varphi$  is homeomorphic to  $\text{Fr}(S^n - C)$ . Namely, we prove that

$$(4.4) \quad \text{SED}(\varphi) = \text{Fr}(E^n - h(C^-)).$$

In fact, let  $\lim_{k \rightarrow \infty} \xi_k = p$ . Then  $\lim_{k \rightarrow \infty} h^{-1}(\xi_k) = x \in C$  if the limit exists.

According to (v),  $h^{-1}(\xi_k) \notin C^-$  for sufficiently large  $k$ . Hence  $\lim h^{-1}(\xi_k) = x^- \in C^-$ . Therefore  $\lim \varphi(\xi_k) = \lim h \lambda h^{-1}(\xi_k) = h(x^-) \in h(C^-)$ . Thus the inclusion

$$\text{SED}(\varphi) \subset \text{Fr}(E^n - h(C^-))$$

is proved.

To prove the converse inclusion consider an arbitrary point  $\xi$  of  $\text{Fr}(E^n - h(C^-))$ . There exists a sequence  $\{\xi_k\}$  converging to  $\xi$  such that all  $\xi_k$  belong to  $E^n - h(C^-)$ . We have  $x_k = h^{-1}(\xi_k) \notin C^-$  and  $h^{-1}(\xi) \in C^-$ . Therefore, by (4.3),  $\lim \varphi(\xi_k) = \lim h \lambda(x_k) = p$ , as  $\lim x_k \in C^-$ . Hence  $\xi \in \text{SED}(\varphi)$ . Thus (4.4) is proved.

Finally, note that if  $n > 2$ , then there exist two-to-one continuous functions on  $E^n = S^n - p$  without SED-points. In fact, let  $S^2$  be the "meridian" of  $S^n$  containing  $p$ . Let  $\varphi$  be a semicontinuous involution on  $S^2$  without fixed points as in Theorem 4. Let  $s$  be the reflection of  $S^n$  with respect to  $S^2$ . Then the function  $s\varphi: S^n - p \rightarrow S^n - p$  is a semicontinuous involution without fixed points. It is easy to verify that the set  $\text{SED}(s\varphi)$  is empty.

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