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ON THE PHAM EXAMPLE AND THE UNIVERSAL TOPOLOGICAL STRATIFICATION OF SINGULARITIES

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This will describe joint work with Andre Galligo on the universal topological stratification for the Pham example as a prototype for multi-modal singularities.

From several directions there has arisen a desire to understand how the versal deformation $\pi: \mathscr{V} \to T$ of a singularity $(X, 0)$ can be stratified by the topological type of the germ of π at points of $\mathscr{V}$. This includes the results of Looijenga [L 1, II], [L 2], [L 3] on the complement of the discriminant for simple elliptic and unimodal hypersurface singularities, extending the results of Brieskorn [B] and Slodowy [S]. In the case of C^∞ germs where $(X, 0)$ is a complete intersection with isolated singularity, this is a major question in understanding topologically stable mappings. Thom and Mather have proven the existence of such a stratification and used it to prove the density of topologically stable mappings in all dimensions [T], [M 1], [M 2]. In [D 4] and [D 5] it is shown that the Thom–Mather stratification agrees with the stratification by analytic (or C^∞) type for simple singularities; however beyond this range, the work of Wall, Bruce, Giblin, and Gibson ([BG], [BW], [GG], [Wa]) has shown just how difficult it is to determine the Thom–Mather stratification even in the case of unimodal hypersurface singularities.

Part of this difficulty was foreseen by Pham around 1970. In [Ph] he showed that constant topological type in a family of singularities does not imply constant topological type of the corresponding families of versal deformations. He found an example of a complex curve singularity $f_0(x, y) = y^3 + x^9$ which has a two parameter family of deformations (with moduli (s, t))

$$f_1(x, y, s, t) = y^3 + tyx^6 + syx^7 + x^9$$

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with constant Milnor number; and hence the family is topologically trivial. However, the versal deformation of f_0 is not topologically a product along the t-axis. In fact, he showed that for $t = 0$ there are fibers arbitrarily close to $f_0^{-1}(0)$ with both E_6 and E_8 singularities in the fiber while this does not happen for values $t \neq 0$.

In his work on the simple elliptic hypersurface singularities, Looijenga [L 1, I] introduced an alternate approach to explicitly determining the universal topological stratification. He directly proved the topological triviality of the versal deformation along the direction of the modal parameter. This was extended to other unimodal hypersurface singularities by Wirthmüller [W] and to unimodal complete intersections by Ronga [R] and [D 1], [D 3]. The success of the results depends largely on the Gorenstein properties of the Jacobian algebra for hypersurface singularities, and on a generalization of this to a duality result for T_X^1 in [D 3]. These results can all be understood in terms of the finite $\mathscr{A}$-determinacy of the deformations which are versal except for the absence of the modal parameter [D 1]. It is exactly the general failure of this condition for multi-modal singularities which forces a deeper analysis.

Here we describe this analysis for the Pham example. In particular, we determine how the (s, t) parameter space is stratified by the topological type of the versal deformation. We believe this provides a framework for understanding the other bimodal singularities and an approach to investigating the higher-modality singularities.

Let $\mathscr{V} \to T$ denote the projection of the versal deformation for the Pham example. Since $\mathscr{V}$ is smooth, with respect to appropriate local coordinates, π is given by the function

$$F(x, y, s, t, u, v) = \left(\bar{F}(x, y, s, t, u, v), s, t, u, v\right)$$

where

$$\bar{F}(x, y, s, t, u, v) = f_1(x, y, s, t) + \sum_{i=0}^{5} u_i x^i y + \sum_{i=1}^{7} v_i x^i .$$

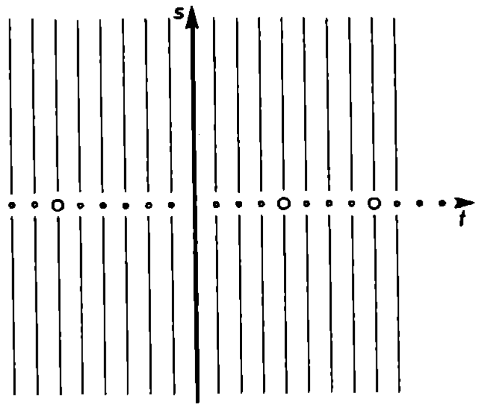

Fig. 1

Since the versal deformation is with respect to $\mathscr{K}$-equivalence, we begin with the $\mathscr{K}$-orbit structure of the (s, t)-subspace given by Figure 1. The missing points on the t-axis correspond to the values $4t^3+27=0$, where finite $\mathscr{K}$-determinacy fails.

We shall prove in either the smooth case or holomorphic case

THEOREM. *The stratification of the (s, t)-subspace, such that the versal unfolding (as a germ of a mapping) is topologically a product on strata is given by: the s-axis, the punctured lines (for $4t^3+27=0$), and the complement* (see Fig. 2).

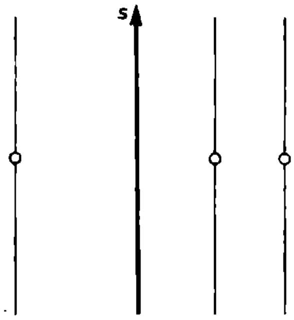

Fig. 2

By a result of Wirthmüller [W], the versal deformation is topologically a product along any line parallel to the s-axis with $4t^3+27 \neq 0$. Hence, we may assume $s=0$ and ask when the unfolding restricted to this subspace is locally topologically a product along the t-axis. We denote this restricted unfolding with $s=0$ by f. If we further restrict $t=t_0$ we denote the unfolding by f_{t_0}. Then f, viewed as an unfolding of f_{t_0} by the parameter t, is an unfolding of weight 0. If f_{t_0} were finitely $\mathscr{A}$-determined, we could use [D 1, I] to deduce topological triviality along the t-axis. This turns out not be the case and is the point of departure from the analysis of the unimodal singularities.

1. The versality discriminant and topological triviality

For the germ f_{t_0}, we let V_0 denote the set of points in the target at which f_{t_0} is not infinitesimally stable, i.e., not versal. Then V_0 is the germ of an analytic set [D 1, I; 5.1]. More generally let V be the union of $V_{t'}$ for t' near t. Again V is an analytic set. Then V_0, respectfully V, is called the *versality discriminant* of f_{t_0}', respectfully the unfolding f of f_{t_0}. To prove that f is topologically trivial along the t-axis near t, it is sufficient to prove, using the result in [D 2, I], that the restriction of f to a "conical neighborhood" of the versality discriminant is topologically trivial in a certain stratified sense.

This requires three steps:

1) geometrically identifying a candidate for the versality discriminant;
2) proving algebraically that this candidate is correct;
3) establishing the stratified topological triviality of the restriction of f.

To see that the versality discriminant is more than just a point (which would imply finite $\mathscr{A}$-determinacy), we consider the family ($t \neq 0$)

$$y^3 + t(x-x_0)^4 \cdot (x+2x_0)^2 \cdot y + (x-x_0)^6 \cdot (x+2x_0)^3.$$

Near $x = x_0$ with $\bar{x} = x - x_0$, we have the germ

$$y^3 + t(3x_0)^2 \cdot \bar{x}^4 + (3x_0)^3 \cdot \bar{x}^6$$

which is an $\tilde{E}_8$. While near $x = -2x_0$ with $\bar{x} = x + 2x_0$, we obtain

$$y^3 + t(3x_0)^4 \cdot \bar{x}^2 + (3x_0)^6 \cdot \bar{x}^3.$$

Thus, along a curve $u_i = tc_i x_0^{6-i}$, $v_i = b_i x^{9-i}$, and $z = b_0 x_0^9$, where z denotes the coordinate for f and x_0 denotes the parameter for the curve, there are $\tilde{E}_8$ and D_4 singularities in a fiber. However, for fixed $t \neq 0$, the dimension of the target space is 14 while the codimensions of $\tilde{E}_8$ and D_4 are 10 and 4 respectively. Thus, if the multi-germ in this fiber were multi-transverse, the set of points where it occurred would be isolated and not along a curve. Thus the curve belongs to the versality discriminant of f_t. The surprising fact is

Proposition 1. *The versality discriminant for f_t, $t \neq 0$, $4t^3 + 27 \neq 0$ is exactly the curve described above.*

Remark. In fact as t varies ($\neq 0$) this curve is analytically trivial and a simple change of coordinates makes it constant.

Using [D 1, II; 4.1] and the preparation theorem, Proposition 1 will follow by showing for fixed t

$$(*) \qquad g \cdot yx^6,\ g \cdot yx^7 \in \mathscr{G}_{x,y,u,v}\left\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\} + \mathscr{G}_{z,u,v}\{1, x, \ldots, x^7, y, \ldots, yx^5\}$$

for a set of generators g of an ideal I defining the curve. Here $\mathscr{G}_{x,y,u,v}$ denotes the algebra of holomorphic (respectively smooth) germs in x, y, u, v, etc. and $R\{w_1, \ldots, w_r\}$ denotes the R-module generated by $w_1, \ldots, w_r$. On the other hand, by the preparation theorem,

$$(**) \qquad \mathscr{G}_{x,y,u,v} = \mathscr{G}_{x,y,u,v}\left\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\} + \mathscr{G}_{u,v}\{1, x, \ldots, x^7, y, \ldots, yx^7\}.$$

Thus the terms $f^i \cdot \varphi_j$ for $\varphi_j \in \{1, \ldots, yx^5\}$, $i > 0$, together with $(**)$ generate relations

$$h_{ij}^{(1)} \cdot yx^6 + h_{ij}^{(2)} yx^7 \in \text{R.H.S. of } (*).$$

Thus, the 2×2 determinants of such elements $g = \det(h_{i_jj_l}^{(k)})$ satisfy $(*)$. It is sufficient to compute enough such g to show they generate an ideal defining the curve. This is carried out via symbolic computations using the computer language MACSYMA and the generalized division theorem of Hironaka (see [G 1] or [G 2]).

2. Stratified topological triviality

For fixed t we have the following picture of the versality discriminant (Fig. 3).

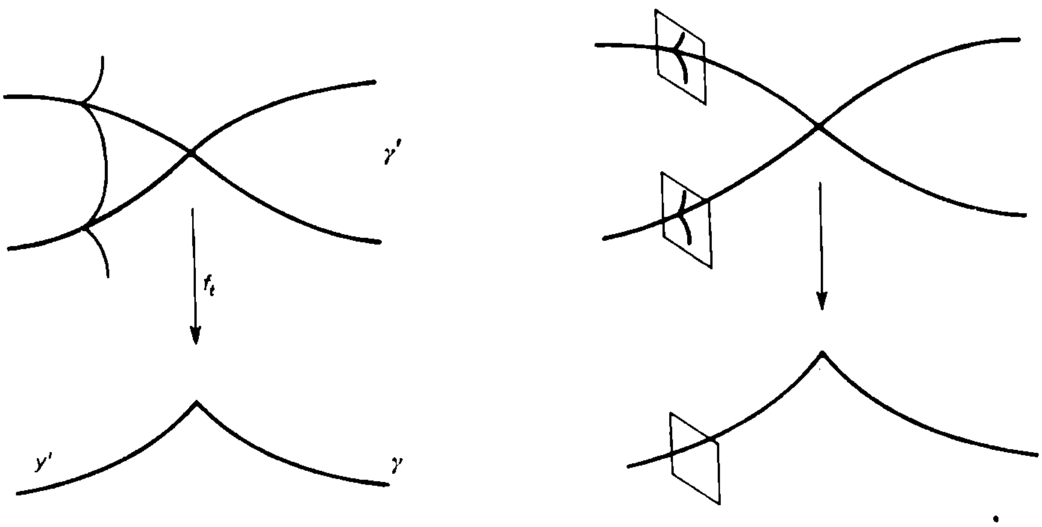

Fig. 3

Its inverse image in the critical set of f consists of a pair of nonsingular curves. In a fiber over the point y' there are two points in γ'. If we take a section to γ and its inverse image in γ' we have a multi-germ of f at x', x''

$$\begin{tikzcd} k^{14}, x' \arrow[dr] & \\ & k^{13}, y' \\ k^{14}, x'' \arrow[ur] & \end{tikzcd}$$

The structure of this multi-germ is given by the following proposition.

PROPOSITION 2. *By a local change of coordinates the multi-germ may be written*

$$z = \bar{y}^3 + (t + c\bar{w}_6)\,\bar{y}\bar{x}^4 + \bar{x}^6 + t\bar{y}(\bar{w}_1 + \ldots + \bar{w}_4\,\bar{x}^3) + \bar{s}_1\,\bar{x} + \ldots + \bar{s}_4\,\bar{x}^4$$

and

$$z = \bar{y}'^3 + t\bar{y}'\bar{x}'^2 + \bar{x}'^3 + \bar{y}'(\bar{w}_5 + \bar{w}_6\bar{x}') + \bar{s}_5 + \bar{s}_6 x'$$

modulo terms of weight > 6, *and* ≥ 6 *in* $(\bar{x}, \bar{y})$ *(respectively* $(\bar{x}', \bar{y}')$*), where:* $wt(\bar{x}) = 1$, $wt(\bar{y}) = 2$, $wt(\bar{w}_i) = 5-i$ *and* $wt(\bar{s}_i) = 6-i$ *for* $1 \leq i \leq 4$, $wt(\bar{x}') = wt(\bar{y}') = 2$, $wt(\bar{w}_i) = 2(7-i)$ *and* $wt(\bar{s}_i) = 2(8-i)$, $i = 5, 6$. *Also,* $c \neq 0$.

To complete the proof of the theorem we extend Looijenga's calculation [L 1, I] for $\tilde{E}_8$ to obtain that the multi-germ has finite graded codimension (in an appropriate sense) with respect to the weight filtration. Thus, Theorem 10.5 of [D 2, II] implies that the multi-germ is topologically trivial along the t-axis. Moreover, this topological trivialization is constructed from stratified vector fields in the sense of [D 2, I]. The vector fields are stratified relative to the stratifications of the $\tilde{E}_8$-stratum by multi-germs involving $\tilde{E}_8$-germs and germs in the versal unfolding of D_4.

First we may extend both the stratified vector fields and the control functions defined on the sections to conical neighborhoods of γ and γ' using the k^*-action (see Fig. 4). Next we extend the sections to compact manifolds intersecting each of the $\boldsymbol{R}_+$-orbits once and take the product of these with the t-axis. Then we may further extend the control functions for the stratified vector fields to these manifolds so they are non-zero outside of smaller conical neighborhoods of γ and γ'. Finally using the $\boldsymbol{R}_+$-action we may extend the control functions to a neighborhood of 0.

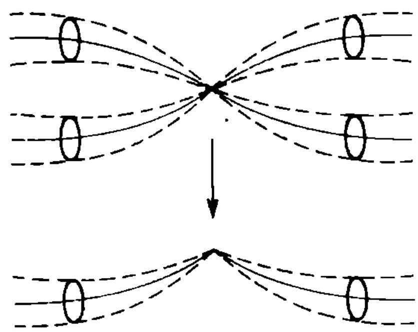

Fig. 4

It can be checked that this data satisfies the condition for f to be stratified topologically trivial in a conical neighborhood of the versality discriminant. Thus, by Theorem 1 of [D 2, I], f itself is topologically trivial along the t-axis. Very briefly, the topological triviality is proven by constructing stratified vector fields in neighborhoods of 0 which can be integrated to give the topological trivializations. These vector fields are constructed from three pieces: those described above in the conical neighborhoods, vector fields defined on a neighborhood of the discriminant off of the versality discriminant (defined using (*)), and a vector field defined off the critical set using the $\mathscr{K}$-finite determinacy of f. These are finally patched together using a partition of unity.

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