PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 19
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1987

SOME COMPACTNESS METHODS IN THE THEORY OF NONLINEAR EVOLUTION EQUATIONS WITH APPLICATIONS TO P.D.E.

IOAN I. VRABIE

Department of Mathematics, Polytechnic Institute, Iaşi, Romania

1. Introduction

The aim of these lectures is to present an existence result for a class of O.D.E. in general Banach spaces whose proof is mainly based on a very recent compactness criterion due to the author (cf. Vrabie [20]).

It is by now classical that there is no infinite-dimensional Banach space in which Peano's local existence theorem holds true in the absence of some additional hypotheses (cf. Godunov [11]). Roughly speaking, the explanation of this nonexistence phenomenon is quite simple and consists mainly in that infinite-dimensional Banach spaces are not locally compact. Thus, when trying to extend Peano's result from finite to infinite dimensions, one has to supply this lack of compactness by something else. At first glance a good idea in this respect might be to assume from the very beginning some "nice compactness" properties on the right-hand side. For example, if we suppose that the right-hand side carries bounded subsets into relatively compact subsets, i.e. if it is a compact operator, then the proof of the corresponding existence result follows exactly the same lines as in the finite-dimensional case. Unfortunately, this condition (and even a more general one involving some measure of noncompactness) is very hard to check in infinite dimensions. As far as we know, the sole example which falls into this frame is that of some O.D.E. arising from functional differential equations of neutral type (cf. Hale [12]). The situation is even worse if we look for abstract existence results applying to O.D.E. which include as specific cases some classes of P.D.E. Indeed, in this case the right-hand side is neither compact nor continuous with respect to the corresponding norm topology.

In order to overcome these additional difficulties introduced by the lack of continuity a first method is to work not in a single Banach space, but in two or more (cf. Brézis [5], Browder [7], Lions [13] and others). If this framework is adopted, with the help of some monotonicity conditions and/or compactness imbedding assumptions, one can quite easily obtain satisfactory existence results. Nevertheless, this advantage is heavily counterbalanced by the incomplete information on smoothness properties of solutions of the corresponding P.D.E. with respect to their "spatial arguments".

A second method is to work in a single Banach space by adding some assumptions on the right-hand side which substitute for the compactness. A good such substitute which is motivated by practice is m-dissipativity (cf. Bénilan [3] and Crandall–Liggett [9]).

It is our belief that whenever one may use both these methods a better policy is to choose the latter because along with a good existence result one can also get enough information concerning the smoothness of the solutions.

In all that follows we shall show how an effective interplay between m-dissipativity (in fact m-accretiveness) and compactness can be used in order to obtain new existence results applying to strongly nonlinear perturbed evolution equations with applications to P.D.E. The main idea which goes back to Pazy [14] consists in that the compactness assumption we use refers to the m-accretive operator involved but not to its continuous perturbation. This idea proves very useful because it reveals one of the simplest ways to translate the compactness machinery from finite to infinite dimensions in order to handle both O.D.E. and nonlinear P.D.E. of parabolic type. For other related results not included here we refer to Vrabie [17, 18, 19, 21].

Acknowledgements. I take this opportunity to express my warmest thanks to the organizers and especially to Professors Bojarski and Olech for their kind invitation to give these lectures during the Semester on P.D.E. held at Stefan Banach International Mathematical Center in Warsaw, September 12–December 12, 1984.

2. Preliminaries

Let X be a real Banach space with the norm $\|\cdot\|$. The closure of a subset C in X is denoted by $\bar{C}$. If $x, y \in X$, (x, y) represents the right directional derivative of the norm calculated at x in the direction y, i.e.

$$(x, y) := \lim_{h \downarrow 0} h^{-1}(\|x+hy\| - \|x\|).$$

By an *operator* we mean a function $A\colon X \to 2^X$. If A is an operator we denote by

$$D(A) := \{x \in X;\ Ax \neq \emptyset\} \quad - \quad \text{the domain of } A$$

and by

$$R(A) := \bigcup_{x \in D(A)} Ax \quad - \quad \text{the range of } A.$$

Thus each function a: $D(a) \subset X \to X$ can be identified with an operator A: $X \to 2^X$ defined by

$$Ax := \begin{cases} \emptyset & \text{if } x \in X - D(a), \\ \{a(x)\} & \text{if } x \in D(a). \end{cases}$$

An operator A is called *accretive* if

$$(x_1 - x_2, y_1 - y_2) \geqslant 0$$

for all $x_i \in D(A)$, $y_i \in Ax_i$, $i = 1, 2$.

An accretive operator is called *m-accretive* if for every $h > 0$, $R(\mathrm{Id} + h \cdot A) = X$, where Id is the identity on X.

Let A: $X \to 2^X$ be an m-accretive operator and let us consider the Cauchy problem

$$\frac{du}{dt}(t) + Au(t) \ni f(t), \quad 0 \leqslant t \leqslant T, \qquad u(0) = u_0, \tag{1}$$

where $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; X)$ are given.

By an *integral solution* of (1) on $[0, T]$ we mean a continuous function u: $[0, T] \to \overline{D(A)}$ with $u(0) = u_0$ satisfying

$$\|u(t) - x\| \leqslant \|u(s) - x\| + \int_a^t (u(\theta) - x, f(\theta) - y) d\theta \tag{2}$$

for all $x \in D(A)$, $y \in Ax$ and $0 \leqslant s \leqslant t \leqslant T$. This concept has been introduced by Bénilan–Brézis [4] in the case when X is a real Hilbert space and by Bénilan [3] in the general case.

Remark 1. We note that whenever A is linear a function u: $[0, T] \to X$ is an integral solution of (1) iff it is a *mild solution* of (1) in the sense of Browder [8], i.e. iff u is given by the variation of parameters formula

$$u(t) = S(t)u_0 + \int_0^t S(t-s) f(s) ds$$

for every $t \in [0, T]$, where $S(t)$ is the C_0-semigroup of linear operators generated by A on X.

The next two results due to Bénilan will be very frequently used in all that follows.

THEOREM 1. *Let A: $X \to 2^X$ be an m-accretive operator. Then for each $(u_0, f) \in \overline{D(A)} \times L^1(0, T; X)$ there exists a unique integral solution of the problem* (1) *on* $[0, T]$.

THEOREM 2. *Let A: $X \to 2^X$ be an m-accretive operator, let (u_0, f), (v_0, g)*

be arbitrary in $\overline{D(A)} \times L^1(0, T; X)$ *and let* u, v *be the integral solutions of* (1) *on* $[0, T]$ *corresponding to* (u_0, f) *and* (v_0, g) *respectively. Then the following inequality*

$$(3) \qquad \|u(t)-v(t)\| \leqslant \|u(s)-v(s)\| + \int_s^t \|f(\theta)-g(\theta)\|\, d\theta$$

holds for all $0 \leqslant s \leqslant t \leqslant T$.

For the proof of Theorems 1 and 2 see Bénilan [3], or Barbu [2].

Let I: $\overline{D(A)} \times L^1(0, T; X) \to C(0, T; X)$ be given by

(4) $\quad I(u_0, f) := u \quad -\quad$ the unique integral solution of (1) on $[0, T]$ corresponding to (u_0, f).

By Theorem 1, I is everywhere defined and single-valued. Moreover, from (3) we easily conclude that I is Lipschitz continuous.

Let C be a closed subset in X. By a *C_0-semigroup of nonexpansive mappings on* C we mean a family $\{S(t);\ S(t)\colon C \to C,\ t \geqslant 0\}$ satisfying

(S_1) $\quad S(t+s)x = S(t)S(s)x$, for all $x \in C$, $t \geqslant 0$, $s \geqslant 0$;

(S_2) $\quad S(0)x = x$, for every $x \in C$;

(S_3) $\quad \lim_{t \downarrow 0} S(t)x = x$, for every $x \in C$;

(S_4) $\quad \|S(t)x - S(t)y\| \leqslant \|x-y\|$, for all $t \geqslant 0$ and $x, y \in C$.

Remark 2. We note that the restriction of I to $\overline{D(A)} \times \{0\}$ defines a C_0-semigroup of nonexpansive mappings on $\overline{D(A)}$ by

$$S(t)u := I(u, 0)(t)$$

for all $u \in \overline{D(A)}$ and $t \geqslant 0$. This semigroup is exactly the C_0-semigroup generated by A on $\overline{D(A)}$ via the Crandall–Liggett exponential formula [9]

$$S(t)u := \lim_n \left(\mathrm{Id} + \frac{t}{n} A\right)^{-n} u$$

for all $u \in \overline{D(A)}$ and $t \geqslant 0$.

More details concerning the theory of m-accretive operators may be found in Barbu [2].

3. The compactness criterion

Let A: $X \to 2^X$ be an m-accretive operator, U an open subset in X such that $U \cap \overline{D(A)} \neq \emptyset$, f: $[0, T] \times U \to X$ a given function, and let us consider the following perturbed Cauchy problem:

$$\frac{du}{dt}(t)+Au(t)\ni f\big(t, u(t)\big), \qquad 0\leqslant t\leqslant T, \tag{5}$$
$$u(0)=u_0,$$

where $u_0\in U\cap\overline{D(A)}$.

By an *integral solution* of (5) on $[0, T]$ we mean a continuous function u: $[0, T]\to U\cap\overline{D(A)}$ such that the function g: $[0, T]\to X$ defined by $g(t) := f\big(t, u(t)\big)$ for a.e. $t\in\,]0, T[$ lies in $L^1(0, T; X)$ and

$$u = I(u_0, g)$$

where I is given by (4).

Thus an integral solution of (5) on $[0, T]$ is a fixed point of the mapping Q defined (if possible) on a suitable subset K in $C(0, T; X)$ by

$$Qu[t] := I\big(u_0, f(\cdot, u(\cdot))\big)(t) \tag{6}$$

for all $u\in K$ and $t\in[0, T]$. Here and subsequently we use the notation

$$Qu[t] := (Qu)(t)$$

for all $u\in K$ and $t\in[0, T]$.

Now, it is quite transparent that when trying to prove a local existence result concerning integral solutions of (5) we have to seek for appropriate conditions on A and on f in order that the operator Q be well defined on a certain subset K in $C(0, T; X)$ and have at least one fixed point in K.

Assume for a moment that we have constructed a nonempty, closed, convex and bounded subset K in $C(0, T; X)$ such that Q maps K into itself and is continuous. At this point, in order to appeal to Schauder's fixed point theorem we have to check the compactness of the mapping Q given by (6). To do this, in some circumstances, it would be enough to exhibit some "good compactness" properties of the mapping I defined by (4).

The main goal of this section is to analyse the mapping I from the viewpoint of compactness.

We begin by recalling that a subset F in $L^1(0, T; X)$ is called *uniformly integrable* if for every $\varepsilon>0$ there is a $\delta(\varepsilon)>0$ such that

$$\int_E \|f(s)\|\,ds\leqslant\varepsilon$$

for every measurable subset E in $[0, T]$ whose Lebesgue measure $m(E)\leqslant\delta(\varepsilon)$ and uniformly with respect to $f\in F$.

Remark 3. A simple argument involving Hölder's inequality shows that each bounded subset F in $L^p(0, T; X)$ with $1<p\leqslant\infty$ is uniformly integrable. Moreover, if F is a subset in $L^1(0, T; X)$ for which there exists a g in

$L^1(0, T; \mathbf{R}_+)$ such that

$$\|f(t)\| \leqslant g(t)$$

for every $f \in F$ and for a.e. $t \in]0, T[$, then F is uniformly integrable.

Now we are able to state the main result of this section.

THEOREM 3. *Let A: $X \to 2^X$ be an m-accretive operator, F a uniformly integrable subset in $L^1(0, T; X)$ and u_0 a fixed element in $\overline{D(A)}$. Then the following conditions are equivalent*:

(i) *$I(u_0, F)$ is relatively compact in $C(0, T; X)$.*

(ii) *There exists a dense subset D in $[0, T]$ such that for every $t \in D$ the set $I(u_0, F)(t) := \{I(u_0, f)(t); f \in F\}$ is relatively compact in X.*

For the proof of Theorem 3 see [20].

In order to understand the exact meaning of Theorem 3 let us examine some examples.

First of all let us assume that A: $X \to 2^X$ is given by $Au = \{0\}$ for every $u \in X$. Since in this case the C_0-semigroup of nonexpansive mappings contains only the identity on X, by Remark 1 we have

$$I(u_0, f)(t) = u_0 + \int_0^t f(s)\, ds$$

for all $f \in L^1(0, T; X)$, $u_0 \in X$ and $t \in [0, T]$. Thus if F is a uniformly integrable subset in $L^1(0, T; X)$ one can easily deduce that $I(u_0, F)$ is equicontinuous on $[0, T]$. Consequently, by Ascoli's theorem it follows that (ii) implies (i). This argument is still valid if A: $X \to 2^X$ is linear continuous. Indeed, by Remark 1 we have

$$I(u_0, f)(t) = S(t)u_0 + \int_0^t S(t-s) f(s)\, ds$$

for all $f \in L^1(0, T; X)$, $u_0 \in X$ and $t \in [0, T]$. On the other hand, since A is linear continuous, the mapping $t \mapsto S(t)$ is uniformly continuous at $t = 0$, i.e. $\lim_{t \downarrow 0} \|S(t) - \mathrm{Id}\| = 0$ (see Pazy [15, Theorem 1.2, p. 2]). Hence, if F is uniformly integrable we conclude that $I(u_0, F)$ is equicontinuous on $[0, T]$, and thus (ii) implies (i).

Consequently, the proof of Theorem 3 is trivial whenever A is linear continuous because in this case the equicontinuity of $I(u_0, F)$ is an intrinsic property of A.

Next, we shall see by analysing a very simple example inspired from Schechter [16] that even for linear operators which are discontinuous the equicontinuity of $I(u_0, F)$ is not automatically satisfied.

Indeed, let $X := C_{2\pi}(\mathbf{R}; \mathbf{R})$ be the space of all 2π-periodic continuous functions from $\mathbf{R}$ into $\mathbf{R}$ endowed with the usual sup-norm, and let A: X

$\to 2^X$ be given by

$$Au := \left\{\frac{du}{dx}\right\}$$

for every $u \in D(A)$, where $D(A) := \{u \in C_{2\pi}(\boldsymbol{R}; \boldsymbol{R});\ du/dx \in C_{2\pi}(\boldsymbol{R}; \boldsymbol{R})\}$.

It is well known that A is m-accretive. Let $u_0 = 0$ and

$$F = \{\sin(nt - nx);\ n \in N\}$$

which is uniformly integrable in $L^1(0, T; C_{2\pi}(\boldsymbol{R}; \boldsymbol{R}))$ being bounded in $L^\infty(0, T; C_{2\pi}(\boldsymbol{R}; \boldsymbol{R}))$ (see Remark 3). A simple computation shows that

$$I(0, F) = \{t \cdot \sin(nt - nx);\ n \in N\}$$

which is not equicontinuous on $[0, T]$ except for $t = 0$.

4. The main existence result

Let A: $X \to 2^X$ be an m-accretive operator and $p \in [1, \infty]$.

Definition 1. A is called a *p-compact generator* if for every $u_0 \in \overline{D(A)}$ and every $F \subset L^p(0, T; X)$ which is bounded in $L^p(0, T; X)$ if $p > 1$ and uniformly integrable in $L^1(0, T; X)$ if $p = 1$ there exists a dense subset D in $[0, T]$ (depending on u_0 and F) such that

$$I(u_0, F)(t) := \{I(u_0, f)(t);\ f \in F\}$$

is relatively compact in X for every $t \in D$.

Remark 4. By Theorem 3 it follows that A is a p-compact generator iff for every $u_0 \in \overline{D(A)}$ and every $F \subset L^p(0, T; X)$ which is bounded if $p > 1$ and uniformly integrable if $p = 1$ the set $I(u_0, F)$ is relatively compact in $C(0, T; X)$.

We note that all m-accretive operators acting on finite-dimensional Banach spaces are 1-compact generators.

A broader class of 1-compact operators is given by

Theorem 4. *Let A: $X \to 2^X$ be an m-accretive operator generating a C_0-semigroup of nonexpansive mappings $\{S(t);\ S(t)$: $\overline{D(A)} \to \overline{D(A)},\ t \geqslant 0\}$ with $S(t)$ compact for $t > 0$. Then A is a 1-compact generator.*

Proof. Let $u_0 \in \overline{D(A)}$ and F uniformly integrable in L^1 $(0, T; X)$ be fixed. Let $t \in]0, T]$ and $h > 0$ be such that $t - h \in [0, T]$. Since the mapping $s \mapsto S(s) I(u_0, f)(t-h)$ is the unique integral solution of

$$\frac{dv}{ds}(s) + Av(s) \ni 0, \qquad 0 \leqslant s \leqslant h,$$

$$v(0) = I(u_0, f)(t-h),$$

by (3) we get

$$\|S(h)I(u_0,f)(t-h)-I(u_0,f)(t)\| \leqslant \int_0^h \|f(t-h+s)\|\,ds$$

for every $h>0$ with $t-h\in[0, T]$. Inasmuch as F is uniformly integrable, this inequality shows that

$$\lim_{h\downarrow 0} \|S(h)I(u_0,f)(t-h)-I(u_0,f)(t)\| = 0$$

uniformly with respect to $f\in F$. Finally, as $\{I(u_0,f)(t);\, f\in F,\ t\in[0, T]\}$ is bounded in X and $S(h)$ is compact for every $h>0$, we easily conclude that $I(u_0, F)(t)$ is relatively compact in X for every $t\in\,]0, T]$. ∎

Remark 5. A necessary and sufficient condition in order that A generate a semigroup containing only compact operators except the identity may be found in Brézis [6].

Remark 6. Combining Theorem 4 with Remark 4 we deduce an extension of some previous results due to Baras [1] and to the author (Vrabie [17]).

Now we can state the main existence result.

THEOREM 5. *Let A: $X\to 2^X$ be an m-accretive operator, U a nonempty open subset in X and f: $[0, T]\times U\to X$ a given function. Assume that there exists $p\geqslant 1$ such that*

(i) *A is a p-compact generator;*

(ii) *$f(t,\cdot)$: $U\to X$ is continuous for a.e. $t\in\,]0, T[$;*

(iii) *$f(\cdot, u)$: $[0, T]\to X$ is measurable for every $u\in U$;*

(iv) *for every $u_0\in U$ there exist $r>0$ and $g\in L^p(0, T; \boldsymbol{R}_+)$ such that*

$$\|f(t, u)\| \leqslant g(t)$$

for every $u\in U$, $\|u-u_0\|\leqslant r$, and for a.e. $t\in\,]0, T[$.

Then for each $u_0\in U\cap\overline{D(A)}$ there exists $T_0\in\,]0, T]$ such that the problem (5) *has at least one integral solution on $[0, T_0]$.*

Proof. Fix $u_0\in U\cap\overline{D(A)}$ and choose $r>0$, $T_0\in\,]0, T]$ small enough and $g\in L^p(0, T; \boldsymbol{R}_+)$ such that

$$B(u_0, r) := \{u\in X;\ \|u-u_0\|\leqslant r\}\subset U, \tag{7}$$

$$\|f(t, u)\|\leqslant g(t) \tag{8}$$

for every $u\in B(u_0, r)$ and for a.e. $t\in\,]0, T[$, and

$$\|S(t)u_0-u_0\| + \int_0^{T_0} g(s)\,ds \leqslant r \tag{9}$$

for every $t \in [0, T_0]$. This is always possible since U is open, f satisfies (iv) and $S(\cdot)u_0$ is strongly continuous at $t = 0$.

Now, let us define the subset K in $C(0, T_0; X)$ by

$$K := \{u \in C(0, T_0; X);\ \|u(t) - u_0\| \leqslant r \text{ for } t \in [0, T_0]\}.$$

Clearly K is bounded, closed, convex and nonempty in $C(0, T_0; X)$. Moreover, by (ii)–(iv) and Theorem 1, the mapping Q given by (6) is everywhere defined on K. Let us observe that by (3), (8) and (9) we have

$$\begin{aligned} \|Qu[t] - u_0\| &\leqslant \|Qu[t] - S(t)u_0\| + \|S(t)u_0 - u_0\| \\ &\leqslant \|S(t)u_0 - u_0\| + \int_0^{T_0} g(s)\,ds \leqslant r \end{aligned}$$

for all $u \in K$ and $t \in [0, T_0]$. Thus Q maps K into itself. Also by (3) it follows that

$$\sup\{\|Qu[t] - Qv[t]\|;\ t \in [0, T_0]\} \leqslant \int_0^{T_0} \|f(s, u(s)) - f(s, v(s))\|\,ds$$

for all $u, v \in K$. Consequently, from (ii)–(iv) and Lebesgue's dominated convergence theorem we deduce that Q is continuous from K into itself.

Finally, since A is a p-compact generator, by Remark 4 we conclude that $Q(K)$ is relatively compact in $C(0, T_0; X)$ and thus, from Schauder's fixed point theorem Q has at least one fixed point which is an integral solution of (5) on $[0, T_0]$. ■

5. Applications to P.D.E.

To illustrate the effectiveness of Theorem 5 let us consider the strongly nonlinear porous medium equation

$$\begin{aligned} &u_t - (h(u))_{xx} = f(t, x, u), \quad 0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant 1, \\ (10)\qquad &u(t, 0) = u(t, 1) = 0, \quad 0 \leqslant t \leqslant T, \\ &u(0, x) = u_0(x), \quad 0 \leqslant x \leqslant 1, \end{aligned}$$

where $u_t := \partial u/\partial t$ and $(h(u))_{xx} := \partial^2 h(u)/\partial x^2$.

We make the following assumptions.

(H_1) $h \in C(\boldsymbol{R}; \boldsymbol{R}) \cap C^1(\boldsymbol{R} - \{0\}; \boldsymbol{R})$, $h' \geqslant 0$, $h^{-1} \in C(\boldsymbol{R}; \boldsymbol{R})$.

(H_2) $f\colon [0, T] \times [0, 1] \to \boldsymbol{R}$ is jointly continuous.

Using Theorem 5 we can prove

THEOREM 6. *Assume that* (H_1), (H_2) *are satisfied. Then for each* u_0 *in* $L^\infty(0, 1; \boldsymbol{R})$ *there exists* $T_0 \in]0, T]$ *such that the problem* (10) *has at least one* *solution* u: $[0, T_0] \to L^\infty(0, 1; \mathbf{R})$ *satisfying*

$$u \in C(0, T_0; L^1(0, 1; \mathbf{R})), \quad u_t \in L^2(0, T_0; H^{-1}(0, 1; \mathbf{R})).$$

In addition, the mapping $t \mapsto \int_0^1 j(u(t, x))dx$ *is absolutely continuous from* $[0, T_0]$ *into* $\mathbf{R}$, *where* j: $\mathbf{R} \to \mathbf{R}$ *is defined by* $j(r) = \int_0^r h(s)ds$ *for each* $r \in \mathbf{R}$.

For the proof of Theorem 6 which seems to be new see Diaz–Vrabie [10]. Other related results may be found in Vrabie [19, 20, 22].

References

[1] P. Baras, *Compacité de l'opérateur* $f \mapsto u$ *solution d'une équation non linéaire* $(du/dt) + Au \ni f$, C. R. Acad. Sci. Paris 289 (1978), 1113–1116.
[2] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden 1976.
[3] P. Bénilan, *Équations d'évolution dans un espace de Banach quelconque et applications*, Thèse, Orsay 1972.
[4] P. Bénilan et H. Brézis, *Solutions faibles d'équations d'évolution dans les espaces de Hilbert*, Ann. Inst. Fourier (Grenoble) 22 (2) (1972), 311–329.
[5] H. Brézis, *Équations et inéquations non linéaires dans les espaces vectoriels en dualité*, ibid. 18 (1) (1968), 115–175.
[6] –, *New results concerning monotone operators and nonlinear semigroups*, in: Analysis of Nonlinear Problems, RIMS, Kyoto Univ. 258 (1974), 2–27.
[7] F. E. Browder, *Nonlinear initial value problems*, Ann. of Math. 82 (1965), 51–87.
[8] –, *Nonlinear equations of evolution*, ibid. 80 (1964), 485–523.
[9] M. G. Crandall and T. Liggett, *Generation of semigroups of nonlinear transformations on general Banach spaces*, Amer. J. Math. 93 (1971), 265–298.
[10] J. I. Diaz and I. I. Vrabie, paper in preparation.
[11] A. N. Godunov, *On Peano's theorem in Banach spaces*, Funktsional. Anal. i Prilozhen. 9 (1) (1975), 59–60 (in Russian).
[12] J. K. Hale, *α-contractions and differential equations*, in: Équations Différentielles et Fonctionnelles Nonlinéaires, P. Janssens, J. Mawhin, N. Rouche (eds.), Hermann, Paris 1973, 15–41.
[13] J. L. Lions, *Sur certaines équations paraboliques non linéaires*, Bull. Soc. Math. France 93 (1965), 155–175.
[14] A. Pazy, *A class of semi-linear equations of evolution*, Israel J. Math. 20 (1975), 23–36.
[15] –, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Math. Sci. 44, Springer, New York 1983.
[16] E. Schechter, *Perturbations of regularizing maximal monotone operators*, Israel J. Math. 43 (1982), 49–61.
[17] I. I. Vrabie, *The nonlinear version of Pazy's local existence theorem*, Israel J. Math. 32 (1979), 221–235.
[18] –, *Compactness methods for an abstract Volterra integro-differential equation*, Nonlinear Anal. 5 (1981), 355–371.
[19] –, *An existence result for a class of nonlinear evolution equations in Banach spaces*, ibid. 6 (1982), 711–722.

[20] I. I. Vrabie, *A compactness criterion in $C(0, T; X)$ for subsets of solutions of nonlinear evolution equations governed by accretive operators*, Rend. Sem. Mat. Univ. Politec. Torino 43 (1) (1985), 149–157.

[21] –, *Compact perturbations of m-accretive operators in Hilbert spaces*, Bul. Inst. Politehn. Iaşi, Supliment CERCETARI MATEMATICE, 1985, 83–84.

[22] –, *Compactness Methods for Nonlinear Evolutions*, Pitman Monographs and Surveys in Pure and Appl. Math. 32, Longman Scientific & Technical, 1987.