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Biholomorphic invariants related to the Bergman function

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Preliminary remarks

Introduction

Biholomorphic transformations play an important role in the theory of several complex variables. In 1907 H. Poincaré stated the following question of central importance. Given two domains D_1 and D_2 in C^n, does there exist a biholomorphic transformation f which maps D_1 onto D_2. When $n = 1$, the Hilbert mapping theorem states that every domain $D \subset C$ can be mapped onto the Riemann sphere with a number of parallel slits, and in particular every simply connected domain $D \subset C$ whose boundary contains two points at least can be mapped onto the unit disc. When $n \geqslant 2$ the situation is more complicated. This is indicated by a well-known theorem of Poincaré: the unit ball $\{|z_1|^2 + |z_2|^2 < 1\}$, and the unit polycylinder $\{|z_1| < 1,\ |z_2| < 1\}$ in C^2 cannot be mapped onto each other by a biholomorphic transformation.

In 1920 a new approach to the problem was found by Bergman [5]. He considered the Hilbert space $L^2H(D)$ of all holomorphic and square integrable functions in D, and introduced the function

$$K_D(z, \bar{t}) = \sum_{m=1}^{\infty} \varphi_m(z)\overline{\varphi_m(t)}, \qquad z, t \in D,$$

defined in terms of a complete orthonormal system φ_m, $m = 1, 2, \ldots$, in the space $L^2H(D)$. This Bergman function is independent of the choice of the orthonormal system. Furthermore, it is a relative biholomorphic invariant, and this fact was used by Bergman in the construction of numerous biholomorphic invariants and the invariant Kaehler metric in D. The introduction of this metric made possible the application of methods of differential geometry to the study of biholomorphic mappings in the papers of Kobayashi [21] and Lichnerowicz [22].

The geodesic distance induced by the Kaehler structure in D is closely related to another invariant distance, introduced by the present author in [30] and given by the explicit formula

$$(1) \qquad \varrho(z, t) = \left[1 - \left(\frac{K_D(z, \bar{t})K_D(t, \bar{z})}{K_D(z, \bar{z})K_D(t, \bar{t})}\right)^{1/2}\right]^{1/2}$$

In the present author's doctoral thesis [30] the expression (1) was used to obtain a local characterization of biholomorphic mapping under suitable general condition. It turns out that for every two domains D_1 and D_2 which satisfy these conditions, and for every two subdomains $U_1 \subset D_1$, $U_2 \subset D_2$ a biholomorphic mapping $f\colon U_1 \to U_2$ extends to a biholomorphic mapping between D_1 and D_2 if and only if f respects the rule of transformation of the Bergman function

$$K_{D_1}(z, \bar{t}) = K_{D_2}\big(f(z), \overline{f(t)}\big) \frac{\partial f}{\partial z} \left(\overline{\frac{\partial f}{\partial t}}\right)$$

for $z, t \in U_1$.

In view of this theorem, as well as for other reasons, the study of the Bergman function and the invariant distance (1) seems well justified. In the present paper a number of results in this direction is presented. For the sake of completeness results of other authors and of [30] are occasionally included.

Chapter 0 contains basic definitions and examples. The domains for which the Bergman function is known in a closed form are listed in this chapter. Among them we indicate an interesting example of B. S. Zinoviev (0.9.c).

In Chapter I we recall the properties of the invariant distance $\varrho(z, t)$ see [32]. It is easy to note that this distance is never greater than one, and is strictly less than one if and only if $K_D(z, \bar{t})$ does not attain zero value. In the latter case the domain is called a *Lu Qi-keng domain.* The question whether there exists a bounded domain D such that $K_D(z, \bar{t}) = 0$ for some $z, t \in D$ was raised by Lu Qi-keng in [23]. We will prove a theorem (stated without proof in [30]) which gives a positive answer to this question. In particular all doubly connected Lu Qi-keng domains in the complex plane are biholomorphically equivalent to the unit disc with deleted center. We shall also prove that the critical points of the invariant distance are related to the notion of a representative domain introduced by Bergman in [3].

In Chapter II we state sufficient conditions for the Bergman metric tensor to be positive definite, see [21]. We also give an example of an unbounded domain $D \subset C^2$ such that $K_D(z, \bar{z}) = 0$ for some $z \in D$, although $K_D(z, \bar{z}) \not\equiv 0$.

In Chapter III we reformulate a conjecture due to Kobayashi as follows: If D is complete with respect to the invariant distance, then $\lim_{m\to\infty} \varrho(q, p_m) = 1$ for every point $q \in D$ and every sequence $q_m \in D$, $m = 1, 2, \ldots$, without an accumulation point in D. This conjecture is still open. We prove that every bounded, strictly starlike Reinhardt domain of holomorphy $D \subset C^n$ is complete with respect to the invariant distance, and that the Kobayashi conjecture holds for D.

In Chapter IV we generalize the results of [30] to obtain a local characterization of a real analytic mapping which preserves the invariant distance. We also indicate a possibility of further generalization to complex manifolds [33].

In Chapter V we study the situation, where the sequence of domains $D_m \subset C^n$, $m = 1, 2, \ldots$, approximates a domain $D \subset C^n$. Here we mention the following elegant result of Ramadanov [27]: If D_m is an increasing sequence of domains, then K_{D_m} converges locally uniformly to K_D, where D is the union of all domains D_m. This suggest the study of decreasing sequences of domains. In this case the situation is entirely different; nevertheless, for $n = 1$ a complete solution is possible. It turns out that the class of domains D for which a theorem of the Ramadanov type is true can be characterized in terms of fine topology and logarithmic capacity. The proof uses some results of L. Hedberg on mean square approximation. The above characterization yields an example of an infinitely connected domain D such that $\bar{D} = \bigcap_{m=1} D_m$, $D_{m+1} \subset D_m$, $m = 1, 2, \ldots$, and $K_D \neq \lim_{m\to\infty} K_{D_m}$.

Finally in Chapter VI we study some further potential theoretic aspects of the Bergman function. It was shown in [34] that with the aid of the Bergman function one can introduce a natural compatification $\hat{D}$ of a bounded Lu Qi-keng domain $D \subset C^n$. This compactification is invariant under a biholomorphic mapping, and is analogous to the Martin compactification in classical potential theory. In the present paper we extend the definition of compactification $\hat{D}$ and ideal boundary $\hat{D}\setminus D$ to larger class of domains, which are not necessarily Lu Qi-keng domains but instead admit a finite covering by so-called distinguished subdomains. We prove that for each bounded, circular, strictly starlike domain D, the ideal boundary of D is canonically homeomorphic with the Euclidean boundary of D. In particular this result holds for the unit circle, and we see that the compactification $\hat{D}$ generalizes Carathéodory compactification. Furthermore, for each domain D which can be mapped onto the unit disc by a biholomorphic transformation the set of all prime ends is canonically homeomorphic with the ideal boundary of D.

In the general case $D \subset C^n$ we establish characteristic properties of compactification $\hat{D}$. It is shown that sufficient conditions in order that the ideal boundary of D be equal to the Euclidean boundary can be stated in terms of the boundary behaviour of the Bergman function $K_D(z, \bar{t})$. We use this conditions to show that for every domain $D \subset C$ bounded by a finite number of simple closed curves the ideal boundary is equal to the Euclidean boundary. This property holds also in a strictly pseudoconvex domain $D \subset C^n$ with smooth boundary due to the fact that in this case for each $t \in \partial D$

$$\lim_{z\to t} |K_D(z, \bar{t})| = \infty.$$

In this place I feel it apropriate to thank the referee for many useful remarks, in particular Remark 3.14 and for his very careful reading of the paper. To Professors Z. Charzyński, L. Hedberg, J. Ławrynowicz, and to dr. E. Ligocka I am thankful for numerous discussions.

I would like to dedicate this paper to the memory of my teacher.

Basic definitions, examples and facts

In the present paper we shall be concerned mostly with subsets of the n-dimensional complex space C^n; the occasional generalizations to complex manifolds are rather trivial. A point in C^n will be denoted by $z = (z_1, \ldots, z_n)$. Obviously C^n is a Banach space under the norm

$$|z| = \max_{1 \leqslant i \leqslant n} |z_i|$$

and a Hilbert space under the norm

$$\|z\| = \Big(\sum_{i=1}^{n} |z_i|^2\Big)^{1/2}.$$

Every norm in C^n induces the same topology. By a domain we shall understand a non-void, open, connected subset $D \subset C^n$. The notation $D_1 \Subset D_2$ means that $\bar{D}_1$ is a compact subset of D_2. For example, a domain D is bounded if and only if $D \Subset C^n$. If D is any domain in C^n, we shall write

$$D^* = \{\bar{z}\colon z \in D\},$$

where $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$. The notation $a_k z^k$ is an abbreviation for the monomial

$$a_{k_1,\ldots,k_n} z_1^{k_1} \ldots z_n^{k_n}.$$

The set of all functions holomorphic in a domain D will be denoted by $H(D)$. It is a linear space over complex numbers.

0.1. Definition. We say that the series

$$\sum_{k=0}^{\infty} f_k, \tag{1}$$

where $f_k\colon D \to C$, *converges normally in a domain* D if the series

$$\sum_{k=0}^{\infty} |f_k|$$

converges locally uniformly in D. We say that $z^0 \in D$ is a *point of normal convergence* of series (1) if this series converges normally in some neighbourhood of z^0.

0.2. DEFINITION. A domain $D \subset C^n$ is called a *Reinhardt domain* with respect to $0 \in C^n$ if

$$(s_1 z_1, s_2 z_2, \ldots, s_n z_n) \in D$$

for every $z \in D$ and $s \in C^n$ such that $|s| \leqslant 1$.

0.3. EXAMPLE. Assume that D is the set of points of normal convergence of some Taylor series

$$\sum_{m=0}^{\infty} a_m z^m.$$

Then D is a Reinhardt domain with the following property: the open subset of R^n

$$\{\log|z_1|, \ldots, \log|z_n|;\ z \in D, z_1 z_2 \ldots z_n \neq 0\}$$

is convex. The inverse statement is also true, see [17].

0.4. THEOREM. *Assume that $D \subset C^n$ is a Reinhardt domain. Then for every $f \in H(D)$ the Taylor series*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

converges normally in D to the sum $f(z)$.

The set of all functions $f \in H(D)$ for which the integral with respect to the Lebesgue measure

$$\|f\|_D = \Big(\int_D |f|^2 dm\Big)^{1/2} \tag{2}$$

is finite will be denoted by $L^2H(D)$

0.5. THEOREM (Bergman [5]). *The space $L^2H(D)$ is a Hilbert space with the scalar product*

$$(f, g)_D = \int_D f\bar{g}\, dm.$$

Moreover, for each $t \in D$ the evaluation functional

$$L^2H(D) \ni f \to f(t) \in C, \tag{3}$$

is bounded, and its norm depends on the distance from t to the complement of D. In particular the convergence in the norm (2) *implies locally uniform convergence.*

By the Riesz–Fisher theorem the bounded functional (3) can be represented as a scalar product with the unique element $g_t \in L^2H(D)$. Therefore for every $f \in L^2H(D)$

$$f(t) = (f, g_t). \tag{4}$$

0.6. DEFINITION. The function

$$K_D(z, \bar{t}) = (g_t, g_z), \quad z, t \in D,$$

is called the *Bergman function of D*. The Bergman function is holomorphic in the Cartesian product $D \times D^*$.

We may now rewrite formula (4) in the form

$$f(t) = \int_D f(z)\overline{K_D(z, \bar{t})}\, dm(z), \quad f \in L^2H(D).$$

This is "the reproducing property" of the Bergman function. With the aid of the following theorem the Bergman function can sometimes be computed in a closed form.

0.7. THEOREM [5]. *Suppose that $L^2H(D)$ is not trivial. Let h_m be an arbitrary orthonormal system, linearly dense in the space $L^2H(D)$. Then*

$$K_D(z, \bar{t}) = \sum_{m=0}^{\infty} h_m(z)\overline{h_m(t)}, \quad z, t \in D,$$

and the series converges normally in $D \times D^*$.

When an orthogonal, linearly dense system in $L^2H(D)$ is known, the system h_m can be obtained simply by normalizing all elements of the orthogonal system. Therefore the following theorem permits the computation of the Bergman function in the case of a complete circular domain.

0.8. THEOREM [5]. *Let D be a Reinhardt domain in C^n. The family of all monomials z^m which belong to $L^2H(D)$ is orthogonal and linearly dense in $L^2H(D)$.*

Proof. It is easy to check by computation that different monomials z^m are orthogonal in the space $L^2H(D)$. It is therefore sufficient to show that an element $f \in L^2H(D)$ orthogonal to all monomials in $L^2H(D)$ must be equal to the zero element. We shall base the proof on an idea of S. Zaremba, see [13] and [40]. By Theorem 0.4 the function f can be represented as the sum of the normally convergent Taylor series in D. Supose that, contrary to our statement, $f \neq 0$. Therefore there exists an index m such that in the series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

the coefficient a_m is different from zero. Consider the bounded Reinhardt domain

$$D_r = \{z \in rD, |z| < (1-r)^{-1}\}, \quad 0 < r < 1.$$

By orthogonality

$$\|f\|^2_{D_r} \geqslant |a_k|^2 \|z^k\|^2_{D_r}$$

and

$$(f, z^k)_{D_r} = a_k \|z^k\|^2_{D_r}.$$

When r approaches (1), the first inequality shows that $z^k \in L^2H(D)$. Therefore we can pass to the limit in the last equality, and by our assumption we obtain

$$0 = a_k \|z^k\|^2_D.$$

Since $\|z^k\|^2_D > 0$, this is contradiction and the proof is completed.

0.9. Examples of the Bergman function.

a. The unit disc $U \subset C$

$$K_U(z, \bar{t}) = \frac{1}{\pi(1-z\bar{t})^2}.$$

b. The unit ball $B \subset C^n$

$$K_B(z, \bar{t}) = \frac{n!}{\pi^n(1-z_1\bar{t}_1 - \ldots - z_n\bar{t}_n)^{n+1}}.$$

c. $(p_1, p_2, \ldots, p_n)$ — circular domain, p_j — natural numbers

$$D_p = \{z \in C^n \mid |z_1|^{2/p_1} + |z_2|^{2/p_2} + \ldots + |z_n|^{2/p_n} < 1\},$$

$$K_{D_p}(z, \bar{t}) = \frac{1}{\pi^n p_1 \ldots p_n} \frac{\partial^n}{\partial q_1 \ldots \partial q_n} \sum \frac{1}{1-v_1 - \ldots - v_n},$$

where $q = (z_1\bar{t}_1, z_2\bar{t}_2, \ldots, z_n\bar{t}_n)$ and each v_i assumes the values of all p_i roots of p_i-th degree of q_i. The expression under differentiation is actually a rational function of variables $q_1, \ldots, q_n$, Zinoviev [41].

d. Some particular cases of c

$$K_{D(1,\ldots,p_m,1,\ldots,1)}(z, \bar{t}) = \frac{1}{\pi^n} \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \frac{(1-x_1 - \ldots - \hat{x}_m - \ldots - x_n)^{p_m - 1}}{(1-x_1 - \quad - \hat{x}_m - \ldots - x_n)^{p_m} - x_m},$$

Zinoviev [41].

$$K_{D_{2,2}}(z, \bar{t}) = \frac{1}{\pi^2} \frac{\partial^2}{\partial q_1 \partial q_2} \frac{1-q_1-q_2}{(1-q_1-q_2)^2 - 4q_1q_2},$$

Zinoviev [41].

$$K_{D_{3,3}}(z, \bar{t}) = \frac{1}{\pi^2} \frac{\partial^2}{\partial q_1 \partial q_2} \frac{(1-q_1-q_2)^2 - 9q_1q_2}{(1-q_1-q_2)^3 - 27q_1q_2},$$

Zinoviev [41].

e. Another particular case of c — domain $R_p \subset C^2$, $R_p = D_{p,1}$

$$K_{R_p}(z, \bar{t}) = \frac{1}{\pi^2} \cdot \frac{(1-q_2)^{p-2}[(p+1)(1-q_2)^p + (p-1)q_1]}{[(1-q_2)^p - q_1]^3},$$

Bergman [5].

f. A generalization of e, consider

$$R_p(n, v) = \left\{z \in C^n, \left(\sum_{k=1}^{v-1} |z_k|^2\right)^{1/p} + \sum_{k=v}^{n} |z_k|^2 < 1\right\},$$

$$p > 0,\ 2 \leqslant v \leqslant n.$$

Then

$$K_{R_p(n,v)}(z, \bar t) = A \frac{\partial^{v-1}}{\partial w^{v-1}} \left(\frac{\partial^{n-v+1}}{\partial s^{n-v+1}} \left(\frac{s^{n+(p-1)(v-1)}}{1-s^p}\right)\right),$$

where

$$A = \frac{1}{\pi^n \left(1 - \sum_{k=v}^{n} z_k \bar t_k\right)^{n+1+(p-1)(v-1)}}$$

and $w = s^p$;

$$s = \frac{\left(\sum_{k=1}^{v-1} z_k \bar t_k\right)^{1/p}}{1 - \sum_{k=v}^{n} z_k \bar t_k},$$

Chalmers [9].

g. A particular case of f

$$K_{R_p(n,n)}(z, \bar t) = \frac{(n-1)!(1-z_n \bar t_n)^{p-2}\left\{[p(n-1)+1](1-z_n \bar t_n)^p + (p-1)\sum_{k=1}^{n-1} z_k \bar t_k\right\}}{\pi^n \left[(1-z_n \bar t_n)^p - \sum_{k=1}^{n-1} z_k \bar t_k\right]^{n+1}},$$

Chalmers [9].

h.

$$E(s) = \{z \in C^2 |\ |z_2|^2 < e^{-s|z_1|^2}\},$$

$$K_{E(s)}(z, \bar t) = \frac{s e^s z_1 \bar t_1}{\pi^2} \frac{1 + z_2 \bar t_2 e^{s z_1 \bar t_1}}{(1 - z_2 \bar t_2 e^{s z_1 \bar t_1})^3},$$

Springer [35].

When a biholomorphic mapping g maps D onto $\tilde D$, the Bergman function for D can be computed with the help of the Bergman function for $\tilde D$.

0.10. THEOREM. *Rule of transformation*

$$K_D(z, \bar t) = K_{\tilde D}(\tilde z, \bar{\tilde t}) \frac{\partial \tilde z}{\partial z} \left(\overline{\frac{\partial \tilde t}{\partial t}}\right), \quad \tilde z = g(z),\ \tilde t = g(t).$$

For the proof see Bergman [5].

If the Bergman functions for D_1, D_2 are known, the Bergman function for the product $D = D_1 \times D_2$ can be computed by means of the following

0.11. THEOREM (Bremermann [7]).

$$K_D(z_1, z_2; \bar{t}_1, \bar{t}_2) = K_{D_1}(z_1, \bar{t}_1) K_{D_2}(z_2, \bar{t}_2).$$

0.12. EXAMPLE. For the polydisc $U^n = \{z \in C^n: |z| < 1\}$

$$K_{U^n}(z, \bar{t}) = \frac{1}{\pi^n} \frac{1}{[(1-z_1\bar{t}_1) \ldots (1-z_n\bar{t}_n)]}.$$

0.13. DEFINITION. A domain $D \subset C^n$ is called *homogeneous* if for every two points $p, q \in D$ there exists a biholomorphic mapping $g: D \to D$ (automorphism) such that $q = g(p)$.

If D is homogeneous, then by Theorem 0.10

$$K_D(p, \bar{p}) = K_D(q, \bar{q}) \frac{\partial q}{\partial p} \left(\overline{\frac{\partial q}{\partial p}}\right). \tag{5}$$

0.14. THEOREM (Ramadanov [27]). *For every increasing sequence of domains D_m such that $D = \bigcup_{m=1}^{\infty} D_m$,*

$$\lim_{m\to\infty} K_{D_m} = K_D$$

and the convergence is locally uniform in $D \times D^*$.

The original proof of Ramadanov easily generalizes to $n > 1$ (see Chapter V).

I. Lu Qi-keng domains

Some properties of Lu Qi-keng domains

In [23] Lu Qi-keng indicated that in many concrecte examples of bounded domains, $K_D(z, \bar{t}) \neq 0$ for all $z, t \in D$, and considered the open problem whether the above property is generally true. Therefore we shall introduce

I.1. DEFINITION. A domain $D \subset C^n$ is called a *Lu Qi-keng domain* if $K_D(z, t) \neq 0$ for all $z, t \in D$.

The class of Lu Qi-keng domains has some interesting properties.

I.2. THEOREM. *A biholomorphic image of a Lu Qi-keng domain is a Lu Qi-keng domain.*

Proof. The proof follows from Theorem 0.10.

I.3. THEOREM. *A Cartesian product of two Lu Qi-keng domains is a Lu Qi-keng domain.*

Proof. The proof follows from Theorem 0.11.

I.4. THEOREM. *If* $K_D \neq \text{const}$ *and* D *is the sum of an increasing sequence of Lu Qi-keng domains* D_m, *then* D *is a Lu Qi-keng domain.*

Proof. The proof follows from Theorem 0.14 and the Hurwitz theorem.

All bounded homogeneous Reinhardt domains are Lu Qi-keng domains, see Bell [2]. However, it is not always easy to determine if a given domain is a Lu Qi-keng domain.

I.5. PROBLEM. Is every D_p (see 0.9.c) a Lu Qi-keng domain?

An example of a bounded non-Lu Qi-keng domain

I.6. THEOREM. *Consider an annulus in the complex plane*

$$D = \{r < |z| < 1\}$$

such that $0 < r < e^{-2}$. *Then* D *is not a Lu Qi-keng domain.*

Proof. The Bergman function for D is given by the series

$$K_D(z, \bar{t}) = \frac{-1}{2\pi z\bar{t}\ln r} + \frac{1}{\pi z\bar{t}} \sum_{m=-\infty}^{\infty} \frac{m z^m \bar{t}^m}{1-r^{2m}},$$

where in the sum the index $m = 0$ is omitted, see [5]. It can also be written in the form

$$K_D(z, \bar{t}) = \frac{1}{\pi w}\left[-\frac{1}{\ln\varrho} + \sum_{m=0}^{\infty}\left(\frac{w\varrho^m}{(1-w\varrho^m)^2} + \frac{\frac{\varrho}{w}\varrho^m}{\left(1-\frac{\varrho}{w}\varrho^m\right)^2}\right)\right],$$

where $w = z\bar{t}$, $\varrho = r^2$, $0 < \varrho < |w| < 1$.

Note that in the above formula the expression in brackets is real for real w, and for w on the distinguished line $|w| = r$ since in the latter case $\varrho/w = \bar{w}$. This expression is a continuous function of w. Denoting its value at w by $h(w)$, we see that $h(w) > 0$ for $w > 0$, and $h < 0$ for real w, close to -1. Indeed

$$h(-1) = \frac{-1}{\ln\varrho} - \sum_{m=0}^{\infty} \frac{\varrho^m}{(1+\varrho^m)^2} - \sum_{m=0}^{\infty} \frac{\varrho^{m+1}}{(1+\varrho^{m+1})^2}$$
$$< -\frac{1}{\ln\varrho} - \frac{1}{4} < 0.$$

By continuity h must vanish at some interior point of the w-annulus, and the proof is completed.

Doubly connected Lu Qi-keng domains in the plane

The assumption that $r < e^{-2}$ in Theorem I.6 is not essential. Making use of some properties of elliptic functions, Rosenthal [28] proved the following

I.7. THEOREM [28]. *Every doubly connected Lu Qi-keng domain in C is biholomorphically equivalent to a disc with the deleted center.*

We shall briefly indicate the proof [28]. Since the Bergman function for the complex plane C is identically zero, it is sufficient to show that the Bergman function for the domain

$$D = \{r < |z| < 1\},$$

$0 < r < 1$, assumes zero value at some points of $D \times D^*$. As was shown by Zarankiewicz in [40] (see also [5], p. 10),

$$K_D(z, \bar{t}) = \frac{1}{\pi z\bar{t}}\left[\wp\big(\log(z\bar{t})\big) + \frac{\eta}{\omega} - \frac{1}{2\omega'}\right]. \tag{1}$$

Here $\wp$ denotes the Weierstrass doubly periodic elliptic function of the second order with periods $2\omega = 2\pi i$ and $2\omega' = \log r^2$. Its Laurent expansion in the neighbourhood of $u = 0$ has the form

$$\wp(u) = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + \ldots$$

The corresponding ζ function is characterized by the equation

$$\zeta'(u) = -\wp(u)$$

and the expansion

$$\zeta(u) = \frac{1}{u} - \frac{g_2}{60}u^3 + - \frac{g_3}{140}u^5 + \ldots$$

According to standard notation $\eta = \zeta(\omega)$, $\eta' = \zeta(\omega')$. If we apply the Legendre equation $\dfrac{\eta}{\omega} - \dfrac{\eta'}{\omega'} = \dfrac{\pi i}{2\omega\omega'}$ and note that in our case $\dfrac{\pi i}{2\omega\omega'}$ $\dfrac{1}{2\omega'}$ formula (1) takes the form

$$K_D(z, \bar{t}) = \frac{1}{\pi z\bar{t}}\left\{\wp\big(\log(z\bar{t})\big) + \frac{\eta'}{\omega'}\right\}.$$

It is more convenient, however, to use another pair of primitive periods $2\tilde{\omega} = -\log r^2$ and $2\omega' = 2\pi i$. Since ζ is an odd function $\dfrac{\tilde{\eta}}{\tilde{\omega}} = \dfrac{\eta'}{\omega'}$, finally

$$K_D(z, \bar{t}) = \frac{1}{\pi z \bar{t}} \left\{ \mathscr{P}(\log(z\bar{t})) + \frac{\tilde{\eta}}{\tilde{\omega}} \right\},$$

where $\tilde{\eta} = \zeta(\tilde{\omega})$.

When $z\bar{t} = q$ varies in the q-annulus $r^2 < |q| < 1$, the expression $u = \log(z\bar{t})$ assumes all possible values in the closed period parallelogram with the exception of values u such that $\operatorname{Re} u = 0$ or $\operatorname{Re} u = -\log r^2$. Now recall that the Weierstrass function with periods $\omega = a$, $\omega' = bi$, $a, b > 0$, maps the rectangle $0 < \operatorname{Re} u < a$, $0 < \operatorname{Im} u < b$, onto the lower half plane, and at points symmetric with respect to $\tilde{\omega}$ or $\tilde{\omega}'$ the values of Weierstrass functions are equal ([25], vol. II, p. 355). It follows that all exceptional values of the Bergman function form the set

$$\left\{ \mathscr{P}(u) + \frac{\tilde{\eta}}{\tilde{\omega}},\ u = ti,\ 0 \leqslant t \leqslant \pi \right\}.$$

Since in our case $\dfrac{\tilde{\eta}}{\tilde{\omega}} = \dfrac{\zeta(-\log r)}{-\log r}$ is real, the Bergman function assumes all complex values with the exception of the segment $\left(-\infty, \mathscr{P}(\pi i) + \tilde{\eta}/\tilde{\omega}\right]$ of the real line.

Therefore the theorem will be established if we show that $\mathscr{P}(\pi i) + \tilde{\eta}/\tilde{\omega}$ is negative. This follows easily from the formula

$$\frac{\tilde{\eta}}{\tilde{\omega}} + \mathscr{P}(u) = -\frac{\pi^2}{\tilde{\omega}^2} \left\{ \frac{1}{(s - s^{-1})^2} + \sum_{m=1}^{\infty} \frac{h^{2m} s^{-2}}{(1 - h^{2m} s^{-2})^2} + \sum_{m=1}^{\infty} \frac{h^{2m} s^2}{(1 - h^{2m} s^2)^2} \right\},$$

where $s = e^{\pi i u / 2\tilde{\omega}}$, $h = e^{\pi i \tilde{\omega}' / \tilde{\omega}}$, $\operatorname{Im} \tilde{\omega}'/\omega > 0$.

Indeed, in our case s and h are real, and so the experession in parantheses is positive.

Remark. The Lu Qi-keng conjecture has recently attracted some more attention, see [26] and [36].

II. Representative coordinates

The Bergman metric tensor

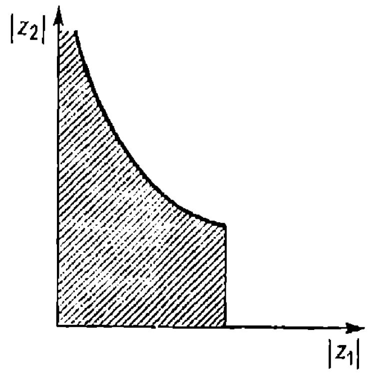
In general it can happen that the evaluation at some point $t \in D$ is trivial, even if $L^2H(D)$ contains non-zero functions.

Consider in particular

II.1. Example. Consider a domain $D \subset C^2$

$$D = \{z \in C\, |z_2| < |z_1|^{-1},\ |z_1| < 1\}$$

with the following Hartogs diagram:



It is a Reinhardt domain. The dimension of $L^2H(D)$ is infinite, and the evaluation at every point of the plane $z_1 = 0$ is a trivial functional on $L^2H(D)$.

Proof. A simple computation shows that

$$\|z_1^{m_1} z_2^{m_2}\|_D^2 = (2\pi)^2 \int_0^1 r_1^{2m_1+1}\left(\int_0^{r_1^{-1}} r_2^{2m_2+1} dr_2\right) dr_1$$

$$= (2\pi)^2 \frac{1}{2m_2+2} \int_0^1 r_1^{2m_1-2m_2-1} dr_1.$$

Therefore a monomial $z_1^{m_1} z_2^{m_2}$ is in $L^2H(D)$ if and only if $m_1 > m_2$. This completes the proof in view of Theorem 0.8.

From now on we shall assume that the evaluation at every point $z \in D$ is not trivial, i.e. $K_D(z, \bar{z})$ is strictly positive in D. If in the real tangent space to D there is defined a Hermitian form H with respect to operator J of the complex structure, i.e. an R-bilinear complex form such that

$$H(JX, Y) = iH(X, Y), \quad H(X, Y) = \overline{H(Y, X)},$$

then the symmetric real tensor $g = \operatorname{Re} H$ is J-invariant and

$$H(X, Y) = g(X, Y) - ig(JX, Y).$$

Also the real skew symmetric tensor $\omega = -\operatorname{Im} H$ is J-invariant. On the other hand, every tensor g or ω with the above properties determines the Hermitian form H, where

$$\omega = g(JX, Y), \quad g = \omega(X, JY).$$

The form H is positive definite if and only if g is positive definite. In this case g is called a *Hermitian structure* and we say that ω is positive.

The form

$$\omega = i\partial\bar{\partial}\log K_D(z, \bar{z}) = i\sum_{j,k=1}^{n} T_{j\bar{k}}dz_j \wedge d\bar{z}_k,$$

where

$$T_{j\bar{k}} = \frac{\partial^2 \log K_D(z, \bar{z})}{\partial z_j \partial \bar{z}_k}$$

is real and J-invariant. The corresponding symmetric tensor field g has the form

$$(1) \qquad g = \tfrac{1}{2}\Big(\sum_{j,k=1}^{n} T_{j\bar{k}}dz_j \otimes d\bar{z}_k + \sum_{j,k=1}^{n} T_{\bar{j}k}d\bar{z}_j \otimes dz_k\Big),$$

where $T_{\bar{j}k} = T_{k\bar{j}}$.

The form ω is positive if and only if the Hessian $(T_{j\bar{k}})$ is positive definite.

We recall the following theorem of Kobayashi [21]

II.2. THEOREM. *The Bergman metric tensor* (1) *is positive definite at a point $p \in D$ if and only if for every holomorphic tangent vector at p*

$$V = \sum_{j=1}^{n} a_j \frac{\partial}{\partial z_j}$$

there exists a function $f \in L^2H(D)$ such that $f(p) = 0$ and $Vf \neq 0$.

The Bergman metric determines actually a Kaehler structure in D. Indeed, by construction, $d\omega = \partial\omega + \bar{\partial}\omega = 0$.

As was proved by Bergman, this structure is invariant under biholomorphic transformations.

Assume now that the Bergman tensor is positive definite in D. In a neighbourhood of any point $t \in D$ the holomorphic functions of variable z

$$(2) \qquad \mu_{\bar{s}}(z) = \frac{\partial}{\partial \bar{t}_s}\log K_D(z, \bar{t}) - \frac{\partial}{\partial \bar{t}_s}\log K_D(t, \bar{t})$$

form a local coordinate system. Indeed, the Jacobian

$$\det\left(\frac{\partial \mu_{\bar{s}}}{\partial z_j}\right) = \det \frac{\partial^2 \log K_D(z, \bar{t})}{\partial z_j \partial \bar{t}_s}$$

is different from zero at the point $z = t$ since $\det(T_{j\bar{s}}) \neq 0$. If D is mapped biholomorphically onto $\tilde{D}$, and z, t are mapped onto $\tilde{z}, \tilde{t}$, respectively, then a simple computation shows that

$$\mu_{\bar{s}}(z) = \tilde{\mu}_{\bar{1}}(\tilde{z})\left(\overline{\frac{\partial \tilde{t}_1}{\partial t_s}}\right) + \ldots + \tilde{\mu}_{\bar{n}}(\tilde{z})\left(\overline{\frac{\partial \tilde{t}_n}{\partial t_s}}\right).$$

Therefore every biholomorphic transformation can be locally described as a linear transformation of coordinates $\mu_{\bar{s}}$. For this reason we shall call $\mu_{\bar{s}}$ *covariant representative coordinates in domain D.*

The idea of representative coordinates is due to Bergman. He considered contravariant representative coordinates

$$\begin{pmatrix} \nu^1 \\ \vdots \\ \nu^n \end{pmatrix} = T^{-1} \begin{pmatrix} \mu_{\bar{1}} \\ \vdots \\ \mu_{\bar{n}} \end{pmatrix},$$

where $T = (T_{i\bar{j}})$. The image of D under the mapping $(z_1, \ldots, z_n) \to (\nu^1(z), \nu^2(z), \ldots, \nu^n(z))$ was called by him a *representative domain*. However, this mapping is generally holomorphic and one-to-one only in a neighbourhood of the point t. The term "representative coordinates" was introduced later by Fuks [12].

A property of representative coordinates

We shall assume the following matrix notation:

$$T(z, \bar{t}) = \big(T_{j,\bar{s}}(z, \bar{t})\big),$$

where

$$T_{j\bar{s}}(z, \bar{t}) = \frac{\partial^2 \log K_D(z, \bar{t})}{\partial z_j \, \partial \bar{t}_s}.$$

Consider the inverse to the biholomorphic mapping $w = \mu(z)$. Define

$$\tilde{K}(w, \overline{w}) = K_D(z, \bar{z}) \frac{\partial z}{\partial w} \left(\overline{\frac{\partial z}{\partial w}}\right)$$

for w sufficiently close to the origin. This function can be regarded as the Bergman function of the covariant representative domain. The corresponding component of the Bergman metric tensor

$$\tilde{T}_{m,\bar{k}}(w, \overline{w}) = \frac{\partial^2 \log \tilde{K}(w, \overline{w})}{\partial w_m \, \partial \overline{w}_k}$$

can also be written as

$$\tilde{T}_{m,\bar{k}}(w, \overline{w}) = \frac{\partial^2 \log K_D(z, \bar{z})}{\partial w_m \, \partial \overline{w}_k}, \qquad z = z(w).$$

By the chain rule

$$\tilde{T}_{m,\bar{k}}(w, \overline{w}) = \sum_{j,s=1}^{n} T_{j,\bar{s}}(z, \bar{z}) \frac{\partial z_j}{\partial w_m} \frac{\partial \bar{z}_s}{\partial \overline{w}_k}.$$

Since

$$\begin{pmatrix} \dfrac{\partial z_1}{\partial w_1} & \dfrac{\partial z_1}{\partial w_n} \\ \\ \dfrac{\partial z_n}{\partial w_1} & \dfrac{\partial z_n}{\partial w_n} \end{pmatrix} = T(z, \bar{t})'^{-1}$$

we have the matrix equation

$$\tilde{T}(w, \bar{w}) = [T(z, \bar{t})^{-1}]T(z, \bar{z})[\overline{T(z, \bar{t})'^{-1}}].$$

Since both sides are holomorphic with respect to w and $\bar{w}$, the formula remains valid also for values $\bar{w}$ independent of w. In particular we may take $\bar{w} = 0$. It follows that $\bar{z} = \bar{t}$ and we have

(3) $$\tilde{T}(w, 0) = [T(z, \bar{t})^{-1}]T(z, \bar{t})[T(t, \bar{t})'^{-1}] = T(t, \bar{t})^{-1}.$$

The above formula proves the following

II.3. Lemma. *The coefficients $\tilde{T}_{m,\bar{k}}(w, \bar{w})$ of the Bergman metric tensor expressed in covariant representative coordinates satisfy the condition*

$$\tilde{T}_{m,\bar{k}}(w, 0) \equiv \text{const}.$$

II.4. Note. The idea of the above proof is due to Lu Qi-keng [23] who gave an analogous result for contravariant representative coordinates

III. An invariant distance

Biholomorphic mappings and canonical isometry

In an abstract Hilbert space H consider the following relation between non-zero elements: $h_1 \sim h_2$ if and only if there exists a complex constant $c \neq 0$ such that $h_1 = ch_2$. The set of all equivalence classes forms (in general infinitely dimensional) projective space $P(H)$. This is a complete metric space with respect to the distance

$$\varrho([h_1], [h_2]) = \text{dist}([h_1]\cap S, [h_2]\cap S),$$

where $S \subset H$ is the unit sphere. Explicitly

$$\begin{aligned} \varrho^2([h_1], [h_2]) &= \inf_{t_1,t_2} \left\| \frac{e^{it_1}h_1}{\|h_1\|} - \frac{e^{it_2}h_2}{\|h_2\|} \right\|^2 \\ &= \inf_{t_1,t_2} \left[2 - 2\,\text{Re}\, \frac{e^{i(t_1-t_2)}(h_1, h_2)}{\|h_1\|\,\|h_2\|} \right] \\ &= 2 - 2\left[\frac{(h_1, h_2)(h_2, h_1)}{(h_1, h_1)(h_2, h_2)} \right]^{1/2} \end{aligned}$$

It is not difficult to check that every linear isometry between two Hilbert spaces

$$L\colon H \to \tilde{H}$$

induces an isometry

$$L\colon P(H) \to P(\tilde{H})$$

given by the formula $L([h]) = [L(h)]$.

Consider now a domain $D \subset C^n$ such that $K_D(z, \bar{z})$ does not vanish at any point $z \in D$. Define $\tau\colon D \to P(L^2H(D))$ by the formula

$$\tau(z) = [g_z].$$

The mapping is injective if and only if for each two different points $p, q \in D$ the functions g_p and g_q are linearly independent. In such a case we say that D admits the invariant distance

$$(1)\qquad \varrho_D(p, q) = 2^{-\frac{1}{2}}\varrho(\tau(p), \tau(q)) = \left(1 - \left(\frac{K_D(p, \bar{q})K_D(q, \bar{p})}{K_D(p, \bar{p})K_D(q, \bar{q})}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

The term invariant distance refers to invariance under biholomorphic transformations. If $\varphi\colon D \to \tilde{D}$ is a biholomorphic mapping, then $L\colon L^2H(\tilde{D}) \to L^2H(D)$ is the canonical isometry defined by the formula

$$Lf = f\circ\varphi \det\varphi'.$$

The rule of transformation of the Bergman function can be written in the form

$$g_p = g_{\tilde{p}}\circ\varphi \det\varphi' \overline{\det\varphi'(p)}, \qquad \tilde{p} = \varphi(p).$$

Therefore $g_p = \mathrm{const}\cdot Lg_{\tilde{p}}$ and $[g_p] = L[g_{\tilde{p}}]$. Finally

$$\begin{aligned}\varrho_D(p, q) &= \varrho_D([g_p], [g_q]) = \varrho_D(L[g_{\tilde{p}}], L[g_{\tilde{q}}]) \\ &= \varrho_{\tilde{D}}([g_{\tilde{p}}], [g_{\tilde{q}}]) = \varrho_{\tilde{D}}(\tilde{p}, \tilde{q}),\end{aligned}$$

where $\tilde{p} = \varphi(p)$, $\tilde{q} = \varphi(q)$. Actually the invariance could be proved also directly from formula (1).

III.1. Example. For the unit disc $U \subset C$

$$\varrho_U(p, q) = \left|\frac{p-q}{1-p\bar{q}}\right|.$$

III.2. Remark. In general the expression (1) is very complicated. Nevertheless, in comparison with the geodesic distance defined by the Bergman metric form, it is still quite explicit and admits numerical computation as soon as the Bergman function is known. Another advantage of the invariant distance lies in the fact that it is uniquely determined by the symmetric quotient

$$H(p,q)=\frac{K_D(p,\bar{q})K_D(q,\bar{p})}{K_D(p,\bar{p})K_D(q,\bar{q})}$$

which is a real analytic function on the whole Cartesian product $D\times D^*$.

III.3. Remark. By the Schwarz inequality the invariant distance is never greater than one. It is always less than one if and only if D is a Lu Qi-keng domain.

Critical points of the invariant distance

Consider now the invariant distance $\varrho_D(p,t)$ as a function of the second variable. We adopt the following

III.4. DEFINITION. A point at which $\varrho_D(p,t)$ has a local extremum or vanishing differential is called a *critical point of the invariant distance.*

We can now state the following

III.5. THEOREM. *In order for the point $t\in D$ to be a critical point of the invariant distance $\varrho(p,t)$ considered as a function of the second variable it is necessary and sufficient that one of the following conditions hold:*

(a) $K(p,\bar{t})=0$,

(b) *In the mapping onto the representative domain (covariant or contravariant) with respect to the point t, the point p corresponds to the origin.*

Proof. The critical points of the invariant distance are exactly stationary points of the real analytic expression

$$Q=\frac{K_D(p,\bar{t})K_D(t,\bar{p})}{K_D(t,\bar{t})}. \tag{2}$$

A direct computation shows that the differential of (2) is equal to

$$Q\sum_{s=1}^{n}(\overline{\mu_{\bar{s}}}\,dt_s+\mu_{\bar{s}}\,d\bar{t}_s)$$

and vanishes either with (a) or when $\mu_{\bar{s}}=0$, $s=1,2,\dots,n$. The latter condition means exactly (b).

Completeness with respect to the invariant distance

A metric space $D\subset C^n$ is complete if every Cauchy sequence $p_m\in D$, $m=1,2,\dots$, converges to some point $p\in D$. The completeness of D with respect to the geodesic distance was studied by Kobayashi in [21]. He proved that a bounded domain $D\subset C^n$ is complete with respect to the geodesic distance if and only if the image $\tau(D)$ in $P\big(L^2\dot{H}(D)\big)$ is complete.

The ideas of [21] can be used also in the study of completeness with respect to the invariant distance.

III.6. THEOREM. *A sequence $p_m \in D$, $m = 1, 2, \ldots$, is Cauchy with respect to the invariant distance if and only if the sequence $\tau(p_m)$ is Cauchy in $P(L^2H(D))$.*

Proof. Immediate from the definition of the invariant distance.

In particular, there is no distinction between completeness with respect to the geodesic distance and completeness with respect to the invariant distance. Therefore some of the following theorems will restate the results known for geodesic distance. For simplicity we restrict our remarks to bounded domains.

III.7. THEOREM. *A sequence $p_m \in D$, $m = 1, 2, \ldots$, is Cauchy if and only if there exists an $f \in L^2H(D)$ such that $\|f\| = 1$ and*

$$\lim_{m\to\infty} \frac{|f(p_m)|^2}{K_D(p_m, \bar{p}_m)} = 1. \tag{3}$$

Proof. By the previous theorem, a sequence p_m is Cauchy in D if and only if $\tau(p_m)$ is Cauchy in $P(L^2H(D))$. Since the metric space $P(L^2H(D))$ is complete, this means that $\tau(p_m)$ converges to some element $[f]$. Since with no loss of generality we may assume that $\|f\| = 1$, the latter condition is equivalent to (3).

III.8. THEOREM. *The Euclidean distance and the invariant distance induce the same topology in $D \Subset C^n$.*

Proof. If $p \in D$ and $\lim_{m\to\infty} |p_m - p| = 0$, then by the continuity of the Bergman function $\lim_{m\to\infty} \varrho(p_m, p) = 0$. Conversely, if $\lim_{m\to\infty} \varrho(p_m, p) = 0$, there exist constants $c_m \neq 0$, $m = 1, 2, \ldots$, such that $c_m g_{p_m}$ converges to g_p in $L^2H(D)$. Since $1 \in L^2H(D)$,

$$\lim_{m\to\infty} c_m = \lim_{m\to\infty} (c_m g_{p_m}, 1) = (g_p, 1) = 1.$$

Since $z_j \in L^2H(D)$, $j = 1, 2, \ldots, n$,

$$\lim_{m\to\infty} z_j(p_m) = \lim_{m\to\infty} \frac{1}{\bar{c}_m} (z_j, c_m g_{p_m}) = (z_j, g_p) = z_j(p)$$

for $j = 1, 2, \ldots, n$, q.e.d.

This theorem is also true for the geodesic distance.

III.9. THEOREM (Kobayashi [13]). *Assume that for every sequence p_m without an accumulation point in D and for every $f \in L^2H(D)$,*

$$\lim_{m\to\infty} \frac{|f(p_m)|^2}{K_D(p_m, \bar{p}_m)} = 0. \tag{4}$$

Then D is complete.

Proof. By contradiction. Suppose that p_m is a Cauchy sequence without limit in D. In view of the theorem p_m has no accumulation point in D. Therefore (4) holds. Now it is enough to note that (4) contradicts (3).

At this point it is worth while to note an interesting

III.10. Conjecture (Kobayashi [13]). If D is complete, then the assumptions of Theorem III.9 are satisfied.

The Kobayashi conjecture can be neatly expressed in terms of the invariant distance.

III.11. Theorem. *The assumptions of Theorem* III.9 *are equivalent to the following: for every point $p \in D$ and every sequence q_m, $m = 1, \ldots,$ without an accumulation point in D,*

$$\lim_{m\to\infty} \varrho(p, q_m) = 1. \tag{5}$$

Proof. Assume that (4) holds and take $f = g_p$. This implies (5). Conversely, note that (5) implies (4) for all f of the form g_p. Denote by $F \subset L^2H(D)$ the subset consisting of all f such that (4) holds. It is easy to see that F is a linear subspace. We show that F is closed. Consider a sequence $f_j \in F$, $j = 1, 2, \ldots,$ which converges to $f \in L^2H(D)$. For every $\varepsilon > 0$ there is a j_0 such that $\|f_{j_0} - f\| < \varepsilon$ and consequently for each m

$$|f_{j_0}(p_m) - f(p_m)|^2 \leqslant \varepsilon^2 K_D(p_m, \overline{p}_m),$$

By assumption the inequality

$$\frac{|f_{j_0}(p_m)|^2}{K_D(p_m, \overline{p}_m)} \leqslant \varepsilon^2$$

holds for all $m > M$. For such m

$$\frac{|f(p_m)|^2}{K_D(p_m, \overline{p}_m)} \leqslant \frac{2\,|f_{j_0}(p_m) - f(p_m)|^2 + 2\,|f_{j_0}(p_m)|^2}{K_D(p_m, \overline{p}_m)} \leqslant 4\varepsilon^2.$$

Therefore $f \in F$. Since F contains the linearly dense set of all functions of the form g_p, $p \in D$, it follows that $F = L^2H(D)$. This completes the proof.

We shall now state two sufficient conditions for the completeness of domain D. In fact, each of these condition implies the Kobayashi condition of Theorem III.9.

III.12. Theorem. *Suppose that for each boundary point p of a bounded domain D there exists a function h holomorphic in D and such that*

(i) $|h(z)| < 1$ *for* $z \in D$,

(ii) $\lim_{z\to p} |h(z)| = 1.$

Then D is complete.

Proof. Let p_m be an arbitrary sequence of points which has no accumulation point in D. We want to show that for every $f \in L^2H(D)$

$$\lim \frac{|f(p_m)|^2}{K_D(p_m, \bar{p}_m)} = 0.$$

With no loss of generality we may assume that p_m converges to $p \in \partial D$. For each $\varepsilon > 0$ there exists a natural number k such that $\|h^k f\|_D^2 < \varepsilon$ by the Lebesgue dominated convergence theorem. For m sufficiently large $(1-\varepsilon) < |h^k(p_m)|^2$, and

$$(1-\varepsilon)|f(p_m)|^2 \leqslant |h^k(p_m)f(p_m)|^2 \leqslant K_D(p_m, \bar{p}_m)\|h^k f\|_D^2.$$

It follows that for sufficiently large m

$$\frac{|f(p_m)|^2}{K_D(p_m, \bar{p}_m)} \leqslant \frac{\varepsilon}{1-\varepsilon},$$

q.e.d.

III.13. COROLLARY (Kobayashi [21]). *Every analytic polyhedron*

$$D = \{z \in G\colon |g_j(z)| < 1,\ j = 1, 2, \ldots, s\},$$

where $g_j \in H(G)$, $D \Subset G$ *is complete.*

Proof. For every $p \in \partial D$ there exists a j such that $|g_j(p)| = 1$. Set $h = g_j$.

III.14. Remark. A set E contained in ∂D is called a *peak set* for the space $A(D)$ (of functions continuous in $\bar{D}$ and holomorphic in D) if there exists a function $h \in A(D)$ such that $h|_E = 1 = \sup_{z \in D} |h(z)|$, and $|h| < 1$ on $\bar{D} \setminus E$. By a result of Davie and Oksendal [10] if D is a strictly pseudoconvex domain with a smooth boundary and $p \in \partial D$, then $\{p\}$ is a peak set. From this and Theorem III.12 it follows that every strictly pseudoconvex domain with a smooth boundary is complete.

A second sufficient condition for completeness has the following form.

III.15. THEOREM. *Suppose that for each boundary point p of a bounded domain D*

(i) $\lim_{z \to p} K_D(z, \bar{z}) = \infty$;

(ii) *The set of all functions bounded in a neighbourhood of p is linearly dense in $L^2H(D)$.*

Then D is complete.

Proof. As before, we shall show that condition (4) of Theorem III.8 is satisfied, and we may assume that p_m converges to $p \in \partial D$. It is evident that the set F of all f satisfying (4) contains the linearly dense set (ii). Since F is a a closed linear subspace, it follows that $F = L^2H(D)$, q.e.d.

III.16. THEOREM. *Every bounded Reinhardt domain of holomorphy D such that $\lambda\bar{D} \subset D$ for $0 < \lambda < 1$ is complete.*

Proof. We shall show that assumptions (i) and (ii) of Theorem III.15 are satisfied. Since D is, in our case, starlike with respect to the origin, the sequence $f_m(z) = f\big((1-1/m)z\big)$ is convergent to f in $L^2H(D)$ and consists of functions bounded in $\bar{D}$. This proves (ii).

In order to prove (i) note that the Hartogs diagram of D

$$D^+ = \{z \in D, z_j > 0\}$$

is mapped onto the convex domain in the transformation

$$(z_1, \ldots, z_n) \to (\log|z_1|, \ldots, \log|z_n|).$$

Condition (i) is certainly satisfied where $n = 1$. Therefore we can apply mathematical induction, and assume that the theorem is true for all domains of lower dimension.

Consider first the case where one component, say p_n, of the boundary point p is equal to zero. The intersection of D with the hyperplane $\{z_n = 0\}$ is a Reinhardt domain $D' \subset C^{n-1}$, and $p' = (p_1, \ldots, p_{n-1})$ is a boundary point in D'. By induction hypotheses $\lim_{m\to\infty} K_{D'}(p'_m, \bar{p}'_m) = \infty$. Let U be a disc in the complex plane with center at the origin such that $D \subset D' \times U$. Then

$$K_D(p_m, \bar{p}_m) \geqslant K_{D'\times U}(p_m, \bar{p}_m) = K_{D'}(p'_m, \bar{p}'_m) K_U\big((p_m)_n, (\bar{p}_m)_n\big)$$

and diverges to infinity with m.

Assume now that $p_j \neq 0$, $j = 1, 2, \ldots, n$. In view of the symmetry of the domain D we may additionally assume that $p_j = r_j$, $r_j > 0$, $j = 1, 2, \ldots, n$, and p is a boundary point in the Hartogs diagram D^+. Since $\log r_1, \ldots, \log r_n$ is a boundary point in the convex domain $\log D^+$, there exist constants A_j, $j = 1, 2, \ldots, n$, such that

$$A_1 \log|z_1| + \ldots + A_n \log|z_n| < A_1 \log r_1 + \ldots + A_n \log r_n$$

for all $z \in D^+$. It is easy to see that all A_j are non-negative. Hence

$$\left|\frac{z_1}{r_1}\right|^{A_1} \cdots \left|\frac{z_n}{r_n}\right|^{A_n} < 1$$

for all $z \in D^+$, and also for all $z \in D$.

For any natural k define in D a continuous function

$$s_k(z) = \left(\left|\frac{z_1}{r_1}\right|^{A_1} \cdots \left|\frac{z_n}{r_n}\right|^{A_n}\right)^k$$

Then $s_k(z) < 1$, $z \in D$.

Let h_k be a function holomorphic in D defined by the formula

$$h_k(z) = \left(\frac{z_1}{r_1}\right)^{E(kA_1)+1} \cdots \left(\frac{z_n}{r_n}\right)^{E(kA_n)+1}$$

Now for all $z \in D$

$$s_k(z) - |h_k(z)| = s_k(z)\left[1 - \left|\frac{z_1}{r_1}\right|^{a_1} \cdots \left|\frac{z_n}{r_n}\right|^{a_n}\right],$$

where $0 < a_j \leqslant 1$, $j = 1, 2, \ldots, n$. Since D is bounded, we obtain

$$\big|s_k(z) - |h_k(z)|\big| \leqslant \text{const}\cdot s_k(z)$$

or

$$|h_k(z)| \leqslant \big||h_k(z)| - s_k(z)\big| + |s_k(z)| \leqslant \text{const}\cdot s_k(z).$$

By the Lebesgue dominated convergence theorem $\lim\limits_{k\to\infty} \|h_k\|_D = 0$. Also $\lim\limits_{z\to p} h_k(z) = 1$ for every fixed k. Therefore for every $\varepsilon > 0$ we can find h_{k_0} with $\|h_{k_0}\| < \varepsilon$, and for m large enough $|h_{k_0}(p_m)| > 1 - \varepsilon$. Hence

$$K_D(p_m, \overline{p}_m) \geqslant \frac{|h_{k_0}(p_m)|^2}{\|h_{k_0}\|^2} \geqslant \frac{(1-\varepsilon)^2}{\varepsilon^2}.$$

Therefore (i) holds and the proof is completed.

IV. Extension theorem

Semiconformal mappings

IV.1. Definition. Consider two domains $D, \tilde{D} \subset C^n$ with a positive definite Bergman metric tensor and the invariant distance function. Let $U \subset D$ be an open connected subset. A mapping

$$h\colon\ U \to \tilde{D}$$

is called a *local semiconformal mapping* with respect to D and $\tilde{D}$ if for all $z, t \in U$

$$\varrho_{\tilde{D}}\big(h(z), h(t)\big) = \varrho_D(z, t). \tag{1}$$

In other words, a local semiconformal mapping h is an isometry of U and $\tilde{U} = h(U)$ with respect to the invariant distance.

IV.2. Remark. Every local semiconformal mapping is a homeomorphism of U and $\tilde{U}$. This follows immediately from Theorem III.8. In particular the inverse of a local semiconformal mapping with respect to U, D, $\tilde{D}$ is a local semiconformal mapping with respect to $\tilde{U}$, $\tilde{D}$, D.

IV.3. Theorem. *Every local semiconformal mapping is real analytic.*

Proof. Let h be such a mapping. We shall prove that h is real analytic in a neighbourhood of an arbitrary point $p \in U$. Denote $\tilde{p} = h(p)$ and $\tilde{z} = h(z)$ for $z \in U$. By assumption

$$\tilde{H}(\tilde{z}, \tilde{p}) - H(z, p) = 0,$$

where H and $\tilde{H}$ denote symmetric quotients in D and $\tilde{D}$, respectively. We look for $2n$ points $\tilde{z}^1, \tilde{z}^2, \ldots, \tilde{z}^{2n}$ such that in a neighbourhood of p the system of $2n$ equations

$$\tilde{H}(\tilde{z}^k, \tilde{t}) - H(z^k, t) = 0, \qquad k = 1, 2, \ldots, 2n,$$

defines a unique real analytic function $\tilde{t} = h(t)$ such that $\tilde{p} = h(p)$.

In view of the implicit mapping theorem it is enough to find $\tilde{z}^k$, $k = 1, \ldots, 2n$, in such a way that

$$\text{(2)} \qquad \det \begin{pmatrix} \dfrac{\partial \tilde{H}(\tilde{z}^1, \tilde{p})}{\partial \tilde{p}_1} & & \dfrac{\partial \tilde{H}(\tilde{z}^1, \tilde{p})}{\partial \bar{\tilde{p}}_n} \\ & & \\ \dfrac{\partial H(\tilde{z}^{2n}, \tilde{p})}{\partial \tilde{p}_1} & \cdots & \dfrac{\partial H(\tilde{z}^{2n}, \tilde{p})}{\partial \bar{\tilde{p}}_n} \end{pmatrix} \neq 0.$$

A simple computation shows that

$$\text{(3)} \qquad \frac{\partial \tilde{H}(\tilde{z}, \tilde{p})}{\partial \bar{\tilde{p}}_s} = \tilde{H}(\tilde{z}, \tilde{p}) \tilde{\mu}_{\bar{s}}(\tilde{z}), \qquad \frac{\partial \tilde{H}(\tilde{z}, \tilde{p})}{\partial \tilde{p}_s} = \tilde{H}(\tilde{z}, \tilde{p}) \overline{\tilde{\mu}_{\bar{s}}, (\tilde{z})},$$

where $\tilde{\mu}_{\bar{s}}(\tilde{z})$ are representative coordinates with respect to $\tilde{p}$. Since representative coordinates form a local coordinate system in a neighbourhood of $\tilde{p}$, we can find in this neighbourhood points $\tilde{z}^k$, $k = 1, 2, \ldots, 2n$, such that $\tilde{\mu}_{\bar{s}}(\tilde{z}^j) = \varepsilon \delta^j_s$ and $\tilde{\mu}_{\bar{s}}(\tilde{z}^{n+j}) = i\varepsilon \delta^j_s$ for $j = 1, 2, \ldots, n$. For sufficiently small ε points $\tilde{z}^k$ belong to $\tilde{U}$, and there are points $z^k \in U$ such that $\tilde{z}^k = h(z^k)$. Also $\tilde{H}(\tilde{z}^k, \tilde{p})$ is close to 1 and therefore does not vanish. It follows that for such small ε determinat (2) differs only by a non-zero factor from

$$\det \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \neq 0, \qquad \text{q.e.d.}$$

The next theorem shows that representative coordinates are very useful for the study of local semiconformal mappings.

IV.4. Theorem. *Suppose that a local semiconformal mapping h takes a point $p \in U$ onto point $\tilde{p} \in \tilde{U}$. Then if both $z \in U$ and $\tilde{z} = h(z) \in \tilde{U}$ belong to a representative coordinate neighbourhood of points p and $\tilde{p}$, respectively,* *the correspondence $z \to \tilde{z}$ can be described in terms of representative coordinates as an R-linear non-singular transformation*

$$\begin{pmatrix} \mu_1(z) \\ \vdots \\ \mu_n(z) \\ \mu_{\bar{1}}(z) \\ \vdots \\ \mu_{\bar{n}}(z) \end{pmatrix} = X' \begin{pmatrix} \tilde{\mu}_1(\tilde{z}) \\ \vdots \\ \tilde{\mu}_n(\tilde{z}) \\ \tilde{\mu}_{\bar{1}}(\tilde{z}) \\ \vdots \\ \tilde{\mu}_{\bar{n}}(\tilde{z}) \end{pmatrix}, \tag{4}$$

where X is the Jacobi matrix of h at p and

$$\mu_s(z) = \overline{\mu_{\bar{s}}(z)}, \quad \tilde{\mu}_s(z) = \overline{\tilde{\mu}_{\bar{s}}(\tilde{z})}.$$

Proof. Since z, and $\tilde{z}$ are in a coordinate neighbourhood, we must have $K_D(z, \bar{p}) \neq 0$ and $K_{\tilde{D}}(\tilde{z}, \bar{\tilde{p}}) \neq 0$ (otherwise the representative coordinate has a pole at z, or $\tilde{z}$). We may differentiate the identity

$$H(z, t) = \tilde{H}(\tilde{z}, \tilde{t})$$

with respect to $\bar{t}_s$ at the point $t = p$. It follows that

$$H(z, p)\mu_{\bar{s}}(z) = \tilde{H}(\tilde{z}, \tilde{p}) \sum_{j=1}^{n} \left(\tilde{\mu}_{\bar{j}}(\tilde{z}) \frac{\partial \bar{\tilde{t}}_j}{\partial \bar{t}_s|_{t=p}} + \overline{\tilde{\mu}_{\bar{j}}(\tilde{z})} \frac{\partial \tilde{t}_j}{\partial \bar{t}_s|_{t=p}} \right). \tag{5}$$

Now we divide both sides by $H(z, p) \neq 0$. As a result we obtain an equation which is equivalent to (4), q.e.d.

IV.5. THEOREM. *Let X be the Jacobi matrix at p of a local semiconformal mapping h. Denote*

$$Q = \begin{pmatrix} 0 & T \\ T' & 0 \end{pmatrix},$$

where $T = [T_{i,\bar{j}}(p, \bar{p})]$. An analogous matrix for $\tilde{D}$ and $\tilde{p} = h(p)$ will be denoted by $\tilde{Q}$.

Then the following matrix equation holds:

$$Q = X'\tilde{Q}X. \tag{6}$$

Proof. At the point p and $\tilde{p}$

$$\frac{D(\mu_1, \ldots, \mu_n, \mu_{\bar{1}}, \ldots, \mu_{\bar{n}})}{D(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)}' = Q', \quad \frac{D(\tilde{\mu}_1, \ldots, \tilde{\mu}_n, \tilde{\mu}_{\bar{1}}, \ldots, \tilde{\mu}_{\bar{n}})}{D(\tilde{z}_1, \ldots, \tilde{z}_n, \bar{\tilde{z}}_1, \ldots, \bar{\tilde{z}}_n)} = \tilde{Q}'.$$

Also in view of (4)

$$\frac{D(\mu_1, \ldots, \mu_n, \tilde{\mu}_{\bar{1}}, \ldots, \tilde{\mu}_{\bar{n}})}{D(\tilde{\mu}_1, \ldots, \tilde{\mu}_n, \tilde{\mu}_{\bar{1}}, \ldots, \tilde{\mu}_{\bar{n}})} = X'$$

Therefore

$$Q' = X'\tilde{Q}'X$$

and the theorem follows by transposition on both sides of this equation.

IV.6. THEOREM. *Let*

$$\lfloor X = \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}$$

be the Jacobi matrix at p of a local semiconformal, holomorphic mapping h. Then the matrix equation (6) *is equivalent to*

$$T = A'\tilde{T}\bar{A}. \tag{7}$$

Proof. Direct computation.

IV.7. DEFINITION. A local semiconformal mapping $h\colon\ U \to \tilde{D}$, $U \subset D$, such that $U = D$, $\tilde{D} = h(D)$ is called a *semiconformal mapping.*

IV.8. Remark. It is clear that a mapping $h\colon\ D \to \tilde{D}$ is biholomorphic if and only if it is holomorphic and semiconformal.

IV.9. EXAMPLE. If $n = 1$, every local semiconformal mapping h is either holomorphic or antiholomorphic. Indeed, the matrix equation (6) yields in this case

$$\begin{pmatrix} 0 & T_{1,\bar{1}} \\ T_{\bar{1},1} & 0 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial h}{\partial p} & \dfrac{\partial \bar{h}}{\partial p} \\ \dfrac{\partial h}{\partial \bar{p}} & \dfrac{\partial \bar{h}}{\partial \bar{p}} \end{pmatrix} \begin{pmatrix} 0 & \tilde{T}_{1,\bar{1}} \\ \tilde{T}_{\bar{1},1} & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial h}{\partial p} & \dfrac{\partial h}{\partial \bar{p}} \\ \dfrac{\partial \bar{h}}{\partial p} & \dfrac{\partial h}{\partial \bar{p}} \end{pmatrix}.$$

In particular

$$\frac{\partial \bar{h}}{\partial p}\frac{\partial h}{\partial p}\tilde{T}_{1,\bar{1}} \equiv 0.$$

Since always $\tilde{T}_{1,\bar{1}} > 0$ and the ring of real analytic functions in U has no zero divisors, it follows that $\partial\bar{h}/\partial p \sim 0$ or $\partial h/\partial p \equiv 0$, q.e.d.

IV.10. EXAMPLE. The mapping of the unit polycylinder onto itself

$$(z_1, z_2) \to (z_1, \bar{z}_2)$$

is clearly semiconformal and is neither holomorphic nor antiholomorphic.

We shall now investigate the possibility of an extension of the local semiconformal mapping.

IV.11. THEOREM. *A real analytic continuation in D of a local semiconformal mapping with values in $\tilde{D}$ is independent of the path of continuation.*

Proof. Assume that the variable z varies along the first path and the variable t varies along the second path of the continuation. We shall prove that

$$(8) \qquad \varrho_{\tilde{D}}\big(h(z), h(t)\big) = \varrho_D(z, t).$$

This is clearly sufficient, since for $z = t$ the right-hand side becomes zero; hence $h(z) = h(t)$. In order to prove (8) note that it is equivalent to

$$(9) \qquad H\big(h(z), h(t)\big) - H(z, t) = 0.$$

The left-hand side of (9) is a real analytic continuation of the function identically zero when $z, t \in U$. In view of the law of permanence for real analytic functions the continuation in question is identically zero, and the proof is completed.

IV.12. COROLLARY. *With the notation as in the previous theorem consider the set $D_0 \subset D$ such that $z \in D_0$ if and only if there exists a real analytic continuation of h along a path in D which ends at z, with values in $\tilde{D}$. Then h extends to a well-defined mapping $h: D_0 \to \tilde{D}$ and the extension is a local semiconformal mapping.*

Proof. We need only to show that the extension is a local semiconformal mapping. This follows from the fact that the left-hand side of (9) vanishes for $z, t \in D_0$.

Extension theorem

Under the additional assumption that $\tilde{D}$ is complete with respect to the invariant distance it can be proved that $D_0 = D$.

IV.13. EXTENSION THEOREM. *Assume that $U \subset D$, $\tilde{D}$ is complete with respect to the invariant distance, and h: $U \to \tilde{D}$ is a local semiconformal mapping with respect to D, $\tilde{D}$. Then h possesses the unique extension to a local semiconformal mapping*

$$h: D \to \tilde{D}.$$

Proof. We shall prove that $D_0 = D$ and the theorem will follow from Corollary IV.12. Assume to the contrary that $p \in D$ is a boundary point of D_0, and let $p_m \in D_0$, $m = 1, 2, \ldots$, be a sequence convergent to p in Euclidian topology. It follows that p_m and hence $h(p_m)$ is a Cauchy sequence with respect to the invariant distance. Since by assumption $\tilde{D}$ is complete, the sequence $\tilde{p}_m = h(p_m)$ converges to some point $\tilde{p} \in D$.

Consider a neighbourhood V of the point p, relatively compact in a representative coordinate neighbourhood of this point, and denote by $\tilde{V}$ a similar neighbourhood of the point $\tilde{p}$. For large m, $K_D(z, \bar{p}_m)$ converges to $K_D(z, p) \neq 0$ uniformly for $z \in V$, and therefore $K_D(z, p_m) \neq 0$ by the Hurwitz theorem. It follows that the representative coordinate mapping with respect to p_m converges uniformly on V to the representative coordinate mapping with respect to p. Analogously the representative coordinate mapping with respect to $\tilde{p}_m$ converges uniformly on $\tilde{V}$ to the representative coordinate mapping with respect to $\tilde{p}$. It follows that for some j the representative coordinate mapping with respect to p_j, say μ, maps biholomorphically V onto an open neighbourhood W of the origin, and the representative coordinate mapping with respect to $\tilde{p}_j$, say $\tilde{\mu}$, maps biholomorphically $\tilde{V}$ onto an open neighbourhood $\tilde{W}$ of the origin. Therefore we can consider V and $\tilde{V}$ as the representative coordinate neighbourhoods of points p_j and $\tilde{p}_j$, respectively.

Consider $z \in D_0$. By Theorem IV.4 there exists an R-linear mapping $L: C^n \to C^n$ determined uniquely by the Jacobi matrix of h at p_j and such that if $z \in V$ and $\tilde{z} = h(z) \in \tilde{V}$, then the mapping h is given as the composition

$$z \to \mu(z) \xrightarrow{L} \tilde{\mu}(\tilde{z}) \to \tilde{z}. \tag{10}$$

In particular for every m such that $p_m \in V$ and $\tilde{p}_m \in \tilde{V}$ we find

$$\mu(p_m) \xrightarrow{L} \tilde{\mu}(\tilde{p}_m).$$

When m goes to infinity, the continuity of L yields

$$\mu(p) \xrightarrow{L} \tilde{\mu}(\tilde{p}).$$

It follows that the real analytic mapping (10) is well defined in a neighbourhood of the point p. Consider in this neighbourhood a point p_m such that $p_m \in V$ and $\tilde{p}_m \in \tilde{V}$. For z close to p_m, $z \in V$, $\tilde{z} \in \tilde{V}$, and the mapping (10) coincides with h. Thus h possesses a real analytic extension to $p \in D_0$ with values in $\tilde{D}$. This contradicts the definition of D_0 and the proof is completed.

Now we can easily prove

IV.14. THEOREM. *Assume that D, $\tilde{D}$ are complete with respect to the invariant distance, $U \subset D$, and*

$$h: U \to \tilde{D}$$

is a local semiconformal mapping. Then h possesses a unique extension to semiconformal mapping of D onto $\tilde{D}$

$$h: D \to \tilde{D}.$$

Proof. In addition to h consider a local semiconformal mapping with respect to $\tilde{D}, D$

$$h^{-1}: \tilde{U} \to D,$$

where $\tilde{U} = h(U)$. By the extension theorem we can extend h and h^{-1} to local semiconformal mappings

$$h: D \to \tilde{D} \quad \text{and} \quad h^{-1}: \tilde{D} \to D.$$

By the law of permanence for real analytic functions both $h\circ h^{-1}$ and $h^{-1}\circ h$ reduce to identity. We conclude that h maps semiconformally D onto $\tilde{D}$ and h^{-1} is the inverse semiconformal mapping. The proof is completed.

Local characterization of a biholomorphic mapping

Consider two domains $D, \tilde{D} \subset C^n$. Let U be an open connected subset of D. Consider a holomorphic mapping

$$h\colon U \to \tilde{D}. \tag{11}$$

IV.15. Definition. We shall say that h respect the *rule of transformation of the Bergman function* if

$$K_D(z, \bar{t}) = K_{\tilde{D}}\big(h(z), \overline{h(t)}\big)\det h'(z)\overline{\det h'(t)}, \tag{12}$$

where h' denotes the holomorphic Jacobi matrix of h.

IV.16. Remark. Assume that h takes values in an open connected subset $\tilde{U}$ of $\tilde{D}$. Consider an arbitrary local coordinate system a in U, and an arbitrary local coordinate system $\tilde{a}$ in $\tilde{U}$. In other words,

$$a\colon U \to a(U), \quad \tilde{a}\colon U \to \tilde{a}(\tilde{U})$$

are biholomorphic mappings. There exists a unique function K^a_D holomorphic in $a(U)\times a(U)^*$, and a unique function $K^{\tilde{a}}_{\tilde{D}}$ holomorphic in $\tilde{a}(\tilde{U})\times \times\tilde{a}(\tilde{U})^*$ such that

$$K_D(z, \bar{t}) = K^a_{\tilde{D}}\big(a(z), \overline{a(t)}\big)\det a'(z)\overline{\det a'(t)}$$

for $z, t \in U$, and

$$K_{\tilde{D}}(\tilde{z}, \bar{\tilde{t}}) = K^{\tilde{a}}_{\tilde{D}}\big(\tilde{a}(\tilde{z}), \overline{\tilde{a}(\tilde{t})}\big)\det \tilde{a}'(\tilde{z})\overline{\det \tilde{a}'(\tilde{t})}$$

for $\tilde{z}, \tilde{t} \in \tilde{U}$.

The mapping h induces the holomorphic mapping $h_{a,\tilde{a}} = \tilde{a}ha^{-1}$

$$h_{a,\tilde{a}}\colon a(U) \to \tilde{a}(\tilde{U})$$

and vice versa. We shall say that $h_{a,\tilde{a}}$ respects the rule of transformation of the Bergman function if

$$K^a_D(z, \bar{t}) = K^{\tilde{a}}_{\tilde{D}}\big(h_{a,\tilde{a}}(z), \overline{h_{a,\tilde{a}}(t)}\big)\det h'_{a,\tilde{a}}(z)\overline{\det h'_{a,\tilde{a}}(t)}. \tag{13}$$

It is not difficult to show that h respects the rule of transformation of the Bergman function if and only if $h_{a,\tilde{a}}$ does.

IV.17. Theorem. *Assume that D and $\tilde{D}$ are complete with respect to the invariant distance, U is an open connected subset of D and*

$$h\colon U \to D$$

is holomorphic. Then h extends to biholomorphic mapping of D onto $\tilde{D}$

$$h\colon D \to \tilde{D}$$

if and only if it respects the rule of transformation of the Bergman function.

Proof. The necessity is obvious. For the sufficiency note that (12) implies that h is local semiconformal mapping. Therefore by Theorem IV.14 h extends to a semiconformal mapping of D onto $\tilde{D}$. Since the real analytic continuation of a holomorphic mapping must be holomorphic, we conclude that h is holomorphic with a differentiable inverse, and hence biholomorphic, q.e.d.

IV.18. Corollary. *Assume that D and $\tilde{D}$ are complete with respect to the invariant distance, and $p \in D$. The domains D and $\tilde{D}$ are biholomorphically equivalent if and only if there exists a holomorphic mapping h in a neighbourhood of p, with values in $\tilde{D}$, such that*

$$K_D(z, \bar{z}) = K_{\tilde{D}}\big(h(z), \overline{h(z)}\big)\,|\det h'(z)|^2. \tag{14}$$

Proof. It is well known that (14) implies (12) for all z and t which take independent values in some small neighbourhood U of p. Of course we may assume that U is connected. Therefore $h\colon U \to \tilde{D}$ respects the rule of transformation of the Bergman function, and by Theorem IV.17 extends to a biholomorphic mapping of D onto $\tilde{D}$. The sufficiency is therefore proved. The necessity is trivial.

In some special local coordinate systems Corollary IV.18 takes an even more tangible form. We shall first prove

IV.19. Lemma. *Consider a domain $D \subset C^n$ with a positive definite Bergman metric tensor. Let p be an arbitrary point in D, and μ the representative coordinate system at p. Then there exists a C-linear transformation with matrix A, such that in the local coordinate system*

$$a = A\mu$$

the coefficients of the Bergman metric tensor at the origin form the unit matrix I.

Proof. Let T be the hermitian matrix associated with the Bergman metric tensor at the point p. It follows from (3), Section II that for the same tensor expressed in representative coordinates the cooresponding matrix is equal to T^{-1}. Actually we are looking for a matrix A such that

$$T^{-1} = A' I \bar{A}.$$

Since the Bergman tensor is positive definite, T^{-1} is a positive definite hermitian matrix and can always be written in the form

$$T^{-1} = A'\bar{A}.$$

This ends the proof of the lemma.

IV.20. Remark. Note that

$$K^{a}_{D}\left(a(z), \overline{a(z)}\right) = \frac{K_D(z, \bar{z})}{|\det T(z, \bar{p})|^2} \det T(p, \bar{p}). \tag{15}$$

We can now state

IV.21. THEOREM. *Consider two domains $D, \tilde{D} \subset C^n$. Assume that both domains are complete with respect to the invariant distance, and the Bergman tensor is positive definite in D and in $\tilde{D}$. Let $p \in D$ and $\tilde{p} \in D$. Consider a local coordinate system a in a neighbourhood of p constructed in Lemma IV.19, and an analogous local coordinate system $\tilde{a}$ in a neighbourhood of the point $\tilde{p}$. Then the domains D and $\tilde{D}$ are biholomorphically equivalent with a point p going into $\tilde{p}$ if and only if there exists a unitary matrix U such that*

$$K^{a}_{D}(a, \bar{a}) = K^{\tilde{a}}_{\tilde{D}}(Ua, \overline{Ua}). \tag{16}$$

for all sufficiently small a.

Proof. *Necessity.* Let h be a biholomorphic mapping of D onto $\tilde{D}$ such that $\tilde{p} = h(p)$. Let $U = h_{a,\tilde{a}}$. Since $h_{\mu,\tilde{\mu}}$ is linear, it follows that U is linear. Now in both local coordinates the coefficients of the Bergman tensor form the unit matrix. In view of the invariance of the Bergman tensor with respect to biholomorphic transformation, we must have

$$I = U'I\bar{U}.$$

Hence U is unitary. Finally (16) follows from the fact that U respects the rule of transformation of the Bergman function.

Sufficiency. From identity (16) it follows, as in Corollary IV.18, that U respects the rule of transformation of the Bergman function. Consider a biholomorphic mapping h of a neighbourhood of p onto neighbourhood of $\tilde{p}$ such that $h_{a,\tilde{a}} = U$. In view of Remark IV.16 h respects the rule of transformation of the Bergman function, and $h(p) = \tilde{p}$. By Theorem IV.17 h extends to a biholomorphic mapping of D onto $\tilde{D}$.

The proof is completed.

IV.22. Remark. Substituting $a = 0$ in formula (16) yields in view of (15)

$$J_D(p) = J_{\tilde{D}}(\tilde{p}),$$

where J_D is the invariant introduced by Bergman, see [5]

$$J_D(p) = \frac{K_D(p, \bar{p})}{\det T(p, \bar{p})}.$$

IV.23. Remark. Most of the results of this chapter can be generalized to complex manifolds, see [33].

V. Domain dependence

Ramadanov theorem

In 1967 I. Ramadanov proved the following theorem:

V.1. THEOREM [27]. *Let the bounded domain $D \subset C$ be the sum of an increasing sequence of domains D_m, $m = 1, 2, \ldots$ Then the sequence of Bergman functions $K_{D_m}(z, \bar{t})$ converges to $K_D(z, \bar{t})$ locally uniformly in $D \times D^*$.*

We shall give a proof of this theorem which is a slight modification of the original proof [27]. Also we shall restate the theorem in a more general form. Our proof will use

V.2. LEMMA. *Denote by S the subset $S \subset L^2H(D)$ consisting of all functions f such that $f(t) \geqslant 0$ and $\|f\| \leqslant f(t)^{1/2}$, t a fixed point in D. Then the Bergman function*

$$\varphi(z) = K_D(z, \bar{t}) \tag{1}$$

is uniquely characterized by the properties

(i) $\varphi \in S$;

(ii) *if $f \in S$ and $f(t) \geqslant \varphi(t)$, then $f = \varphi$.*

Proof. First of all note that there exists at most one element $\varphi \in L^2H(D)$ which satisfies (i) and (ii). It is therefore suficient to show that (1) has both properties. Property (i) is evident. We shall show (ii). Consider first the case where $f(t) = 0$. Then $f \equiv 0$ and $\varphi \equiv 0$. In the other case consider the functions

$$\frac{f}{f(t)} \quad \text{and} \quad \frac{\varphi}{\varphi(t)}. \tag{2}$$

It is well known that the second function is uniquely characterized as an element in the set

$$\{h \in L^2H(D)\colon\ h(t) = 1\}$$

with the minimal norm. On the other hand, the first function is also in this set, and has an even smaller norm since

$$\left\|\frac{f}{f(t)}\right\| = \frac{1}{f(t)^{1/2}}\,\frac{\|f\|}{f(t)^{1/2}} \leqslant \frac{1}{f(t)^{1/2}} \leqslant \frac{1}{\varphi(t)^{1/2}} = \left\|\frac{\varphi}{\varphi(t)}\right\|. \tag{3}$$

It follows that both functions in (2) are equal. Equality in (3) yields $f(t) = \varphi(t)$ and (2) again yields $f = \varphi$. The proof of the lemma is completed.

V.3. THEOREM (Ramadanov). *Let D be a domain in C^n. Assume that D is the sum of an increasing sequence of domains D_m, $m = 1, 2, \ldots$ Then the sequence of Bergman functions $K_{D_m}(z, \bar{t})$ converges to $K_D(z, \bar{t})$ locally uniformly in $D \times D^*$.*

Proof. Let F be an arbitrary compact subset of D. We note that the family of functions $K_m(z, \bar{t}) = K_{D_m}(z, \bar{t})$ is bounded on $F \times F^*$. Indeed, let Q be any domain such that $F \subset Q \subset D$. For sufficiently large m, $Q \subset D_m$ and we have

$$(4) \qquad |K_m(z, \bar{t})| \leqslant K_m(z, \bar{z})^{1/2} K_m(t, \bar{t})^{1/2} \leqslant K_Q(z, \bar{z})^{1/2} K_Q(t, \bar{t})^{1/2} \leqslant \max_{z \in F} K_Q(z, \bar{z}).$$

By the Montels theorem the family $K_m(z, \bar{t})$, $m = 1, 2, \ldots$, is normal in $D \times D^*$. Therefore a certain subsequence $K_{m_j}(z, \bar{t})$ converges locally uniformly to some holomorphic function $k(z, \bar{t})$ in $D \times D^*$. We shall show that $f(z) = k(z, \bar{t}) \in S$. Inequality $k(t, \bar{t}) \geqslant 0$ is obvious. Consider an arbitrary $Q \Subset D$.

For large j, $Q \Subset D_{m_j}$ and

$$\|k(\cdot, \bar{t})\|_Q = \lim_{j \to \infty} \|K_{m_j}(\cdot, \bar{t})\|_Q \leqslant \liminf_{j \to \infty} \|K_{m_j}(\cdot, \bar{t})\|_{D_{m_j}} = \liminf_{j \to \infty} K_{m_j}(t, \bar{t})^{1/2} = k(t, \bar{t})^{1/2}.$$

Since Q is arbitrary it follows that

$$\|k(\cdot, \bar{t})\|_D \leqslant k(t, \bar{t})^{1/2}.$$

Therefore $f \in S$. Now the inequality $K_D(t, \bar{t}) \leqslant K_{m_j}(t, \bar{t})$ implies that $K_D(t, \bar{t}) \leqslant f(t)$. From Lemma V.2 we conclude that

$$K_D(\cdot, \bar{t}) = k(\cdot, \bar{t}).$$

We conclude that an arbitrary subsequence of the sequence $K_m(z, \bar{t})$ contains a subsequence which converges to $K_D(z, \bar{t})$. Therefore the sequence $K_m(z, \bar{t})$ itself converges to $K_D(z, \bar{t})$ locally uniformly in $D \times D^*$, and the proof is complete.

An analogue of Ramadanov theorem for decreasing sequences

The Ramadanov theorem suggests the study of the sequence K_{D_m} for a decreasing sequence of domains.

V.4. THEOREM. *Let D_m be a decreasing sequence of domains such that a domain D is contained in every D_m. The sequence K_{D_m} converges to K_D locally uniformly in $D \times D^*$ if and only if for every $t \in D$*

$$(5) \qquad \lim_{m \to \infty} K_{D_m}(t, \bar{t}) = K_D(t, \bar{t}).$$

Proof. The necessity is obvious. We shall prove the sufficiency of the condition. For an arbitrary compact $F \subset D$ there exists a constant M such that $\max_{z \in F} |K_D(z, \bar{z})| \leqslant M$. Hence for $z, t \in F$

$$|K_{D_m}(z, \bar{t})| \leqslant K_{D_m}(z, \bar{z})^{1/2} K_{D_m}(t, \bar{t})^{1/2} \leqslant K_D(z, \bar{z})^{1/2} K_D(t, \bar{t})^{1/2} \leqslant M.$$

It follows that K_{D_m} is a Montel family on $D \times D^*$. It is now sufficient the show that every convergent subsequence of this family converges to K_D. With no loss of generality we consider the sequence K_{D_m} itself, assuming that it is convergent to a function k, and we want prove that $k = K_D$. For every compact $F \subset D$ and $t \in D$ we have

$$\int_F |k(\cdot, \bar{t})|^2 = \lim_{m\to\infty} \int_F |K_{D_m}(\cdot, \bar{t})|^2 \leqslant \liminf_{m\to\infty} \int_{D_m} |K_{D_m}(\cdot, \bar{t})|^2$$
$$= \lim_{m\to\infty} K_{D_m}(t, \bar{t}) = k(t, \bar{t}).$$

Since F is arbitrary, $\|k(\cdot, \bar{t})\|_D^2 \leqslant k(t, \bar{t}) = K_D(t, \bar{t})$. By Lemma V.2 $k(\cdot, \bar{t}) = K_D(\cdot, \bar{t})$ for every $t \in D$, and the proof is complete.

By $L_a^2(E)$ we shall denote the Hilbert space of all complex functions which are square integrable on a set E, and holomorphic in the interior of E. The set of functions defined in E and possessing a holomorphic extension to an open neighbourhood of E will be denoted by $H(E)$. We shall see that our problem is related to the following

V.5. Property. *$H(E)$ is dense in $L_a^2(E)$.*

In the following we shall assume that E is compact.

V.6. Definition. We shall say that a decreasing sequence of domains D_m approximates E from outside if for each open G such that $E \subset G$ the inclusion

$$E \subset D_m \Subset G$$

holds for all sufficiently large m.

V.7. Theorem. *Suppose that $E = \bar{D}$ has Property V.5. Then for every sequence D_m approximating E from outside*

$$\lim_{m\to\infty} K_{D_m} = K_D$$

locally uniformly in $D \times D^*$.

Proof. Consider a fixed $t \in D$ and an arbitrary $f \in L^2H(D)$. Set $f = 0$ on $E \setminus D$. Then $f \in L_a^2(E)$. Consider any $h \in H(E)$. For large m, $h \in L^2H(D_m)$, and we have

$$|h(t)| \leqslant K_{D_m}(t, \bar{t})^{1/2} \|h\|_{D_m}.$$

In the limit $m \to \infty$ we obtain

$$|h(t)| \leqslant k(t, \bar{t})^{1/2} \|h\|_E,$$

where $k(t, \bar{t}) = \lim_{m\to\infty} K_{D_m}(t, \bar{t})$. Since $H(E)$ is dense in $L_a^2(E)$, the above inequality holds for f. Hence

$$|f(t)| \leqslant k(t, \bar{t})^{1/2} \|f\|_E = k(t, \bar{t})^{1/2} \|f\|_D.$$

It follows that $K_D(t,\bar{t}) \leqslant k(t,\bar{t})$. Since the opposite inequality is obvious, we have

$$\lim_{m\to\infty} K_{D_m}(t,\bar{t}) = K_D(t,\bar{t})$$

for all $t \in D$. In view of Theorem V.4 the proof is completed.

V.8. Theorem. *Suppose that a sequence D_m approximates $E = \bar{D}$ from the outside,* (5) *holds and that the boundary of D has Lebesgue measure zero. Then E has Property* V.5.

Proof. It is well known that the family $K_D(\cdot,\bar{t})$, $t \in D$, is dense in $L^2H(D)$ and therefore in $L^2_a(E)$. The fact that $K_{D_m}(t,\bar{t})$ converges to $K_D(t,\bar{t})$ implies that $K_{D_m}(\cdot,\bar{t})$ converges to $K_D(\cdot,\bar{t})$ in an L^2 norm. Since $K_{D_m}(\cdot,\bar{t}) \in H(E)$, the proof is completed.

A counterexample in the plane

It is easy to give an example of a sequence of domains D_m, $m = 1, 2, \ldots$, approximating from outside the closure of D and such that (5) does not hold. For example we may take $D_m = \{|z| < 1+1/m\}$ and $D = \{|z| < 1, z \notin [0,1)\}$.

In this example an essential role is played by the fact that D has a slit along the positive radius, and therefore is not equal to the interior of its closure. However, it is possible to find a counterexample to (5) which satisfies

$$D = (\bar{D})^0$$

and the boundary of D has measure zero. In view of Theorems V.7 and V.8 this problem is reduced to finding D such that $H(E)$ is not dense in $L^2_a(E)$, where $E = \bar{D}$. We note the following

V.9. Theorem (Havin [14]). *Let E be a compact set in the complex plane. Then $H(E)$ is dense in $L^2_a(E)$ if and only if the set of all points at the boundary of D which do not belong to the fine closure of the complement of E has zero logarithmic capacity.*

This theorem yields

V.10. Corollary. *Let $D = (\bar{D})^0$ be a bounded domain in the complex plane, such that the Lebesgue measure of its boundary is equal to zero. Suppose that the decreasing sequence of domains D_m approximates $\bar{D}$ from the outside. Then* (5) *holds if and only if the set of all points at the boundary of D which do not belong to the fine closure of the exterior of D has zero logarithmic capacity. In particular, if the complement of the closure of D has a finite number of components, then* (5) *holds.*

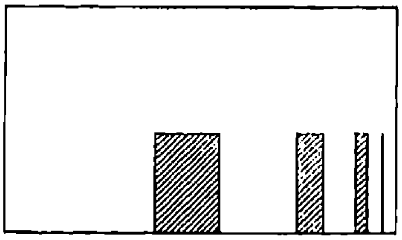
Let us consider the unit disc with a closed segment removed. From this domain we remove a sequence of pairwise disjoint closed discs Q_n

which accumulate at every point of the segment. In this way we obtain a bounded domain D such that $D = (\bar{D})^0$ and ∂D has Lebesgue measure zero. Let r_n be the radius of the disc Q_n. If r_n converges rapidly to zero, no point of the segment belongs to the fine closure of the complement of D. Since a segment has positive logarithmic capacity, (5) does not hold in view of Corollary V.10.

VI. The ideal boundary

Definition of the ideal boundary (1)

Consider a domain $D \subset C$ obtained from the square by removing countably many closed rectangles which converge to a segment on the side of the square (see the figure).



It is well known that a biholomorphic mapping of D onto the unit disc $U \subset C$ does not admit an extension to a homeomorphism between $\bar{D}$ and $\bar{U}$. This example shows that the Euclidean compactification $\bar{D}$ of a domain D is "unnatural" for the purposes of complex analysis. A proper compactification in this case is the classical Carathéodory compactification $\hat{D}$ obtainded by ajdoining to D all prime ends.

In the following we shall use the Bergman function in order to introduce a certain natural compactification for a domain contained in the space C^n. In fact, we shall distinguish in an invariant way a rather large class of domains $D \subset C^n$ which admit a canonically defined compactification $\hat{D}$. This construction is invariant under the biholomorphic transformation $\varphi\colon D \to \tilde{D}$ in the sense that the mapping φ can always be extended to the homeomorphism $\hat{\varphi}\colon \hat{D} \to \hat{\tilde{D}}$. The set $\Delta_D = \hat{D} \setminus D$ will be referred to as the ideal boundary of D. In the case of a simply connected domain D properly contained in the complex plane the compactification $\hat{D}$ is canonically homeomorphic with the Carathéodory compactification, and the ideal boundary Δ_D corresponds to the set of all prime ends in D. We begin with

(1) A same what simpler discussion of the notion of ideal boundary can be found in authors paper *The ideal boundary of a domain in C^n*, Ann. Polon. Math. 39 (to appear).

VI.1. Definition. Let D be a domain in C^n. A subdomain $U \subset D$ together with a point $a \in D$ is called *distinguished* if it satisfies the following conditions:

(i) $K_D(a, \bar{t}) \neq 0$ for all $t \in U$,

(ii) the family of all holomorphic functions of the form

$$f_t(z) = \frac{K_D(z, \bar{t})}{K_D(a, \bar{t})}, \qquad t \in U,$$

is normal in D.

VI.2. Remark. It is easy to see that the above notion is invariant under the biholomorphic mapping $\varphi\colon D \to \tilde{D}$. Indeed, assume that the points a, t, z are mapped onto points $\tilde{a}, \tilde{t}, \tilde{z}$. Then, by the rule of transformation of the Bergman function,

$$f_{\tilde{t}}(\tilde{z}) = \frac{K_{\tilde{D}}(\tilde{z}, \bar{\tilde{t}})}{K_{\tilde{D}}(\tilde{a}, \bar{\tilde{t}})} = \frac{K_D(z, \bar{t})}{K_D(a, \bar{t})} \frac{\partial z}{\partial \tilde{z}} \frac{\partial \tilde{a}}{\partial a}.$$

If $\tilde{U} = \varphi(U)$, then the family $f_{\tilde{t}}(\tilde{z})$, $\tilde{t} \in \tilde{U}$, is normal in $\tilde{D}$, as a product of a normal family by a holomorphic function. This shows that the pair $\tilde{U}, \tilde{a}$ is a distinguished domain in $\tilde{D}$.

We shall be specially interested in domains which admit finite covering by distinguished subdomains, and we adopt the following

VI.3. Definition. A domain $D \subset C^n$ *belongs to the class* S if it can be covered by finitely many distinguished subdomains, and if for every pair of distinct points $t, s \in D$ the evaluation functions $\chi_t\colon L^2H(D) \to C$ and $\chi_s\colon L^2H(D) \to C$ defined by the formula

$$\chi_s(f) = f(s), \qquad \chi_t(f) = f(t),$$

are linearly independent.

VI.4. Remark. It $\varphi\colon D \to \tilde{D}$ is a biholomorphic mapping and $D \in S$, then $\tilde{D} \in S$.

Assume now that a domain $D \in S$ is covered by a finite family θ of distinguished subdomains $\{U_1, a_1\}, \{U_2, a_2\}, \ldots, \{U_k, a_k\}$. For every $i = 1, 2, \ldots, k$, denote by F_i the closure of the family

$$f_t(z) = \frac{K_D(z, \bar{t})}{K_D(a_i, \bar{t})}, \qquad t \in U_i,$$

in the Fréchet space of all holomorphic functions in D. Denote by F the compact space

$$F = F_1 \cup F_2 \cup \ldots \cup F_k.$$

We shall say that two elements $f, g \in F$ are *proportional* if there exists a complex number $c \neq 0$ such that $f = cg$. This is obviously an equivalence relation in F.

The set of all equivalence classes will be denoted by E. Thus

$$E = \{[f], f \in F\}.$$

Our next task will be to introduce the canonical topology in E. We need first some lemmas.

VI.5. Lemma. *Let i be fixed. If $f \in F_i$ and $g \in F_i$ are proportional, then $f \equiv g$.*

Proof. This is obvious in view of the fact that each function in F assumes value 1 at a_i.

VI.6. Lemma. *Let i and j be fixed. Assume that the sequence $f_m \in F_i$ converges to f, and that $g_m \in F_j$ is proportional to f_m for $m = 1, 2, \ldots$ Then the sequence g_m converges, and the limit g is proportional to f.*

Proof. We shall first prove that if g_m converges, then the limit g is proportional to f. By assumption $f_m = c_m g_m$ for $m = 1, 2, \ldots$ By passing to a subsequence we may additionally assume that the sequence c_m converges to an extended complex number c. If $c \neq 0, \infty$ we can pass to the limit to obtain the desired relation $f = cg$. For $z = a_i$ and $z = a_j$ we obtain, respectively,

$$1 = c_m g_m(a_i) \quad \text{and} \quad f_m(a_j) = c_m.$$

Therefore the possibilities $c = 0$ or $c = \infty$ are excluded.

We now return to the general case. We have seen that all convergent subsequences of g_m have proportional limits. These limits obviously are in F_j, so they are all equal by the previous lemma. Therefore all convergent subsequences of g_m have a common limit. Since F_j is compact, this implies that the sequence g_m itself is convergent, and its limit g is proportional to f by the first part of the proof. The proof is complete.

It follows from the previous lemma that each equivalence class $[f] \in E$ has the form $[f] = \{f_i, i \in I\}$, where the set $I = \{i_1, \ldots, i_s\}$ is finite, and $f_i \in F_i$ for $i \in I$. We shall assume that

$$1 \leqslant i_1 < i_2 < \ldots < i_s \leqslant k.$$

VI.7. Theorem. *For $[f] \in E$ and $\varepsilon > 0$ let us consider the subset $U(f, \varepsilon)$ of E consisting of all classes*

$$[g] = \{g_j, j \in J\}$$

such that $J \subset I$ and $\sup_{j \in J} \varrho(g_j, f_j) < \varepsilon$. The family $\mathcal{O}$ of all sets of the form $U(f, \varepsilon)$ is a basis for a topology in E.

Proof. Obviously $[f] \in U(f, \varepsilon)$, and so the family θ covers the set E. Assume that $[f] \in E$ belongs to both $U(h^1, \varepsilon^1)$ and $U(h^2, \varepsilon^2)$, where

$$[h^1] = \{h^1_r, r \in L_1\}, \qquad [h^2] = \{h^2_r, r \in L_2\}.$$

We need to find $\varepsilon > 0$ such that $U(f, \varepsilon) \subset U(h^1, \varepsilon^1) \cap U(h^2, \varepsilon^2)$. Set

$$\varepsilon = \min\left(\varepsilon^1 - \sup_{i\in I} \varrho(f_i, h^1_i), \varepsilon^2 - \sup_{i\in I} \varrho(f_i, h^2_i)\right).$$

If $[g] \in U(f, \varepsilon)$, then $J \subset I \subset L_1$ and

$$\sup_{j\in J} \varrho(g_j, h^1_j) \leqslant \sup_{j\in J} \varrho(g_j, f_j) + \sup_{j\in J} \varrho(f_j, h^1_j)$$
$$< \varepsilon^1 - \sup_{i\in I} \varrho(f_i, h^1_i) + \sup_{j\in J} \varrho(f_j, h^1_j) \leqslant \varepsilon^1.$$

Therefore $[g] \in U(h^1, \varepsilon^1)$. Analogously $[g] \in U(h^2, \varepsilon^2)$. The proof is complete.

VI.8. Remark. Every point $[f] \in E$ has a countable basis of neighbourhoods. Indeed, it is enough to consider the sets $U(f, 1/m)$, $m = 1, 2, \ldots$

VI.9. Remark. The mapping which assigns to each element $f \in F$ its equivalence class $[f] \in E$ is continuous. Indeed, assume that $\lim f_m = f$. We need to show that $\lim [f_m] = [f]$. It is possible to split the sequence f_m into no more than k subsequences with the property that the terms of each subsequence belong to a fixed set F_i. So with no loss of generality we may assume that $f_m \in F_i$. It follows that $f \in F_i$. If

$$[f_m] = \{f^m_i, i \in I_m\}, \qquad [f] = \{f_j, j \in J\},$$

denote by L the set of all l such that F_l contains an element g^m_l proportional to f^m_i for infinitely many values of m. It follows from Lemma VI.6 that the limit

$$g_l = \lim_m g^m_l$$

exists and is proportional to f. Therefore $L \subset J$.

Now it is easy to see that for every $\varepsilon > 0$ and sufficiently large m the set I_m is contained in L, and $[f_m] \in U(f, \varepsilon)$. The proof is complete.

Let us now consider the mapping $p: D \to E$ given by the formula

$$p(t) = \left[\frac{K_D(\cdot, \bar{t})}{K_D(a_i, \bar{t})}\right] \quad \text{if } t \in U_i.$$

The right-hand side does not depend on the choice of U_i which contains t.

VI.10. Remark. The mapping $p: D \to E$ is continuous and one-to-one. Furthermore the set $p(D)$ is dense in E.

To prove continuity, assume that $\lim t_m = t_0$, where $t_m \in D$, $m = 1, 2, \ldots$, and $t_0 \in U_i$. The mapping p_i: $U_i \to F$ given by

$$p_i(t) = \frac{K_D(\cdot, \bar{t})}{K_D(a_i, \bar{t})}$$

is continuous. In U_i the mapping $p = [p_i]$ is continuous as a composition of two continuous mappings. It follows that p is continuous in D. If $p(t_1) = p(t_2)$, then the functions $K_D(\cdot, \bar{t}_1)$ and $K_D(\cdot, \bar{t}_2)$ are proportional. It follows from the reproducing property of the Bergman function that the evaluation functions χ_{t_1} and χ_{t_2} are linearly dependent. Since by assumption the domain D belongs to the class S, this is possible only when $t_1 = t_2$. Therefore p is one-to-one.

To see that $p(D)$ is dense consider an arbitrary element $[f] \in E$. Take i such that $f \in F_i$. By the definition of F_i we can find a sequence $t_m \in U_i \subset D$ such that $\lim p_i(t_m) = f$. Since $p_i(t_m) \in F_i$, we can follow the reasoning in Remark VI.10 to conclude that $\lim p(t_m) = \lim [p_i(t_m)] = [f]$. The proof is complete.

VI.11. Remark. If $D \in S$, then the space E is compact. In fact, it is a continuous image of the compact space F.

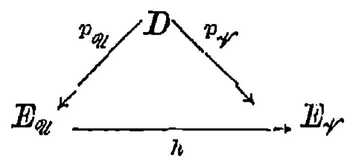
VI.12. Remark. E is a Hausdorff space. In fact, assume that $[f]$, $[g]$ are different elements of E. It follows that $[f]$ and $[g]$ are finite disjoint subsets of the metric space F, and the number

$$\varepsilon = \mathrm{dist}_F([f], [g])$$

is positive. It is easy to see that the neighbourhoods $U(f, \varepsilon/2)$ and $U(g, \varepsilon/2)$ are disjoint.

As the next step we shall compare the spaces E constructed for different coverings of D by distinguished neighbourhoods. If $\mathscr{U} = \{U_1, a_1, \ldots, U_k, a_k\}$ and $\mathscr{V} = \{V_1, b_1, \ldots, V_s, b_s\}$ are two such neighbourhoods, we shall write $E_{\mathscr{U}}$ and $E_{\mathscr{V}}$, respectively. We shall also introduce a similar notation $p_{\mathscr{U}}$ and $p_{\mathscr{V}}$. It is our aim to prove that $E_{\mathscr{U}}$ and $E_{\mathscr{V}}$ are canonically homeomorphic.

VI.13. Theorem. *There exists a unique homeomorphism* h: $E_{\mathscr{U}} \to E_{\mathscr{V}}$ *such that the diagram*



is commutative.

The uniqueness is obvious since $h = p_{\mathscr{V}} \cdot p_{\mathscr{U}}^{-1}$ on a dense subset $p_{\mathscr{U}}(D)$. We shall divide the existence proof into several steps.

Step 1. Assume that for every sequence $t_m \in D$ the sequence $p_{\mathscr{U}}(t_m)$ is convergent in $E_{\mathscr{U}}$ if and only if the sequence $p_{\mathscr{V}}(t_m)$ is convergent in $E_{\mathscr{V}}$. Then the mapping $h = p_{\mathscr{V}} \cdot p_{\mathscr{U}}^{-1}$ can be extended to a homeomorphism $h\colon E_{\mathscr{U}} \to E_{\mathscr{V}}$.

Proof. Our assumption implies that two sequences of the form $p_{\mathscr{U}}(t_m)$ converge to the same point in E if and only if the corresponding sequences of the form $p_{\mathscr{V}}(t_m)$ converge to the same point in $E_{\mathscr{V}}$. Therefore if $[f] \in E_{\mathscr{U}}$ we can select any sequence $t_m \in D$ such that $\lim p(t_m) = [f]$ and define

$$h[f] = \lim p_{\mathscr{V}}(t_m);$$

here the right-hand side is independent of the choice of the sequence t_m. Note that $h\colon E_{\mathscr{U}} \to E_{\mathscr{V}}$ is a one-to-one mapping onto E, and that h^{-1} plays the role of h if we replace $\mathscr{U}$ by $\mathscr{V}$ and vice versa. Therefore we need only to show that h is continuous at every point $[f]$ of $E_{\mathscr{U}}$. Consider an arbitrary sequence $[f_m] \in E_{\mathscr{U}}$ such that $\lim [f_m] = [f]$. We want to show that the sequence $h[f_m]$ converges to $h[f]$ in $E_{\mathscr{V}}$. Since $E_{\mathscr{V}}$ is compact, it is enough to show that if $[g] = \lim h[f_m]$, then $[g] = h[f]$. For each $j = 1, 2, \ldots$ consider neighbourhoods $U(f, 1/j)$ and $U(g, 1/j)$, and select m such that $[f_m] \in U(f, 1/j)$ and $h[f_m] \in U(g, 1/j)$.

Since $p_{\mathscr{U}}(D)$ is dense in $E_{\mathscr{U}}$, there exists a sequence $t_r \in D$ such that $\lim p_{\mathscr{U}}(t_r) = [f_m]$. By the definition of h, $\lim p_{\mathscr{V}}(t_r) = h[f_m]$. Therefore there exists a point q_j in the sequence t_r such that $p_{\mathscr{U}}(q_j) \in U(f, 1/j)$ and $p_{\mathscr{V}}(q_j) \in U(g, 1/j)$.

It follows that $\lim_j p_{\mathscr{U}}(q_j) = [f]$ and $\lim_j p_{\mathscr{V}}(q_j) = [g]$. Hence $[g] = h[f]$ and the proof is complete.

Step 2. The theorem holds if the coverings are of the form

$$\mathscr{U} = \{U_1, a_1, \ldots, U_k, a_k\}, \qquad \mathscr{V} = \{U_1, b_1, \ldots, U_k, b_k\}.$$

Proof. We shall show that the assumptions made in step 1 are satisfied. The role played by both coverings is symmetric, so it is enough to show that the convergence of $p_{\mathscr{U}}(t_m)$ to $[f] \in E_{\mathscr{U}}$ implies the convergence of $p_{\mathscr{V}}(t_m)$ to some element $[g] \in E_{\mathscr{V}}$. Since $E_{\mathscr{V}}$ is compact, we need only to show that all convergent subsequences of $p_{\mathscr{V}}(t_m)$ have a common limit. Of course it will be sufficient to prove that if $p_{\mathscr{V}}(t_m)$ is convergent to $[g] \in E_{\mathscr{V}}$, then g is proportional to f.

By considering a convenient subsequence if necessary, we may assume that there exists a fixed set U_i such that $t_m \in U_i$ for all m.

Assume that $[f] = \{f_j, j \in J\}$ and $[g] = \{g_l, l \in L\}$. It follows that $i \in J \cap L$ and

$$\lim_m \frac{K_D(\cdot, \bar{t}_m)}{K_D(a_i, \bar{t}_m)} = f_i, \qquad \lim_m \frac{K_D(\cdot, \bar{t}_m)}{K_D(b_i, \bar{t}_m)} = g_i.$$

Since the sequences

$$\frac{K_D(b_i, \bar{t}_m)}{K_D(a_i, \bar{t}_m)} \quad \text{and} \quad \frac{K_D(a_i, \bar{t}_m)}{K_D(b_i , \bar{t}_m)}$$

are both convergent, their limits must be different from zero. Hence

$$g_i = \lim \frac{K_D(\cdot, \bar{t}_m)}{K_D(a_i, \bar{t}_m)} \lim \frac{K_D(a_i, \bar{t}_m)}{K_D(b_i, \bar{t}_m)} = f_i c$$

with $c \neq 0$. This proves the statement.

Step 3. Assume that $\mathscr{U}$ and $\mathscr{V}$ are two coverings of D by distinguished subdomains. Then the theorem holds for two coverings of the form

$$\mathscr{U} = \{U_1, a_1, \ldots, U_k, a_k\},$$
$$\mathscr{W} = \{U_i \cap V_j, a_{ij}, i = 1, 2, \ldots, k, j = 1, 2, \ldots, s\},$$

where $a_{ij} = a_i$ for all i, j.

Proof. As before, we are going to prove that the assumptions made in step 1 are satisfied. Note that in the present case $F_{\mathscr{U}} = F_{\mathscr{W}}$, and the underlying sets of the spaces $E_{\mathscr{U}}$ and $E_{\mathscr{W}}$ are equal. Also $p_{\mathscr{U}}(t_m) = p_{\mathscr{W}}(t_m)$. We want to show that if $p_{\mathscr{U}}(t_m)$ converges in $E_{\mathscr{U}}$, then $p_{\mathscr{W}}(t_m)$ converges in $E_{\mathscr{W}}$ and vice versa. Since the spaces $E_{\mathscr{U}}$ and $E_{\mathscr{W}}$ are both compact, the statement will follow if we prove that

$$\lim p_{\mathscr{U}}(t_m) = [f] = \{f_i, i \in I\},$$
$$\lim p_{\mathscr{W}}(t_m) = [g] = \{g_{ij}, (i, j) \in L\},$$

implies $[f] = [g]$. By choosing a convenient subsequence we may assume that there exist fixed sets U_i, V_j such that $t_m \in U_i \cap V_j$ for $m = 1, 2, \ldots$ Then

$$p_{\mathscr{U}}(t_m) = [p_i(t_m)]_{\mathscr{U}}, \quad p_{\mathscr{W}}(t_m) = [p_i(t_m)]_{\mathscr{W}}.$$

Our assumption implies that $p_i(t_m)$ converges locally uniformly in D to f_i and at the same time to g_{ij}. Hence $f_i = g_{ij}$ and $[f] = [g]$. The proof is complete.

Step 4. The theorem holds for arbitrary coverings

$$\mathscr{U} = \{U_1, a_1, U_2, a_2, \ldots, U_k, a_k\}, \quad \mathscr{V} = \{V_1, b_1, V_2, b_2, \ldots, V_s, b_s\}.$$

Proof. Denote by $\mathscr{W}_a$ the covering $\mathscr{W}$ of step 3 and introduce an analogous covering

$$\mathscr{W}_b = \{U_i \cap V_j, b_{ij}, i = 1, 2, \ldots, k, j = 1, 2, \ldots, s\},$$

where $b_{ij} = b_j$. There exists a chain of canonical homeomorphisms which leads from $E_{\mathcal{U}}$ to $E_{\mathcal{V}}$:

$$E_{\mathcal{U}} \to E_{\mathcal{W}_a} \quad \text{by step 3},$$
$$E_{\mathcal{W}_a} \to E_{\mathcal{W}_b} \quad \text{by step 2},$$
$$E_{\mathcal{W}_b} \to E_{\mathcal{V}} \quad \text{by step 3}.$$

The composition of these mappings yields the desired homeomorphism of $E_{\mathcal{U}}$ onto $E_{\mathcal{V}}$. In fact, it is easy to check that this composition maps each element of the form $p_{\mathcal{U}}(t)$, $t \in D$, onto $p_{\mathcal{V}}(t)$. The theorem is completely proved.

In view of the theorem we can identify all spaces $E_{\mathcal{U}}$ with a single space E. It follows that $p\colon D \to E$, where $p = p_{\mathcal{U}}$ is a well-defined, continuous and one-to-one mapping of D onto a dense subset of the compact space E. We do not know whether p is open and therefore E cannot serve as a compactification of D. The standard way to overcome this difficulty makes use of the Alexandrov compactification D_∞ of D. Consider the mapping $p_\infty\colon D \to D_\infty \times E$ given by

$$p_\infty(t) = \big(t, p(t)\big).$$

VI.14. Definition. The topological space $\hat{D}$ is defined as the closure of $p_\infty(D)$ in the space $D_\infty \times E$.

We can now prove

VI.15. Theorem. *The mapping p_∞: $D \to \hat{D}$ is continuous, one-to-one and open.*

Proof. The first two statements are obvious. Assume that p_∞ is not open. Then there exists an open set $W \subset D$ such that for a certain $t \in W$ the point $p_\infty(t) = \big(t, p(t)\big)$ does not belong to the interior of $p_\infty(W)$. It follows that we can find a sequence of points $(t_m, e_m) \in \hat{D}$ such that $(t_m, e_m) \notin p_m(W)$ and $\lim(t_m, e_m) = \big(t, p(t)\big)$. In particular $\lim t_m = t$, and $t_m \in W$ for sufficiently large m. For fixed m we can find points $s_j \in D$, $j = 1, 2, \ldots$, such that

$$\lim p_\infty(s_j) = (t_m, e_m) \in \hat{D}.$$

Since $p_\infty(s_j) = \big(s_j, p(s_j)\big)$, we conclude that $\lim\limits_j s_j = t_m$. Since p is continuous, it follows that $e_m = \lim p(s_j) = p(t_m)$. Therefore

$$(t_m, e_m) = \big(t_m, p(t_m)\big) = p_\infty(t_m) \in p_\infty(W),$$

contradicting the assumption that $(t_m, e_m) \notin p_\infty(W)$. It follows that p must be open, and the proof is complete.

VI.16. Remark. By the previous theorem the mapping p_∞ is a homeomorphism of D onto an open dense subset $p_\infty(D)$ of the compact space $\hat{D}$.

Therefore we may consider the space $\hat{D}$ as a compactification of D by identifying D and $p_\infty(D)$. The set $\hat{D}\setminus D$ will be referred to as the ideal boundary of D.

The most important property of the compactification $\hat{D}$ is its invariance under biholomorphic mappings.

VI.17. THEOREM. *Let φ: $D \to \tilde{D}$ be a biholomorphic mapping of a domain $D \in S$. Then $\tilde{D} \in S$, and φ possesses a unique extension to a homeomorphism $\hat{\varphi}$: $\hat{D} \to \hat{\tilde{D}}$.*

Proof. If A is any quantity constructed for a domain D, the corresponding quantity constructed for $\tilde{D}$ will be denoted by $\tilde{A}$. Consider a distinguished covering $\mathscr{U} = \{U_1, a_1, \ldots, U_k, a_k\}$ of D. Then $\tilde{\mathscr{U}} = \{\tilde{U}_1, \tilde{a}_1, \ldots \ldots, \tilde{U}_k, \tilde{a}_k\}$ is a distinguished covering of $\tilde{D}$. Consider the spaces $F = F_1 \cup \ldots \cup F_k$ and $\tilde{F} = \tilde{F}_1 \cup \ldots \cup \tilde{F}_k$. For $i = 1, 2, \ldots, k$ consider the mapping Φ_i: $H(D) \to H(\tilde{D})$ given by

$$(\Phi_i f)(\tilde{z}) = f\big(z(\tilde{z})\big)\,\frac{\partial z}{\partial \tilde{z}}\,\frac{\partial \tilde{a}_i}{\partial a_i}.$$

It is easy to see that this is a homeomorphism with respect to the Frechet metric in $H(D)$ and in $H(\tilde{D})$. Also the diagram

$$\begin{array}{ccc} U_i & \xrightarrow{\ \varphi\ } & \tilde{U}_i \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle \tilde{p}_i} \\ F_i & \xrightarrow{\ \Phi_i\ } & \tilde{F}_i \end{array}$$

is commutative. It follows that $\Phi_i(F_i) = \tilde{F}_i$. If $f_i \in F_i$ and $f_j \in F_j$ are proportional, then so are $\tilde{f}_i = \Phi_i f_i$ and $\tilde{f}_j = \Phi_j f_j$. Hence there exists a well-defined mapping Φ: $E \to \tilde{E}$, such that

$$\Phi[f] = [\Phi_i f] \quad \text{if } f \in F_i.$$

Also for $t \in U_i$ the element $\tilde{t} = \varphi(t)$ belongs to $\tilde{U}_i$ and

$$\Phi \circ p(t) = \Phi[p_i(t)] = [\Phi_i \circ p_i(t)] = [\tilde{p}_i \circ \varphi(t)] = \tilde{p} \circ \varphi(t).$$

Hence $\Phi \circ p = \tilde{p} \circ \varphi$. To see that Φ is continuous, consider a sequence $[f_m]$ which converges to $[f]$. We can assume with no loss of generality that all f_m belong to a fixed space F_i, for otherwise we would split the sequence into subsequences with this property. It follows that the sequence f_m converges to an element proportional to f, so we can replace f by this limit and assume that

$$\lim f_m = f \in F_i.$$

We want to show that $\lim \Phi[f_m] = \Phi[f]$. This follows now from the continuity of Φ_i, since

$$\lim \Phi[f_m] = \lim [\Phi_i f_m] = [\lim \Phi_i f_m] = [\Phi_i f] = \Phi[f].$$

The continuity of Φ is therefore proved. By reversing the roles of E and $\tilde{E}$ it is easy to see that Φ has a continuous inverse. It follows that $\Phi\colon E \to \tilde{E}$ is a homeomorphism. Of course

$$\varphi \times \Phi\colon D_\infty \times E \to \tilde{D}_\infty \times \tilde{E},$$

where $\varphi(\infty) = \infty$ is a homeomorphism. For $t \in D$ we have

$$\varphi \times \Phi\big(t, p(t)\big) = \big(\varphi(t), \Phi \circ p(t)\big) = \big(\varphi(t), \tilde{p} \circ \varphi(t)\big) = \big(\tilde{t}, p(\tilde{t})\big).$$

Hence

$$\varphi \times \Phi\big(p_\infty(D)\big) = \tilde{p}_\infty(\tilde{D}).$$

It follows that $\varphi \times \Phi$ defines a homeomorphism of $\hat{D}$ onto $\hat{\tilde{D}}$ which agrees with φ on D. The proof is complete.

Characteristic properties

We shall presently describe the properties of the compactification $\hat{D}$ which characterize it among all possible compactifications.

VI.18. THEOREM. *Assume that $D \in S$. The compactification $\hat{D}$ has the following properties:*

(i) *D is an open subset of $\hat{D}$.*

(ii) *There exists a finite covering of D by open sets $U_1, U_2, \ldots, U_k$ and points $a_1, a_2, \ldots, a_k \in D$ such that for each $i = 1, 2, \ldots, k$*

$$K_D(a_i, \bar{t}) \neq 0 \quad \textit{for } t \in U_i$$

and the function

$$\frac{K_D(z, \bar{t})}{K_D(a_i, \bar{t})}, \quad z \in D,\ t \in U_i,$$

admits a continuous extension to $D \times \bar{U}_i$, where $\bar{U}_i$ denotes the closure of U_i in $\hat{D}$.

(iii) *For $t_i \in \bar{U}_i \setminus D$, $t_j \in \bar{U}_j \setminus D$ the functions of the variable z*

$$\frac{K_D(z, \bar{t}_i)}{K_D(a_i, \bar{t}_i)} \quad \textit{and} \quad \frac{K_D(z, \bar{t}_j)}{K_D(a_j, \bar{t}_j)}$$

are not proportional if and only if $t_i \neq t_j$.

These properties are characteristic of the compactification $\hat{D}$. More precisely, if $\hat{D}_0$ is another compactification of D with properties (i)–(iii), *then the identity mapping of D extends to a homeomorphism of $\hat{D}_0$ onto $\hat{D}$.*

Proof. We shall first prove that $\hat{D}$ has all the listed properties. Note that (i) was inherent in the construction of $\hat{D}$.

To prove (ii), note that an element in $\hat{D}$ can be written in the form

$$(t_0, [f]), \quad t_0 \in D_\infty, \ [f] \in E.$$

It follows from the definition of topology in E that if this element belongs to $\overline{U}_i$, then there exists a unique $f_i \in F_i$ proportional to f. The function which assigns to a pair $z_0 \in D$, $(t_0, [f]) \in \overline{U}_i$ the number $f_i(z_0)$ is an extension of

$$\frac{K_D(z, \bar{t})}{K_D(a_i, \bar{t})}, \quad z \in D, \ t \in U_i,$$

to $D \times \overline{U}_i$. For the convenience of notation, in the following this extension will be denoted simply by

$$\frac{K_D(z, \bar{t})}{K_D(a_i, \bar{t})}, \quad z \in D, \ t \in \overline{U}_i.$$

Consider now another element $(t, [g])$ in $\overline{U}_i \subset \hat{D}$. The extended function assigns to a pair $z \in D$, $(t, [g]) \in \overline{U}_i$ the number $g_i(z)$. We want to estimate the expression

$$|g_i(z) - f_i(z_0)| \leqslant |g_i(z) - g_i(z_0)| + |g_i(z_0) - f_i(z_0)|.$$

If $[g]$ is sufficiently close to $[f]$ in E, then g_i is close to f_i in the topology of locally uniform convergence. This convergence is stronger than pointwise convergence, so we can make the second term small. Since the normal family F_i is uniformly continuous at z_0, and $f_i, g_i \in F_i$, we can make the first term small by a choice of z independently of g_i. Therefore the extension is continuous as claimed.

We pass to property (iii). A point $t_i \in \overline{U}_i \setminus D$ has the form $(\infty, [f])$, where f is proportional to $f_i \in F_i$. Also

$$\frac{K_D(z, \bar{t}_i)}{K_D(a_i, \bar{t}_i)} = f_i(z).$$

Similarily, a point $t_j \in \overline{U}_j \setminus D$ can be written as $(\infty, [g])$, where g is proportional to $g_j \in F_j$, and

$$\frac{K_D(z, \bar{t}_j)}{K_D(a_j, \bar{t}_j)} = g_j(z).$$

By assumption, $t_i \neq t_j$. Therefore $[f] \neq [g]$. It follows that f_i is not proportional to g_i. Property (iii) is therefore proved.

Consider now another compactification $\hat{D}_0$ of domain D with properties (i)–(iii). For $i = 1, 2, \ldots, k$ consider the family of functions in D

$$F_i = \left\{ \frac{K_D(z, \bar{t})}{K_D(a_i, \bar{t})}, \ t \in \overline{U}_i \right\}.$$

Let W be a compact neighbourhood of a point in D. The continuous function

$$\frac{K_D(z,\bar{t})}{K_D(a_i,\bar{t})}, \qquad z \in D,\ t \in \overline{U}_i,$$

is bounded on a compact set $W \times \overline{U}_i$. It follows that the family F_i is locally bounded, and therefore normal in D. We see in particular that

$$\mathscr{U} = \{U_1, a_1, \ldots, U_k, a_k\}$$

is a distinguished covering of D.

Note that each F_i, $i = 1, 2, \ldots, k$, is equal to the closure of the family

$$\frac{K_D(z,\bar{t})}{K_D(a_i,\bar{t})}, \qquad t \in U_i,$$

in the Fréchet space $H(D)$. For each $t \in \hat{D}_0$ define

$$\pi(t) = \left[\frac{K_D(z,\bar{t})}{K_D(a_i,\bar{t})}\right] \quad \text{if } t \in \overline{U}_i,\ \overline{U}_i \subset \hat{D}_0.$$

We have to show that $\pi\colon \hat{D}_0 \to E$ is well defined mapping of $\hat{D}_0$ onto E. But for $t \in \overline{U}_i \cap \overline{U}_j \cap D$ the functions of variable z

$$\frac{K_D(z,\bar{t})}{K_D(a_i,\bar{t})} \quad \text{and} \quad \frac{K_D(z,\bar{t})}{K_D(a_j,\bar{t})}$$

are obviously proportional, and for $t \in (\overline{U}_i \cap \overline{U}_j)\setminus D$ the proportionality of these functions follows from property (iii) of $\hat{D}_0$. Therefore π is well defined. Moreover, since F_i is locally bounded one can show that the mapping from $\overline{U}_i$ into $H(D)$, given by

$$t \mapsto \frac{K_D(z,\bar{t})}{K_D(a_i,\bar{t})}$$

is continuous.

It follows that on each $\overline{U}_i$, $i = 1, 2, \ldots, k$, the mapping

$$\pi(t) = \left[\frac{K_D(z,\bar{t})}{K_D(a_i,\bar{t})}\right], \qquad t \in \overline{U}_i,$$

is continuous as a composition of two continuous mappings see Remark VI.9. Therefore π is continuous on $\hat{D}_0$ as $\overline{U}_i$ form a finite covering of $\hat{D}_0$ by closed sets. Define the mapping $\pi_\infty\colon \hat{D}_0 \to \hat{D}$ by

$$\pi_\infty(t) = \begin{cases} (t, \pi(t)) & \text{if } t \in D, \\ (\infty, \pi(t)) & \text{if } t \in \hat{D}_0\setminus D. \end{cases}$$

This mapping is continuous, since each of its compotents is continuous.

Moreover, it is one-to-one, by (iii). It maps a compact space $\hat{D}_0$ onto a compact Hausdorff space $\hat{D}$. Such a mapping is necessarily a homeomorphism, and this completes the proof.

In the case where D is a Lu Qi-keng domain the characteristic properties of compactification D assume a particularly simple form. To proceed we need the following

VI.19. LEMMA. *Assume that D is a Lu Qi-keng domain, and the pair U, a is a distinguished subdomain in D. Then for each $b \in D$ the pair U, b defines a distinguished subdomain in D.*

Proof. We need only to show that the family

$$\frac{K_D(z, \bar{t})}{K_D(b, \bar{t})}, \quad t \in U, \tag{2}$$

is normal in D. Consider an arbitrary sequence $t_m \in U$, $m = 1, 2, \ldots$ Since the family

$$\frac{K_D(z, \bar{t})}{K_D(a, \bar{t})}, \quad t \in U,$$

is normal by assumption, we can choose a subsequence denoted again, by t_m and such that

$$\frac{K_D(z, \bar{t}_m)}{K_D(a, \bar{t}_m)} \tag{3}$$

converges in $H(D)$ to an element f. Note that $f \neq 0$ since $f(a) = 1$. In fact, $f(z) \neq 0$ for all $z \in D$. Indeed, assume to the contrary that $f(z_0) = 0$ for $z_0 \in D$. Then for every neighbourhood V of z_0 and sufficiently large m the function (3) attains value zero at some point $z \in V$, and this contradicts the assumption that D is a Lu Qi-keng domain. It follows in particular that

$$\lim_{m\to\infty} \frac{K_D(b, \bar{t}_m)}{K_D(a, \bar{t}_m)} = f(b) \neq 0.$$

Hence the sequence

$$\frac{K_D(z, \bar{t}_m)}{K_D(b, \bar{t}_m)} = \frac{K_D(z, \bar{t}_m)}{K_D(a, \bar{t}_m)} \frac{K_D(a, \bar{t}_m)}{K_D(b, \bar{t}_m)}$$

converges in $H(D)$. Thus family (2) is normal and the lemma is proved.

We can now prove

VI.20. THEOREM. *If a Lu Qi-keng domain D belongs to the class S, then for each $a \in D$ the pair D, a is a distinguished subdomain of D.*

Proof. Let $\mathscr{U} = \{U_1, a_1, U_2, a_2, \ldots, U_k, a_k\}$ be the covering of D by distinguished subdomains. By the lemma U_i, a is a distinguished subdomain for each $i = 1, 2, \ldots, k$. It follows that D, a is a distinguished subdomain.

Theorems VI.18 and VI.20 yield immediately the following

VI.21. COROLLARY. *The compactification $\hat{D}$ of a Lu Qi-keng domain $D \in S$ is characterized by the following properties:*

(i) *D is an open subset of $\hat{D}$.*

(ii) *There exists a point $a \in D$ such that the function $K_D(z, \bar{t})/K_D(a, \bar{t})$ admits a continuous extension from $D \times D$ to $D \times \hat{D}$.*

(iii) *If $t_1, t_2 \in \hat{D} \setminus D$ and $t_1 \neq t_2$, then*

$$\frac{K_D(z, \bar{t}_1)}{K_D(a, \bar{t}_1)} \neq \frac{K_D(z, \bar{t}_2)}{K_D(a, \bar{t}_2)}.$$

Proof. We may apply Theorem VI.18 with $k = 1$, $U_1 = D$, $a_1 = a$ (an arbitrary point in D). The compactification $\hat{D}$ obviously has property (i). Property (ii) follows since D is dense in $\hat{D}$, and therefore $\bar{D} = \hat{D}$. Finally in our case property (iii) is equivalent to (iii) of Theorem VI.18. Indeed, both functions assume value 1 at a and, if proportional, they are equal. Conversely, any compactification of $\hat{D}$ with properties listed above must be canonically homeomorphic with $\hat{D}$ by the second part of Theorem VI.18.

Remark. The above corollary shows that each function of the variable t in the family

$$\frac{K_D(t, \bar{z})}{K_D(t, \bar{a})}, \quad z, a \in D,$$

possesses a continuous extension to $\hat{D}$, and the family of all such extensions separates the points of $\hat{D} \setminus D$. It follows that if a domain $D \in S$ is Lu Qi-keng, then the compactification $\hat{D}$ is canonically homeomorphic with a compactification introduced for general Lu Qi-keng domain in [34].

The case of a bounded circular domain

For a large class of domain the ideal boundary is equal to the Euclidean boundary

VI.22. THEOREM. *Assume that D is a bounded complete circular domain in C^n (i.e. $z \in D$ $\lambda \in C$, $|\lambda| \leqslant 1$ implies $\lambda z \in D$) such that $\bar{D} \subset \lambda D$ for every $\lambda > 1$. Then the ideal boundary Δ_D is equal to the Euclidean boundary of D.*

Proof. We shall show that the Euclidean compactification satisfies conditions (i)–(iii) of Theorem VI.18 for the covering which consists of a single domain D and a point $a = 0$. By a theorem of H. Cartan

$$K_D(z, \bar{t}) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{\varphi_\alpha^k(z)\overline{\varphi_\alpha^k(t)}}{\beta_\alpha^k}, \qquad z, t \in D, \; \alpha = (\alpha_1, \ldots, \alpha_n),$$

locally uniformly in $D \times D$, where φ_α^k in the above series denotes a homogeneous polynomial of degree k, and β_α^k are positive constants. The system

$$\{\varphi_\alpha^k, |\alpha| = k, k \geqslant 0\}$$

is orthogonal and linearly dense in $L^2H(D)$.

It follows that $K_D(0, \bar{t}) = \text{const} \neq 0$. Furthermore, for $\lambda > 1$ the function defined in the product $\frac{D}{\lambda} \times \bar{D}$ by a locally uniformly convergent series

$$K_\lambda(z, \bar{t}) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{\varphi_\alpha^k(\lambda z)\overline{\varphi_\alpha^k(t/\lambda)}}{\beta_\alpha^k}$$

is continuous. Since $\varphi_\alpha^k(\lambda z)\overline{\varphi_\alpha^k(t/\lambda)} = \varphi_\alpha^k(z)\overline{\varphi_\alpha^k(t)}$, the functions K_{λ_2} and K_{λ_1}, where $\lambda_2 < \lambda_1$ are equal in $\frac{D}{\lambda_1} \times \bar{D}$ and are equal to $K_D(z, \bar{t})$ in $\frac{D}{\lambda_1} \times D$. It follows that there exists a continuous function in $D \times \bar{D}$ which extends $K_D(z, \bar{t})$ and agrees with $K_\lambda(z, \bar{t})$ on $\frac{D}{\lambda} \times \bar{D}$. This proves (ii). For (iii) we need only to show that $t_1, t_2 \in \bar{D}$ and $t_1 \neq t_2$ implies $K_D(z, \bar{t}) \neq K_D(z, \bar{t}_2)$ for a certain z in D. In fact such z can be found in D/λ_0. Assume to the contrary that $K_D(z, \bar{t}_1) = K_D(z, \bar{t}_2)$ for all $z \in D/\lambda_0$. Both functions are represented by uniformly convergent series (with $\lambda < \lambda_0$). Since the system φ_α^k is orthogonal also in D/λ_0, we can view each series as a Fourier expansion in D/λ_0. Such an expansion is unique, and the corresponding coefficients $\varphi_\alpha^k(t_1)$ and $\varphi_\alpha^k(t_2)$ must be equal. Since the linear functions φ_α^1, $|\alpha| = 1$ are linearly independent, it follows that $t_1 = t_2$. Since (i) is obvious, the proof is complete.

In particular, the above theorem says that the ideal boundary of the unit disc in the complex plane is equal to the unit circle. The same property holds for the Carathéodory compactification. Since both compactifications are invariant under a biholomorphic mapping, we obtain

VI.23. Theorem. *The Carathéodory compactification is canonically homeomorphic to the compactification* $\hat{D}$.

Plane domains, and strictly pseudoconvex domains

The following sufficient condition for the equality $\Delta_D = \partial D$ is sometimes useful:

VI.24. Theorem. *Let D be a bounded domain in C^n such that the Berg-*

man function $K_D(z, \bar{t})$ *admits a continuous extension to* $D \times \bar{D}$. *Assume further the following properties*:

(a) $\limsup_{z \to t \in \partial D} |K_D(z, \bar{t})| = \infty$,

(b) $\limsup_{z \to \xi \in D} |K_D(z, \bar{t})| < \infty$ *for each* $\xi \neq t \in \partial D$.

Then the ideal boundary of D *is equal to its Euclidean boundary.*

Proof. Note that by (a) to each $t \in \partial D$ there exists an $a \in D$ such that $|K_D(a, \bar{t})| > \varepsilon > 0$. By assumption this inequality is preserved in some neighbourhood V of t. Consider a finite covering of ∂D by such neighbourhoods $V_1, V_2, \ldots, V_p$. Set $U_i = V_i \cap D$, $i = 1, 2, \ldots, p$. Since D is bounded, we can find further sets U_i and points a_i, $i = p+1, \ldots, k$ such that $U_i \Subset D$ and the sets

$$U_1, a_1, U_2, a_2, \ldots, U_k, a_k$$

form a covering of D. It is easy to see that condition (ii) of Theorem VI.18 is satisfied when U_i is a sufficiently small ball with center a_i for $i = p+1, \ldots, k$. Thus in order to apply Theorem VI.18 we need only to check condition (iii), since (i) is satisfied automatically.

Consider different points $t_i \in \bar{U}_i \setminus D$, $t_j \in \bar{U}_j \setminus D$, and assume to the contrary that

$$\frac{K_D(z, \bar{t}_i)}{K_D(a_i, \bar{t}_i)} \quad \text{and} \quad \frac{K_D(z, \bar{t}_j)}{K_D(a_j, \bar{t}_j)}$$

are proportional as functions of z. It follows that at t_i (or t_j) the upper limit of the modulus of both functions is simultaneously finite or simultaneously infinite. In view of (a) and (b) this is possible only when $t_i = t_j$, a contradiction of the assumption that $t_i \neq t_j$. The proof is complete.

For plane domains we have

VI.25. Theorem. *Let* $D \subset C$ *be a bounded domain whose boundary consists of finitely many simple closed curves. Then the ideal boundary of* D *is equal to the Euclidean boundary.*

Proof. It is well known that D can be mapped onto a domain bounded by analytic curves, by a biholomorphic mapping which can be extended to a homeomorphism between closed domains. Therefore we may assume from the beginning that D is bounded by analytic curves.

In order to apply Theorem VI.24 we shall prove that

(i) $K_D(z, \bar{t})$ is continuous in $\bar{D} \times \bar{D}$ with the exeption of the set $\{(t, t),\ t \in \partial D\}$,

(ii) for each $t \in \partial D$, $\lim_{z \to t} K_D(z, \bar{t}) = \infty$.

Note that conditions (i) and (ii) are invariant under a biholomorphic mapping of D which can be extended to a biholomorphic mapping of some domain containing $\bar{D}$. This follows immediately from the fact that the jacobian of such a mapping is bounded away from zero and infinity, and from the rule of transformation of the Bergman function. In particular conditions (i) and (ii) are invariant under a biholomorphic mapping onto a domain bounded by analytic curves, since such a mapping can be extended with the aid of the Schwarz symmetry principle. Thus in proving (i) we may assume that the unbounded component of $C\setminus\bar{D}$ is equal to the exterior of the unit circle. Consider two different points a and b on ∂D, and small disjoint neighbourhoods V_a and V_b in $\bar{D}$. Let $G_D(z, t)$ be the Green function of D. We will first show that if we set

$$G_D(z, t) = 0 \quad \text{if } (z, t) \in (V_a \times V_b)\setminus(D\times D),$$

then G_D is continuous in $V_a \times V_b$. Since D is contained in the unit circle, we have the inequality

$$0 \leqslant G_D(z, t) \leqslant G(z, t) = \ln\left|\frac{1-z\bar{t}}{z-t}\right|.$$

Since the Green function of the unit circle $G(z, t)$ is continuous in $V_a \times V_b$, so is $G_D(z, t)$. Set $\tilde{V}_a = V_a \cup V_a^*$, $\tilde{V}_b = V_b \cup V_b^*$, where V_a^* and V_b^* are reflections in ∂D of V_a and V_b, respectively. By the Schwarz symmetry principle $G_D(z, t)$ is separately harmonic in $\tilde{V}_a \times \tilde{V}_b$.

Using the continuity of $G_D(z, t)$, it is easy to show that $G_D(z, t)$ is in fact harmonic in $\tilde{V}_a \times \tilde{V}_b$. (Actually one could dispose of the continuity considerations by using a stronger theorem of Lelong on separate harmonicity.)

By a result of Schiffer [5]

$$K_D(z, \bar{t}) = \frac{-2}{\pi}\frac{\partial^2 G_D(z, t)}{\partial z\, \partial \bar{t}}.$$

Hence $K_D(z, \bar{t})$ is regular in $\tilde{V}_a \times \tilde{V}_b$ and condition (i) is proved.

In proving (ii) we shall assume that the unbounded component of $C\setminus\bar{D}$ is equal to the exterior of the unit disc, and that $|t| = 1$. By a result of Schiffer [6] the difference between the Bergman function $K_D(z, \bar{s})$ and the "geometric quantity"

$$\Gamma_D(z, s) = \int_{C\setminus D} \frac{dm(w)}{(w-z)^2\overline{(w-s)}^2}$$

is regular in $\bar{D}\times\bar{D}$. Consider the Bergman function of the unit disc

$$K(z, \bar{s}) = \frac{1}{\pi(1-z\bar{s})^2}$$

and denote by F the sum of bounded components of $C\setminus D$. It follows that the difference

$$K_D(z, \bar{s}) - \frac{1}{\pi(1-z\bar{s})^2} = \int_F \frac{dm(w)}{(w-z)^2(\overline{w-s})^2}$$

is bounded when z and s belong to sufficiently small neighbourhood of t, and so is the difference

$$K_D(z, \bar{t}) - \frac{1}{\pi(1-z\bar{t})^2}.$$

Hence $\lim\limits_{z\to t} K_D(z, \bar{t}) = \infty$ and (ii) is proved.

For the higher-dimensional case we can state

VI.26. THEOREM. *Let $D \subset C^n$ be a strictly pseuconvex domain with a smooth boundary, such that for each $t \in \partial D$*

$$\lim_{z,s\to t} K_D(z, \bar{s}) = \infty.$$

Then the ideal boundary of D is equal to its Euclidean boundary.

Proof. By a theorem of Kerzman [20] the Bergman function $K_D(z, s)$ is smooth on $\bar{D}\times\bar{D}$ with the exception of points of the set $\{(t, t) t \in \partial D\}$. It follows that the assumptions of Theorem VI.24 are satisfied.

VI.27. Remark. From [11], Theorem 2, follows that every strictly pseudoconvex domain with a smooth boundary satisfies the condition of Theorem VI.26.

This yields a new proof of the theorem, that every biholomorphic mapping between two strictly pseudoconvex domains with smooth boundaries extends to a homeomorphism between closed domains, see Margulis [24], Vormoor [37]. (Actually this homeomorphism is of class C^∞, by a deep result of Feferman [11].)

References

[1] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1950), p. 337–404.

[2] D. Bell, *Some properties of the Bergman kernel function*, Compositio Math. 21 (1969), p. 329–330.

[3] S. Bergman, *Über die Existenz von Repräsentantenbereichen*, Math. Ann. 102 (1929), p. 430–446,

[4] — *On pseudo-conformal mappings of circular domains*, Pacific J. Math. 41 (1972), p. 579–585.

[5] — *The kernel function and conformal mapping*, Math. Surveys No. 5, Amer. Math. Soc. (1950).

[6] — and M. Schiffer, *Kernel functions and conformal mapping*, Compositio Math. 8 (1951), p. 205–249.

[7] H. Bremermann, *Holomorphic continuation of the kernel function and the Bergman metric in several complex variables*, Lectures on functions of a complex variable, Ann Arbor, Univ. of Michigan Press, 1955.

[8] H. Cartan, *Les fonctions de deux variable complexes et le problème de la representation analytique*, J. Math. Pures Appl. 10 (1931), p. 1–114.

[9] B. Chalmers, *Sur le comportement, a la frontière de la fonction noyau de Bergman*, Comp. Rend. Acad. Sci. 266, 33 (1968).

[10] A. M. Davie and B. K. Oksendal, *Peak interpolation sets for some algebras of analytic functions*, Pacific J. Math. 41 (1972), p. 81–87.

[11] Ch. Fefermann, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. 26 (1974), p. 1–65.

[12] B. A. Fuks, *Theory of analytic functions of several complex variables*, OGIZ, Moscow 1948.

[13] J. Hadamard, *Sur le principe de Dirichlet*, Bull. Soc. Math. France 34 (1906), p. 135–139.

[14] V. Havin, *Approximation in mean by analytic functions*, Dokl. Akad. Nauk 178 (1968), p. 1025–1029.

[15] L. Hedberg, *Non-linear potentials and approximation in the mean by analytic functions*, Math. Z. 129 (1972), p. 299–319.

[16] L. Helms, *Introduction to potential theory*, Wiley Interscience, New York 1969.

[17] L. Hormander, *An introduction to complex analysis in several variables*, Van Nostrand, 1966.

[18] Hua Lo-Keng, *Harmonic analysis of several complex variables in the classical domains*, Science Press, Peking 1958; Engl. transl., Transl. Math. Monographs 6, Amer. Math. Soc. Providence, R. I., 1963.

[19] T. B. Iwiński and M. Skwarczyński, *The convergence of Bergman functions for a decreasing sequence of domains*, Z. Ciesielski and J. Musielak eds., *Approximation theory*, p. 117–120, P.W.N., Warszawa 1975.

[20] N. Kerzman, *The Bergman kernel function. Differentiability at the boundary*, Math. Ann. 195 (1972), p. 149–158.

[21] S. Kobayashi, *Geometry of bounded domains*, Trans. Amer. Math. Soc. 92 (1959), p. 267–290.

[22] A. Lichnerowicz, *Varietes complexes et tenseur de Bergman*, Ann. Inst. Fourier 15 (1965), p. 345–408.

[23] Lu Qi-keng, *On Kaehler manifolds with constant curvature*, Acta Math. Sinica 16 (1966), p. 269–281.

[24] Г. А. Маргулис, *Соответствие границ при биголоморфных отображениях многомерных областей*, Тезисы докладов Всесоюзной Конференции по теории функций, Харков (1971), p. 137–138.

[25] А. И. Маркушевич, *Теория аналитических функций*, vol. I and II, Наука, Москва 1967.

[26] S. Matsuura, *On the Lu Qi-keng conjecture and the Bergman representative domains*, Pacific J. Math. 49 (1973), p. 407–416.

[27] I. Ramadanov, *Sur une propriété de la fonction de Bergman*, Comt. Rend. Acad. Bulg. Sci. 20 (1967), p. 759–762.

[28] P. L. Rosenthal, *On the zeros of the Bergman function in doubly-connected domains*, Proc. Amer. Math. Soc. 21 (1969), p. 33–35.

[29] S. Sinanian, *Approximation in the mean by analytic functions and polynomials on the complex plane*, Mat. Sbor. 69 (1966), p. 111.

[30] M. Skwarczyński, *The Bergman function and semiconformal mappings*, Thesis, Warsaw University, 1969.

[31] — *Bergman function and conformal mappings in several complex variables*, Bull. Acad. Polon. Sci. 17 (1969), p. 139.

[32] — *The invariant distance in the theory of pseudoconformal transformations and the Lu Qi-keng conjecture*, Proc. Amer. Math. Soc. 22 (1969), p. 305-310.

[33] — *The Bergman function and semiconformal mappings*, Bull. Acad. Polon. Sci. 22 (1974), p. 667–673.

[34] — *A new notion of boundary in the theory of several complex variables*, ibidem 24 (1976), p. 327–330.

[35] G. Springer, *Pseudo-conformal transformations onto circular domains*, Duke Math. J. 18 (1951), p. 411–424.

[36] N. Suita and A. Yamada, *On the Lu Qi-keng conjecture*, Proc. Amer. Math. Soc. 59 (1976), p. 222–224.

[37] N. Vormoor, *Topologische Fortsetzung biholomorpher Funktionen auf dem Rande bei beschränkten streng-pseudo-konvexen Gebieten im C^n mit C^∞-Rand*, Math. Ann. 204 (1973), p. 239–269.

[38] H. Welke, *Über die analytischen Abbildungen von Kreiskörpern und Hartogschen Bereichen*, ibidem 103 (1930), p. 437–449

[39] K. Zarankiewicz, *Über ein numerisches Verfahren zur konformen Abbildung zweifach zusammenhangender Gebiete*, Zeitschr. Angew. Math. Mech. 14 (1934), p. 97–104.

[40] S. Zaremba, *Sur le calcul numérique des fonctions demandées dans le problème de Dirichlet et le problème hydrodynamique*, Bull. Internat. Acad. Sci. Cracovie 1 (1909), p. 125–195.

[41] Б. Зиновьев, *О воспроизводящих ядрах для кратнокруговых областей голоморфности*, Сиб. мат. ж. 15 (1974), p. 35–48.