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Structure properties of D-R **spaces**

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CONTENTS

Introduction

A *D-R* space is, roughly speaking, a vector space X together with a pair (D, R) of endomorphisms such that $DR = I$.

The aim of this dissertation is not to extend the theory in the direction of the solvability of equations with such operators, but to analyse the structure of the space in its relation to the pair (D, R).

The results obtained show, for instance, in the case of the operator $D = d/dt$ relations between the space C^∞, the space of arbitrary differentiable functions (of one variable) and spaces of solutions of linear differential equations with scalar coefficients. Limitation of space make it quite impossible to give all the results here, so we shall concentrate on such topics as the general algebraic structure of a *D-R* space and its homomorphic representation in a suitable sequence space, abstract Taylor expansion in relation to the notion of sequential convergence, and decomposition of a *D-R* space into *D-R* subspaces and the extension of a *D-R* space to a space where the operator $I - tR$ is invertible for all scalars t.

I am grateful to Professor D. Przeworska-Rolewicz for the constant encouragement I got from her while writing this dissertation — an outcome of two years' work. Thanks must also be extended to the Institute of Mathematics of the Polish Academy of Sciences and to Deutscher Akademischer Austauschdienst, who gave me the opportunity to present my dissertation here in Warszawa.

Notations

We denote vector spaces over a field F by X and Y and their elements by x and y. The linear span of a subset C of X is written $\langle C \rangle$ and the trivial vector space is denoted by (0).

The field $\mathscr{F}$ of scalars is supposed to be algebraically closed. $\mathscr{F}[t]$ denotes the ring of polynomials over $\mathscr{F}$ and $\mathscr{F}(t)$ the field of rational functions.

Id or I is used to denote the identical mapping and i denotes (in

general) an isomorphism or monomorphism of structures (i.e. vector spaces, modules, algebras).

The terms *linear operator* and *homomorphism* are used synonymously (in relation to a given structure).

Let $f: X \to Y$ be a mapping. $f^{-1}(y)$ denotes the inverse image of y under f, i.e. $f^{-1}(y) = \{x \in X \mid f(x) = y\}$.

The end of a proof (or absence of a proof) is marked with □ in the case of a lemma or intermediate result, and with ■ if it concerns a theorem or proposition.

The text is divided into sections, subsections, and sub-subsections. To avoid excessive numeration and notation the numbering of theorems, statements, formulas, etc., is made with reference to sections and the following symbols and abbreviations are used:

Number of formula (n); number of section (m.); Theorem (T); Proposition (P); Corollary (C); Lemma (L); Definition (D); Remark (R); Example (E). Thus (T3C2) means Corollary 2 of Theorem 3 of the current section. For cross-references outside the current section, the number of the section referred to is written first, thus (7-21) means formula (21) of § 7. All other notations are standard or introduced in the course of the text.

§ 1. Preliminaries

In this section we give the basic definitions, notions and notations necessary for an analysis of D-R spaces. There will be very few results but mainly an adaptation of the language of linear algebra to special requirements via appropriate definitions. In general, this will be denoted by the prefix D-R. So we shall speak of D-R subspaces, D-R homomorphisms, etc. We begin with the definition of right invertible operators, the basis of all that follows.

1. Right invertible operators. Let X be an arbitrary vector space over an (algebraically closed) field $\mathscr{F}$. We consider linear operators (or homomorphisms) defined on a linear subspace $\mathscr{D}_A$ of X (called the *domain of A*), which map $\mathscr{D}_A$ into X. We denote by $L(X)$ the collection of all such operators and by $L_0(X)$ those operators where $\mathscr{D}_A = X$. Finally, the *kernel* of $A \in L(X)$ will be denoted by Z_A, i.e. $Z_A = \{x \in \mathscr{D}_A \mid A(x) = 0\}$.

DEFINITION 1 (Right invertible operator).

(a) An operator $D \in L(X)$ is said to be *right invertible* if there is an operator $R \in L(X)$, called a *right inverse of D*, such that

(i) $R(X) \subset \mathscr{D}_D$ and $\mathscr{D}_R = X$,

(ii) $DR = I$ on X,   i.e. $DR(x) = x$, $x \in X$.

The set of all right invertible operators of $L(X)$ will be denoted by $R(X)$.

(b) An operator $A \in L_0(X)$ is called a *pre-Volterra operator* if A has *no* eigenvalues and a *Volterra operator* if I-tA is invertible for all $t \in \mathscr{F}$.

(c) An operator $D \in R(X)$ is called *V-invertible* if D has a right inverse R which is a Volterra operator. We then call R a *V-right inverse of* D and the set of all *V*-right invertible operators is denoted by V-$R(X)$.

2. D-R vector spaces.

DEFINITION 2 (*D-R* vector space).

(a) A vector space X together with a pair $D, R \in L(X)$ is called a *D-R vector space* (or *D-R space*) if $D \in R(X)$ and $DR = I$. For brevity we shall very often write (X, D, R); this notation is convenient in order to distinguish between different *D-R* spaces.

(b) (X, D, R) is called a *D-R pre-Volterra space* if R is a pre-Volterra operator and, accordingly, *D-R Volterra space* if R is a Volterra operator.

The chosen name *D-R* space will be justified by the fact that the pair (D, R) gives the underlying space X a very particular structure.

We note down two simple results which immediately follow from (D1a):

(1) $DR = I$ implies $D: \mathscr{D}_D \to X$ surjective and $R: X \to X$ injective.

(2) $D^n R^n = I$ on X, $\forall n$.

3. Basic types of D-R spaces. For a given *D-R* space X we have to consider some cases due to the alternative $\mathscr{D}_D = X$ or $\mathscr{D}_D \subsetneqq X$:

(I) $\mathscr{D}_D = X$ $\begin{cases}\text{(a) } Z_D = (0), & \text{this implies } D = R^{-1} \text{ on } X \text{ by (1)}, \\ \text{(b) } Z_D \neq (0),\end{cases}$

(II) $\mathscr{D}_D \subsetneqq X$ $\begin{cases}\text{(a) } Z_D = (0), \\ \text{(b) } Z_D \neq (0).\end{cases}$

Case I(a) is not interesting in general, with the exception of certain proper subspaces of X to be discussed later in (2-3.).

For the remaining cases we have $R(X) \subsetneqq X$ and we shall call such spaces *true D-R spaces*. They cannot be of finite dimension as is shown by

PROPOSITION 1. X *is a true D-R space only if* $\dim (X) = \infty$.

Proof. Dim $(X) = n$ and $DR = I \overset{(1)}{\Rightarrow} R$ surjective, hence $R(X) = X$. ∎

Remark 1. If $\dim (X) = \infty$, X can be made a *D-R* space in many ways. We simply need an injective but not surjective operator R (which always exists) and can then construct a suitable D with $DR = I$.

In the sequel we deal almost exclusively with true *D-R* spaces, hence the spaces under consideration will always be true if not otherwise stated.

3.1. Examples.

I(a): Any vector space with a pair of mutually inverse linear operators (on X).

I(b): $X = C_\infty [0, 1]$ with $D := d/dt$ and R defined by $R(x)(t)$

$$= \int_0^t x(s)\, ds; \quad t \in [0, 1].$$

The space of distributions with differentiation operator D given by $D(F) = F'$, where $(F', \phi) = -(F, \phi')$. A suitable integration operator R is defined by $(R(F), \phi) := -(F(t), \int_{-\infty}^t P\phi(s)\, ds)$ with $F \in \vartheta'$, $\phi \in \vartheta$ and P a given (continuous) projection of ϑ onto ϑ_0.

II(a): $X = \{x \in C_0 [0, 1] \mid x(0) = 0\}$. D and R as in I(b) but d/dt acting on $\mathscr{D}_D = X \cap C_1 [0, 1]$.

II(b): $X = C_0 [0, 1]$, D and R as in I(b), $\mathscr{D}_{d/dt} \doteq C_1 [0, 1]$.

4. Subspaces.

DEFINITION 3. Let X be a D-R space and U a subspace of X.

(a) U is called a D-R *subspace* of X if there is a subspace U_1 such that

$$(3) \qquad U_1 \subset \mathscr{D}_D, \quad R(U) \subset U_1 \subset U, \quad D(U_1) \subset U$$

and write $(U, D_{|U_1}, R_{|U})$. If, additionally, $U_1 = U$, then U is said to be a D-R *invariant subspace* (of X).

(b) Let V be an arbitrary vector space and $T \in L(X)$. A subspace U of X is called T-*invariant* (or *invariant under* T) if $T(U) \subset U$, and T-*hyperinvariant* if $T(U) \subset U$ and $T(v) \in U \Rightarrow v \in U$.

We notice that, according to (a), $(U, D_{|U_1}, R_{|U})$ is a D-R space in the restriction to U and furthermore

(4)   if U is a D-R invariant subspace of X, then D and R are mutually
       inverse on U iff $R(U) = U$ iff $Z_D \cap U = (0)$.

DEFINITION 4. A D-R space X is said to be D-R *decomposable* if there are nontrivial D-R subspaces U and V of X such that $X = U \oplus V$. Accordingly, such a decomposition is said to be a D-R *decomposition*.

It may well happen that in a D-R decomposition of a true D-R space one of the factors is not true. Later on in (2-3.) we shall see that there are in fact many D-R spaces which contain a canonical non-true D-R subspace different from (0).

5. Homomorphisms.

DEFINITION 5. A mapping $f: (X, D, R) \to (X', D', R')$ is called a D-R *homomorphism* (*isomorphism*) if $f: X \to X'$ is a homomorphism (isomorphism) such that

$$(5) \qquad fR = R'f \text{ and } fD = D'f \quad \text{with} \quad f(\mathscr{D}_D) \subset \mathscr{D}_{D'}.$$

PROPOSITION 2 (Properties of D-R homomorphisms). *Let* (X, D, R) *and* (X', D', R') *be given*.

(i) *Let U be a D- or R-invariant subspace of X and f a D-R homomorphism; then $f(U)$ is a D'- or R'-invariant subspace of X'.*

(ii) *The D-R homomorphic image of a D-R subspace is a D'-R'-subspace. The kernel and the range of a D-R homomorphism are D-R (resp. D'-R') subspaces of X (resp. X').*

(iii) *The D-R homomorphisms of (X, D, R) into (X', D', R') form a subspace of $L_0(X)$.*

(iv) *The composition of D-R homomorphisms (isomorphisms) is a D-R homomorphism (isomorphism).*

Proof. The condition $f(\mathcal{D}_D) \subset \mathcal{D}_{D'}$ guarantees that $fD = D'f$ is well defined on $\mathcal{D}_D$. With the exception of (ii) all assertions follow easily from the corresponding definitions.

(ii) Let U be a D-R subspace of X. According to (3) we have

$$\left(U_1 \subset \mathcal{D}_D, \quad R(U) \subset U_1 \subset U, \quad D(U_1) \subset U\right)$$

$$\Rightarrow \left(f(U_1) \subset f(\mathcal{D}_D), \quad f(R(U)) \subset f(U_1) \subset f(U), \quad f(D(U_1)) \subset f(U)\right)$$

$$\overset{(5)}{\Rightarrow} \left(f(U_1) \subset \mathcal{D}_{D'}, \quad R'(f(U)) \subset f(U_1) \subset f(U), \quad D'(f(U_1)) \subset f(U)\right).$$

Hence $f(U)$ is a D'-R' subspace. (X, D, R) is its own D-R subspace, and so $f(X)$ is a D'-R' subspace (of X'). For the kernel of f we obtain by (5) with $U_1 = Z_f \cap \mathcal{D}_D$

$$U_1 \subset \mathcal{D}_D, \quad R(Z_f) \subset U_1 \subset Z_f, \quad D(U_1) \subset Z_f. \quad \blacksquare$$

It may be added that for a given D-R homomorphism f the inverse image of a D'-R' subspace of X' is also a D-R subspace of X.

Further properties of D-R subspaces and D-R homomorphisms will be studied later. It remains to show a sum and intersection property.

PROPOSITION 3 (Sum and intersection of D-R subspaces). *Let $(U, D_{|U_1}, R_{|U})$ and $(W, D_{|W_1}, R_{|W})$ be subspaces of (X, D, R). Then*

(sum) $\qquad (U, D_{|U_1}, R_{|U}) + (W, D_{|W_1}, R_{|W}) = (U + W, D_{|U_1 + W_1}, R_{|U + W}),$

(intersection) $\qquad (U, D_{|U_1}, R_{|U}) \cap (W, D_{|W_1}, R_{|W}) = (UW, D_{|U_1 \cap W_1}, R_{|U \cap W}).$

The proof is quite obvious and the result remains true for arbitrary intersections and finite sums. $\blacksquare$

The properties of D-R homomorphy give rise to a *isomorphism principle* for D-R spaces: D-R isomorphic spaces are considered $(D$-$R)$ equivalent and they are said to be *D-R compatible.*

DEFINITION 6. (X', D', R') is called a *D-R extension* of (X, D, R) if there is a D-R monomorphism $i\colon X \to X'$

Equivalently, (X, D, R) is said to be *D-R embedded* in (X', D', R') and the monomorphism i is called a *D-R embedding* of (X, D, R) in (X', D', R').

5.1. Quotient spaces and homomorphisms. Let X be a D-R space and U a D-R invariant subspace of X. The *quotient space* X/U can be made a D-R space in a natural way via the induced quotient mappings

(6) $\bar{D}(x+U) := D(x)+U$ and $\bar{R}(x+U) := R(x)+U,$ $x+U \in X/U.$

This is quite obvious as well as the next proposition, which sums up what can be expected.

PROPOSITION 4. *Let U be a D-R invariant subspace of (X,D,R). With the induced mappings $\bar{D}$ and $\bar{R}$ of* (6) *the quotient space X/U is a $\bar{D}$-$\bar{R}$ space and the canonical projection*

$$P\colon (X, D, R) \to (X/U, \bar{D}, \bar{R})$$

is a D-R homomorphism. ∎

DEFINITION 7. The D-R space described in the preceding proposition is called a *D-R quotient space.*

We finally show that the well-known splitting of a homomorphism into a canonical projection and an isomorphism has a corresponding D-R version if the kernel of the D-R homomorphism is contained in the domain of D.

PROPOSITION 5. *Let $f\colon (X, D, R) \to (X', D', R')$ be a D-R homomorphis·n with $Z_f \subset \mathcal{D}_D$. Then f can be decomposed canonically into a D-R homomorphism as shown in the diagram:*

(7)

Proof. For the proof we use the following general lemma:

LEMMA 1 (A general result on homomorphisms). *Let V_i and V_i' be vector spaces, $f_i \in L_0(V_i, V_i')$, $i = 1, 2$, $L \in L_0(V_1, V_2)$ and $L' \in L_0(V_1', V_2')$ such that $f_2 L = L' f_1$. Then the following diagram of homomorphisms commutes:*

(8)

where $L(v_1 + Z_{f_1}) := L(v_1) + Z_{f_2}$, P_i are the canonical projections and j_i the isomorphisms given by $j_i(v_i + Z_{f_i}) := f_i(v_i)$ for $i = 1, 2$.

Proof. L_1. The "wings" of (8) are given by the elementary homomorphism theorem. The commutativity of the upper part is obvious. For the lower part we have

$$V_1/Z_{f_1} \ni v_1 + Z_{f_1} \xrightarrow{L} L(v_1) + Z_{f_2} \xrightarrow{i_2} f_2 L(v_1)$$

$$= L'f_1(v_1) \xleftarrow{L'} f_1(v_1) \xleftarrow{i_1} v_1 + Z_{f_1} \in V_1/Z_{f_1}. \quad \Box$$

Hence, if f is a D-R homomorphism, we put in (8) $f_1 = f = f_2$ and specialize for R in the following way $V_1 = V_2 = X$, $V_2 = V_2' = X'$, $L = R$ and $L' = R'$. For D we choose $V_1 = \mathscr{D}_D$, $V_1' = \mathscr{D}_{D'}$, $V_2 = X, V_2' = X', L = D$ and $L' = D'$. Thus we see that $D'f = D'ip = i\bar{D}p = ipD = fD$ and the same is true for R. ∎

§ 2. The general Taylor theorem

This section is mainly devoted to the so-called *general Taylor theorem* which is essential for the whole theory of D-R spaces. It makes evident a strong relation between right invertible operators, their right inverses and the *structure of the underlaying space*. The theorem can be found already in [9] and [2] as a generalization of an abstract Taylor formula given in [8], but it seems to be quite new in the algebraic aspects to be presented here.

We begin with an elementary version of the theorem, which already shows some of the structure properties of the spaces we are going to deal with.

1. The elementary Taylor theorem.

THEOREM 1 (The elementary Taylor theorem). *Let $D \in L(X)$ be right invertible and R a right inverse of D. With $F := I - RD$ we have on $\mathscr{D}_D$*

(i) $I = F + RD$.

(ii) $F + RD$ *is a direct sum of projections acting on $\mathscr{D}_D$.*

(iii) $\mathscr{D}_D = Z_D \oplus R(X)$ *where $F(\mathscr{D}_D) = Z_D$ and $RD(\mathscr{D}_D) = R(X)$.*

(iv) $DF = 0$ *on $\mathscr{D}_D$, $FR = 0$ on X and $Z_D \cap Z_F = (0)$.*

The main part of the proof can be found in [8], (2) and what remains is very easy to check. ∎

Of special importance is the operator F, as we shall see in the next section. Following [8], we call F an *initial operator for D corresponding to R*. It is shown in [8] that if F is defined on X by

$$F^2 = F, \quad F(X) = Z_D, \quad FR = 0,$$

this is necessary and sufficient for F to satisfy the identity

$$F = I - RD \quad \text{on } \mathscr{D}_D.$$

Remark 1. The name initial operator is due to the fact that by means of F abstract initial value problems can be formulated; thus F plays an important role in abstract operator theory, as can be seen in e.g. [1], [8].

As an immediate consequence of the elementary Taylor theorem we obtain a characterization of right invertible operators which are invertible. Moreover, the relations between F, Z_D, D, etc. become more obvious.

PROPOSITION 1. *Let* $D \in R(X)$ *with the right inverse* R. D *is invertible iff the following equivalent assertions hold*: (i) $F = I - RD = 0$ *on* $\mathscr{D}_D$, (ii) D *is injective*, (iii) $Z_D = (0)$, (iv) $\mathscr{D}_D = R(X)$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. (iii) $\Rightarrow$ (iv) by $\mathscr{D}_D = Z_D \oplus R(X)$, and (iv) $\Rightarrow$ (i) as $F(\mathscr{D}_D) = FR(X) = (0)$. ∎

So, if X is a true D-R space, i.e. $R(X) \subsetneqq X$, then we have for invertible D:

$$DR = I \text{ on } X,$$

$$RD = I \text{ precisely on } \mathscr{D}_D = R(X),$$

as shown in the commutative diagram of isomorphisms.

EXAMPLE 1 (Invertible operator D with $\mathscr{D}_D \subsetneqq X$). Let

$$X = \{x \in C_0[0,1] \mid x(0) = 0\}, \quad D := \frac{d}{dt}, \quad R(x(t)) := \int_0^t x(s)\,ds.$$

We choose as domain $\mathscr{D}_D := (X \cap C_1[0,1]) \subsetneqq X$. Then $DR = I$ on X, $Z_D = (0)$ and consequently $DR = RD = I$ only for the elements of $\mathscr{D}_D$.

As another application of (T1) we show that in (1-5) the condition $f(\mathscr{D}_D) \subset \mathscr{D}_{D'}$ can be replaced by $f(Z_D) \subset Z_{D'}$. Namely, we have

(1) $f(Z_D) \subset Z_{D'}$ iff $f(\mathscr{D}_D) \subset \mathscr{D}_{D'}$.

Indeed, one direction is trivial and for the other it is clear that

$$f(\mathscr{D}_D) = f(R(X) \oplus Z_D) = f(R(X)) + f(Z_{D'})$$
$$= R'(f(X)) \oplus f(Z_D) \subset R'(X') \oplus Z_D = \mathscr{D}_{D'}.$$

1.1. Bands of subspaces. In (1-D3) we defined a D-R subspace as a triple $(U, D_{|U_1}, R_{|U})$ where $U_1 \subset \mathscr{D}_D, R(U) \subset U_1 \subset U, D(U_1) \subset U$. Clearly, U is a D-R space in the restriction to U, so, by (T1) the domain U_1 has the representation

(2) $U_1 = R(U) \oplus Z_{D_{|U_1}} = R(U) \oplus (Z_D \cap U_1).$

Obviously, there is a certain variety of possible domains U_1 of D with respect to U. To determine this variety we give

DEFINITION 1. Let U be an R-invariant subspace of the D-R space X. A subspace U_1 of X which satisfies the condition

$$(3) \qquad U_1 \subset \mathscr{D}_D, \quad R(U) \subset U_1 \subset U, \quad D(U_1) \subset U$$

is called an *admissible domain of D in U*. According to (1-D3), every admissible domain U_1 determines a D-R subspace $(U, D_{|U_1}, R_{|U})$ and we call the collection of all such D-R subspaces the U_R-*band (of D-R subspaces)*.

Clearly, $R(U)$ satisfies (3) and (1-3), and so the U_R-band contains at least the space $(U, D_{|R(U)}, R_U)$ and the above definition makes sense. The next proposition shows in which form the admisible domains (and hence the elements of the U_R-band) can vary between certain extremes.

PROPOSITION 2. *Let U be an R-invariant subspace of the D-R space X. The admissible domains U_1 of D in U are given by*

$$R(U) \subset U_1 \subset \left(R(U) \oplus (Z_D \cap U)\right) \subset (\mathscr{D}_D \cap U) \subset U$$

in such a way that

(i) $R(U)$ *is the lower bound and* $R(U) \oplus (Z_D \cap U)$ *the upper bound for the lattice of admissible domains.*

(ii) $Z_D \cap U_1 = Z_D \cap U_2$ *iff* $(U, D_{|U_1}, R_{|U}) = (U, D_{|U_2}, R_{|U})$.

(iii) $\mathscr{D}_D \cap U = R(U) \oplus (Z_D \cap U)$ *iff* $(R(X) \oplus Z_D) \cap U = (R(X) \cap U) \oplus \oplus (Z_D \cap U)$ *and* $R(X) \cap U = R(U)$.

$R(U) \oplus (Z_D \cap U) = U$ *iff* $U \subset \mathscr{D}_D$ *and* $D(U) \subset U$; *hence, in particular,* $R(U) = U$ *iff* $Z_D \cap U = (0)$.

Proof. (i) Clearly, $R(U)$ is the lower bound for the U_1. For $U_1 = R(U) \oplus (Z_D \cap U)$ we have $D(U_1) = U$ and $U_1 \subset \mathscr{D}_D$, and so $R(U) \oplus \oplus (Z_D \cap U)$ is an admissible domain. Let $(U, D_{|U_0}, R_{|U})$ be an arbitrary D-R subspace. By (2) we have $U_0 = R(U) \oplus (Z_D \cap U_0)$ where $U_0 \subset U$. But then $Z_D \cap U_0 \subset Z_D \cap U$; hence U_0 is contained in $R(U) \oplus (Z_D \cap U)$ and thus admissible.

(ii) $(U, D_{|U_1}, R_{|U}) = (U, D_{|U_2}, R_{|U})$ iff $R(U) \oplus (Z_D \cap U_1) = U_1 = U_2 = R(U) \oplus (Z_D \cap U_2)$. We apply F to both sides, which gives, by (T1), $Z_D \cap U_1 = Z_D \cap U_2$.

(iii) We prove only the first part. As $R(U) \subset R(X) \cap U$ we have $\left(R(U) \oplus (Z_D \cap U)\right) \subset \left((R(X) \cap U) \oplus (Z_D \cap U)\right) \subset \left((R(X) \oplus Z_D) \cap U\right)$ and the rest follows easily. ∎

Thus any D-R subspace $(U, D_{|U_0}, R_{|U})$ is an element of the U_R-band determined by the variation of U_1 between the extremes $R(U)$ and

$R(U)\oplus(Z_D \cap U)$. We finally remark that the elements of the main band of a *D-R* space are determined by those subspaces U_1 which satisfy $R(X) \subset U_1 \subset \mathscr{L}_D \subset X$.

2. The general Taylor theorem.

THEOREM 2 (General Taylor theorem). *Let* $(D_i)_i$ *be a family of right invertible operators of the vector space* X, $(R_i)_i$ *a family of right inverses with* $D_i R_i = I$ *on* X, *and* $(F_i = I - R_i D_i)_i$ *the corresponding family of induced initial operators on* $\mathscr{L}_{D_i}$, $i = 1, 2, \ldots$ *With* $D'' := D_n \ldots D_1$, $R'' := R_1 \ldots R_n$, $n \geqslant 1$ *and* $F'' = I - R''D''$ *the following assertions hold on* $\mathscr{L}_{D''} = \mathscr{L}_{D_n \ldots D_1}$.

(i) $I = F_1 + \left(\sum\limits_{i=1}^{n-1} R_1 \ldots R_i F_{i+1} D_i \ldots D_1 \right) + R_i \ldots R_n D_n \ldots D_1$ *is a direct sum of projections defined on* $\mathscr{D}_{D_n \ldots D_1}$.

(ii) *The associated decomposition of* $\mathscr{D}_{D''}$ *is given by*

$$\mathscr{D}_{D''} = Z_{D_1} \oplus R_1 (Z_{D_2}) \oplus \quad \oplus R_1 \ldots R_{n-1} (Z_{D_n}) \oplus R_1 \ldots R_n (X)$$

where

(a) $F_1 (\mathscr{D}_{D''}) = Z_{D_1}$,

(b) $R_1 \ldots R_i F_{i+1} D_i \ldots D_1 (\mathscr{L}_{D''}) = R_1 \ldots R_i (Z_{D_{i+1}})$, $1 \leqslant i \leqslant n-1$,

(c) $R_1 \ldots R_n D_n \ldots D_1 (\mathscr{L}_{D''}) = R_1 \ldots R_n (X)$.

(iii) $F'' = F_1 + \left(\sum\limits_{i=1}^{n-1} R_1 \ldots R_i F_{i+1} D_i \ldots D_1 \right)$ *and*

$$F'' (\mathscr{L}_{D''}) = Z_{D_n \ldots D_1} = Z_{D_1} \oplus \ldots \oplus R_1 \ldots R_{n-1} (Z_{D_n}).$$

Proof. The proof will be accomplished in several steps. As there are complications with the domains of superposed right invertible operators, we state first some lemmas concerning the behaviour of these domains. All the lemmas are stated under the hypothesis of the theorem to be proved.

LEMMA 1. $\mathscr{L}_{D_n \ldots D_1} = Z_{D_n \ldots D_1} \oplus R_1 \ldots R_n (X)$.

Proof L1. $D'' R'' = I$ on X; hence (T1) yields

$$Z_{D''} \oplus R'' (X) = \mathscr{L}_{D''} = Z_{D_n \ldots D_1} \oplus R_1 \ldots R_n (X). \quad \square$$

LEMMA 2. *We have*

$$D_i \ldots D_1 (Z_{D_n \ldots D_1}) = \begin{cases} Z_{D_n \ldots D_{i+1}}, & 1 \leqslant i \leqslant n-1, \\ (0), & i = n. \end{cases}$$

Proof L2. $D_n \ldots D_1 (Z_{D_n \ldots D_1}) = (0)$ obviously, and so let $1 \leqslant i \leqslant n-1$. Now, $x \in Z_{D_n \ldots D_1} \Rightarrow D_n \ldots D_1(x) = 0 \Rightarrow (D_n \ldots D_{i+1})(D_i \ldots D_1)(x) = 0 \Rightarrow D_i \ldots \ldots D_1(x) \in Z_{D_n \ldots D_{i+1}}$, which shows the map

(4) $\qquad\qquad (D_i \ldots D_1) \colon Z_{D_n \ldots D_1} \to Z_{D_n \ldots D_{i+1}}$

to be *into*.

Onto. Let $x \in Z_{D_n \ldots D_{i+1}}$ and define $y := R_1 \ldots R_i(x)$.

$$D_i \ldots D_1(y) = (D_i \ldots D_1)(R_1 \quad R_i)(x) = x.$$

$$D_n \ldots D_1(y) = (D_n \ldots D_{i+1})(D_i \ldots D_1)(R_1 \ldots R_i)(x) = D_n \ldots D_{i+1}(x) = 0$$

and thus y is a preimage of x in (4). $\square$

LEMMA 3. *We have*

$$D_i \ldots D_1(\mathscr{D}_{D_n \ldots D_1}) = \begin{cases} \mathscr{D}_{D_n \ldots D_{i+1}}, & 1 \leqslant i \leqslant n-1, \\ X, & i = n. \end{cases}$$

Proof L3. Let $1 \leqslant i \leqslant n-1$. By (L1) we have

(5) $$\mathscr{D}_{D_n \ldots D_1} = Z_{D_n \ldots D_1} \oplus R_1 \ldots R_n(X).$$

As $\mathscr{L}_{D_n \ldots D_1} \subset \mathscr{L}_{D_i \ldots D_1}$, we can apply $D_i \ldots D_1$ to both sides of (5) and obtain

$$D_i \ldots D_1(\mathscr{L}_{D_n \ldots D_1})$$

$$\overset{\text{L1}}{=} (D_i \ldots D_1)(Z_{D_n \ldots D_1}) + (D_i \ldots D_1)(R_1 \ldots R_i)(R_{i+1} \ldots R_n)(X)$$

$$\overset{\text{L2}}{=} Z_{D_n \ldots D_{i+1}} \oplus R_{i+1} \ldots R_n(X) \overset{\text{L1}}{=} \mathscr{L}_{D_n \ldots D_{i+1}}.$$

(As $D_j R_j = I$ for $i+1 \leqslant j \leqslant n$, we have: $(D_n \ldots D_{i+1})(R_{i+1} \ldots R_n) = I$ on X.)
$i = n$: $(D_n \ldots D_1)(R_1 \ldots R_n) = I$; hence (1-1) implies $D_n \ldots D_1(\mathscr{L}_{D_n \ldots D_1})$
$= X$.

LEMMA 4. $F_{i+1}(\mathscr{L}_{D_n \ldots D_{i+1}}) = Z_{D_{i+1}}, 0 \leqslant i \leqslant n-1$.

Proof L4. $0 \in \mathscr{L}_{D_n \ldots D_{i+2}} \Rightarrow Z_{D_{i+1}} \subset \mathscr{L}_{D_n \ldots D_{i+1}} \subset \mathscr{L}_{D_{i+1}}$. Now we can apply F_{i+1} and obtain by (T1)

$$Z_{D_{i+1}} = F_{i+1}(Z_{D_{i+1}}) \subset F_{i+1}(\mathscr{L}_{D_n \ldots D_{i+1}}) \subset F_{i+1}(\mathscr{L}_{D_{i+1}}) = Z_{D_{i+1}}. \square$$

We can now show

(I) (a) $F_1(\mathscr{L}_{D_n \ldots D_1}) = Z_{D_1}$,

    (b) $R_1 \ldots R_i F_{i+1} D_i \ldots D_1(\mathscr{L}_{D_n \ldots D_1}) = R_1 \ldots R_i(Z_{D_{i+1}})$, $1 \leqslant i \leqslant n-1$,

    (c) $R_1 \ldots R_n D_n \ldots D_1(\mathscr{L}_{D_n \ldots D_1}) = R_1 \ldots R_n(X)$.

Proof (I). (a) by (L4) and (c) by (L3).

(b) $F_{i+1}(D_i \ldots D_1)(\mathscr{L}_{D_n \ldots D_1}) \overset{\text{L3}}{=} F_{i+1}(\mathscr{L}_{D_n \ldots D_{i+1}}) \overset{\text{L4}}{=} Z_{D_{i+1}}$. $\square$

(II) *If we put* $Q_i = R_1 \ldots R_i D_i \ldots D_1, 1 \leqslant i \leqslant n$, *then*

(i) $Q_i: \mathscr{L}_{D_n \ldots D_1} \to \mathscr{L}_{D_n \ldots D_1}$ *is a well defined linear operator.*

(ii) $Q_i Q_j = Q_j Q_i = Q_l$ *on* $\mathscr{L}_{D_n \ldots D_1}$, *where* $l = \max(i, j), 1 \leqslant i, j \leqslant n$.
The Q_i *are projections.*

Proof (II). (i) $R_1 ... R_i D_i ... D_1 (\mathscr{D}_{D_n...D_1}) \overset{\mathrm{LJ}}{=} R_1 ... R_i (\mathscr{D}_{D_n...D_{i+1}}) \subset \mathscr{D}_{D_n...D_1}$.

(ii) By (i) the superposition $Q_i Q_j$ is well defined on $\mathscr{D}_{D_n...D_1}$.

$i \leqslant j.$ $Q_i Q_j = (R_1 ... R_i D_i ... D_1)(R_1 ... R_j D_j ... D_1)$

$= R_1 ... R_i (D_i ... D_1)(R_1 ... R_i) R_{i+1} ... R_j D_j ... D_1$

$= R_1 ... R_j D_j ... D_1 = Q_j.$

$j \leqslant i.$ $Q_i Q_j = R_1 ... R_i D_i ... D_{j+1} (D_j ... D_1)(R_1 ... R_j) D_j ... D_1$

$= R_1 ... R_i D_i ... D_1 = Q_i$

because $(D_i ... D_1)(R_1 ... R_i) = I = (D_j ... D_1)(R_1 ... R_j).$ $\square$

(III) *Putting* $P_0 = F_1$ *and* $P_i = R_1 ... R_i F_{i+1} D_i ... D_1$, $1 \leqslant i \leqslant n-1$,
gives

(i) $P_i = Q_i - Q_{i+1}$, $0 \leqslant i = n-1$, $Q_0 := I$.

(ii) $P_i P_j = \delta_{ij} P_i$, $0 \leqslant i, j \leqslant n-1$.

(iii) $P_i Q_j = Q_j P_i = \begin{cases} P_i, & j \leqslant i, \\ 0, & j > i, \end{cases}$ $1 \leqslant j \leqslant n$ *on* $\mathscr{D}_{D_1...D_n}$.

Proof (III). (i) (I) shows that P_i is defined on $\mathscr{D}_{D_n...D_1}$. Furthermore,
$D_i ... D_1 (\mathscr{D}_{D_n...D_1}) = \mathscr{D}_{D_n...D_{i+1}} \subset \mathscr{D}_{D_{i+1}}$. Hence

$Q_i - Q_{i+1} = (R_1 ... R_i)(D_i ... D_1) - (R_1 ... R_i)(R_{i+1} D_{i+1})(D_i ... D_1)$

$= (R_1 ... R_i)(I - R_{i+1} D_{i+1})(D_i ... D_1)$

$= R_1 ... R_i F_{i+1} D_i ... D_1 = P_i$, $1 \leqslant i \leqslant n-1$.

(ii) $P_i P_j = (Q_i - Q_{i+1})(Q_j - Q_{j+1}) = \begin{cases} 0, & i \neq j, \\ P_i, & i = j, \end{cases}$ by (II) (ii).

(iii) $P_i Q_j = (Q_i - Q_{i+1}) Q_j = \begin{cases} Q_i - Q_{i+1} = P_i, & j \leqslant i, \\ Q_j - Q_j = 0, & j > i, \end{cases}$ by (II) (ii)

and, analogously

$Q_j P_i = \begin{cases} P_i, & j \leqslant 1, \\ 0, & j > i. \end{cases}$ $\square$

Now we can put all the pieces together and obtain on $\mathscr{D}_{D_n...D_1}$.

$I = (I - Q_1) + (Q_1 - Q_2) + \quad + (Q_{n-1} - Q_n) + Q_n$

$= P_0 + P_1 + \quad + P_{n-1} + Q_n$

$= F_1 + \sum_{i=1}^{n-1} R_1 ... R_i F_{i+1} D_i ... D_1 + R_1 ... R_n D_n ... D_1$

as a sum of projections defined on $\mathscr{D}_{D_n...D_1}$, which immediately implies the remaining assertions. ∎

Because of its generality this theorem is very flexible, i.e. by special-
isation we can study a great many cases and problems, see e.g. [8].

We mention only one result, which is basic for the next section. For $R = R_i$ and $D = D_i$ we obtain the "simple" Taylor formula

(6) $$I = \sum_{i=0}^{n-1} R^i FD^i \oplus R^n D^n \quad \text{on} \quad \mathscr{D}_{D^n}.$$

§ 3. Structure elements of D-R spaces

NOTE. From now on we refer to an arbitrary but fixed D-R space X.

In the sequel it will turn out that the simple Taylor formula (2-6) is an adequate means to analyse the structure of a D-R space. So we restate this formula (or theorem) in all details. No proofs are needed since we are dealing with a special case of the general theorem (2-T2).

1. The simple Taylor formula.

THEOREM 1. (The simple Taylor formula). *Let* X *be a* D-R *space and* $F = I - RD$ *the induced initial operator. With* $F_n := I - R^n D^n$ *and* $P_i := R^i FD^i, 0 \leqslant i \leqslant n-1$ *the following assertions hold on* $\mathscr{D}_{D^n}$:

(i) $I = \sum_{i=0}^{n-1} P_i + R^n D^n$.

(ii) $P_i P_j \Rightarrow \delta_{ij} P_i$, $(R^n D^n)^2 = R^n D^n$ and $R^n D^n P_i = P_i R^n D^n = 0$.

(iii) $\mathscr{D}_{D^n} = Z_D \oplus \quad \oplus R^i(Z_D) \oplus \quad \oplus R^{n-1}(Z_D) \oplus R^n(X)$.

(iv) $P_i(\mathscr{D}_{D^n}) = R^i(Z_D), 0 \leqslant i \leqslant n-1$; $R^n D^n(\mathscr{D}_{D^n}) = R^n(X)$.

(v) $F_n = I - R^n D^n = \sum_{i=0}^{n-1} P_i$ *is the initial operator induced by* D^n *and* R^n.

$$F_n(\mathscr{D}_{D^n}) = Z_{D^n} = Z_D \oplus \quad \oplus R^i(Z_D) \oplus \quad \oplus R^{n-1}(Z_D).$$

If $\mathscr{D}_D = X$, we have to replace $\mathscr{D}_{D^n}$ by X. To stress more strongly the direct sum property of (i) we shall sometimes write

(1) $$I = \bigoplus_{i=0}^{n-1} P_i \oplus R^n D^n. \quad \blacksquare$$

The next definitions will serve to classify the result of Theorem 1.

DEFINITION 1 (The basic decomposition).

(2) $$\mathscr{D}_{D^n} = Z_D \oplus \quad \oplus R^i(Z_D) \oplus \quad \oplus R^{n-1}(Z_D) \oplus R^n(X)$$

is called the *basic decomposition of the* domain of D^n (or *of the space* X if $\mathscr{D}_D = X$).

Thus every element $x \in \mathscr{D}_{D^n}$ (or X if $\mathscr{D}_D = X$) has a unique representation

(3) $x = z_0 + R(z_1) + \quad + R^i(z_i) + \quad + R^{n-1}(z_{n-1}) + R^n(x'), \quad x' \in X, \ z_i \in Z_D$

where the z_i are uniquely determined by $z_i = FD^i(x) = D^i P_i(x)$, $0 \leqslant i \leqslant n-1$, while $x' = D^n(x)$.

DEFINITION 2 (Basic projection operators of $\mathscr{D}_{D^n}$).

(i) The projection operator P_i of (T1 (ii)) is called the *i-th component projection* for $\mathscr{D}_{D^n}$. It projects $\mathscr{D}_{D^n}$ onto $R^i(Z_D)$; hence $P_i(x)$ is called the *i-th component* of x for $x \in \mathscr{D}_{D^n}$, $i < n$.

(ii) The projection operator $R^n D^n$ is called the *n-th remainder projection*. It projects $\mathscr{D}_{D^n}$ onto $R^n(X)$, and so we call $R^n D^n(x)$ the *n-th remainder* of x for $x \in \mathscr{D}_{D^n}$.

With this terminology every $x \in \mathscr{D}_{D^n}$ can be written uniquely as the sum of its components and the remainder term as

(4) $$x = \sum_{i=0}^{n-1} P_i(x) + R^n D^n(x).$$

2. Distinguished subspaces and subspace chains. There are three "natural" chains of nested subspaces which will serve us to get more information about the structure of a D-R space.

(5) $\begin{array}{llll} X \supset \mathscr{D}_D \supset ... \supset \mathscr{D}_{D^n} \supset & X \supset R(X) \supset & \supset R^n(X) \supset \\ & (0) \subset Z_D \subset & \subset Z_{D^n} \subset \end{array}$

The chains are not independent of each other. They are related, according to (3-T1), by the fact that

$$Z_{D^n} \oplus R^n(X) = \mathscr{D}_{D^n} \quad \forall n.$$

2.1. Canonical subspaces of a D-R space. We are now going to define certain subspaces characteristic for the structure of a D-R space.

DEFINITION 3 (Canonical subspaces). Let X be a D-R space.

(i) $D_l := \bigcap_{i=0}^{\infty} \mathscr{D}_{D^i}$ is called the *space of D^∞-elements of X*.

(ii) $S := \bigcup_{i=0}^{\infty} Z_{D^i}$ is called the *space of finite elements of X*.

(iii) $Q := \bigcap_{i=0}^{\infty} R^i(X)$ is called the *space of singular elements of X*.

All the spaces have very particular properties, as will be seen in the sequel.

2.2. The space D_I.

PROPOSITION 1. D_I is the largest D-R invariant subspace of X and contains S and Q.

Proof. As an intersection of subspaces, D_I clearly is a subspace of X. If U is a D-R invariant subspace, then by (1-D3) U is contained in D_I. But $Z_{D^n} \subset \mathscr{L}_{D^i} \ \forall i \Rightarrow Z_{D^n} \subset D_I \Rightarrow S \subset D_I$ and $Q \subset R^i(X) \subset \mathscr{L}_{D^i} \ \forall i \Rightarrow Q \subset D_I$.

It remains to show that D_I is invariant under D and R. Making use of (5), we obtain

$$D(D_I) = D\Big(\bigcap_{i=0}^{\infty} \mathscr{L}_{D^i}\Big) \subset \bigcap_{i=0}^{\infty} D(\mathscr{L}_{D^i}) \overset{2-L3}{=} \bigcap_{i=0}^{\infty} \mathscr{L}_{D^{i-1}} = D_I,$$

$$R\Big(\bigcap_{i=0}^{\infty} \mathscr{L}_{D^i}\Big) = \bigcap_{i=0}^{\infty} R(\mathscr{L}_{D^i}).$$

But

$$R(\mathscr{L}_{D^i}) \subset \mathscr{L}_{D^{i+1}}, \quad \text{so} \quad \bigcap_{i=0}^{\infty} R(\mathscr{L}_{D^i}) \subset \bigcap_{i=1}^{\infty} \mathscr{L}_{D^i} = D_I. \quad \blacksquare$$

2.3. The space S.

PROPOSITION 2. (i) S is the smallest D-R invariant subspace of X containing Z_D.

(ii) $S = \bigcup_{i=0}^{\infty} Z_D \oplus \dots \oplus R^i(Z_{D^i})$.

(iii) $S = \{x \in X \mid \exists\, n \geqslant 0 : R^n D^n(x) = 0\}$.

Proof. (ii) (T1 (v)). (i) If U is a D-R invariant subspace which contains Z_D, then U contains $R^i(Z_D) \ \forall i$; hence $S \subset U$ by (ii). To show the D-R invariance we observe first that $D(Z_{D^i}) \subset Z_{D^{i-1}}$ and $R(Z_{D^i}) \subset Z_{D^{i+1}}, i \geqslant 1$. Hence

$$D(S) = D\Big(\bigcup_{i=0}^{\infty} Z_{D^i}\Big) = \bigcup_{i=0}^{\infty} D(Z_{D^i}) \subset \bigcup_{i=1}^{\infty} Z_{D^{i-1}} = S, \quad Z_{D^0} := (0), \dots$$

$$R\Big(\bigcup_{i=0}^{\infty} Z_{D^i}\Big) = \bigcup_{i=0}^{\infty} R(Z_{D^i}) \subset \bigcup_{i=1}^{\infty} Z_{D^{i+1}} \subset S.$$

(iii) $x \in S \Leftrightarrow \exists\, n : D^n(x) = 0 \Leftrightarrow R^n D^n(x) = 0. \quad \blacksquare$

<center>2-T2II</center>

As $R^n D^n(x) = 0 \Rightarrow R^m D^m(x) = 0 \ \forall m \geqslant n$, it makes sense to associate with every nontrivial $x \in S$ a number $n(x)$ according to

DEFINITION 4. Let $x \in S$ and $x \neq 0$. The lowest n such that $R^n D^n(x) \neq 0$ but $R^{n+1} D^{n+1}(x) = 0$ is called the degree of x. The element 0 has no degree. Obviously every $x \in S, x \neq 0$, has a finite degree, and so the name

given to S in (D3) is justified. S can be considered as a space of generalized polynomials. Example E1 in (6.) shows the elements of S to be indeed polynomial functions in $t \in [0, 1]$.

2.4. The space Q. This is a most interesting and curious space, as is shown by

PROPOSITION 3. Q can be characterized by the following properties:

(i) $Q = \bigcap_{i=0}^{\infty} R^i(D_I)$.

(ii) $Q = \{x \in X \mid R^n D^n(x) = x, \forall n\}$.

(iii) Q is the largest D-R invariant subspace where D and R are mutually inverse.

In other words, every true D-R space finally contracts to a D-R invariant subspace which is no longer a true D-R space (1-3.), or, equivalently, every right invertible operator D which is not invertible, $(Z_D \neq (0))$, or not defined on the whole space $X, (\mathscr{D}_D \subsetneqq X)$, becomes finally defined, stable and invertible on a certain well-defined remainder space. Clearly this remainder space Q can be trivial, see (4-P2C1), but in general every dimension is possible, as is shown by (6.) E2.

Proof. (ii) $R^i D^i(0) = 0 \, \forall i \Rightarrow \{..\} \neq \emptyset$. $x = R^i D^i(x) \, \forall i \Rightarrow x \in R^i(X) \, \forall i$ $\Rightarrow x \in Q$. For $x \in Q$ we have $x = R^i(x_i) \, \forall i$; hence $R^i D^i(x) = R^i D^i(R^i(x_i))$ $= R^i(x_i) = x \, \forall i$.

(i) $\bigcap_{i=0}^{\infty} R^i(D_I) \subset \bigcap_{i=0}^{\infty} R^i(X)$. To show the other inclusion we use (ii).

$$x = R^i D^i(x) \, \forall i \Rightarrow x \in D_I \Rightarrow x = R^i D^i(x) \in R^i D^i(D_I) = R^i(D_I) \, \forall i$$

by (1-1) and (P1).

(iii) By (D3) we have

$$Q = \bigcap_{i=1}^{\infty} R^i(X); \quad D(Q) = D\left(\bigcap_{i=1}^{\infty} R^i(X)\right) \subset \bigcap_{i=1}^{\infty} D(R^i(X)) = \bigcap_{i=1}^{\infty} R^{i-1}(X) = Q.$$

For R we obtain

$$R\left(\bigcap_{i=0}^{\infty} R^i(X)\right) = \bigcap_{i=0}^{\infty} R^{i+1}(X) = Q;$$

hence Q is D-R invariant. $Z_D \cap Q = (0)$ is easy to see, for $R^i D^i(x) = x$ $\forall i$ on Q and $x \in Z_D$ implies $x = 0$. Thus by (1-4) we have $D = R^{-1}$. Let U be a D-R invariant subspace where D and R are mutually inverse. Then $R^n D^n(u) = u \, \forall n$ and $\forall u \Rightarrow U \subset Q$. ∎

The relations between the D-R subspaces S and Q are restated in

PROPOSITION 4. S and Q have a trivial intersection and $S \oplus Q$ is a D-R invariant subspace of D_I.

Proof. (P2), (P3) and the proof of (P3 (iii)). ∎

Thus we see that, roughly speaking, the D-R space D_I (in the restriction) is built around two D-R invariant subspaces, namely S and Q.

3. Extension of the domain of D. For an analysis of chains (5) it is useful to extend the domain of $D \in R(X)$ to the whole space X (presupposed $\mathscr{L}_D \subsetneqq X$). We do this by means of

DEFINITION 5. A D-R space X is called (D', K)-*extended* (*to* X) if for a subspace K of X and a linear operator $D' \in L_0(X)$ we have

(i) $X = \mathscr{L}_D \oplus K$.

(ii) $D' = D_{|\mathscr{L}_D} \oplus 0_{|K}$, i.e. D is trivially extended over K to X.

The next proposition is quite obvious and states what is to be expected.

PROPOSITION 5. *The* (D', K)-*extension of a* D-R *space* X *is a* D'-R *space with*

$$\mathscr{L}_{D'} = X, \quad Z_{D'} = Z_D \oplus K, \quad D'_{|\mathscr{L}_D} = D.$$

The induced initial operator F' *is given by* $F' = F_{|\mathscr{L}_D} \oplus 0_{|K}$, *and so* $F'_{|\mathscr{L}_D} = F$. *Moreover,* (X, D', R) *is a* D-R *extension of* (X, D, R) *with* $i = \mathrm{id}$. ∎

Trivially a (D', K)-extension is unique iff $\mathscr{L}_D = X$. So if $\mathscr{L}_D \subsetneqq X$ and if there are no other restrictions, the selection of the (isomorphic) complement K is quite arbitrary. Thus any D-R space with $\mathscr{L}_D \subsetneqq X$ can be (D', K) extended in a great many ways.

Remark 1. As a dual form to the (D', K)-extension we can define a (D', K)-*restriction* (*to* $\mathscr{L}_{D'}$) of a D-R space by restricting D to $R(X) \oplus K$ where $K \subset Z_D$. As $\mathscr{L}_{D'} = R(X) \oplus K$, we then have $\mathscr{L}_{D'} \subset \mathscr{L}_D$, etc.; such a restriction will be used in (§ 4).

The next result shows the relations between a (D', K)-extension and the original space.

THEOREM 2. *Let* (X, D', R) *be a* (D', K)-*extension of* (X, D, R). *Then the following relations hold between the two structures*:

(i) $X = K \oplus \ldots \oplus R^{n-1}(K) \oplus \mathscr{L}_{D^n}$, $\forall n$.

(ii) $Z_{D'^n} = K \oplus \quad \oplus R^{n-1}(K) \oplus Z_{D^n}$.

(iii) *Every element* $x \in X$ *can be written in a unique way as*

(6) $\quad x = \sum_{i=0}^{n-1} R^i(k_i) + \sum_{i=0}^{n-1} R^i(z_i) + R^n(\bar{x})$ *with* $\bar{x} = D'^n(x)$; $k_i \in K$, $z_i \in Z_D$.

Proof. (i) We apply (T1) to (X, D', R) and use (P5). This gives

(7) $\quad X = Z_{D'} \oplus \ldots \oplus R^{n-1}(Z_{D'}) \oplus R^n(X)$

$\qquad = (Z_D \oplus K) \oplus \quad \oplus R^{n-1}(Z_D \oplus K) \oplus R^n(X)$

$\qquad = (K \oplus \quad \oplus R^{n-1}(K)) \oplus (Z_D \oplus \ldots \oplus R^{n-1}(Z_D)) \oplus R^n(X)$

$\qquad \overset{\mathrm{T1}}{=} (K \oplus \quad \oplus R^{n-1}(K)) \oplus \mathscr{L}_{D^n}$.

To prove (ii) we observe that $Z_{D,n} = Z_D \oplus \quad \oplus R^{n-1}(Z_{D'})$ and proceed as in (i). (iii) follows from (7) together with (3). ∎

We are now able to exploit sufficiently the properties of chains (5) and to specify another chain, determinant for the structure of a D-R space.

4. The structure chain. Chains (5) can be extended by adding the corresponding canonical subspaces D, Q and S. Thus we have

(Ch 1) $X \supset \mathscr{D}_D \supset \quad \supset \mathscr{D}_{D^n} \supset \quad \supset D_I$,

(Ch 2) $X \supset R(X) \supset \quad \supset R^n(X) \supset \quad \supset Q$,

(Ch 3) $(0) \subset Z_D \subset \ldots \subset Z_{D^n} \subset \quad \subset S$.

Although, according to (5), $R^n(X) \oplus Z_{D^n} = \mathscr{D}_{D^n}$, this direct sum property is *not* inherited in general by the "limit" spaces D_I, Q and S. As was mentioned in (P4), we can ensure only

$$Q \oplus S \subset D_I \subset X.$$

It is still necessary to introduce another chain which is bounded too by Q according to (P3(i)), namely

(Ch 4) $D_I \supset R(D_I) \supset \quad \supset R^n(D_I) \supset \quad \supset Q$.

We are now going to determine the corresponding *ratios* of *chain ascent* and *descent*.

Chain 1. We use (T2) and obtain

$$X = K \oplus \ldots \oplus R^{n-1}(K) \oplus \mathscr{D}_{D^n},$$

$$X = K \oplus \ldots \oplus R^{n-1}(K) \oplus R^n(K) \oplus \mathscr{D}_{D^{n+1}}.$$

$\mathscr{D}_{D^{n+1}}$ and $R^n(K)$ are contained in $\mathscr{D}_{D^n}$; hence $R^n(K) \oplus \mathscr{D}_{D^{n+1}} \subset \mathscr{D}_{D^n}$ and by comparing the two expressions for X we obtain as ratio of chain descent

(8) $\mathscr{D}_{D^n} = R^n(K) \oplus \mathscr{D}_{D^{n+1}} . \Rightarrow \mathscr{D}_{D^n}/\mathscr{D}_{D^{n+1}} \cong K$.

Chain 2. By (D',K)-extension X is a D'-R space. By (2-T1) we have

$$X = Z_{D'} \oplus R(X);$$

hence the application of R^n to both sides gives the ratio of chain descent

(9) $R^n(X) = R^n(Z_{D'}) \oplus R^{n+1}(X) \Rightarrow R^n(X)/R^{n+1}(X) \cong Z_{D'}$

where $Z_{D'} = Z_D \oplus K$.

Chain 3. According to (P1), D_I is a D-R space in the restriction, and so by the same reasoning as in (9) we obtain as *ratio of chain descent*

(10) $R^n(D_I) = R^n(Z_D) \oplus R^{n+1}(D_I) \Rightarrow R^n(D_I)/R^{n+1}(D_I) \cong Z_D$.

Chain 4. By comparison of Z_{D^n} and $Z_{D^{n+1}}$ according to (T1) we obtain

(11) $Z_{D^{n+1}} = Z_{D^n} \oplus R^n(Z_D) \Rightarrow Z_{D^{n+1}}/Z_{D^n} \cong Z_D$

as *ratio of chain ascent*.

We remark that, similary to (5), chains 3 and 4 are related by $Z_{D^n} \oplus R^n(D_I) = D_I \ \forall \ n$. $\square$

We combine chains 1 and 4 and sum up the last results in

THEOREM 3 (The *D-R* structure chain). *Every D-R space X gives rise to the chain*

(12) $X \supset \mathcal{D}_D \supset \quad \supset D_I \supset R(D_I) \supset \ldots \supset Q$
                                             U
                   (a) S (b)

With K as an algebraic complement of $\mathcal{D}_D$ *in X we have* (i) $K = (0)$ *iff* $X = \mathcal{D}_D$ *iff* $X = D_I$, (ii) $Z_D = (0)$ *iff* $D_I = Q$, (iii) $S = (0)$ *iff* $Z_D = (0)$.

If K and Z_D *are trivial, then* $X = Q$, *i.e. X is of no interest as a D-R space (see* (1-3.)).

If $\mathcal{D}_D \subsetneqq X$ *or* $Z_D \neq (0)$, *then X is a true D-R space. Parts* (a) *and* (b) *of* (12) *are nontrivial and the corresponding chain descent is given by* (8) *and* (10), *respectively, i.e. we have*

$$\mathcal{D}_{D^n}/\mathcal{D}_{D^{n+1}} \cong K; \quad R^n(D_I)/R^{n+1}(D_I) \cong Z_D.$$

D is invertible on $\mathcal{D}_D$ *(see* (2-P1)) *iff* $\mathcal{D}_D \subsetneqq X$ *and* $Z_D = (0)$. *In this case* D_I *is the largest D-R invariant subspace of X where D and R are mutually inverse.*

Finally, if $\mathcal{D}_D = X$, *then X is a true D-R space iff* $Z_D \neq (0)$ *and this are all possible cases according to* (1-3). $\blacksquare$

5. Components and formal component series. The components $P_i(x)$, see (D2), will serve very well to describe more structure properties of a *D-R* space. They provide a means for defining later on, in (§ 5), the notion of abstract Taylor expansion.

If $\mathcal{D}_D \subsetneqq X$, we can sometimes use a (D', K)-extension to have all $x \in X$ components of any order. The relations between components in (X, D, R) and in a $(D'-K)$-extension (X, D', R) are given by

PROPOSITION 6. *Let* (X, D', R) *be a* (D', K)-*extension of the D-R space X. With the notations of* (P5) *we have*

(13) $x \in \mathcal{D}_{D^n}$ *iff* $P_i'(x) \in \mathcal{D}_{D^n}$,   *whence* $P_i'(x) = P_i(x)$, $0 \leqslant i \leqslant n-1$.

Thus we have in particular

(14) $x \in D_I$ *iff* $P_i'(x) \in D_I$,   *consequently* $P_i'(x) = P_i(x) \ \forall \ i$,

where P'_i and P_i are the component operators, (D2), in (X, D', R) and (X, D, R), respectively.

Proof. In (T2) it was shown that

$$x = \sum_{i=0}^{n-1} R^i(k_i) + \sum_{i=0}^{n-1} R^i(z_i) + R^n(\bar{z})$$

and

$$X = (K \oplus \quad \oplus R^i(K) \oplus \quad \oplus R^{n-1}(K)) \oplus \mathcal{D}_{D^n}.$$

As $R^n(\bar{x}) \in \mathcal{S}_{D^n}$, we have therefore $x \in \mathcal{D}_{D^n}$ iff $\sum_{i=0}^{n-1} R^i(k_i) = 0$ iff $R^i(k_i)$ $= 0$, $0 \leq i \leq n-1$. The component operators P'_i in turn yield $P'_i(x)$ $= R^i(k_i) + R^i(z_i)$, which means $P'_i(x) \in \mathcal{S}_{D^n}$ iff $R^i(k) = 0$, but then $P'_i(x)$ $= P_i(x)$, $0 \leq i \leq n-1$. ∎

DEFINITION 6. A *D-R* space X is said to be *of type D_I* if $\mathcal{D}_D = X$.

In the case of $\mathcal{S}_D \subsetneqq X$ we have at least a properly contained *D-R* space (in the restriction) which is of type D_I, namely $\bigcap_{i=0}^{\infty} \mathcal{D}_{D^i}$ according to (P1).

With this definition we show the usefulness of the concept of components.

THEOREM 4 (Components mod Q). *Let X be a space of type D_I. The elements of X can be characterized mod Q by their components*

$$(15) \qquad\qquad x = x' (\text{mod } Q) \quad \textit{iff} \quad P_i(x) = P_i(x') \ \forall \ i.$$

Proof. We prove the equivalent assertion $x \in Q$ iff $P_i(x) = 0 \ \forall \ i$. By (T1) and (P3) we have $P_i = R^i D^i - R^{i+1} D^{i+1}$ and $x \in Q$ iff $R^i D^i(x) = x \ \forall \ i$. Clearly $x \in Q \Rightarrow P_i(x) = 0 \ \forall \ i$. To see the other direction we use (4) and have $x = \sum_{i=0}^{n-1} P_i(x) + R^n D^n(x) \ \forall \ n$. Hence $P_i(x) = 0 \ \forall \ i$ implies $x \in Q$. ∎

This immediately implies

COROLLARY 1 (Classification of elements in spaces of type D_I).

(i) $x \in S \oplus Q$ *iff* $P_i(x) = 0$ *for* $i \geq n \geq 0, n = n(x)$. *Elements of finite degree*: $x \in S$ *iff additionally* $R^{n+1} D^{n+1}(x) = 0$. *Singular elements*: $x \in Q$ *iff* $P_i(x) = 0 \ \forall \ i$.

(ii) *Elements of infinite degree*: x *has infinitely many nontrivial components $P_i(x)$ iff x is contained in the complement of $S \oplus Q$.*

Proof. (ii) is the negation of the first assertion in (i).

(i) Let $s + q \in S \oplus Q$. $\exists \ n \ \forall \ i \geq n$: $R^i D^i(s) = 0$. So $R^i D^i(s+q) = R^i D^i(q)$ $= q \Rightarrow P_i(s+q) = 0 \ \forall \ i \geq n$. $P_i(x) = 0 \ \forall \ i \geq n \Rightarrow R^i D^i(x) = R^{i+1} D^{i+1}(x)$ $\Rightarrow R^n D^n(x) = R^{n+k} D^{n+k}(x) \ \forall \ k$. Now,

$$R^i D^i (R^n D^n) = \begin{cases} R^n D^n & \text{if} \quad i \leq n, \\ R^i D^i & \text{if} \quad i \geq n, \end{cases}$$

and so $R^i D^i(R^n D^n(x)) = R^n D^n(x) \forall i$; hence $R^n D^n(x) \in Q$. But x can be written as $x = (I - R^n D^n)(x) + R^n D^n(x)$ where $(I - R^n D^n)(x) \in Z_{D^n} \subset S$ by (T1) and (D3). Therefore $x \in S \oplus Q$. The second assertion follows from the last argument and the third one is the proof of Theorem 4. ∎

In an arbitrary D-R space X the elements of S can also be described by

$$x = \sum_{i=0}^{n} P_i(x); \quad n = n(x),$$

as follows from (4) and (T4 (1)). It is natural to go a step further and associate with every element of D_I the formal expression $\sum_{i=0}^{\infty} P_i(x)$ and ask for conditions for x to be "equal" to $\sum_{i=0}^{\infty} P_i(x)$. This problem will be treated later, in (§ 6). At present we can give

DEFINITION 7. Let X be a space of type D_I. The formal expression $\sum_{i=0}^{\infty} P_i(x)$ associated with every $x \in X$,

$$(16) \qquad\qquad x \rightarrow \sum_{i=0}^{\infty} P_i(x),$$

is called the *formal component series of* x.

Theorem (4) makes sure that association (16) is indeed well defined and this series turns out to be basic for the notion of abstract Taylor expansion. At the moment we can only say that

$$x = \bar{x} \pmod{Q} \quad \text{iff} \quad \sum_{i=0}^{\infty} P_i(x) = \sum_{i=0}^{\infty} P_i(\bar{x}).$$

6. Examples.

EXAMPLE 1 ($C_0[0, 1]$ as a D-R space). Let $X = C_0[0, 1]$. With $D := d/dt$ and R defined by $R(x)(t) := \int_0^t x(s)\,ds, t \in [0, 1]$, $C_0[0, 1]$ clearly is a D-R space.

$\mathscr{L}_D = C_1[0, 1]$ and the induced initial operator F is given by $F(x) = x(0)$. The kernel of D, i.e. Z_D, is the space of constant functions. Other interesting subspaces are listed below:

$$S = \left\{ \sum_{i=0}^{n} a_i t^i \,\middle|\, n \in N, a_i \in R \right\}$$

are the polynomials in t;

$$\mathscr{L}_{D^n} = C_n[0, 1] = \left(\bigoplus_{i=0}^{n-1} Rt^i \right) \oplus \{x \in C_n[0, 1] \mid x^{(k)}(0) = 0; \ 0 \leqslant k \leqslant n-1\};$$

$$Z_{D^n} = \left\{ \sum_{i=0}^{n-1} a_i t^i \,\middle|\, a_i \in R \right\}.$$

The Taylor formula on $\mathcal{L}_{D^n}$ is of the form

$$I = \sum_{i=0}^{n-1} \frac{t^i}{i!} F \frac{d^i}{dt^i} + R^n \frac{d^n}{dt^n}$$

and gives just the ordinary finite Taylor expansion for $x(t)$ if applied to $x(t) \in C_n[0, 1]$.

$D_I = C_\infty[0, 1]$. With any $x \in C_\infty[0, 1]$ is associated the formal Taylor series

$$x \rightarrow \sum_{i=0}^{\infty} \frac{t^i}{i!} x^{(i)}(0).$$

$Q = \{x \in C_\infty[0, 1] | \ x^{(k)}(0) = 0, \forall \, k\}$, and so Q consists of functions which behave as, for example, $\exp(-1/t^2)$.

EXAMPLE 2 (Adjunction of singular elements). Let X be a D-R space, W an arbitrary vector space and $f: W \rightarrow W$ an automorphism with $g := f^{-1}$. If we define on the product space $U = X \times W$ the linear transformations

$$D'(x, w) := (D(x), g(w)) \quad \text{and} \quad R'(x, w) := (R(x), f(w)), \quad (x, w) \in U,$$

then X is D-R embedded in the D'-R' space U. The space W consists only of singular elements of U and $W = Q$ in U if X has *no* singular elements.

Remark 2. In (1-D3) we made a distinction between D-R invariant subspaces and D-R subspaces. While D-R invariant subspaces obviously do play an important role, we have somewhat neglected the D-R subspaces and shall do so in the future. All the results which hold for D-R spaces also hold of course for D-R subspaces in the respective restrictions.

As for the behaviour of a D-R space X under a D-R homomorphism $f: (X, D, R) \rightarrow (X', D', R')$, see (1-D3), it is quite easy to show that

$$f(Z_{D^n}) \subset Z_{D'^n}, \quad f(\mathcal{L}_{D^n}) \subset \mathcal{L}_{D'^n},$$

$$f(D_I) \subset D_I', \quad f(S) \subset S', \quad f(Q) \subset Q'$$

Hence the homomorphic image of the structure chain of (X, D, R) is "contained" in the structure chain of the image space (X', D', R').

Remark 3. There are still many open problems concerning the relation between the structure elements of D-R spaces if

(i) D is fixed and R varies over all possible right inverses of D,

(ii) R is fixed and D varies over all right invertible operators which have R as a right inverse and are defined on the same domain.

In the first case we see that Z_D and hence S are fixed while Q varies. In the second case obviously Q is fixed and S varies. This shows again the complementary role of the two spaces Q and S.

§ 4. The D-R homomorphism theorem

D-R spaces and a suitable sequence space to be introduced now are in a very close relation. As we shall see, this space can be viewed as the simplest model of a "good" D-R space. For this reason all the definitions and results, although very elementary, will be explained in some detail.

I. The D-R reference space X_0. Let Z be an arbitrary vector space and let

$$X(Z) := \overset{\infty}{\underset{i=0}{X}} (Z)$$

be the Cartesian product of countably many copies of Z endowed with the vector space structure of the *strong sum*. In other words,

$$X(Z) = \{x \mid x = (z_0, z_1, \ldots, z_i, \ldots) \mid z_i \in Z\},$$

(1)
$$(z_i)_{i \in I} + (w_i)_{i \in I} = (z_i + w_i)_{i \in I},$$
$$I = 0, 1, 2, \ldots$$
$$c(z_i)_{i \in I} = (c z_i)_{i \in I}, \quad c \in \mathfrak{F},$$

$X(Z)$ contains a canonical substructure: the *weak sum* $\Sigma(Z)$, i.e.

(2)
$$\Sigma(Z) = \{(z_i)_{i \in I} \in X(Z) \mid \text{almost all } z_i = 0\}.$$

With $X(Z)$ there are associated *canonical projections* p_j given by

(3)
$$p_j: X(Z) \to Z, \quad (z_i)_{i \in I} \to z_j, \quad \text{so} \quad p_j(X(Z)) = Z;$$

as well as *canonical injections* f; with

(4)
$$f_j: Z \to X(Z), \quad z \to (0, \ldots, 0, \overset{j}{z}, 0, \ldots),$$

$$\text{and so} \quad f_j(Z) = (0, \ldots, 0, \overset{j}{Z}, 0, \ldots) =: Z_j, \quad j = 0, 1, 2, \ldots$$

Obviously $f_j p_j(x) = (0, \ldots, 0, z_j, 0, \ldots)$ and $p_j f_j = I$ on $Z, \forall j$. For this reason

(5)
$$f_j p_j(x) \text{ is called the } j\text{-th component of } x.$$

With (4) the weak sum (2) can be written in a more convenient way

(6)
$$\Sigma = \overset{\infty}{\underset{I=0}{\bigcup}} Z_0 \oplus \quad \oplus Z_I.$$

1.1. $X(Z)$ as a D_0-R_0 space with $\mathscr{D}_{D0} = X(Z)$. It is very easy to make the sequence space $X(Z)$ a D_0-R_0 space. We simply define the following shift operators D_0 and R_0 on $X(Z)$:

$$
\begin{aligned}
D_0(x_0, \ldots, \overset{\downarrow i}{x_i}, \ldots) &:= (x_1, \ldots, \overset{\downarrow i}{x_{i+1}}, \ldots), \\[6pt]
R_0(x_0, \ldots, \overset{\downarrow i}{x_i}, \ldots) &:= (0, x_1, \ldots, \overset{\downarrow i}{x_{i-1}}, \ldots).
\end{aligned}
\tag{7}
$$

Trivially $D_0 R_0 = I$ on $X(Z)$ and the linearity is also clear.

PROPOSITION 1. *The sequence space $X(Z)$ together with the shift operators* (7) D_0 *and* R_0 *is a D_0-R_0 space with* $\mathscr{D}_{D0} = X(Z)$. ∎

DEFINITION 1. The space $X(Z)$ as a D_0-R_0 space is called a *D-R reference space for Z* and is denoted by $X_0(Z)$ or, more briefly, by X_0.

The D_0-R_0 structure of $X(Z)$ enables us to express properties of the sequence space $X(Z)$ in terms of a D-R space and we can see how the general concepts and results of (§ 3) work out for X_0.

PROPOSITION 2. *Let X_0 be a reference space for Z. With the notations of* (4), (5), (3-T1) *and* (3-D2) *we have*

(i) $Z_D = Z_0$ and $R_0^n(Z_{D0}) = Z_n \forall n$; *hence*

$$
Z_{D_0^n} \overset{3\text{-}T1}{=} Z_{D0} \oplus \quad \oplus R_0^{n-1}(Z_{D0}) = (Z_0 \oplus \ldots \oplus Z_n) = (Z, \ldots, Z, \overset{\downarrow n}{0}, \ldots).
$$

(ii) $R_0^n(X_0) = (0, \ldots, 0, \overset{\downarrow n}{Z}, Z, \ldots)$.

(iii) $R_0^n D_0^n - R_0^{n+1} D_0^{n+1} = R_0^n F_0 D_0^n = P_{0,n} = f_n p_n$. $F_0 = I - R_0 D_0$ *is the initial operator for D_0 with respect to R_0.*

The proof is quite obvious. ∎

COROLLARY 1. *For the canonical subspaces S and Q, (see* (3-D3)), *we have in the case of the D_0-R_0 space $X_0(Z)$*

(i) $S_0 = \Sigma(Z) = \bigcup_{n=0}^{\infty} Z_0 \oplus \ldots \oplus Z_n$, *i.e. the finite elements are given by the weak sum.*

(ii) X_0 *has no singular elements, i.e. $Q_0 = (0)$.*

The structure chain (3-T3) *is very simple and looks as follows*

$$
X_0 = (Z, Z, \ldots) \supsetneq (0, Z, Z, \ldots) \supsetneq \ldots \supsetneq (0, \ldots, \overset{\downarrow n-1}{0}, Z, Z, \ldots) \supsetneq \quad \supsetneq (0) = Q_0,
$$

$$
S_0 = (Z, 0, \ldots) \cup \ldots \cup (Z, \ldots, Z, \overset{\downarrow n}{0}, \ldots) \cup \ldots \quad ∎
$$

1.2. The d_0-convergence. In view of the simple D-R structure of X_0, it is convenient to study certain convergence problems first in X_0 before going over to the more complicated case of an arbitrary D-R space, all

the more as every *D-R* space of type D_I with $Z_D \neq (0)$ turns out to be *homomorphically representable* in the reference space $X_0(Z_D)$.

The notion of convergence to be introduced in the sequel is very simple but suitable for treating the problem of *abstract Taylor expansion*, etc., already mentioned in (3-5.). Naturally, such concepts as convergence, etc., call for the wider setting of an adequate topology, but we shall refer to this aspect only in a brief note at the end of this sub-subsection.

DEFINITION 2 (The d_0-convergence in $X_0(Z)$). A sequence x_n of elements of X_0 is d_0-*convergent to* $x \in X_0$ iff

(8) $\forall\, i \geqslant 0 \; \exists\, N(i) \geqslant 0 \;\, \forall\, n \geqslant N(i): \; x - x_n \in R_0^i(X_0).$

x is called the d_0-*limit of* x_n and this is denoted by

$$x_n \xrightarrow{d_0} x \quad \text{or} \quad d_0\text{-}\lim_{n \to \infty}(x_n) = x.$$

Intuitively, this convergence means that the tail of the sequence y_n $= x_n - x$ is swallowed by any $(0, ..., \overset{\downarrow i-1}{0}, Z, ...) = R_0^i(X_0)$. The d_0-limit has all the properties of a "nice" limit.

PROPOSITION 3 (Properties of the d_0-limit).

(i) *The d_0-limit is unique.*

(ii) $x_n \xrightarrow{d_0} x$ *and* $y_n \xrightarrow{d_0} y \Rightarrow x_n + y_n \to x + y$; $x_n \to x \Rightarrow cx_n \to cx$; $c \in \mathscr{F}$. *Hence the d_0-limit is compatible with the linear structure of X_0.*

(iii) *Every element* $x \in X_0$ *can be written as a d_0-limit of elements of S_0:*

$$x_n := (z_0, ..., z_n, 0, ...) \xrightarrow{d_0} (z_0, ..., z_n, z_{n+1}, ...) = x.$$

The proof is very simple and will be omitted. ∎

If there is already convergence, it is convenient to introduce the notion of continuity.

DEFINITION 3 (d_0-continuity). Let $A: X_0 \to X_0$ be an arbitrary mapping. A is called d_0-*continuous* iff A transforms any d_0-convergent sequence into a d_0-convergent sequence, i.e. $x_n \to x \Rightarrow A(x_n) \to A(x)$ for all sequences (x_n).

In the linear case we shall call A a d_0-*homomorphism.*

Note. If there is no confusion, the symbol d_0 will be omitted.

The next proposition shows that the shift operators (7) D_0 and R_0 are d_0-continuous.

PROPOSITION 4. *Let X_0 be a reference space for Z. Then D_0 and R_0 are d_0-continuous.*

Proof. By the linearity of D_0 and R_0 it suffices to show that $x_n \to 0$ implies $D_0(x_n) \to 0$ and $R_0(x_n) \to 0$. So, let $i \geqslant 0$;

$$x_n \in R_0^{i-1}(X_0) \; \forall \, n \geqslant N(i-1) \; \Rightarrow \; R_0(x_n) \in R_0^i(X_0) \; \forall \, n \geqslant N.$$

$$x_n \in R_0^{i+1}(X_0) \; \forall \, n \geqslant N(i+1) \; \Rightarrow \; D_0(x_n) \in D_0 R_0^{i+1}(X_0) = R_0^i(X_0) \; \forall \, n \geqslant N. \; \blacksquare$$

Note. By (4) we have $Z \overset{f_0}{\leftrightarrow} (Z, 0, ...) = Z_{D_0}$, and so we shall identify the element $z \in Z$ with the element $(z, 0, ...) \in Z_{D_0}$.

For the end of this section we will refer to an arbitrary but fixed reference space $X_0(Z)$ if not otherwise stated.

The next result is obvious by (P3) and shows how D_0-R_0 structure and d_0-processes fit together on X_0.

PROPOSITION 5.

(i) $\quad \displaystyle\sum_{i=0}^{n} R_0^i(z_i) = (z_0, ..., z_n, 0, ...) \overset{d_0}{\twoheadrightarrow} (z_0, ..., z_n, z_{n+1}, ...) = \sum_{i=0}^{\infty} R_0^i(z_i)$

for every sequence $(z_i)_{i \in I}$, $I = 0, 1, 2, ...$

(ii) $\quad \displaystyle\sum_{i=0}^{\infty} R_0^i(c_1 z_i + c_2 y_i) = c_1 \sum_{i=0}^{\infty} R_0^i(z_i) + c_2 \sum_{i=0}^{\infty} R_0^i(y_i), \quad z_i, y_i \in Z_{D_0}, \quad c_1, c_2 \in \mathfrak{F},$

(iii) $\quad \displaystyle R_0 \Big(\sum_{i=0}^{\infty} R_0^i(z_i) \Big) = \sum_{i=0}^{\infty} R_0^{i+1}(z_i), \quad D_0 \Big(\sum_{i=0}^{\infty} R_0^i(z_i) \Big) = \sum_{i=0}^{\infty} R_0^i(z_{i+1}). \; \blacksquare$

We are now in a position to introduce in X_0 the notion of abstract Taylor expansion by means of the d_0-convergence. Let $x \in X_0$. Using the simple Taylor formula (3-4), we can write

$$x = \sum_{i=0}^{n-1} P_{0,i}(x) + R_0^n D_0^n(x) \quad \text{with} \quad P_{0,i} = R_0^i F_0 D_0^i \; \forall \, n.$$

Consequently, by the definition of the d_0-convergence

$$x - \sum_{i=0}^{n-1} P_{0,i}(x) = R_0^n D_0^n(x) \overset{d_0}{\twoheadrightarrow} 0 \quad \text{for} \quad n \to \infty.$$

Hence

(9) $\qquad\qquad\qquad x = \displaystyle\sum_{i=0}^{\infty} P_{0,i}(x) \quad$ in the d_0-sense.

So, in the special case of a reference space $X_0(Z)$ the formal component series $\sum P_i(x)$ of (3-5.) not only converges but represents x in X_0 and we are allowed to call (9) the *abstract Taylor expansion of x in X_0* (*with respect to the d_0-convergence*). Similarly, we shall call $\displaystyle\sum_{i=0}^{\infty} R_0^i(z_i)$, $z_i \in Z_{D_0}$ an *abstract power series in X_0*.

THEOREM 1.

(i) *Every element $x \in X_0$ is represented by its abstract Taylor expansion,*

i.e. $x = \sum_{i=0}^{\infty} P_{0,i}(x).$

(ii) *Every abstract power series* $\sum_{i=0}^{\infty} R_0^i(z_i)$ *converges in* X_0 *(in the* d_0-sense) and is its own Taylor expansion, $z_i \in Z_{D_0}$.

(iii) *Let* $x = \sum_{i=0}^{\infty} R_0^i(z_i)$. D_0 *and* R_0 *can be applied termwise, and hence*

$$R_0^k(x) = \sum_{i=0}^{\infty} R_0^{i+k}(z_i), \qquad D_0^k(x) = \sum_{i=0}^{\infty} D_0^k(R_0^i(z_i)) = \sum_{i=0}^{\infty} R_0^i(z_{i+k}).$$

(iv) $I = (\bigoplus_{i=0}^{n-1} P_{0,i}) \oplus R_0^n D_0^n \xrightarrow[\text{pointwise}]{d_0} \bigoplus_{i=0}^{\infty} P_{0,i} = I$ *on* X_0.

Proof. (i) is already proved and (iv) follows from the proof of (i). (iii) follows by (P4).

(ii): $\sum_{i=0}^{\infty} R_0^i(z_i) = (z_0, \dots, z_i, \dots) = x$ and by (i) we have $R_0^i(z_i)$
$= P_{0,i}(x)$. ∎

Although this result is rather trivial, it gives nevertheless an idea how to proceed in the case of arbitrary D-R spaces and what can at most be expected with a similar approach.

1.3. The Volterra property of X_0 and eigenspaces of D_0. We have seen that X_0 indeed answers the requirements of a simply structured D-R space. What is still lacking is whether or not X_0 is a D_0-R_0 Volterra space (1-2.). It is indeed.

THEOREM 2 (Volterra property of X_0). $I - tR_0 \colon X_0 \to X_0$ is a $(d_0$-continuous) isomorphism for $\forall t \in \mathscr{F}$.

Proof. Let $(I - tR_0)(v_0, \dots, v_i, \dots) = (z_0, \dots, z_i, \dots) \in X_0$. Applying $I - tR_0$ to $(v_i)_{i \in J}$, we obtain by comparison

$$(I - tR_0)^{-1} (z_i)_{i \in J} = \left(\sum_{n=0}^{i} z_{i-n} t^n \right)_{i \in J}, \qquad J = 0, 1, 2, \dots$$

The injectivity follows in a similar way.

We finish this subsection with the possible eigenspaces of D_0.

PROPOSITION 6 (Eigenvalues of D_0).

(i) *For $t = 0$ the corresponding eigenspace is Z_{D_0}.*

(ii) *Any $t \neq 0$ is an eigenvalue. The corresponding eigenspaces are*

$$E_{D_0,t} = \left\{ \sum_{i=0}^{\infty} t^i R_0^i(z_0) \mid z_0 \in Z_{D_0} \right\} = (I - tR_0)^{-1}(Z_{D_0}).$$

Proof. (ii): Let $D_0(z_0, ..., z_i, ...) = (tz_0, ..., tz_i, ...)$. By comparison we obtain $z_i = t^i z_0$, $i = 0, 1, 2, ...$ Every element of $E_{D_{0,t}}$ is in fact an eigenvector. The remaining assertion is proved in [8], Theorem 1.2. ∎

Remark 2. If we restrict the domain of D_0 by restricting the kernel $Z_{D_0} = Z_0$ of D_0 to a subspace $K \subset Z_0$, the eigenspaces become accordingly "smaller".

2. $\mathscr{D}_{D_0} \subsetneqq X_0(Z)$. We still have to look at reference spaces X_0 where D_0 is not defined on the whole space X_0. There are two reasons for doing this: first, it is good to have examples of "simple" *D-R* spaces with this property, second we need such spaces as reference spaces in the case of *D-R* spaces X where $\mathscr{D}_D \subsetneqq X$.

Let $X_0(W)$ be a D_0-R_0 space with $\dim(W) \geqslant 1$ and let the vector space W be decomposed into $W = Z \oplus K$. Then $X_0(W)$ can be written as

$$X_0(W) = X_0(Z) \oplus X_0(K), \quad K \neq (0),$$

and is thus D_0-R_0 decomposable into the reference spaces $X_0(Z)$ and $X_0(K)$. According to (2-P2) and (3-R1) we select an element of the main band of $X_0(W)$ which is given by the restriction of D_0 to $U_{0,1} := R_0(X_0(W)) \oplus Z_0$ where $Z_0 = (Z, 0, ...) \subset X_0(Z)$. Let

$$(10) \qquad\qquad D_{0,1} := D_{0|U_{0,1}};$$

then, evidently

$$\mathscr{D}_{D_{0,1}} = (Z, W, W, ...) \quad \text{and} \quad Z_{D_{0,1}} = (Z, 0, ...) = Z_0.$$

By (3-T1) we immediately have

$$\mathscr{D}_{D_{0,1}^n} = Z_{D_{0,1}} \oplus \cdots \oplus R_0^{n-1}(Z_{D_{0,1}}) \oplus R_0^n(X_0(W)) = (Z, ..., Z, \overset{\downarrow n}{W}, W, ...)$$

and consequently

$$D_{I,1} = \bigcap_{i=0}^{\infty} \mathscr{D}_{D_{0,1}} = (Z, ..., Z, ...) = X_0(Z).$$

The structure chain (3-T3) is now as follows (compare with (P2C1)):

$$X_0(W) \subsetneqq (Z, W, ...) \subsetneqq \quad \subsetneqq (Z, ..., Z, \overset{\downarrow n}{W}, ...) \subsetneqq \quad \subsetneqq X_0(Z) = D_{I,1},$$

$$X_0(Z) \subsetneqq (0, Z, ...) \subsetneqq \quad \subsetneqq (0, ..., 0, \overset{\downarrow n}{Z}, ...) \subsetneqq \quad \subsetneqq (0) = Q_{0,1},$$

and the space $S_{0,1}$ of abstract polynomials is given by

$$S_{0,1} = \bigcup_{i=0}^{\infty} (Z, ..., Z, \overset{\downarrow i+1}{0}, ...).$$

We remark that evidently the original operator D_0 is obtained again if we extend the kernel Z_0 of $D_{0,1}$ to $W_0 = \text{kernel}(D_0) = (W, 0, ...)$. If we write the elements of $X_0(W)$ as sequences

$$x = (w_1, ..., w_i, ...) = (z_1 + k_1, ..., z_i + k_i, ...), \quad z_i \in Z, \; k_i \in K, \; Z \oplus K = W,$$

we see that obviously, see (3-P6),

(11)
$$x \in \mathscr{D}_{D_{0,1}^n} \quad \text{iff} \quad x = (z_1, ..., z_{n-1}, \overset{\downarrow n}{z_n + k_n}, ...),$$

$$x \in D_{I,1} \quad \text{iff} \quad x = (z_1, ..., z_i, ...).$$

Remark 3. All the results which are based only on R_0 remain of course correct if we change from D_0 to $D_{0,1}$, but those which depend on D_0 must be reconsidered accordingly: see for instance (R2).

3. The D-R homomorphism theorem. This theorem plays a cen[*]
in the theory of D-R spaces and gives the essential rea*
a reference space; namely it establishes a canonical D-K 'lo*   ,*
representation of any true D-R space X of type D_I in $X_0(Z_D)$. (In this context we remember that a true D-R space means $R(X) \subsetneq X$.) We shall first deal with spaces of the type D_I, i.e. $\mathscr{D}_D = X$ or D_I if $\mathscr{D}_D \subsetneq X$ and suppose $Z_D \neq (0)$. The remaining cases, see (3-T3), will be considered afterwards since the* present some intrinsic difficulties.

THEOREM 3 (The D-R homomorphism theorem, $\mathscr{D}_D = X$). Let X be D-R space of type D_I with $Z_D \neq (0)$. The mapping

(12)
$$\phi: X \to X_0(Z_D)$$

given by

$$\phi(x) := \sum_{i=0}^{\lambda} f_i FD^i(x) = (F(x, ..., FD^i(x), ...)$$

with $f_i(z) = (0, 0, \frac{i}{2}, 0 ...)$, $z \in Z_D$, has the following properties:

(i) With the notations of (1-P5)

$$(X, D, R) \xrightarrow{\quad \varphi \quad} (\varphi(X), D_0, R_0) \subset (X_0, D_0, R_0)$$

$$(X/Q, \bar{D}, \bar{R})$$
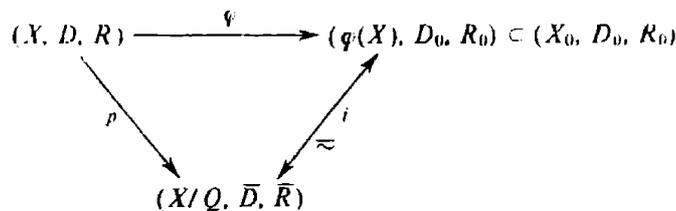
is a commutative diagram of D-R homomorphisms.

(ii) The kernel of ϕ is Q and $\phi^{-1}(R_0^n(X_0)) = R^n(X)$.

(iii) $\phi: S \to S_0$ is a D-R embedding of S in X_0, where

$$\sum_{i=0}^{n} R^i(z_i) \overset{\text{id}}{\leftrightarrow} \sum_{i=0}^{n} f_i(z_i) = \sum_{i=0}^{n} R_0^i(z_i) = (z_0, \ldots, z_i, 0, \ldots)$$

$$Z_D \ni z_i \overset{\text{id}}{\leftrightarrow} (z_i, 0, \ldots) \in Z_0 = Z_{D_0}.$$

(iv) $\phi(X)$ is a D_0-R_0 subspace of X_0 with $(0) \subsetneqq S_0 \subset \phi(X) \subset X_0(Z_D)$.

Proof. (i): According to (1-5) we have to verify that $\phi R = R_0 \phi$ and $\phi D = D_0 \phi$, but we show first that ϕ is a well-defined homomorphism.

$$x = x' \Rightarrow FD^i(x) = FD^i(x') \Rightarrow f_i FD^i(x) = f_i FD^i(x') \Rightarrow \phi(x) = \phi(x').$$

The linearity of ϕ is obvious by the structure of $X_0(Z_D)$ and the linearity of the FD^i. $\square$

By the identification $z = (z, 0, \ldots)$, $z \in Z_D$, we have

$$\phi(x) = \sum_{i=0}^{\infty} f_i FD^i(x) = \sum_{i=0}^{\infty} R_0^i(FD^i(x)),$$

$$D_0 \phi(x) = D_0 \left(\sum_{i=0}^{\infty} R_0^i(FD^i(x)) \right) \overset{\text{T1}}{=} \sum_{i=0}^{\infty} R^{i-1}(FD^i(x)) = (FD(x), \ldots, \overset{i!}{FD^{i+1}}(x), \ldots),$$

$$\phi D(x) = \sum_{i=0}^{\infty} R_0^i \left(FD^i(D(x)) \right) = \sum_{i=0}^{\infty} R_0^i(FD^{i+1}(x)) = (FD(x), \ldots, \overset{i!}{FD^{i+1}}(x), \ldots).$$

Similarly $\phi R = R_0 \phi$.

(ii): $Z_\phi \ni x$ iff $FD^i(x) = 0 \; \forall \, i$ iff $R^i FD^i(x) = P_i(x) = 0 \; \forall \, i \overset{3\text{-T4Cl}}{\Rightarrow} x \in Q$. $\square$

Let $x \in \phi^{-1}(R_0^n(X_0))$. By (3-3), x can be written as $x = \sum_{i=0}^{n-1} R^i(z_i) + R^n(x')$. Then

$$\phi(x) = \sum_{i=0}^{n-1} \phi(R^i(z_i)) + \phi(R^n(x')) = \sum_{i=0}^{n-1} R_0^i(z_i) + R_0^n(\phi(x'))$$

by (i). We have

$$R_0^n(\phi(x')) \in R_0^n(X_0)$$

$$\Rightarrow \sum_{i=0}^{n-1} R_0^i(z_i) \in R_0^n(X_0) \overset{3\text{-T1}}{\Rightarrow} \sum_{i=0}^{n-1} R_0^i(z_i) = 0 \overset{3\text{-3}}{\Rightarrow} z_i = 0, \; 0 \leqslant i \leqslant n-1.$$

Hence $x = R^n(x') \in R^n(X)$.

(iii): Clearly the D-R invariant subspace S is mapped isomorphically onto the D_0-R_0 invariant subspace S_0 because of $S \cap Q = (0)$.

(iv) follows from (1-P2) and (iii). ∎

In fact, we have obtained a rather good representation of X in X_0 with the great advantage of having a more concrete space at our disposal now.

The disadvantage is the loss of information about Q, but on the other hand we see that for good reasons Q is a distinguished subspace.

3.1. Eigenvectors of D and R. The role of Q is still more stressed by a corollary about the possible eigenvectors of R and D. Clearly an eigenvector is an element of D_I, and so we can apply the preceding theorem.

COROLLARY 1.

(i) R has at most eigenvectors in Q.

(ii) D has eigenvectors in Q iff R has eigenvectors in Q.

(iii) Let x be an eigenvector of D to the eigenvalue $t \neq 0$.

If $x \notin Q$, then x has the components $P_i(x) = t^i R^i(z)$ with $z = F(x) \in Z_D$, $i = 0, 1, \dots$

Proof. If x is an eigenvector of D or R, then $\phi(x)$ is an eigenvector of D_0 or R_0 and (i) follows by (T2).

(ii): Q is D-R invariant and $DR = RD = I$ on Q.

(iii): Let $D(x) = tx$. Then $P_i(x) = R^i FD^i(x) = R^i F(t^i x) = t^i R^i(F(x))$. ∎

4. The D-R homomorphism theorem for $\mathscr{D}_D \subsetneqq X$. If we try to formulate the D-R homomorphism theorem in the case of $\mathscr{D}_D \subsetneqq X$, we are confronted with the difficulty that the canonical homomorphism (12) requires spaces of type D_I. If a D-R space is *not* of type D_I, this means according to (1-3.)

(IIa) *General case*: $\mathscr{D}_D \subsetneqq X$ and $Z_D \neq (0)$ and hence $(0) \subsetneqq D_I \subsetneqq X$ by (3-T3) and Theorem 3 can be applied only to the D-R invariant subspace

$$D_I = \bigcap_{i=0}^{\infty} \mathscr{D}_{D^i}.$$

(IIb) *Special case*: $\mathscr{D}_D \subsetneqq X$ and $Z_D = (0)$. This implies $Q = D_I$ by (3-T3) and Theorem 3 becomes trivial (on D_I).

For both cases it would be nice if we could establish a representation of the *entire* space X in a *reference space*, provided the existing representation of the D_I-part remains untouched.

The idea we apply to handle this problem is precisely that of (3-D5) and thus somewhat arbitrary.

As in (3-3.), we choose an arbitrary but fixed algebraic complement K of $\mathscr{D}_D$ in X, i.e.

$$X = \mathscr{D}_D \oplus K,$$

and continue D to X by extending the domain of D according to (3-D5); hence

$$D' := \mathscr{D}_{|\mathscr{D}_D} \oplus O_{|K}.$$

Clearly, D' is now defined on the whole space X and

X is a D'-R space with $Z_{D'} = Z_D \oplus K$ (Z_D can be trivial).

According to (1.1.) we form the reference space

(13) $$\left(X_0\left(Z_D\oplus K\right), D_0', R_0'\right)$$

and can now apply the homomorphism theorem to the (D'-K) extension (X, D', R) of (X, D, R), where

(13a)
$$\phi': (X, D', R) \to \left(X_0\left(Z_{D'}\right), D_0', R_0'\right),$$

$$\phi'(x) = \sum_{i=0}^{\infty} f_i\left(F'D'^i(x)\right), \qquad F' := I - RD'.$$

Now we have to look for a D_0'-R_0' subspace of (13) which can serve as a ϕ'-copy (mod Q) of (X, D, R).

We proceed as in (2.) and choose that D_0'-R_0' subspace of (13) which is given by the restriction of D_0' to a suitable subspace $U_{0,1}$, i.e.

$$D_0 := D_0'|_{U_{0,1}},$$

where $U_{0,1}$ is specified by

$$U_{0,1} := R_0'\left(Z_{D'}\right)\oplus f_0(Z_D), \quad \text{where} \quad f_0(Z_D) = (Z_D, 0, ...).$$

In analogy to (2.) we have

(14) $$\mathcal{D}_{D_0} = (Z_D, Z_D \oplus K, ...); \quad Z_{D_0} = (Z_D, 0, ...)$$

So, by (2-T1) or (3-T1)

(15) $$\mathcal{S}_{D_0^n} = Z_{D_0^n} \oplus R_0'^n\left(X_0\left(Z_{D'}\right)\right) = (Z_D, ..., Z_D, \overset{\downarrow n}{0}, ...)\oplus$$

$$\oplus (0, ..., 0, \overset{\downarrow n}{Z_D \oplus K}, ...) = (Z_D, ..., Z_D, \overset{\downarrow n}{Z_D \oplus K}, ...)$$

and

(16) $$D_{0,I} = (Z_D, ..., Z_D, ...) = X_0\left(Z_D\right).$$

By means of the homomorphism theorem and the Taylor formula (3-T1) we can relate $\mathcal{S}_{D^n}$ to $\mathcal{S}_{D_0^n}$, $n \geqslant 0$, and D_I to $D_{0,I}$. The details are given in

LEMMA 1. With the notations of (3-3.) we have

(i) $P_i'(x) = P_i(x)$, $0 \leqslant i \leqslant n-1$ on $\mathcal{S}_{D^n}$.

(ii) $\phi'^{-1}\left(\mathcal{S}_{D_0^n}\right) = \mathcal{S}_{D^n}$ and $\phi'D^n = D_0^n\phi'$ on $\mathcal{S}_{D^n}$.

(iii) ϕ' is a continuation of ϕ with $Z_\phi = Z_{\phi'}$ and $\phi'^{-1}(D_{0,I}) = D_I$.

Proof L1. (i): (3-13).

(ii): By (3-13) we have $x \in \mathcal{S}_{D^n}$ iff $P_i'(x) \in \mathcal{S}_{D^n}$, $0 \leqslant i \leqslant n-1$. We show first that, equivalently,

(17) $$x \in \mathcal{S}_{D^n} \quad \text{iff} \quad F'D'^i(x) \in Z_D, \quad 0 \leqslant i \leqslant n-1.$$

Indeed, $\quad x \in \mathscr{D}_{D^n} \Rightarrow P_i'(x) \in \mathscr{D}_{D^n} \Rightarrow R^i F' D''(x) \in \mathscr{D}_{D^n} \overset{\text{2-L3}}{\Rightarrow} F' D''(x) \in D^i(\mathscr{D}_{D^n})$

$= \mathscr{D}_{D^{n-i}} \subset \mathscr{D}_D$. But $F' D''^i(x) \in Z_{D'} = Z_D \oplus K$ and $\mathscr{D}_D \cap Z_D \overset{\text{2-T1}}{=} (R(X) \oplus Z_D) \cap$

$\cap (Z_D \oplus K) = Z_D$ by the selection of K. Hence $F' D''^i(x) \in Z_D$, $0 \leqslant i \leqslant n-1$.

$F' D''^i(x) \in Z_D$

$\Rightarrow R^i F' D''^i(x) = P_i'(x) \in R^i(Z_D) \subset \mathscr{D}_{D^n}$, $0 \leqslant i \leqslant n-1 \overset{\text{3-13}}{\Rightarrow} x \in \mathscr{D}_{D^n}$. $\square$

$$x \in \phi'^{-1}(\mathscr{D}_{D'^n}) \text{ iff } \phi'(x) \in \mathscr{D}_{D_0^n} \overset{(15)}{\Rightarrow} (F' D''^i(x))_{i \in J} \in (Z_D, ..., \overset{n-1}{\underset{\downarrow}{Z_D}}, Z_D \oplus K, ...),$$

$$J = 0, 1, ..., \text{ iff } F' D''^i(x) \in Z_D, \, 0 \leqslant i \leqslant n-1 \overset{(17)}{\Rightarrow} x \in \mathscr{D}_{D^n}. \square$$

By the *D-R* homomorphism theorem we have $\phi' D''^n = D_0''^n \phi'$

Let $x \in \mathscr{D}_{D^n}$; then $\phi' D''^n(x) = \phi'(x)$. But $\phi'(x) \in \mathscr{D}_{D_0^n}$, and hence $D_0''^n \phi'(x) = D_0^n \phi'(x)$.

(iii): $\phi'^{-1}(D_{0,I}) = \phi'^{-1}\left(\bigcap_{i=0}^{\infty} \mathscr{D}_{D_0^n} \right) = \bigcap_{i=0}^{\infty} \phi'^{-1}(\mathscr{D}_{D_0^n}) \overset{(ii)}{=} \bigcap_{i=0}^{\infty} \mathscr{D}_{D^i} = D_I$.

By (3-14) we have $x \in D_I$ iff $R^i F' D''^i(x) = P_i'(x) = P_i(x) = R^i F D^i(x) \, \forall \, i$

$\Rightarrow \phi'(x) = (F' D''^i(x))_{i \in J} = (F D^i(x))_{i \in J} = \phi(x)$. $\square$

$\phi'^{-1}(0) \subset \phi'^{-1}(D_{0,I}) = D_I$. But $\phi = \phi'$ on D_I, and hence $Z_\phi = Z_{\phi'} = Q$. ∎

We sum up and state the extended version of Theorem 3

THEOREM 4 (The *D-R* homomorphism theorem for $\mathscr{D}_D \subsetneqq X$). *Let X be a D-R space with $\mathscr{D}_D \subsetneqq X$ and let (X, D', R) be a $(D'-K)$-extension of (X, D, R).*

(i) $\quad (D_I, D, R) \overset{\text{id}}{\to} (\mathscr{D}_{D^n}, D, R) \overset{\text{id}}{\to} (X, D, R) \overset{\text{id}}{\to} (X, D', R)$

and

$(D_{0,I}, D_0, R_0) \overset{\text{id}}{\to} (\mathscr{D}_{D_0^n}, D_0, R_0') \overset{\text{id}}{\to} (X_0(Z_{D'}), D_0, R_0') \overset{\text{id}}{\to} (X_0(Z_{D'}), D_0', R_0')$

are sequences of D-R embeddings where

$(\mathscr{D}_{D^n}, D, R) := (U, D_{|U_1}, R_{|U})$ with $\quad U = \mathscr{D}_{D^{n+1}}$ and $U_1 = \mathscr{D}_{D^n}$,

$\mathscr{D}_{D_0^n} = (Z_D, ..., Z_D, Z_D \oplus K, ...), \quad n \geqslant 1, \quad Z_{D_0'} = (Z_D \oplus K, 0, ...),$

$D_{0,I} = X_0(Z_D) = (Z_D, ..., Z_D, ...), Z_{D_0} = (Z_D, 0, ...)$

according to (14)–(16).

$(X_0(Z_{D'}), D_0', R_0')$ *is a* $(D_0'-K_0)$ *extension of* $(X_0(Z_{D'}), D_0, R_0')$ *with* $K_0 = (K, 0, ...)$.

(ii) *The D'-R homomorphism*

$$\phi': (X, D', R) \to \bigl(X_0(Z_{D'}), D'_0, R'_0\bigr), \qquad Z_{D'} = Z_D \oplus K,$$

$$x \to \sum_{i=0}^{\infty} f_i F' D'^i(x), \qquad F' = I - RD',$$

is a continuation of the D-R homomorphism

$$\phi: (D_I, D, R) \to \bigl(X_0(Z_D), D_0, R_0\bigr)$$

such that •

(a) $P'_i = P_i, 0 \leqslant i \leqslant n-1$, *and* $\phi' D^n = D_0^n \phi'$ *on* $\mathscr{D}_{D^n}$; $\phi'^{-1}(\mathscr{D}_{D_0^n}) = \mathscr{D}_{D^n}$,

(b) $\phi'^{-1}(D_{0,I}) = D_I$ *and* $Z_\phi = Z_{\phi'} = Q$.

(iii) *The ϕ'-image of (X, D, R) is contained as a $D_0 - R'_0$ subspace in* $\bigl(X_0(Z_{D'}), D_0, R'_0\bigr)$ *in such a way that*

$$\phi': (\mathscr{D}_{D^n}, D, R) \to (\mathscr{D}_{D_0^n}, D_0, R'_0) \quad \forall \, n$$

is a D-R homomorphism with the properties (a).

Proof. $(\mathscr{D}_{D^n}, D, R)$ is indeed a D-R subspace of (X, D, R), see (1-3), as

$$\mathscr{D}_{D^{n+1}} \subset \mathscr{D}_D, \quad R(\mathscr{D}_{D^n}) \subset \mathscr{D}_{D^{n+1}} \subset \mathscr{D}_{D^n}, \quad D(\mathscr{D}_{D^{n+1}}) \subset \mathscr{D}_{D^n}.$$

All the other properties are either proved or a consequence of the D-R homomorphy (1-D5). ∎

The theorem just proved refers to the general case II(a). The special case II(b) follows by the specialisation $Z_D = (0)$ and leads to the corresponding simplifications $D_I = Q, \phi = 0, Z_{D'} = K$, etc. The only serious drawback of this approach is the arbitrary selection of the complement K. So we have to check what happens if we choose another complement, say $\bar{K}$, and form the corresponding right inverse $\bar{D}'$. The D-R structures (X, D', R) and $(X, \bar{D}', R)$ are different from each other as D-R structures and it is not yet known under which conditions they can be related by a D-R isomorphism which leaves D_I pointwise fixed.

5. d_0-topology. We finish this section with a brief note on the d_0-topology. In (1.2.) it was pointed out that the d_0-convergence in $X_0(Z)$ could be related to a topology. This topology is given if we take the subspaces

$$U_i := R_0^i(X_0) = (0, \ldots, \overset{\mu}{Z}, Z, \ldots), \qquad i = 1, 2, \ldots,$$

of X_0 as a *neighbourhood base of zero*.

This topology, we shall call it the *d_0-topology*, has many interesting properties. It is compatible with linear structure and makes X_0 a complete metrizable Hausdorff space with S_0 dense in X_0. There are countably many base elements U_i and they are both open and closed. Continuity is thus equivalent to sequential continuity (d_0-continuity (D3)) and d_0-continuous linear operators defined only on S_0 can be extended uniquely to X_0.

On the other hand, we have seen in (T3) and (T4) that a *D-R* space X can be represented homomorphically in a reference space $X_0(Z_D)$ or $X_0(Z_{D'})$; so, if we give the space X the "*d*-topology" determined by the neighbourhood base $\{R^i(X)\}_{i=0,1,\ldots}$, then ϕ is continuous by (T3(ii)) and a *d*-continuous operator $H \in L_0(X)$ induces a d_0-continuous linear operator H_0 on $\phi(X)$ such that

$$\phi H = H_0 \phi.$$

As $S_0 \subset \phi(X) \subset X_0$ and S_0 is dense, H_0 can be extended continuously to $H_0' \in L_0(X_0)$, so that finally

$$\phi H = H_0' \phi \quad \text{where} \quad H_0 \in B_0(X) \text{ and } H_0' \in B_0(X_0).$$

By this relation we can for instance transform the equation $H(x) = y$ in X into the equation $H_0'(\phi(x)) = \phi(y)$ in X_0, i.e.

$$H(x) = y \xrightarrow{\phi} H_0 \phi(x) = \phi(y),$$

and use the "better" structure of the (topologized) reference space X_0 for an analysis of possible solutions, at least mod. Q in X.

There are already many interesting results, but to give them here would by far exceed the scope of this dissertation.

§ 5. Convergence in *D-R* spaces

1. B_S-spaces and the notion of the abstract Taylor expansion. Let X be a *D-R* space of type D_I. If $\mathscr{D}_D \subsetneqq X$, we have seen in (4-T4) how X can be made a *D-R* space with $\mathscr{D}_D = X$, and so we refer to an arbitrary but fixed *D-R* space, a space X with $\mathscr{D}_D = X$ and $Z_D \neq (0)$. The Taylor formula (3-4) yields

$$(1) \qquad x = \sum_{i=0}^{n-1} P_i(x) + R^n D^n(x)$$

$$\text{with} \quad P_i = R^i(FD^i(x)) = R^i(z_i); \; z_i = FD^i(x) \forall n$$

for any $x \in X$. So we can try to give a meaning to

$$(2) \qquad \sum_{i=0}^{\infty} P_i(x) \quad \text{and} \quad \sum_{i=0}^{\infty} R^i(z_i), \qquad z_i \in Z_D,$$

as well as to

$$(3) \qquad R^n D^n(x) \to 0, \qquad n \to \infty,$$

in terms of an adequate convergence.

We will see that we need at least a subspace B of X where (1) holds and for this it is sufficient that all expressions $R^i(z_i), z_i \in Z_D,\ i \geqslant 0$, i.e. S, be contained in B, as is shown by

PROPOSITION 1. *Let B be a subspace of X with $S \subset B$. This implies*

(4) $R^n D^n$ *projects B into B* $\forall\, n$.

Proof. In (3-T1) it was shown that $R^n D^n$ is a projection on X. By (1) we can write $b \in B$ as $b = \sum_{i=0}^{n-1} P_i(b) + R^n D^n(b)$ and according to (3-T1) the component projections P_i map X onto $R^i(Z_D) \subset S\ \forall\, i$. Hence $P_i(X) \subset B\ \forall\, i \Rightarrow R^n D^n(b) \in B$. ∎

Remark 1. We have proved more, namely

(5) $x \in B$ iff $R^n D^n(x) \in B$.

DEFINITION 1. Subspaces B of X with $S \subset B$ will be called B_S-*spaces*.

It is rather surprising that B_S-spaces have properties which are very similar to those of D-R subspaces, see (§ 3).

PROPOSITION 2 (Properties of B_S-space).

(i) $\bigcap\limits_{n=0}^{\infty} R^n D^n(B) = B \cap Q$, *and hence* $\bigcap\limits_{n=0}^{\infty} R^n D^n(B) = (0)$ iff $B \cap Q = (0)$.

(ii) $B = Z_D \oplus R(Z_D) \oplus \ldots \oplus R^{n-1}(Z_D) \oplus R^n D^n(B)\ \forall\, n$.

(iii) $R^n D^n(B) = R^n(Z_D) \oplus R^{n+1} D^{n+1}(B)$.

Proof. (i): $R^n D^n(B) \subset R^n(X) \Rightarrow \bigcap\limits_{n=0}^{\infty} R^n D^n(B) \subset \bigcap\limits_{n=0}^{\infty} R^n(X) = Q$. But $R^n D^n(B) \subset B$ by (P1), so $\bigcap\limits_{n=0}^{\infty} R^n D^n(B) \subset B$ and one inclusion is proved.

$$b \in B \cap Q \overset{3\text{-}P3}{\Rightarrow} R^n D^n(b) = b\ \forall\, n \Rightarrow b \in \bigcap\limits_{n=0}^{\infty} R^n D^n(B).$$

(ii): $b = \sum\limits_{i=0}^{n-1} P_i(b) + R^n D^n(b)$ for all $b \in B$. But all P_i and $R^n D^n$ are projections which map B into B, and so by (3-T1) and with $S \subset B$

$$B = Z_D \oplus \quad \oplus R^{n-1}(Z_D) \oplus R^n D^n(B)\ \forall n.$$

(iii): $B = Z_D \oplus \quad \oplus R^{n-1}(Z_D) \oplus R^n(Z_D) \oplus R^{n+1} D^{n+1}(B)$.

Let $b \in R^n(Z_D) \oplus R^{n+1} D^{n+1}(B) \Rightarrow b = R^n(z) + R^{n+1} D^{n+1}(b') = R^n(z + RD^{n+1}(b')) \Rightarrow R^n D^n(b) = b \Rightarrow b \in R^n D^n(B)$. Thus we have shown that $R^n(Z_D) \oplus R^{n+1} D^{n+1}(B) \subset R^n D^n(B)$ and the above direct sum representations imply equality. ∎

From (iii) we immediately infer

COROLLARY 1. *A B_S-space has the structure chain*

$$(6) \qquad B \gneqq RD(B) \gneqq \quad \gneqq R^n D^n(B) \gneqq \quad \gneqq B \cap Q. \quad \blacksquare$$

For $B \cap Q = (0)$ (P2) shows that (6) shrinks to (0). By this we have a means of defining $R^n D^n(x) \to 0$ in the same way as in $X_0(Z_D)$, see (4-T1), but we give first the following general definition of abstract power series, Taylor series and analytic elements in a B_S-space:

DEFINITION 2. Let X be an arbitrary D-R space and $B \subset X$ a subspace such that

(a) There is a notion of *sequential convergence* with *unique* limit in B.

(b) $S \subset B$, i.e. $R^i(z) \in B \ \forall z \in Z_D, \ i = 0, 1, \dots$

(i) We call $\sum\limits_{i=0}^{\infty} R^i(z_i)$ an *abstract power series* (APS) if

$$\sum_{i=0}^{\infty} R^i(z_i) \text{ converges in } B.$$

(ii) The (APS) $\sum\limits_{i=0}^{\infty} R^i(z_i)$ is called an *abstract analytic element* (AAE) if

$$R^k \left(\sum_{i=0}^{\infty} R^i(z_i) \right) = \sum_{i=0}^{\infty} R^{k+i}(z_i)$$

$$(7) \qquad\qquad\qquad\qquad\qquad \text{for all } k = 0, 1, 2,$$

$$D^k \left(\sum_{i=0}^{\infty} R^i(z_i) \right) = \sum_{i=0}^{\infty} R^i(z_{i+k}), \quad z_i \in Z_D,$$

(iii) We say that $x \in X$ has an *abstract Taylor expansion* (ATE) if the associated formal series of x: $\sum\limits_{i=0}^{\infty} P_i(x) = \sum\limits_{i=0}^{\infty} R^i(FD^i(x))$ *converges* in B and is *equal* to x.

We notice that in (i) every finite sum $\sum\limits_{i=0}^{n} R^i(z_i)$ belongs to B, and so it makes sense to ask if the limit exists in B. Likewise we have in (iii) $B \supset S \in P_i(x) = R^i(z_i)$ with $z_i = FD^i(x) \in Z_D$ and we can ask whether or not the formal series $\sum\limits_{i=0}^{\infty} P_i(x)$ converges. In any case, this convergence clearly depends on the space B.

2. The b-convergence in B_S-spaces. Let X be a D-R space and B a *subspace* such that $S \subset B$ and $B \cap Q = (0)$. By (P2) chain (6) shrinks to (0), i.e.

$$(8) \qquad B \gneqq RD(B) \gneqq R^2 D^2(B) \gneqq \quad \gneqq R^n D^n(B) \gneqq \quad \gneqq (0).$$

Similarly to (4-D2) we give

DEFINITION 3 (b-Convergence). Let b_n be a sequence of B. We say that b_n converges to $b \in B$, $b_n \xrightarrow{b} b$, iff

$$\forall\, i \geqslant 0 \;\exists\, N(i) \geqslant 0 \;\forall\, n \geqslant N(i):\; b - b_n \in R^i D^i(B).$$

PROPOSITION 3. *The b-convergence has the following properties.*

(i) *The limit is unique.*

(ii) *The limit is compatible with the linear structure of* B.

Proof. (i): Suppose that $b_n \to b$ and $b_n \to b'$. Then, for $i \geqslant 0$, there exists an N such that $b - b_n \in R^i D^i(B)$ and $b' - b_n \in R^i D^i(B)$ $\forall\, n \geqslant N$. As $R^i D^i(B)$ is a subspace of X, it follows that $b - b' = (b - b_n) - (b' - b_n) \in R^i D^i(B)$ and (8) implies $b = b'$.

(ii): We have to show that

$$b_n \to b \text{ and } b'_n \to b' \;\Rightarrow\; b_n + b'_n \to b + b'.$$

$$b_n \to b \Rightarrow cb_n \to cb, \quad c \in \mathscr{F}.$$

Let $n \geqslant N(i) = \max(N_1, N_2)$. Then $b - b_n$ and $b' - b'_n$ are contained in $R^i D^i(B)$ for given $i \geqslant 0 \;\Rightarrow\; (b + b') - (b_n + b'_n) \in R^i D^i(B)$ as $R^i D^i(B)$ is a subspace. $\square$

For $n \geqslant N$ we have $b - b_n \in R^i D^i(B) \Rightarrow cb - cb_n \in R^i D^i(B)$ for any given i. $\blacksquare$

THEOREM 1. *If* $\displaystyle\sum_{i=0}^{\infty} R^i(z_i)$ *converges in* B, *i.e.* $\displaystyle\sum_{i=0}^{\infty} R^i(z) = b \in B$, *then*

$$P_i(b) = R^i(z_i) \quad \forall\, i.$$

Consequently, $\displaystyle\sum_{i=0}^{\infty} R^i(z_i) = \sum_{i=0}^{\infty} P_i(b)$ *and, according to* (D2), *a b-convergent abstract power series is its own Taylor series.*

Proof. Let $k >_{.} 0$ and $i > k$. As $\displaystyle\sum_{i=0}^{n} R^i(z_i) \xrightarrow[n \to \infty]{b} b$, we have

$$b - \sum_{i=0}^{n} R^i(z_i) \in R^i D^i(B) \quad \text{for} \quad n \geqslant N_i.$$

Let $n = \max(K, N_i)$; then $b - \displaystyle\sum_{i=0}^{n} R^i(z_i) \in R^i D^i(B)$ and $R^k(z_k)$ appears as a term of the finite sum. But

$$b = \sum_{i=0}^{n} R^i(z_i) + \left(b - \sum_{i=0}^{n} R^i(z_i)\right),$$

and so we can apply P_k on both sides, which gives

$$P_k(b) = \sum_{i=0}^{n} P_k R^i(z_i) + P_k\left(b - \sum_{i=0}^{n} R^i(z_i)\right) = R^k(z_k),$$

for $P_k R^n D^n = 0$ as $k < n$ by (3-T1), and the component projection P_k reproduces exactly $R^k(z_k)$. $\blacksquare$

We define b-continuity in the same way as in (4-D2)

DEFINITION 4 (b-continuity). A mapping T which maps B into a linear space B' endowed with the notion of sequential convergence is called *b-continuous* if

$$b_n \xrightarrow{b} b \;\Rightarrow\; T(b_n) \xrightarrow{b'} T(b)$$

holds for every sequence of B.

Remark 1. The definition of d_0-convergence (4-D2) should have been given in the wider sense of the above definition.

LEMMA 1. $B \cap R^i(X) = R^i D^i(B)$.

Proof. As $S \subset B$, we have by (P1) $R^i D^i(B) \subset B$, but $R^i\big(D^i(B)\big) \subset R^i(X)$ and "$\supset$" is proved.

$b = z_0 + \;\; + R^{i-1}(z_{i-1}) + R^n D^n(b)$ where $z_0 + \;\; + R^{i-1}(z_{i-1}) \in Z_{D^i}$ and $Z_{D^i} \oplus R^i(X) = X$, as $\mathscr{D}_D = X$ by (3-T1). So, if $b \in R^i(X)$, then $z_0 + \;\; + R^{i-1}(z_{i-1}) = 0$; hence $b = R^i D^i(b) \in R^i D^i(B)$. $\square$

The next result shows the close relation between b-convergence in B and d_0-convergence in the reference space $X_0(Z_D)$ via the canonical homomorphism ϕ (4-12).

THEOREM 2 (Equivalence of B and $\phi(B)$).

$$B \xleftrightarrow{\phi} \phi(B) \text{ is a bi-continuous isomorphism.}$$

Proof. Clearly ϕ is an isomorphism as $Z_\phi = Q$ and $B \cap Q = (0)$ by hypothesis.

(a) Let $R_0^i(X_0)$ be given. $b - b_n \in R^i D^i(B), n \geqslant N_0 \;\Rightarrow\; \phi(b - b_n) \in \phi R^i D^i(B) = R_0^i D_0^i \phi(B) \subset R_0^i(X)$; hence ϕ is continuous; $b, b_n \in B$.

(b) Let $R^i D^i(B)$ be given and let $\phi(b) - \phi(b_n) = \phi(b - b_n) \in R_0^i(X_0)$. Then $\phi_{|B}^{-1} \phi(b - b_n) \in \phi_{|B}^{-1}\big(R_0^i(X_0)\big) \subset \phi^{-1}\big(R_0^i(X_0)\big)$

$$\Rightarrow\; b - b_n \in R^i(X) \cap B \overset{L1}{=} R^i D^i(B) \;\; \forall n \geqslant N_i.$$

Hence $\phi_{|B}^{-1}$ is continuous. ∎

It is now very easy to justify the identification

$$B \ni \sum_{i=0}^{\infty} R^i(z_i) \;\leftrightarrow\; \sum_{i=0}^{\infty} f_i(z_i) = (z_0, \ldots, z_i, \ldots) \in \phi(B) \subset X_0.$$

COROLLARY 1. $\displaystyle\sum_{i=0}^{\infty} R^i(z_i)$ converges in B iff $\displaystyle\sum_{i=0}^{\infty} f_i(z_i) \in \phi(B)$.

$$\sum_{i=0}^{\infty} R^i(z_i) = \phi_B^{-1}\left(\sum_{i=0}^{\infty} f_i(z_i)\right).$$

Proof. The bi-continuity of ϕ implies

$$\phi\left(\sum_{i=0}^{\infty} R^i(z_i)\right) = \sum_{i=0}^{\infty} \phi(R^i(z_i)) = \sum_{i=0}^{\infty} R_0^i \phi(z_i) = \sum_{i=0}^{\infty} f_i(z_i).$$

If $\displaystyle\sum_{i=0}^{\infty} f_i(z_i) \in \phi(B)$, we apply $\phi_{|B}^{-1}$ and obtain the remaining assertions. ∎

THEOREM 3 (Taylor expansion in a B_S-space).

$$\sum_{i=0}^{\infty} P_i(x) = x \quad \text{iff} \quad x \in B \ \forall x \in X$$

Proof. $\Rightarrow$: $\sum_{i=0}^{\infty} P_i(x) = x$ means $\sum_{i=0}^{\infty} P_i(x)$ converges in B and is equal to x; so, according to (D2(iii)), $x \in B$ trivially.

$\Leftarrow$: Let $x \in B$. By the Taylor formula we have

$$x - \sum_{i=0}^{n-1} P_i(x) = R^n D^n(x) \ \forall n.$$

As chain (8) strictly descends to (0), we have $R^n D^n \cdot \xrightarrow{b} 0$, i.e.

$$x = \sum_{i=0}^{\infty} P_i(x).$$

So precisely the elements of B have a Taylor expansion which represents them and, see (4-P3(iii)) and (T1), every element $b \in B$ can be written as a d-limit of elements of S. ∎

Next we shall see that if $b \in B_S$, then

$$\sum_{i=0}^{\infty} R^i(z_i) = b \Rightarrow \sum_{i=0}^{\infty} DR^i(z_i) = D(b) \text{ or } \sum_{i=0}^{\infty} RR^i(z_i) = R(b)$$

is *not* true in general, but only for a certain subspace of B_S.

PROPOSITION 4 (Continuity of a linear operator $A \in L_0(X)$ on B). *Let* $A \in L_0(X)$ *and* $A_0 \in L_0(X_0(Z_D))$ *be such that*

(9) $\qquad\qquad \phi A = A_0 \phi \text{ on } X$

and A_0 *is* d_0-*continuous on* $\phi(B)$. *Then the following assertions hold*:

(i) $b_n \xrightarrow{b} b$ *and* $A(b_n), A(b) \in B \ \forall n \Rightarrow A(b_n) \xrightarrow{b} A(b)$. *Hence* A *is continuous on every* A-*invariant subspace* U *of* B.

(ii) $B_A := \{b \in B | \ A^n(b) \in B, \ \forall n\}$ *is the largest* A-*invariant subspace of* B.

Proof. (i): $A(b_n), A(b) \in B \ \forall n \Rightarrow \phi A(b_n), \phi A(b) \in \phi(B) \ \forall n$; so $b_n \xrightarrow{b} \xrightarrow{b} b \overset{T2}{\Rightarrow} \phi(b_n) \xrightarrow{d_0} \phi(b) \Rightarrow A_0 \phi(b_n) \xrightarrow{d_0} A_0 \phi(b) \overset{(9)}{\Rightarrow} \phi A(b_n) \xrightarrow{d_0} \phi A(b) \overset{T2}{\Rightarrow} A(b_n) \xrightarrow{b} \xrightarrow{b} A(b)$.

(ii): B_A evidently contains any A-invariant subspace of B and is, by the linearity of A^n, itself a subspace of B.

Remark 3. $b, A^n(b) \in B$ imply $A^n(b) = \phi_{|B}^{-1} A_0^n \phi(b)$.

Indeed, $A^n(b) \in B \Rightarrow A^n(b) = \phi_{|B}^{-1} \phi A^n(b) = \phi_{|B}^{-1} A_0^n \phi(b)$ by (9). □

Closely related to part (i) is the following

COROLLARY 1 (Nearly closed graph property of an operator $A \in L_0(X)$ on B).

$$b_n \xrightarrow{b} b \text{ and } A(b_n) \xrightarrow{b} b' \Rightarrow A(b) = b' + q, \ q \in Q.$$

Proof. $b_n \xrightarrow{b} b \xRightarrow{T2} \phi(b_n) \xrightarrow{d_0} \phi(b) \Rightarrow A_0 \phi(b_n) \xrightarrow{d_0} A_0 \phi(b) \xRightarrow{(9)} \phi A(b_n) \xrightarrow{d_0} \phi A(b)$ by the continuity of A_0.

On the other hand, $A(b_n) \xrightarrow{b} b' \xRightarrow{T2} \phi A(b_n) \xrightarrow{d_0} \phi(b')$; hence the d_0-sequence $\phi A(b_n)$ has the limits $\phi A(b)$ and $\phi(b')$ in $\phi(B) \Rightarrow \phi(A(b))$ $= \phi(b') \xRightarrow{4\text{-}T3} A(b) = b' +$ ∎

After these preparations   can see how abstract power series, Taylor series and analytic el. :nents ('' ') behave in a B_S-space under D and R.

THEOREM 4 (Abstract po⋅   series in a B_S-space). *Let B be a subspace of X with $S \subset B$ and $B \cap Q = (0)$.*

(i) *If* $\displaystyle\sum_{i=0}^{\infty} R^i(z_i) = b \in B$ *and if* $D(b) \in B$ *or* $R(b) \in B$, *then*

$$\sum_{i=0}^{\infty} DR^i(z_i) = D(b) \quad and \quad \sum_{i=0}^{\infty} RR^i(z_i) = R(b).$$

(ii) *The space $B_{D,R}$ of abstract analytic elements* (AAE), *given by*

$$(10) \qquad B_{D,R} = \{b \in B \mid R^n(b), D^n(b) \in B, \forall n\},$$

contains S and is the largest D-R invariant subspace (of (X, D, R)) contained in B.

Proof (i): Any finite section $\displaystyle\sum_{i=0}^{n} R^i(z_i)$, $n \geqslant 0$, is contained in S. Bu⋅ S is D-R invariant and contained in B, and hence $D\left(\displaystyle\sum_{i=0}^{n} R^i(z_i)\right)$ $= \displaystyle\sum_{i=0}^{n} DR^i(z_i) \in S \subset B$ and the same is true for R. As $D(b) \in B$ or $R(b) \in B$, the assertion follows by (P4).

(ii): The intersection of B_R and B_D together with (P4) implies that $B_{D,R}$ is of the stated form and invariant under D and R. Clearly $S \subset B_{D,R}$

If $\displaystyle\sum_{i=0}^{l} R^i(z_i) = b \in B_{D,R}$, then the continuity of D and R on $B_{D,R}$ due to the D-R invariance, implies relation (7).

But according to (T3) every element of B and thus every element of $B_{D,R}$ has a Taylor expansion.

$$b = \sum_{i=0}^{\infty} P_i(b) = \sum_{i=0}^{\infty} R^i(FD^i(b)) = \sum_{i=0}^{\infty} R^i(z_i), \quad z_i = FD^i(b) \in Z_D.$$

Hence every element of $B_{D,R}$ is an (AAE). Let $\displaystyle\sum_{i=0}^{\infty} R^i(z_i) = b$ be an (AAE). The (AAE) condition (7) implies that

$$\sum_{i=0}^{\infty} R^{i+n}(z_i) = R^n(b) \in B \quad and \quad \sum_{i=0}^{\infty} R^i(z_{i+n}) = D^n(b) \in B \; \forall n \Rightarrow b \in B_{D,R}. \; ∎$$

Remark 4. For the elements b of $B_{D,R}$ we have $R^i D^j (b)$
$= \phi_{|B}^{-1} R_0^i D_0^j \phi (b), i,j \geqslant 0$.

The last theorem shows clearly a problem related to this notion of abstract convergence:

Although *every* element of B_S *has* a Taylor expansion, *only* the elements of $B_{D,R}$ have the property that D and R or any product $R^i D^j$ can be applied term by term to the Taylor expansion, i.e.

$$b = \sum_{i=0}^{\infty} P_i(b) \Rightarrow R^i D^j (b) = \sum_{i=0}^{\infty} R^i D^j P_i(b) = \sum_{i=0}^{\infty} P_i(R^i D^j(b)).$$

There is a similar problem, namely if x is an eigenvector of D, then *only* the eigenvectors $x \in B$ are (AAE), whereas nothing can be said if $x \notin B$.

PROPOSITION 5. *Let x be an eigenvector of D to the eigenvalue $t \neq 0$. If $x \in B$, then $x \in B_{D,R}$, i.e. x is an (AAE).*

Proof. $D^n(x) = t^n x, \forall n$, and hence $x \in B_D$ (for B_D see (P4)). As $x \in B$, we infer by $R^n D^n(B) \subset B \ \forall n$ that

$$R^n D^n(x) = R^n(t^n x) = t^n R^n(x) \in B \Rightarrow R^n(x) \in B \ \forall n;$$

so $x \in B_R$, and hence $x \in B_{D,R}$ by (T4). ∎

Both problems can be avoided if R has no eigenvalues on Q, for in § 7 it will be shown that there exists a space B which contains all the eigenspaces of D and is invariant under D and R.

'§ 6. Taylor expansion in a D-R space

Note. In this section we refer to an arbitrary but fixed D-R space X of type D_I with $Z_D \neq (0)$.

1. The spaces E.

DEFINITION 1. Let E be an algebraic complement of Q in X which contains S, i.e.

(1) $X = E \oplus Q$ and $S \subset E$

together with the associated projections P_E and P_Q, i.e.

(2) $P_E \oplus P_Q = I, \quad P_E(e+q) = e, \quad P_Q(e+q) = q \quad \forall x = e+q \in X \oplus Q;$

then X is said to be (E, Q)-*decomposed* and $E \oplus Q$ is called a (E, Q)-*decomposition* of X.

E is by definition a B_S-space. All the results of § 5 carry over to E

and it is our aim to introduce a Taylor expansion for "nearly" all the elements of X. Clearly this is not possible for the elements of Q, because

$$P_i(q) = 0 \; \forall \, i \quad \text{and} \quad R^n D^n(q) = q \; \forall \, n.$$

Trivially the associated component series (3-16) converges (in E) but is never equal to q.

Depending on the selection of E we will now give a description of a possible abstract Taylor expansion in X.

THEOREM 1. *Let* X *be a D-R space of type* D_I *and suppose* X *to be* (E, Q)-*decomposed. Then precisely the elements of* E *have a Taylor expansion, i.e.*

$$\sum_{i=0}^{\infty} P_i(x) = x \quad \text{iff} \quad x \in E,$$

and every element $x \in X$ *can be written in a unique way as*

$$x = \sum_{i=0}^{\infty} P_i(x) + q, \quad q \in Q.$$

The space of abstract analytic elements is given by

$$E_{D,R} = \{ e \in E | \; R^n(e) \in E \quad \text{and} \quad D^n(e) \in E, \; \forall n \}.$$

Proof. E is a B_S-space, and so the first and third assertion follow from (5-T3) and (5-T4).

Let $x \in X$. By (1) $x = e + q$ uniquely; so, if we apply P_i to x, we obtain $P_i(x) = P_i(e + q) = P_i(e) \, \forall \, i$ by (3-T4C1). Let $e \in E$. Then $e = \sum_{i=0}^{\infty} P_i(e)$ uniquely by (5-T3). Hence

$$x = e + q = \sum_{i=0}^{\infty} P_i(e) + q = \sum_{i=0}^{\infty} P_i(e + q) + q. \quad \blacksquare$$

Clearly it is of high interest to know whether or not an (E, Q)-decomposition can be found where $E_{D,R} = E$, i.e. where E is invariant under D and R. The existence of such a decomposition will be proved in (§ 7) under the rather general assumption that R has no eigenvalues. At present we can only state

PROPOSITION 1. *Under the hypothesis of the preceding theorem we have*

$$R(E) \subset E \quad \text{iff} \quad D(E) \subset E.$$

So the D-R invariance of E *is already given if the space is invariant under* D *or* R.

Proof. $\Rightarrow$: Let $D(e) = e' + q \in E \oplus Q = X$. Applying R gives

$$R(e') + R(q) = RD(e) \in E \quad \text{by (5-5) as } S \subset E;$$

hence $RD(e) - R(e') = R(q) \in E \cap Q$ by (3-P3). But then $R(q) = 0$, for $E \cap Q$ $= (0)$ and thus $q = 0$.

$\Leftarrow$: The proof is similar. We put $R(e) = e' + q$ and apply D. This gives

$$e = DR(e) = D(e') + D(q)$$

$e - D(e') = D(q) \in E \cap Q$ by (3-P3). Thus $D(q) = 0$, whence $q = 0$, as ι injective on Q by (3-P3). ■

2. The extension of the simple T   rm . view of the results
of (§ 5) and (1.) the finite Taylor formula

$$I = \left(\bigoplus_{i=0}^{n-1} P_i \right) \oplus R^n Q^n, \quad n \geq 0,$$

leads naturally to the question how to associate a meaning with the expression

$$\bigoplus_{i=0}^{\infty} P_i, \quad P_i P_j = \delta_{ij} P_i.$$

The problem is that the P_i are defined on the whole space X, whereas, according to (T1), $\sum_{i}^{\alpha}$ r) is only de . . on E (with reference a fi /

(E, Q)-decomposition i   so i   . . . arently not vet quite legitimat to say that   .   to   true at most in the restriction tc E where a well-dei ed I h-convergence ca   ensures that $e = \sum_{i=0}^{\prime} P_i(e)$.

We solve this problem by the in oduc n of a suitable $L R$ space X togethei with a suitable (F, Q) decomposition where   .rs as an element oi E which can be written as

$$P_L = \sum_{i=0}^{\prime} P_i$$

in the E-convergence (i.e. h-convergence in E).

2.1.   as a D-R space. The space we use for our purpose is the space $L_0(X)$. To brevity we put

$$X := L_0(X)$$

and assume a *fixed* (E, Q)-decomposition of the D-R space X to be given. The next definition gives X the necessary D-R structure.

DEFINITION 2.

$$\begin{array}{ll} D: X \to X: & T \to DT, \\ R: X \to X: & T \to RT, \end{array} \qquad T \in X = L_0(X).$$

Trivially, D and R are linear operators on $L_0(X)$ such that

$$DR(T) = D(RT) = (DR)T = T \quad \forall\, T \in L_0(X).$$

Concerning the higher powers of D and R we see that

$$D^n(T) = D^n T \quad \text{and} \quad R^n(T) = R^n T, \quad T \in L_0(X);$$

hence the operators $R^n D^n$, P_i and, in particular, $F = P_0$ are given by

$$R^n D^n(T) = R^n D^n T, \quad P_i(T) = P_i T \quad \text{and} \quad F(T) = FT, \quad T \in L_0(X),$$

as is checked immediately. $(P_i = R^i D^i - R^{i+1} D^{i+1}, \ i \geqslant 0.)$

We can state as the first result

PROPOSITION 2. *The space* $X = L_0(X)$ *together with the operators* D *and* R *of* (D2) *is a D-R space with* $\mathcal{D}_D = X$. *Furthermore*

(i) $Z_D = FX = \{T \in L_0(X) \mid T(X) \subset Z_D\}$; *hence* $Z_D = (0)$ *iff* $Z_D = (0)$.

(ii) *Let* S *be the space of finite elements of* X; *then*

$$S = \{T \in L_0(X) \mid \exists n\colon T(X) \subset Z_{D^n}\}.$$

Proof. (ii): $T \in S$ iff $D^n(T) = D^n T = 0$ iff $T(X) \subset Z_{D^n}$.

(i): $T \in Z_D \Rightarrow T = F(T) = FT \in FX$. $FT \in FX \Rightarrow D(FT) = DFT = 0$ $\Rightarrow FT \in Z_D$. ∎

The next proposition guarantees the existence of an adequate (E, Q)-decomposition of X, so that E can be endowed with the E-convergence (of (T1)).

PROPOSITION 3. *Let* X *be a D-R space together with an* (E, Q)-*decomposition of* X. *This induces on* X *the* (E, Q)-*decomposition*

$$X = E \oplus Q$$

with $E := P_E X$ *and* $Q = P_Q X = \{T \in L_0(X) \mid T(X) \subset Q\}$.

Q *is the space of singular elements of* X, *and so* $Q = (0)$ *iff* $Q = (0)$.

Proof. $I = P_E \oplus P_Q$, so $T = P_E T + P_Q T$, $\forall\, T \in L_0(X)$. Let $T \in P_E X \cap \cap P_Q X$; then $P_E T_1 = P_Q T_2$, and hence $0 \overset{(2)}{=} P_Q P_E T_1 = P_Q T_2$ and $X = P_E X \oplus P_Q X$.

$Q = P_Q X$: Obviously $P_Q X = \{\ldots\}$. $T \in Q = \bigcap_{i=0}^{\infty} R^i(X)$ iff $R^i D^i(T) = T$ $\forall i$ iff $R^i D^i T = T \, \forall i$ iff $R^i D^i T(x) = T(x) \, \forall x$ and $\forall i$ iff $T(x) \in Q \, \forall x$ iff $T(X) \subset Q$.

$S \subset P_E X$: $T \in S \overset{P2}{\Rightarrow} T(X) \subset Z_{D^n} \subset S \subset E \Rightarrow P_E T = T$ so $T \in P_E X$. ∎

COROLLARY 1. *If* E *is a D-R invariant subspace of* (X, D, R), *then* E *is a D-R invariant subspace of* (X, D, R).

Proof: $D(P_E T) = DP_E T$. $P_E T(X) \subset E \Rightarrow DP_E T(X) \subset D(E) \subset E$ $\Rightarrow P_E DP_E T = DP_E T \Rightarrow D(P_E T) \in P_E X = E$ and the assertion follows by (P1). ∎

With the E-convergence available we are now in a position to give a satisfactory justification of the formula $\bigoplus_{i=0}^{\infty} P_i = P_E$. We can show even more.

THEOREM 2. *Under the hypothesis of the preceding proposition every $P_E T$ with $T \in L_0(X) =: X$ has a Taylor expansion (5-D2 (iii)) (in the E-convergence of (X).*

$$P_E T = \sum_{i=0}^{\infty} P_i T, \quad \forall\, T \in L_0(X).$$

For $T = P_E$ we have in particular

$$(3) \qquad\qquad P_E = \bigoplus_{i=0}^{\infty} P_i.$$

The compatibility between E-convergence and E-convergence is given by

$$\sum_{i=0}^{\infty} (P_i T(x)) = \left(\sum_{i=0}^{\infty} P_i T \right)(x)$$

and we have especially

$$\sum_{i=0}^{\infty} P_i(x) = \left(\sum_{i=0}^{\infty} P_i \right)(x).$$

Proof. We show first

$$(4) \qquad\qquad P_i P_E = P_i \;\forall\, i.$$

In fact, let $e + q \in E \oplus Q = X$: then $P_i(e + q) \overset{3\text{-}T4}{=} P_i(e) = P_i P_E(e) \overset{(2)}{=} P_i P_E(e + q)$. $\square$

Now, $P_E T \in E$, and so by (T1) the element $P_E T$ has in X the Taylor expansion

$$P_E T = \sum_{i=0}^{\infty} P_i(P_E T).$$

But $P_i(P_E T) = P_i P_E T \overset{(4)}{=} P_i T$, and so $P_E T = \sum_{i=0}^{\infty} P_i T$ as claimed. Putting $T = P_E$, we have

$$P_E = \sum_{i=0}^{\infty} P_i P_E = \sum_{i=0}^{\infty} P_i = \bigoplus_{i=0}^{\infty} P_i, \quad \text{as} \quad P_i P_j = \delta_{ij} P_i.$$

Compatibility: $P_E T(x) \in E$, so $P_E T(x)$ has a Taylor expansion by (T1) with

$$P_E T(x) = \sum_{i=0}^{\infty} P_i(P_E T(x)) = \sum_{i=0}^{\infty} P_i T(x).$$

But $P_E T(x) = \left(\sum_{i=0}^{\infty} P_E T \right)(x)$ and the proof is complete. ∎

COROLLARY 1. $Q = (0) \Rightarrow I = \bigoplus_{i=0}^{\infty} P_i.$ ∎

2.2. The extended Taylor formula.

By means of the representation $P_E = \bigoplus_{i=0}^{\infty} P_i$ it is now quite easy to extended the finite Taylor formula

$$I = \bigoplus_{i=0}^{n-1} P_i \oplus R^n D^n.$$

Before we can do this, we first need an equivalent form of this formula related to the (E, Q)-decomposition of X.

PROPOSITION 4. *Let X be a D-R space of type D_I together with a (E, Q)-decomposition. The following identity holds on X:*

$$(5) \qquad I = \left(\bigoplus_{i=0}^{n-1} P_i\right) \oplus R^n D^n P_E \oplus P_Q.$$

Proof.

LEMMA 1. (i) $R^n D^n P_Q = P_Q R^n D^n = P_Q.$ (ii) $P_n P_Q = P_Q P_n = 0.$ $\forall n$

Proof. L1 (i): Let $e + q \in E \oplus Q = X.$ $R^n D^n P_Q (e+q) = R^n D^n(q) \overset{3\text{-}P3}{=} q = P_Q(e+q).$

$$R^n D^n(E) \overset{5\text{-}P1}{\subset} E \Rightarrow P_Q R^n D^n(E) = (0).$$

$$P_Q R^n D^n(e+q) = P_Q R^n D^n(q) = P_Q(q) = P_Q(e+q).$$

(ii): $P_n(X) \overset{3\text{-}T1}{=} R^n(Z_D) \subset S \subset E$ and $E \cap Q = (0)$ implies (ii). □

By hypothesis $I = P_E + P_Q,$ and hence

$$I = \sum_{i=0}^{n-1} P_i + R^n D^n = \left(\sum_{i=0}^{n-1} P_i + R^n D^n\right)(P_E + P_Q)$$

$$= \left(\sum_{i=0}^{n-1} P_i\right) P_E + \left(\sum_{i=0}^{n-1} P_i\right) P_Q + R^n D^n P_E + R^n D^n P_Q = \sum_{i=0}^{n-1} P_i + R^n D^n P_E + P_Q,$$

according to L1 and (4). As $R^n D^n(E) \subset E,$ we have $P_E R^n D^n(e) = R^n D^n(e).$ Hence

$$(R^n D^n P_E)^2 (e+q) = R^n D^n P_E R^n D^n(e) = (R^n D^n)^2(e) = R^n D^n(e) = R^n D^n P_E(e+q)$$

and $R^n D^n P_E$ is a projection.

$$P_i R^n D^n P_E \overset{3\text{-}T1}{=} 0 \overset{3\text{-}T1}{=} R^n D^n P_i \overset{(4)}{=} R^n D^n P_E P_i.$$

$P_Q R^n D^n P_E \overset{L1}{=} P_Q P_E = 0 = R^n D^n P_E P_Q.$ This together with (3-T1) implies (5). ∎

We sum up the last results in the main theorem of this section.

THEOREM 3 (Taylor expansion in a *D-R* space). *Let X be a* (E, Q)-*decomposed D-R space. The identity on X can be decomposed into a direct sum of projections*

(6)
$$I = \left(\bigoplus_{i=0}^{\infty} P_i \right) \oplus P_Q, \qquad P_i = R^i F D^i.$$

The projection operator $R^n D^n P_E$ *has the representation*

$$R^n D^n P_E = \bigoplus_{i=n}^{\infty} P_i$$

and (6) *is an extension of the finite Taylor formula*

$$I = \bigoplus_{i=0}^{n-1} P_i \oplus R^n D^n.$$

All the elements of X have a finite Taylor expansion

$$x = \sum_{i=0}^{n-1} P_i(x) + R^n D^n P_E(x) + P_Q(x)$$

(finite part)   (remainder term)   (singular part)

such that

$$R^n D^n P_E(x) \xrightarrow{E} 0 \quad \text{iff} \quad \sum_{i=0}^{n} P_i(x) \xrightarrow[n \to \infty]{E} P_E(x) = \left(\sum_{i=0}^{\infty} P_i \right)(x).$$

Consequently

$$\sum_{i=0}^{n-1} P_i(x) + R^n D^n P_E(x) \xrightarrow[n \to \infty]{E} \sum_{i=0}^{\infty} P_i(x) = P_E(x)$$

and every element x of X can be written as

$$x = \sum_{i=0}^{\infty} P_i(x) + P_Q(x),$$

where $\sum_{i=0}^{\infty} P_i(x)$ *may be called the* "*Taylor series of x*".

The convergence is to be understood in the sense of E-convergence in X and E-convergence in $X = L_0(X)$ *(i.e. b-convergence on E and Ê).*

Proof. By the (E, Q)-decomposition '(D1) we have $I = P_E + P_Q$; so, by (T2),

$$I = \bigoplus_{i=0}^{\infty} P_i + P_Q.$$

But

$$\bigoplus_{i=0}^{\infty} P_i + P_Q = \bigoplus_{i=0}^{n-1} P_i + \bigoplus_{i=n}^{\infty} P_i + P_Q \overset{P4}{=} \bigoplus_{i=0}^{n-1} P_i + R^n D^n P_E + P_Q,$$

and thus $R^n D^n P_E = \bigoplus\limits_{l=n}^{\infty} P_l$ as claimed. The remaining assertions are obvious by the previous results. ∎

§ 7. *D-R* decomposition and *D-R* extension

In this section we show the existence of an (E, Q)-decomposition of a *D-R* Volterra space (X, D, R) such that E is a *D-R invariant* subspace.

We assume the field $\mathscr{F}$ to be algebraically closed and (X, D, R) to be of type D_I, see (3-D6), with $Z_D \neq (0)$. Throughout this section we refer to a *fixed* space and consider only operator polynomials in D with *scalar* coefficients.

Clearly, $Q(R): X \to X$ is an isomorphism if the free constant a_0 of $Q(t) \in \mathscr{F}[t]$ is different from zero. This follows from the assumptions and immediately implies

PROPOSITION 1. *With the scalar product* $(Q(R)^{-1} P(R), x) \to Q(R)^{-1} \times$ $\times P(R)(x)$ *the linear space* X *is a unitary module over the ring*

(1) $\mathscr{F}^+ R := \{Q(R)^{-1} P(R)/Q(t), P(t) \in \mathscr{F}[t]$ and $Q(0) \neq 0\}$. ∎

Remark 1. Likewise, X can be considered a module over the ring of fractions $P(t)/Q(t)$, $Q(0) \neq 0$, if we define the product by $(P(t)/Q(t)) \cdot x := Q(R)^{-1} P(R)(x)$.

Although modular aspects are interesting in themselves we shall not exploit them but concentrate mainly on those facts which are necessary to establish the *D-R* decomposition theorem. The proof of this theorem requires the application of the lemma of Kuratowski–Zorn, so the decomposition is not unique unless additional structure is given to X.

A complementary question is posed if we suppose (X, D, R) to be a *pre-Volterra* space, namely whether or not we can find a canonical D^0-R^0 extension space (X^0, D^0, R^0) of X such that the extension R^0 of R is a *Volterra* right inverse of D^0 in X^0. The answer is, in fact, affirmative and given by the *D-R* extension theorem, which is based on the construction of a distribution module [1].

1. The *D*-hull of a *D-R* Volterra space. If not otherwise stated, all subspaces are supposed to be *invariant* under D. These spaces can be "closed" by a *hull operation* to be introduced now.

DEFINITION 1. Let U be a subspace of X.

(2) $H_D(U) := \{x \mid \exists P(t) \in \mathscr{F}[t]$ with $P(D)(x) \in U\}$

is called the *D-hull of* U

This definition also makes sense if $D \notin V\text{-}R(X)$, but some interesting results depend on whether or not R is a Volterra right inverse. The most important property of the D-hull is given by

PROPOSITION 2. *Let U be a subspace of X and $H_D(U)$ its D-hull.*

(i) $U \subset H_D(U)$.

(ii) $H_D(U)$ *is a D-R invariant subspace.*

(iii) $I\text{-}tR$ *is invertible on* $H_D(U)$ *for* $\forall\, t \in \mathscr{F}$

Hence $H_D(U)$ *is invariant under* $R^m(a_0 + \quad + a_n R^n)^{-1}$; $a_0 \neq 0$; $n, m \geqslant 0$, $a_i \in \mathscr{F}$.

Proof. (i): $D(U) \subset U \Rightarrow U \subset H_D(U)$.

(ii): $x \in H_D(U) \Rightarrow P(D)(x) \in U \Rightarrow P(D)(ax) \in U \Rightarrow ax \in H_D(U)$, $a \in \mathscr{F}$, $P \in \mathscr{F}[t]$.

$x_1, x_2 \in H_D(U)$ iff $P_1(D)(x_1)$, $P_2(D)(x_2) \in U$. Hence $P_1(D)P_2(D)(x_1 + x_2)$
$= P_2(D)P_1(D)(x_1) + P_1(D)P_2(D)(x_2) \in U \Rightarrow x_1 + x_2 \in H_D(U)$ by the invariance of U under a polynomial in D with scalar coefficients. Thus $H_D(U)$ is a subspace of X.

$$x \in H_D(U) \Rightarrow P(D)(x) \in U \Rightarrow DP(D)(R(x)) \in U \Rightarrow R(x) \in H_D(U).$$

(iii): $D - tI = D(I - tR)$; hence $(D - tI)(I - tR)^{-1} = D$ on X.

$$x \in H_D(U) \Rightarrow P(D)(x) \in U$$
$$\Rightarrow P(D)(D - tI)(I - tR)^{-1}(x) \in U \Rightarrow (I - tR)^{-1}(x) \in H_D(U).$$

So $Q(R): H_D(U) \to H_D(U)$ is an isomorphism if $Q(0) \neq 0, Q \in \mathscr{F}[t]$. ∎

The name *D-hull* is justified by the next proposition.

PROPOSITION 3 (Hull operations). *Let U and V be subspaces of X. The following hull operations hold:*

(i) $U \subset V \Rightarrow H_D(U) \subset H_D(V)$. *Hence* $H_D(0) \subset H_D(V)$.

(ii) $H_D(H_D(U)) = H_D(U)$.

(iii) $H_D(U \cap V) = H_D(U) \cap H_D(V)$. *Hence* $H_D\left(\bigcap_{i \in I} V_i\right) \subset \bigcap_{i \in I} H_D(V_i)$.

(iv) $H_D(U + V) = H_D(U) + H_D(V)$.

Proof. (i): $x \in H_D(U) \Rightarrow P(D)(x) \in U \subset V \Rightarrow x \in H_D(V)$.

(ii): $H_D(U) \subset H_D(H_D(U))$ by (P1 (i)). $x \in H_D(H_D(U)) \Rightarrow P(D)(x) \in H_D(U) \Rightarrow Q(D)(P(D)(x)) \in U \Rightarrow x \in H_D(U)$.

(iii): $x \in H_D(U \cap V) \Rightarrow P(D)(x) \in U \cap V \Rightarrow x \in H_D(U) \cap H_D(V)$.

$x \in H_D(U) \cap H_D(V) \Rightarrow P(D)(x) \in U$ and $Q(D)(x) \in V \Rightarrow P(D)Q(D)(x) \in U$ and $P(D)Q(D)(x) \in V \Rightarrow P(D)Q(D)(x) \in U \cap V \Rightarrow x \in H_D(U \cap V)$.

(iv) will be proved later. ∎

COROLLARY 1. $U \cap V = (0) \Rightarrow H_D(U) \cap H_D(V) = H_D(0)$. ∎

Of special interest are the properties of the zero-hull.

PROPOSITION 4 (0-hull properties).

(i) $H_D(0) = H_D(S)$.

(ii) $H_D(0) = \bigcup_P Z_{P(D)}$, $P \in \mathscr{F}[t]$; so $H_D(0) = (0)$ iff $Z_D = (0)$.

(iii) $S \subsetneqq H_D(S)$.

Proof. (i): $(0) \subset S \Rightarrow H_D(0) \subset H_D(S)$ by (P3 (i)).

$x \in H_D(S) \Rightarrow P(D)(x) = s \in S \Rightarrow D^n P(D)(x) = D^n(s) \overset{3\text{-D3}}{=} 0$, $n = n(s)$
$\Rightarrow x \in H_D(0)$.

(ii): $x \in H_D(0)$ iff $P(D)(x) = 0$ iff $x \in Z_{P(D)} \subset \bigcup_P Z_{P(D)}$.

The remaining assertion is proved by (T1 (i)), which is needed as an auxiliary result.

THEOREM 1 (Abstract differential equations with scalar coefficients). *Let* $P(D)$ *be an operator polynomial with scalar coefficients. There exists an associated invertible polynomial* $P_0^*(R)$ *with scalar coefficients such that*

(i) $Z_{P(D)} = \left(P_0^*(R)\right)^{-1}(Z_{D^n})$ *where* $Z_{D^n} = Z_D \oplus \ldots \oplus R^{n-1}(Z_D)$ *by* (3-T1).

(ii) *All the solutions of the equation* $P(D)(x) = y$ *are of the form*

$$x = \left(P_0^*(R)\right)^{-1} R^n(y) + z_n \quad with \quad z_n \in Z_{P(D)}.$$

Proof. [8], Theorem 1.2. $\square$

(iii) Suppose $S = H_D(S)$. By (T1 (i)) we have $Z_{D-tI} = (I-tR)^{-1}(Z_D)$ $\neq (0)$. $x \in Z_{D-tI} \overset{(ii)}{\Rightarrow} x \in H_D(S)$. This implies $D^n(x) = t^n x \neq 0$ $\forall n$. Contradiction. ∎

We now prove (P3 (iv)): $P(D)(x) = u + v \in U + V \Rightarrow x = u_1 + v_1 + z_n$ where u_1, v_1 and z_n are determined by (T1. (ii)). But $u \in U \subset H_D(U)$, $v \in V \subset H_D(V)$ and $z_n \in H_D(0) \subset H_D(U) \cap H_D(V) \overset{P2(iii)}{\Rightarrow} x \in H_D(U) + H_D(V)$.

$\left. \begin{array}{l} x_1 \in H_D(U) \Rightarrow P_1(D)(x_1) \in U \\ x_2 \in H_D(V) \Rightarrow P_2(D)(x_2) \in V \end{array} \right\} \Rightarrow \left. \begin{array}{l} P_1(D) P_2(D)(x_1) \in U \\ P_1(D) P_2(D)(x_2) \in V \end{array} \right\}$

$\Rightarrow P_1(D) P_2(D)(x_1 + x_2) \in U + V \Rightarrow x_1 + x_2 \in H_D(U + V)$. $\square$

We need some information about the space of singular elements Q in a *D-R* Volterra space.

PROPOSITION 5. *Let* $H(R) = \sum\limits_{i=0}^{n} a_i R^i$ *be a polynomial in* R *with* $a_0 \neq 0$ *and* $P(D) = \sum\limits_{i=0}^{m} b_i D^i$ *a polynomial in* D. *This implies*

(i) $H(R)$: $R^i(X) \to R^i(X)$ $\forall i$;

(ii) $R^n H(R)$: $Q \to Q$ $\forall n$ *are isomorphisms*;

(iii) $P(D)$: $Q \to Q$.

Proof. (i): R is a Volterra right inverse; hence $H(R)$ is invertible on X. But $H(R)$ commutes with R^l $\forall\, i$, which implies (i).

(ii): $H(R)(Q) = H(R) \bigcap\limits_{i=0}^{\infty} R^i(X) \overset{\text{(i)}}{=} \bigcap\limits_{i=0}^{\infty} H(R)R^i(X) \overset{\text{(i)}}{=} \bigcap\limits_{i=0}^{\infty} R^i(X) = Q.$

This, together with (3-P3(iii)), proves (ii).

(iii): Let $P(D) = (a_0 + \quad + a_n D^n)$; then $P(D) = D^n(a_n + a_{n-1}R + + a_i R^{n-i})$, where a_i is the *first* coefficient of $P(D)$ different from zero. By (i) and (3-P3 (iii)) $P(D)$: $Q \to Q$ is an isomorphism. ∎

As a by-result we obtain interesting information about the "bigness" of Q; see (3-6.E2) and (3-2.4.)

THEOREM 2. *In a D-R Volterra space we have* $\dim(Q) = \infty$ *if* $Q \neq (0)$.

Proof. Suppose $\dim(Q) = n$. Then R has a characteristic polynomial and this contradicts (P5(ii)). ∎

The D-hull of Q has some particular properties.

PROPOSITION 6 (D-hull of Q).

(i) $Q \cap U = (0) \Leftrightarrow Q \cap H_D(U) = (0)$.

(ii) $H_D(Q) = Q \oplus H_D(0)$.

Proof. (i): $x \in Q \cap H_D(U) \Rightarrow x \in Q$ and $x \in H_D(U) \overset{\text{P5(iii)}}{\Longrightarrow} P(D)(x) \in Q$ and $P(D)(x) \in U \Rightarrow P(D)(x) = 0 \Rightarrow x = 0$ by (P5(iii)).

(ii): $Q + H_D(0) \subset H_D(Q)$ by (P2(i)) and (P3(i)).

$$x \in H_D(Q) \Rightarrow P(D)(x) = q \overset{\text{T1(ii)}}{\Longrightarrow} x = (P_0^*(R))^{-1} R^n(q) + z_n.$$

By (P5(ii)) and (P4(ii)) follows $x \in Q + H_D(0)$.

$$Q \cap (0) = (0) \overset{\text{(ii)'}}{\Rightarrow} Q \cap H_D(0) = (0). \blacksquare$$

The D-hull can be related to the notion of D-*closure* given in

DEFINITION 2. A subspace U is called D-*closed* iff

(3) $\qquad\qquad P(D)(x) \in U \Leftrightarrow x \in U,\ P(t) \in \mathscr{F}[t].$

PROPOSITION 7 (Properties of the D-closure).

(i) *Let U be a subspace. U is D-closed iff $H_D(U) = U$. Hence the D-hull is D-closed.*

(ii) *Let $\{V_i\}_{i \in I}$ be a chain of D-closed subspaces ordered by inclusion; then $M := \bigcup\limits_{i \in I} V_i$ is D-closed.*

(iii) *The intersection of D-closed subspaces is D-closed.*

Proof. (i): The direction "$\Leftarrow$" in (3) is trivial as U is supposed to be D-invariant.

$\Rightarrow$: $U \subset H_D(U), x \in H_D(U) \Rightarrow P(D)(x) \in U \Rightarrow x \in U$ by (3).

$\Leftarrow$: $P(D)(x) \in U \Rightarrow x \in H_D(U) = U$

(ii): M clearly is a subspace by the chain property. $x \in M$ iff $x \in M_{i'}$ $\overset{(3)}{\Leftrightarrow}$ $P(D)(x) \in M_{i'}$ iff $P(D)(x) \in M$; hence $H_D(M) = M$.

(iii): $P(D)(x) \in \bigcap_{i \in I} V_i$ iff $P(D)(x) \in V_i$ $\forall i$ $\overset{(3)}{\Leftrightarrow}$ $x \in V_i$ $\forall i$ iff $x \in \bigcap_{i \in I} V_i$. ∎

We can give the D-hull still another interesting interpretation, for the D-hull can be characterized also in terms of *abstract differential equations* with scalar coefficients, see (T2).

THEOREM 3. *Let* U *be a subspace of* X. $H_D(U)$ *is the smallest extension space of* U *where every differential equation* $P(D)(x) = y \in H_D(U)$ *has a solution and all the solutions remain in* $H_D(U)$.

Proof. $P(D)(x) = y$ has solutions x for every $y \in X$ according to (T1 (ii)). Hence $P(D)(x) = y \in H_D(U)$ implies $x \in H_D(H_D(U)) = H_D(U)$. Let $V \supset U$ be a (D-invariant) extension space. As above, $P(D)(x) = y$ has solutions x according to (T1 (ii)). So, if $P(D)(x) = y \in V$, then (by hypothesis) $x \in V$ $\overset{(3)}{\Leftrightarrow}$ V is D-closed, i.e. $H_D(V) = V$.

$$\left(\bigcap_{U \subset V} V = H_D(V) \right) \subset H_D(U).$$

$$U \subset \left(\bigcap_{U \subset V} V = H_D(V) \right) \Rightarrow H_D(U) \subset H_D\left(\bigcap_{U \subset V} V \right) \overset{P3(iii)}{\subset} \left(\bigcap_{U \subset V} H_D(V) = V \right). ∎$$

Before we close this subsection with a theorem about the behaviour of the closure in a D-R *extension* space, we still need

PROPOSITION 8. *Let* U *and* V *be subspaces of* X. *If* $U \cap V = (0)$ *and* $H_D(0) \subset V$, *then* $U \cap H_D(V) = (0)$.

Proof: $x \in U \cap H_D(V)$ $\Rightarrow$ $x \in U$ and $x \in H_D(V)$ $\Rightarrow$ $P(D)(x) \in U$ and $P(D)(x) \in V$ $\Rightarrow$ $P(D)(x) = 0$ $\overset{P4(ii)}{\Longrightarrow}$ $x \in H_D(0) \subset V$ $\Rightarrow$ $x = 0$. ∎

THEOREM 4. *Let* U *be a* (D-invariant) *subspace of the* D-R *Volterra space* X *and let* X^0 *be a* D^0-R^0 *pre-Volterra extension space of* X *with* $Z_{D^0} \subset X$. *This implies that*

U is D-closed in X iff U is D^0-closed in X^0.

Before we prove (T4), we remark that (D2) clearly makes sense in any D-R space.

Proof. If $D(U) \subset U$ in X, then $D^0(U) \subset U$ in X^0, for D^0 is an extension of D. By the definition of the D-closure it remains to show that

"$\Rightarrow$" $P(D^0)(x^0) = u \in U \Rightarrow x^0 \in U$; if U is D-closed in X,

"$\Leftarrow$" $P(D)(x) \in U \Rightarrow x \in U$; if U is D^0-closed in X^0.

$\Leftarrow$ is very easy, as $P(D)(x) \in U$ and $x \in X \subset X^0$ implies $P(D^0)(x) = P(D)(x)$ and hence $x \in U$ by hypothesis.

$\Rightarrow$: We use (T1), but in a more explicit way.

Let $P(D^0) = a_0 + \cdots + a_n(D^0)^n$. $P(D^0)$ can be transformed in the following way:

$$P(D^0) = (D^0)^n \left(a_n + a_{n-1}R^0 + \cdots + a_{n-m}(R^0)^m\right) = (D^0)^n P_0^*(R^0),$$

with

$$P_0^*(R^0) = b(I - t_1 R^0) \dots (I - t_m R^0), \quad m \leqslant n$$

decomposed into its linear factors where t_i are the (not necessarily different) roots of $P_0^*(t) \in \mathcal{F}[t]$.

$$P(D^0)(x^0) = (D^0)^n P_0^*(R^0)(x^0) = u \in U$$

$$\overset{\text{T1}}{\underset{\uparrow}{\Rightarrow}} \ P_0^*(R^0)(x^0 + \bar{z}^0) = (R^0)^n(u) + z^0 + \cdots + (R^0)^{n-1}(z_{n-1}^0).$$

Now, $H_D(U) = U \overset{\text{P2}}{\underset{\uparrow}{\Rightarrow}} R(U) \subset U$ and $I - tR$ is invertible on U. By hypothesis $R^0 = R$ on X; hence

(a) $(R^0)^n = R^n$ and $I - tR^0 = I - tR$ on U;

(b) $D^0 = D$ on X and $Z_{D^0} \subset X \Rightarrow Z_D = Z_{D^0} \overset{\text{3-T1}}{\Rightarrow} Z_{(D^0)^n} = Z_{D^n} \overset{\text{P4}}{\subset} H_D(0)$ $\subset U$ as U is D-closed, $\forall\, n$. So we can conclude that

$$b(I - t_1 R^0) \dots (I - t_m R^0)(x^0) = u_1 \in U$$

where $(I - t_i R^0)$ are injective on X^0 by hypothesis.

But this implies that $P_0^*(R^0)$ is injective on X^0 and invertible on U; consequently $x^0 \in U$. ∎

COROLLARY 1. $U = X \Rightarrow H_{D^0}(X) = X$.

Proof. The proof is exactly the same as for "$\Rightarrow$". ∎

We conclude this subsection with the remark that similar notions and concepts can be introduced with respect to the operator R. Thus, D-R Volterra spaces have plenty of structure under the aspect of module theory.

2. The D-R decomposition theorem. This section is entirely devoted to one of the main results of this exposition and enhances the strong role of Volterra right inverses. We repeat therefore the full hypothesis and state

THEOREM 5 (The D-R decomposition theorem). *Let X be a D-R Volterra space of type D_I with $Z_D \neq (0)$. There is an (E, Q)-decomposition of X such that*

(i) *E is a D-R Volterra space (in the restriction) with $S \subset E$.*

(ii) *Q is a vector space over the field $\{P(R)^{-1} Q(R) \mid P, Q \in \mathcal{F}[t]\} \cong \mathcal{F}(t)$.*

(iii) *E is D-R embedded in the reference space $X_0(Z_D)$ by the canonical homomorphism ϕ of (4-T3); $\phi(E) = \phi(X)$.*

Proof. The proof is carried out in several steps and is based on the rather complicated construction of a chain which guarantees a maximal element.

(A) Let $0 \neq x \in X$. The subspace $\langle x \rangle_D := \langle \{ D^i(x) \mid i \geqslant 0 \} \rangle$ is a non-trivial *D-invariant* subspace with elements of the form $P(D)(x)$; $P(t) \in \mathscr{F}[t]$.

The proof is obvious. $\square$

LEMMA 1. (i) $x \notin H_D(V) \Rightarrow P(D)(x) \neq (0) \; \forall \, P \in \mathscr{F}[t]$. *Hence* $\mathscr{F}[t]$ *and* $\langle x \rangle_D$ *are isomorphic as vector spaces by* $P(t) \leftrightarrow P(D)(x)$.

(ii) $x \notin H_D(V)$ *iff* $\langle x \rangle_D \cap H_D(V) = (0)$ *and* $x \neq 0$.

Proof L1. (i): $P(D)(x) = 0 \in V \overset{\text{D1}}{\Rightarrow} x \in H_D(V)$. So, $P_1(D)(x) = P_2(D)(x)$ with $P_1 \neq P_2$ leads to a contradiction and the remaining assertion is obvious.

(ii) $\Rightarrow$: $P(D)(x) \in H_D(V) \overset{\text{P7(i)}}{\Rightarrow} x \in H_D(V)$, where $P(D)(x) = 0 \overset{\text{(i)}}{\Leftrightarrow} x = 0$.

$\Leftarrow$: $x \in H_D(V) \Rightarrow x \in \langle x \rangle_D \cap H_D(V)$, as $V \subset H_D(V)$. $\square$

(B) Let V be a *D-invariant* subspace.

$$ x \notin H_D(V) \Rightarrow H_D(V) \subsetneqq H_D(\langle x \rangle_D \oplus H_D(V)). $$

This follows by (L1 (ii)) and (P2 (i)). $\square$

(C) $x \notin Q + H_D(V) \Rightarrow Q \cap H_D(\langle x \rangle_D \oplus H_D(V)) = (0)$.

Proof (C). We first prove $x \notin Q + H_D(V) \Rightarrow Q \cap (\langle x \rangle_D \oplus H_D(V)) = (0)$.

$x \notin Q + H_D(V) \overset{\text{B}}{\Rightarrow} \langle x \rangle_D \cap H_D(V) = (0)$. So $P(D)(x) + h = 0 \overset{\text{L1}}{\Rightarrow} P = 0$, $P \in \mathscr{F}[t]$, $h \in H_D(V)$.

Suppose now that $P(D)(x) + h = q$ where $q \in Q$ and $q \neq 0$. (T1 (ii)) implies

$$ x = (P_0^*(R))^{-1} R^n(-h) + (P_0^*(R))^{-1} R^n(q) + z_n, \qquad z_n \in Z_{P(D)}; $$

hence $x \in Q + H_D(V)$ by (P2(iii)) (P3(ii)) and (P5(ii)). Contradiction. $\square$

$$ Q \cap (\langle x \rangle_D \oplus H_D(V)) = (0) \overset{\text{P8}}{\Rightarrow} Q \cap H_D(\langle x \rangle_D \oplus H_D(V)) = (0), $$

which completes the proof of (C). $\square$

Combining (B) and (C), we obtain the key result

(D) Let $Q \oplus V_1 \subsetneqq X$ *and let* V_1 *be D-closed. There exists a D-closed proper extension space* V_2 *of* V_1 *such that* $Q \cap V_2 = (0)$, *i.e.* $Q \oplus V_1 \subsetneqq Q \oplus V_2$.

In fact, $Q \oplus V_1 \subsetneqq X \Rightarrow \exists \, x \in X$: $x \notin Q + V_1 \overset{\text{B}}{\Rightarrow} V_1 \subsetneqq H_D(\langle x \rangle_D \oplus V_1)$. With $V_2 := H_D(\langle x \rangle_D \oplus V_1)$ we thus have $V_1 \subsetneqq V_2$ where $V_2 \cap Q = (0)$ by (C) and V_2 is *D*-closed. $\square$

Now we can use the above-mentioned chain argument to show the existence of a maximal element.

(E) Let $W := \{ W_l \}_{l \in L}$ be the family of *D*-closed subspaces of X with $W_l \cap Q = (0)$. Clearly, each W_l contains S by (P4) and (P-3(i)). W is *not* empty because $H_D(0) \in W (0 = Q \cap H_D(0)$ and $S \subset H_D(0)$ by (P6) and (P4)).

Under the set-theoretical inclusion, W can be given a half-order and we can form chains, i.e. totally ordered subsets of W. Let $\{ W_k \}_{k \in K} \subset W$ be a chain.

We define an element M' by $M' := \bigcup_k W_k$ and see that

(i) M' is D-closed by (P7(ii)).

(ii) $M' \cap Q = (0)$ (for $M \cap Q \neq (0)$ contradicts (E)); hence $M' \subset W$.

(iii) M' is an upper bound of the chain $\{W_k\}$.

Thus W is a non-empty inductively ordered set and by Kuratowski's lemma we can infer that W has a *maximal* element M with $M \cap Q = (0)$.

(F) $M \oplus Q = X$.

Suppose that $M \oplus Q \subsetneqq X$. By (D) there exists a D-closed *proper* extension space M_1 of M which belongs to W. Contradiction.

As $M \in W$, we infer by (P2) that M is D-R invariant and $I - tR$ is invertible (on M) $\forall\, t \in \mathscr{F}$ S is contained in M; so we put $E := M$ and the proof of (i) is complete.

(ii): The Volterra property of R implies by (P5(ii)) that every operator polynomial $P(R) = a_0 + \cdots + a_m R^m$, $a_i \in \mathscr{F}$ is invertible on Q. With the scalar product of (P1) X is a module over the ring $\{P(R)^{-1} V(R)/P,\ V \in \mathscr{F}[t]\}$, which clearly is a field isomorphic to $\mathscr{F}(t)$ under $P(R)^{-1} V(R) \leftrightarrow V(t)/P(t)$.

(iii): E is D-R invariant and $\phi_{|E}$ a D-R isomorphism with $\phi(E) = \phi(X)$ by (i) and (4-T3). Hence according to (1-D6)

$$\phi: E \to X_0(Z_D) \text{ is a } D\text{-}R \text{ embedding.} \quad \blacksquare$$

We now see clearly that the difficulties mentioned at the end of (§ 5) do not occur in the case of D-R Volterra spaces. With the terminology of (5-D2) *every* element of E is an (AAE) and all the eigenspaces of D are contained in $H_D(0) \subset E$.

3. The D-R extension theorem. The aim is to solve the following problem:

Given an arbitrary D-R space X over an algebraically *closed* field $\mathscr{F}$ such that R is *without* eigenvalues (i.e. X is a D-R *pre-Volterra* space (1-D2)), construct a D^0-R^0 extension space X^0 of (X, D, R) such that

(i) $X \overset{i}{\to} X^0$ is a canonical D-R embedding,

(ii) R^0 is a Volterra right inverse of D^0 in X^0, i.e. (X^0, D^0, R^0) has to be a D^0-R^0 Volterra extension space of (X, D, R).

We begin with a D-R Volterra space X and look for a way to proceed if R is *not* a Volterra right inverse. Clearly, every operator polynomial $P(R) = a_0 + \cdots + a_m R^m$, $a_0 \neq 0$, is invertible on X; hence the inverse of $x \in X$ under $P(R)$ can be written (up to the scalar factor a_0) as

(4)
$$(I - H(R)R)^{-1}(x) = x + RH(R)(I - H(R)R)^{-1}(x) \quad \text{or}$$

$$(I - H(R)R)^{-1}(x) = \sum_{i=0}^{n-1} R^i (H(R))^i + R^n (H(R))^n (I - H(R)R)^{-1}(x)$$

(where $H(R) := (-1/a_0)(a_1 + \cdots + a_n R^{n-1})$) according to the following general lemma.

LEMMA 2 (An inverse image formula). *Let X be a linear space,*

$A \in L_0(X)$ and $x \in X$. If x has a preimage y under $I - A$, the following relation between the coset $(I - A)^{-1}(x) = y + Z_{I-A}$ and $A^i(x)$, $0 \leqslant i \leqslant n - 1$, holds:

$$(5) \qquad (I - A)^{-1}(x) = \sum_{i=0}^{n} A^i(x) + A^{n+1}(I - A)^{-1}(x),$$

where $A^n(Z_{I-A}) = Z_{I-A}$, $\forall n$.

Proof L2. It is sufficient to show

$$(6) \qquad (I - A)^{-1}(x) = x + A(I - A)^{-1}(x).$$

Substituting $(I - A)^{-1}(x)$ in (6) gives

$$(I - A)^{-1}(x) = x + A(x + A(I - A)^{-1}(x)) = x + A(x) + A^2(I - A)^{-1}(x).$$

If we repeat this process n-times, we obtain (5). To prove (6) we observe first that $x = (I - A)(x) + A(x)$.

$\supset$: Applying $I - A$ to $x + A(I - A)^{-1}(x)$ gives

$$(I - A)(x + A(I - A)^{-1}(x)) = (I - A)(x) + (I - A)A(I - A)^{-1}(x)$$
$$= (I - A)(x) + A(I - A)(I - A)^{-1}(x) = (I - A)(x) + A(x) = x;$$

hence $x + A(I - A)^{-1}(x) \subset (I - A)^{-1}(x)$.

$\subset$: If $y \in (I - A)^{-1}(x)$, then $(I - A)(y) - x = x - x = 0$. Considering the identity $y = (x + (I - A)(y) - x) + A(y)$, we have $x + (I - A)(y) - x = x$ and $A(y) \in A(I - A)^{-1}(x)$, and so $(I - A)^{-1}(x) \subset (x + A(I - A)^{-1}(x))$. With $x = 0$ in (5) the other assertion is obvious. $\square$

If we apply R and D to $(I - H(R)R)^{-1}(x)$, we obtain with the first line of (4)

$$(7) \qquad \begin{aligned} R(I - H(R)R)^{-1}(x) &= (I - H(R)R)^{-1}(R(x)), \\ D(I - H(R)R)^{-1}(x) &= D(x) + (I - H(R)R)^{-1}(H(R)(x)). \end{aligned}$$

When we pass to the problem posed, the difficulty consists in that certain elements $(I - H(R)R)^{-1}(x)$ are *not* defined and (7) makes no sense in general.

We can try to introduce those elements as formal fractions by

$$(8) \qquad x/(I - H(R)R)$$

in the hope that the formula $x/(I - H(R)R) = x + RH(R)/(I - H(R)R)$ can be justified.

According to (7) it is natural to define

$$(9) \qquad \begin{aligned} R^0\big(x/(I - H(R)R)\big) & \quad \text{by} \quad R(x)/(I - H(R)R), \\ D^0\big(x/(I - H(R)R)\big) & \quad \text{by} \quad D(x) + H(R)/(I - H(R)R). \end{aligned}$$

Thus we have at least a formal analogy to (7).

The idea sketched above can indeed be realized and is carried out by

the construction of a *distribution module* [1]. The construction is somewhat complicated, so we shall quote the main steps briefly.

The *D-R* space we are going to extend can be viewed, similarly to (P1), as a unitary module over the ring (algebra)

$$\mathscr{F}[R] := \{P(R)/P(t) \in \mathscr{F}[t]\}$$

of operator polynomials in R. Let

$$\mathscr{H}[R] := \{Q(R)/Q(t) \in \mathscr{F}[t] \text{ with } Q(0) = 1\}.$$

Clearly, by hypothesis, every $Q(R) \in \mathscr{H}[R]$ is injective on X; hence $\mathscr{H}[R]$ is a multiplicative *half-group of regular elements* (trivially) contained in the centre of $\mathscr{F}[R]$.

Consequently, we can embed $\mathscr{F}[R]$ canonically in $\mathscr{F}^0[R]$, the *ring of fractions* with respect to $\mathscr{H}[R]$:

$$\mathscr{F}[R] \xrightarrow{i} \mathscr{F}^0[R] = \{(P(R)/Q(R)): P(R) \in \mathscr{F}[R] \text{ and } Q(R) \in \mathscr{H}[R]\},$$

$$P(R) \xrightarrow{i} Q(R)P(R)/Q(R).$$

We use a similar construction to establish an *extension module* X^0 of X over $\mathscr{F}^0[R]$ and define

(10) $X^0 := \{x/Q(R): x \in X \text{ and } Q(R) \in \mathscr{H}[R]\}.$

As in quotient rings, we define the *equality* of two elements of X^0 by

(11) $x/Q(R) = x/P(R) \quad \text{iff} \quad P(R)(x) = Q(R)(x).$

The *sum* and the *scalar product* are now defined by

$$x/Q(R) + y/T(R) := (T(R)(x) + Q(R)(y))/Q(R)T(R),$$
(12) $T(R), Q(R) \in \mathscr{H}[R],$

$$(P(R)/T(R)) \cdot (x/Q(R)) := P(R)(x)/T(R)Q(R), \quad P(R) \in \mathscr{F}[R]; \ x, y \in X,$$

and the *canonical embedding* of X in X^0 is given by

(13) $X \xrightarrow{i} X^0, \quad x \xrightarrow{i} Q(R)(x)/Q(R).$

$\mathscr{F}(R)$ being an algebra over $\mathscr{F}$, we finally define the *product* with a *scalar* $c \in \mathscr{F}$ by

(14)
$$c \cdot (P(R)/Q(R)) := cP(R)/Q(R) \quad \text{in } \mathscr{F}^0[R],$$
$$c \cdot (x/Q(R)) := cx/Q(R) \quad \text{in } X^0,$$

making thus $\mathscr{F}^0[R]$ into an $\mathscr{F}$-algebra and X^0 into a *vector space* over $\mathscr{F}$, i.e. X^0 is an $(\mathscr{F}^0[R].\mathscr{F})$ double-module.

DEFINITION 3. X^0 is called the "*distribution module* or *distributional extension* of X with respect to $\mathscr{H}[R]$" The elements $x^0 = x/Q(R) \in X^0$ are called *distributions* and *true distributions* if they are *not* contained in X.

$I := Q(R)/Q(R)$ (with $Q(R) \in \mathcal{H}[R]$) is the identity operator on X^0 and every element $x^0 = x/Q(R)$ can be written as $Q(R)^{-1}(x)$ where $Q(R)^{-1} = T(R)/T(R)Q(R)$ and $x \overset{\text{id}}{=} T(R)(x)/T(R)$.

Remark 1. We see in the construction that the fractions $P(R)/Q(R) \in \mathcal{F}^0[R]$ and $x/Q(R) \in X^0$ are such that $Q(R) = 1 + a_1 R + \cdots + a_n R^n \in \mathcal{H}[R]$. But since (14) holds, we can be sure that indeed *all* denominators of the form $V(R) = b_0 + \cdots + b_n \cdot R^n$, $0 \neq b_0$ and hence all fractions $x/V(R)$ do occur. Therefore: The ring $\mathcal{F}^+[R]$ of (P1) can be identified with $F^0[R]$.

We are now going to give X^0 a D^0-R^0 extension space structure.

LEMMA 3. The element $x^0 = (x/Q(R)) \in X^0$, $Q(R) = I + a_1 R + \cdots + a_n R^n \in \mathcal{H}[R]$ can be written as

(15)
$$x/(I-HR) = \sum_{i=0}^{n-1} R^i H^i(x) + R^n H^n(x)/(I-HR),$$

with

$$R^i H^i(x) \overset{\text{id}}{=} Q_i R^i H^i(x)/Q_i; \quad Q_i \in \mathcal{H}[R],$$

$$\text{where} \quad H := H(R) := (-1)(a_1 + \cdots + a_{m-1} R^{m-1}).$$

Hence every element $x^0 = x/V(R) \in X^0$ can be written as in (15), up to a non-zero constant factor $b_0 \in \mathcal{F}$.

Proof: $x = \sum_{i=0}^{n-1} (I-RH)R^i H^i(x) + R^n H^n(x)$ and all follows easily with (11). □

We use (15) with $n = 1$, i.e.

$$x/(I-HR) = x + RH(x)/(I-HR)$$

and choose the following definition of D^0 and R^0:

(16)
(i) $R^0(x/(I-HR)) := R(x)/(I-HR)$,
(ii) $D^0(x/(I-HR)) := D(x) + H(x)/(I-HR)$.

There are several unpleasant details to verify.

LEMMA 4. D^0 and R^0 are well-defined linear operators on X^0.
Proof. (I) D^0 is well defined: We have to show that

$$x/P(R) = y/Q(R) \quad \text{implies} \quad D^0(x/P(R)) = D^0(x/Q(R)).$$

We write $P(R) = I - RL$ and $Q(R) = I - RN$ where $L = L(R)$ and $N = N(R)$.

$$x/(I-RL) = y/(I-RN) \quad \text{iff} \quad (I-RN)(x) \overset{(a)}{=} (I-RL)(y).$$

(A) Applying D to (a) gives $D(x) + L(y) \overset{(b)}{=} D(y) + N(x)$.
(B) (a) implies the relations

$$x = (I-RL)(y) + RN(x), \quad y = (I-RN)(x) + RL(y).$$

(C) Let

Eq1. $D^0\big(x/(I-RL)\big) = D(x)+L(x)/(I-RL),$

Eq2. $D^0\big(y/(I-RN)\big) = D(y)+N(x)/(I-RN).$

Using (B), we substitute x and y on the right side of Eq1 and Eq2. This, after some simplifications, gives

$$D^0\big(x/(I-RL)\big) = \big(D(y)+N(x)\big)+RLN(x)/(I-RL),$$

$$D^0\big(y/(I-RN)\big) = \big(D(x)+L(y)\big)+RLN(y)/(I-RN).$$

The hypothesis together with (b) proves (I).

(II) R^0 is well defined. This is very easy to show and the proof is omitted.

D^0 and R^0 are linear operators.

(III) We show only the linearity of D^0, which presents some difficulties, whereas the linearity of R^0 is easy to verify.

With $x^0 = x/(I-RK)$ and $y^0 = y/(I-RL)$ we have by (12)

$$x^0+y^0 = \big((I-RK)(x)+(I-RH(y))\big)/(I-RH)(I-RK),$$

where

$$(I-RH)(I-RK) = I-R(H+K-HK).$$

We need as an intermediate result

(c) $(K+H-RKH)(x)/(I-RH) = \big(H(x)/(I-RH)\big)+(I-RH)K(x)/(I-RH)$

$= H(x)/(I-RH)+K(x)$

and

(d) $(K+H-RKH)(y)/(I-KH) = K(y)/(I-RK)+H(y).$

Now, formula (15) ($n = 1$) together with the definition of D^0 gives

$$D(x^0+y^0) = D(x)-K(x)+D(y)-H(y)+(K+H-RKH)(x)/(I-RH)+$$
$$+(K+H-RKH)(x)/(I-RK)$$

$$\overset{(c)(d)}{=} \big(D(x)+H(x)/(I-RH)\big)+\big(D(y)+K(x)/(I-RK)\big)$$

$$= D^0(x^0)+D^0(y^0).$$

The homogeneity of D^0 is obvious. $\square$

LEMMA 5. (i) $D^0 R^0 = I$ on X^0. $D^0_{|X} = D$ and $R^0_{|X} = R$.
(ii) $I-tR^0$ is invertible on X^0 for all $t \in \mathscr{F}$
Proof. (i): Let $x^0 = x/(I-RH)$.

$$D^0 R^0\big(x/(I-RH)\big) = D^0\big(R(x)/(I-RH)\big)$$

$$= D^0\big(R(x)+RHR(x)/(I-RH)\big)$$

$$= x+RH(x)/(I-RH) \overset{(15)}{=} x/I-RH$$

Thus $D^0 R^0(x^0) = x^0 \ \forall \ x^0 \in X^0$.

We have to prove that i is a D-R embedding, but we show only for D^0 that $iD = D^0 i$ on X.

$$D^0 i(x) = D^0 \big((I - RH)(x)/(I - RH)\big) = D^0 \big(x/(I - RH)\big) - D^0 \big(RH(x)/(I - RH)\big)$$

$$= D(x) + H(x)/(I - RH) - H(x)/(I - RH)$$

$$= (I - RH)\big(D(x)\big)/(I - RH) = iD(x),$$

where $D(x) \overset{\text{id}}{=} (I - RH)\big(D(x)\big)/(I - RH)$.

(ii): $(I - tR^0)(x^0) = (I - tR^0)\big(x/(I - RH)\big) = x/(I - RH) - tR(x)/(I - RH)$

$$= (I - tR)(x)/(I - RH).$$

Thus, given $x^0 = x/(I - RH)$, we see that $x_1^0 = x/(I - RH)(I - tR)$ is a preimage of x^0 under $(I - tR^0)$. Hence $I - tR^0$ is *surjective* $\forall \, t \in \mathscr{F}$.

Injectivity: $(I - tR^0)\big(x/(I - RH)\big) = 0 \Rightarrow (I - tR)(x)/(I - RH) = \big(0/(I - RK)\big)$ $\Rightarrow (I - tR)(I - RK)(x) = 0 \Rightarrow x = 0$, for R has no eigenvalues. $\square$

We have thus proved the central result of this subsection:

THEOREM 6. *Let X be a D-R pre-Volterra space (i.e. R has no eigenvalues). There exists a canonical D^0-R^0 Volterra extension space X^0 of X and this space is given by* (D3) *together with* (16). ∎

The elements of X^0 which are not contained in X, i.e. the "true" distributions (D3), have an interesting permanence property under D^0 and R^0.

PROPOSITION 9. *A distribution x^0 of X^0 is "true" iff $D^0(x^0)$ or $R^0(x^0)$ is "true".*

Proof:

LEMMA 6. $x/(I - RH) \in X$ *iff* $H(x)/(I - RH) \in X$.

Proof L6. $\Rightarrow$: $x/(I - RH) = x + RH(x)/(I - RH) \in X$ iff $RH(x)/(I - RH) \in$ $\in X$. Hence

$$D^0\big(RH(x)/(I - RH)\big) = H(x)/(I - RH) \in D^0(X) \subset X.$$

The other direction follows by the R^0 invariance of X. $\square$

We have thus proved that $x/(I - RH)$ is true iff $H(x)/(I - RH)$ is true. Let $x^0 = x/(I - RH) \in X^0$.

$$D^0\big(x/(I - RH)\big) = D(x) + H(x)/(I - RH).$$

Hence

$$D^0\big(x/(I - RH)\big) \text{ is true iff } H(x)/(I - RH) \text{ is true } \overset{\text{L6}}{\Leftrightarrow} x/(I - RH) \text{ is true.}$$

$R^0\big(x/(I - RH)\big) = R(x)/(I - RH)$, hence

$$R(x)/(I - RH) \in X \quad \text{iff} \quad x/(I - RH) \in X$$

by the D^0-R^0 invariance of X. ∎

Higher powers of D^0. If we want to calculate $(D^0)^n(x^0)$ with $x^0 = x/$ $/(I - RH)$, we have to use formula (15). Applying $(D^0)^n$ on both sides gives

$$(17) \qquad (D^0)^n\big(x/(I - RH)\big) = \sum_{i=0}^{n-1} D^{n-i} H^i(x) + H^n(x)/(I - RH).$$

The remainder projection $(R^0)^n (D^0)^n$ acting on x^0, see (3,D2), can then be written as

$$(18) \qquad (R^0)^n (D^0)^n (x/(I-RH)) = \sum_{i=0}^{n-1} R^n D^{n-i} H^i(x) + R^n H^n(x)/(I-RH).$$

Given an *arbitrary* D-R Volterra space X, we can write D in place of D^0 and $P(R)^{-1} = (I-RH)^{-1}$ in place of $(x/(I-RH))$ and have, with (17) and (18), useful formulas for $D^n(P(R))^{-1}(x)$ and $R^n D^n(P(R))^{-1}(x)$.

The next two propositions give us some information about what the distributional extension does to interesting subspaces of X, such as, for instance, Z_D, S, Q, $R^n(X)$.

PROPOSITION 10. *Let* X_0 *be the distributional extension of the D-R space* X. *This implies*

$$•(19) \qquad Z_{D0} = Z_D; \quad hence \quad Z_{(D0)^m} = Z_{D^m} \text{ and } S^0 = S,$$

where S^0 *denotes the space of elements of finite order of* X^0.

Proof. Obviously, $Z_D \subset Z_{D0}$. Let $x^0 = x/(I-RH)$ and $D^0(x^0)$ $= D^0(x/(I-RH)) = 0$. By (P9) x^0 is not "true" and, as $D^0 = D$ on X, we conclude that $x^0 \in Z_D$.

$$Z_D = Z_{D0} \Rightarrow R^i(Z_D) = (R^0)^i(Z_{D0}) \Rightarrow S = S^0 \text{ by (3-P2)}.$$

PROPOSITION 11. *Under the same hypothesis as above we have*

(i) $(R^n(X))^0 = (R^0)^n(X^0) \ \forall n$.

(ii) $Q_{(R^0)} = (Q_R)^0$, *i.e.* $\bigcap_{i=0}^{\infty} (R^0)^i(X^0) = (\bigcap_{i=0}^{\infty} R^i(X))^0$,

where $(...)^0$ *denotes the distributional extension of subspaces of* X.

Proof. (i): $(R^n(X))^0 \ni R^n(x)/(I-RH) = (R^0)^n(x/(I-RH)) \in (R^0)^n(X^0)$.

(ii): According to (3-P3 (ii)) we must show that

$$(R^0)^n(D^0)^n(x/(I-RH)) = x/(I-RH) \quad \text{iff} \quad x \in Q, \ \forall n.$$

Let $q \in Q$.

$$q/(I-RH) \overset{3\text{-}P3}{=} R^n D^n(q)/(I-RH)$$

$$= (R^0)^n(D^n(q)/(I-RH)) \in (R^0)^n(X^0) \ \forall n \Rightarrow q/(I-RH) \in Q_{(R^0)}.$$

The other direction is more complicated.

LEMMA 7. *Let* $P(R) = \sum_{i=0}^{m} a_i R^i$ *and* $R^n D^n(x) = x$ *for a fixed* n. *This implies*

$$(20) \qquad R^k D^k P(R)(x) = P(R)(x) \ \forall k \leq n$$

and consequently we have

$$(21) \qquad R^{k+i} D^k P(R)(x) = R^i P(R)(x) \ \forall k \leq n, i \geq 0.$$

Proof L7. By (2-T2 proof II) we have $R^k D^k R^n D^n = R^n D^n$; hence

(22) $$R^k D^k(x) = x, \quad 0 \leqslant k \leqslant n.$$

(a) $n \geqslant i$: $R^n D^n\big(R^i(x)\big) = R^n D^{n-i}(x) = R^i(R^{n-i} D^{n-i})(x) = R^i(x)$ by (22).

(b) $n < i$: $R^i(X) \subset R^n(X)$. By (3-T1) the projection $R^n D^n$ maps $R^n(X)$ onto $R^n(X)$, so $R^n D^n\big(R^i(x)\big) = R^i(x)$. Thus

$$R^n D^n \Big(\sum_{i=0}^m a_i R^i(x)\Big) = \sum_{i=0}^m a_i (R^n D^n) R^i(x) = \sum_{i=0}^m a_i R^i(x)$$

and the assertion follows by (22) with $P(R)(x)$ in place of x. $\square$

We now prove by induction

$$(R^0)^n (D^0)^n\big(x/(I-RH)\big) = x/(I-RH) \ \forall \, n \Rightarrow x \in Q,$$

where $x^0 = x/(I-RH)$.

$n = 1$: $x/(I-RH) = x + RH(x)/(I-RH)$. $R^0 D^0(x^0) = RD(x) + RH(x)/(I-RH)$.

The condition $R^0 D^0(x^0) = x^0$ implies $x = RD(x)$.

n: Suppose now that $(R^0)^n (D^0)^n(x^0) = x^0$ implies $R^n D^n(x) = x$.

$n \to n+1$: With formula (15) we write $x/(I-RH)$ in the form

(23) $$x/(I-RH) = x + \quad + R^i H^i(x) + \quad + R^n H^n(x) + R^{n+1} H^{n+1}(x)/(I-RH)$$

and thus have by formula (18)

(24) $$(R^0)^{n+1}(D^0)^{n+1}\big(x/(I-RH)\big) = R^{n+1} D^{n+1}(x) +$$
$$\ldots + R^{n+1} D^{n+1-i} H^i(x) + \quad + R^{n+1} H^{n+1}(x)/(I-RH).$$

(L7) together with the induction hypothesis yields

$$R^{n+1} D^{n+1-i} H^i(x) = R^i H^i(x), \quad 1 \leqslant i \leqslant n,$$

and the assertion $R^{n+1} D^{n+1}(x) = x$ follows from the condition

$$(R^0)^{n+1}(D^0)^{n+1}(x^0) = (x^0)$$

together with (23) and (24).

Thus $R^n D^n(x) = x \ \forall \, n$ and $x \in Q$ as claimed. ∎

The last two propositions can be summed up under structural aspects. We apply (3-T1) and (3-T3) and obtain

(25) $$X^0 = Z_D \oplus R(Z_D) \oplus \ldots \oplus R^{n-1}(Z_D) \oplus (R^n(X))^0.$$
$$X^0 \supsetneq (R(X))^0 \supsetneq \quad \supsetneq (R^n(X))^0 \supsetneq \ldots \supsetneq (Q_R)^0 = Q_{(R^0)}.$$

We are now in a position to show that the *distributional extension* X^0 is strongly related to the *D-hull* (D1), and is the most "economical" Volterra extension possible.

THEOREM 7. *Let X be a D-R pre-Volterra space and let X^+ be any D^+-R^+ pre-Volterra extension space of the distributional extension (X^0, D^0, R^0) of (X, D, R) such that $Z_{D^+} \subset X^0$. This implies*

$$H_{D^+}(X) = X^0.$$

COROLLARY 1. $(X^0)^0 = X^0.$

The distributional extension is thus the smallest extension such that R has the Volterra property.

Proof. We first observe that $Z_{D^+} \subset X^0$ implies $Z_D = Z_{D^0} = Z_{D^+}$ as $D^+ = D^0 = D$ on X and $Z_D = Z_{D^0}$ by (19).

We can thus apply (T4) and obtain in (X^+, D^+, R^+)

$$X \subset X^0 \Rightarrow H_{D^+}(X) \subset H_{D^+}(X^0) = X^0,$$

because X^0 is closed in itself (as a subspace in X^+) and hence closed in X^+ by (T4C1).

$X^0 \subset H_{D^+}(X)$: Let $x^0 = x/(I - RH)$, where $Q(R) := I - RH = I + a_1 \times \times R + \quad + a_m R^m$ and $x \in X$.

We define an operator polynomial $P(D^+)$ by

$$P(D^+) := a_m + a_{m-1} D^+ + \quad + (D^+)^m$$

and see that $P(D^+)$ can be written as

$$P(D^+) = (D^+)^m(I + a_1 R^+ + \quad + a_m(R^+)^m) = (D^+)^m Q(R^+).$$

Hence

$$P(D^+)(x^0) = (D^+)^m Q(R^+)(x/Q(R)) = (D^+)^m Q(R^0)(x/Q(R))$$
$$= (D^+)^m(Q(R)(x)/Q(R)) = (D^+)^m(x) = D^m(x) \in X.$$

Thus, by (D1), $x^0 \in H_{D^+}(X)$. ∎

Proof of the corollary. We consider the following sequence of distributional extensions:

$$(X^0, D^0, R^0) \subset ((X^0)^0, (D^0)^0, (R^0)^0) \subset (X^+, D^+, R^+).$$

$X^0 \subset (X^0)^0$ implies $H_{D^+}(X^0) = (X^0)^0$ by the theorem.

X^0 is $(D^0)^0$ closed in $(X^0)^0$ by (T4C1); so, by (T4), X^0 is D^+-closed in X^+, whence $H_{D^+}(X^0) = X^0$. ∎

By the extension theorem (T6) we are able to embed the pre-Volterra space (X, D, R) in a Volterra extension space (X^0, D^0, R^0), which in turn can be D^0-R^0 decomposed into $X^0 = (Q_R)^0 \oplus \hat{E}$, according to the decomposition theorem (T5) and (P11).

Here a word of caution is necessary: $\hat{E}$ is in general *not* the distributional extension $(E)^0$ of a subspace $E \subset X$ with $E \oplus Q = X$.

(P10) shows that $Z_D = Z_{D^0}$, and hence by the homomorphism theorem (4-T3) $\phi^0(X^0)$ is contained in the reference space $X_0 = X_0(Z_D)$ and the

homomorphism ϕ^0, given by $\phi^0(x^0) = \sum_{i=0}^{\infty} f_i F^0 (D^0)^i(x^0)$, is a continuation of the canonical homomorphism ϕ. Moreover, $\phi(X) \subset \phi^0(X^0)$ and $\hat{E}$ is D^0-R^0 embedded in $X_0(Z_D)$, where $\phi^0(\hat{E}) = \phi^0(X^0)$.

The next and last theorem sums up all these relations and gives a further result about hull properties.

THEOREM 8. *Let* X *be a D-R pre-Volterra space and* X^0 *its distributional extension.*

(i) *The following diagram of D-R homomorphisms commutes*
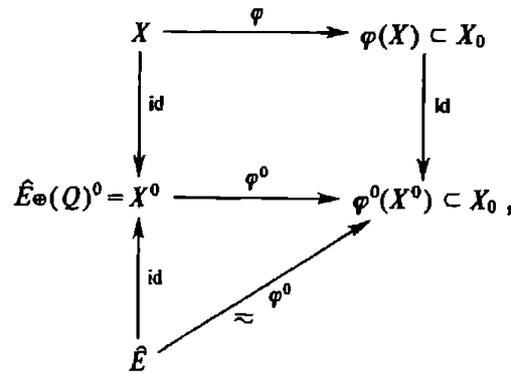
where $Z_{\phi 0} = (Q)^0 \supset Q = Z_\phi$, $\hat{E}$ *is a* D^0-R^0 *invariant Volterra subspace of* X^0, *and* $\phi^0(\hat{E}) = \phi^0(X^0)$ *is a* D_0-R_0 *invariant Volterra subspace of* X_0.

(To be precise, one should write (X, D, R) instead of X, etc.)

(ii) $\phi^0(H_{D^0}(U^0)) = H_{D_0}(\phi^0(U^0))$, *i.e. the* D^0-*hull operation is* ϕ^0-*compatible.* $(U^0$ *a* D^0 *invariant subspace of* X^0.)

In particular,

$$\phi^0(X^0) = H_{D^0}(\phi(X)) \quad and \quad H_{D^0}(0) \overset{\phi}{\cong} H_{D_0}(0),$$

and there is a 1:1 *correspondence between the* D^0-*closed subspaces of* $\hat{E}$ *in* X^0 *and the* D_0-*closed subspaces of* $\phi^0(E)$ *in* $X_0(Z_D)$.

Proof. (i): $D^0 = D$ on X; hence $F^0 = I - RD^0 = F$ on X, and so $F^0(D^0)^i = FD^i$ on $X \forall i$ and consequently $\phi^0 = \phi$ on X. The other assertions are already explained in the preceding text and the Volterra space property of $\phi^0(X^0)$ will be shown in (ii).

(ii): As $(\hat{E}, D^0, R^0)$ is a Volterra space, we infer by the D^0-R^0 embedding via ϕ^0 that $(\phi^0(\hat{E}), D_0, R_0)$ is a Volterra subspace of the Volterra space $(X_0(Z_D), D_0, R_0)$ with $Z_{D_0} \subset \phi^0(\hat{E})$. We can apply (T4C1) and thus obtain

$$H_{D_0}(\phi^0(\hat{E})) = \phi^0(\hat{E}) \quad (\text{in } X_0(Z_D)).$$

Consequently,

$$H_{D_0}(\phi^0(X^0)) = \phi^0(X^0) \quad as \quad \phi^0(X^0) = \phi^0(\hat{E}).$$

Let U^0 be a D^0-invariant subspace of X^0; then $\phi^0(U^0)$ is a D_0-invariant subspace of X_0 with $\phi^0(U^0) \subset \phi^0(X^0)$, and thus

$$(26) \qquad H_{D_0}(\phi^0(U^0)) \subset H_{D_0}(\phi^0(X^0)) = \phi^0(X^0). \quad \square$$

$\phi^0(H_{D^0}(U^0)) = H_{D_0}(\phi^0(U^0))$:

$\subset$: $y^0 \in H_{D^0}(U^0) \Rightarrow P(D^0)(y^0) \in U^0 \overset{4\text{-}T3}{\Longrightarrow} P(D_0)(\phi^0(y^0)) \subset \phi^0(U^0) \Rightarrow \phi^0(y^0) \in H_{D_0}(\phi^0(U^0))$.

$\supset$: $y_0 \in H_{D_0}(\phi^0(U^0)) \Rightarrow P(D_0)(y_0) = \phi^0(u^0)$. Now, $y_0 \in \phi^0(X^0)$ by (26) $\Rightarrow y_0 = \phi^0(x^0)$ and $P(D_0)(y_0) = P(D_0)(\phi^0(x^0)) \overset{4\text{-}T3}{\Longrightarrow} \phi^0(P(D^0))(x^0) = \phi^0(u^0)$.

Hence $P(D^0)(x^0) = u^0 + q^0$, $q^0 \in Q^0 = Z_{\phi^0}$. But $P(D^0)$ is an isomorphism on Q^0 by (P5 (iii)) and we thus have $q^0 = P(D^0)(\bar{q}^0)$ with a certain $\bar{q}^0 \in Q^0$. Therefore $P(D)(x^0 - \bar{q}^0) = u^0 \in U^0$, whence $x^0 - \bar{q}^0 \in H_{D^0}(U^0)$ and $\phi(x^0 - \bar{q}^0) = y_0$. $\square$

$$\phi^0(X^0) = \phi^0(H_{D^0}(X)) = H_{D_0}(\phi^0(X)) \overset{(i)}{=} H_{D_0}(\phi(X)),$$

because $H_{D^0}(X) = X^0$ by (T7).

Every ϕ^0-image of a D^0-closed subspace of $\hat{E}$ is, by a similar argument to that used above, D_0-closed in $\phi^0(\hat{E})$ and hence D_0-closed in $X_0(Z_D)$. The D-R isomorphism ϕ^0 establishes thus a $1:1$ correspondence between the D_0-closed subspaces in $\phi^0(\hat{E})$ and the D^0-closed subspaces in E. Thus we have in particular

$$(H_{D^0}(0)) \overset{\phi}{\cong} H_{D_0}(0).$$

List of symbols and definition

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