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SURVEY OF RECENT RESULTS ON SINGULARITIES OF EQUIVARIANT MAPS

C. T. C. WALL

Department of Pure Mathematics, University of Liverpool, Liverpool, United Kingdom

In order to introduce my viewpoint, before discussing equivariant maps — or even maps — I will begin with the familiar case of singularities of functions of several variables. Most of the important ideas here are due to René Thom.

1. Functions

The earliest result, however, goes back to Morse: that any smooth function on N can be approximated by one all of whose critical points are nondegenerate (hence isolated); moreover, one may suppose the critical values distinct. Nondegeneracy is a condition on second derivatives: it can also be characterised by reducibility to normal form

$$f(x_1, \ldots, x_n) = c+\sum_1^n \varepsilon_i x_i^2 \quad (\text{each } \varepsilon_1 = \pm 1)$$

in local coordinates $x = (x_1, \ldots, x_n)$. This result was elaborated by Cerf [4]. Suppose N compact. Then the functions as above may be considered as constituting open strata of $C^\infty(N)$; one can define strata of codimension 1 by allowing either 2 critical points with the same critical value or one degenerate critical point of type

$$c+x_1^3+\sum_2^n \varepsilon_i x_i^2$$

in suitable local coordinates. Introducing also critical points of types corresponding to Thom's elementary catastrophes allows us to define explicitly strata up to codimension 4. Moreover the list is complete in the sense that any 4-parameter family of functions can be approximated by one meeting only these strata (and, moreover, meeting them transversely in an appropriate sense).

In proceeding further, we observe that the detailed considerations needed are all local. The same will turn out to be the case for maps in general, and we shall usually work in convenient local coordinates in N. We find that the singularities of functions of several variables that are important for the theory can be characterised in several alternative ways. I will first state the result, then give the explanations.

THEOREM 1.1. *The following conditions on $f(x)$ are equivalent*:
(1) *f has a (finite dimensional) versal unfolding*;
(2) *the singularity (of f at $\mathbf{0}$) is algebraically isolated*;
(3) *f is finitely determined (for right equivalence).*

Here, an *unfolding* of f is a mapping

$$\mathbf{R}^n \times \mathbf{R}^a \xrightarrow{F} \mathbf{R} \times \mathbf{R}^a$$

of the form $(x, u) \xrightarrow{F} (F^1(x, u), u)$, where $F^1(x, \mathbf{0}) = f(x)$. A *morphism* of unfoldings is a commutative diagram

$$(*) \qquad \begin{array}{ccc} \mathbf{R}^n \times \mathbf{R}^b & \xrightarrow{G} & \mathbf{R}^p \times \mathbf{R}^b \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle K} \\ \mathbf{R}^n \times \mathbf{R}^a & \xrightarrow{F} & \mathbf{R}^p \times \mathbf{R}^a \end{array} \qquad \text{where } p=1$$

where $K(t, u) = (t, K^1(u))$ and $H(x, u) = (H^1(x, u), K^1(u))$ with $H^1(x, \mathbf{0}) = x$.

An unfolding is *versal* if any unfolding has a morphism to it (is induced from it). Several slight variants on these definitions are important, but I am sticking to the simplest.

In the case when f is a polynomial map, it also defines a complex-analytic map $\mathbf{C}^n \to \mathbf{C}$. If this has an isolated singularity at the origin, we say that the singularity of f is algebraically isolated. This is equivalent to requiring the ideal J_f generated by the partial derivatives $\partial f/\partial x_i$ to have finite codimension μ in the ring $\mathscr{E}_n$ of germs of functions at $\mathbf{0}$ (it turns out not to matter whether we use C^∞ functions or restrict to analytic ones). Indeed, if the functions $\Phi_1, \ldots, \Phi_\mu \in \mathscr{E}_n$ define a basis of $\mathscr{E}_n/J_f$, we can take

$$F^1(x, u_1, \ldots, u_\mu) = f(x) + \sum_1^\mu u_i \Phi_i(x)$$

as a versal unfolding.

We say that f is *finitely determined* — or, more precisely, *k-determined* — if for any function g such that $g - f$ has order $> k$ (i.e. all derivatives of order $\leqslant k$ vanish at the origin) we can find a local diffeomorphism h of $\mathbf{R}^n$ at 0 with $g = f \circ h$. Here h defines a *right equivalence* between f and g.

The culmination of this theory is the result (due to Mather [15], [16]) that all function-germs at $\mathbf{0} \in \mathbf{R}^n$ satisfy the conditions of the Theorem, with the exception of a subset of infinite codimension. This permitted Looijenga [13] to extend the partial stratification of $C^\infty(N)$ described above to include a set whose complement has infinite codimension. Unfortunately, his construction is no longer explicit: serious difficulties in describing it even in codimension 6 are encountered in Wall [26].

We shall not discuss in this article the more delicate parts of singularity theory of functions (Milnor fibration and lattice, monodromy, period map etc); nor the corresponding results for invariant functions.

2. Mappings

For mappings $N^n \to P^p$ with $p > 1$, right-equivalence is not enough to be useful: one has to introduce at least diffeomorphisms of the target, P, giving left-equivalence or $\mathscr{L}$-equivalence; also $\mathscr{A} = \mathscr{R} \times \mathscr{L}$. For the study of the most interesting case, $\mathscr{A}$, it is helpful also to define the groups $\mathscr{C}$, of germs of N-parameter families of diffeomorphisms of P fixing a point $Y \in P$, and $\mathscr{K} = \mathscr{R} \cdot \mathscr{C}$. Roughly speaking $\mathscr{K}$-equivalence measures equivalence of the pre-image $f^{-1}(Y)$ under diffeomorphisms of N. We shall sometimes use $\mathscr{B}$ to denote an unspecified one of $\mathscr{R}$, $\mathscr{L}$, $\mathscr{A}$, $\mathscr{C}$ or $\mathscr{K}$. Recall that all these considerations, and those below, are local.

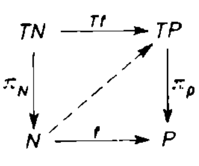
Results analogous to the Theorem above hold in all these cases: we next describe these. The notion of unfolding is the same in each case, but morphisms are not: e.g., for $\mathscr{A}$-equivalence, a morphism is a commutative square (*) but where now $p \neq 1$ in general and

$$K(y, \mathbf{u}) = \left(K^2(y, \mathbf{u}),\ K^1(\mathbf{u})\right)$$

with $K^2(y, \mathbf{0}) = y$ (so for each $\mathbf{u}$ near $\mathbf{0}$, $y \mapsto K^2(y, \mathbf{u})$ is a local diffeomorphism).

If we have a 1-parameter unfolding ($a = 1$) and differentiate with respect to u at $u = 0$, we obtain an element of the "tangent space" $\theta(f)$ at f to the space $C^\infty(N, P)$ of map-germs $N \to P$. Thus $\theta(f)$ consists of germs of vector fields along f. In local coordinates, we can identify $\theta(f)$ with the set of p tuples of elements of $\mathscr{E}_n$.

We can regard $\mathscr{B}$ as a group acting on the manifold $C^\infty(N, P)$ and obtain the tangent spaces $T\mathscr{B}(f)$ to the orbits: also important are certain "extended" tangent spaces $T_e\mathscr{B}(f)$ (the distinction arises according as base points are kept fixed or not). In the diagram



we can identify θf with the set of germs of C^∞ maps $\alpha: N \to TP$ with $\pi_p \circ \alpha = f$ (vector fields along f). Then if θN, θP denote the spaces of germs of sections of π_N and π_P, we define

$$tf: \theta N \to \theta f,$$

$$\omega f: \theta P \to \theta f$$

by composition with Tf (on the right) and f (on the left) respectively. Then

$$T_e \mathscr{R}(f) = tf(\theta N),$$

$$T_e \mathscr{L}(f) = \omega f(\theta P),$$

$$T_e \mathscr{A}(f) = T_e \mathscr{R}(f) + T_e \mathscr{L}(f).$$

Here, θN and θf are modules over the ring $\mathscr{E}_n$, and tf a map of $\mathscr{E}_n$-modules; θP is only a module over $\mathscr{E}_p$, and ωf a map over f^*: $\mathscr{E}_p \to \mathscr{E}_n$. Write $\mathfrak{m}_n$, $\mathfrak{m}_p$ for the maximal ideals (consisting of germs vanishing at the origin) in $\mathscr{E}_n$, $\mathscr{E}_p$. Then

$$T\mathscr{R}(f) = tf(\mathfrak{m}_n \cdot \theta N),$$

$$T\mathscr{L}(f) = \omega f(\mathfrak{m}_p \cdot \theta P), \qquad T\mathscr{A}(f) = T\mathscr{R}(f) + T\mathscr{L}(f),$$

$$T\mathscr{C}(f) = f^* \mathfrak{m}_p \cdot \theta f, \qquad T\mathscr{K}(f) = T\mathscr{R}(f) + T\mathscr{C}(f).$$

Also, for uniformity, set

$$T_e \mathscr{C}(f) = T\mathscr{C}(f), \qquad T_e \mathscr{K}(f) = T_e \mathscr{R}(f) + T_e \mathscr{C}(f).$$

Then we can assert Mather [16, III].

DETERMINACY THEOREM 2.1. *For $\mathscr{B} = \mathscr{R}$, $\mathscr{L}$, $\mathscr{A}$, $\mathscr{C}$ or $\mathscr{K}$ the following properties of the map-germ f are equivalent*:

(1) *f has a finite-dimensional $\mathscr{B}$-versal unfolding*;

(2) $\dim(\theta f / T_e \mathscr{B}(f)) < \infty$;

(2′) $\dim(\mathfrak{m}_n \cdot \theta f / T\mathscr{B}(f)) < \infty$;

(3) *f is finitely $\mathscr{B}$-determined.*

We shall call map-germs satisfying these conditions *$\mathscr{B}$-finite*.

A more precise statement of the relation between (2) and (3) is:

If f is r-$\mathscr{B}$-determined, then $T\mathscr{B}f \supset \mathfrak{m}_n^{r+1} \cdot \theta(f)$. If $T\mathscr{B}f \supset \mathfrak{m}_n^{r+1} \cdot \theta(f)$, then f is $(\varepsilon r + 1)$-$\mathscr{B}$-determined, where $\varepsilon = 1$ for $\mathscr{B} = \mathscr{R}$, $\mathscr{C}$, $\mathscr{K}$ and $\varepsilon = 2$ for $\mathscr{B} = \mathscr{L}$, $\mathscr{A}$ (the estimates in these two cases are due to Gaffney [9]).

As before, we obtain a $\mathscr{B}$-versal unfolding (of least possible dimension) by choosing a basis of $\theta f / T_e \mathscr{B}(f)$, and using the same construction. In the special case when $T_e \mathscr{B}(f) = \theta f$, f is a $\mathscr{B}$-versal unfolding of itself: all other unfoldings are induced from a morphism to f (such unfoldings are said to be trivial). When this is so, f is said to be *$\mathscr{B}$-stable*: or rather, the germ of f at the point in question is so.

The particular cases of this notion are as follows.

$\mathscr{R}$-stable: f is a submersion,

$\mathscr{L}$-stable: f is an embedding,

$\mathscr{C}$-stable: $f(\mathbf{0}) \neq \mathbf{0}$,

$\mathscr{K}$-stable: either $f(\mathbf{0}) \neq \mathbf{0}$ or f is a submersion.

The interesting case is $\mathscr{A}$-stability, which cannot be defined so simply, and was closely studied in an important series of papers by Mather [16, I–VI] which laid the foundations for the whole subject.

There are two key results which describe $\mathscr{A}$-stable maps. First, each jet extension $j^k f$: $N \to J^k(N, P)$ must be transverse to each "$\mathscr{K}$-orbit", and similarly for multijets. Conversely, these conditions (at least if $k \geqslant p+1$, and multijets are allowed multiplicity $(p+1)$) suffice for stability. Secondly, there is a "normal form" for $\mathscr{A}$-stable germs. It is the same form as for an unfolding, versal for a category closely related to that of $\mathscr{K}$-unfoldings. In particular, a map-germ has an $\mathscr{A}$-stable unfolding if and only if it is $\mathscr{K}$-finite.

Now if f is analytic, it can be complexified and we can add the following condition to the theorem.

GEOMETRIC CRITERION. (4) *For some neighbourhood U of* $\mathbf{0}$ *in* $N \otimes C, f_C$ *is $\mathscr{B}$-stable at all points (and finite subsets) of* $U - \{\mathbf{0}\}$.

(I omit discussion of germs at finite subsets for the sake of brevity.) This leads to the philosophy that (roughly) *the important properties of f are determined by the sets $U_{\mathscr{B}}(f)$ of its $\mathscr{B}$-unstable points.*

In addition to Mather's foundational papers, I would like to refer the reader to my recent survey [27] of this area for a fuller account of these notions and an extensive bibliography.

3. The equivariant case

We now suppose G a compact Lie group acting smoothly on the manifolds N (usually compact) and P, and seek to study the typical singularities of maps f: $N \to P$ which are equivariant for G. This seems a natural extension of the above: frequently one finds a problem with natural symmetries which are preserved by the maps of interest.

The overall plan is to follow as closely as possible the results above. Let me first recall the basic structure of compact group actions. At each point $X \in N$ we form the isotropy subgroup G_X of X. Then, by an important result (which is, however, not difficult in the differentiable case) there is always a slice at X: namely, a smooth submanifold S (e.g. a disc), invariant under G_X, transverse at X to the G-orbit $G \cdot X$, and of complementary dimension. Thus a neighbourhood N of the orbit is equivariantly diffeomorphic to the fibre product $G \times_{G_X} S$, and there is a natural bijection between G-equivariant maps

$N \to P$ and G_X-equivariant maps $S \to P$. We always use this construction to reduce to the case when X is a fixed point for G. Next, the local linearisation theorem due to Bôchner states that if X is fixed by G then we can choose local coordinates at X with respect to which G acts linearly. As any equivariant map takes fixed points to fixed points, this shows that for the local theory we can take N, P to be vector spaces with G acting linearly.

Next, one needs some finiteness results. If $S(N)$ denotes the algebra of polynomial functions on N, then according to Hilbert the subring $S^G(N)$ of invariant polynomials is finitely generated. The generators can be taken as homogeneous: write A for the highest of their degrees. If there are l such generators, they define a map $N \xrightarrow{\pi} \mathbf{R}^l$ such that $\pi^* S(\mathbf{R}^l) = S^G(N)$. We shall denote $\pi(N)$ by N/G. By a theorem of Schwarz [25], the same holds for C^∞ functions:

$$\pi^* \mathscr{E}_l = \mathscr{E}^G(N).$$

Similarly, G acts on the $S(N)$ module $S(N, P)$ of polynomial maps $N \to P$: the $S^G(N)$-submodule $S^G(N, P)$ left fixed by G (which correspond to the equivariant maps) is (Poenaru [19]) finitely generated, and we can write B for the highest degree of a generator. Again, these polynomial generators also generate the $\mathscr{E}^G(N)$-module $\mathscr{E}^G(N, P)$ of C^∞ equivariant maps. These finiteness results allow one to set up basic machinery extending that of Mather: this is done ad hoc in Poenaru [19] and explained from a more general viewpoint in Damon [5].

For each group $\mathscr{G}$ as defined above, we write $\mathscr{G}^G$ for the subgroup of those maps which respect the G-action. The corresponding tangent spaces and extended tangent spaces $T\mathscr{G}^G(f)$, $T_e\mathscr{G}^G(f)$ are just the G-invariant parts of those we had previously. The determinacy theorem as described above now goes over to these new cases: the general result seems to have been first stated by Roberts [20], [21] and Damon [5]. There is even still a geometric characterisation, to which we will return below. One point needs clarification: in the definition of unfolding, we insist that G acts trivially on the parameter space $\mathbf{R}^a$ (this is what is needed for the theorem: other cases are not without interest).

For some of the results below, we need somewhat different equivariant versions of Mather's equivalence relations. First set

$$T\mathscr{R}^{*G}(f) = \mathfrak{m}_n^G \cdot T\mathscr{R}_e^G(f),$$
$$T\mathscr{L}^{*G}(f) = f^* \mathfrak{m}_p^G \cdot T\mathscr{L}_e^G(f),$$
$$T\mathscr{A}^{*G}(f) = T\mathscr{R}^{*G}(f) + T\mathscr{L}^{*G}(f),$$

which always have finite codimensions in $T\mathscr{R}_e^G(f)$, $T\mathscr{L}_e^G(f)$, $T\mathscr{A}_e^G(f)$ respectively; and

$$T\mathscr{C}^{*G}(f) = f^* \, \mathfrak{m}_p^G \cdot \theta(f)^G,$$

$$T\mathscr{K}^{*G}(f) = T\mathscr{R}^{*G}(f) + T\mathscr{C}^{*G}(f),$$

which need *not* have finite codimensions in $T\mathscr{C}_e^G(f)$, $T\mathscr{K}_e^G(f)$. These definitions are due to Roberts [20]. It is shown in the appendix to Wall [28] how to construct groups of equivalences with these as the tangent spaces to the orbits. Now the determinacy theorem extends (Roberts [21]) to these cases also, though only for $\mathscr{C}^{*G}$, $\mathscr{K}^{*G}$ does this yield anything new. The relations between these groups are shown in the diagram

$$\begin{array}{ccccccc} 1 & \subset & T\mathscr{L}^{*G}(f) & \subset & T\mathscr{C}^{*G}(f) & \subset & T\mathscr{C}^{G}(f) \\ \cap & & \cap & & \cap & & \cap \\ T\mathscr{R}^{*G}(f) & \subset & T\mathscr{A}^{*G}(f) & \subset & T\mathscr{K}^{*G}(f) & \subset & T\mathscr{K}^{G}(f) \end{array}$$

The proof of the theorem yields explicit estimates, but these are much inferior to those obtained in the absence of a group action. In fact:

for $\mathscr{B} = \mathscr{R}$, $\mathscr{C}$, $\mathscr{K}$, $\mathscr{C}^*$ or $\mathscr{K}^*$ if $T\mathscr{B}^G(f) \supset (\mathfrak{m}^k \theta(f))^G$ then f is $\{(k+1)A + B\}$-$\mathscr{B}^G$-determined;

for $\mathscr{B} = \mathscr{L}$ or $\mathscr{A}$, if $T\mathscr{B}^{*G}(f) \supset (\mathfrak{m}^k \theta(f))^G$, then f is $\{2kA + B\}$-$\mathscr{B}^G$-determined.

These compare unfavourably with the estimates available when G is absent; however it follows from Bruce et al. [3] that if we restrict to the subgroup of $\mathscr{B}^G$ of elements with trivial 1-jet (which is unipotent) we obtain a sharp determinacy criterion which can be used to improve the above.

A further difference from the previous case is that there we had rather simple characterisations of $\mathscr{B}$-stability for $\mathscr{B} = \mathscr{R}$, $\mathscr{C}$, $\mathscr{K}$ and $\mathscr{L}$. These do not persist in the equivariant case; however (Roberts [20], [21]) we do have:

f is $\mathscr{C}^{*G}$-stable $\Leftrightarrow f(\mathbf{0}) \neq \mathbf{0}$;

f is $\mathscr{K}^{*G}$-stable $\Leftrightarrow f$ is $\mathscr{R}^{*G}$-stable or $\mathscr{C}^{*G}$-stable;

f has an $\mathscr{A}^G$-stable unfolding $\Leftrightarrow$ it is $\mathscr{K}^{*G}$-finite.

We shall return to the relation between stability and jet transversality when we have defined equivariant jets.

We observe first, that the geometric characterisations of stability hold for $\mathscr{C}^{*G}$ and $\mathscr{K}^{*G}$ rather than for the — at first sight more natural — $\mathscr{C}^G$ and $\mathscr{K}^G$. Secondly we note that Mather's theory of stability and of topological stability makes essential use of the result that $\mathscr{K}$-finiteness holds in general. A careful analysis of the conditions for finiteness to hold in general is given in Roberts [20], [23]: he gives there examples where this fails even for $\mathscr{K}^G$-finiteness. Such examples are even easier to find for $\mathscr{K}^{*G}$-finiteness: we will return to this below.

To pursue the notions of stability and determinacy further, we next need to study the geometry of a complex singularity. Again we first consider the case when there is no group action.

4. Complex geometry

The best way to approach the geometric criterion for $\mathscr{B}$-finiteness seems to be by sheaf theory. We shall now assume N and P to be complex manifolds though, as usual, we shall be concerned only with germs.

Define $\mathscr{V}(N)$ to be the sheaf of germs of (holomorphic) tangent vector fields to N; similarly $\mathscr{V}(P)$; then $\mathscr{V}(f) = f^* \mathscr{V}(P)$ is the sheaf of germs of tangent vector fields along f. There is a natural map tf defining an exact sequence

$$\mathscr{V}(N) \xrightarrow{tf} \mathscr{V}(f) \to \mathscr{S}_R(f) \to 0:$$

for the stalks at $X \in N$, this reduces to the sequence

$$\theta(N) \xrightarrow{tf} \theta(f) \to \theta(f)/T_e\mathscr{R}(f) \to 0$$

previously considered. If $Y \in P$, and $\mathscr{M}_Y$ denotes the subsheaf of $\mathscr{O}(P)$ of functions vanishing at Y, then $f^*.\mathscr{M}_Y$ defines a subring of $\mathscr{O}(N)$, and we define

$$\mathscr{S}_C(f) = \mathscr{V}(f)/f^*.\mathscr{M}_Y \cdot \mathscr{V}(f),$$

$$\mathscr{S}_K(f) = \mathscr{V}(f)/f^*.\mathscr{M}_Y \cdot \mathscr{V}(f) + tf\,\mathscr{V}(N).$$

All of these are coherent sheaves over N, and the stalk of $\mathscr{S}_B(f)$ is $\theta(f)/T_e\mathscr{B}(f)$. Now for coherent sheaves we have the zero theorem: X is an isolated point of support of $\mathscr{S}$ if and only if the stalk of $\mathscr{S}$ at X is finite dimensional. Applying this in the above cases gives the geometric criterion for finite determinacy (note that the stalk is zero at $\mathscr{B}$-stable points).

If f is $\mathscr{C}$-finite, then $f^{-1}(Y)$ is (locally) reduced to the origin. It follows by semicontinuity that for some neighbourhood U of X in N, all fibres $f^{-1}(Z) \cap U$ are finite: $f|U$ is a finite map in the usual sense. Loosely speaking, f is $\mathscr{C}$-finite $\Leftrightarrow$ f is a finite map. It is easily shown (Wall [27]) that $\mathscr{R}$-finite germs (other than submersions) can only exist when $\dim N = 1$. In general we write Σ_f for the set of critical points (in N) of f. These are just the points at which the germ of f is not $\mathscr{R}$-stable. Now arguing as above using the geometric criterion for $\mathscr{K}$-finiteness and the characterisation of $\mathscr{K}$-stability shows that

$$f \text{ is } \mathscr{K}\text{-finite} \quad \Leftrightarrow \quad f|\Sigma_f \text{ is a finite map.}$$

We now turn to $\mathscr{L}$-finiteness and $\mathscr{A}$-finiteness: these lie somewhat deeper. For f to be $\mathscr{L}$-finite, it must certainly be $\mathscr{C}$-finite, hence (as above) finite in the sense of (finite-to-one). By Grauert's theorem if this is so, f_* preserves coherence. In particular, $f_* \mathscr{V}(N)$ is coherent over P. Now composing with f induces a map

$$\omega f\colon \mathscr{V}(P) \to f_* \mathscr{V}(N)$$

whose cokernel we denote by $\mathscr{S}_L(f)$. It is coherent by the above, and the stalk at a point $Y \in P$ is

$$\bigoplus \{\theta_X(f)\colon X \in f^{-1}(Y)\}/\omega f \theta_Y(P)$$

so its vanishing reflects $\mathscr{L}$-stability of the germ of f at the finite set $f^{-1}(Y)$ (rather than at a single point) — in fact here, vanishing implies that $f^{-1}(Y)$ must be a single point X (or empty) and $df X$ injective; as already mentioned.

Similarly for f to be $\mathscr{A}$-finite, it must be $\mathscr{K}$-finite; but this implies $f | \Sigma_f$ a finite germ, hence preserving coherence. As $\mathscr{S}_R(f)$ is supported on Σ_f, this implies $f_* \mathscr{S}_R(f)$ coherent, hence also

$$\mathscr{S}_A(f) = \operatorname{Coker} \overline{\omega f}\colon \mathscr{V}(P) \to f_* \mathscr{S}_R(f).$$

Again, the support of $\mathscr{S}_A(f)$ is the set of points Y such that the germ of f at $f^{-1} Y \cap \Sigma_f$ (a finite set since f is $\mathscr{K}$-finite) is $\mathscr{A}$-unstable. Now applying the zero theorem to $\mathscr{S}_L(f)$ and $\mathscr{S}_A(f)$ implies the geometric criteria for $\mathscr{L}$- and $\mathscr{A}$-finiteness given earlier.

These arguments appear first in Gaffney [8], where they are attributed to Mather; an exposition is also given in Wall [27].

In the equivariant case, when we regard N and P as complex it is also desirable to regard G as complex. It is convenient to work in the context of reductive algebraic groups G acting (algebraically) on affine algebraic varieties N and P (though results can be generalised for complex analytic actions on Stein manifolds). Invariant theory yields a "categorical" quotient $\pi\colon N \to N/G$, though this is not a pointwise quotient (nor is π proper — except when G is finite). However, for each $\xi \in N/G$, the preimage $\pi^{-1}(\xi)$ contains a unique closed G-orbit T_ξ, which lies in the closure of every orbit in this preimage.

All the sheaves defined above now admit actions of G (this is defined so that the sheaf of sections of a G-bundle is an example). Now for a G-sheaf $\mathscr{S}$ on N we define $\pi_*^G(\mathscr{S})$ to be the sheaf on N/G whose sections over an open set U are the G-invariant sections of $\mathscr{S}$ over $\pi^{-1}(U)$. Roberts [22] proves two fundamental results for coherent G-sheaves $\mathscr{S}$ over N:

THEOREM 4.1. (a) *$\pi_*^G(\mathscr{S})$ is coherent over N/G.*

(b) *If X is a fixed point of G, the stalk of $\pi_*^G(\mathscr{S})$ at $\pi(X)$ is the G-invariant part of the stalk of $\mathscr{S}$ at X.*

Now let $\mathscr{B} = \mathscr{R}$, $\mathscr{C}$ or $\mathscr{K}$. Then $\pi_*^G \mathscr{S}_B(f)$ is a coherent sheaf over N/G, whose support contains $\pi(X)$, for X a fixed point, if and only if the germ of f

at X is not $\mathscr{B}^G$-stable. For a general point ξ we take the closed orbit T_ξ; a point $X \in T_\xi$ with isotropy group H, say; an H-slice M at X, and then consider stability of the germ at X of $f \mid M$. Thinking (for simplicity) of a fixed point, we deduce in turn equivalence of

(i) the germ of f at X is $\mathscr{B}^G$-finite;
(ii) $T_e \mathscr{B}^G(f)$ has finite codimension in $\theta^G(f)$ (at X);
(iii) the stalk of $\pi_*^G \mathscr{S}_B(f)$ at $\pi(X)$ is finite dimensional;
(iv) $\pi(X)$ is an isolated point of support of $\pi_*^G \mathscr{S}_B(f)$;
(v) there is a neighbourhood U of $\pi(X)$ in N/G such that if $\xi \in U$, $\xi \neq \pi(X)$, the germ of f along the closed orbit T_ξ is $\mathscr{B}^G$-stable.

This gives the appropriate form of the geometric criterion in these cases.

We can extend the argument to $\mathscr{C}^{*G}$ and $\mathscr{K}^{*G}$. For if $\mathscr{M}^G$ is the subsheaf of $\mathscr{O}(P/G)$ of germs of functions vanishing at $\eta \in P/G$, and $\bar{f}$: $N/G \to P/G$ the induced map, we set

$$\mathscr{S}(\mathscr{C}^{*G}, f) = \pi_*^G \mathscr{V}(f)/\bar{f}^* \mathscr{M}^G \cdot \pi_*^G \mathscr{V}(f),$$

$$\mathscr{S}(\mathscr{K}^{*G}, f) = \pi_*^G \mathscr{S}_R(f)/\bar{f}^* \mathscr{M}^G \cdot \pi_*^G \mathscr{S}_R(f).$$

Again, these are coherent and supported at the unstable points; the geometric criterion follows as before. Now recall that f is $\mathscr{C}^{*G}$-stable at $X \Leftrightarrow \bar{f}(\pi(X)) \neq \eta$. It follows that f is $\mathscr{C}^{*G}$-finite at T_ξ if and only if ξ is isolated in $\bar{f}^{-1}(\eta)$. As before, this is equivalent to the germ of $\bar{f}$ at η being a finite map-germ. Similarly, if we define $\Sigma^G(f)$ to be the support of $\mathscr{S}(\mathscr{R}^G, f)$: the set of $\mathscr{R}^G$-unstable points (of N/G), we find using the characterisation of $\mathscr{K}^{*G}$-stability that

$$f \text{ is } \mathscr{K}^{*G}\text{-finite} \quad \Leftrightarrow \quad \bar{f} \mid \Sigma^G(f) \text{ is a finite map-germ.}$$

Now as before if f is $\mathscr{C}^{*G}$-finite, $\bar{f}_*$ preserves coherence; in particular, $\bar{f}_* \pi_*^G \mathscr{V}(N)$ (which is the same as $\pi_*^G f_* \mathscr{V}(N)$) is coherent, hence so is

$$\mathscr{S}(\mathscr{L}^G, f) = \text{Coker } \omega^G f\colon \pi_*^G \mathscr{V}(P) \to \bar{f}_* \pi_*^G \mathscr{V}(N).$$

Similarly $\mathscr{K}^{*G}$-finiteness of f leads to coherence of

$$\mathscr{S}(\mathscr{A}^G, f) = \text{Coker } \bar{\omega}^G f\colon \pi_*^G \mathscr{V}(P) \to \bar{f}_* \pi_*^G \mathscr{S}_R(f).$$

Now applying the zero theorem again leads to geometric criteria for determinacy. Thus we have

THEOREM 4.2. *f is $\mathscr{A}^G$-finite at a fixed point X if and only if it is $\mathscr{K}^{*G}$-finite and for some invariant neighbourhood U of X, f is $\mathscr{A}^G$-stable at each closed orbit in $U - \{X\}$ and at each finite set of such orbits with a common image in P/G.*

We remark that it is not necessary to restrict to the case of a fixed point. Moreover, the standard method of dealing with other cases (pick a closed orbit T_ξ, a point $X \in T_\xi$ with isotropy group H, and an H-slice at X) does *not* reduce to a straightforward analogue for H-equivariant maps at a fixed point of H. It is necessary also to consider the isotropy group $K \supset H$ of $f(X) \in P$, and a K-slice at $f(X)$. Although these actions can be linearised, the linear problems encountered are more complicated than in the simple case of a fixed point, though they can be handled by the same methods (see Wall [28]). The sheaf approach has the advantage of by-passing these complications, and giving the "correct" result without trouble.

5. Equivariant jets

Ordinary singularity theory makes heavy use of jet bundles, submanifolds thereof, transversality etc. It is not immediate how to carry over this apparatus to the equivariant case. For example, the subset of ordinary jet space consisting of jets of equivariant maps is neither open nor closed, and yields no relation between the jets of a function f at different points in the same G-orbit.

Consider the quotient N/G (in the real case with G compact, this will be the usual pointwise quotient; in the complex case, as above). There is a canonical stratification of this corresponding to the partition of (closed) orbits in N into orbit types, given by taking the (conjugacy class of the) stabiliser H together with (the isomorphism class of) its representation on the tangent space. Denote by Λ the set of orbit types; by $\{(N/G)_\lambda : \lambda \in \Lambda\}$ the stratification. Over a stratum $(N/G)_\lambda$ it is easy to define jet bundles: indeed, there are two versions corresponding at a fixed point X to taking quotients of $\theta(f)^G$ by $(\mathfrak{m}_X^k \cdot \theta(f))^G$ or by $(\mathfrak{m}_X^G)^k \cdot \theta(f)^G$. These families of subgroups define the same topology, so are essentially equivalent. Either leads to satisfactory notions of jet bundles over each fixed stratum. These are good for results involving jet transversality, since this only makes sense when restricted to a stratum which is a manifold. Roberts [20], [23] gives criteria for stability analogous to Mather's, involving such jet transversality conditions.

However, for some purposes it is necessary to have a jet space mapping to the whole of N/G, and this cannot be constructed simply by piecing together the above bundles. Instead, we adapt a method of Bierstone [1]. We first reduce (taking a slice) to the case that X is fixed under G, then take coordinates in which the action is linear. Then take a minimal set $\{\psi_i : 1 \leqslant i \leqslant m\}$ of generators of $S^G(N, P)$ as $S^G(N)$-module. Then in both the analytic and differentiable cases we can express the equivariant map germ f as $f = \sum_1^m h_i \psi_i$, with the h_i invariant function germs. These are the coordinates of a map germ of N/G to $\boldsymbol{R}^m$ (or $\boldsymbol{C}^m$) – say of N/G to M – which extend to a differentiable (or analytic) germ K: $\boldsymbol{R}^l \to \boldsymbol{R}^m$ ($\boldsymbol{C}^l \to \boldsymbol{C}^m$). We now consider the jet bundles for K, restricted to the subset N/G: these are the desired bundles. We denote them by $J^k(N/G, M)$. They have the unsatisfactory feature that to find the jet of f we have to make choices of functions h_i and an extension K; however it is easy to verify that the operations one performs are independent of these choices.

THEOREM 5.1 (Wall [28]). *Let $\mathscr{B}$ be one of $\mathscr{R}$, $\mathscr{C}$, $\mathscr{K}$, $\mathscr{C}^*$ and $\mathscr{K}^*$. Then*
(i) *the condition for $\mathscr{B}^G$-stability of f depends only on the equivariant* 1-*jet of f;*
(ii) *any $\mathscr{B}^G$-stable germ is determined (up to $\mathscr{B}^G$-equivalence) by its equivariant* 1-*jet;*
(iii) *the set* $\mathrm{Uns}(\mathscr{B}^G)$ *of* 1-*jets of $\mathscr{B}^G$-unstable germs is a (closed) algebraic set in $J^1(N/G, M)$.*

Remark 1. In order to obtain such results as (iii) it is essential to have a jet space defined over N/G.

Remark 2. In the cases of $\mathscr{C}$, $\mathscr{C}^*$ it is even sufficient to work with equivariant 0-jets. Also, it is shown in Bruce et al. [3] that an $\mathscr{K}^G$-stable map is $(\mathfrak{m}_X^2 \cap \mathfrak{m}_X^G) \cdot \mathscr{E}^G(N, P)$-$\mathscr{K}^G$-determined.

In the cases of $\mathscr{L}$ and $\mathscr{A}$ analogous results hold, but we need multijets.

THEOREM 5.2 (Wall [28]). *Let $\mathscr{B}$ denote $\mathscr{L}$ or $\mathscr{A}$. There is an (easily computable) integer k such that*
(i a) *The condition for $\mathscr{B}^G$-stability of a (multi) germ depends only on its equivariant k-jet.*
(i b) *A multigerm is $\mathscr{B}^G$-stable if its restriction to each k-point subset is so.*
(ii) *Any $\mathscr{B}^G$-stable (multi) germ is determined up to $\mathscr{B}^G$-equivalence by its equivariant k-jet.*
(iii) *There is a closed algebraic subset* $\mathrm{Uns}(\mathscr{B}^G)$ *of* $(J^k(N/G, M))^k$ *such that a multigerm is $\mathscr{B}^G$-stable if and only if its k-jet avoids this subset.*
(iv a) $\mathrm{Uns}(\mathscr{L}^G)$ *contains all k tuples with two source points coincident.*
(iv b) *A k tuple $(\phi_1, \ldots, \phi_k)$ fails to belong to* $\mathrm{Uns}(\mathscr{A}^G)$ *if and only if those ϕ_i which are jets of $\mathscr{K}^G$-unstable germs have distinct source points, and the multijet they determine is $\mathscr{A}^G$-stable.*

6. Real geometry

In the case of function-germs, Kuo [11] showed in 1968 that a function f is topologically determined by its r-jet provided an inequality of the form

$$\sum \left| \frac{\partial f}{\partial x_i} \right| > C \|x\|^{r-1}$$

holds in a neighbourhood of the origin. This inequality should be seen as quantifying the condition that $\mathbf{0}$ is an isolated critical point of f. Numerous developments by several authors (see Wall [27] for references) on this theme led to a major advance by Wilson [30], [31] in conceptualisation. We shall give the results in the terminology of Wall [27].

First observe that each $\mathscr{B}(=\mathscr{R}, \mathscr{L}$, etc.), though originally defined as a group of C^∞-maps, has a natural C^k-version. (Of course if two C^∞-maps are C^k-equivalent, then the equivalence must itself satisfy stringent conditions.) Next, denote by $\mathfrak{m}_n^\infty$ the ideal in $\mathscr{E}_n$ of flat functions $\mathfrak{m}_n^\infty = \bigcap \{\mathfrak{m}_n^k : k \in N\}$: correspondingly, $\mathscr{E}_n/\mathfrak{m}_n^\infty$ is the ring of formal power series, or ∞-jets, and we can ask whether f is determined (for some equivalence relation) by its ∞-jet.

Now introduce the conditions (for each $\mathscr{B}$)

(a_k) f *is* ∞-C^k-$\mathscr{B}$-determined;

(b_k) f is finitely-C^k-$\mathscr{B}$-determined;

(t) $T\mathscr{B}(f) \supset \mathfrak{m}_n^\infty \cdot \theta(f)$;

(g) f has $\mathscr{B}$-stable germs on a punctured neighbourhood of $\mathbf{0}$ in $\mathbf{R}^n$.

Condition (g) is appropriate for analytic germs, but in the C^∞ case needs to be strengthened to a quantitative version (e), corresponding to the above condition of Kuo for right equivalence.

For $\mathscr{B} = \mathscr{R}$, $\mathscr{C}$ or $\mathscr{K}$ we define an ideal $I_B(f)$ in $\mathscr{E}_n$ as follows. $I_R(f)$ is generated by the $p \times p$ minors of the matrix representing df; $I_C(f) = f^* \mathfrak{m}_p \cdot \mathscr{E}_n$; $I_K(f) = I_R(f) + I_C(f)$. These are the stalks of sheaves which (in complex geometry) are supported at the points where f is $\mathscr{B}$-unstable: $\Sigma(f)$, $f^{-1}(\mathbf{0})$ or $\Sigma(f) \cap f^{-1}(\mathbf{0})$ respectively. A finitely generated ideal $I = \langle \phi_1, \ldots, \phi_t \rangle$ in $\mathscr{E}_n$ is said to be *elliptic* if it satisfies the following equivalent conditions

(Ei) $I \supset \mathfrak{m}_n^\infty$;

(Eii) some inequality $\sum_1^t \phi_i(x)^2 \geqslant C\|x\|^\alpha$ ($C, \alpha > 0$) holds on some neighbourhood of $\mathbf{0}$.

Now for $\mathscr{B} = \mathscr{R}$, $\mathscr{C}$ or $\mathscr{K}$ we can define the condition

(e) $I_B(f)$ is elliptic.

In the cases $\mathscr{B} = \mathscr{L}$ or $\mathscr{B}$ a more complicated condition (e) is needed, since stability is a condition on multigerms: we defer the details. Now we have

THEOREM 6.1. *If* $\mathscr{B} = \mathscr{R}, \mathscr{L}, \mathscr{C}$ *or* $\mathscr{K}$, *the following are equivalent*: (a_k) ($0 \leqslant k \leqslant \infty$), ($b_k$) ($0 \leqslant k < \infty$), (t) *and* (e).

Moreover if f is analytic, these are equivalent to (g).

The case $G = \mathscr{A}$ is much more delicate. Wilson [30], [31] shows that, provided f is analytic and $\mathscr{K}$-finite, conditions (a_∞), (t) and (g) are equivalent.

In a preprint du Plessis and Brodersen [6] the hypothesis of analyticity (but not that of $\mathscr{K}$-finiteness) is dispensed with, and equivalence of (a_k) $(k_0 \leqslant k \leqslant \infty)$, (b_k) $(k_0 \leqslant k < \infty)$, (t) and (e) is proved, for $k_0 = \min(n+1, p+1)$. More recently, Brodersen [1] has established the equivalence of (a_k) $(p+1 \leqslant k \leqslant \infty)$, (b_k) $(p+1 \leqslant k < \infty)$, (t) and (e) without imposing any condition of f. The proof – involving careful estimates at all stages – is very difficult. The least possible value of k_0 is not precisely known, but (except in the nice dimensions) is certainly strictly greater than 0.

All these notions and results (except the final ones of Brodersen) can be generalised to the equivariant case. The following are taken from Wall [29]. We suppose $\mathbf{0}$ a fixed point of G, and sometimes use local coordinates here.

First, we have

$$\mathfrak{m}_n^\infty \cap \mathscr{E}^G(N) = \bigcap \{(\mathfrak{m}_n^G)^k : k \in N\}$$

and denote this ideal $\mathfrak{m}_n^{G\infty}$. Corresponding results hold for modules, so that ∞-jets are well defined and correspond (as before) to formal power series. For a finitely generated ideal $I = \langle \phi_1, \ldots, \phi_t \rangle$ in $\mathscr{E}^G(N)$, I contains $\mathfrak{m}_n^{G\infty}$ if and only if (Eii) above holds: again, we call such ideals *elliptic*.

For $\mathscr{B} = \mathscr{R}, \mathscr{C}, \mathscr{K}, \mathscr{C}^*, \mathscr{K}^*, \mathscr{L}$ or $\mathscr{A}$ define

(a_k) f is C^k-$\mathscr{B}^G$-determined by its infinite jet;

(b_k) f is C^k-$\mathscr{B}^G$-determined by a finite jet;

(t) $T\mathscr{B}^G(f) \supset \mathfrak{m}_n^{G\infty} \cdot \theta^G(f)$;

(g) for some neighbourhood U of $\mathbf{0}$ in N, all multigerms of f in $U - \{\mathbf{0}\}$ are $\mathscr{B}^G$-stable.

For $\mathscr{B} = \mathscr{R}, \mathscr{C}, \mathscr{K}, \mathscr{C}^*$ or $\mathscr{K}^*$, define $I_B^G(f)$ to be the 0th *fitting ideal of* $\theta^G(f)/T_e\mathscr{B}^G(f)$. Then

(e) $I_B^G(f)$ is elliptic.

For $\mathscr{B} = \mathscr{R}, \mathscr{C}, \mathscr{K}, \mathscr{C}^*$ or $\mathscr{K}^*$, define $I_B^G(f)$ to be the 0th *fitting ideal* of this condition using

$$I'^G_{C*}(f) = f^* \mathfrak{m}_p^G \cdot \mathscr{E}^G(N),$$
$$I'^G_{K*}(f) = f^* \mathfrak{m}_p^G \cdot \mathscr{E}^G(N) + I_R^G(f).$$

A further condition is easily seen to be equivalent to these. Let $j_1^G f\colon N/G \to J_G^1(N/G, M)$ be an equivariant 1-jet of f; write $\mathrm{Uns}(\mathscr{B}^G)$ for the set of 1-jets of $\mathscr{B}^G$-unstable germs. Then define

(e′) There exist a neighbourhood U of $\mathbf{0}$ in N and positive constants C, α such that the distance

$$d\big(j_1^G f(\pi x), \mathrm{Uns}(\mathscr{B}^G)\big) \geqslant C\|x\|^\alpha \quad \text{for } x \in U.$$

THEOREM 6.2. *For* $\mathscr{B} = \mathscr{R}, \mathscr{C}, \mathscr{K}, \mathscr{C}^*$ *or* $\mathscr{K}^*$, *the following conditions on equivariant germs f are equivalent*: (a_k) $(k_0 \leqslant k \leqslant \infty)$, (b_k) $(k_0 \leqslant k < \infty)$, (t), (e) *and* (e′).

Estimates for k_0 can be obtained, but are not at present good enough to be of interest.

The definition of conditions (e) for the remaining cases hinges on the above analysis of the sets $\mathrm{Uns}(\mathscr{L}^G)$ and $\mathrm{Uns}(\mathscr{A}^G)$. Let U be a neighbourhood of $\mathbf{0}$ in N; k as in Theorem 5.2. Define D_L^G to be the subset of U^k of k tuples such that either (1) one of the points x_i is at the origin or (2) two of the points coincide. Define $D_A^G(f)$ to be the set of k tuples such that either

(1) one of the $x_i = \mathbf{0}$, or

(2) for some $i \neq j$, $x_i = x_j \in \Sigma_G(f)$.

We can now define, for $\mathscr{B} = \mathscr{L}$ or $\mathscr{A}$,

(e) There exist a neighbourhood U of $\mathbf{0}$ in N and constants C, $\alpha > 0$ such that for $x_1, \ldots, x_k \in U$,

$$d\big((j_G^k f\pi(x_1), \ldots, j_G^k f\pi(x_k)), \mathrm{Uns}(\mathscr{B}^G)\big) \geqslant Cd\big((x_1, \ldots, x_k), D_B^G(f)\big)^\alpha.$$

THEOREM 6.3. *Let f be an equivariant analytic germ. Suppose that either $\mathscr{B} = \mathscr{L}$ and f is $\mathscr{C}^{*G}$-finite or $\mathscr{B} = \mathscr{A}$ and f is $\mathscr{K}^{*G}$-finite. Then the following conditions are equivalent*: (a_∞), (t), (g), (e).

It seems likely that, with considerable extra effort, the hypothesis of analyticity could be dispensed with, and the conclusion strengthened to include (a_k) and (b_k) for $k \geqslant k_0$ for some k_0: indeed, the proof already gives somewhat more than is stated above. However, the assumption of $\mathscr{K}^{*G}$-finiteness is much more difficult to abandon, and is by no means so anodyne as in the case where G is absent, as we now discuss.

7. Counterexamples

7.1. (Roberts [20], [23]) **An example to show that K^G-finiteness does not hold in general.** (As this is the weakest property in the C^∞-theory, nothing else holds either.) Let $n > s \geqslant 6$; let the cyclic group Z/nZ act on $N \cong C^3$ by $t(x, y, z) = (tx, ty, z)$; on $P \cong C$ by $t(u) = (t^s u)$.

We claim there are no $\mathscr{K}^G$-finite germs $N \to P$. Now equivariance forces f to map the fixed point set in N (the z-axis) to that in P (the origin). It is enough (by the geometric criterion) to show that all germs of f along the z-axis are $\mathscr{K}^G$-unstable; for this it suffices to consider germs at $\mathbf{0}$. But $T\mathscr{R}^G(f)$ is spanned as $\mathscr{E}^G$-module by 5 elements $\left(x\frac{\partial f}{\partial x}, x\frac{\partial f}{\partial y}, y\frac{\partial f}{\partial x}, y\frac{\partial f}{\partial y} \text{ and } \frac{\partial f}{\partial z}\right)$ and $T\mathscr{K}^G(f)$ by these together with f. Since $\theta^G(f)$ requires $(s+1)$ generators (e.g. $x^s, x^{s-1}y, \ldots, y^s$) the claim is established.

7.2. A large class of examples where there are no K^{*G}-finite germs. Suppose G any nontrivial compact Lie group; N, P G-vector spaces with $\dim N^G > \dim P^G$, $\dim N < \dim P$, and the principal isotropy group of G on

N trivial. Let $f\colon N \to P$ be an analytic equivariant map (defined on some open neighbourhood U of $\mathbf{0}$ in N).

At a point with trivial isotropy group, $\mathscr{R}^G$-stability reduces to $\mathscr{R}$-stability, i.e. df surjective. This cannot hold, since $\dim N < \dim P$. Thus $\Sigma_G f$ contains all points with trivial isotropy group; but these are dense, hence $\Sigma_G f = N$.

Equivariance of f forces $f(N^G) \subset P^G$. Since $\dim N^G > \dim P^G$, the fibres of $f \mid N^G$ have positive dimension. Thus for the complexification f_C, $\mathbf{0}$ is not isolated in $f_C^{-1}\{\mathbf{0}\} \cap N^G$. As the germ of f at any of these points is $\mathscr{K}^{*G}$-unstable, we deduce that the germ at $\mathbf{0}$ is not $\mathscr{K}^{*G}$-finite. As the existence of a $\mathscr{K}^{*G}$-finite germ would imply the existence of an analytic one, we conclude there are none.

Since the C^∞-properties studied fail to hold in general, we turn instead to topological properties. Here I must first refer to Bierstone [1] (see also, Field [7]) on "equivariant general position". From the viewpoint of this survey, we can loosely interpret this result as stating that we can approximate any germ by one which is topologically $\mathscr{K}^G$- (or perhaps $\mathscr{K}^{*G}$-) stable. This leaves the interesting question of topological ($\mathscr{A}^G$-) stability.

7.3. Examples where no topologically $\mathscr{A}^G$-stable maps exist. It is shown by Nakai [17] that for $n \geqslant 4$ the topological type of a general map $C^3 \to C^3$ of the form

$$(x, y, z) \mapsto (za(x, y), zb(x, y), c(x, y)),$$

where each of a, b, c is homogeneous of degree n, depends on a (at least one) continuous parameter. If we take $G = C^x \times C^x$ acting on source and target by

$$(t, u)(x, y, z) = (tx, ty, uz),$$

$$(t, u)(X, Y, Z) = (t^n uX, t^n uY, t^n Z);$$

then the above is the general form for an equivariant map. Thus none is topologically stable.

Next forget the complex structure, and identify source and target each with $\boldsymbol{R}^6$; also replace $C^x \times C^x$ by its (maximal) compact subgroup $S^1 \times S^1$. An extension of the same argument yields a modulus for the generic equivariant map here also. A further extension allows us to restrict to a finite subgroup $\boldsymbol{Z}_N \times \boldsymbol{Z}_K$, provided $N > 2n$ is prime to n and $K \geqslant 3$.

These latter cases are not so rigid; in particular, as we move the source point along the z-axis, the topological modulus will not stay constant. The map from the z-axis to the moduli space is thus also a topological invariant. It is now easy to see that we have in general infinitely many topological moduli, and that a generic map is not even finitely determined for topological equivalence.

Such results were also obtained by Nakai [18] for Z_2-actions, in the case $\dim V = \dim W$, $\dim V^G = 1 + \dim W^G$, $\operatorname{codim} W^G \geqslant 3$. The method of proof here is different, and is too long to summarise here; the argument uses the geometry of webs.

To conclude, let me speculate a little on what might be true. The above counterexamples certainly have a good many properties in common. In paticular, it seems likely that an analogue of Thom–Mather topological stability theory (see e.g. Gibson *et al.* [10]) could be pushed through in the equivariant case subject to a $\mathscr{K}^{*G}$-finiteness hypothesis (though there are significant difficulties). On the other hand, in the situation of 7.2 it is easy to see that (unless f is completely degenerate) f cannot be stratified as a Thom map, (it presents "éclatement" in the sense of Thom), so that the theory certainly will not work here. So perhaps genericity of $\mathscr{K}^{*G}$-finiteness is necessary and sufficient for such a theory to apply (and conditions for this to hold were worked out by Roberts [23]). In the remaining cases, one must look at some relation weaker than $\mathscr{A}^{*G}$-equivalence. Perhaps some form of "equivalence after blowing-up" (cf. Kuo [12], Sabbah [24]) will be the most appropriate.

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