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RUNGE–KUTTA METHODS FOR PARABOLIC EQUATIONS WITH NONSMOOTH INITIAL DATA

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1. Introduction

We shall consider the time-semidiscrete approximation to the abstract parabolic equation of the form:

$$\partial_t u + A_t u = 0, \tag{1.1}$$

$$u(0) = U_0. \tag{1.2}$$

The solution of (1.1) is the function u: $[0, T] \to X$ where X is a reflexive Banach space. The operator A_t for each $t \in [0, T]$ is a closed operator with the domain $D(A_t) = D_t$ dense in X. In the abstract setting the parabolicity means that the following assumptions are imposed:

ASSUMPTION 1. *There exists $\Theta \in (0, \pi/2)$ such that the set $\Sigma = \{\lambda; \arg\lambda \notin [-\Theta, \Theta]\} \cup \{0\}$ belongs to the resolvent set of the operator A_t for each $t \in [0, T]$ and the following inequality*

$$\|(\lambda I - A_t)^{-1}\| \leqslant c_1/(1+|\lambda|) \tag{1.3}$$

is valid for $\lambda \in \Sigma$ and $t \in [0, T]$.

Assumption 1 implies that for each t the operator A_t is a generator of the holomorphic semigroup $G_t(\tau) = \exp(-\tau A_t)$ and the following inequality is valid

$$\|A_t^\alpha G_t(\tau)\| = c_2/\tau^\alpha \tag{1.4}$$

for positive α, uniformly with respect to t [4].

The properties of the solution of (1.1) were considered in the literature for various assumptions concerning the regularity of the operator function A_t with respect to t [1–3]. The review of the results is presented in the monograph [4]. In our paper we shall define the regularity classes $K(\mu, \alpha)$ of the operator function A_t. The assumptions used in the theory of the equations of type (1.1) correspond to the assumption $A_t \in K(\mu,\alpha)$ with μ, α properly chosen for each particular assumption set.

In Section 2 we shall introduce the definition of the class $K(\mu,\alpha)$ and give some examples elucidating the meaning of the assumptions hidden in the definition of $K(\mu,\alpha)$. The more detailed discussion of the properties of the classes $K(\mu,\alpha)$ will be published in the separate paper [5]. We shall give the regularity results for the solution of (1.1) depending on μ, α, leaving the proof again to paper [5]. The results of [5] show that using the definition of $K(\mu,\alpha)$ one can construct in a unified way the theory for (1.1) covering the previously known results. Moreover, having at hand the value of μ, α in the intervals $\mu > 0$, $0 < \alpha \leqslant 1$, we can choose them in the optimal way in the practical applications and get the stronger regularity results for the solution.

In Section 3 we shall define approximate methods of the Runge-Kutta type and exhibit their properties pertinent to equation (1.1). We shall prove stability and some smoothing properties of discrete analogue of (1.1).

Section 4 is devoted to the error estimate analysis for nonsmooth initial data.

The approximate methods for the parabolic equations have been considered by several authors. We shall mention only the papers closely related to the problem investigated in this paper. M.-N. le Roux [8] obtained the error estimates for time-independent operator and proved the stability of multistep methods under the assumption that the domain of the operator is time--independent [9]. The error estimates in the case of nonsmooth initial data were derived for the backwards Euler method and for regularly accretive operator by Mingou and Thomme [10], Suzuki [11], Luskin and Rannacher [12]. Sammon [13] considered the various finite difference schemes for the equation obtained by applying first space discretization to (1.1) with selfadjoint operators A_t having constant domain.

2. Regularity of the solution

We start with assumptions concerning the regularity of the operator function A_t with respect to t. For $\mu > 0$, let $[\mu]$ and $\{\mu\}$ be numbers such that $[\mu]+\{\mu\} = \mu$; $[\mu]$ is an integer and $0 < \{\mu\} \leqslant 1$. By $C^{\mu}(X)$ we shall denote the space of functions v: $[0,T] \to X$ having the bounded strong derivatives up to order $[\mu]$ and with $\partial_t^{[\mu]} v$ being $\{\mu\}$-Hölder continuous. The symbol $\mathscr{L}(X,Y)$ will stand for the Banach algebra of bounded operators acting from X to Y. We shall use the symbols $D(t)_{\alpha,p}$ and $\| \ \|_{t,\alpha,p}$ for the interpolation space obtained by means of real α, p-K-interpolation functor [14] applied to the interpolation couple $\{D_t, X\}$ and the norm in this space, respectively.

The regularity of A_t with respect to t will be classified by the following definition:

DEFINITION 1. We shall say that $A_t \in K(\mu, \alpha)$ for $0 < \alpha < 1$, $\mu > 0$ if and only if there exists a space Y such that
(i) for $t \in [0, T]$; $D_t \subset Y \subset D(t)_{\alpha,\infty}$,
(ii) $A_t^{-1} \in C^\mu(\mathscr{L}(X, Y))$.

One can define the class $K(\mu, \alpha)$ in the following alternative way:

DEFINITION 1′. $A_t \in K(\mu, \alpha)$ if and only if

$$A_t^{-1} \in C^\mu(\mathscr{L}(X, X)) \tag{2.1}$$

and for $\lambda \in \Sigma$; $r, t, s \in [0, T]$ the following inequalities

$$\|A_r(\lambda I - A_r)^{-1}(\partial_t^{[\mu]} A_t^{-1} - \partial_s^{[\mu]} A_s^{-1})\| \leqslant c_2 |\lambda|^{-\alpha} |t - s|^{\{\mu\}}, \tag{2.2}$$

$$\|A_r(\lambda I - A_r)^{-1} \partial_t^i A_t^{-1}\| \leqslant c_3 |\lambda|^{-\alpha}, \quad i = 0, 1, \ldots, [\mu], \tag{2.3}$$

are valid with the constants c_2, c_3 independent of λ, r, s, t. The equivalence follows from the following expression for the norm in $D(t)_{\alpha,\infty}$

$$|x|_{t,\alpha,\infty} = \sup_{0 < \tau < \infty} \|\tau^\alpha A_t (\tau I + A_t)^{-1} x\|. \tag{2.4}$$

Starting from Definition 1′, we can define Y by the formula

$$Y = \bigcap_t D(t)_{\alpha,\infty}$$

and prove (i) and (ii) in Definition 1.

Now we shall consider some examples.

EXAMPLE 1. The operators are regularly accretive operators in Hilbert space H associated with the form $a_t(u, v)$ defined on $V \times V$. One has $D_t \subset V \subset X$. It is assumed that

$$\operatorname{Re} a_t(u, u) \geqslant c \|u\|_V^2, \tag{2.5}$$

$$a_t \in C^\mu[\mathscr{L}(V \times V, \boldsymbol{C})], \tag{2.6}$$

where $\boldsymbol{C}$ stands for the set of complex numbers. It can easily be shown that $A_t^{-1} \in C^\mu[\mathscr{L}(V', V)]$ where V' denotes the space dual to V $(H' = H)$. From the conditions [4]

$$\|A_t(\lambda I - A_t)^{-1}\|_{\mathscr{L}(V, V')} \leqslant c/|\lambda|, \tag{2.7}$$

$$\|A_t(\lambda I - A_t)^{-1}\|_{\mathscr{L}(V, X)} \leqslant c/|\lambda|^{1/2}, \tag{2.8}$$

it follows that $A_t, A_t^* \in K(\mu, 1/2)$ in H and $A_t, A_t^* \in K(\mu, 1)$ in V'.

EXAMPLE 2. Let us consider the operators

$$\mathscr{A}_t = \sum_{|\beta| < 2m} a_\beta(X, t) D^\beta, \tag{2.9}$$

$$\mathscr{B}_t^j = \sum_{|\beta| \leqslant m} b_{j\beta}(x, t) D^\beta, \quad j = 1, \ldots m, \tag{2.10}$$

written in the standard notation, where β is a multiindex and $D^\beta = \partial_{x_1}^{\beta_1} \dots \partial_{x_n}^{\beta_n}$. Let Ω be a bounded domain in $\boldsymbol{R}^n$ with the boundary sufficiently regular, $\mathscr{A}_t$ uniformly strongly elliptic differential operator, $\mathscr{B}_t^j$ normal boundary operators complementing $\mathscr{A}_t$ [4, 14]. By $W^{s,p}$ we shall denote Sobolev spaces [4] (for integer s, $W^{s,p}$ consists of functions having the generalized derivatives up to order s integrable with pth power). For $a_\beta, b_{j\beta}$ sufficiently regular we can define the operator A_t in $L_p = W^{0,p}$ with the domain

$$D(A_t) = W_{\mathscr{B}_t}^{2m,p} = \{u \in W^{2m,p}; \mathscr{B}_t^j u|_{\partial\Omega} = 0,\ j = 1, \dots, m\}$$

and putting $A_t u = \mathscr{A}_t u$ for $u \in D(A_1)$. Denoting by $W_{\mathscr{B}_t}^{s,p}$ the subspace of $W^{s,p}$ defined by

$$W_{\mathscr{B}_t}^{s,p} = \{u \in W^{s,p}; \mathscr{B}_t^j u|_{\partial\Omega} = 0 \text{ if } m_j > s - 1/p,\ j = 1, 2, \dots m\},$$

one has the Grisvard interpolation theorem [15]

$$[L_p, W_{\mathscr{B}_t}^{s,p}]_{\alpha,p} = W_{\mathscr{B}_t}^{s\alpha,p}.$$

Taking $\alpha < \inf_{m_j > 0} ((m_j + 1/p)/2m)$ and $Y = W^{2\alpha m,p}$ if $m_j \neq 0$ for all j and $Y = \{u \in W^{2m\alpha,p}; u|_{\partial D} = 0\}$ otherwise, we obtain from the Grisvard theorem

$$D_t \subset Y \subset D_{t,\alpha,\infty}. \tag{2.11}$$

In order to prove that $A_t \in K(\mu, \alpha)$ it is sufficient to show that $A_t^{-1} \in C^\mu[\mathscr{L}(X, Y)]$. For $\mu < 1$ one can consider the equations

$$\mathscr{A}_t(u_t - u_s) = (\mathscr{A}_s - \mathscr{A}_t)u_s, \tag{2.12}$$

$$\mathscr{B}_t^j(u_t - u_s) = (\mathscr{B}_s^j - \mathscr{B}_t^j)u_s, \quad j = 1, \dots, m, \tag{2.13}$$

where $u_t = A_t^{-1} f$. From (2.12) and (2.13) it follows [4] that

$$\|(u_t - u_s)\|_{W^{2m,p}} \leqslant \|(\mathscr{A}_s - \mathscr{A}_t)u_s\|_{L_p} + \sum_j \|(\mathscr{B}_s^j - \mathscr{B}_t^j)u_s\|_{W^{2m-m_j-1/p,p}(\partial\Omega)}.$$

If the coefficients a_β, $b_{j\beta}$ are sufficiently regular one gets the Hölder continuity of u_t in $W^{2m,p}$ norm. In order to consider the regularity for higher value of μ one has to differentiate formally the equations

$$\mathscr{A}_t u_t = f, \quad \mathscr{B}_t^j u_t = 0, \quad j = 1, \dots, m,$$

with respect to t. After the first differentiation one gets

$$\mathscr{A}_t \partial_t u_t = -\partial_t \mathscr{A}_t u_t,$$
$$\mathscr{B}_t^j \partial_t u_t = -\partial_t \mathscr{B}_t^j u_t,$$

and

$$\mathscr{A}_t(\partial_t u_t - \partial_s u_s) = (\mathscr{A}_s - \mathscr{A}_t)\partial_s u_s + \partial_t \mathscr{A}_t(u_s - u_t) + (\partial_s \mathscr{A}_s - \partial_t \mathscr{A}_t)u_s,$$
$$\mathscr{B}_t^j(\partial_t u_t - \partial_s u_s) = (\mathscr{B}_s^j - \mathscr{B}_t^j)\partial_s u_s + \partial_t \mathscr{B}_t^j(u_s - u_t) + (\partial_s \mathscr{B}_s^j - \partial_t \mathscr{B}_t^j)u_s,$$

where $\partial_t \mathscr{A}_t$, $\partial_t \mathscr{B}_t^j$ are operators obtained by formal differentiation of the coefficients of the operators. The equations above allow for showing that $A_t^{-1} \in K(\mu, \alpha)$ for $1 < \mu < 2$ under obvious assumptions imposed on the coefficients a_β and b_β^j. The analysis for higher value of μ can be accomplished in the similar manner by successive differentiation of equations defining u_t.

Remarks. In case of the second order differential operator, one gets $A_t \in K(\mu, \alpha)$ for $\alpha < 1/2 + 1/(2p)$ in case of time-dependent boundary condition with $m_1 = 1$ and $\alpha = 1$ for $m_1 = 0$. In [6] we have extended the definition of $K(\mu, \tau)$ to $\alpha > 1$. Under this definition it was shown that $A_t \in K(\mu, \alpha)$ for $\alpha < 1 + 1/(2p)$ in the case $m_1 = 0$.

Now we shall present the regularity results for the solution of equation (1.1) [5]. The solution can be represented by the family of the evolution operators defined by the relation

$$u(t) = U(t, s)\, u(s) \tag{2.14}$$

where u is the solution of (1.1). In terms of evolution operators we can state the results in the theorem.

THEOREM 1. *If $A_t \in K(\mu, \alpha)$ for $\mu > 0$, $0 < \alpha \leqslant 1$ then the bounded evolution operators $U(t, s)$ exist for $0 \leqslant s \leqslant t \leqslant T$ under the assumption that $\mu + \alpha > 1$, and for*

$$\beta < \min(1, \mu) + \alpha - 1, \qquad \varrho < \min(\{\mu + \alpha\}, \{\mu\}, \alpha), \qquad \varrho' < \alpha, \qquad t' < t$$

the following inequalities are valid

$$\begin{aligned}
\|\partial_t^i u(t, s)\| &\leqslant c/|t - s|^i, \qquad i = 1, \ldots, [\mu + \alpha],\\
\|A_t^{1+\beta} U(t, s)\| &\leqslant c/|t - s|^{1+\beta},\\
\|\partial_t^{[\mu+\alpha]} U(t, s) - \partial_{t'}^{[\mu+\alpha]} U(t', s)\| &\leqslant c|t - t'|^{\varrho}/|t' - s|^{[\mu+\alpha]+\varrho},\\
\|A_t^{\varrho'} \partial_t^i u(t, s)\| &\leqslant c/(t - s)^{i+\varrho'}, \qquad i = 1, \ldots, [\mu + \alpha] - 1,\\
\|A_t^{\varrho} \partial_t^{[\mu+\alpha]} u(t, s)\| &\leqslant c/|t - s|^{[\mu+\alpha]+\varrho}.
\end{aligned}$$

In the error analysis we shall use the operators $U^*(t, s)$ adjoint to $U(t, s)$. The operators $U^*(t, s)$ correspond to the following equation

$$\partial_\tau v = -A_\tau^* v$$

with the reversed time direction. The following relation holds

$$v(s) = U^*(t, s)\, v(t).$$

From the assumption that $A_t^* \in K(\mu^*, \alpha^*)$ we can deduce the similar regularity results for $U(t, s)$ with obvious change of the role of variables t and s. Since X is reflexive Banach space, we obtain in this way the regularity of both families $U(t, s)$ and $U^*(t, s)$ with respect to both variables. One has to understand the corresponding relation in which the operator of the form $U(t,s)A_s^{1+\beta^*}$ is involved e.g.

$$\|U(t,s)A_s^{1+\beta^*}\| \leqslant c/(t-s)^{1+\beta^*}$$

in the following sense. The operator $U(t,s)A_s^{1+\beta^*}$ defined on $D(A_s^{1+\beta^*})$ being dense in X has the bounded continuous extension on X. In the further consideration we shall not distinguish between the operator and its extension in the above sense.

3. Stability and smoothing properties of Runge–Kutta methods

The class of Runge–Kutta schemes considered in the paper can be described by the following equations

$$y_{n,\nu} = y_n + \Delta t \sum_{\nu'=1}^{q} \mathfrak{a}_{\nu\nu'} A_{n\Delta t + t_{\nu'}} y_{n,\nu'}, \qquad \nu = 1, \ldots, q, \tag{3.1}$$

$$y_{n+1} = y_n + \Delta t \sum_{\nu=1}^{q} b_\nu A_{n\Delta t + t_\nu} y_{n,\nu}, \tag{3.2}$$

where $t_\nu = \tau_\nu \Delta t$, $0 < \tau_\nu \leqslant 1$, y_n correspond to approximate value of solution at $t = n\Delta t$ defined by (3.1) are auxiliary values. We shall introduce the following notation:

$$(\tilde{\mathscr{A}}_n)_{\nu\nu'} = A_{n\Delta t + t\nu'}\delta_{\nu\nu'}, \qquad (\bar{\mathscr{A}}_n)_{\nu\nu'} = A_{n\Delta t}\cdot\delta_{\nu\nu'},$$

$$(\tilde{\tau})_{\nu\nu'} = \tau_\nu \delta_{\nu\nu'}, \qquad e = [1, 1, \ldots, 1]^T, \qquad b = [b_1, \ldots, b_q].$$

Assuming that $(I + \Delta t \mathfrak{a} \tilde{\mathscr{A}}_n)^{-1}$ exists, we can write using matrix notation

$$y_{n+1} = W(n+1, n) y_n = [I + \Delta t \mathfrak{b} \tilde{\mathscr{A}}_n (I + \Delta t \mathfrak{a} \tilde{\mathscr{A}}_n)^{-1} e] y_n.$$

Moreover, we shall introduce

$$W_k(n+1, n) = [I + \Delta t \mathfrak{b} \tilde{\mathscr{A}}_k (I + \Delta t \mathfrak{a} \tilde{\mathscr{A}}_k)^{-1} e],$$

$$W(j,k) = W(j, j-1)\, W(j-1, j-2) \ldots W(k+1, k),$$

$$W_l(j,k) = W_l(j, j-1)\, W_l(j-1, j-2) \ldots W_l(k+1, k).$$

By $\omega(z)$ we shall denote the rational function defined by the relation

$$\omega(\Delta t) = W(n+1, n) \quad \text{for } A_t = -Z.$$

It will be assumed that the scheme satisfies the following assumption.

ASSUMPTIONS 2.

(i) $\quad |\omega(z)| \leqslant 1; \quad \operatorname{Re} z \leqslant 0,$

(ii) $\quad \lim_{z\to\infty} \omega(z) = \omega(\infty) < 1,$

(iii) $\quad \mathfrak{a}^{-1}$ exists.

In other words, Assumptions (i) and (ii) means that the scheme is strongly A-stable.

Let us now recall certain properties of the scheme of this type applied to (1.1) wich constant operator. One has the following error estimate

$$\|\exp(-A(l-k')\Delta t)-W(l,k')\| \leqslant c\Delta t^k/((l-k')\Delta t)^k \tag{3.4}$$

where k is the order of the methods. One observes that (3.4) is equivalent to the following properties:

$$\begin{aligned} &W(l,k') = W^{(1)}(l,k')+W^{(2)}(l,k'),\\ &\|W^{(1)}(l,k')\| \leqslant c\Delta t^{\alpha}/((l-k')\Delta t)^{\alpha}, \quad \alpha \leqslant k, \end{aligned} \tag{3.5}$$

$$\|A^{\alpha} W^{(2)}(l,k')\| \leqslant c/((l-k')\Delta t)^{\alpha}, \quad \alpha \leqslant k, \tag{3.6}$$

Indeed, putting $W^1(l,k') = -\exp(-A(l-k')\Delta t)+W(l,k')$, we get (3.5) from (3.4). Inequality (3.6) is the direct consequence of the properties of the semigroup. From (3.5) and (3.6) and semigroup properties after some algebra one can obtain (3.4). The properties described by (3.5) and (3.6) will be called *smoothing properties* of the scheme in further discussion. We shall prove similar smoothing properties for time-dependent operator.

We start by the following lemma.

LEMMA 1. *If $A_t \in K(\mu,\alpha)$ for $\alpha+\mu > 1$ then under Assumption 2 there exists h_0 such that $(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_n)^{-1}$ exists for $\Delta t < h_0$ and*

$$\|\tilde{\mathscr{A}}_n(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_n)^{-1}\| \leqslant c/\Delta t, \tag{3.7}$$

$$\|(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_n)^{-1}\| \leqslant c. \tag{3.8}$$

Proof. We can write

$$\begin{aligned}(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_n) &= (\mathscr{A}_n^{-1}\mathfrak{a}^{-1}+\Delta t I)\mathfrak{a}\tilde{\mathscr{A}}_n\\ &= (\tilde{\mathscr{A}}_n^{-1}\mathfrak{a}^{-1}+\Delta t I)[I+(\tilde{\mathscr{A}}_n^{-1}\mathfrak{a}^{-1}+\Delta t I)^{-1}(\mathscr{A}_n^{-1}-\tilde{\mathscr{A}}_n^{-1})\mathfrak{a}^{-1}]\mathfrak{a}\tilde{\mathscr{A}}_n\end{aligned}$$

where $(\mathscr{A}_n^{-1}\mathfrak{a}^{-1}+\Delta t I)^{-1} = \mathfrak{a}\mathscr{A}_n(I+\Delta t\,\mathfrak{a}\mathscr{A}_n)^{-1}$.

The term in square brackets has bounded inverse due to the equality

$$\begin{aligned}Q &= \mathfrak{a}\mathscr{A}_n(I+\Delta t\,\mathfrak{a}\mathscr{A}_n)^{-1}(\tilde{\mathscr{A}}_n^{-1}-\mathscr{A}_n^{-1})\\ &= \mathfrak{a}\left(\frac{1}{\Delta t}I-\mathscr{A}_n\right)(I+\Delta t\,\mathfrak{a}\mathscr{A}_n)^{-1}\tilde{\mathscr{A}}_n\left(\frac{1}{\Delta t}I+\mathscr{A}_n\right)^{-1}(\mathscr{A}_n^{-1}-\tilde{\mathscr{A}}_n^{-1})\end{aligned}$$

and the inequality

$$\|Q\| \leqslant c\Delta t^{\mu+\alpha-1}.$$

The remaining part of the lemma follows from the formula

$$(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_n)^{-1} = \tilde{\mathscr{A}}_n^{-1}\mathfrak{a}^{-1}[I+Q]^{-1}(\mathscr{A}_n^{-1}\mathfrak{a}^{-1}+\Delta t I)^{-1}$$

defining the inverse and the equality

$$\tilde{\mathscr{A}}_n(I+\mathfrak{a}\tilde{\mathscr{A}}_n\Delta t)^{-1} = \frac{1}{\Delta t}\mathfrak{a}^{-1}[I-(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_n)^{-1}].$$

For the operator function A_t we can prove the following smoothing properties.

THEOREM 2. *If $A_t \in K(\mu,\alpha)$ for $\alpha+\mu>1$ then the discrete evolution operators have the following properties*

$$\|W(n,k)\| \leqslant c(1+|\ln(n-k)|),$$

$$W(n,k) = W^{(1)}(n,k)+W^{(2)}(n,k),$$

$$\|W^{(1)}(n,k)\| \leqslant c\Delta t^{1+\beta}/((n-k)\Delta t)^{1+\beta}\cdot(1+|\ln(n-k)|), \tag{3.9}$$

$$\|A_{n\Delta t}^{1+\beta} W^2(n,k)\| \leqslant c\Delta t^{1+\beta}/((n-k)\Delta t)^{1+\beta}\cdot(1+|\ln(n-k)|), \tag{3.10}$$

for $\beta<\min(1,\mu)+\alpha-1$.

Proof. We can write

(3.11)

$$W(n,k) = W_n(n,k)+\sum_{i=k+1}^{n} W_n(n,i)[W(i,i-1)-W_n(i,i-1)]W(i-1,k)$$

where

$$W(i,i-1)-W_n(i,i-1) = \Delta t\, b\tilde{\mathscr{A}}_n(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_n)^{-1}(\tilde{\mathscr{A}}_n^{-1}-\tilde{\mathscr{A}}_i^{-1})\tilde{\mathscr{A}}_i(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_i)^{-1}e.$$

Equation (3.11) multiplied by $\tilde{\mathscr{A}}_{n+1}(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_{n+1})^{-1}$ can be written in the form

$$\Phi(n,k) = \Phi_0(n,k)+\sum_{k+1}^{n}\Delta t\, K(n,i)\,\Phi(i,k), \tag{3.12}$$

where

(3.13)

$$\Phi(n,k) = \tilde{\mathscr{A}}_{n+1}(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_{n+1})^{-1}e[W(n,k)-W_n(n,k)],$$

$$K(n,i) = \tilde{\mathscr{A}}_{n+1}(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_{n+1})^{-1}e\,W_n(n,i)b\tilde{\mathscr{A}}_n(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_n)^{-1}(\tilde{\mathscr{A}}_n^{-1}-\tilde{\mathscr{A}}_i^{-1}),$$

$$\Phi_0(n,k) = \sum_{i=k+1}^{n}\Delta t\, K(n,i)\varphi(i,k),$$

$$\varphi(i,k) = \tilde{A}_i(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_i)^{-1}e\,W_{i-2}(i-1,k).$$

We can split the operators $K(n,i)$ into the sum of two terms defined by the expressions

(3.14)

$$K^{(i)}(n,i) = \tilde{\mathscr{A}}_{n+1}(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_{n+1})^{-1}e\,W^{(i)}(n,i)b\tilde{\mathscr{A}}_n(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_n)^{-1}(\tilde{\mathscr{A}}_n^{-1}-\tilde{\mathscr{A}}_i^{-1}).$$

From the equality

$$(3.15)\quad \tilde{\mathscr{A}}_n(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_n)^{-1}(\tilde{\mathscr{A}}_n^{-1}-\tilde{\mathscr{A}}_i^{-1})$$
$$=\left(\frac{1}{(n-i)\Delta t}+\tilde{\mathscr{A}}_n\right)(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_n)^{-1}\tilde{\mathscr{A}}_n\left(\frac{1}{(n-i)\Delta t}+\tilde{\mathscr{A}}_n\right)^{-1}(\tilde{\mathscr{A}}_n^{-1}-\tilde{\mathscr{A}}_i^{-1})$$

and (3.5) for $\alpha = 2$ one gets

$$\|K^{(1)}(n,i)\| = c/((n-i)\Delta t)^{2-\alpha-\min(1,\mu)}.$$

In order to get the same estimate for K^2 one uses the identity

$$\tilde{\mathscr{A}}_{n+1}(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_{n+1}) = (I+\mathscr{B})\tilde{\mathscr{A}}_n(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_n)^{-1}$$

where

$$\mathscr{B}=\left(\frac{1}{\Delta t}I+\tilde{\mathscr{A}}_{n+1}\right)(I+\Delta t\,\mathfrak{a}\tilde{\mathscr{A}}_{n+1})^{-1}\left(\frac{1}{\Delta t}I+\tilde{\mathscr{A}}_{n+1}\right)^{-1}\tilde{\mathscr{A}}_{n+1}(\tilde{\mathscr{A}}_{n+1}^{-1}-\tilde{\mathscr{A}}_n^{-1})$$

and $\|\mathscr{B}\| \leqslant c\Delta t^{\min(\mu,1)+\alpha-1}$.

The inequality above together with (3.6) for $\alpha = 2$ shows the desired inequality for K^2. Combining the results, we have

$$(3.16)\qquad \|K(n,i)\| \leqslant c/((n-i)\Delta t)^{2-\alpha-\min(1,\mu)}.$$

Using identity (3.15) with $\left(\frac{1}{(n-i)\Delta t}+\tilde{\mathscr{A}}_n\right)$ replaced by $\frac{1}{\Delta t}+\tilde{\mathscr{A}}_n$, one gets

$$\|K(n,n)\| \leqslant c/\Delta t^{2-\alpha-\min(1,\mu)}.$$

Now we shall consider the function $\varphi(i,k)$. Using the similar tricks, one can show that

$$(3.17)\qquad \|\varphi(i,k)\| \leqslant c\frac{1}{(i-k)\Delta t},$$

$$(3.18)\qquad \|\varphi(i,i)\| \leqslant c/\Delta t.$$

From (3.17), (3.18), (3.16) it follows that

$$(3.19)\qquad \|\Phi_0(n,k)\| \leqslant \frac{1}{[(n-k)\Delta t]^{2-\alpha-\min(1,\mu)}}(1+\ln(n-k)).$$

Now (3.19) together with (3.16) implies that the function Φ satisfies the inequality of the form

$$(3.20)\qquad \|\Phi(n,k)\| \leqslant \frac{1}{((n-k)\Delta t)^{2-\alpha-\min(1,\mu)}}(1+\ln(n-k)).$$

From (3.11) it follows that

$$W(n,k) = W_n(n,k) + \Delta t \sum_{i=k+1}^{n} W_n(n,i)\mathscr{A}_n(I+\Delta t\,\mathfrak{a}\mathscr{A}_n)^{-1} \times$$
$$\times (\tilde{\mathscr{A}}_n^{-1} - \tilde{\mathscr{A}}_i^{-1})[\Phi(i-1,k) + \varphi(i-1,k)].$$

Again one can use the splitting of $W_n(n,i)$ and (3.5), (3.6). The contribution $W^{(1)}(n,k)$ from $W_n^1(n,k)$ satisfies (3.9). In order to get (3.10) for the remaining part of $W(n,k)$ one can use the inequality

$$\|\tilde{\mathscr{A}}_n^{\gamma}(\tilde{\mathscr{A}}_n^{-1} - \tilde{\mathscr{A}}_i^{-1})\| \leqslant c((n-i)\Delta t)^{\mu}, \quad \gamma < \alpha, \tag{3.21}$$

which follows from the inclusion

$$D(A_{n\Delta t}^{\gamma}) \supset D(t)_{\alpha,\infty} \supset Y \tag{3.22}$$

and the definition of class $K(\mu,\alpha)$. This ends the proof.

In [6] the theorem above is extended to the value greater than 1 with the suitable changing of the definition of class $K(\mu,\alpha)$. Some smoothing properties have been shown for the operator function A_t which does not belong to $K(\mu,\alpha)$. But all this required a lot more sophisticated proof.

In the error analysis we shall need the operators adjoint to $W(n,k)$. Similarly as in the continuous case, the adjoint operator $W^*(n,k)$ can be interpreted as some Runge–Kutta method applied to (3.15) with the reversed time direction. This scheme can be defined by (3.3) with $\mathfrak{a}$, $\tilde{\tau}$, b, e replaced by $\mathfrak{a}^T$, $1-\tilde{\tau}$, b^T, e^T. The scheme obtained in this way has the same order of approximation. Assuming that $A_t^* \in K(\mu^*,\alpha^*)$ we can establish the similar smoothing properties for the dual scheme. Again, since X is reflexive Banach space, we can get the appropriate smoothing properties for both $W(n,k)$ and $W^*(n,k)$ with the estimates for the extension of the operators $W(n,k)A_{k\Delta t}$ and $W^*(n,k)A_{n\Delta t}^*$.

4. Error estimates

Using the properties of the evolution operator $U(t,s)$ and the properties of its discrete analogue $W(n,k)$, we can show the following error estimates.

THEOREM 3. *If $A_t \in K(\mu,\alpha)$ in X for $\mu > 1+\alpha$ and $A_t^* \in K(\mu^*,\alpha^*)$ in X^* for $\mu^* > 1+\alpha^*$ then for Runge–Kutta scheme satisfying Assumptions 2 and of order $k \geqslant 2$ the following error estimate is valid*

$$\|U(n\Delta t, k\Delta t) - W(n,k)\| \leqslant c\left(\frac{\Delta t}{(n-k)\Delta t}\right)^{\beta} \tag{4.1}$$

where $\beta < (1+\alpha, 1+\alpha^*)$. If, moreover, $\mathfrak{a}e = \tilde{\tau}e$ then (4.1) is valid with $\beta < 1+\alpha^*$.

Remark. The condition $\mathfrak{a}e = \tilde{\tau}e$ is not restrictive one. All schemes used in practice satisfy this condition. For the second order partial differential equation with the first order boundary condition, by taking $X = L_p, X = L_q$ we obtain $\alpha^* = \frac{1}{2} + 1/(2q)$ which corresponds to $\beta \leqslant 2 - 1/(2p)$ in (4.1).

Proof. The proof will be divided into three steps.
Step 1. We shall estimate the differences:

$$\varepsilon_l = \left[U\left((l+1)\Delta t, l\Delta t\right) - W(l+1, l)\right] u(l\Delta t),$$

$$\varepsilon_l^* = \left[U^*\left((l+1)\Delta t, l\Delta t\right) - W^*(l+1, l)\right] u^*((l+1)\Delta t),$$

where u and u^* are exact solutions of (1.1) and (2.15), respectively. One can easily show the inequalities:

$$\|A_{(l+1)\Delta t}^{-1} \varepsilon_l\| \leqslant c\Delta t \|u(l\Delta t)\|,$$

$$\|(A_{(l+1)\Delta t}^*)^{-1} \varepsilon_l^*\| \leqslant c\Delta t \left\|u^*((l+1)\Delta t)\right\|.$$

Indeed, one gets

$$\|A_{l\Delta t}^{-1}[W(l, l+1) - I]\| \leqslant \Delta t \| b(\tilde{\mathscr{A}}_l^{-1} - \mathscr{A}_l^{-1})\tilde{\mathscr{A}}_l(I + \Delta t \mathfrak{a}\tilde{\mathscr{A}}_l)^{-1} e\|$$
$$+ \Delta t \| b(I + \Delta t \mathfrak{a}\tilde{\mathscr{A}}_l)^{-1} e\| \leqslant c\Delta t$$

and

$$B = A_{(l+1)\Delta t}^{-1}\left((U(l+1)\Delta t, l\Delta t) - I\right) = \int_{l\Delta t}^{(l+1)\Delta t} A_{(l+1)\Delta t}^{-1} A_\tau U(\tau, l\Delta t) d\tau$$

$$= \int_{l\Delta t}^{(l+1)\Delta t} (A_{(l+1)\Delta t}^{-1} - A_\tau^{-1}) A_\tau U(\tau, l\Delta t) dz + \int_{l\Delta t}^{(l+1)\Delta t} U(\tau, l\Delta t) d\tau$$
$$+ (A_{(l+1)\Delta t}^{-1} - A_{l\Delta t}^{-1})\left[U\left((l+1)\Delta t, l\Delta t\right) - I\right],$$

showing that $\|B\| < c\Delta t$. Using the identities

$$\Delta t \mathfrak{a}\mathscr{B}(I + \Delta t \mathfrak{a}\mathscr{B})^{-1} = I - (I + \Delta t \mathfrak{a}\mathscr{B})^{-1},$$

$$\mathscr{B}(I + \Delta t \mathfrak{a}\mathscr{B})^{-1} - \mathscr{C}(I + \Delta t \mathfrak{a}\mathscr{C})^{-1} = \mathscr{B}(I + \Delta t \mathfrak{a}\mathscr{B})^{-1}(\mathscr{B}^{-1} - \mathscr{C}^{-1})\mathscr{C}(I + \Delta t \mathfrak{a}\mathscr{C})^{-1},$$

we can write

$$W(l+1, l)u(l\Delta t) = u(l\Delta t) + \Delta t \partial_t u|_{t=l\Delta t} + \sum_{i=1}^{5} I_i,$$

where

$$I_1 = \Delta t^2 b \mathfrak{a} \mathscr{A}_{l-1}(I+\Delta t \mathfrak{a} \mathscr{A}_{l-1})^{-1} e A_{l\Delta t} u(l\Delta t),$$

$$I_2 = \Delta t^2 b \mathscr{A}_{l-1}(I+\Delta t \mathfrak{a} \mathscr{A}_{l-1})^{-1} \hat{\tau} e \partial_t A_t^{-1}|_{t=l\Delta t} A_{l\Delta t} u(l\Delta t),$$

$$I_3 = \Delta t b \tilde{\mathscr{A}}_l(I+\Delta t \mathfrak{a} \tilde{\mathscr{A}}_l)(\tilde{\mathscr{A}}_l^{-1} - \bar{\mathscr{A}}_l^{-1} - \Delta t \hat{\tau} \partial_t A_t^{-1} I)(I+\Delta t \mathfrak{a} \mathscr{A}_l)^{-1} e A_{l\Delta t} u(l\Delta t),$$

$$I_4 = \Delta t^3 b \tilde{\mathscr{A}}_l(I+\Delta t \mathfrak{a} \tilde{\mathscr{A}}_l)^{-1} \hat{\tau} \partial_t A_t^{-1}|_{t=l\Delta t} \mathfrak{a} \bar{A}_l (I+\Delta t \mathfrak{a} \bar{\mathscr{A}}_l)^{-1} e A_{l\Delta t} u(l\Delta t),$$

$$I_5 = \Delta t^2 b \tilde{\mathscr{A}}_l(I+\Delta t \mathfrak{a} \tilde{\mathscr{A}}_l)^{-1}(\tilde{\mathscr{A}}_l - \bar{\mathscr{A}}_l^{-1})(I+\Delta t \mathfrak{a} \bar{\mathscr{A}}_l)\hat{\tau}(\partial_t A_t^{-1})_{t=l\Delta t} \times$$
$$\times (I+\Delta t \mathfrak{a} \bar{\mathscr{A}}_l)^{-1} e A_{l\Delta t} u(l\Delta t).$$

We can easily estimate the terms $I_1 - I_5$ using the following inequalities:

$$\|\tilde{\mathscr{A}}_l^{-1} - \bar{\mathscr{A}}_l^{-1} - \hat{\tau} \Delta t \partial_t A_t^{-1}|_{t=l\Delta t} I\| \leqslant c \Delta t^{\min(\mu,2)}, \tag{4.2}$$

$$\|\tilde{\mathscr{A}}_l(I+\Delta t \mathfrak{a} \tilde{\mathscr{A}}_l)^{-1} \hat{\tau} \partial_t A_t^{-1} I\| \leqslant c \Delta t^{\alpha-1}, \tag{4.3}$$

$$\|\tilde{\mathscr{A}}_l(I+\Delta t \mathfrak{a} \bar{\mathscr{A}}_l)^{-1}(\tilde{\mathscr{A}}_l^{-1} - \bar{\mathscr{A}}_l^{-1})\| \leqslant c \Delta t^{\alpha}, \tag{4.4}$$

$$\|\bar{\mathscr{A}}_l(I+\Delta t \mathfrak{a} \bar{\mathscr{A}}_l)^{-1} e A_{l\Delta t}^{-\beta}\| \leqslant c \Delta t^{\beta-1}, \quad \beta < \alpha. \tag{4.5}$$

We obtain

$$\Big\|\sum_{i=1}^{5} I_i\Big\| \leqslant c \Delta t^{1+\beta} \|A_{l\Delta t}^{1+\beta} u(l\Delta t)\|, \quad \beta < \alpha, \tag{4.6}$$

$$\Big\|A_{(l+1)\Delta t}^{-1} \sum_{i=3}^{5} I_i\Big\| \leqslant c \Delta t^{2+\beta} \|A_{l\Delta t}^{1+\beta} u(l\Delta t)\|, \quad \beta < \alpha. \tag{4.7}$$

For the terms $I_3 - I_5$ it is possible to get other estimates using the assumption $A_t^* \in K(\mu^*, \alpha^*)$ which implies

$$\|\partial_t A_t^{-1} \mathscr{A}_{l-1}(I+\Delta t \mathfrak{a} \mathscr{A}_{l-1})^{-1}\| \leqslant c \Delta t^{\alpha^*-1}, \tag{4.8}$$

$$\|(\mathscr{A}_{l-1}^{-1} - \bar{\mathscr{A}}_l^{-1}) \mathscr{A}_{l-1}(I+\Delta t \mathfrak{a} \mathscr{A}_{l-1})^{-1} \leqslant c \Delta t^{\alpha^*}. \tag{4.9}$$

Instead of (4.7) and (4.8) we have

$$\Big\|\sum_{i=3}^{5} I_i\Big\| \leqslant c \Delta t^{1+\beta^*} \|A_{l\Delta t} u(l\Delta t)\|, \quad \beta^* < \alpha^*, \tag{4.10}$$

$$\Big\|A_{(l+1)\Delta t}^{-1} \sum_{i=3}^{5} I_3\Big\| \leqslant c \Delta t^{2+\beta^*} \|A_{l\Delta t} u(l\Delta t)\|, \quad \beta^* < \alpha^*. \tag{4.11}$$

Now we shall estimate $I_1 + I_2$. Writing $I_1 + I_2 = b \bar{A}_l z$ we can split $A_{(l+1)\Delta t}^{-1}(I_1 + I_2)$ as follows:

$$A_{(l+1)\Delta t}^{-1} b \bar{\mathscr{A}}_l z = b(\mathscr{A}_{l+1}^{-1} - \bar{\mathscr{A}}_l^{-1}) \bar{\mathscr{A}}_l z + bz, \tag{4.12}$$

where

$$\|b(\mathscr{A}_{l+1}^{-1} - \bar{\mathscr{A}}_l^{-1}) \bar{\mathscr{A}}_l z\| \leqslant c \Delta t^{1+\beta} \|A_{l\Delta t}^{1+\beta} u(l\Delta t)\|, \quad \beta < \alpha. \tag{4.13}$$

We can write

$$bz = \frac{(\Delta t)^2}{2}(A_{l\Delta t} + \partial_t A_t^{-1}|_{l\Delta t} A_{l\Delta t} u) + R_1 \tag{4.14}$$

$$= \frac{(\Delta t)^2}{2}(A_{l\Delta t})^{-1} \partial_t^2 u|_{t=l\Delta t} + R_1,$$

where

$$\|R_1\| \leqslant c\Delta t^{2+\beta} \|A_{l\Delta t}^{1+\beta} u(l\Delta t)\|, \qquad \beta < \alpha. \tag{4.15}$$

Under additional assumption $\alpha e = \hat{\tau} e$ we can obtain, using similar tricks,

$$(I_1 + I_2) = \tfrac{1}{2}\Delta t^2 \partial_t^2 u_t|_{t=t_l} + R_2, \tag{4.16}$$

$$A_{(l+1)\Delta t}^{-1}(I_1 + I_2) = \tfrac{1}{2}\Delta t^2 A_{2\Delta t}^{-1} \partial_t^2 u_t|_{t=t_l} + R_3, \tag{4.17}$$

where $\|R_2\| \leqslant c\Delta t^2 \|\partial_t^2 u\|$ and $\|R_3\| \leqslant c\Delta t^3 \|\partial_t^i u|_{t=l\Delta t}\|$. Combining the above results, we obtain

$$\left\|A_{(l+1)\Delta t}^{-i}\big(U((l+1)\Delta t, l\Delta t) - W(l+1,l)u(l\Delta t)\big)\right\| \tag{4.18}$$
$$\leqslant c\Delta t^{i+j} \|\partial_t^j u\|; \qquad i, j = 0, 1,$$

$$\left\|A_{(l+1)\Delta t}^{-1}\big(U((l+1)\Delta t, l\Delta t) - W(l+1,l)u(l\Delta t)\big)\right\| \tag{4.19}$$
$$\leqslant c\Delta t^{2+\beta} \|A_t^{1+\beta} u\| + c\Delta t^3 \|\partial_t^2 u\|, \qquad \beta < \alpha.$$

Under the assumption $\alpha e = \hat{\tau} e$ we obtain

$$\left\|A_{(l+1)\Delta t}^{-i}\big(U((l+1)\Delta t, l\Delta t) - W(l+1,l)\big)\right\| \tag{4.20}$$
$$\leqslant c\Delta t^{i+1+\beta^*} \|A_t u\| + c\Delta^{i+2} \|\partial_t^2 u\| \qquad \text{for } i = 0, 1,\ \beta^* < \alpha^*.$$

The same estimate can be obtained for $W^*(n,k)$ leading to the inequalities corresponding to (4.18) and (4.19).

Step 2. The identity

$$F_{nk} = U(j\Delta t, k\Delta t) - W(j,k)$$
$$= \sum_{i=k+1}^{j} W(j,i)\big[W(i,i-1) - U(i\Delta t,(i-1)\Delta t)\big]\, U((i-1)\Delta t, k\Delta t)$$

together with (4.18) leads to the estimate

$$\|E_{jk}\| \leqslant c\left[\frac{\Delta t}{(j-k)\Delta t}\right]^{1-\varepsilon}; \qquad \varepsilon > 0. \tag{4.21}$$

Step 3. Using the standart technique we can expressed the error in the form

$$E_{jk} = E_{jl} U(l\Delta t, k\Delta t) + E_{jl} E_{lk} + U(j\Delta t, l\Delta t) E_{lk}$$

with the following inequality

$$\|E_{jk}\| \leqslant \|E_{jl} U(l\Delta t, k\Delta t)\| + c\left[\frac{\Delta t}{(j-k)\Delta t}\right]^{2-\varepsilon} + \|E_{lk}^{*} U^{*}(j\Delta t, l\Delta t)\|.$$

The results of Step 1 imply now the conclusions of the theorem.

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