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A COMPARISON OF SOME WEIGHTED SIEVES

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Richert's paper [14] on the weighted linear sieve has been deservedly popular since its publication; it provided a readily applicable general result of the type we describe below, which many subsequent writers (including the present author: see [4], [5], for example) were painlessly able to use. At the same time it was clear, then as now, that the field was one in which much work remained to be done.

Approximately one decade later, a number of papers appeared containing devices which can be viewed as improvements of the weighting device of Richert. It is our purpose in this article to discuss the relationships between three of these: the continuous form of the weighting device of Buchstab (for which see [1], [2]) that has been described by Laborde [13], the weighting device described in a paper [7] by Halberstam, Heath-Brown and Richert, and the author's own device introduced in [6] (see also the article by Halberstam and Richert in these Proceedings) which, it appears, is the most powerful of the three. We hope also to make it clear that this device gives results which are far from what may reasonably be conjectured to be best possible in this subject.

Let $\mathscr{A} = \mathscr{A}(X)$ be a set of positive integers, depending on a real parameter X which, throughout, is to be supposed to be as large as may be required. Denote by

$$\mathscr{A}_d = \{a \in \mathscr{A}\colon a \equiv 0 \bmod d\}$$

the set of members of $\mathscr{A}$ that are divisible by d. We suppose that the number of these members of $\mathscr{A}$ satisfies the "special linear sieve" axiom

$$(1) \qquad |\mathscr{A}_d| = \frac{X}{d} + O(1),$$

although everything in this article is relevant to the general linear case where we write

(2)
$$|\mathcal{A}_d| = \frac{X}{d}\varrho(d) + R(X, d),$$

for a suitable multiplicative function ϱ satisfying

(3)
$$-L < \sum_{w \leqslant p < z} \frac{\varrho(p)\log p}{p-\varrho(p)} - \log\frac{z}{w} < K.$$

The axiom (1) is satisfied in the example of the integers in an interval,

$$\mathcal{A} = \{n:\ Y-X \leqslant n < Y\},$$

about which, however, there is a great deal more than (1) and its consequences to be said: cf. [7], [10] for example.

In this context we may proceed as follows. Let $\lambda(d)$ be a function defined when $d \leqslant D$ and form

(4)
$$m(a) = \sum_{d|a} \lambda(d),$$

where we may suppose $|\lambda(d)| \leqslant 1$ without loss of generality. Then in the restricted context of (1) we have

(5)
$$\sum_{a\in\mathcal{A}} m(a) = X \sum_{d \leqslant D} \frac{\lambda(d)}{d} + O(D).$$

In the methods we shall describe the choice of λ will satisfy

$$\sum_{d \leqslant D} \frac{\lambda(d)}{d} \gg \frac{1}{\log D}.$$

Then the coefficient of X in (5) retains its relevance if, for example,

$$D = X/\log^2 X,$$

in the sense that we find in particular that some a in $\mathcal{A}$ has $m(a) > 0$, a statement that can be put on a more quantitative basis. Needless to say we are only interested in constructions where the property $m(a) > 0$ is itself of some interesting arithmetical significance.

Define

(6)
$$S(a, z) = \begin{cases} 1 & \text{if } p|a \Rightarrow p > z \quad (p \text{ prime}), \\ 0 & \text{otherwise.} \end{cases}$$

In the sieve method of Viggo Brun [3] and his successors the construction is arranged so that (in the "lower bound" aspect of the method) we have

$$m(a) > 0 \Rightarrow S(a, z) > 0,$$

provided the parameter z is chosen suitably. Indeed we have the following result.

THE LINEAR SIEVE THEOREM. *Under the hypothesis* (1) *we have*

$$\sum_{a\in\mathscr{A}} S(a, D^{1/s}) \geqslant X\Big\{\prod_{p<D^{1/s}}\Big(1-\frac{1}{p}\Big)\Big\}\Big\{f(s)+O\Big(\frac{1}{\log^c D}\Big)\Big\}+O(D),$$

for a certain $c > 0$ *and a function* f *whose most important property is*

(7) $$f(s) > 0 \quad \text{if} \quad s > 2.$$

The function f is actually defined via the system

$$\frac{d}{ds}\{sf(s)\} = F(s-1)\ (s>2); \quad \frac{d}{ds}\{sF(s)\} = f(s-1)\ (s>1),$$

$$F(s) = 1+O(e^{-s}); \quad f(s) = 1+O(e^{-s}),$$

which can be shown to have a solution satisfying

$$sf(s) = 2e^{\gamma}\log(s-1) \quad (2\leqslant s\leqslant 4),$$
$$sF(s) = 2e^{\gamma} \quad (1\leqslant s\leqslant 3),$$

γ being Euler's constant.

For a proof of the linear sieve theorem, by what is the case $w(p) = 0$ of the more general weighted sieve that we discuss below, the reader may consult [8], where the theorem is obtained with the constant $c = 1/3$. Alternatively the procedure described in [9] (which would differ only in its treatment of error terms) may be adapted to the present context to yield the result with $c = 1/5$, or slightly larger. We should perhaps remark that the first published proof of the theorem (by a method that yields slightly large O-terms) was by Jurkat and Richert in [11].

For the remainder of this article we shall refer to a number g (the "degree") with the property

(8) $$a\in\mathscr{A}\Rightarrow a<D^g.$$

Reading between the lines of Brun's famous paper [3] gives the impression that at one stage he had hopes of discovering the existence of primes in certain interesting contexts (Goldbach's problem) by his sieve method. However, it now appears clear that in general one cannot expect to do better, by this approach, than as follows. Choose an integer $R\geqslant 2$ and suppose

$$g<(R+1)/2.$$

In the linear sieve theorem choose $s = 2+\varepsilon$ ($\varepsilon>0$), so that $f(s)>0$. Thus some a in $\mathscr{A}$ has $S(a, z)>0$, if X (and hence D) is large enough. Then we conclude

(9) $$a \in P_R,$$

in the sense that the number of distinct prime factors does not exceed R. This is because the product of $R+1$ (or more) primes all exceeding $D^{1/s} = D^{1/(2+\varepsilon)}$ would exceed D^g, if ε is sufficiently small.

In this situation it appears we should change our objective, and ask not that the size of the prime factors of some a should be large but that the number of these prime factors should be small. Kuhn [12] was the first to modify the sieve method with this revised objective in mind; his procedure would be that obtained by replacing $w(p)$ in (15) by an appropriate constant. Since the appearance of the paper of Richert [14] (or of those [1], [2] of Buchstab) we have been able to conclude that (9) holds for some a in $\mathscr{A}$ satisfying (1), (8) provided

(10) $$g < R - \delta_R = \Lambda_R$$

for certain numbers δ_R in the range $0 < \delta_R < 1/3$; the aim is to draw such conclusions for numbers δ_R as small as possible. The author's method succeeds whenever $\delta_R < 1/8$, and for somewhat smaller δ_R for small values of R. It is generally conjectured that such results should hold for any $\delta_R \geqslant 0$, but a proof of this conjecture does not, at the time of writing, appear to be an immediate prospect. Such a proof, if the conjecture is correct, would correspond to the result (7) on the linear sieve without weights in that examples due to Selberg would indicate that the result proved would be best possible, in the general situation described by (2), (3), (8) and the condition

$$\sum_{d \leqslant D} |R(X, d)| < X/\log^2 X.$$

One might attempt to attack the "$\delta_R = 0$" conjecture by modifying the sieve so as to show that some a, if it has k prime factors less than D, has them satisfying

(11) $$q_1 q_2 \ldots q_k > D^{k-1}.$$

Richert's procedure in [14] was as follows. Choose reals U, V satisfying

(12) $$V < U; \quad 1/2 < U < 1, \quad V + RU \geqslant g$$

(for success is not possible with $V = 0$, $U = 1$), and specify that w will satisfy

(13) $$W(1) = U - V, \quad 0 \leqslant w(p) \leqslant \frac{\log p}{\log D} - V \quad \text{if} \quad D^V \leqslant p \leqslant D^U,$$
$$w(p) = 0 \quad \text{if} \quad p < D^V.$$

For convenience we will also define

$$w(p) = W(1) \quad \text{if} \quad p > D^U. \tag{14}$$

Then take

$$m(a) = W(1) - \sum_{\substack{p|a \\ p^V \leqslant p \leqslant D^U}} \{W(1) - w(p)\}, \tag{15}$$

so that, introducing extra negative summands, we have

$$m(a) \leqslant U - V - \sum_{\substack{p|a \\ p > D^V}} \left(U - \frac{\log p}{\log D}\right) \leqslant U\{1 - \omega(a) + R\},$$

where we have used (8), (12). Thus if $m(a) > 0$ we have $\omega(a) < R+1$ whence, the number $\omega(a)$ of prime factors of a being an integer, we have

$$\omega(a) \leqslant R \quad \text{when} \quad m(a) > 0.$$

This function $m(a)$ is not suitable for direct application of (5) since we would have

$$\sum_{d \leqslant D} \frac{\lambda(d)}{d} \sim -U \log\log D;$$

Richert applies his device only to those a having no prime factors smaller than D^T for a suitably chosen $T > 0$, using the linear sieve theorem to complete the estimations.

This procedure (and, let it be said at once, all the others considered in this article) has the undesirable feature that some of the numbers a in the class P_R that we are seeking are not counted; namely those having one of their R prime factors smaller than D^T. One may speculate that the best-possible conjecture "$\delta_R = 0$" is unlikely to be proved by a method that shares this defect to any significant extent; see the considerations at the end of this article.

We proceed now to a unified discussion, in so far as it is possible, of the three weighted sieves described in [6], [7], [13]. The quantity $S(a, z)$ defined in (6) is expressible by Möbius's formula

$$S(a, z) = \sum_{d|(a, P(z))} \mu(d),$$

where

$$P(z) = \prod_{p < z} p$$

denotes the product of all primes strictly less than z. Hence we obtain the

"Buchstab" identity

$$S(a, z) = 1 - \sum_{p<z} \sum_{d_1 | (a, P(p))} \mu(d_1),$$

where the rôle of p is that of the greatest prime factor of $d = pd_1$. Thus we obtain

$$S(a, y) - S(a, z) = \sum_{y \leqslant p < z} S\left(\frac{a}{p}, p\right) \quad \text{if} \quad y \leqslant z. \tag{16}$$

The procedure of Brun, like that below, is based upon a suitable iterated use of this identity. Thinking first only of the case $k = 1$ in (11) we define

$$A = (a, P(D)) \tag{17}$$

and

$$T(a, D) = W(1) S(A, D) + \sum_{p|a; p<D} w(p) S\left(\frac{A}{p}, D\right),$$

where $W(1) > 0$ and $w(p) \geqslant 0$ are to be specified. Whatever $g \leqslant R$ we have in (8) it is the case that

$$T(a, D) > 0 \Rightarrow a \in P_R, \tag{19}$$

so that $T(a, D)$ is certainly one suitable object to study. By successive applications of (16) we have

$$T(a, D)$$
$$= W(1) S(A, 1) - \sum_{p<D} \{W(1) - w(p)\} S\left(\frac{A}{p}, p\right) - \sum_{p<p'<D} w(p) S\left(\frac{A}{pp'}, p'\right),$$

the case $p = p'$ not occurring because A is squarefree. Hence we obtain

$$T(a, D) = \Sigma_1 + \Sigma_{11},$$

where

$$\Sigma_1 = W(1) - \sum_{p_1 < D} \{W(1) - w(p_1)\} S\left(\frac{A}{p_1}, 1\right),$$

$$\Sigma_{11} = \sum_{p_2 < p_1 < D} \{W(1) - w(p_1)\} S\left(\frac{A}{p_1 p_2}, p_2\right) - \sum_{p_2 < p_1 < D} w(p_2) S\left(\frac{A}{p_1 p_2}, p_1\right).$$

Applying (16) to the last terms only if $p_2^3 p_1 \leqslant D$, we now have

$$T(a, D) = \Sigma_1 + \Sigma_2 + \Sigma_3^* + \bar{\Sigma}_1 - \bar{\Sigma}_2$$

where

$$\Sigma_2 = \sum_{\substack{p_2 < p_1 < D \\ p_2^3 p_1 \leqslant D}} \{W(1) - w(p_1) - w(p_2)\} S\left(\frac{A}{p_1 p_2}, p_2\right), \tag{20}$$

$$(21)\quad \Sigma_3^* = \sum_{\substack{p_2 < p < p_1 < D \\ p_2^3 p_1 \leqslant D}} w(p_2) S\left(\frac{A}{p_1 p_2 p}; p\right) = \sum_{\substack{p_3 < p_2 < p_1 \\ p_3^3 p_1 \leqslant D}} w(p_3) S\left(\frac{A}{p_1 p_2 p_3}, p_2\right),$$

$$\bar{\Sigma}_1 = \sum_{\substack{p_2 < p_1 < D \\ D < p_2^3 p_1}} \{W(1) - w(p_1)\} S\left(\frac{A}{p_1 p_2}, p_2\right),$$

$$\bar{\Sigma}_2 = \sum_{\substack{p_2 < p_1 < D \\ D < p_2^3 p_1}} w(p_2) S\left(\frac{A}{p_1 p_2}, p_1\right).$$

The condition $p_2^3 p_1 \leqslant D$ and its negation are borrowed from the "combinatorial" proof (i.e. the proof by the development of Brun's method; see e.g. [8]) of the linear sieve theorem.

As is usual in this type of argument, we will dismiss $\bar{\Sigma}_1$ with the remark

$$\bar{\Sigma}_1 \geqslant 0;$$

such treatment cannot, however, be given to the term $-\bar{\Sigma}_2$. Accordingly we write

$$T^*(a, D) = \Sigma_1 + \Sigma_2 + \Sigma_3^*,$$

and argue differently in the two cases below. Case 1 is when $\bar{\Sigma}_2 = 0$ and we have

$$(22)\qquad T(a, D) \geqslant T^*(a, D).$$

Case 2 is when $\bar{\Sigma}_2 \neq 0$. Then $Q_1^3 Q_2 > D$, where $Q_1 < Q_2$ are the two smallest prime factors of a. Then the sums defining Σ_2 and Σ_3^* are empty and we have

$$T^*(a, D) = \Sigma_1.$$

Accordingly we specify $W(1)$, $w(p)$ to be as in (13). Then Σ_1 is precisely the expression (15), so that we have in case 2 that

$$(23)\qquad T^*(a, D) > 0 \Rightarrow a \in P_R.$$

This conclusion also holds in case 1 because of (19) and (22).

This argument, which is contained in the content of §§ 2, 3 of [5], has been arranged to facilitate the comparison of the sieves that we wish to discuss. We will choose U, V so that $U + V < 1$; then $\Sigma_2 \geqslant 0$ because $p_1 p_2 \leqslant D$ guarantees

$$W(1) - w(p_1) - w(p_2) \geqslant U + V - 1.$$

We thus see that our inequality (23) improves on that used in [14] to the extent of the additional contribution derived from Σ_2 and Σ_3^*.

The somewhat weaker remark

$$\Sigma_1 + \Sigma_2 > 0 \Rightarrow a \in P_R$$

can be seen to be equivalent to the continuous form of Buchstab's weights described by Laborde [13]. In the first place no benefit would be derived by including contributions from p_1, p_2 satisfying $p_2^3 p_1 > D$ because the procedure described by Laborde is equivalent to estimating terms involving $S(A/(p_1 p_2), p_2)$ from below using the linear sieve theorem, so that the estimates are in terms of

$$f\left(\frac{\log\{D/(p_1 p_2)\}}{\log p_2}\right),$$

which, by (7), is equal to zero when $p_2^3 p_1 > D$. Thus $\Sigma_1 + \Sigma_2$ is equivalent, for this purpose, to $\Sigma_1 + \Sigma_2^*$, where

$$\Sigma_2^* = \sum_{\substack{p_2 < p_1 < D \\ w(p_1) + w(p_2) \leqslant W(1)}} \{W(1) - w(p_1) - w(p_2)\} S\left(\frac{A}{p_1 p_2}, p_2\right).$$

In the second place $\Sigma_1 + \Sigma_2^*$ can be rewritten as

$$\Sigma_1 + \Sigma_2^* = W(1) - \sum_{p|a} l(p),$$

where

$$l(p) = \begin{cases} W(1) - w(p) & \text{if} \quad p = Q_1, \\ \min\{W(1) - w(p),\, w(Q_1)\} & \text{otherwise.} \end{cases}$$

Apart from more straightforward changes of notation and scale this is the expression considered by Laborde [13].

An expression equivalent to Σ_3^* also occurs in the paper [7] by Halberstam, Heath-Brown and Richert (in their expression (2.19), along with other terms which, as they write, "may safely be omitted"). Their paper thus contains what is essentially a proof of (23) somewhat different in aspect from the one described in this article.

The inequality (23) does not yet constitute a sieve inequality, because we have not yet shown how to estimate Σ_3^* from below by an expression of the type (4). Halberstam, Heath-Brown and Richert estimate Σ_3 directly from below (in their Proposition 3) using the linear sieve theorem, so that their estimate is in terms of

$$f\left(\frac{\log\{D/(p_1 p_2 p_3)\}}{\log p_2}\right),$$

which, by (7), is non-zero only when $p_1 p_2^3 p_3 < D$.

The author's procedure outlined below takes non-trivial account of the contribution to Σ_3 from the somewhat larger set of $p_1 p_2 p_3$ satisfying the weaker inequality $p_2^3 p_1 < D$. Accordingly we record at once the fact that

$$\Sigma_3^* \geqslant \Sigma_3,$$

where

$$\Sigma_3 = \sum_{\substack{p_3 < p_2 < p_1 < D \\ p_2^3 p_1 \leqslant D}} w(p_3) S\left(\frac{A}{p_1 p_2 p_3}, p_2\right). \tag{24}$$

We deal with Σ_3 by applying the "Buchstab" identity (16) to it in such a way that the expressions obtained interact, not with Σ_2 itself, but (and herein lies the essence of the author's approach) with the expressions obtained when Σ_2 is estimated from below by the method used in the proof [8] of the linear sieve theorem. We obtain from (20), (24)

$$\Sigma_2 + \Sigma_3 = \sum_{\substack{p_2 < p_1 < D \\ p_2^3 p_1 \leqslant D}} \{W(1) - \sum_{i=1}^{2} w(p_i)\} S\left(\frac{a}{p_1 p_2}, 1\right) + \Sigma_{31},$$

where

$$\Sigma_{31} = -\sum_{\substack{p_3 < p_2 < p_1 < D \\ p_2^3 p_1 \leqslant D}} \{W(1) - \sum_{i=1}^{3} w(p_i)\} S\left(\frac{a}{p_1 p_2 p_3}, p_3\right) -$$

$$- \sum_{\substack{p_4 < p_3 < p_2 < p_1 \\ p_2^3 p_1 \leqslant D}} w(p_4) S\left(\frac{a}{p_1 p_2 p_3 p_4}, p_3\right)$$

$$= -\sum_{\substack{p_3 < p_2 < p_1 < D \\ p_2^3 p_1 \leqslant D}} \{W(1) - \sum_{i=1}^{3} w(p_i)\} S\left(\frac{a}{p_1 p_2 p_3}, 1\right) + \Sigma_4 + \bar{\Sigma}_4,$$

with

$$\bar{\Sigma}_4 = \sum_{\substack{p_4 < p_3 < p_2 < p_1 < D \\ p_2^3 p_1 \leqslant D \\ D < p_4^3 p_3 p_2 p_1}} \{W(1) - \sum_{i=1}^{4} w(p_i)\} S\left(\frac{a}{p_1 p_2 p_3 p_4}, p_4\right) +$$

$$+ \sum_{\substack{p_5 < p_4 < p_3 < p_2 < p_1 < D \\ p_2^3 p_1 \leqslant D \\ D < p_4^3 p_3 p_2 p_1}} w(p_5) S\left(\frac{a}{p_1 p_2 p_3 p_4 p_5}, p_4\right),$$

while Σ_4 is the similar expression obtained when the condition $D < p_4^3 p_3 p_2 p_1$ is replaced by its negation $p_4^3 p_3 p_2 p_1 \leqslant D$. We will cast $\bar{\Sigma}_4$ out with the remark

$$\bar{\Sigma}_4 \geqslant 0,$$

and deal with Σ_4 in the same style as we did with $\Sigma_2+\Sigma_3$. In this construction, the terms involving the expressions

$$S\left(\frac{A}{p_1 p_2}, 1\right), S\left(\frac{A}{p_1 p_2 p_3}, 1\right), \ldots$$

(which simply count when $p_1 p_2 | A$, $p_1 p_2 p_3 | A$, ...) will contribute to the expression (4).

If we proceed appropriately in this fashion we will arrive at the construction described in [6]. We should mention the fact that requirements such as

$$p_4 < p_3 < p_2 < p_1, \quad p_2^3 p_1 \leqslant D \Rightarrow \sum_{i=1}^{4} w(p_i) \leqslant W(1)$$

which is needed to infer $\bar{\Sigma}_4 \geqslant 0$, follow from (13) only in the case when $U+3V \geqslant 1$; the further conditions to be imposed on w in the contrary case are described in [6].

In conclusion, we remark that it is the author's opinion that the weakness of this argument lies in its neglect (by merely dismissing them as being non-negative) of such terms as the contribution

$$\Sigma_3^{\dagger} = \sum_{\substack{p_3 < p_2 < p_1 < D \\ p_3 p_2^2 p_1 \leqslant D < p_2^3 p_1}} w(p_3) S\left(\frac{A}{p_1 p_2 p_3}, p_2\right) \tag{25}$$

to Σ_3^*. We say this because inspection of an example described by Selberg (cf. [15]) suggest that $\Sigma_3^{\dagger}$ (and other related terms) make a significant contribution in what may be conjectured to be an extremal example for the problem discussed in this article.

Lastly, we draw attention to the fact that the estimate obtained in [6] for the expression of the shape (5) is of the type described by the equation

$$\sum_{d \leqslant D} \frac{\lambda(d)}{d} = 2e^{\gamma} \left\{ \prod \left(1 - \frac{\varrho(p)}{p}\right) \right\} \left\{ \mathfrak{M}(W) + O\left(\frac{1}{\log^c D}\right) \right\},$$

where, if we write $w(p) = W(\log p/\log D)$, we have

$$\mathfrak{M}(W) = -\int_{1/2}^{1} \frac{W(1)-W(t)}{1-t}\frac{dt}{t} + \int_{0}^{1/2} \frac{W(t)}{t}\left\{\frac{1}{1-t} - h(t)\right\} dt,$$

for a certain $h(t) > 0$ when $0 < t < 1/4$. If such a result could be proved with $h(t)$ replaced by 0 then the conjecture "$\Lambda_R = R$" (see (10)) would follow by a suitable choice of W close to $W(t) = t$. It thus follows that methods such as those discussed in this article must, at best, lead to results involving a non-zero h, because of considerations relating to the neglected sum (25), for example. Indeed in [6] the function h satisfied

$$h(t) > \frac{1}{1-t} \quad \text{for} \quad 0 < t < T_0,$$

for a certain constant T_0 close to 0.074368 ... This means that the method cannot take effective account of the sought-for numbers a lying in the classes $\mathscr{A}$ and P_R if they are divisible by a prime smaller than D^{T_0} (for choosing non zero $W(t)$ for $t < T_0$ makes $\mathfrak{M}(W)$ less positive than it would otherwise be). This is the defect shared (with somewhat larger values of T_0) of all weighted sieves whose behaviour is known to the author. Like the failure to prove "$\Lambda_R = R$", it is linked to the non-vanishing of the function h.

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