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THE RECOGNITION PROBLEM FOR TOPOLOGICAL MANIFOLDS *

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1. Introduction

A geometric topologist often has difficulties when trying to determine whether a space he has constructed in the course of his investigations, is a topological manifold. Therefore there has been for a long time a need for a practical list of topological properties which would be reasonably easy to check and would characterize topological manifolds among topological spaces. ([3], [13], [37], [39], [47], [55], [62], [75], [78], [79], [100], [101], [102], [123], [124].)

This recognition problem has long been solved for manifolds of dimensions 1 and 2. For example, S^1 is the only compact, connected metric space containing at least 2 points, which is separated by every pair of its points [94], and S^2 is the only nondegenerate locally connected, connected, compact metric space which is separated by no pair of its points but is separated by each of its simple closed curves [7].

Until the late seventies nothing as elegant was even suspected to exist in higher dimensions. In 1977, first J. W. Cannon and then R. D. Edwards solved the celebrated double suspension problem – they proved that the double suspension of every homology 3-sphere is S^5 ([38], [54]).

Central to both proofs was a new device for detecting general position, called the *disjoint disks property* (DDP) – a higher dimensional analogue of a concept introduced in the late fifties by R. H. Bing [13] for the study of upper semicontinuous decompositions of R^3: it requires that pairs of disks can be pushed aparat by arbitrarily small moves [38]. Having discovered the power of this relatively simple condition, J. W. Cannon conjectured that the DDP could distinguish topological manifolds from other generalized manifolds in dimensions greater than 4 [37], and hence a simple geometric characterization of higher dimensional manifolds would follow. His intuition was quickly proven to be correct – first, R. D. Edwards exhibited the full

* This paper is in final form and no version of it will be submitted for publication elsewhere.

power of the DDP by proving that ANR cell-like quotients of n-manifolds ($n \geqslant 5$) with the DDP are n-manifolds [54]. Only a year later F. S. Quinn announced that every generalized n-manifold ($n \geqslant 5$) is a cell-like image of an n-manifold. Although there still has to appear a complete proof of this claim, Quinn has by now verified it for a large class of generalized manifolds ([98], [99]).

The subject of this paper is to give a survey of recent work on the recognition problem in dimension three. We shall give only a brief summary of the developments in higher dimensions because there already exist expositions on that subject ([37], [39], [55], [78]).

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2. Preliminaries

Let F be a covariant (resp. contravariant) functor defined on some topological category $\mathscr{C}$ and let $T\colon F(X) \to F(Y)$ (resp. $F(Y) \to F(X)$) be a morphism, where $X \subset Y$ are any two objects of $\mathscr{C}$. Then it will always be assumed that $T = F(\text{incl.})$ unless otherwise specified.

We shall be working in the category of locally compact Hausdorff spaces and continuous mappings throughout this paper. Manifolds will be assumed to have no boundary unless specified. Homology (resp. homotopy) equivalences will be denoted by $\sim$ (resp. $\simeq$). Isomorphisms (resp. TOP homeomorphisms) will be denoted by $\cong$ (resp. $\approx$). The singular (resp. Čech) (co)homology over a principal ideal domain (PID) R will be denoted by $H(\_\,;R)$ (resp. $\check{H}(\_\,;R)$). Whenever $R = \boldsymbol{Z}$ we shall not write the coefficients. The euclidean n-space (resp. the closed n-ball, the standard n-sphere, the n-cube $= [0,1]^n$) will be denoted by $\boldsymbol{R}^n$ (resp. B^n, S^n, I^n). A *homotopy* (resp. *R-homology*) *n-cell* is a compact n-manifold with boundary M such that $M \simeq B^n$ (resp. $M \sim B^n$ over R). The definition of a *homotopy* (*R-homology*) *n-sphere* is analogous.

A compact subset K of an n-manifold M is *cellular* in M if K is the intersection of a *properly nested* decreasing sequence of n-cells in M, $K = \bigcap_{i=1}^{\infty} B_i^n$ (i.e. for every i, $B_{i+1}^n \subset \operatorname{int} B_i^n$). A space X is *cell-like* if there exist a manifold N and an embedding $f\colon X \to N$ such that $f(X)$ is cellular in N. A map defined on a space (resp. an ANR, a manifold) X is *monotone* (resp. *cell-like*, *cellular*) if its point-inverses are continua (resp. cell-like sets, cellular sets) in X. A closed map is *proper* if its point-inverses are compact. A map $f\colon X \to Y$ is *one-to-one over* $Z \subset Y$ if for every $z \in Z$, $f^{-1}(z)$ is a point.

A compactum K in a manifold is *point-like* if $M - K \approx M - \{pt\}$. A space X is k-lc(R) (resp. $\mathrm{lc}^k(R)$, $\mathrm{lc}^\infty(R)$) at $x \in X$ ($k \in Z_+$, R a PID) if for every neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that $H_k(V; R) \to H_k(U; R)$ is trivial (resp. $H_j(V; R) \to H_j(U; R)$ is trivial for every $0 \leqslant j \leqslant k$, $H_j(V; R) \to H_j(U; R)$ is trivial for all $j \geqslant 0$). A compactum K in an ANR X has the k-uv(R) (resp. $\mathrm{uv}^k(R)$, $\mathrm{uv}^\infty(R)$) *property* ($k \in \boldsymbol{Z}_+$, R a PID) if for each neighborhood $U \subset X$ of K there is a neighborhood $V \subset U$ of K such that $H_k(V; R) \to H_k(U; R)$ is trivial (resp. $H_j(V; R) \to H_j(U; R)$ is trivial for every $0 \leqslant j \leqslant k$, $H_j(V; R) \to H_j(U; R)$ is trivial for all $j \geqslant 0$). The uv properties are related to the Čech cohomology: if a compactum K has the properties j-uv(R) ($j = k-1, k$) then $\check{H}^k(K; R) \cong 0$ and conversely, if $\check{H}^j(K; R) \cong 0$ ($j = k, k+1$) then K has the property k-uv(R) [77]. If instead of homology R-modules one uses homotopy groups one gets the corresponding definitions of the k-LC, LC^k, LC^∞ and k-UV, UV^k, UV^∞ properties [77]. A map defined on an ANR is $\mathrm{uv}^k(R)$ (resp. UV^k) ($k \in \boldsymbol{Z}_+$, R a PID) if its point-inverses have the $\mathrm{uv}^k(R)$ (resp. UV^k) property.

A subset $Z \subset X$ is *locally simply coconnected* ($1-$LCC) if for every $x \in X$ and every neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that $\pi_1(V - Z) \to \pi_1(U - Z)$ is trivial. A metric space X is *uniformly locally simply connected* (1-ULC) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that each loop in X of diameter less than δ bounds a disk in X of diameter less than ε.

Let G be a decomposition of a space X into compact and connnected subsets and let $\pi\colon X \to X/G$ be the corresponding quotient map, H_G the collection of all *nondegenerate* (i.e. $\neq$ pt) elements of G, and N_G their union. A set $U \subset X$ is *G-saturated* if $U = \pi^{-1}\pi(U)$. A decomposition G is *upper semicontinuous* if for each $g \in G$ and for each open neighborhood $U \subset X$ of g there exists a G-saturated open neighborhood $V \subset U$ of g. Equivalently, π is a closed map. A decomposition G of a separable metric space X is *k-dimensional* (resp. *closed k-dimensional*), $k = -1, 0, 1, \ldots$, if $\dim \pi(N_G) = k$ (resp. $\dim \pi(\overline{N_G}) = k$). A decomposition G of a metric space X is *weakly shrinkable* if for each $\varepsilon > 0$ and each neighborhood $U \subset X$ of N_G there is a homeomorphism $h\colon X \to X$ such that $h|X - U = \mathrm{id}$ and for each $g \in G$, $\operatorname{diam} h(g) < \varepsilon$. A decomposition G of a space X is *shrinkable* if for every G-saturated open cover $\mathcal{U}$ of N_G and every open cover $\mathcal{V}$ of X there is a homeomorphism $h\colon X \to X$ such that:

(i) $h|X - \mathcal{U}^* = \mathrm{id}$ where $\mathcal{U}^* = \bigcup\{U \in \mathcal{U}\}$;

(ii) for each $x \in X$ there exists $U \in \mathcal{U}$ such that $\{x, h(x)\} \subset U$;

(iii) for each $g \in G$ there exists $V \in \mathscr{V}$ such that $h(g) \subset V$.
Let $f\colon X \to Y$ be a map. The *nondegeneracy set* of f is defined by $N(f) = \{x \in X \mid f^{-1}f(x) \neq x\}$ and its image $S(f) = f(N(f))$ is called the *singular set* of f. Let $f\colon M \to X$ be a proper, cell-like map from a manifold onto an ENR. Then the associated decomposition $G(f) = \{f^{-1}(x) \mid x \in X\}$ of M is upper semicontinuous and cell-like. Moreover, $H_{G(f)} = f^{-1}(S(f))$ and $N_{G(f)} = N(f)$. For more on decompositions see [47].

A countable collection of pairwise disjoint compacta $\{C_i\}$ in a metric space X is a *null-sequence* if for every $\varepsilon > 0$ all but finitely many among the C_i's have diameter less than ε. A compactum $K \subset \boldsymbol{R}^m$ has *embedding dimension* $\leqslant n$, $\operatorname{dem} K \leqslant n$, if for every closed subpolyhedron $L \subset \boldsymbol{R}^m$ with $\dim L \leqslant m-n-1$, there exists an arbitrarily small ambient isotopy of $\boldsymbol{R}^m$, with support arbitrarily close to $K \cap L$ which moves L off K. This concept is due to M. A. Štan'ko [112] – for an exposition see [53].

A *crumpled cube* is the complementary domain of an open n-cell in S^n. A *fake* cube is a homotopy 3-cell which is not homeomorphic to B^3. The classical Poincaré conjecture asserts that there are no fake cubes [63]. A space X is said to have the *Kneser finiteness* (KF) if no compact subset of X contains more than finitely many pairwise disjoint fake cubes. Kneser finiteness theorem [70] says that every 3-manifold has the KF. A *homotopy handlebody* is a regular neighborhood of a wedge of finitely many circles in some 3-manifold.

Let X be a locally compact space and present it as the union $X = \bigcup_{i=1}^{\infty} K_i$ of a properly nested increasing sequence of compact subsets $K_i \subset X$. An *end* of X is a sequence $e = \{U_i\}$ of properly nested decreasing sequence of components of $X - K_i$. The *Freudenthal compactification* $\hat{X}$ of X is $X \cup \{e\}$ with $\{U_i\}$ as the basis of topology at the end e ([59], [60], [107]). For example, if X is a generalized n-manifold with 0-dimensional singular set $S(X)$ (see Chapter 3 for definitions) then X is the Freudenthal compactification of the open n-manifold $X - S(X)$.

A space X is 1-*acyclic at* ∞ if for every compact set $K \subset X$ there exists a compact set $K' \supset K$ such that $H_1(X-K') \to H_1(X-K)$ is trivial.

3. Generalized manifolds: Preliminaries

Generalized manifolds were introduced around 1930. For some history of this class of spaces see the surveys [37], [78], [79], [100], [124].

Different definitions of a generalized manifold were used at different times in the past. We shall adopt the following modern definition [35]: A space X is a *generalized n-manifold* ($n \in N$) if:

(i) X is a *euclidean neighborhood retract* (ENR), i.e. for some integer m, X embedds in $\mathbf{R}^m$ as a retract of an open subset of $\mathbf{R}^m$; and

(ii) X is a *homology n-manifold*, i.e. for every $x \in X$,

$$H_*(X, X - \{x\}; \mathbf{Z}) \cong H_*(\mathbf{R}^n, \mathbf{R}^n - \{0\}; \mathbf{Z}).$$

Note that condition (i) is equivalent to: X is a locally compact, finite dimensional separable metrizable ANR [20].

One of the most important features of generalized manifolds is that although they are defined by a set of *local* properties of topological manifolds, they nevertheless satisfy most of the basic *global* properties of manifolds, e.g. the Poincaré duality (in its most general form), the invariance of domain, standard separation properties, and they admit linking and intersection theory ([6], [17], [19], [21], [22]).

So let X be a generalized n-manifold. If $n \leqslant 2$ then it has long been known that then X is actually a genuine n-manifold. In all higher dimensions X may fail to be locally euclidean at some points (or perhaps at all) — we call such exceptions *singularities* of X and they together form the *singular set* of X, $S(X) = \{x \in X \mid$ *x does not have a neighborhood in X homeomorphic to an open subset of* $\mathbf{R}^n\}$. Its complement, $M(X) = X - S(X)$ is called the *manifold set* of X. Note that $S(X)$ is always closed and if $S(X) \neq X$ then $M(X)$ is an open n-manifold.

4. Generalized manifolds: Examples

One reason why many topologists are interested in generalized manifolds is that they arise in a multitude of situations. In this chapter we present the main groups of examples.

We begin with cell-like, upper semicontinuous *decompositions of manifolds*: every proper, cell-like surjection from a (generalized) n-manifold onto a finite dimensional metric space yields a generalized n-manifold [77]. In fact, theory of decompositions of euclidean spaces, developed by the school of R. H. Bing, has provided several rather bizarre examples of generalized manifolds, thus revealing some of their unknown pathologies. Let us mention only three examples — for more, the interested reader can consult the rich bibliographies in ([13], [47]).

In 1957, R. H. Bing exhibited his dogbone space [10] — a quotient of $\mathbf{R}^3$ by a Cantor set worth of tame arcs. Although each individual arc is as nice as possible, the quotient $\mathbf{R}^3/G$ becomes a manifold only after the stabilization by $\mathbf{R}$: $\mathbf{R}^3/G \times \mathbf{R} \approx \mathbf{R}^4$ [12].

Shrinking a wild arc in $\mathbf{R}^3$ to a point one gets a generalized 3-manifold with exactly one singularity. Already in 1957, K. W. Kwun showed how to place only a countable collection of wild arcs in $\mathbf{R}^3$ so densely that the corresponding quotient R^3/G is totally singular, i.e. $S(R^3/G) = R^3/G$, although again $R^3/G \times R \approx R^4$ [74].

One need not restrict to dimension 3 to generate bad examples. Probably the most striking examples of ghastly generalized n-manifolds ($n \geq 3$) are those constructed by R. J. Daverman and J. J. Walsh [48]:

(i) X is a quotient of S^n by a cell-like, upper semicontinuous decomposition G such that each $g \in G$ is nondegenerate, noncellular, 1-dimensional, and contains a wild Cantor set;

(ii) $X \times R \approx S^n \times R$;

(iii) Every mapping $f\colon B^2 \to X$ such that $f|\partial B^2$ is an embedding, has nonempty interior;

(iv) X does not admit a cell-like map onto any topological n-manifold; and

(v) X contains no ANR's of dimension ≥ 1.

The three decomposition spaces above also represent another class of examples of generalized manifolds – *manifold factors*. Using the Künneth formula it is not too difficult to show that given locally compact n_i-dimensional Hausdorff spaces X_i ($i = 1, 2$), their product $X_1 \times X_2$ is a generalized $(n_1 + n_2)$-manifold if and only if each X_i is a generalized n_i-manifold [18]. We shall study the relationship between these two classes of examples in later chapters where we shall discuss methods of *desingularization*, i.e. when a generalized n-manifold X is either *resolved* ($f\colon M \to X$ with M a topological n-manifold and f a proper cell-like map) or *stabilized* ($X \times R^k$ becomes a topological $(n+k)$-manifold for some $k > 0$). We also intend to explore the questions of existence and uniqueness of the two methods and the relationship between them.

The third class of examples comes from the *transformation groups* theory. P. E. Conner and E. E. Floyd discovered in 1959 that the Smith manifolds [110] are nothing but (classical) generalized manifolds [44]: the fixed point set of a toral group action (resp. a Z_p action with p any prime) on a manifold is a generalized manifold.

As we have already observed above, manifold factors do not always retain their locally euclidean character whereas generalized manifold factors do keep all of their defining properties. This fact is crucial for making generalized manifolds an indispensable tool in the *theory of slices* of actions of compact groups on manifolds: let G be a compact Lie group acting on a generalized n-manifold X. Then the orbit $G(x)$ of any $x \in X$ is a base of a fiber bundle $S_x \times_{G_x} G$ and its fibers, the slices S_x are generalized manifolds (usually of dimension $< n$) and the orbit space X/G looks close to x like the quotient of S_x by the isotropy group G_x at x ([18], [100]). Therefore one may use *induction* in the analysis of the action of G on X, since G_x is a proper, closed subgroup of G.

Another way of going beyond the class of topological manifolds and still staying in the class of generalized manifolds is to *suspend homology spheres*: the k-fold suspension of a generalized n-manifold with the singular homology of S^n is always a generalized $(n+k)$-manifold.

Our list of examples is still slightly incomplete. One could, for example, add those generalized manifolds which arise as the ENR's which admit maps onto closed manifolds with arbitrarily small point-inverses [83] or the Freudenthal compactifications of certain open manifolds ([23], [25], [27], [122]).

One of the most important questions in the theory of generalized manifolds today is to what extent do all these classes of examples overlap. Some answers will be discussed in the forthcoming chapters.

5. Generalized n-manifolds ($n \geqslant 4$): Resolutions

A *resolution* of an n-dimensional ANR X is a pair (M, f) consisting of a topological n-manifold M and a proper, cell-like map $f\colon M \to X$. Consequently, if X has a resolution then X is a generalized n-manifold [77]. A resolution (M, f) of X is called *conservative* if f is one-to-one over $M(X)$.

A proof of the existence of resolutions for higher dimensional generalized manifolds would be one of the key steps in the proof of the Cannon conjecture (the other one being R. D. Edwards' Shrinking theorem – see Theorem (6.1)). In 1978 F. Quinn announced such a proof [98]. However, in 1984 it was discovered by S. Cappell and S. Weinberger that Quinn's argument [98] contains a mistake – see [99]. The present status of the affairs is described below:

5.1. THEOREM. (F. S. Quinn ([98], [99])). *Let X be a generalized n-manifold.*

(a) *If $n \geqslant 4$, then X admits a conservative resolution if $X \times \mathbf{R}$ resolves.*

(b) *If $n \geqslant 5$ and if a certain integer obstruction $\sigma(X)$ vanishes, then X admits a conservative resolution.*

Moreover, if (M_i, f_i) are any two conservative resolutions of X, $n \geqslant 4$, and $U \subset X$ is a neighborhood of $S(X)$, then there is a homeomorphism $h\colon M_1 \to M_2$ such that $f_1(x) = f_2 h(x)$, for every $x \notin U$.

Quinn's theorem implies all previously known results in dimension $n \geqslant 4$ ([29], [30], [31], [35], [40], [58], [78], [117], [118]). In particular, if X is a generalized n-manifold, $n \geqslant 4$, and X is not totally singular, i.e. $S(X) \neq X$, then by [99] X always has a (conservative) resolution.

To outline the proof of Theorem (5.1) we need the following result which is a consequence of Edwards' shrinking theorem [54], Quinn's end theorems ([96], [97]), and R. J. Daverman's observation [45] that all generalized manifolds adopt the DDP after having been crossed by $\mathbf{R}^2$. Note that this result also shows that the two, in the past most useful methods of desingularizing a generalized n-manifold ($n \geq 4$), resolving and stabilizing, are equivalent.

5.2. THEOREM. *Let X be a generalized n-manifold, $n \geq 4$. Then the following statements are equivalent*:

(i) *X has a resolution*;

(ii) *$X \times \boldsymbol{R}^k$ has a resolution, for some $k \in \boldsymbol{N}$; and*

(iii) *$X \times \boldsymbol{R}^2$ is a manifold.*

Proof. (i) $\Rightarrow$ (ii). If f: $M \to X$ is a resolution of X then $f \times \mathrm{id}$: $M \times \boldsymbol{R}^k \to X \times \boldsymbol{R}^k$ is a resolution of $X \times \boldsymbol{R}^k$.

(ii) $\Rightarrow$ (iii). Suppose that $k > 2$ and let f: $M \to X \times \boldsymbol{R}^k$ be a resolution of $X \times \boldsymbol{R}^k$. Consider the canonical projection e: $M \to X \times \boldsymbol{R}^{k-1}$. The map e has two ends [96]. Since e is cell-like, both ends are 1-LC and tame [77]. By [96] ([97] if $n = 4$), there is a manifold with boundary $M' \supset M$ and a proper map e': $M' \to X \times \boldsymbol{R}^{k-1}$ such that $\mathrm{int}\, M' = M$ and $e'|M = e$. It follows that $e'|\partial M$ is a resolution of $X \times \boldsymbol{R}^{k-1}$. By 5.1, $X \times \boldsymbol{R}^2$ has a resolution. Since $X \times \boldsymbol{R}^2$ also has the DDP [45] the assertion follows by [54].

(iii) $\Rightarrow$ (i) Similar argument as in the preceding paragraph.

Proof of Theorem (5.1). First observe that the case $n = 4$ follows from the other case ($n \geq 5$): if X^4 is a generalized 4-manifold and $X^4 \times \boldsymbol{R}^k$ is resolvable for some $k \geq 1$ then by Theorem (5.2) X^4 admits a resolution. Next, note that it suffices to show that for some $k \in \boldsymbol{N}$, $X \times \boldsymbol{R}^k$ is locally resolvable. For then by (5.2), $X \times \boldsymbol{R}^n$ is locally euclidean for all $n \geq k+2$, hence a manifold, so by (5.2), X must admit a resolution. We may also assume that $n+k \equiv 0 \bmod 4$.

Choose now an arbitrary point $x \in X \times \boldsymbol{R}^k$ and find a neighborhood $U \subset X \times \boldsymbol{R}^k$ such that for every $i \in \boldsymbol{N}$ there exist:

(i) a $(n+k)$-manifold M_i and a proper $\frac{1}{2^i}$-homotopy equivalence f_i: $M_i \to U$; and

(ii) a homeomorphism g_i: $M_i \to M_{i+1}$ such that $d(f_i, f_{i+1} g_i) < \frac{1}{2^i}$.

One then verifies that for a fixed j, the maps $f_i g_{i-1} g_{i-2} \ldots g_j$: $M_j \to U$ converge to some proper cell-like map f: $M_j \to U$.

The surgery obstructions to the existence of $\{(M_i, f_i)\}_{i \geq 1}$ are ε-versions of the ordinary ones. They are all reduced to a single integer obstruction σ defined by transversality on a manifold degree one normal map with the same form as f: $M \to U$. In order to get (ii) one must arrange for the M_i's to be normally bordant and use surgery to make the normal bordism an (ε, h)-cobordism. Finally, one must apply Quinn's thin h-cobordism theorem [96] to obtain the homeomorphisms g_i.

So we have a resolution. In order to trim it into a conservative resolution we invoke the cell-like approximation theorems of F. Quinn [97], if $n = 4$ and L. C. Siebenmann [108], if $n \geqslant 5$. The uniqueness then follows easily by the thin h-cobordism theorem – [97], if $n = 4$ and [96], if $n \geqslant 5$.

6. Generalized n-manifolds ($n \geqslant 5$): The DDP

Topological n-manifolds ($n \geqslant 5$) have the following simple general position property: any two (singular) 2-disks can be pushed apart by arbitrarily small moves. It turns out that this property, called the *disjoint disks property* (DDP) is also characteristic for higher dimensional manifolds (Theorem (6.2)). In this chapter we shall give a brief account of the results which culminated in the following definite higher dimensional shrinking theorem:

6.1. Theorem (R. D. Edwards [54]). *Let G be a cell-like, upper semicontinuous decomposition of an n-manifold M ($n \geqslant 5$) such that* $\dim M/G < \infty$. *Then G is shrinkable if and only if M/G has the* DDP.

A metric space X has the DDP if for every pair of maps $f, g\colon B^2 \to X$ and every $\varepsilon > 0$ there exist maps $f', g'\colon B^2 \to X$ such that $d(f, f') < \varepsilon > d(g, g')$ and $f'(B^2) \cap g'(B^2) = \emptyset$ [38]. In an arbitrary generalized n-manifold ($n \geqslant 5$) the DDP can fail badly [48] (see also Chapter 7). But if it is valid it *detects* topological manifolds:

6.2. Theorem (F. S. Quinn [98], [99]). *A space X is a topological n-manifold ($n \geqslant 5$) if and only if X is a generalized n-manifold, has the* DDP, *and $\sigma(X)$ vanishes.*

Proof. Follows by Theorems (5.1) and (6.1).

6.3. Corollary (J. W. Cannon [38]). *The double suspension of every homology n-sphere is homeomorphic to S^{n+2}.*

Proof. Let H be a homology n-sphere. If $n \leqslant 2$ the assertion is clear since then $H \approx S^n$. Assume therefore $n \geqslant 3$. Then $\Sigma^2 H$ is a generalized ($n+2$)-manifold, hence it suffices by Theorem (6.2) to verify that $\Sigma^2 H$ has the DDP. Clearly, $\Sigma^2 H$ is a manifold at all points except possibly at the suspension circle C. Now, given any two maps $f, g\colon B^2 \to \Sigma^2 H$, they can be deformed near C to the union of finitely many cones, each of which intersects C in $\leqslant 1$ point, so we can separate $f(B^2)$ and $g(B^2)$ at C by pushing these cones along C as to make them disjoint. So if then $f(B^2) \cap g(B^2) \neq \emptyset$, all the intersections occur away from C and we can apply standard general position arguments [68].

The concept of a disjoint disks property was introduced into topology by R. H. Bing. He used a version of it to prove that his dogbone space was not a manifold ([10], [13]). This and most of the subsequent applications of disjoint disks properties were aimed at distinguishing certain pathological (usually decomposition) spaces from true manifolds, i.e. they were used mostly for diagnosing nonshrinkable decompositions (see e.g. [15]). R. J. Daverman and W. T. Eaton began to use general position properties to prove that certain decomposition spaces were indeed manifolds ([46], [49], [50], [51]) (see [42] for an exposition of Eaton's Mismatch theorem and its extension to higher dimensions, showing a connection between decomposition theory and taming problems). In its present form the DDP first appeared in J. W. Cannon's proof of (6.3) – the double suspension problem [38]. Therein, he used the DDP as a hypothesis sufficient to single out manifolds inside a class of certain generalized manifolds. (Note that Cannon's original proof [38] does not use (6.1) or (6.2) since it preceded both [54] and [98]. Instead, Cannon proved a weaker form of (6.1) and used his earlier resolution theorem [36] which was sufficient for this case.) Completely independently of Cannon, H. Toruńczyk discovered a higher dimensional analogue of the DDP, the disjoint cells property, and has proved that it detects Hilbert cube manifolds among ANR's (cf. [55, § 10]).

For a discussion of the disjoint disks properties in dimension 3 see Chapter 9.

We now turn to a brief discussion of Edwards' theorem. This result is one of the most impressive in modern geometric topology, both in its simplicity and its elegance of the argument. It is a sweeping generalization of many earlier related results, e.g. ([38], [108], [116]). The proof is a prime example of the Bing school of topology. Its basic ingredients are classical shrinking techniques of R. H. Bing, radial engulfing and some fundamental taming results of R. H. Bing and J. M. Kister [16] and J. L. Bryant and C. L. Seebeck [33].

The original manuscript [54] was never completed for publication. Instead, Edwards prepared an outline of the proof in his survey article [55]. Complete versions can be found in [82] and (with more details) in [47]. The following is, by [84], an equivalent formulation of (6.1):

6.1.* Theorem (R. D. Edwards [54]). *A proper, cell-like map $f\colon M \to X$ from an n-manifold M ($n \geq 5$) onto an* ANR *X can be approximated by homeomorphisms if and only if X has the* DDP.

6.4. Corollary (L. C. Siebenmann [108]). *A proper, cell-like map between topological n-manifolds ($n \geq 5$) can be approximated by homeomorphisms.*

This corollary is also known in lower dimensions: $n = 2$ ([106], [120]) (for S^2 already [93]), $n = 3$ (f cellular) ([2], [108]) and for $n = 4$ [97].

7. Generalized 3-manifolds: Preliminaries

Dimension 3 is in many respects peculiar: (i) this is the lowest dimension in which singularities occur (recall that generalized 1- and 2-manifolds are always locally euclidean [124]), and (ii) this is the only dimension in which we still don't know if there can exist any *exotic* homotopy spheres, i.e. $\neq S^3$ ([57], [109]).

A consequence of (i) is that generalized 3-manifolds cannot have "cone" singularities which are quite common in higher dimensions. (This fact was first observed by K. W. Kwun and F. Raymond [76] and recorded again in [32]. See also [62].) Take for example, the suspension ΣH^n of any nonsimply connected homology n-sphere H^n (such exist for all $n \geqslant 3$). Clearly, ΣH^n is a generalized $(n+1)$-manifold with the suspension points as the only two singularities. As a corollary of (i), if a generalized 3-manifold X admits a PL structure then X has no singularities.

A consequence of (ii) is that, assuming the existence of fake 3-spheres one can construct strange examples of generalized 3-manifolds which in many respects still behave like 3-manifolds and yet they may be totally singular. The simplest example is (7.1) below, more sophisticated ones are (7.2)–(7.4) further on.

Let X be a generalized 3-manifold with 0-dimensional singular set and let $p \in X$. Then p has arbitrarily small compact neighborhoods $N \subset X$ such that:

(i) $X - \operatorname{int} N$ is a compact 3-manifold with boundary;
(ii) $\operatorname{int} N$ is orientable; and
(iii) $\partial(X - \operatorname{int} N) \subset M(X)$.

(For a proof see [27].) We say that X has *genus* $\leqslant n$ *at* p ($n \in \mathbb{N}$) if p has arbitrarily small such neighborhoods N with $\partial(X - \operatorname{int} N)$ a (closed orientable) surface of genus $\leqslant n$. We say that X has *genus* n *at* p if X has genus $\leqslant n$ at p and doesn't have genus $\leqslant n-1$ at p. If X doesn't have genus $\leqslant n$ at p for any n we say that X has *genus* ∞ *at* p. We shall denote the genus of X at p by $g(X, p)$ [79].

7.1. Example (R. L. Wilder [124]). Suppose fake cubes exist and consider in S^3 a null-sequence of pairwise disjoint 3-cells $\{B_i\}$ converging to a point $p \in S^3$. Replace each B_i by a fake cube F_i and choose a metric in $W^3 = (S^3 - \bigcup_{i=1}^{\infty} \operatorname{int} B_i) \cup (\bigcup_{i=1}^{\infty} F_i)$ so that the F_i's also converge to p. Then W^3 is a compact generalized 3-manifold with the following properties:

(i) $S(W) = \{p\}$;
(ii) W does not have a resolution;
(iii) $W \simeq S^3$; and
(iv) $g(W, p) = 0$.

Proof. (i) Follows by the Kneser finiteness theorem [70], (ii) by Theorem (7.8), while (iv) is clear. To see that (iii) holds, consider the map $f: W \to S^3$ which shrinks out all the F_i's. Clearly f is cell-like hence a homotopy equivalence [77].

Generalized 3-manifolds which are obtained via an operation (or a sequence of countably many of such) as described above and whose singular set is 0-dimensional, shall be called *Wilder manifolds* (in [115] they are termed "near manifolds"). Note that if the Poincaré conjecture is true then every Wilder manifold is a genuine 3-manifold.

7.2. EXAMPLE (W. Jakobsche and D. Repovš [67]). Suppose the Poincaré conjecture is false. Then there exists a compact homogeneous ANR X with the following properties:

(i) X is a generalized 3-manifold and $S(X) = X$;
(ii) X does not admit a resolution;
(iii) X has the Dehn's lemma property;
(iv) X has the map separation property;
(v) $X \times S^1 \approx S^3 \times S^1$.

(For the definitions of properties in (iii) and (iv) see Chapters 9 and 10.) The construction is a modification of W. Jakobsche's earlier example of a homogeneous 3-dimensional ANR which fails to be a manifold [66]: X is obtained as the inverse limit $X = \varprojlim \{X_k, f_k\}$, where $X_k = X_{k-1} \# \# H^3 \# \ldots \# H^3$, H^3 = homotopy 3-sphere, $X_1 = S^3$, and $f_k: X_k \to X_{k-1}$ are spine maps. For details see [67].

7.3. EXAMPLE (M. G. Brin [25]). Suppose the Poincaré conjecture is false. Then there exists a compact generalized 3-manifold X with the following properties:

(i) $S(X) = \{p\}$ and $g(X, p) = 1$;
(ii) $M(X)$ is an irreducible 3-manifold (i.e., every 2-sphere in $M(X)$ bounds a 3-cell in $M(X)$); and
(iii) X does not admit a resolution.

Brin's example arises as the endpoint compactification of an open 3-manifold N which he constructs as follows: in a fake 3-sphere P he considers a certain link (J_1, J_2), where the simple closed curve J_2 lies in no 3-cell in P. The fundamental building block of N is the complement Q (in P) of an open tubular neighborhood of this link in P. Let $\{Q_i\}$ be an infinite collection of copies of Q. Denote by A_i and B_i the two boundary tori of Q_i. Brin now glues, in a certain pattern, first, a solid torus T to Q_1 using A_1, and then for every $i > 1$, he matches up Q_i with Q_j, $|i-j| = 1$, glueing B_{i-1} onto A_i and then B_i onto A_{i+1}. Finally, $N = T \cup (\bigcup_{i=1}^{\prime} Q_i)$ and $X = \hat{N}$. For details see [25]. (Note that this construction can easily be modified as to get as the singular set of X any (necessarily wild) Cantor set (rather than just one point p).)

7.4. EXAMPLE (M. G. Brin and D. R. McMillan, Jr. [27]). Suppose the Poincaré conjecture is false. Then there exists a compact generalized 3-manifold X such that:

(i) $S(X) = \{p\}$ and $g(X, p) = \infty$;

(ii) $M(X)$ is the union of an increasing, properly nested sequence of handlebodies; and

(iii) X doesn't admit a resolution.

The construction is based on a homotopy 3-sphere $H \not\approx S^3$ with a Heegaard splitting $H = A \cup B$ into 2 handlebodies. Earlier, McMillan had found a homeomorphism h: $H \to H$, isotopic to the identity, such that $h(A) \subset \operatorname{int} A$ and this inclusion is null homotopic [92]. Consequently, $C = \bigcap_{i=1}^{\infty} h^k(A)$ is a cell-like set [89], the open 3-manifold complement $U = H - C$ is acyclic [89], and its endpoint compactification $\hat{U}$ is a resolvable generalized 3-manifold, $\hat{U} = H/C$. The desired example X is then obtained as the endpoint compactification of $p^{-1}(H-(C \cup T \cup J))$, where p: $V \to H - (\operatorname{int} T)$ is an infinite cyclic cover, $T \subset H - C$ an appropriately chosen solid torus, and $J \subset \overline{H - C \cup T}$ is an arc from C to ∂T. For details see [27]. (It is an open problem whether there is such an example with $g(X, p) \leqslant n_0$ for some fixed integer n_0, i.e. whether the family of handlebodies can be modified as to have genus $\leqslant n_0$.)

We continue our discussion with some geometric properties of generalized 3-manifolds. We begin by a taming theorem of J. L. Bryant and R. C. Lacher [32] which is the 3-dimensional analogue of the 1-LCC shrinking theorem from [40] for higher dimensions ($n \geqslant 5$). As already Lacher has pointed out in [79], Theorem (7.5) has limited application in the recognition process since many potential singular sets may be *wildly* embedded. (Recall that this occurs also in higher dimensions – the only points which Cannon had to check in [38] to prove that the double suspension $\Sigma^2 H^3$ of a homology 3-sphere H^3 is the 5-sphere, were those at the suspension circle C, a wildly embedded simple closed curve in $\Sigma^2 H^3$.)

7.5. THEOREM (J. L. Bryant and R. C. Lacher [32]). *Let X be a compact generalized 3-manifold, satisfying the Kneser finiteness, and suppose that $S(X)$ is a 1-LCC subset of X. If $\dim S(X) \leqslant 0$ then X is a 3-manifold.*

The special case $Z = \{pt\}$ was proved already in the early 1960's by C. H. Edwards, Jr. [52] and, independently, by C. T. C. Wall [121]. Using the connection between tameness and genera of points in X (see (7.7) below) we can restate (7.5) as follows:

7.5.* THEOREM (J. L. Bryant and R. C. Lacher [32]). *Let X be a* *generalized 3-manifold with* $\dim S(X) \leqslant 0$. *If X satisfies the Kneser finiteness and if* $g(X, x) = 0$ *at all* $x \in X$, *then X has no singularities.*

7.6. COROLLARY. *Let X be a generalized 3-manifold with* $\dim S(X) = 0$. *If* $g(X, p) = 0$ *at all* $p \in X$ *then X is a Wilder manifold.*

Proof of Theorem (7.5)*. By hypothesis there is a closed, 0-dimensional set $Z \subset X$ such that $S(X) \subset Z$ and Z is 1-LCC in X. We may assume that Z is compact. First construct arbitrarily small compact neighborhoods $N_i \subset X$ of Z with orientable interiors and such that for every i, $N_{i+1} \subset \operatorname{int} N_i$ and $X - \operatorname{int} N_i$ is a compact 3-manifold, with boundary a collection of 2-spheres, PL embedded in $X - Z$. We may also assume that the only homology 3-cells in $X - Z$ are the real 3-cells (use the Kneser finiteness theorem and the Gruško theorem [63]) and that for every component of N_i, $\overline{X - \operatorname{int} N_i}$ has connected boundary (drill to join the components of $\partial(X - \operatorname{int} N_i)$). Then every component of $N_i - \operatorname{int} N_{i+1}$ is a punctured 3-cell. It is therefore possible to replace N_1 by a 3-cell so that the new space is a 3-manifold, homeomorphic to X. This proves (7.5)*.

To complete the argument for (7.5) it remains to prove the following proposition whose proof may be found, e.g. in [102].

7.7. PROPOSITION. *Let X be a generalized 3-manifold with* $\dim S(X) \leqslant 0$. *Then* $S(X)$ *is 1-LCC in X if and only if, for every* $x \in X$, $g(X, x) = 0$.

We could roughly *classify* the singularities of generalized 3-manifolds X as follows [79]: Suppose $S(X) = \{p\}$. Then p can be, for example:

(i) *tame*, e.g. S^3/C, $C =$ the Whitehead continuum [122], or

(ii) *wild*, e.g. S^3/A, $A =$ the Fox–Artin wild arc [56], or

(iii) *Wilder type*, e.g. our Example (7.1).

Like manifolds, generalized 3-manifolds also satisfy certain algebraic finiteness properties:

7.8. THEOREM (J. L. Bryant and R. C. Lacher [32]). *Let X be a compact generalized 3-manifold (resp. with a resolution). Then there is an integer* $k \in \mathbf{N}$ *such that among any* $k+1$ *pairwise disjoint* $\mathbf{Z}_2$*-homology 3-cells in X at least one is contractible (resp. a 3-cell).*

Proof. An excercise using van Kampen and Gruško's theorems [63].

8. Generalized 3-manifolds: Resolutions

In higher dimensions the existence of a resolution immediately implies the existence of a conservative one (see Chapter 6). In dimension 3 few more arguments are needed:

8.1. THEOREM (J. L. Bryant and R. C. Lacher [32]). *Every resolvable generalized 3-manifold admits a conservative resolution.*

Proof. Take any resolution $f\colon M \to X$. Then the set $C \subset M(X)$ of those points over which f fails to be cellular is locally finite in $M(X)$ [88], hence we may assume ([2], [108]) that f is one-to-one over $X-(C \cup S(X))$, so eventual fake cubes in $X-C$ lift to M. By the Kneser finiteness theorem [70] we conclude that C is locally finite in X. Shrinking out the collection $\{f^{-1}(c) \mid c \in C\}$ we get a proper, cell-like map $q\colon M \to M'$ onto a 3-manifold M' and an induced proper, cell-like map $f'\colon M' \to X$ which is one-to-one over $M(X)$ and such that $f = f'q$.

We now turn to the existence of resolutions. The first resolution theorem in dimension 3 was obtained by J. L. Bryant and R. C. Lacher [32] in 1978. We state here the improved version [105] (in [32] M was assumed to be orientable). The advantage of this result over other resolution theorems which we shall discuss later on, is that it contains no bound on dimension of the singular set. Note that the set Z in (8.2) can possibly be even *dense* in X.

8.2. THEOREM (D. Repovš and R. C. Lacher [105]). *Let $f\colon M \to X$ be a closed, monotone mapping from a 3-manifold onto a locally simply connected $\boldsymbol{Z}_2$-homology 3-manifold. Suppose that there is a 0-dimensional set $Z \subset X$ such that $\check{H}^1(f^{-1}(x); \boldsymbol{Z}_2) \cong 0$ for all $x \in X-Z$. Then the set $C = \{x \in X \mid f^{-1}(x)$ is not cell-like$\}$ is locally finite in X. Moreover, X is a resolvable generalized 3-manifold.*

An elementary example when C is nonempty can be obtained by considering any spine map [77]. An easy modification of the construction of the Whitehead continuum [122] shows that on the other hand, the set $D = \{x \in X \mid f^{-1}(x)$ *is not cellular in* $M\}$ may be uncountable even when C happens to be empty.

The hypothesis that X be locally simply connected cannot be omitted from (8.2): consider the decomposition G of $\boldsymbol{R}^3$ into points and a Cantor set worth of dyadic solenoids (to get G take the standard construction of a single dyadic solenoid in $\boldsymbol{R}^3$ and "double" the defining sequence of solid tori at every stage – compare [101]). Consider now $f\colon M \to X$ where $M = \boldsymbol{R}^3$, $X = \boldsymbol{R}^3/G$ and $f =$ the decomposition map. Since $\check{H}^1(f^{-1}(x); \boldsymbol{Z}_2) \cong 0$ for all $x \in X$, X is a $\boldsymbol{Z}_2$-homology 3-manifold [77]. However, f is not even strongly acyclic over $S(f)$ hence C is uncountable. For a "ghastlier" example see [48]: there, a strongly acyclic map (over $\boldsymbol{Z}$) from S^3 onto a finite dimensional $\boldsymbol{Z}$-homology 3-manifold is constructed such that $C = X$.

Neither can the hypothesis $\dim Z = 0$ be weakened in (8.2): consider Bing's figure eights decomposition of S^3. Then C is uncountable, in fact, $C = Z \approx (0,1)$.

As a corollary, we obtain a partial converse in dimension 3 to the well-known fact that a cell-like, upper semicontinuous decomposition G of an n-manifold always yields a generalized n-manifold (if $n \geqslant 4$ one must assume, in addition, that M/G is finite dimensional). A special case of this question was studied in the late 1960's namely, whether monotone, 0-dimensional, upper semicontinuous decompositions of S^3 which yield S^3 must have point-like elements. Partial results were obtained by several people – see e.g. ([1], [5], [81], [85], [91]). Note that the restriction on the dimension of G in (8.3) comes in naturally due to Bing's example in the preceding paragraph.

8.3. COROLLARY. *Let G be a monotone, 0-dimensional, upper semicontinuous decomposition of a compact 3-manifold M such that M/G is a generalized 3-manifold. Then G is almost cell-like, i.e. the set $C = \{g \in G \mid g$ is not cell-like$\}$ is finite.*

The proof of Theorem (8.2) combines algebra with geometry. It relies on some fundamental theorems from the 3-manifolds topology. The first necessary technical result we need to develop is (8.4) below. In [72] T. E. Knoblauch proved that in a closed, orientable 3-manifold there can be but a finite number of pairwise disjoint compact sets that do not have a neighborhood embeddable in $\boldsymbol{R}^3$. It is easy to manufacture examples to show that this does not always hold for nonorientable 3-manifolds. In the next result we give an additional (minimal?) condition which is sufficient to generalize Knoblauch's finiteness theorem to the nonorientable case.

8.4. THEOREM (D. Repovš and R. C. Lacher [105]). *For every closed, nonorientable 3-manifold M there exists an integer k such that if $X_1, \ldots, X_{k+1} \subset M$ are pairwise disjoint compact sets and each X_i has a neighborhood $U_i \subset M$ such that $H_1(U_i - X_i; \boldsymbol{Z}_2) \to H_1(M; \boldsymbol{Z}_2)$ is trivial then at least one X_i has a neighborhood in M which embeds in $\boldsymbol{R}^3$.*

8.5. COROLLARY. *Let G be a cell-like decomposition of a closed 3-manifold M. Then all but finitely many elements $g \in G$ are intersections of properly nested decreasing sequences of cubes-with-handles in M.*

Proof of (8.5). Follows by [72] if M is orientable (resp. by (8.4) if it is not) and by [89].

Proof of (8.4). We may assume that the U_i's are pairwise disjoint. A straightforward computation shows that for every $n \geqslant 1$, $\operatorname{im}\big(H_1(\bigcup_{i=1}^{i=n} U_i) \to H_1(M)\big) \cong \bigoplus_{i=1}^{n} \operatorname{im}(H_1(U_i) \to H_1(M))$; hence at least $n - \operatorname{rank} H_1(M)$ among the U_i's are orientable [80] and thus they lift to the orientable 3-manifold double cover $\tilde{M}$ of M as two homeomorphic copies. The conclusion now follows by [72] applied in $\tilde{M}$. For details see [105].

Although Theorem (8.4) is clearly invalid over $\boldsymbol{Z}_p$ (p any odd prime), e.g. take $M = P^2 \times S^1$ and $X_t = P^2 \times \{t\}$ where $P^2 =$ the projective plane, it is true over the integers: by the Universal coefficients theorem, the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1(U_i - X_i; Z)\otimes Z_2 & \xrightarrow{f} & H_1(U_i - X_i; Z_2) & \longrightarrow & \operatorname{Tor}(H_0(U_i - X_i; Z), Z_2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j_*\otimes id} & & \downarrow{\scriptstyle j'_*} & & \downarrow & & \\
0 & \longrightarrow & H_1(M; Z)\otimes Z_2 & \xrightarrow{g} & H_1(M; Z_2) & \longrightarrow & \operatorname{Tor}(H_0(M; Z), Z_2) & \longrightarrow & 0
\end{array}$$

thus if $j_* = 0$ then $j'_* = 0$.

The next step in the proof of (8.2) is the following neighborhood theorem (below) – it describes neighborhoods of peripherally acyclic continua in 3-manifolds (for a study of peripheral acyclicity see [103]). Certain parts of (8.6) were proved earlier by D. R. McMillan, Jr. [91], A. H. Wright [125], and J. L. Bryant and R. C. Lacher [32]. The proof of (8.6) relies heavily on the Haken finiteness theorem [61] and J. W. Milnor's prime decomposition theorem for 3-manifolds [63]. Since it is quite elaborate we refer the interested reader to [101].

8.6. Theorem (D. Repovš and R. C. Lacher [105]). *Let K be a compact connected subset of the interior of a 3-manifold M. Suppose that K doesn't separate its connected neighborhoods and that for every neighborhood $U \subset M$ of K there exists a neighborhood $V \subset U$ of K such that $H_1(V-K; Z_2) \to H_1(U; Z_2)$ is trivial. Then $K = \bigcap_{i=1}^{\infty} N_i$ where each $N_i \subset \operatorname{int} M$ is a compact 3-manifold with boundary, satisfying the following properties: for every i,*

(i) $N_{i+1} \subset \operatorname{int} N_i$;

(ii) *N_i is obtained from a compact 3-manifold Q_i with a 2-sphere boundary, by adding to ∂Q_i a finite number of orientable (solid) 1-handles;*

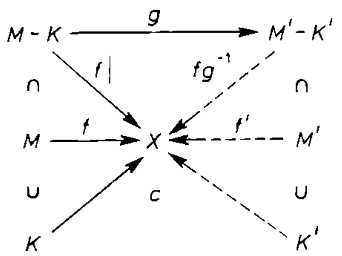
(iii) *$H_1(\partial N_{i+1}; Z_2) \to H_1(N_i; Z_2)$ is trivial; and*

(iv) *there is a homeomorphism $h_i\colon N_i \to N_i$ such that $h_i|\partial N_i = \mathrm{id}$ and $h_i(Q_i^*) = Q_{i+1}$, where $Q_i^* \subset \operatorname{int} Q_i$ is formed by pushing Q_i into $\operatorname{int} Q_i$ along a collar of ∂Q_i.*

Proof of Theorem (8.2). We present an argument whose first part is different from the original one in [105]. Let $A = \{x \in X \mid \check{H}^1(f^{-1}(x)) \neq 0\}$. By Assertion 1 on p. 315 of [32], A is locally finite in X. Since by [90] and [80], every $f^{-1}(x)$, $x \in X - A$, has an orientable neighborhood in M, it follows by Assertion 3 on p. 316 in [32] that $C - A$ is locally finite in $X - A$. It thus remains to show that no limit point of $C - A$ can belong to A. Let $a \in A$ and suppose that for some sequence $\{x_n\} \subset X - A$, $\lim_{n\to\infty} x_n = a$. By Assertion 2 on p. 316 in [32], every $f^{-1}(x)$ is strongly Z_2-acyclic hence by [89] the intersection of a nested sequence of Z_2-homology 3-cells with handles. Thus for each $n \geq 1$ there exists an orientable neighborhood $U_n \subset M$ of $f^{-1}(x_n)$ and a Z_2-homology 3-cell with handles $H_n \subset U_n$ such that $f^{-1}(x_n) \subset \operatorname{int} H_n$. We may also assume that if $i \neq j$ then $U_i \cap U_j = \emptyset$. It is a well-known corollary of the Gruško–Neumann theorem [63] that in a compact 3-manifold there is but a finite number of pairwise disjoint

Z_2-homology 3-cells which fail to be real 3-cells. Therefore, by [91] all but finitely many among $\{f^{-1}(x_n)\}$ are $1-UV$ hence cell-like [77]. Thus $x_n \notin C$ for all but a finite number of indices n. Consequently, the set $C-A$ (hence also the set C) is locally finite in X. (For a proof independent of Assertions 1-3 from [32] see [105].)

It now remains to find a resolution for X. We construct it by improving f over the points of C. We may assume that $C = \{c\}$. It is not difficult to check that $K = f^{-1}(c)$ satisfies all the hypotheses of Theorem (8.6). Hence K can be expressed as $K = \bigcap_{i=1}^{\infty} N_i$ where the N_i's are as described in (8.6). Let $M' = M/Q_1^*$ and let h_i': $N_i \to N_i$ be a homeomorphism such that $h_i'|\partial N_i = \text{id}$ and $h_i'(Q_{i+1}) = Q_{i+1}^*$. Define a homeomorphism h_i^*: $M \to M$ by letting $h_i^* = h_i' h_i$ on N_i and the identity elsewhere. Let g_0: $M \to M'$ be the quotient map. Define inductively $g_i = g_{i-1}(h_i^*)^{-1}$: $M \to M'$, $i \geqslant 1$. Finally, let $K' = \bigcap_{i=0}^{\infty} g_i(N_{i+1})$. It can be shown that K' is cell-like. The proof is then completed by defining a map g: $M-K \to M'$ by letting $g = g_i$ on $M - \text{int}\, N_i$, $i \geqslant 1$, and f': $M' \to X$ by $f' = fg^{-1}$ on $M'-K'$ and $f'(K') = c$.



It is not difficult to verify that f' is cell-like. For details see [105].

After [32] the search for a different resolution theorem narrowed down to the class $\mathscr{C}$ of generalized 3-manifolds with 0-dimensional singular set. The first one to prove a resolution theorem for such spaces was M. G. Brin: in his thesis [23] he showed that, modulo the Poincaré conjecture, every $X \in \mathscr{C}$ such that $S(X) = \{p\}$ admits a resolution, provided $g(X, p) \leqslant 1$. (Note that if $g(X, p) = 0$ then X is immediately a 3-manifold by Theorem (7.5)*.) He soon generalized this result to those $X \in \mathscr{C}$ for which $g(X, x) \leqslant 1$ at all singular points [25]. The philosophy of the attack on the resolution problem in [23] and [25] was predetermined by an important earlier observation of D. R. McMillan, Jr. – see [25] (and also [27]) that resolving an $X \in \mathscr{C}$ is nothing else than embedding the open 3-manifold $M(X)$ into some compact 3-manifold. This follows from a slightly more general statement (8.7) below. We remark that there exist many examples of open n-manifolds ($n \geqslant 3$) which embed in no compact n-manifold ([69], [87], [114]).

8.7. Theorem (M. G. Brin and D. R. McMillan, Jr. [27]). *Let X be a* *compact generalized* 3*-manifold with* $\dim S(X) = 0$. *Then the following statements are equivalent*:

(i) X *admits a resolution*;
(ii) $M(X)$ *embeds in some compact* 3*-manifold*;
(iii) $S(X)$ *has a neighborhood* $U \subset X$ *such that* $U \cap M(X)$ *embeds in* $\mathbf{R}^3$;
(iv) $S(X)$ *has a neighborhood* $U \subset X$ *such that* $U \cap M(X)$ *embeds in some compact* 3*-manifold*.

Proof. (i) $\Rightarrow$ (ii) Follows by Theorem (8.1).

(ii) $\Rightarrow$ (iii) Suppose that $M(X)$ embeds in the compact 3-manifold N. We may assume that $\partial N = \emptyset$. There exists an open neighborhood $U \subset X$ of $S(X)$ such that U is orientable [27]. The assertion now follows by applying [71] in the orientable 3-manifold double covering of N.

(iii) $\Rightarrow$ (iv) Clear.

(iv) $\Rightarrow$ (i) Let N be the compact 3-manifold in which $U \cap M(X)$ embeds via some h: $U \cap M(X) \to N$. We may assume that U is compact and that N is closed. Let f: $N \to U$ be the map (induced by h^{-1}) whose only nondegenerate point-inverses are the components of $N - h(U \cap M(X))$. The assertion now follows by Theorem (8.2).

The idea of the proof in [23] and [25] was therefore to present the manifold set $M(X)$ of an $X \in \mathscr{C}$ as a union $M(X) = \bigcup_{i=1}^{\infty} K_i$ of an increasing, properly nested sequence of compact 3-manifolds with boundary K_i and then use the hypothesis ($g(X, p) = 1$ in [23] and $g(X, x) = 1$ for all $x \in X$ in [25]) to embed $M(X)$ in some compact 3-manifold and then invoke Theorem (8.7).

Together with D. R. McMillan, Jr., Brin improved his resolution theorem – they demonstrated in [27] that, modulo the Poincaré conjecture, every $X \in \mathscr{C}$ whose singular set $S(X)$ has arbitrarily small neighborhoods with torsion free fundamental group, admits a resolution. The "torsion free" hypothesis which replaced the condition on the genera of singularities in [25] was inherited from Brin's version of the Loop theorem for the class $\mathscr{C}$ [26]. The proof in [27] is again (like in [23] and [25]) an embedding procedure for $M(X)$. Since the argument is pretty technical, we refer the interested reader to [27].

Few years later, T. L. Thickstun identified the suspected red herring nature of the "no π_1-torsion" in [27]:

8.8. THEOREM (T. L. Thickstun [115]). *Let* X *be a generalized* 3*-manifold with* $\dim S(X) = 0$. *Then there is a Wilder manifold* Y *and a proper, cell-like map* f: $Y \to X$.

8.9. COROLLARY. *Let* $\mathscr{C}$ *be the class of all compact generalized* 3*-manifolds* X *with* $\dim S(X) = 0$ *and let* $\mathscr{C}_0 \subset \mathscr{C}$ *be the subclass of all* $X \in \mathscr{C}$ *such that* $S(X) = \{\mathrm{pt}\}$ *and* $X \simeq S^3$. *Then the following conjectures are equivalent*:

(i) *The Poincaré conjecture.*

(ii) *Every* $X \in \mathscr{C}$ *admits a resolution.*

(iii) *Every* $X \in \mathscr{C}_0$ *admits a resolution.*

Proof of (8.9). (i) $\Rightarrow$ (ii). Follows by Theorem (8.8).

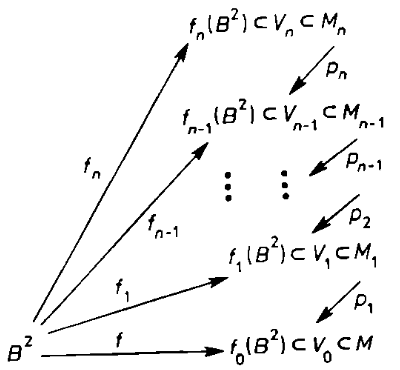
(ii) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (i). Example (7.1).

The key device needed for (8.8) is a version of the Loop theorem for the class $\mathscr{C}$ *without* the "no π_1-hypothesis" (as in [26]). A similar result was proved (using entirely different methods) by A. J. Casson and C. Mc A. Gordon [43].

8.10. THEOREM (T. L. Thickstun [115]). *Let M be an orientable 3-manifold with compact boundary, $F \subset \partial M$ a surface and G a normal subgroup of $\pi_1(F)$. Suppose that M is 1-acyclic at ∞ and that f: $(B^2 - C, \partial B^2) \to (M, F)$ is a proper Dehn map, where $C \subset \operatorname{int} B^2$ is of dimension $\leqslant 0$, and such that $[f|\partial B^2] \notin G$. Then there exists a proper embedding g: $(B^2 - D, \partial B^2) \to (M, F)$, where $D \subset \operatorname{int} B^2$ is of dimension $\leqslant 0$, such that $[g|\partial B^2] \notin G$. Furthermore, if $N \subset M$ is a neighborhood of $S(f)$ then we can choose g, so that $g(B^2 - D) \subset f(B^2 - C) \cup N$.*

The proof of (8.10) is different from the proof of the classical Loop theorem [63]. Recall that a classical proof would use Papakyriakopulos' finite tower of maps:



where V_k is a regular neighborhood of $f_k(B^2)$ in M_k and p_k: $M_k \to V_{k-1}$ a connected double covering such that $p_k f_k = f_{k-1}$. The complexity of the map f_0 is reduced at each level until f_n can be turned into an embedding. Using Dehn cuts this embedding is then transferred from the top to the bottom, one level at a time [63].

M. G. Brin used in [26] another technique: instead of constructing a tower of maps as above he considered a single (the largest) cover M to which f: $B^2 \to M$ still lifted. By the cut-and-paste he would then get an embedding in M, disjoint from its translates, so it would project to M without singularities. In order for these operations to be possible, the covering translations had to be *torsion free*. This is how the condition "no π_1-torsion" came into [27]. (Compare [95] and [86].)

Thickstun introduced a new idea — he still uses the tower construction but he does not remove *all* singularities by the time he reaches the top. After performing Dehn cuts at the top level he gets an embedding there and then he sends it to the bottom *without* stopping at any intermediate level. He does not get an embedding in this way but rather a map, *less* singular than the original.

He then repeats this procedure *transfinitely* many times to eventually produce an embedding into M. All these operations are perfomed on some monotone sequence of large compact pieces filling up B^2-C. So Thickstun is really desingularizing an entire sequence of compact planar maps and then piecing them together in a clever way (essentially using methods of [28]) to get the desired noncompact planar surface in M.

Proof of Theorem (8.8). Let $N \subset X$ be a compact neighborhood of $S(X)$ such that $N^* = \overline{X-N}$ is a compact 3-manifold with orientable boundary. Consider an essential loop γ on ∂N^* such that γ bounds a singular disk in N. (Such γ's exist because X is an ENR; hence, in particular, 1-LC [65].) Using Theorem (8.10), get a proper, embedded planar surface γ^*: $(B^2-D, \partial B^2) \to (N-S(X), \partial N^*)$ where $D \subset \operatorname{int} B^2$ is of dimension $\leqslant 0$. Extending γ^* over D we get a (singular) Dehn disk f: $B^2 \to N$ such that $\overline{S(f)} \subset S(X)$.

The idea is now to thicken (blow up) this disk in order to get a larger generalized 3-manifold Y_1 in which f shall induce a compression on N away from (i.e. missing) the singular set. So consider a regular neighborhood W of $f(B^2-D)$ in the 3-manifold $N-S(X)$. Cutting W out of N and glueing on $\overline{\partial(N-W)}$ the "product" $f(B^2\times(-1, 1))$ in the obvious way yields a new generalized 3-manifold Y_1 and a proper, cell-like map g_1: $Y_1 \to X$ which shrinks out the fibers of $f(D\times(-1, 1))$. Note that in Y_1, N can be compressed along the embedded disk $f(B^2\times\{0\})$, missing the singular set of Y_1.

This procedure reduces the genus of ∂N^* and thus we eventually end up with 2-spheres. Hence $g(Y_n, y) = 0$ for all $y \in Y_n$, for some $n \geqslant 1$. By Corollary (7.6), Y_n is a Wilder manifold. Since the composition $f_n \dots f_1$: $Y_n \to Y$ is cell like [77], the assertion of (8.8) follows. (This proof was also effected by R. J. Daverman in the Spring of 1981 (unpublished).)

9. Generalized 3-manifolds: The DDP

Due to the lack of general position, Cannon's DDP which was so successfully exploited in higher dimensions (as we described in Chapter 6) does not apply in dimensions below 5 (not even $\mathbf{R}^4$ satisfies it). There is presently no appropriate version of the DDP for dimension 4 which would yield the analogue of Edwards' shrinking theorem (6.1) and, consequently, a recognition theorem for 4-manifolds. Following M. H. Freedman's fundamental paper on 4-manifolds [57] there has recently been a breakthrough of activity in this area. Perhaps we may thus soon expect some results concerning manifold characterization in this dimension, too.

In Chapter 8 we have discussed present status of the resolution problem in dimension 3. We have seen that all work has been done mainly on the class $\mathscr{C}$ of all generalized 3-manifolds with at most 0-dimensional singular set. In this chapter we shall prove that for this class there exists an appropriate version of the DDP, called the *map separation property* (MSP) which, modulo the Poincaré conjecture, yields the 3-dimensional analogue of Edwards' Theorem (6.1):

9.1. THEOREM. (D. Repovš and R. C. Lacher [104]), *Let $\mathscr{C}$ be the class of all compact generalized 3-manifolds X with $\dim S(X) \leqslant 0$, and let $\mathscr{C}_0 \subset \mathscr{C}$ be the subclass of all $X \in \mathscr{C}$ such that $S(X) \subset \{pt\}$ and $X \simeq S^3$. Then the following conjectures are equivalent:*

(i) *The Poincaré conjecture.*
(ii) *Every $X \in \mathscr{C}$ with the* MSP *is a 3-manifold.*
(iii) *Every $X \in \mathscr{C}_0$ with the* MSP *is homeomorphic to S^3.*

We begin with some history on this subject. Various DDP's have been used in the past 30 years – notably by R. H. Bing and his school of decompositions of $\boldsymbol{R}^3$. It all began with the celebrated dogbone space $D^3 = \boldsymbol{R}^3/G$ [10] – here G is a closed, 0-dimensional cellular, upper semicontinuous decomposition. Although each nondegenerate element of G is a tame arc, they are tangled in such a clever way that collectively they destroy the manifold-likeness of the quotient D^3. As Bing has pointed out in [13], one of the crucial facts about D^3 is that it fails to possess a "general position" property of $\boldsymbol{R}^3$ (and all 3-manifolds, as we shall see later on) that embedded disks which intersect at their interior points only, can be separated.

Most other versions of the DDP were defined for the *decomposition* G itself (rather than for the *quotient* $\boldsymbol{R}^3/G$), e.g. the DDP's of M. Starbird [113]. Since we are working with generalized 3-manifolds X, we do not *à priori* have any (preferred) decomposition of some 3-manifold, $f\colon M \to X$, playing the role of a resolution for X. In fact, we do not even know in general, if X has *any* resolution to begin with (see Chapter 8). Therefore, for the purpose of the recognition problem we want a DDP which is a property of the *range* M/G, rather than of the decomposition G and assures that M/G be a 3-manifold.

Having been inspired by [13], H. W. Lambert and R. B. Sher introduced in 1966 a version of the DDP which they named the *map separation property*: a space X is said to have the MSP if given any collection of maps $f_1, \ldots, f_k$: $B^2 \to X$ such that for every i, $\overline{N(f_i)} \cap \partial B^2 = \emptyset$ and if $i \neq j$ then $f_i(\partial B^2) \cap f_j(B^2) = \emptyset$, and given a neighborhood $U \subset X$ of $\bigcup_{i=1}^{i=k} f_i(B^2)$ there exist maps $F_1, \ldots, F_k$: $B^2 \to U$ such that for every i, $F_i | \partial B^2 = f_i | \partial B^2$ and if $i \neq j$ then $F_i(B^2) \cap F_j(B^2) = \emptyset$ [81]. They used the MSP to detect shrinkability of point-like, closed 0-dimensional, upper semicontinuous decompositions of S^3. Some 15 years later we generalized their main result to cell-like (closed 0-dimensional) decompositions of *arbitrary* 3-manifolds:

9.2. THEOREM (D. Repovš and R. C. Lacher [104]). *Let G be a cell-like, closed 0-dimensional, upper semicontinuous decomposition of a 3-manifold M. Then M/G is a 3-manifold if and only if M/G has the* MSP.

It is not obvious at all that even $\boldsymbol{R}^3$ should have the MSP because the singular disks $f_i(B^2)$ can be, in general, wildly embedded (in $\boldsymbol{R}^3$). Hence our first task is to verify the forward implication in (9.2). We shall need the following topological version of D. W. Henderson's refinement of the Dehn lemma [64]:

9.3. THEOREM (D. Repovš and R. C. Lacher [104]). *Let f: $B^2 \to M$ be a continuous mapping of the standard 2-cell into a 3-manifold (possibly with boundary) and let $U \subset M$ be a neighborhood of $\overline{S(f)}$. Suppose that $\overline{N(f)} \subset \operatorname{int} B^2$. Then there is an embedding F: $B^2 \to M$ such that*:

(i) $F(B^2) - U = f(B^2) - U$; *and*

(ii) $F | \partial B^2 = f | \partial B^2$.

Proof. Note first, that by attaching a collar to the boundary of M we may assume that $f(B^2)$ lies in $\operatorname{int} M$. Put $\overline{N(f)}$ inside pairwise disjoint PL disks with holes $C_1, \ldots, C_m \subset f^{-1}(U)$. Let $C = \bigcup_{i=1}^{i=m} C_i$. Assume that on some neighborhood of ∂C, f is a locally PL embedding. Consider the surface $H = f(B^2 - \operatorname{int} C)$ and use [9] to make it PL. Apply [126] to make the surface $L = f(C)$ PL. Then $H \cup L$ is a PL singular disk with singularities far away from the boundary so we can invoke [64] to replace it by an embedded (PL) disk $T \subset M$. Finally, substitute the portions of T which stick out of U by the corresponding pieces of H.

In general, the simple closed curves of $f(\partial C)$ are going to be wildly embedded in M so additional care must be taken to improve f near ∂C. This is achieved by using several "concentric" families of pairwise disjoint PL disks with holes rather than just one (C above). Details are technical and we refer the interested reader to [104].

9.4. THEOREM (D. Repovš and R. C. Lacher [104]). *Let* $f_1, \ldots, f_k$: B^2

$\to M$ *be continuous mappings of the standard 2-cell into a 3-manifold (possibly with boundary) such that for every* i, $\overline{N(f_i)} \cap \partial B^2 = \emptyset$ *and if* $i \neq j$ *then* $f_i(\partial B^2) \cap f_j(B^2) = \emptyset$. *Then for every neighborhood* $U \subset M$ *of* $\bigcup_{i=1}^{i=k} f_i(B^2)$ *there exist embeddings* $F_1, \ldots, F_k \colon B^2 \to U$ *such that for every* i,

(i) $F_i | \operatorname{int} B^2$ *is locally* PL;

(ii) $F_i | \partial B^2 = f_i | \partial B^2$; *and*

(iii) *if* $i \neq j$ *then* $F_i(B^2) \cap F_j(B^2) = \emptyset$.

9.5. COROLLARY. *Every 3-manifold (possibly with boundary) has the* MSP.

Proof of Theorem (9.4). As before, assume that for every i, $f_i(B^2) \subset \operatorname{int} M$. The proof is by induction on k. The assertion for $k = 1$ follows by Theorem (9.3) and [9]. The inductive step is a combination of [9], general position arguments and some cut-and-paste. For details see [104].

The next result is a generalization of Corollary (9.5). Note that by Theorem (7.5)*, the only generalized 3-manifolds in (9.6) not included in (9.5) are the Wilder manifolds.

9.6. THEOREM (D. Repovš and R. C. Lacher [104]). *Let* X *be a generalized 3-manifold with* $\dim S(X) \leqslant 0$ *and suppose that for every* $x \in X$, $g(X, x) = 0$. *Then* X *has the* MSP.

Proof. Assume that $\operatorname{Fr} U \cap S(X) = \emptyset$ and also that each $f_i(\partial B^2)$ misses $S(X)$. Cover $S(X) \cap U$ by pairwise disjoint compact neighborhoods $N_1, \ldots, N_m \subset U$ such that each $\operatorname{Fr} N_i$ is a PL embedded 2-sphere in $M(X) - \bigcup_{j=1}^{j=k} f_j(\partial B^2)$. The idea is now to somehow "cut" the $f_i(B^2)$'s off at the boundary of $\bigcup_{j=1}^{j=m} N_j$ thus transferring the problem to the 3-manifold $M(X)$ where Corollary (9.5) would apply. Details are technical mainly because the singular disks $f_i(B^2)$ may be wildly embedded and so the intersections with $\bigcup_{j=1}^{j=m} \operatorname{Fr} N_j$ may be quite messy. We refer to [104] for a complete proof and also for the argument in the general case.

Proof of Theorem (9.2). The forward implication follows by Corollary (9.5). The argument for the other implication can be split into a sequence of assertions.

ASSERTION 1. *If every* $g \in G$ *has a neighborhood in* M *embeddable in* $\boldsymbol{R}^3$ *then* G *is shrinkable (hence* $M/G \approx M$).

Proof. By [119] it suffices to show that G is weakly shrinkable (see Chapter 2 for the definition). Choose an $\varepsilon > 0$ and a neighborhood $U \subset M$ of $\overline{N_G}$. By [89] there are pairwise disjoint cubes with handles $F_1, \ldots, F_k \subset U$ such that $\overline{N_G} \subset \bigcup_{i=1}^{i=k} \operatorname{int} F_i$. Let $W_1, \ldots, W_k \subset U$ be pairwise disjoint open neighborhoods of $F_1, \ldots, F_k$, respectively. We may restrict our attention to $F_1 \subset W_1$ only. Let $C_1 = \overline{N_G} \cap F_1$. As far as F_1 is concerned it suffices to find a homeomorphism h_1: $M \to M$ which will shrink C_1 and will stay the identity off W_1. We shall get such an h_1 as the composition of two homeomorphisms f_1, t_1: $M \to M$ which we now proceed to describe.

The first one, f_1: $M \to M$ shrinks F_1 towards its 1-dimensional spine so that $f_1(F_1)$ can be split up into adjacent 3-cell "chambers" of size less than $\varepsilon/2$. Pull this chamber partition up into F_1. It is clear that if $g \in G$ lies in at most two adjacent chambers, it will get shrunk under f_1 to a size less than ε. So it remains to make each $g \in G$ hit at most one of the walls separating the chambers.

Observe that the separating walls go down to M/G as singular disks whose nondegeneracy set stays far away from ∂B^2, and if they intersect in M/G they do so at their interior points. We can therefore use the MSP to separate them. Lift the new disks to M using [77] and apply the Dehn lemma to get the new walls. Pick now any homeomorphism t_1: $M \to M$ which maps the new walls on the corresponding old ones and rests off F_1. Finally, let $h_1 = f_1 t_1$.

The only reason why this argument doesn't work in the general case is that an arbitrary cell-like, closed 0-dimensional, upper semicontinuous decomposition of M is definable by homotopy (hence possibly fake) cubes with handles [89]. Consequently, some of the partitioning chambers may be fake 3-cells so no homeomorphism such as our t_1 above can be produced. However, as our finiteness Theorem (8.6) suggests, there cannot be too many bad nondegenerate elements in G after all: if $G_0 = \{g \in G |$ *g has no neighborhood in M embeddable in $\mathbf{R}^3$*$\}$ then $\pi(G_0)$ must be locally finite in M/G, where π: $M \to M/G$ is the decomposition map.

ASSERTION 2. *For every $g \in G$ and every neighborhood $U \subset M$ of g there is a homotopy 3-cell $H \subset U$ such that $g \subset \operatorname{int} H$.*

Proof. By [89] there is a homotopy cube with handles $H \subset U$ such that $g \subset \operatorname{int} H$. We may also assume that on some neighborhood of ∂H in M, π: $M \to M/G$ is an embedding (recall that G is closed 0-dimensional). The idea of the proof is to cut the handles of H along pairwise disjoint compressing disks (in H) which miss g. We detect such disks by passing to M/G and using the MSP. The point is that if D_1 and D_2 are disjoint compressing disks in H then $\pi(D_1)$ and $\pi(D_2)$ satisfy the requirements for the MSP in M/G. Thus $\pi(D_1)$ and $\pi(D_2)$ can be made disjoint and we can lift them back into M, using [77] and the Dehn lemma. Clearly, at least one of the new disks (in H) will now miss g and so we can compress H, avoiding g.

Consider now the quotient $M_0 = M/G_0$ and let π_0: $M \to M_0$ be the corresponding decomposition map. Since the elements of G_0 are cell-like, M_0 is a generalized 3-manifold. Next, $S(M_0) \subset \pi_0(G_0)$, where $S(M_0)$ denotes the singular set of M_0. Finally, by Theorem (7.8), M_0 satisfies the Kneser finiteness since it has a resolution.

ASSERTION 3. *For every* $p \in M_0$, $g(M_0, p) = 0$.

Proof. The assertion is clear for $p \in M_0 - \pi_0(G_0)$. So suppose that $p \in \pi_0(G_0)$. Given a neighborhood $U \subset M_0$ of p, we may assume that $U \cap \pi_0(G_0) = \{p\}$ (since $\pi(G_0)$ is locally finite in M/G). By Assertion 2 there is a homotopy 3-cell $H \subset \pi_0^{-1}(U)$ such that $\pi_0^{-1}(p) \subset \operatorname{int} H$ and $\partial H \subset M - \bigcup \{g \in G_0\}$. Therefore $\pi_0(H)$ is the desired neighborhood of p in M_0.

By Assertion 3 and Theorem (7.5)*, M_0 is a 3-manifold since $\dim S(M_0) \leqslant \dim \pi_0(G_0) \leqslant 0$. Let $G_1 = (G - G_0) \cup \pi_0(G_0)$ be the induced decomposition of M_0. Then G_1 is upper semicontinuous, closed 0-dimensional and cellular, by Assertion 2. Clearly, $M_0/G_1 = (M/G_0)/G_1 = M/G$, so M_0/G_1 has the MSP and therefore G_1 is shrinkable, by Assertion 1. This completes the proof of Theorem (9.2). For details see [104].

Proof of Theorem (9.1). The implication (i) $\Rightarrow$ (ii) follows by Theorems (8.8) and (9.2) while (ii) $\Rightarrow$ (iii) is obvious. To see that (iii) $\Rightarrow$ (i) suppose (i) is false and consider Example (7.1). Apply (9.6) to conclude that (iii) can't be true, either.

10. Epilogue

We now fairly well understand the class of generalized 3-manifolds X with $\dim S(X) \leqslant 0$ but we know almost nothing about the other generalized manifolds in this dimension. The following open problems are among the most important for the recognition of 3-manifolds:

10.1. PROBLEM. *Let X be a generalized 3-manifold with* $\dim S(X) \geqslant 1$. *Modulo the Poincaré conjecture, does there exist a 3-manifold M and a monotone, closed map $f\colon M \to X$ such that $\check{H}^1(f^{-1}(x); \mathbf{Z}_2) \cong 0$ for every $x \in X - Z$, where $Z \subset X$ is some (dense) subset of dimension $\leqslant 0$?*

Should (10.1) prove to be too hard one could try resolving only the generalized 3-manifolds with the MSP:

10.2. PROBLEM. *Let X be a generalized 3-manifold satisfying the Kneser finiteness. Suppose that $S(X) \subset Z$ for some graph $Z \subset X$ and that X has the* MSP. *Does then X have a resolution? Is X a 3-manifold?*

There is another general position property in dimension 3, called *Dehn's lemma property* (DLP) [81]: a space X is said to have the DLP if for every map $f\colon B^2 \to X$ such that $\overline{N(f)} \cap \partial B^2 = \emptyset$ and for every neighborhood

$U \subset X$ of $\overline{S(f)}$ there exists an embedding $F\colon B^2 \to f(B^2) \cup U$ such that $F(\partial B^2) = f(\partial B^2)$. It follows by Theorem (9.3) that every 3-manifold (possibly with boundary) has the DLP. As it was demonstrated in [104] all results from Chapter 9 involving the MSP (either as a hypothesis or as a conclusion) remain valid if the MSP is substituted by the DLP. What remains open is the following question:

10.3. Problem. *Let X be a generalized 3-manifold. If X has the* MSP *does it then also have the* DLP, *and vice versa?*

Another question is a possible relationship between M. Starbird's DDP's [113] and the MSP (note that they are equivalent if G is cell-like, closed 0-dimensional, and every $g \in G$ possesses a neighborhood embeddable in $\boldsymbol{R}^3$ – see Theorem (3.1) of [113] and our Theorem (9.2)):

10.4. Problem. *Let G be a cell-like, upper semicontinuous decomposition of a 3-manifold M and suppose that G has one of Starbird's* DDP's. *Does M/G then have the* MSP *(and vice versa)?*

Finally, we have already remarked that at present there is no known analogue of the DDP in dimension 4. Therefore the following problem is the central issue in this dimension:

10.5. Problem. *Find a* DDP *for 4-manifolds. Such a property should satisfy the following "minimal" criteria:*

(i) *every 4-manifold should possess this* DDP;

(ii) *it should be a "homotopical" or a "map approximating" property;*

(iii) *it should be relatively easy to verify in practice;*

(iv) *given a cell-like, upper semicontinuous decomposition G of a 4-manifold M such that the quotient space M/G has this* DDP, *G should turn out to be shrinkable (or, equivalently, the decomposition map $q\colon M \to M/G$ should be approximable by homeomorphisms).*

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