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Fixed-point theorems for multi-valued functions and their applications to functional equations

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CONTENTS

Introduction

In this paper we present some fixed point theorems for multi-valued functions (m.v. functions for short), generalizing well-known results (Chapter 2).

Next, in Chapter 3, we consider continuous solutions of equation (3.1). First we characterize the continuous solutions of this equation, and then we give some theorems on extending continuous solutions.

Chapter 4 contains theorems on the existence of continuous solutions when this equation is of order n, i.e. if the set of indices is finite.

The last chapter shows how we can apply the results of Chapters 3 and 4 to a double functional inequality (5.1).

In this paper we denote all m.v. function by capital letters.

1. Some properties of multi-valued functions

In this part we give some definitions and notions which are associated with multi-valued functions (m.v. functions for short), i.e. functions of the type $F\colon X \to 2^Y$, where X and Y are arbitrary sets and for every $x \in X$ we have $F(x) \subset Y$.

For a given m.v. function $F\colon X \to 2^Y$ and sets $A \subset X$ or $B \subset Y$ we can define the sets

$$F(A) := \{y \in Y\colon\ y \in F(x) \text{ for a certain } x \in A\},\ (^1)$$

$$F_+^{-1}(B) := \{x \in X\colon\ F(x) \subset B\}, \qquad F_-^{-1}(B) := \{x \in X\colon\ F(x) \cap B \neq \emptyset\},$$

which we shall call respectively, the image of set A, the *upper inverse image of* B and the *lower inwerse image of* B under the m.v. function F.

DEFINITION 1.1. The m.v. function $F\colon X \to 2^Y$, where X and Y are topological spaces, is said to be *lower semicontinuous* (resp. *upper semicontinuous*) if the set $F_-^{-1}(U)$ (resp. $F_+^{-1}(U)$) is open in X for every open set $U \subset Y$ (see [15]).

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(1) We use the symbol ":=" to indicate definitions.

Let Y be an arbitrary topological space. We will define the following classes of subsets of Y:

$$\mathrm{Cl}(Y) := \{A \subset Y\colon A \text{ is closed and nonempty}\};$$
$$\mathrm{C}(Y) := \{A \subset Y\colon A \text{ is compact and nonempty}\}.$$

If, moreover, Y is a metric space, then

$$\mathrm{CB}(Y) := \{A \subset Y\colon A \text{ is bounded, closed and nonempty}\};$$

and if Y is a subset of a linear space over the field of real or complex numbers, then

$$\mathrm{CC}(Y) := \{A \subset Y\colon A \text{ is nonempty, closed in } Y \text{ and convex}\};$$
$$\mathrm{CCB}(Y) := \{A \in \mathrm{CC}(Y)\colon A \text{ is bounded}\}.$$

In the metric space (Y, d), for $A \in \mathrm{Cl}(Y)$ and $r > 0$, let

$$K(A, r) := \{y \in Y\colon d(y, A) < r\},$$

where $d(y, A) := \inf\{d(y, x)\colon x \in A\}$.

We will call $K(A, r)$ the *open-ball* with centre A and radius r.

In the space $\mathrm{Cl}(Y)$ (consisting of sets) we should like to define a generalized metric, but for this purpose we shall need the notion of a generalized metric space.

DEFINITION 1.2 ([8]). The pair (X, d) will be called a *generalized metric space* if X is an arbitrary nonempty set and a function $d\colon X \times X \to [0, \infty]$ fulfils all the standard conditions for a metric.

In this space, the generalized metric d is allowed to take on $+\infty$ as well. For every set $X \neq \emptyset$, we can define a discrete generalized metric as follows: $d(x, y) = 0$ if $x = y$ and $d(x, y) = +\infty$ if $x \neq y$; and in any generalized metric space (X, d) we can define a metric d' topologically equivalent to d by the formula $d' = \min\{1, d\}$. In a generalized metric space, just as in a metric space, we can define open and closed balls, convergence of sequences, completeness of the space, etc.

C. K. Jung gave in [11] the following characterization of the generalized metric space by metric spaces. Let (X_s, d_s), $s \in S$, be a nonempty family of disjoint metric spaces. Then the set $X = \bigcup_{s \in S} X_s$ with the function d defined by

$$d(x, y) = \begin{cases} d_s(x, y) & \text{if there exists an } s \in S \text{ such that } x, y \in X_s, \\ +\infty & \text{otherwise,} \end{cases}$$

is a generalized metric space, but the converse is also true.

THEOREM 1.1 ([11]). *Let (X, d) be a generalized metric space. Then the relation ϱ defined as*

$$x\varrho y \Leftrightarrow d(x, y) < \infty \quad \text{for} \quad x, y \in X, \tag{1.1}$$

is an equivalence relation and if $\{X_s\colon s \in S\}$ are the equivalence classes under ϱ, then $d(x, y) = +\infty$ whenever $x \in X_s$, $y \in X_t$, $s \neq t$. Also, if we let $d_s := d|_{X_s \times X_s}$, then (X_s, d_s) is a metric space (for each $s \in S$).

DEFINITION 1.3. The partition of a generalized metric space (X, d) into a family of disjoint metric spaces (X_s, d_s), $s \in S$, constructed in Theorem 1.1, will be called the *canonical partition* of the space (X, d) (see [11]).

Now we can define the generalized Hausdorff metric D in the set $\mathrm{Cl}(Y)$, even if (Y, d) is only a generalized metric space, by

$$D(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\} \quad \text{for} \quad A, B \in \mathrm{Cl}(Y) \tag{1.2}$$

(see also [8]).

The generalized metric space $(\mathrm{Cl}(Y), D)$ has by Theorem 1.1 a canonical partition.

If (Y, d) is a metric space, then it follows from definition (1.1) of ϱ that one element of the canonical partition of $(\mathrm{Cl}(Y), D)$ is the metric space $(\mathrm{CB}(Y), D)$.

The metric space $(\mathrm{CB}(Y), D)$ has a whole series of properties: it is a complete metric space (resp. compact) if so is (Y, d), but need not be separable when Y is separable. Moreover, two equivalent metrics in Y do not necessarily induce equivalent metrics in $\mathrm{CB}(Y)$ (see [15], [16] and [9]).

The problem of describing the canonical partition of the space $(\mathrm{Cl}(Y), D)$ is difficult, even when $Y = \boldsymbol{R}$ ($\boldsymbol{R} = \{\text{real numbers}\}$) and $d(x, y) = |x - y|$. We can do it for the subspace $(\mathrm{CC}(\boldsymbol{R}), D)$ of the space $(\mathrm{Cl}(\boldsymbol{R}), D)$. This space has an application in the theory of functional inequalities and we shall need its canonical partition in Chapter V. The form of the canonical partition of $(\mathrm{CC}(\boldsymbol{R}), D)$ is given by

LEMMA 1.1. *Consider $(\mathrm{CC}(\boldsymbol{R}), D)$ as a generalized metric space, where D is defined by (1.2) from the metric (on $\boldsymbol{R}$) $d(x, y) = |x - y|$. Then the sets $X_i \subset \mathrm{CC}(\boldsymbol{R})$, $i = 1, 2, 3, 4$, defined as*

$$\begin{aligned} X_1 &:= \{[a, b]\colon a, b \in \boldsymbol{R}\}, & X_2 &:= \{[a, \infty)\colon a \in \boldsymbol{R}\}, \\ X_3 &:= \{(-\infty, a]\colon a \in \boldsymbol{R}\}, & X_4 &:= \{\boldsymbol{R}\} \end{aligned} \tag{1.3}$$

form the canonical partition of this space.

Proof. The form of the canonical partition follows by definition (1.1) from the fact that the elements of $\mathrm{CC}(\boldsymbol{R})$ are closed intervals for which the

generalized metric D is given by

$$
(1.4)\quad
\begin{aligned}
&D([a, b], [c, d]) = \max\{|a-c|, |b-d|\} \quad \text{for } a,b,c,d \in \boldsymbol{R};\\
&D([a, \infty), [b, \infty)) = |a-b| \quad \text{for } a,b \in \boldsymbol{R};\\
&D((-\infty, a], (-\infty, b]) = |a-b| \quad \text{for } a,b \in \boldsymbol{R};\\
&D(A, B) = +\infty \quad \text{for } A \in X_i,\ B \in X_j,\ i \neq j,\ i,j = 1, 2, 3, 4,
\end{aligned}
$$

which concludes the proof.

Remark 1.1. In the same way we can find the canonical partition of the space $(\mathrm{CC}(Y), D)$, where Y is a nonempty, closed and convex subset of $\boldsymbol{R}$.

Making use of form (1.3) of the canonical partition of $(\mathrm{CC}(\boldsymbol{R}), D)$ or in general, from its form for $(\mathrm{CC}(Y), D)$, where Y is a nonempty and closed interval in $\boldsymbol{R}$, we get the next lemma, characterizing the m.v. functions $H\colon X \to \mathrm{CC}(Y)$, which are lower semicontinuous.

LEMMA 1.2. *Let X be an arbitrary set and let $Y \subset \boldsymbol{R}$ be a closed and nonempty interval.*

(a) *Then an arbitrary m.v. function $H\colon X \to \mathrm{CC}(Y)$ may be written in the form*

$$H(x) = [h_1(x), h_2(x)] \cap Y \quad \textit{for } x \in X,$$

where h_1 and h_2 are functions such that

$$h_1\colon X \to \boldsymbol{R} \cup \{+\infty\}, \quad h_2\colon X \to \boldsymbol{R} \cup \{-\infty\}, \quad h_1(x) \leqslant h_2(x) \quad \textit{for } x \in X.$$

(b) *Moreover, if X a topological space, then H is a lower semicontinuous m.v. function if and only if h_1 is upper semicontinuous and h_2 is lower semicontinuous.*

Proof. In part (a) we define the functions h_1 and h_2 as follows:

$$(1.5)\qquad h_1(x) := \inf H(x), \quad h_2(x) := \sup H(x) \quad \text{for } x \in X;$$

we see that these functions satisfy all the required conditions. In particular, the formula for the m.v. function H follows from the fact that its values are intervals. Part (b) is an elementary consequence of Definition 1.1 and part (a) (cf. [20], Example 1.2*).

Now we prove some lemmas about the properties of the generalized metric D and of m.v. functions continuous with respect to D.

Directly from (1.2) we obtain

LEMMA 1.3 (see also [22], Remark 1). *Let the m.v. function $F\colon Y \to \mathrm{Cl}(Y)$ be defined on a generalized metric space (Y, d). Then for every $\varepsilon > 0$ and for every $y_1, y_2 \in Y$, $z_1 \in F(y_1)$ there exists a $z_2 \in F(y_2)$ such that*

$$d(z_1, z_2) \leqslant D(F(y_1), F(y_2)) + \varepsilon,$$

where D is defined by (1.2).

LEMMA 1.4. *Let X be a topological space and (Y, d) an arbitrary metric space. Then any m.v. function $F\colon X \to \mathrm{Cl}(Y)$, continuous with respect to the generalized Hausdorff metric D in $\mathrm{Cl}(Y)$, is lower semicontinuous.*

Proof. Let $U \subset Y$ be an arbitrary open set and let $x_0 \in F_-^{-1}(U) = \{x \in X\colon F(x) \cap U \neq \emptyset\}$ be an arbitrary point. Then there exists a $y_0 \in F(x_0) \cap U$ with which we define a positive number

$$r := \begin{cases} d(y_0, Y \backslash U) & \text{if } U \neq Y, \\ 1 & \text{if } U = Y. \end{cases}$$

Then by the continuity of F the set

$$V := \{x \in X\colon D(F(x_0), F(x)) < \tfrac{1}{2}r\}$$

is an open neighbourhood of x_0, and by Lemma 1.3 we obtain $V \subset F_-^{-1}(U)$, which shows that the set $F_-^{-1}(U)$ is open in X.

The converse of Lemma 1.4 is false, e.g.:

EXAMPLE 1.1. Let $X = Y = [0, 1]$, $d(x, y) = |x - y|$ and let $F\colon X \to \mathrm{Cl}(Y)$ be given by

$$F(x) = \begin{cases} [0,1] & \text{for } x \in (0,1], \\ \{0\} & \text{for } x = 0. \end{cases}$$

The m.v. function F is lower semicontinuous by Lemma 1.2, but is not continuous with respect to the Hausdorff metric D, because $D(F(0), F(x)) = 1$ for $x \in (0, 1]$.

A statement analogous to Lemma 1.4 for upper semicontinuous m.v. functions is false, i.e. there exists an m.v. function $F\colon X \to \mathrm{Cl}(Y)$ which is continuous with respect to D but is not upper semicontinuous. This theorem will be true if F takes compact values, e.g. $F(x) \in \mathrm{C}(Y)$ for $x \in X$ (see [13], Proposition 1 and Example 1).

The additional assumption of the continuity of the m.v. function gives

LEMMA 1.5. *If X is a compact topological space, and (Y, d) is a metric space, then for every continuous m.v. function $F\colon X \to \mathrm{CB}(Y)$ the set $F(X) \subset Y$ is bounded.*

Proof. First we define the family of sets $\{U_x\colon x \in X\}$ as follows:

$$U_x := \{z \in X\colon D(F(x), F(z)) < 1\} \quad \text{for } x \in X.$$

By the continuity of F, the sets U_x, $x \in X$ are open and this family covers the compact space X. Hence there exist $x_1, \ldots, x_k \in X$ such that $X = \bigcup_{i=1}^{k} U_{x_i}$. By the definition of U_x, $x \in X$, and of the generalized metric D, this yelds

$$F(z) \subset \bigcup_{i=1}^{k} K(F(x_i), 1) \quad \text{for } z \in X,$$

where $K(F(x_i), 1)$, $i = 1, \ldots, k$, are open balls with centre at $F(x_i)$. These balls are bounded, and so we have our assertion.

Similarly, an upper semicontinuous m.v. function with values in $\mathrm{CB}(Y)$ is bounded on compact sets:

LEMMA 1.6. *Let X be a compact space and (Y, d) a metric space; then for every upper semicontinuous m.v. function $F\colon X \to \mathrm{CB}(Y)$ the set $F(X) \subset Y$ is bounded.*

Proof. The family of open subsets

$$U_x := \{z \in X\colon F(z) \subset K(F(x), 1)\} \quad \text{for } x \in X,$$

is by the upper semicontinuity of F, an open cover of X and hence there exist $x_1, \ldots, x_k \in X$ such that $X = \bigcup_{i=1}^{k} U_{x_i}$; thus $F(X) \subset \bigcup_{i=1}^{k} K(F(x_i), 1)$, which gives the boundedness of F.

In further chapters we will need a certain kind of metric spaces, known as *metricaly convex spaces.*

DEFINITION 1.4. The metric space (Y, d) will be said to be *metrically convex* if for every distinct $y_1, y_2 \in Y$ there exist a $y \in Y$ such that

$$d(y_1, y_2) = d(y_1, y) + d(y, y_2) \quad \text{and} \quad y_1 \neq y \neq y_2.$$

For example every normalized space (or any of its convex subsets) is metrically convex. The next lemma gives an important property of such spaces.

LEMMA 1.7 (Menger [6], cf. [12] for a short proof). *If a complete metric space (Y, d) is metrically convex, then for every $y_1, y_2 \in Y$, $y_1 \neq y_2$ and for every $\lambda \in (0, 1)$ there exists a $y \in Y$ such that*

$$d(y, y_1) = \lambda \cdot d(y_1, y_2) \quad \textit{and} \quad d(y, y_2) = (1-\lambda) \cdot d(y_1, y_2).$$

To finish this chapter we define *selections* and *continuous selections* from a m.v. function and give a sufficient condition for the existence of a continuous selection.

DEFINITION 1.5. Let $F\colon X \to 2^Y$ be an arbitrary m.v. function. A function $f\colon X \to Y$ will be called a *selection* from the m.v. function F if $f(x) \in F(x)$ for $x \in X$. Any continuous function $f\colon X \to Y$ which is a selection from F will be called (briefly) a *continuous selection.*

Now we can give a theorem which guarantees the existence of continuous selections and the possibility of their extension.

This theorem is a special case of E. Michael's theorem 1.2 from [21] (see also [20]).

THEOREM 1.2. *If X is a paracompact Hausdorff topological space and Y is a closed and complete metrizable subset of a complete locally convex space over the fields of real or complex numbers, then*

(a) *any lower semicontinuous m.v. function* $F\colon X \to \mathrm{CC}(Y)$ *admits a continuous selection.*

(b) *Moreover, if* $F|_A$ *is the restriction of a lower semicontinuous m.v. function* $F\colon X \to \mathrm{CC}(Y)$ *to a closed subset* $A \subset X$ *and* $f\colon A \to Y$ *is a continuous selection from* $F|_A$ *defined on* A, *then* f *can always be extended to a continuous selection of* F *defined on the whole set* X.

2. Fixed-point theorems for multi-valued functions

In this chapter we consider some fixed-point theorems for m.v. functions. First we generalize the theorem of Covitz–Nadler ([8], Corollary 1) for m.v. functions with closed values, and then we give a generalization of the theorems of C. S. Wong ([28], Theorem 1) and J. Matkowski ([17], Theorem 1.2) to m.v. functions with compact values. We also generalize to these m.v. functions Theorem 2 from [18], which has been proved for single-valued functions only.

These theorems will next be applied in Chapter 4 to the proof of the existence of continuous solutions of functional equation with m.v. functions.

1. Now we shall consider the m.v. functions $F\colon Y \to \mathrm{Cl}(Y)$, where (Y,d) and $(\mathrm{Cl}(Y),D)$ are generalized metric spaces with a generalized metric D defined by (1.2). For these m.v. functions we define the sequence of iterates at a point and the fixed point.

DEFINITION 2.1. The sequence $\{y_k\}_{k=1}^{\infty}$, $y_k \in Y$, for $k = 1,\ldots$, is called a *sequence of iterates of* F *at* $y_0 \in Y$ iff

$$y_k \in F(y_{k-1}) \quad \text{for } k = 1,2,\ldots$$

A point $y \in Y$ is a fixed point of F iff $y \in F(y)$.

For a m.v. function F we assume that

(2.1) there exists a function $\psi\colon [0,\infty) \to [0,\infty)$ such that

$$D\big(F(x),F(y)\big) \leqslant \psi\big(d(x,y)\big) \quad \text{for } d(x,y) < \infty,\ x,y \in Y.$$

If F fulfils (2.1) with the function ψ, then it is said to be a *ψ-contraction.*

In the sequel, by ψ^k we denote the k-th iterates of ψ.

We have the following

THEOREM 2.1. *Let* $F\colon X \to \mathrm{Cl}(Y)$ *be an m.v. function mapping a complete generalized metric space* (Y,d) *to the space* $(\mathrm{Cl}(Y),D)$ *and fulfilling condition* (2.1) *with some strictly increasing function* ψ *for which*

$$\sum_{k=1}^{\infty} \psi^k(t) < \infty \quad \textit{for } t > 0.$$

Then for every fixed $y_0 \in Y$ *one of the following two possibilities occurs:*

(a) *each sequence of iterates* $\{y_k\}_{k=1}^{\infty}$ *of* F *at* y_0 *satisfies*

$$d(y_k, y_{k-1}) = \infty \quad \text{for } k = 1, 2, \ldots,$$

or

(b) *there exists a sequence of iterates of* F *at* y_0 *which converges to some fixed point of* F.

Proof. If condition (a) does not hold, then there exists a finite sequence $y_1, \ldots, y_N$ such that

$$y_i \in F(y_{i-1}) \quad \text{for } i = 1, \ldots, N \text{ and } d(y_{N-1}, y_N) < \infty.$$

Put $t_0 = d(y_{N-1}, y_N) + 1$; then it follows from the strict monotonicity of ψ that $\psi(d(y_{N-1}, y_N)) < \psi(t_0)$. Hence, by Lemma 1.3, there exists a $y_{N+1} \in F(y_N)$ such that $d(y_N, y_{N+1}) < \psi(t_0)$.

Suppose that there exist $y_1, \ldots, y_N, \ldots, y_{N+k} \in Y$ for $k \geqslant 1$ such that

$$y_i \in F(y_{i-1}) \quad \text{for } i = 1, \ldots, N+k, \tag{2.2}$$

and

$$d(y_{N+i-1}, y_{N+i}) < \psi^i(t_0) \quad \text{for } i = 0, \ldots, k \tag{2.3}$$

hold; then, making use of Lemma 1.3, conditions (2.1), (2.2) and (2.3), we can find a $y_{N+k+1} \in F(y_{N+k})$ such that

$$d(y_{N+k+1}, y_{N+k}) < \psi^{k+1}(t_0).$$

So, by induction, we define a sequence of iterates $\{y_k\}_{k=1}^{\infty}$ of F at $y_0 \in Y$ which is a Cauchy sequence in Y, since the series $\sum_{k=1}^{\infty} \psi^k(t_0)$ converges. The limit of this sequence must be a fixed point of F, as easily follows from Lemma 1.3 the continuity of F and the fact that F's values are closed sets (cf. the proof of Theorem 1 from [8]).

Remark 2.1. Theorem 2.1 generalizes the result of Covitz–Nadler, which assumed that the m.v. function F was a ψ-contraction with the function $\psi(t) = s \cdot t$, where $s \in (0, 1)$ (see [8], Corollary 1).

Theorem 2.1 then implies

THEOREM 2.2. *If* (Y, d) *is a complete metric space, and* $F\colon Y \to \mathrm{Cl}(Y)$ *is an m.v. function which fulfils* (2.1) *with some strictly increasing function* ψ *such that* $\lim_{k\to\infty} \psi^k(t) = 0$ *for every* $t > 0$, *then*

(a) *for every* $y_0 \in Y$ *and for every fixed point* $y \in Y$ *of* F *there exists a sequence of iterates of* F *at* y_0 *which converges to* y,

(b) *moreover, if*

$$\sum_{k=1}^{\infty} \psi^k(t) < \infty \quad \textit{for } t > 0,$$

then the set of fixed points of F is nonempty.

Proof. Let $y \in F(y)$ be an arbitrary fixed point (if such a point exists in this case, see Remark 2.3), and let $y_0 \in Y$ be an arbitrary point. Put $t_0 = d(y, y_0) + 1$. Then, by Lemma 1.3, the monotonicity of ψ, and (2.1), there exists a $y_1 \in F(y_0)$ such that $d(y, y_1) < \psi(t_0)$.

In the same way we can construct by induction a whole sequence of iterates at y_0 for which

$$d(y, y_k) < \psi^k(t_0) \quad \text{for } k = 0, 1, \ldots$$

Hence, from $\lim_{k \to \infty} \psi^k(t) = 0$ we get the first part of the theorem. The existence of a fixed point of F in part (b) follows from Theorem 2.1, because (Y, d) is a metric space, and then just case (b) of Theorem 2.1 can occur.

The assumptions of Theorem 2.2 can be weakened if the metric space (Y, d) is metrically convex. For this purpose we need a theorem which gives the equivalence of some other contraction conditions found in fixed-point theorems (see [18], [19], [7], and [23]), namely

THEOREM 2.3 (see [18], Theorem 1). *Let (Y, d) be a metrically convex complete metric space and let (X, ϱ) be an arbitrary metric space. Moreover, let $f\colon Y \to X$ be an arbitrary function. Then*

(a) *the following conditions are equivalent:*

(2.4) *for every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\varepsilon < d(x, y) < \varepsilon + \delta \Rightarrow \varrho(f(x), f(y)) \leqslant \varepsilon, \quad x, y \in Y;$$

(2.5) *for every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\varepsilon \leqslant d(x, y) < \varepsilon + \delta \Rightarrow \varrho(f(x), f(y)) < \varepsilon, \quad x, y \in Y;$$

(2.6) *there is a nonincreasing function $\alpha\colon [0, \infty) \to [0, 1]$, $\alpha(t) < 1$ for $t > 0$ and*

$$\varrho(f(x), f(y)) \leqslant \alpha(d(x, y)) \cdot d(x, y), \quad x, y \in Y;$$

(2.7) *there is a function $\gamma\colon [0, \infty) \to [0, \infty)$, $\gamma(t) < t$ for $t > 0$ and*

$$\varrho(f(x), f(y)) \leqslant \gamma(d(x, y)), \quad x, y \in Y.$$

(b) *Furthermore, if f fulfils one of the above conditions, then the function γ in (2.7) is strictly increasing, concave and continuously differentiable in $[0, \infty)$, and the function α in (2.6) is continuous.*

Using the above theorem, we have the following

COROLLARY 2.1. *If (Y, d) is a metrically convex complete metric space and $F\colon Y \to \mathrm{Cl}(Y)$ is an m.v. function which fulfils* (2.1) *with some function ψ such that $\psi(t) < t$ for $t > 0$, then:*

(a) *for every $y_0 \in Y$ and for every fixed point $y \in Y$ of F there exists a sequence of iterates of F at y_0 which converges to y,*

(b) *moreover, if there exists a $t_0 > 0$ such that the function ψ is strictly increasing in the interval $[0, t_0]$ and for which the series $\sum_{k=1}^{\infty} \psi^k(t_0)$ converges, then the set of fixed points of F is nonempty.*

Proof. If the m.v. function F satisfies (2.1), then by the definition of relation ϱ in (1.1), it can take its values from only one of the metric spaces belonging to the canonical partition of $(\mathrm{Cl}(Y), D)$. Hence, by part (b) of Theorem 2.3, there exists a strictly increasing and continuous function γ which satisfies condition (2.7) with $\varrho = D$. Making use of Theorem 0.4 from [14], we have $\lim_{k\to\infty} \gamma^k(t) = 0$ for $t > 0$, and hence by part (a) of Theorem 2.2 we get the first part of the corollary.

For the proof of Corollary 2.1 (b), let $y_0 \in Y$ and $y_1 \in F(y_0)$ be arbitrary points; as in the proof of Theorem 2.1, there exists a sequence of iterates of F at y_0 such that

$$d(y_{k+1}, y_k) < \gamma^k(d(y_0, y_1)+1) \quad \text{for } k = 0, 1, \ldots$$

Since $\lim_{k\to\infty} \gamma^k(t) = 0$ for $t > 0$ there exists an integer N such that $d(y_{N-1}, y_N) < t_0$. Extending $y_0, \ldots, y_N$ to a sequence of iterates $y_0, \ldots, y_N, y'_{N+1}, y'_{N+2}, \ldots$ such that

$$d(y'_{N+k+1}, y'_{N+k}) < \psi^k(t_0) \quad \text{for } k = 1, 2, \ldots,$$

we conclude that a fixed point exists by the same reasoning as that used in Theorem 2.1.

The above results are (in essence) a generalization of theorem of Covitz–Nadler, according to which

EXAMPLE 2.1. Let $Y = [0, \infty)$, $d(x, y) = |x-y|$ and let the m.v. function $F\colon Y \to \mathrm{Cl}(Y)$ be given by

$$F(y) = \begin{cases} [y - y^{\frac{3}{2}}, \infty) & \text{for } 0 \leqslant y \leqslant \frac{4}{9}, \\ [\frac{4}{27}, \infty) & \text{for } \frac{4}{9} < y. \end{cases}$$

Then F fulfils condition (2.1) with the function $\psi\colon [0, \infty) \to [0, \infty)$ defined by

$$\psi(t) = \begin{cases} t - t^{\frac{3}{2}} & \text{for } 0 \leqslant t \leqslant \frac{4}{9}, \\ \frac{4}{27} & \text{for } \frac{4}{9} < t, \end{cases}$$

i.e.

$$D(F(x), F(y)) \leqslant \psi(|x-y|) \quad \text{for } x, y \in Y = [0, \infty),$$

where D is the generalized Hausdorff metric in $\mathrm{Cl}(Y)$. This fact is an easy consequence of (1.4) and the concavity of ψ.

Moreover, the function ψ is strictly increasing on the interval $[0, t_0] = [0, \frac{4}{9}]$, $\psi(t) < t$ for $t > 0$ and $\sum_{k=1}^{\infty} \psi^k(\frac{4}{9}) < \infty$.

In particular, the convergence of the series follows from the theorem of Thron in [27]. The function ψ gives the best estimation of $D(F(x), F(y))$ since

$$D(F(y), F(0)) = \psi(y) - \psi(0) = \psi(|y-0|) \quad \text{for } y \in [0, \infty),$$

and because $\psi'(0) = 1$ we find that F does not fulfil a Lipschitz condition with $s \in (0, 1)$. So in this case we cannot apply a Covitz–Nadler theorem, but we can use Corollary 2.1.

2. In this section we consider m.v. functions which take closed and bounded values, in particular, those which take compact values and give fixed point theorems for them.

First we generalize the theorem of J. Matkowski ([17], Theorem 1.2), and also the theorem of C. S. Wong ([28], Theorem 1), to m.v. function with compact values as follows:

THEOREM 2.4. *If an m.v. function $F\colon Y \to \mathrm{Cl}(Y)$ is defined on a complete metric space (Y, d) and fulfils* (2.1) *with a nondecreasing function ψ such that $\lim_{k\to\infty} \psi^k(t) = 0$ for $t > 0$, then*

(a) *the function $\hat{F}\colon \mathrm{CB}(Y) \to \mathrm{CB}(Y)$ defined by*

$$\hat{F}(A) := \overline{F(A)} \quad \text{for } A \in \mathrm{CB}(Y), \tag{2.8}$$

where $\overline{F(A)}$ denotes the closure of $F(A)$ in Y, fulfils

$$D(\hat{F}(A), \hat{F}(B)) \leqslant \psi(D(A, B)) \quad \text{for } A, B \in \mathrm{CB}(Y);$$

(b) *$\hat{F}$ has exactly one fixed point $A_0 \in \mathrm{CB}(Y)$ such that*

$$A_0 = \overline{\bigcup \{F(y)\colon y \in A_0\}}, \tag{2.9}$$

and for any $A \in \mathrm{CB}(Y)$ the sequence of iterates $\{\hat{F}^k(A)\}_{k=1}^{\infty}$ of $\hat{F}$ converges to A_0, i.e. $D(\hat{F}^k(A), A_0) \to 0$ if $k \to \infty$;

(c) *the set of fixed points B_0 of F is closed and*

$$B_0 \subset A_0. \tag{2.10}$$

(d) *Moreover, if the values of F are compact, then the set B_0 is nonempty,*

sets B_0 and A_0 are compact, and for every $y \in B_0$ and for every $y_0 \in Y$ there exists a sequence of iterates of F at y_0 which converges to y.

Proof. Part (a) was proved in [28] or [26]. For the proof of (b) we can make use of the fixed point theorem from [17] (Theorem 1.2), since we know that $(\mathrm{CB}(Y), D)$ is a complete metric space and F fulfils the required conditions. The closedness of B_0 in (c) follows from Lemma 1.3 and the continuity of F.

Inclusion (2.10) follows from (b). Namely, if $y \in B_0$, then from (2.8) we infer by induction that $y \in \hat{F}^k(\{y\})$ for $k = 1, \ldots$, which, with $D(\hat{F}^k(\{y\}), A_0) \to 0$ if $k \to \infty$, gives $y \in A_0$.

The proof of the existence of fixed points of F and the compactness of B_0 and A_0 in part (d) is quite similar to the one given in Theorem 1 from [26], and is based on Theorem 4 from [10].

We let $y \in B_0$ and $y_0 \in Y$ be arbitrary elements; then using (1.2) and (2.1) we have

$$d(y, F(y_0)) \leqslant D(F(y), F(y_0)) \leqslant \psi(d(y, y_0)). \tag{2.11}$$

Since $F(y_0) \in \mathrm{C}(Y)$, there exists a $y_1 \in F(y_0)$ such that $d(y, y_1) = d(y, F(y_0))$ and from (2.11) we get $d(y, y_1) \leqslant \psi(d(y, y_0))$.

Continuing this procedure, we can define a sequence of iterates $\{y_k\}_{k=1}^{\infty}$ of F at y_0 such that

$$d(y, y_k) \leqslant \psi^k(d(y, y_0)) \quad \text{for } k = 0, 1, \ldots$$

Hence we infer that the sequence $\{y_k\}$ converges to y.

Immediately from the above theorem and Theorem 2.3, for a metrically convex space, we obtain the next result, which generalizes Theorem 2 from [19].

COROLLARY 2.2. *If $F\colon Y \to \mathrm{CB}(Y)$ is defined on a complete and metrically convex metric space (Y, d) and F fulfils one of conditions (2.4)–(2.7), where $(X, \varrho) = (\mathrm{CB}(Y), D)$, then F fulfils each of the above conditions, and condition (2.1), with some function ψ which is strictly increasing, concave and continuously differentiable in $[0, \infty)$; and $\psi(t) < t$ for $t > 0$.*

Moreover, with this function ψ, every conclusion of Theorem 2.4 holds for F.

Remark 2.2. The preceding corollary generalizes to m.v. functions with compact values Theorem 2 from [18], which considered only single-valued functions. For m.v. functions with compact values which satisfy (2.5), S. Reich proved the fixed point theorem even without the metric convexity of Y (see [24], Theorem 10).

C. S. Wong in [28] proved a fixed point theorem similar to Corollary 2.2 from condition (2.7), but supposed additionally that the function γ is

nondecreasing. The authors of [26] omitted this additional condition, but gave an erroneous proof (see [26], Theorem 2).

Remark 2.3. It remains an open question whether an m.v. function $F\colon Y\to \mathrm{CB}(Y)$ has a fixed point if (Y, d) is a complete and metrically convex metric space and if F fulfils one of conditions (2.4)–(2.7) within $(X, \varrho) = (\mathrm{CB}(Y), D)$.

In particular, the existence of fixed points for such m.v. functions is given by Corollary 2.1, but we can omit in it the convergence of the series of iterates of the function ψ.

To conclude, we give a known example of an m.v. function which shows that the inclusion in (2.10) is not necessarily an equality.

Example 2.2 (S. B. Nadler, [22]). Let $Y=[0, 1)$, $d(x, y) = |x-y|$, and $F\colon Y\to \mathrm{CB}(Y)$ be given by

$$F(y) = \{f(y)\}\cup\{0\} \quad \text{for } y\in[0, 1],$$

where the function f is defined as follows:

$$f(y) = \begin{cases} \frac{1}{2}\cdot y+\frac{1}{2} & \text{for } y\in[0, \frac{1}{2}], \\ -\frac{1}{2}\cdot y+1 & \text{for } y\in(\frac{1}{2}, 1]. \end{cases}$$

It is obvious that the m.v. function F fulfils a Lipschitz condition with the constant $s = \frac{1}{2}$.

Moreover, from (2.8) and from the compactness of sets belonging to $\mathrm{CB}(Y)$ we have

$$\hat{F}(A) = \overline{F(A)} = F(A) \quad \text{for } A\in\mathrm{CB}(Y).$$

For this function F the set B_0 is $\{0, \frac{2}{3}\}$, but the set A_0 is $\{0, \frac{2}{3}, f(0), f^2(0), \ldots\}$, which follows from the uniqueness of the fixed point of $\hat{F}$ and from the equality $A_0 = \hat{F}(A_0)$.

3. On the characterization and extension of continuous solutions of functional equation with multi-valued functions

In this chapter we shall consider the characterization of continuous solutions of the functional equation

$$\varphi(x)\in H(x, \triangle_{s\in S}\varphi[f_s(x)]), \tag{3.1}$$

where we are given an m.v. function H and functions f_s, $s\in S$, and φ is an unknown function. Here, $\triangle_{s\in S}\varphi\circ f_s$ denotes the diagonal of a family $\{\varphi\circ f_s\colon s\in S\}$ of functions.

In a special case, we can characterize the continuous solutions of (3.1) by the continuous solutions of

$$\varphi(x) = h(x, \triangle_{s\in S} \varphi[f_s(x)]), \tag{3.2}$$

where the function h is a continuous selection from the m.v. function H.

We assume the following hypotheses for the given functions H and f_s, $s\in S$ (S — is an arbitrary set):

(3.3) $H\colon X\times Y^S \to \mathrm{CC}(Y)$, where X is a Hausdorff topological space, and Y is a closed and complete metrizable subset of a complete locally convex space over the field of real or complex numbers;

(3.4) H — is lower semicontinuous;

(3.5) $f_s\colon X\to X$, $s\in S$, are continuous functions.

Then the following lemma is true.

LEMMA 3.1. *Let hypotheses* (3.3), (3.4), (3.5) *be fulfilled. If $X\times Y^S$ is a paracompact topological space and $A\subset X$ is a closed subset such that $f_s(A)\subset A$ for $s\in S$, then for every continuous solution $\varphi_0\colon A\to Y$ of equation* (3.1) *there exists a continuous selection h from the m.v. function H such that φ_0 satisfies equation* (3.2) *(with h) on A.*

Proof. The function $\triangle_{s\in S}\varphi_0\circ f_s\colon A\to Y^S$ is continuous, and hence its graph $G_\triangle\subset X\times Y^S$ is a closed subset of $X\times Y^S$, which follows from the fact that A is a closed subset of X (see [9], p. 114).

The function $h_0\colon G_\triangle\to Y$ defined by

$$h_0(x, \triangle_{s\in S} y_s) := \varphi_0(\Pi_X(x, \triangle_{s\in S} y_s)) \quad \text{for } (x, \triangle_{s\in S} y_s)\in G_\Delta, \tag{3.6}$$

where Π_X is the projection of $X\times Y^S$ onto the X-axis, is continuous. Moreover, h_0 is a continuous selection from the m.v. function H on the closed set G_Δ. Thus by Theorem 1.2 the function h_0 can be extended onto $X\times Y^S$ to a continuous selection from H.

Hence from (3.6) we find that φ_0 fulfils equation (3.2) on A.

The following characterization of continuous solutions of equation (3.1) results directly from Lemma 3.1.

THEOREM 3.1. *Let hypotheses* (3.3), (3.4), (3.5) *be fulfilled. If $X\times Y^S$ is a paracompact topological space, then a continuous function $\varphi\colon X\to Y$ is a solution of equation* (3.1) *if and only if there exists a continuous selection $h\colon X\times Y^S\to Y$ from H such that φ fulfils equation* (3.2) *with h.*

Proof. The sufficiency is obvious, while the necessity follows immediately from Lemma 3.1 if we take $A = X$.

Now we pass to the problem of the extension of continuous solutions of equation (3.1). Here we use the following two theorems on the extension of continuous solutions of equation (3.2).

THEOREM 3.2 (see [4], Theorem 1). *Let X and Y be arbitrary topological spaces, let $U \subset X$ – be an open subset, $h\colon X \times Y^S \to Y$, and let $f_s\colon X \to Y$, $s \in S$, be arbitrary continuous functions satisfying the following conditions:*

(3.7) $$f_s(U) \subset U \quad \text{for } s \in S;$$

(3.8) *for every $x \in X$ there exists a positive integer k such that, for every $s_1, \ldots, s_k \in S$, $f_{s_1} \circ \ldots \circ f_{s_k}(x) \in U$;*

and

(3.9) *for every open set V, $U \subset V \subset X$, the set $\bigcap \{f_s^{-1}(V)\colon s \in S\}$ is open.*

Then for every continuous solution $\varphi_0\colon U \to Y$ of the functional equation (3.2) *there exists exactly one continuous solution $\varphi\colon X \to Y$ of this equation such that*

(3.10) $$\varphi(x) = \varphi_0(x) \quad \text{for } x \in U.$$

THEOREM 3.3 ([4] and [5]). *Let X be a closed and convex subset of a finite-dimensional Banach space, $U \subset X$ an open (in X) subset, and Y – a topological space. Let $h\colon X \times Y^S \to Y$ and $f_s\colon X \to X$, $s \in S$, be arbitrary continuous functions. Moreover, if $\{f_s\colon s \in S\}$ is a locally equicontinuous family of functions such that*

(3.11) $$\sup\{\|f_s(x) - \xi\|\colon s \in S\} < \|x - \xi\|, \quad x \in X,\ x \neq \xi,$$

holds for a certain $\xi \in U$, then for every continuous solution $\varphi_0\colon U \to Y$ of equation (3.2) *there exists exactly one continuous solution $\varphi\colon X \to Y$ of this equation such that condition* (3.10) *is fulfilled.*

Making use of Theorems 3.2 and 3.3, we get the following two theorems on extending of continuous solutions of equation (3.1).

THEOREM 3.4. *Let hypotheses* (3.3), (3.4), (3.5) *be fulfilled, let $X \times Y^S$ be a paracompact space and let $U \subset X$ be an open subset such that conditions* (3.7), (3.8), (3.9) *are fulfilled. Then for every continuous solution $\varphi_0\colon \bar{U} \to Y$ of equation* (3.1) *there exists a continuous selection $h\colon X \times Y^S \to Y$ from H (not unique) for which there exists a unique continuous function $\varphi\colon X \to Y$ fulfilling equation* (3.2), *with this function h and condition*

(3.12) $$\varphi(x) = \varphi_0(x) \quad \text{for } x \in \bar{U}.$$

Proof. The inclusions

$$f_s(\bar{U}) \subset \overline{f_s(U)} \subset \bar{U} \quad \text{for } s \in S$$

follow directly from (3.5) and (3.7). Hence, making use of Lemma 3.1 and putting $A = \bar{U}$, we get the existence of a continuous selection $h\colon X \times Y^S \to Y$ from H such that φ_0 fulfils equation (3.2), with h, in $\bar{U}$.

Now, by Theorem 3.2, for the function $\varphi_0|_U$ there exists a unique continuous function $\varphi\colon X \to Y$ fulfilling equation (3.2) and condition (3.10). Thus by the continuity of φ and φ_0 on $\bar{U}$ condition (3.12) also holds.

THEOREM 3.5. *Let hypotheses* (3.3), (3.4), (3.5) *be fulfilled, where X is a closed and convex subset of a finite-dimensional Banach space and $X \times Y^S$ is a paracompact space. Moreover, let $U \subset X$ be an open subset of X fulfilling* (3.7), *and let $\{f_s\colon s \in S\}$ be a locally equicontinuous family such that* (3.11) *holds for a certain $\xi \in U$.*

Then for every continuous solution $\varphi_0\colon \bar{U} \to Y$ of equation (3.1) *there exists a continuous selection $h\colon X \times Y^S \to Y$ from H (not unique) for which there exists a unique continuous function $\varphi\colon X \to Y$ fulfilling equation* (3.2), *with this function h and condition* (3.12).

Proof. This theorem can be proved analogously to Theorem 3.4 by using Theorem 3.3 instead of Theorem 3.2.

COROLLARY 3.1. *In the last two theorems we see, in particular, that an arbitrary continuous solution $\varphi_0\colon \bar{U} \to Y$ of equation* (3.1) *can be extended (nonuniquely) on X to a continuous solution $\varphi\colon X \to Y$ of this equation.*

Remark 3.1. Among the assumptions Theorems 3.1, 3.4 and 3.5 was the paracompactness of the product $X \times Y^S$. This assumption is more restrictive in our considerations, for example in the case where Y is a complete metrizable locally convex space, because then the product can be paracompact only if S is a countable set of indices or if $Y = \{0\}$ (i.e. Y is trivial).

This follows from the fact that, if Y is non-trivial and S is infinite, then the set

$$A := \{k \cdot y\colon k = 1, 2, \ldots\}$$

defined for a certain $y \in Y$, $y \neq 0$, is a closed subset of Y, homeomorphic to N, N being the set of natural numbers (see [25], p. 21).

Hence A^S is a closed subset of Y^S and is not normal, since the set N^S is not normal ([9], p. 160). This shows that Y^S (and $X \times Y^S$) is not a paracompact space.

The product $X \times Y^S$ will be paracompact in Theorems 3.1, 3.4, 3.5, for instance, if X is a compact or a metric space and Y is a compact space or S is a countable set.

In Theorems 3.4 and 3.5 we extended a continuous solution $\varphi_0\colon \bar{U} \to Y$ of equation (3.1) from the closure of a certain open subset $U \subset X$, while in (3.2) we could extend continuous solutions just from U (see Theorems 3.2 and 3.3).

The next example shows that under the assumptions of Theorems 3.4 or

3.5 there does not even exist, in general, any continuous extension of a continuous solution φ_0: $U \to Y$ of (3.1) which is defined only in an open subset U.

EXAMPLE 3.1. Let $X = [0, \infty)$, $Y = [0, \infty)$, H: $X \times Y \to \mathrm{CC}(Y)$, $H(x, y) = [y, \infty)$, f: $X \to X$, $f(x) = \frac{1}{2}x$, $U = [0, \frac{1}{2}\pi)$, and $\varphi_0(x) = \tan x$.

The function φ_0 is continuous and fulfils the equation

$$\varphi_0(x) \in H\big(x, \varphi_0(f(x))\big) \quad \text{for } x \in U.$$

It is easy to see that all the assumptions of Theorem 3.4 or Theorem 3.5 are fulfilled, except for φ_0 being defined on $\bar{U}$.

However, φ_0 is not a restriction of any continuous function defined on the whole of X. Hence φ_0 cannot be extended onto X to a continuous solution of the above equation.

4. On the existence of continuous solutions of functional equations of n-th order with multi-valued functions

Here we shall prove some theorems about the existence of continuous solutions of the functional equation of n-th order

$$\varphi(x) \in H\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big), \tag{4.1}$$

where φ is an unknown function, and an m.v. function H and functions f_i, $i = 1, \ldots, n$, are given.

The problem of the existence of continuous solutions of (4.1) with a single-valued function H was investigated e.g. in [1], [2], [3], cf. also monograph [14].

We assume the following conditions on the given functions in equation (4.1):

(4.2) H: $X \times Y^n \to \mathrm{CC}(Y)$, where X is an arbitrary compact and Hausdorff topological space, (Y, d) is a complete metric space, and Y is furthermore a closed topological subspace of a complete locally convex space over the field of real or complex numbers, such that all open balls in the metric space (Y, d) are convex;(1)

(4.3) there exists a function $\mathscr{B}$: $[0, \infty)^n \to [0, \infty)$, nondecreasing with respect to each variable such that

$$D\big(H(x, y_1, \ldots, y_n), H(x, \bar{y}_1, \ldots, \bar{y}_n)\big) \leqslant \mathscr{B}\big(d(y_1, \bar{y}_1), \ldots, d(y_n, \bar{y}_n)\big)$$
$$\text{for } x \in X, \text{ and } y_k, \bar{y}_k \in Y,\ k = 1, \ldots, n,$$

(1) Thus obviously Y must also be a convex subset.

where D is the generalized Hausdorff metric in $\mathrm{CC}(Y)$ defined by (1.2);

(4.4) $f_k\colon X \to X$, $k = 1,\dots,n$, are arbitrary continuous functions.

Put

$$\gamma(t) := \mathscr{B}(t,\dots,t) \quad \text{for } t \geqslant 0. \tag{4.5}$$

The set

$$\mathscr{C} := \{\varphi\colon X \to Y\colon \varphi \text{ — is continuous}\}, \tag{4.6}$$

with metric $\bar{d}(\varphi_1, \varphi_2) = \sup\{d(\varphi_1(x), \varphi_2(x))\colon x \in X\}$ is a complete metric space.

Moreover, let $\bar{D}$ be the generalized Hausdorff metric in $\mathrm{Cl}(\mathscr{C})$ which is defined by (1.2) (d being replaced by $\bar{d}$ and D by $\bar{D}$).

LEMMA 4.1. *Let hypotheses* (4.2), (4.3), (4.4) *and* (3.4) *be fulfilled. Then the m.v. function* $T\colon \mathscr{C} \to \mathrm{CC}(\mathscr{C})$ *given by*

$$T(\varphi) := \{\psi \in \mathscr{C}\colon \psi(x) \in H(x, \varphi[f_1(x)],\dots,\varphi[f_n(x)]),\ x \in X \text{ for } \varphi \in \mathscr{C}\}, \tag{4.7}$$

fulfils the condition

$$\bar{D}(T(\varphi_1), T(\varphi_2)) \leqslant \gamma(\bar{d}(\varphi_1, \varphi_2)) \quad \text{for } \varphi_1, \varphi_2 \in \mathscr{C}, \tag{4.8}$$

where γ *is defined by* (4.5).

Proof. It follows from (4.2), (3.4), (4.4) that the m.v. function

$$H(\cdot, \varphi[f_1(\cdot)],\dots,\varphi[f_n(\cdot)])\colon X \to \mathrm{CC}(Y)$$

is lower semicontinuous for every $\varphi \in \mathscr{C}$, as it is the composition of a lower semicontinuous m.v. function H with a continuous function.

Hence, making use of Theorem 1.2, we obtain $T(\varphi) \neq \emptyset$ for every $\varphi \in \mathscr{C}$. The values of T are evidently a closed and convex subset of $\mathscr{C}$, which follows from (4.2) and (4.7).

By the definition of $\bar{D}$, in order to prove (4.8) it is enough to show that

$$\bar{d}(\psi_1, T(\varphi_2)) \leqslant \gamma(\bar{d}(\varphi_1, \varphi_2)) \quad \text{for every function } \psi_1 \in T(\varphi_1), \tag{4.9}$$

where $\bar{d}(\psi_1, T(\varphi_2)) = \inf\{\bar{d}(\psi_1, \psi_2)\colon \psi_2 \in T(\varphi_2)\}$ and, analogously, that $\bar{d}(\psi_2, T(\varphi_1)) \leqslant \gamma(\bar{d}(\varphi_1, \varphi_2))$ for $\psi_2 \in T(\varphi_2)$.

We shall prove (4.9) only, showing that

(4.10) for every $\varepsilon > 0$ there exists a $\psi_2 \in T(\varphi_2)$ such that

$$\bar{d}(\psi_1, \psi_2) \leqslant \gamma(\bar{d}(\varphi_1, \varphi_2)) + \varepsilon,$$

where ψ_1 is an arbitrary fixed function belonging to $T(\varphi_1)$.

From the definition of D in $\mathrm{Cl}(Y)$ and from (4.3), (4.5), and (4.7) we infer that for a given function $\psi_1 \in T(\varphi_1)$,

$$\begin{aligned} & d\big(\psi_1(x), H(x, \varphi_2[f_1(x)], \ldots, \varphi_2[f_n(x)])\big) \\ & \quad \leqslant D\big(H(x, \varphi_1[f_1(x)], \ldots, \varphi_1[f_n(x)]), H(x, \varphi_2[f_1(x)], \ldots, \varphi_2[f_n(x)])\big) \\ & \quad \leqslant \mathscr{B}\big(d(\varphi_1[f_1(x)], \varphi_2[f_1(x)]), \ldots, d(\varphi_1[f_n(x)], \varphi_2[f_n(x)])\big) \\ & \quad \leqslant \mathscr{B}(\bar{d}(\varphi_1, \varphi_2), \ldots, \bar{d}(\varphi_1, \varphi_2)) \\ & \quad = \gamma(\bar{d}(\varphi_1, \varphi_2)) \quad \text{for } x \in X. \end{aligned}$$

Thus, by Lemma 1.3, for a fixed $\varepsilon > 0$ and for every $x \in X$

(4.12) there exists a $y_x \in H(x, \varphi_2[f_1(x)], \ldots, \varphi_2[f_n(x)])$ such that

$$d(\psi_1(x), y_x) < \gamma(\bar{d}(\varphi_1, \varphi_2)) + \varepsilon.$$

Then by Theorem 1.2 (on the extension of continuous selections from the closed subset $\{x\} \subset X$) there exists a family $\{\psi_x \in \mathscr{C}: x \in X\}$ of continuous functions such that

$$\psi_x(x) = y_x \quad \text{and} \quad \psi_x \in T(\varphi_2) \quad \text{for } x \in X. \tag{4.13}$$

Using (4.12) and (4.13), we find that the inequality

$$d(\psi_1(x), \psi_x(x)) = d(\psi_1(x), y_x) < \gamma(\bar{d}(\varphi_1, \varphi_2)) + \varepsilon$$

holds for every $x \in X$. By the continuity of ψ_1, ψ_x, $y \in X$, and of the metric d, the above sharp inequality is fulfilled in an open neighbourhood $U_x \subset X$ of x, i.e.

$$d(\psi_1(z), \psi_x(z)) < \gamma(\bar{d}(\varphi_1, \varphi_2)) + \varepsilon \quad \text{for } z \in U_x. \tag{4.14}$$

The family $\mathscr{U} := \{U_x: x \in X\}$ is an open cover of the space X. Let $p_s: X \to [0, 1]$ be a locally finite partition of unity which is subordinated to $\mathscr{U}$, where the support of the function p_s is contained in a certain U_{x_s} from the cover $\mathscr{U}$, i.e. $\operatorname{supp} p_s \subset U_{x_s}$ for $s \in S$ (see [9], p. 368).

Then we assert that the function

$$\psi_2(z) := \sum_{s \in S} p_s(z) \psi_{x_s}(z) \quad \text{for } z \in X, \tag{4.15}$$

fulfils the required condition (4.10) for our fixed $\varepsilon > 0$.

Indeed, obviously as follows from (4.15), (4.13), (4.7) and from the convexity of the values of H the function ψ_2 is continuous and $\psi_2 \in T(\varphi_2)$.

The function ψ_2 also fulfils the inequality in (4.10), which in turn follows from (4.15), (4.14) and the convexity of the open balls $K(\psi_1(z), \gamma(\bar{d}(\varphi_1, \varphi_2)) + \varepsilon)$ for $z \in X$:

$$d(\psi_1(z), \psi_2(z)) = d\Big(\psi_1(z), \sum_{s \in S} p_s(z) \psi_{x_s}(z)\Big) < \gamma(\bar{d}(\varphi_1, \varphi_2)) + \varepsilon \quad \text{for } z \in X.$$

Taking the supremum in the above inequality, we have

$$\bar{d}(\psi_1, \psi_2) \leqslant \gamma(\bar{d}(\varphi_1, \varphi_2)) + \varepsilon,$$

which completes the proof.

Making use of Lemma 4.1, we have the following theorem on the existence of continuous solutions of equation (4.1).

THEOREM 4.1. *Let hypotheses* (4.2), (4.3), (3.4), (4.4) *be fulfilled, and suppose that there exists a strictly increasing function* $\psi\colon [0, \infty) \to [0, \infty)$ *such that*

$$\gamma(t) \leqslant \psi(t) \quad \text{and} \quad \sum_{k=1}^{\infty} \psi^k(t) < \infty \quad \text{for } t > 0. \tag{4.16}$$

Then the set of continuous solutions of (4.1) *is nonempty.*

Moreover, for every continuous solution φ *of equation* (4.1) *and for every function* $\varphi_0 \in \mathscr{C}$ *there exists a sequence of functions* $\varphi_k \in \mathscr{C}$, $k = 1, 2, \ldots$, *such that*

$$\varphi_{k+1}(x) \in H(x, \varphi_k[f_1(x)], \ldots, \varphi_k[f_n(x)]) \quad \text{for } x \in X \text{ and } k = 0, 1, \ldots,$$

which converges uniformly to φ.

Proof. By Lemma 4.1 the m.v. function $T\colon \mathscr{C} \to \mathrm{CC}(\mathscr{C})$, defined by (4.7) on a complete metric space $(\mathscr{C}, \bar{d})$, fulfils condition (4.8), i.e. is a γ-contraction as well as a ψ-contraction.

From Theorem 2.2 we get the existence of continuous solutions of equation (4.1), because the set of fixed points of T is nonempty. The second part of the theorem follows from part (a) of Theorem 2.2 and the definitions of T and $\bar{d}$.

Remark 4.1. If Y in the above theorem is a nonempty, closed and convex subset of a Banach space, then the metric space $(\mathscr{C}, \bar{d})$ is metrically convex, as a convex subset of some Banach space. In this case we can weaken the conditions on ψ to the following ones: the function $\psi\colon [0, \infty) \to [0, \infty)$ is strictly increasing in some interval $[0, t_0]$, $t_0 > 0$, and such that $\sum_{k=1}^{\infty} \psi^k(t_0) < \infty$ and $\gamma(t) \leqslant \psi(t) < t$ for $t > 0$.

Similarly, using the fixed point theorems for m.v. functions with closed and bounded values, we get:

THEOREM 4.2. *Let hypotheses* (4.2), (4.3), (4.4) *be fulfilled. Moreover, let the m.v. function* H *have bounded values, and let* H *be either continuous with respect to* D *or else upper and lower semicontinuous. If there exists a strictly increasing function* $\psi\colon [0, \infty) \to [0, \infty)$ *such that* (4.16) *holds, then Theorem* 4.1 *is fulfilled, and the set of continuous solutions of equation* (4.1) *is bounded in* $\mathscr{C}$.

Proof. It suffices to prove that the set of continuous solutions of equation (4.1) is bounded in $\mathscr{C}$.

From Lemmas 1.4 and 1.5, or in the second case from Lemma 1.6, it follows that the set

$$H(X, \varphi[f_1(X)], \ldots, \varphi[f_n(X)]) \subset Y$$

is bounded for any $\varphi \in \mathscr{C}$. Hence the m.v. function T given by (4.7) maps $\mathscr{C}$ onto $\mathrm{CCB}(\mathscr{C})$, and its set of fixed points must be bounded, which follows from Theorem 2.4, parts (b) and (c).

Remark 4.2. As in Remark 4.1, we can weaken the assumptions about the function ψ when Y is a nonempty closed and convex subset of a Banach space.

5. On the continuous solutions of a functional inequality

The present chapter contains applications of previous results to the functional double inequality (5.1). The following lemma shows that functional equation (4.1) is (in a particular case) equivalent to double inequality (5.1).

LEMMA 5.1. *Let X and S be arbitrary sets, and let Y be an arbitrary set belonging to* $\mathrm{CC}(\boldsymbol{R})$.

Moreover, let $H\colon X \times Y^S \to \mathrm{CC}(Y)$ and $f_s\colon X \to X$, $s \in S$, be arbitrary functions. Then any function $\varphi\colon X \to Y$ is a solution of (3.1) *if and only if it fulfils the inequalities*

$$h_1(x, \triangle_{s\in S} \varphi[f_s(x)]) \leqslant \varphi(x) \leqslant h_2(x, \triangle_{s\in S} \varphi[f_s(x)]) \quad \text{for } x \in X, \tag{5.1}$$

where the functions $h_1\colon X \times Y^S \to Y \cup \{-\infty\}$ and $h_2\colon X \times Y^S \to Y \cup \{+\infty\}$ are given by (1.5).

Proof. This lemma is a consequence of Lemma 1.2, part (a).

Now we can consider the continuous solutions of double inequality (5.1), making use of earlier results for equation (3.1).

First we characterize the continuous solutions of (5.1).

THEOREM 5.1. *Let X be a Hausdorff topological space and S an arbitrary set, and let $Y \in \mathrm{CC}(\boldsymbol{R})$ be such that $X \times Y^S$ is a paracompact space. Let h_1: $X \times Y^S \to Y \cup \{-\infty\}$ and $h_2\colon X \times Y^S \to Y \cup \{+\infty\}$ be, respectively, an arbitrary upper and an arbitrary lower semicontinuous, function, with $h_1 \leqslant h_2$, and let $f_s\colon X \to X$, $s \in S$, be continuous functions.*

Then a continuous function $\varphi\colon X \to Y$ is a solution of a double inequality (5.1) *if and only if there exists a continuous function $h\colon X \times Y^S \to Y$, $h_1 \leqslant h \leqslant h_2$, such that φ fulfils (with h) equation* (3.2).

Proof. This theorem results from Theorem 3.1 and Lemma 5.1, since φ is a solution of (5.1) if and only if it fulfils equation (3.1) with an m.v. function $H\colon X\times Y^S\to \mathrm{CC}(Y)$ given by

$$H(x, \underset{s\in S}{\triangle} y_s) = [h_1(x, \underset{s\in S}{\triangle} y_s), h_2(x, \underset{s\in S}{\triangle} y_s)]\cap Y.$$

In the same way, we can easily apply Theorems 3.4 and 3.5 to the extension of continuous solutions of (5.1), but we will omit these theorems.

Now we pass to the question of the existence of continuous solutions of (5.1). It will be assumed that the order of inequality (5.1) is finite, i.e.

$$h_1(x, \varphi[f_1(x)],\ldots,\varphi[f_n(x)]) \leqslant \varphi(x) \leqslant h_2(x, \varphi[f_1(x)],\ldots,\varphi[f_n(x)]). \tag{5.2}$$

We start with a lemma which shows when an m.v. function $H\colon X\times Y^n \to \mathrm{CC}(Y)$, $Y\in \mathrm{CC}(\boldsymbol{R})$, given by

$$H(x, y_1,\ldots,y_n) = [h_1(x, y_1,\ldots,y_n), h_2(x, y_1,\ldots,y_n)]\cap Y, \quad \text{for } (x, y_1,\ldots,y_n)\in X\times Y^n, \tag{5.3}$$

fulfils condition (4.3) with a function $\mathscr{B}$.

LEMMA 5.2. *Let X and $Y\in \mathrm{CC}(\boldsymbol{R})$ be arbitrary sets, and let $\mathscr{B}\colon [0,\infty)^n \to [0,\infty)$, $h_1\colon X\times Y^n\to Y\cup\{-\infty\}$, $h_2\colon X\times Y^n\to Y\cup\{+\infty\}$, $h_1\leqslant h_2$, be arbitrary functions.*

Then the m.v. function $H\colon X\times Y^n\to \mathrm{CC}(Y)$ defined by (5.3) fulfils condition (4.3), with a metric $d(x, y) = |x-y|$ and a generalized Hausdorff metric D in $\mathrm{CC}(Y)$, if and only if

(5.4) *there exist subsets $I_i\subset X$, $i = 1, 2$, such that*

$$h_i(x, y_1,\ldots,y_n)\in Y \quad \textit{for } (x, y_1,\ldots,y_n)\in I_i\times Y^n,\ i = 1, 2,$$

$$|h_i(x, y_1,\ldots,y_n)| = +\infty \quad \textit{for } (x, y_1,\ldots,y_n)\in (X\setminus I_i)\times Y^n,\ i = 1, 2,$$

and

$$|h_i(x, y_1,\ldots,y_n)-h_i(x, \bar{y}_1,\ldots,\bar{y}_n)| \leqslant \mathscr{B}(|y_1-\bar{y}_1|,\ldots,|y_n-\bar{y}_n|)$$
$$\textit{for } (x, y_1,\ldots,y_n), (x, \bar{y}_1,\ldots,\bar{y}_n)\in I_i\times Y^n,\ i = 1, 2.$$

Proof. We prove the necessity only, because the converse easily follows from Lemma 1.1 and from (1.4).

So let the m.v. function defined by (5.3) fulfil condition (4.3); then, for every fixed $x\in X$, the values of the m.v. function H must belong to only one element of the canonical partition of $(\mathrm{CC}(Y), D)$ (see the form of this partition for $Y=\boldsymbol{R}$, e.g. (1.3), and see also Remark 1.1).

Hence the sets

$$I_i := \{x\in X\colon h_i(x, y_1,\ldots,y_n)\in Y\}, \quad i = 1, 2,$$

are well defined and by (4.3), (5.3), (1.4) condition (5.4) is fulfilled.

So we obtain our final

THEOREM 5.2. *Let X be a Hausdorff compact space, let $Y \in CC(\boldsymbol{R})$ be an arbitrary fixed set, and let $h_1\colon X \times Y^n \to Y \cup \{-\infty\}$, $h_2\colon X \times Y^n \to Y \cup \cup \{+\infty\}$, $h_1 \leqslant h_2$, be arbitrary upper (resp. lower) semicontinuous functions fulfilling* (5.4) *with a function $\mathscr{B}\colon [0, \infty)^n \to [0, \infty)$, nondecreasing with respect to each variable, and let $f_i\colon X \to X$, $i = 1, \ldots, n$, be continuous functions. Moreover, let there be a function $\psi\colon [0, \infty) \to [0, \infty)$ and a positive number t_0 such that ψ is strictly increasing on $[0, t_0]$, $\gamma(t) \leqslant \psi(t) < t$ for $t > 0$ and $\sum_{k=1}^{\infty} \psi^k(t_0) < \infty$, where γ is defined by* (4.5).

Then there exists at least one continuous solution $\varphi\colon X \to Y$ of (5.2). *Furthermore, for every continuous function $\varphi_0\colon X \to Y$ and for every continuous solution φ of* (5.2) *there exists a sequence of continuous functions $\varphi_k\colon X \to Y$, $k = 1, 2, \ldots$, uniformly converging to φ on X and such that*

$$h_1(x, \varphi_k[f_1(x)], \ldots, \varphi_k[f_n(x)]) \leqslant \varphi_{k+1}(x) \leqslant h_2(x, \varphi_k[f_1(x)], \ldots, \varphi_k[f_n(x)])$$
$$\text{for } x \in X \text{ and } k = 0, 1, \ldots$$

Proof. The product $X \times Y^n$ is obviously a Hausdorff paracompact space. Moreover, the m.v. function H defined by (5.3) is lower semicontinuous by Lemma 1.2, part (b), and by Lemma 5.2 fulfils condition (4.3).

Hence, by Theorem 4.1, Remark 4.1 and Lemma 5.1, we obtain our assertion.

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References

[1] K. Baron, *Continuous solutions of a functional equation of n-th order*, Aeq. Math. 9 (1973), pp. 257–259.

[2] –, *Note on the existence of continuous solutions of a functional equation of n-th order*, Ann. Polon. Math. 30 (1974), pp. 77–80.

[3] –, *Note on continuous solutions of a functional equation*, Aeq. Math. 11 (1974), pp. 267–269.

[4] –, *On extending of solutions of functional equation*, ibidem 13 (1975), pp. 285–288.

[5] –, *On extending of solutions of functional equations in a single variable*, Recueil des travaux de l'Institut Math., Nouvelle série, No. 1 (9) (1976), pp. 23–24.

[6] L. Blumenthal, *Theory and applications of distance geometry*, Oxford 1953.

[7] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. 20 (1969), pp. 458–464.

[8] H. Covitz and S. B. Nadler jr., *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math. 8 (1970), pp. 5–11.

[9] R. Engelking, *General topology*, Monografie Mat. t. 60, Warszawa 1976.

[10] R. B. Fraser jr. and S. B. Nadler jr., *Sequences of contractive maps and fixed points*, Pacific J. Math. 31 (1969), pp. 659–667.

[11] C. K. Jung, *On generalized complete metric spaces*, Bul. Amer. Math. Soc. 75 (1969), pp. 113–116.

[12] W. A. Kirk, *Caristi's fixed point theorem and metric convexity*, Colloq. Math. 36 (1976), pp. 81–86.

[13] H. M. Ko, *Fixed points theorems for point to set mappings and the set of fixed points*, Pacific J. Math. 42 (1972), pp. 369–379.

[14] M. Kuczma, *Functional equations in a single variable*, PWN, Monografie Mat. 46, Warszawa 1968.

[15] K. Kuratowski, *Topology I*, Acad. Press, New York 1966.

[16] –, *Topology II*, ibidem 1968.

[17] J. Matkowski, *Integrable solutions of functional equations*, Diss. Math. 127, PWN, Warszawa 1975.

[18] – and R. Węgrzyk, *On equivalence of some fixed point theorems for selfmappings of metrically convex space*, Boll. Un. Mat. Ital (5) 15-A (1978), pp. 359–369.

[19] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. 28 (1969), pp. 326–329.

[20] E. Michael, *Continuous selections I*, Ann. of Math. 63 (1956), pp. 361–382.

[21] –, *A selection theorem*, Proc. Amer. Math. Soc. 17 (1966), pp. 1404–1406.

[22] S. B. Nadler jr., *Multivalued contraction mappings*, Pacific J. Math. 30 (1969), pp. 475–488.

[23] E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc. 13 (1962), pp. 459–465.

[24] S. Reich, *Fixed points of contractive functions*, Boll. Un. Math. Ital. (4) 5 (1972), pp. 26–42.

[25] H. H. Schaefer, *Topological vector spaces*, New York 1966.

[26] C. Shiau, K. K. Tan and C. S. Wong, *Quasi-nonexpansive multi-valued maps and selections*, Fund. Math. 87 (1975), pp. 109–119.

[27] W. J. Thron, *Sequences generates by iteration*, Trans. Amer. Math. Soc. 96 (1960), pp. 38–53.

[28] C. S. Wong, *Fixed point theorems for point to set mappings*, Canad. Math. Bull. 17 (4) (1974).