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Induced contraction semigroups and random ergodic theorems

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CONTENTS

§ 1. Introduction

Let $(X, \mathfrak{B}, m)$ be a σ-finite measure space and let $\mathfrak{X}$ be a Banach space with the norm $|||\cdot|||$. By $L_p(m, \mathfrak{X})$, $1 \leqslant p < \infty$, we denote, as usual, the Banach space of all strongly $\mathfrak{B}$-measurable $\mathfrak{X}$-valued functions f defined on X with $|||f(x)|||^p$ integrable with respect to m and by $L_\infty(m, \mathfrak{X})$ the Banach space of all strongly $\mathfrak{B}$-measurable, $\mathfrak{X}$-valued, m-essentially bounded functions defined on X. Two functions in $L_p(m, \mathfrak{X})$ or $L_\infty(m, \mathfrak{X})$ which coincide with each other m-a.e. will be identified. If $\mathfrak{X}$ is the linear space $\boldsymbol{R}^N$ or $\boldsymbol{C}^N$ of all N-tuples of real or complex numbers, we shall adopt the notations $L_p(m)$ and $L_\infty(m)$ instead of $L_p(m, \mathfrak{X})$ and $L_\infty(m, \mathfrak{X})$, respectively.

Given a measure preserving transformation φ of X and a $(x\,y)$-measurable family $\{\psi_x\colon x \in X\}$ of measure preserving transformations of another σ-finite measure space $(Y, \mathfrak{F}, \mu)$, the so-called *skew product* T of φ and $\{\psi_x\colon x \in X\}$ is given by the formula $T(x, y) = (\varphi x, \psi_x y)$.

If we write

$$\psi_{(0,x)} = \text{identity}, \qquad \psi_{(n,x)} = \psi_{\varphi^{n-1}x} \cdots \psi_x \quad \text{for} \quad n \geqslant 1,$$

then $\{\psi_{(n,x)}\colon x \in X\}$ satisfies the relation

$$\psi_{(n+m,x)} = \psi_{(n,\varphi^m x)} \psi_{(m,x)} \quad \text{for} \quad n, m \geqslant 0. \tag{1.1}$$

In general, we call $\{\psi_{(n,x)}\colon x \in X\}$ satisfying (1.1) a *quasi semigroup associated with* φ.

Also, given a measurable semiflow $\{\varphi_t\colon t \geqslant 0\}$ on X and a (t, x, y)-measurable quasi semigroup $\{\psi_{(t,x)}\colon x \in X\}$ of measure preserving transformations of Y associated with $\{\varphi_t\}$, the skew product $\{T_t\colon t \geqslant 0\}$ of $\{\varphi_t\colon t \geqslant 0\}$ and $\{\psi_{(t,x)}\colon x \in X, t \geqslant 0\}$ is given by

$$T_t(x, y) = (\varphi_t x, \psi_{(t,x)} y).$$

For the skew products of dynamical systems, see, for example, E. Kin [15].

Every measurable semiflow $\{T_t\colon t \geqslant 0\}$ on $X \times Y$ has the continuity: for all $f \in L_p(m \times \mu, \mathfrak{X})$ with $1 \leqslant p < \infty$,

$$\lim_{t \to s} \|f \circ T_t - f \circ T_s\|_{L_p(m \times \mu, \mathfrak{X})} = 0.$$

Furthermore, we shall denote the contraction operators induced by measure preserving transformations by the same notations as the transformations where no confusion can arise.

The fundamental theorems of random ergodic theory, which emerged in the epoch 1942–56, were recognized as those pertaining to quasi semigroups of measure preserving transformations. Indeed, random ergodic theorems formulated by H. R. Pitt, S. Ulam, J. von Neumann and S. Kakutani were statements about the limiting behaviors of random operator averages for quasi semigroups of measure preserving transformations and have since extensively been studied by several investigators (cf. [5], [6], [13], [16], [17], [20], [21], [23]). Some further operator-theoretical generalizations were given by Beck and Schwartz [5] and Cairoli [6] in the discrete parameter case.

The purpose of the present paper is to give a consistent operator-theoretical treatment of random ergodic theorems.

It is now very natural to make use of ergodic theorems in dealing with random ergodic theorems. As skew product transformations become effective for quasi semigroups of measure preserving transformations, so induced contraction semigroups become effective for quasi semigroups of contraction operators.

In § 2 we introduce a notion of strongly measurable, a.e. strongly continuous quasi semigroups of contraction operators associated with a measurable semiflow, some examples of which are provided. This concept will take added interest in dealing with the limiting behaviors of random operator averages in a regular fashion. We also touch upon the infinitesimal generators of such quasi semigroups.

In § 3 we introduce a concept of induced contraction semigroups and show how to construct the induced semigroup from a given quasi semigroup. This semigroup is especially well-suited to handle one crucial step in the proofs of random ergodic theorems.

§ 4 is devoted to the discussions of discrete random ergodic theorems.

In a recent paper of the author [23], there were presented several random ergodic theorems with weighted averages in the one dimensional parameter case. We here generalize and extend some of them and some results due to Gładysz [13] and Beck and Schwartz [5]. We also give a generalization of the so-called "non-commuting" ergodic theorem due to Dunford and Schwartz [10].

Continuous random ergodic theorems for measurable semiflows were extensively studied by Kin ([16], [17]). We generalize these results to those at the operator theoretic level in the last § 5.

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§ 2. Contraction quasi semigroups associated with a semiflow

A notion of quasi semigroups of operators associated with a measurable semiflow introduced below plays an important role in the systematical study of the theory of ergodic theorems.

Two different σ-finite measure spaces $(X, \mathfrak{B}, m)$ and $(Y, \mathfrak{F}, \mu)$ are considered.

Let R_+^N denote the set of all N-tuples of non-negative real numbers and let $\{\varphi_t: t \in R_+^N\}$ be a measurable N-parameter semiflow on X. To each $(t, x) \in R_+^N \times X$ there corresponds a linear operator $U_{(t,x)}$ on $L_p(\mu, \mathfrak{X})$ with $\|U_{(t,x)}\|_{L_p(\mu,\mathfrak{X})} \leqslant 1$. We shall call $\{U_{(t,x)}: (t, x) \in R_+^N \times X\}$ a *contraction quasi semigroup* on $L_p(\mu, \mathfrak{X})$ associated with $\{\varphi_t: t \in R_+^N\}$ provided that for any $t, s \in R_+^N$,

$$U_{(0,x)} = \text{identity}, \qquad U_{(t+s,x)} = U_{(s,\varphi_t x)} U_{(t,x)} \ m\text{-a.e.} \tag{2.1}$$

A contraction quasi semigroup $\{U_{(t,x)}: (t, x) \in R_+^N \times X\}$ is said to be *strongly $\mathfrak{B}$-measurable* if for each $t \in R_+^N$, $U_{(t,\cdot)}$ is strongly $\mathfrak{B}$-measurable as an $L_p(\mu, \mathfrak{X})$-operator valued function defined on X, that is to say, for any $g \in L_p(\mu, \mathfrak{X})$, there are countably $L_p(\mu, \mathfrak{X})$-valued functions $h_n^{(t)}(x, \cdot)$ defined on X such that

$$\lim_{n\to\infty} \|h_n^{(t)}(x, \cdot) - (U_{(t,x)}g)(\cdot)\|_{L_p(\mu,\mathfrak{X})} = 0 \ \ m\text{-a.e.}$$

If for any $s \in R_+^N$ and all $g \in L_p(\mu, \mathfrak{X})$,

$$\lim_{t\to s} \|U_{(t,x)}g - U_{(s,x)}g\|_{L_p(\mu,\mathfrak{X})} = 0 \ \ m\text{-a.e.}, \tag{2.2}$$

then we say that $\{U_{(t,x)}: (t, x) \in R_+^N \times X\}$ is *m-a.e. strongly continuous.* If (2.2) holds for all x, it is said to be *strongly continuous.*

In the above definition, we must notice that the function $(U_{(t,x)}g)(y)$ is not necessarily measurable in (t, x, y). However, we shall demonstrate in the next section that the combination of the m-a.e. strong continuity and the strong $\mathfrak{B}$-measurability of quasi semigroups is a sufficient condition for choosing the functions $[U_{(t,x)}g](y)$ measurable in (t, x, y) from their equivalence classes.

Before going on to the construction of induced semigroups, we provide some examples of contraction quasi semigroups a.e. strongly continuous in t. It will be plain that all the arguments apply equally well in the case of multi-dimensional quasi semigroups, but we simplify, in the following examples, the notations by confining ourselves to the one parameter case to avoid complexity.

EXAMPLE 1. Take $X = Y = \boldsymbol{R}^1$ and define

$$a(t, x) = \int_0^t a(\varphi_u x)\, du \tag{2.3}$$

for a bounded positive continuous function $a(x)$ defined on X, where the semiflow $\{\varphi_t\colon t \geqslant 0\}$ is given by $\varphi_t x = x+t$. Then the function $a(t, x)$ is obviously measurable in (t, x) and possesses the quasi additivity:

$$a(t+s, x) = a(s, \varphi_t x)+a(t, x). \tag{2.4}$$

With the quasi semigroup $\{\psi_{(t,x)}\colon (t, x)\in R_+ \times X\}$ defined on Y by $\psi_{(t,x)}y = y+a(t, x)$, we put, for $g\in L_p(\mu, \mathfrak{X})$, $(U_{(t,x)}g)(y) = g(\psi_{(t,x)}y)$ which is measurable in (t, x, y). Then $\{U_{(t,x)}\colon (t, x)\in R_+ \times X\}$ becomes an m-a.e. strongly continuous contraction quasi semigroup on $L_p(\mu, \mathfrak{X})$ associated with $\{\varphi_t\colon t \geqslant 0\}$.

EXAMPLE 2. Let $Y = \boldsymbol{R}^1$ and let $a(t, x)$ be a function given by (2.3) in which $a(x)$ is a bounded positive measurable function defined on X. Except on a suitable set of m-measure zero, put

$$N_{(t,x)}(u) = \big(2\pi a(t, x)\big)^{-1/2}\exp\left(-\frac{u^2}{2a(t, x)}\right), \qquad -\infty < u < \infty,\ t > 0,$$

$$(U_{(t,x)}g)(y) = \begin{cases} \int_{-\infty}^{\infty} N_{(t,x)}(y-u)g(u)\,du & \text{for} \quad t > 0,\\ g(y) & \text{for} \quad t = 0,\end{cases}$$

for every $g\in L_1(\mu, \mathfrak{X})$. Then $\{U_{(t,x)}\colon (t, x)\in R_+ \times X\}$ defines an m-a.e. strongly continuous contraction quasi semigroup on $L_1(\mu, \mathfrak{X})$ associated with a measurable semiflow $\{\varphi_t\colon t \geqslant 0\}$ for which the function $a(t, x)$ is defined (see (2.3)).

In fact, it is clear that every $U_{(t,x)}$ is a linear contraction operator on $L_1(\mu, \mathfrak{X})$. To show the m-a.e. strong continuity, we first observe that for $g\in L_1(\mu, \mathfrak{X})$,

$$\begin{aligned} &(U_{(t,x)}g)(y)-g(y) \\ &\quad = \big(2\pi a(t, x)\big)^{-1/2}\int_{-\infty}^{\infty} \exp\left(-\frac{(y-u)^2}{2a(t, x)}\right)\{g(u)-g(y)\}\,du \\ &\quad = \pi^{-1/2}\int_{-\infty}^{\infty} \exp(-u^2)\big\{g\big(y-u\sqrt{2a(t, x)}\big)-g(y)\big\}\,du \quad m\text{-a.e.}\end{aligned} \tag{2.5}$$

so that, by Fubini's theorem,

$$\begin{aligned} &\|U_{(t,x)}g-g\|_{L_1(\mu,\mathfrak{X})} \\ &\quad \leqslant \pi^{-1/2}\int_{-\infty}^{\infty} \exp(-u^2)\Big\{\int_Y \big\|g\big(y-u\sqrt{2a(t, x)}\big)-g(y)\big\|\,d\mu(y)\Big\}\,du \quad m\text{-a.e.}\end{aligned} \tag{2.6}$$

A priori,

$$\int_Y \left\|\left|g\left(y-u\sqrt{2a(t,x)}\right)-g(y)\right|\right\| d\mu(y) \leqslant 2\,\|g\|_{L_1(\mu,\mathfrak{X})}.$$

Now it is worth to note that the exceptional sets (for which (2.5) and (2.6) do not hold) may depend upon t, but in fact (2.5) and (2.6) hold for every x. Hence, applying the Lebesgue's dominated convergence theorem, we get

$$\lim_{t\to 0+} \|U_{(t,x)}g-g\|_{L_1(\mu,\mathfrak{X})} = 0$$

which is enough to show (2.2). In proving the quasi semigroup property we need the following equation

$$N_{(t+s,x)}(u) = \int_{-\infty}^{\infty} N_{(s,\varphi_t x)}(u-v)\,N_{(t,x)}(v)\,dv \tag{2.7}$$

from which (2.1) follows after an easy computation. The Fourier transform of the left-hand side of (2.7) is equal to

$$\begin{aligned}
&\hat{F}_{(t,s,x)}(\beta)\\
&= (2\pi)^{-1/2}\int_{-\infty}^{\infty} N_{(t+s,x)}(u)\exp(-i\beta u)\,du\\
&= \left(4\pi^2 a(t+s,x)\right)^{-1/2}\int_{-\infty}^{\infty}\exp\left\{-\left(i\beta u+\frac{u^2}{2a(t+s,x)}\right)\right\}du\\
&= \left(4\pi^2 a(t+s,x)\right)^{-1/2}\exp\left(-\frac{\beta^2 a(t+s,x)}{2}\right)\int_{-\infty}^{\infty}\exp\left\{-\frac{\left(u+i\beta a(t+s,x)\right)^2}{2a(t+s,x)}\right\}du\\
&= \left(4\pi^2 a(t+s,x)\right)^{-1/2}\exp\left(-\frac{\beta^2 a(t+s,x)}{2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{u^2}{2a(t+s,x)}\right)du\\
&= (2\pi)^{-1/2}\exp\left(-\frac{\beta^2 a(t+s,x)}{2}\right).
\end{aligned}$$

On the other hand, since the Fourier transform of a convolution equals to the product of Fourier transforms after omitting the normalizing multipliers and then using the above calculation of the Fourier transform of $N_{(t,x)}$ to apply the quasi additivity (2.4) of the function $a(t,x)$, the Fourier transform of the right-hand side of (2.7) is equal to

$$\begin{aligned}
\hat{G}_{(t,s,x)}(\beta) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\exp(-i\beta u)\left\{\int_{-\infty}^{\infty} N_{(s,\varphi_t x)}(u-v)N_{(t,x)}(v)\,dv\right\}du\\
&= (2\pi)^{-1/2}\exp\left(-\frac{\beta^2 a(t+s,x)}{2}\right).
\end{aligned}$$

Therefore,

$$\hat{F}_{(t,s,x)}(\beta) = \hat{G}_{(t,s,x)}(\beta)$$

which leads us to the equation (2.7).

EXAMPLE 3. Let X, Y and $a(t,x)$ be as in Example 1, and define for $\lambda > 0$ and $\beta > 0$,

$$(U_{(t,x)}g)(y) = \exp\big(-\lambda a(t,x)\big) \sum_{k=0}^{\infty} \frac{\big(\lambda a(t,x)\big)^k}{k!} g(y-k\beta)$$

for $g \in L_1(\mu, \mathfrak{X})$. Then $\{U_{(t,x)}: (t,x) \in R_+ \times X\}$ turns out to be an m-a.e. strongly continuous contraction quasi semigroup on $L_1(\mu, \mathfrak{X})$ associated with $\{\varphi_t: t \geqslant 0\}$. The m-a.e. strong continuity follows from the following inequality:

$$\|U_{(t,x)}g - g\|_{L_1(\mu,\mathfrak{X})} \leqslant \exp\big(-\lambda a(t,x)\big) \sum_{k=0}^{\infty} \frac{\big(\lambda a(t,x)\big)^k}{k!} \int_Y |||g(y-k\beta) - g(y)|||\,d\mu(y).$$

As for the quasi semigroup property, we have

$$\begin{aligned}
&(U_{(s,\varphi_t x)} U_{(t,x)} g)(y)\\
&= \exp\{-\lambda\big(a(s,\varphi_t x) + a(t,x)\big)\} \sum_{k=0}^{\infty} \frac{\big(\lambda a(s,\varphi_t x)\big)^k}{k!} \sum_{j=0}^{\infty} \frac{\big(\lambda a(t,x)\big)^j}{j!} g\big(y-(k+j)\beta\big)\\
&= \exp\big(-\lambda a(t+s,x)\big) \sum_{p=0}^{\infty} \frac{1}{p!}\Big[p! \sum_{j=0}^{p} \frac{\big(\lambda a(t,x)\big)^j}{j!} \frac{\big(\lambda a(s,\varphi_t x)\big)^{p-j}}{(p-j)!}\Big] g(y-p\beta)\\
&= (U_{(t+s,x)}g)(y).
\end{aligned}$$

EXAMPLE 4. With X, Y and $a(t,x)$ given in Example 1, put for $t > 0$,

$$\eta_{(t,x)}(u) = \{a(t,x)\}^{-2} \eta\big(u\{a(t,x)\}^{-2}\big),$$

where

$$\eta(u) = \begin{cases} \{2\pi^{1/2} u^{3/2} \exp(1/4u)\}^{-1} & \text{for} \quad u > 0,\\ 0 & \text{for} \quad u = 0. \end{cases}$$

Then $\{\eta_{(t,x)}\}$ has the following properties:

(i) $\int_{-\infty}^{\infty} \eta_{(t,x)}(u)\,du = 1$ for $t > 0$,

(ii) $\eta_{(t+s,x)} = \eta_{(s,\varphi_t x)} * \eta_{(t,x)}$ for $t, s > 0$,

(iii) $\lim_{t\to 0+} \int_s^{\infty} \eta_{(t,x)}(u)\,du = 0$ for $s > 0$.

Here $\eta_1 * \eta_2$ stands for the convolution of η_1 and η_2 (see Dunford and Schwartz [10], Lemma 12, p. 160, Terrell [22], Lemma 2.1, p. 269).

Now for a strongly continuous contraction semigroup $\{U_t\colon t \in R_+^2\}$ on $L_1(\mu, \mathfrak{X})$ with two dimensional parameter and for any $g \in L_1(\mu, \mathfrak{X})$, let

$$(U_{(t,x)}g)(y) = \begin{cases} \int_0^\infty \int_0^\infty \eta_{(t,x)}(u_1)\eta_{(t,x)}(u_2)(U_{(u_1,u_2)}g)(y)\,du_1 du_2 & \text{for} \quad t > 0, \\ g(y) \quad \text{for} \quad t = 0. \end{cases}$$

Then $\{U_{(t,x)}\colon (t, x) \in R_+ \times X\}$ defines an m-a.e. strongly continuous contraction quasi semigroup on $L_1(\mu, \mathfrak{X})$ associated with $\{\varphi_t\}$.

Example 4 is useful in producing new quasi semigroups from a given semigroup.

Now it is an interesting problem to study the infinitesimal generators of contraction quasi semigroups. Let $\{U_{(t,x)}\colon (t, x) \in R_+ \times X\}$ be an m-a.e. strongly continuous quasi semigroup of bounded linear operators in a B-space $\boldsymbol{B}$.

With the exception of an appropriate m-null set, we define the (*strong*) *infinitesimal generator* A_x of $\{U_{(t,x)}\colon t \in R_+\}$ by

$$A_x g = \boldsymbol{B}\text{-}\lim_{t \to 0+} \frac{(U_{(t,x)} - I)g}{t}, \tag{2.8}$$

that is to say, A_x is the linear operator whose domain is the set

$$D(A_x) = \{g \in \boldsymbol{B}\colon (2.8) \text{ exists}\},$$

which is non-empty since it contains at least the zero vector (cf. Yosida [25], IX).

If in Example 1, we take $X = Y = R_+$ and replace $L_p(\mu, \mathfrak{X})$ by $C([0, \infty], \mathfrak{X})$ being the space of bounded uniformly continuous $\mathfrak{X}$-valued functions defined on $[0, \infty)$, then a simple calculation shows that the strong infinitesimal generator A_x of $\{U_{(t,x)}\colon t \in R_+\}$ is given by the formula

$$(A_x g)(y) = a(x)g'(y)$$

for $g \in C([0, \infty], \mathfrak{X})$ with its first derivative $g' \in C([0, \infty], \mathfrak{X})$.

If in Example 3, $L_1(\mu, \mathfrak{X})$ is replaced by $C([-\infty, \infty], \mathfrak{X})$, the strong infinitesimal generator A_x of $\{U_{(t,x)}\colon t \in R_+\}$ is given by

$$(A_x g)(y) = \lambda a(x)\{g(y-\beta) - g(y)\} \quad \text{for} \quad g \in C([-\infty, \infty], \mathfrak{X}).$$

§ 3. Induced contraction semigroups

An induced contraction semigroup constructed below is of service to the study of random ergodic theorems, and the role to be played by this concept will be potent in the later sections. From now on, let $\{\varphi_n: n \in Z_+^N\}$ be a discrete semiflow on X, where Z_+^N denotes the set of all N-tuples of non-negative integers, $\{\psi_{(n,x)}: (n, x) \in Z_+^N \times X\}$ a (x, y)-measurable quasi semigroup of measure preserving transformations of Y associated with $\{\varphi_n\}$ and $\{T_n: n \in Z_+^N\}$ the skew product of $\{\varphi_n\}$ and $\{\psi_{(n,x)}\}$. Also, let $\{\varphi_t: t \in R_+^N\}$ be a measurable semiflow on X, $\{\psi_{(t,x)}: (t, x) \in R_+^N \times X\}$ a (t, x, y)-measurable, μ-measure preserving quasi semigroup on Y associated with $\{\varphi_t\}$ and $\{T_t: t \in R_+^N\}$ the skew product of $\{\varphi_t\}$ and $\{\psi_{(t,x)}\}$.

Unless otherwise noted, we write $\varphi = \varphi_1$ and $T = T_1$ in the case of $N = 1$.

When we want to regard $f(x, y)$ as a function defined on Y for an x arbitrarily fixed in X, we shall write $f_x(y)$ for $f(x, y)$ in what follows.

THEOREM 1. *Suppose that m is finite and μ is σ-finite. Let $\{U_{(n,x)}: (n, x) \in Z_+^N \times X\}$ be a strongly $\mathfrak{B}$-measurable discrete contraction quasi semigroup on $L_p(\mu, \mathfrak{X})$ $(1 \leqslant p < \infty)$ associated with $\{\varphi_n\}$. Then there exists a discrete contraction semigroup $\{U_n^*: n \in Z_+^N\}$ on $L_p(m \times \mu, \mathfrak{X})$ such that for any $f \in L_p(m \times \mu, \mathfrak{X})$, there is a set $E(f)$ of m-measure zero such that for any $x \in X - E(f)$,*

$$(U_n^* f)_x(y) = \big(U_{(n,x)}(T_n f)_x\big)(y), \qquad n \in Z_+^N,$$

almost everywhere on Y.

We shall call $\{U_n^*: n \in Z_+^N\}$ the *discrete contraction semigroup induced on $L_p(m \times \mu, \mathfrak{X})$ by $\{U_{(n,x)}: (n, x) \in Z_+^N \times X\}$* (more briefly, *induced (discrete) contraction semigroup*).

The proof of Theorem 1 will be accomplished in a series of the following lemmas.

LEMMA 1. *Let $f \in L_p(m \times \mu, \mathfrak{X})$ with $1 \leqslant p < \infty$ and $n \in Z_+^N$. Then the function $\big(U_{(n,x)}(T_n f)_x\big)(\cdot)$ is strongly $\mathfrak{B}$-measurable as an $L_p(\mu, \mathfrak{X})$-valued function defined on X.*

Proof. Let us denote by $\mathfrak{A}_p$ the set of functions f of the form:

$$f(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \chi_{A_i \times B_j}(x, y),$$

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |||a_{ij}|||^p m(A_i) \mu(B_j) < \infty,$$

$$a_{ij} \in \mathfrak{X}, \quad i, j = 1, 2, \ldots,$$

where $\{A_i\}$ and $\{B_j\}$ are measurable partitions of X and Y, respectively and $\chi_E(\cdot)$ stands for the indicator function of the set E. It follows then that

$$L_p(m \times \mu, \mathfrak{X}) = \text{the closed linear hull of } \mathfrak{A}_p.$$

If the function $f(x, y)$ is of the form $a \cdot \chi_{A \times B}(x, y)$ with $\mu(B) < \infty$, then

$$\big(U_{(n,x)}(a \cdot \chi_{A \times B})_x\big)(\cdot) = \chi_A(x)\big(U_{(n,x)}(a \cdot \chi_B)\big)(\cdot). \tag{3.1}$$

Since $\{U_{(n,x)}\}$ is strongly $\mathfrak{B}$-measurable and since $\chi_A(\cdot)$ is a finite numerically valued function which is measurable, the function (3.1) is strongly $\mathfrak{B}$-measurable (Hille and Phillips [14], Theorem 3.5.4).

Now for any $f \in L_p(m \times \mu, \mathfrak{X})$, we can choose a sequence $\{g_k^{(n)}\}$ of $\mathfrak{A}_p$-functions such that

$$\lim_{k \to \infty} \|g_k^{(n)} - T_n f\|_{L_p(m \times \mu, \mathfrak{X})} = 0.$$

Thus, finding an appropriate subsequence $\{g_{k'}^{(n)}\}$ of $\{g_k^{(n)}\}$, we get

$$\lim_{k' \to \infty} \|(g_{k'}^{(n)})_x(\cdot) - (T_n f)_x(\cdot)\|_{L_p(\mu, \mathfrak{X})} = 0 \quad m\text{-a.e.}$$

and so

$$\lim_{k' \to \infty} \left\|\big(U_{(n,x)}(g_{k'}^{(n)})_x\big)(\cdot) - \big(U_{(n,x)}(T_n f)_x\big)(\cdot)\right\|_{L_p(\mu, \mathfrak{X})} = 0 \quad m\text{-a.e.} \tag{3.2}$$

However, since every function $\big(U_{(n,x)}(g_{k'}^{(n)})_x\big)(\cdot)$ is strongly $\mathfrak{B}$-measurable, there is a countably $L_p(\mu, \mathfrak{X})$-valued $\mathfrak{B}$-measurable function $(h_{k'}^{(n)})_x(\cdot)$ satisfying

$$\lim_{k' \to \infty} \left\|(h_{k'}^{(n)})_x(\cdot) - \big(U_{(n,x)}(g_{k'}^{(n)})_x\big)(\cdot)\right\|_{L_p(\mu, \mathfrak{X})} = 0 \quad m\text{-a.e.} \tag{3.3}$$

Hence from (3.2) and (3.3), we have

$$\lim_{k' \to \infty} \left\|(h_{k'}^{(n)})_x(\cdot) - \big(U_{(n,x)}(T_n f)_x\big)(\cdot)\right\|_{L_p(\mu, \mathfrak{X})} = 0 \quad m\text{-a.e.},$$

which proves the lemma.

A function f in $L_p(m \times \mu, \mathfrak{X})$ is regarded as a representative of an equivalence class $\tilde{f}$. And then, for almost all x, $U_{(n,x)}(T_n f)_x$ is a representative of an equivalence class $U_{(n,x)}(T_n \tilde{f})_x$. As mentioned in § 2, a function $\big(U_{(n,x)}(T_n f)_x\big)(y)$ may not be measurable with respect to (x, y). But, as will be shown below, we can choose a function $[U_{(n,\cdot)}(T_n f).](\cdot)$ from the equivalence class $U_{(n,x)}(T_n \tilde{f})_x$ in such a way that the function $[U_{(n,x)}(T_n f)_x(y)]$ is measurable in (x, y).

Lemma 2. *For any $f \in L_p(m \times \mu, \mathfrak{X})$ with $1 \leq p < \infty$ and $n \in Z_+^N$, there exists a (x, y)-measurable version $[U_{(n,\cdot)}(T_n f).](\cdot)$ of the function $\big(U_{(n,x)}(T_n f)_x\big)(y)$ such that*

$$\left\|[U_{(n,x)}(T_n f)_x](\cdot) - \big(U_{(n,x)}(T_n f)_x\big)(\cdot)\right\|_{L_p(\mu, \mathfrak{X})} = 0 \quad m\text{-a.e.}$$

Such a version $[U_{(n,\cdot)}(T_nf).](\cdot)$ *is uniquely determined up to sets of* $m\times\mu$-*measure zero.*

Proof. For the function $\big(U_{(n,x)}(T_nf)_x\big)(\cdot)$, there are countably $L_p(\mu,\mathfrak{X})$-valued $\mathfrak{B}$-measurable functions $(g_k^{(n)})_x(\cdot)$ which satisfy

$$\lim_{k\to\infty}\left\|(g_k^{(n)})_x(\cdot)-\{U_{(n,x)}(T_nf)_x\}(\cdot)\right\|_{L_p(\mu,\mathfrak{X})}=0 \tag{3.4}$$

uniformly on $X-E(f)$, where $E(f)$ is a set of m-measure zero, since $\big(U_{(n,x)}(T_nf)_x\big)(\cdot)$ is strongly $\mathfrak{B}$-measurable in compliance with Lemma 1 (Hille and Phillips [14], III, 3.5). Thus

$$\lim_{i,j\to\infty}\|(g_i^{(n)})_x(\cdot)-(g_j^{(n)})_x(\cdot)\|_{L_p(\mu,\mathfrak{X})}=0$$

uniformly on $X-E(f)$. Noticing that every $g_k^{(n)}(x,y)$ is $\mathfrak{B}\times\mathfrak{F}$-measurable and considering the finiteness of the measure m, one obtains

$$\lim_{i,j\to\infty}\|g_i^{(n)}-g_j^{(n)}\|_{L_p(m\times\mu,\mathfrak{X})}=0,$$

so that there exist a function $g_f^{(n)}\in L_p(m\times\mu,\mathfrak{X})$ and a subsequence $\{g_{k'}^{(n)}\}$ of $\{g_k^{(n)}\}$ satisfying

$$\lim_{k'\to\infty}\|(g_{k'}^{(n)})_x(\cdot)-(g_f^{(n)})_x(\cdot)\|_{L_p(\mu,\mathfrak{X})}=0\quad m\text{-a.e.} \tag{3.5}$$

Therefore, it results from (3.4) and (3.5) that

$$\left\|(g_f^{(n)})_x(\cdot)-\big(U_{(n,x)}(T_nf)_x\big)(\cdot)\right\|_{L_p(\mu,\mathfrak{X})}=0\quad m\text{-a.e.}$$

It is now facile to verify that such a function $g_f^{(n)}$ is uniquely determined except for an $m\times\mu$-null set. Finally, taking the function $[U_{(n,x)}(T_nf)_x]\ (y)$ to be $g_f^{(n)}(x,y)$, we have the desired one. The proof of Lemma 2 is hereby completed.

With the function $g_f^{(n)}$ chosen in Lemma 2, we define a mapping U_n^* from $L_p(m\times\mu,\mathfrak{X})$ to itself as follows:

$$U_n^*f=g_f^{(n)}\ (=[U_{(n,\cdot)}(T_nf).])\qquad n\in Z_+^N.$$

Next, starting with the function U_n^*f, we see that $U_m^*U_n^*f=g_{U_n^*f}^{(m)}$ for $m\in Z_+^N$. Moreover, as is easily seen, it holds that if we write $L_n(f)=g_f^{(n)}$ then in $L_p(m\times\mu,\mathfrak{X})$

$$\begin{aligned}L_{m+n}(f)&=L_m(U_n^*f)\quad(=L_n(U_m^*f)),\\ L_m(c_1f_1+c_2f_2)&=c_1L_m(f_1)+c_2L_m(f_2).\end{aligned} \tag{3.6}$$

Lemma 3. $\{U_n^*:\ n\in Z_+^N\}$ *is a discrete contraction semigroup on* $L_p(m\times\mu,\mathfrak{X})$. *Furthermore,*

$$\|U_n^*\|_{L_\infty(m\times\mu,\mathfrak{X})}\leqslant 1\qquad\textit{whenever}\qquad\|U_{(n,x)}\|_{L_\infty(\mu,\mathfrak{X})}\leqslant 1.$$

Proof. Let $f \in L_p(m \times \mu, \mathfrak{X})$. Then by virtue of Lemma 2,

$$\|(U_n^* f)_x(\cdot) - \big(U_{(n,x)}(T_n f)_x\big)(\cdot)\|_{L_p(\mu,\mathfrak{X})} = 0 \quad m\text{-a.e.},$$

$$(3.7) \qquad \begin{aligned} &\|(U_m^* U_n^* f)_x(\cdot) - \big(U_{(m,x)}(T_m U_n^* f)_x\big)(\cdot)\|_{L_p(\mu,\mathfrak{X})} \\ &\quad = \|(U_m^* U_n^* f)_x(\cdot) - \big(U_{(m,x)} U_{(n,\varphi_m x)}(T_{m+n} f)_x\big)(\cdot)\|_{L_p(\mu,\mathfrak{X})} = 0 \quad m\text{-a.e.}, \\ &\|(U_{m+n} f)_x(\cdot) - \big(U_{(m+n,x)}(T_{m+n} f)_x\big)(\cdot)\|_{L_p(\mu,\mathfrak{X})} = 0 \quad m\text{-a.e.}, \end{aligned}$$

wherefore, in view of (2.1) and (3.7),

$$\|(U_{m+n}^* f)_x(\cdot) - (U_m^* U_n^* f)_x(\cdot)\|_{L_p(\mu,\mathfrak{X})} = 0 \quad m\text{-a.e.}$$

Consequently, considering that $g_f^{(m+n)}(x, y)$ and $g_{U_n^* f}^{(m)}(x, y)$ are $\mathfrak{B} \times \mathfrak{F}$-measurable, we get, in $L_p(m \times \mu, \mathfrak{X})$,

$$U_{m+n}^* f = U_m^* U_n^* f, \qquad U_0^* f = f,$$

which implies that $\{U_n^*: n \in Z_+^N\}$ is a semigroup. It is apparent from (3.6) that for each $n \in Z_+^N$, U_n^* is additive and homogeneous. Lastly, the contraction properties of U_n^*

$$\|U_n^* f\|_{L_p(m\times\mu,\mathfrak{X})} \leqslant \|f\|_{L_p(m\times\mu,\mathfrak{X})} \quad \text{for} \quad f \in L_p(m \times \mu, \mathfrak{X}),$$

$$\|U_n^* f\|_{L_\infty(m\times\mu,\mathfrak{X})} \leqslant \|f\|_{L_\infty(m\times\mu,\mathfrak{X})} \quad \text{for} \quad f \in L_p(m \times \mu, \mathfrak{X}) \cap L_\infty(m \times \mu, \mathfrak{X}),$$

are implied by the following inequalities:

$$\|(U_n^* f)_x(\cdot)\|_{L_p(\mu,\mathfrak{X})} \leqslant \|(T_n f)_x(\cdot)\|_{L_p(\mu,\mathfrak{X})} \quad m\text{-a.e.}$$

$$\|(U_n^* f)_x(\cdot)\|_{L_\infty(\mu,\mathfrak{X})} \leqslant \|(T_n f)_x(\cdot)\|_{L_\infty(\mu,\mathfrak{X})} \quad m\text{-a.e.}$$

The proof of Lemma 3 is now complete. We have herewith carried out the proof of Theorem 1.

THEOREM 2. *Suppose the measures m and μ are finite. Let $\{U_{(t,x)}: (t, x) \in R_+^N \times X\}$ be a strongly $\mathfrak{B}$-measurable, m-a.e. strongly continuous, contraction quasi semigroup on $L_p(\mu, \mathfrak{X})$ $(1 \leqslant p < \infty)$ associated with $\{\varphi_t\}$. Then there exists a strongly continuous semigroup $\{U_t^*: t \in R_+^N\}$ on $L_p(m \times \mu, \mathfrak{X})$ such that for any $f \in L_p(m \times \mu, \mathfrak{X})$ and any $t \in R_+^N$, there is an m-null set $E(f, t)$ such that for all $x \in X - E(f, t)$,*

$$(U_t^* f)_x(y) = \big(U_{(t,x)}(T_t f)_x\big)(y)$$

almost everywhere on Y.

We shall say that $\{U_t^*: t \in R_+^N\}$ is the (*continuous*) *contraction semigroup induced on $L_p(m \times \mu, \mathfrak{X})$ by $\{U_{(t,x)}: (t, x) \in R_+^N \times X\}$* (more shortly, *induced contraction semigroup*). Now the same argument as that in the discrete case is valid for the continuous parameter case.

Let $f \in L_p(m \times \mu, \mathfrak{X})$ and let t be arbitrarily fixed in R_+^N. In view of Lemma 1, $\big(U_{(t,x)}(T_t f)_x\big)(\cdot)$ is strongly $\mathfrak{B}$-measurable as an $L_p(\mu, \mathfrak{X})$-valued function defined on X. According to Lemma 2, there exists a (x, y)-measurable version $\langle U_{(t,\cdot)}(T_t f)_\cdot\rangle(\cdot)$ of the function $\big(U_{(t,x)}(T_t f)_x\big)(y)$ to be uniquely determined except for an $m\times\mu$-null set, such that

$$\big\|\langle U_{(t,x)}(T_t f)_x\rangle(\cdot)-\big(U_{(t,x)}(T_t f)_x\big)(\cdot)\big\|_{L_p(\mu,\mathfrak{X})}=0 \quad m\text{-a.e.}$$

We define a mapping U_t^* from $L_p(m\times\mu,\mathfrak{X})$ to itself by

$$U_t^* f = g_f^{(t)} \quad (=\langle U_{(t,\cdot)}(T_t f)_\cdot\rangle),$$

$g_f^{(t)}$ being the function of $L_p(m\times\mu,\mathfrak{X})$ chosen by Lemma 2. If we start with U_t^*, then $U_s^* U_t^* f = g^{(s)}_{U_t^* f}$ for $s\in R_+^N$. In addition,

$$L_{t+s}(f)=L_s(U_t^* f)(=L_t(U_s^* f)),\qquad L_t(c_1f_1+c_2f_2)=c_1L_t(f_1)+c_2L_t(f_2),$$

in $L_p(m\times\mu,\mathfrak{X})$ by taking $L_t(f)$ to be $g_f^{(t)}$. Lemma 3 guarantees that $\{U_t^*:\ t\in R_+^N\}$ is a contraction semigroup on $L_p(m\times\mu,\mathfrak{X})$ such that

$$\|U_t^*\|_{L_\infty(m\times\mu,\mathfrak{X})}\leqslant 1 \qquad \text{whenever} \qquad \|U_{(t,x)}\|_{L_\infty(\mu,\mathfrak{X})}\leqslant 1.$$

In order to complete the proof of Theorem 2, we must show the strong continuity of $\{U_t^*:\ t\in R_+^N\}$. To do this, the following lemmas are needed.

LEMMA 4. *For any $f\in L_p(m\times\mu,\mathfrak{X})$ and $s\in R_+^N$, it holds that*

$$\lim_{n\to\infty}\big\|\big(U_{(t_n,x)}(T_{t_n}f)_x\big)(\cdot)-\big(U_{(s,x)}(T_s f)_x\big)(\cdot)\big\|_{L_p(\mu,\mathfrak{X})}=0 \quad m\text{-a.e.}$$

for a sequence $\{t_n\}$ with $t_n\to s$ $(n\to\infty)$ for which

$$\lim_{n\to\infty}\|(T_{t_n}f)_x(\cdot)-(T_s f)_x(\cdot)\|_{L_p(\mu,\mathfrak{X})}=0 \quad m\text{-a.e.} \tag{3.8}$$

Proof. There is a sequence $\{t_n\}$ with $t_n\to s$ $(n\to\infty)$ such that (3.8) holds. Then the m-a.e. strong continuity of $\{U_{(t,x)}\}$ shows that

$$\lim_{n\to\infty}\big\|\big(U_{(t_n,x)}(T_s f)_x\big)(\cdot)-\big(U_{(s,x)}(T_s f)_x\big)(\cdot)\big\|_{L_p(\mu,\mathfrak{X})}=0 \quad m\text{-a.e.} \tag{3.9}$$

Using the inequality

$$\begin{aligned}&\big\|\big(U_{(t_n,x)}(T_{t_n}f)_x\big)(\cdot)-\big(U_{(s,x)}(T_s f)_x\big)(\cdot)\big\|_{L_p(\mu,\mathfrak{X})}\\ &\quad\leqslant \|(T_{t_n}f)_x(\cdot)-(T_s f)_x(\cdot)\|_{L_p(\mu,\mathfrak{X})}+\\ &\qquad\qquad +\big\|\big(U_{(t_n,x)}(T_s f)_x\big)(\cdot)-\big(U_{(s,x)}(T_s f)_x\big)(\cdot)\big\|_{L_p(\mu,\mathfrak{X})},\end{aligned}$$

the desired conclusion follows immediately from (3.8) and (3.9).

LEMMA 5. *$\{U_t^*:\ t\in R_+^N\}$ is strongly measurable in t.*

Proof. Let $f\in L_\infty(m\times\mu,\mathfrak{X})$ and $t=(t_1,\ldots,t_N)\in R_+^N$. Write

$$\sigma_n(t)=\left(\frac{[nt_1]}{n},\ldots,\frac{[nt_N]}{n}\right),$$

where $[nt_i]$ means the intergal part of the number nt_i. Clearly, $U^*_{\sigma_n(\cdot)}f$ is a countably $L_\infty(m\times\mu,\mathfrak{X})$-valued function defined on R^N_+. By Lemma 4, it holds that

$$\lim_{n'\to\infty}\left\|\left(U_{(\sigma_{n'}(t),x)}(T_{\sigma_{n'}(t)}f)_x\right)(\cdot)-\left(U_{(t,x)}(T_tf)_x\right)(\cdot)\right\|_{L_p(\mu,\mathfrak{X})}=0$$

m-almost everywhere on X, for a subsequence $\{\sigma_{n'}(t)\}$ of $\{\sigma_n(t)\}$ such that

$$\lim_{n'\to\infty}\|(T_{\sigma_{n'}(t)}f)_x(\cdot)-(T_tf)_x(\cdot)\|_{L_p(\mu,\mathfrak{X})}=0\ \ m\text{-a.e.}$$

On account of Lemma 2,

$$\|(U^*_{\sigma_{n'}(t)}f)_x(\cdot)-(U^*_tf)_x(\cdot)\|_{L_p(\mu,\mathfrak{X})}$$
$$\leqslant\left\|\left(U_{(\sigma_{n'}(t),x)}(T_{\sigma_{n'}(t)}f)_x\right)(\cdot)-\left(U_{(t,x)}(T_tf)_x\right)(\cdot)\right\|_{L_p(\mu,\mathfrak{X})}\ m\text{-a.e.},$$

whereupon,

$$\lim_{n'\to\infty}\|(U^*_{\sigma_{n'}(t)}f)_x(\cdot)-(U^*_tf)_x(\cdot)\|_{L_p(\mu,\mathfrak{X})}=0\ \ m\text{-a.e.}$$

But since μ is finite and

$$\left\|\|(U^*_{\sigma_{n'}(t)}f)_x(\cdot)-(U^*_tf)_x(\cdot)\|_{L_p(\mu,\mathfrak{X})}\right\|_{L_\infty(\mu,\mathfrak{X})}\leqslant 2\cdot\mu(Y)\cdot\|f\|_{L_\infty(m\times\mu,\mathfrak{X})},$$

it follows from the Lebesgue's dominated convergence theorem that

$$\lim_{n'\to\infty}\|U^*_{\sigma_{n'}(t)}f-U^*_tf\|_{L_p(m\times\mu,\mathfrak{X})}=0.$$

For any $f\in L_p(m\times\mu,\mathfrak{X})$, there are functions $f_n\in L_\infty(m\times\mu,\mathfrak{X})$ satisfying

$$\lim_{n\to\infty}\|f_n-f\|_{L_p(m\times\mu,\mathfrak{X})}=0,$$

$$\lim_{k\to\infty}\|U^*_{\sigma_k^{(n)}(t)}f_n-U^*_tf_n\|_{L_p(m\times\mu,\mathfrak{X})}=0,\qquad n\geqslant 1,$$

for suitable sequences $\{\sigma_k^{(n)}(t)\}$, $n\geqslant 1$, with

$$\{\sigma_k(t)\}\supset\{\sigma_k^{(1)}(t)\}\supset\{\sigma_k^{(2)}(t)\}\supset\ldots$$

Thus, finding a common subsequence $\{\sigma_{k''}(t)\}$ of $\{\sigma_k^{(n)}(t)\}$, $n\geqslant 1$, by making use of the diagonal procedure, we have

$$\lim_{k''\to\infty}\|U^*_{\sigma_{k''}(t)}f_n-U^*_tf_n\|_{L_p(m\times\mu,\mathfrak{X})}=0,\qquad n\geqslant 1.$$

However,

$$\text{(3.10)}\qquad \|U^*_{\sigma_{k''}(t)}f-U^*_tf\|_{L_p(m\times\mu,\mathfrak{X})}\leqslant\|U^*_{\sigma_{k''}(t)}(f_n-f)\|_{L_p(m\times\mu,\mathfrak{X})}+$$
$$+\|U^*_{\sigma_{k''}(t)}f_n-U^*_tf_n\|_{L_p(m\times\mu,\mathfrak{X})}+\|U^*_t(f_n-f)\|_{L_p(m\times\mu,\mathfrak{X})}$$
$$\leqslant 2\cdot\|f_n-f\|_{L_p(m\times\mu,\mathfrak{X})}+\|U^*_{\sigma_{k''}(t)}f_n-U^*_tf_n\|_{L_p(m\times\mu,\mathfrak{X})}.$$

Therefore, letting first k'' and then n tend to infinity in (3.10), we obtain

$$\lim_{k''\to\infty} \|U^*_{\sigma_{k''}(t)} f - U^*_t f\|_{L_p(m\times\mu,\mathfrak{X})} = 0$$

as was to be shown.

LEMMA 6. *The strong measurability of $\{U^*_t\colon t\in R^N_+\}$ implies its strong continuity on t.*

For the proof of this lemma, see Dunford and Schwartz [11], VIII, Lemma 1.3.

We have hereby completed the proof of Theorem 2.

Now for almost all x, $\big(U_{(t,x)}(T_t f)_x\big)(\cdot)$ is the representative of an equivalence class $U_{(t,x)}(T_t\tilde{f})_x$ whenever f is the representative of an equivalence class $\tilde{f}$. Once more, we note that the function $\langle U_{(t,x)}(T_t f)_x\rangle(y)$ may not be measurable with respect to (t, x, y). But we can choose a function $[U_{(\cdot,\cdot)}(T_\cdot f)_\cdot](\cdot)$ from the equivalence class $\langle U_{(t,\cdot)}(T_t\tilde{f})_\cdot\rangle$ in such a way that the function $[U_{(t,x)}(T_t f)_x](y)$ is measurable in (t, x, y). In fact, for the function U^*_t, we can find a (t, x, y)-measurable function $[U^*_\cdot f](\cdot,\cdot)$ from the equivalence class $U^*_t\tilde{f}$, and such a function is uniquely determined up to sets of $dt\times m\times\mu$-measure zero (Dunford [9], Dunford and Schwartz [10], Ornstein [19]). Therefore, one may obtain the desired one by taking $[U_{(t,x)}(T_t f)_x](y)$ to be $[U^*_t f](x, y)$. Summing up the above, we have

THEOREM 3. *Let $\{U_{(t,x)}\colon (t, x)\in R^N_+\times X\}$ be as in Theorem 2. Then for every $f\in L_p(m\times\mu,\mathfrak{X})$ with $1\leqslant p<\infty$, there exists a (t, x, y)-measurable version $[U_{(\cdot,\cdot)}(T_\cdot f)_\cdot](\cdot)$ of the function $\big(U_{(t,x)}(T_t f)_x\big)(y)$, and such a version is uniquely determined except for a set in $R^N_+\times X\times Y$ with $dt\times m\times\mu$-measure zero.*

Note: The induced contraction semigroup $\{U^*_t\colon t\in R^N_+\}$ given in Theorem 2 may be understood as the skew product of $\{\varphi_t\}$ and $\{U_{(t,x)}\}$.

§ 4. Discrete random ergodic theorems

In this section we shall present a general treatment of the discrete random ergodic theorems including the results obtained by Gładysz [13] and the author [23].

We consider in the sequel a finite measure space $(X, \mathfrak{B}, m)$ and a σ-finite measure space $(Y, \mathfrak{F}, \mu)$. Let $\{w_k\colon k\geqslant 1\}$ be a sequence of non-negative numbers whose sum is one and let $\{u_k\colon k\geqslant 0\}$ be the sequence defined by

$$u_0 = 1,\qquad u_k = w_1 u_{k-1} + \ldots + w_k u_0,\qquad k\geqslant 1.$$

Let $\{\beta_k\colon k\geqslant 1\}$ be the sequence defined by $\beta_1 = w_1$ $(= u_1)$ and, for

$k \geqslant 2$, by

$$\beta_k = \begin{cases} w_k/(1-w_1-\ldots-w_{k-1}) & \text{if} \quad w_1+\ldots+w_{k-1}<1, \\ 0 & \text{if} \quad w_1+\ldots+w_{k-1}=1. \end{cases}$$

Then it is clear that $0 \leqslant u_k \leqslant 1$, $0 \leqslant \beta_k \leqslant 1$ for every k.

We consider a measure space $(N, \mathscr{A}, \lambda)$ obtained by taking N to be the positive integers and $\mathscr{A}$ the σ-algebra of all subsets of N and λ the measure given by

$$\lambda(\{1\}) = 1, \quad \lambda(\{k\}) = 1-w_1-\ldots-w_{k-1}, \quad k \geqslant 2.$$

Let W be the linear operator on $L_1(\lambda)$ such that

$$W\delta_1 = \sum_{k=1}^{\infty} \beta_k \cdot \delta_k, \qquad W\delta_k = (1-\beta_{k-1})\cdot \delta_{k-1}, \qquad k \geqslant 2,$$

where $\delta_k(i)$ denotes the Kronecker delta. Then W is a positive linear contraction on $L_1(\lambda)$ as well as on $L_\infty(\lambda)$ and $(W^k\delta_1)(1) = u_k$ for all $k \geqslant 0$.

THEOREM 4. *Let $\mathfrak{X}$ be reflexive and let $\{U_{(n,x)}: (n, x) \in Z_+ \times X\}$ be a strongly $\mathfrak{B}$-measurable contraction quasi semigroup on $L_1(\mu, \mathfrak{X})$ associated with φ such that*

$$\|U_{(n,x)}\xi\|_{L_\infty(\mu,\mathfrak{X})} \leqslant \|\xi\|_{L_\infty(\mu,\mathfrak{X})} \quad \textit{for} \quad \xi \in L_1(\mu, \mathfrak{X}) \cap L_\infty(\mu, \mathfrak{X}).$$

If $f \in L_p(m \times \mu, \mathfrak{X})$ with $1 \leqslant p < \infty$ and g is a complex valued $\mathfrak{B} \times \mathfrak{F}$-measurable function defined on $X \times Y$ with $|g(x, y)| \leqslant 1$, then there exist an m-null set E and a function $f^* \in L_p(m \times \mu, \mathfrak{X})$ such that for any $x \in X - E$, the strong limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} u_k \cdot \{g_x U_{(1,x)}(T_1 f)_x\}^{*k}(y) = f_x^*(y) \tag{4.1}$$

almost everywhere on Y, and if $1 < p < \infty$,

$$\lim_{n\to\infty} \left\| f_x^*(\cdot) - \frac{1}{n} \sum_{k=1}^{n} u_k \cdot \{g_x U_{(1,x)}(T_1 f)_x\}^{*k}(\cdot) \right\|_{L_p(\mu,\mathfrak{X})} = 0, \tag{4.2}$$

and if μ is finite and $f(x, y) \equiv h(y)$ for $h \in L_1(\mu, \mathfrak{X})$,

$$\lim_{n\to\infty} \left\| f_x^*(\cdot) - \frac{1}{n} \sum_{k=1}^{n} u_k \cdot \{g_x U_{(1,x)}(T_1 f)_x\}^{*k}(\ ) \right\|_{L_1(\mu,\mathfrak{X})} = 0. \tag{4.3}$$

Here

$$\{g_x U_{(1,x)}(T_1 f)_x\}^{*k} = \{g_x U_{(1,x)}\{(T_1 g)_x U_{(1,\varphi_1 x)}\{\ldots\{(T_{k-1} g)_x U_{(1,\varphi_{k-1} x)}(T_k f)_x\}\ldots\}.$$

Proof. According to Theorem 1, there exists the contraction semigroup $\{U_n^*\colon n \in Z_+\}$ induced on $L_1(m\times\mu, \mathfrak{X})$ by $\{U_{(n,x)}\colon (n,x)\in Z_+\times X\}$ such that for every $f\in L_1(m\times\mu,\mathfrak{X})$,

$$\left\|(U_n^* f)_x(\cdot) - \left(U_{(n,x)}(T_n f)_x\right)(\cdot)\right\|_{L_1(\mu,\mathfrak{X})} = 0 \quad m\text{-a.e.}$$

and such that by Lemma 3,

$$\|U_n^* f\|_{L_\infty(m\times\mu,\mathfrak{X})} \leqslant \|f\|_{L_\infty(m\times\mu,\mathfrak{X})} \quad \text{for} \quad f\in L_1(m\times\mu,\mathfrak{X})\cap L_\infty(m\times\mu,\mathfrak{X}).$$

For each $n\in Z_+$, we have

$$\|\{gU_1^* f\}_x^{*n}(\cdot) - \{g_x U_{(1,x)}(T_1 f)_x\}^{*n}(\cdot)\|_{L_1(\mu,\mathfrak{X})} = 0 \;\; m\text{-a.e.}, \tag{4.4}$$

where

$$\{gU_1^* f\}^{*n} = \overbrace{\{gU_1^*\{gU_1^*\{\ldots\{gU_1^* f\}}^{n}\ldots\}.$$

Let us now define, for $f\in L_1(m\times\mu,\mathfrak{X})$ and $n\in Z_+$,

$$(V_n f)(x,y) = \{gU_1^* f\}^{*n}(x,y).$$

Then each V_n is evidently additive and homogeneous as a mapping of $L_1(m\times\mu,\mathfrak{X})$ to itself, and

$$V_n V_m f = \{gU_1^*(V_m f)\}^{*n} = \{gU_1^* f\}^{*(n+m)} = V_{n+m},$$

$$\|V_n f\|_{L_1(m\times\mu,\mathfrak{X})} = \|\{gU_1^* f\}^{*n}\|_{L_1(m\times\mu,\mathfrak{X})} \leqslant \|f\|_{L_1(m\times\mu,\mathfrak{X})} \quad \text{for} \quad f\in L_1(m\times\mu,\mathfrak{X}),$$

$$\|V_n f\|_{L_\infty(m\times\mu,\mathfrak{X})} = \|\{gU_1^* f\}^{*n}\|_{L_\infty(m\times\mu,\mathfrak{X})} \leqslant \|f\|_{L_\infty(m\times\mu,\mathfrak{X})}$$
$$\text{for} \quad f\in L_1(m\times\mu,\mathfrak{X})\cap L_\infty(m\times\mu,\mathfrak{X}).$$

Accordingly, $\{V_n\colon n\in Z_+\}$ turns out to be a contraction semigroup on $L_1(m\times\mu,\mathfrak{X})$ as well as on $L_\infty(m\times\mu,\mathfrak{X})$. Thus we may apply Theorem 1 of the author [23] with $\{V_n\colon n\in Z_+\}$, to conclude that for any $f\in L_p(m\times\mu,\mathfrak{X})$, $1\leqslant p<\infty$, there is a function $f^*\in L_p(m\times\mu,\mathfrak{X})$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} u_k\cdot\{gU_1^* f\}^{*k}(x,y) = f^*(x,y) \tag{4.5}$$

strongly in $\mathfrak{X}$ almost everywhere on $X\times Y$, and if $1<p<\infty$,

$$\lim_{n\to\infty}\left\|f^* - \frac{1}{n}\sum_{k=1}^{n} u_k\cdot\{gU_1^* f\}^{*k}\right\|_{L_p(m\times\mu,\mathfrak{X})} = 0, \tag{4.6}$$

and if μ is finite,

$$\lim_{n\to\infty}\left\|f^* - \frac{1}{n}\sum_{k=1}^{n} u_k\cdot\{gU_1^* f\}^{*k}\right\|_{L_1(m\times\mu,\mathfrak{X})} = 0. \tag{4.7}$$

Going into details, (4.5) is implied by Theorem 1 of Chacon [7] applied to the direct product $\{\tau_k\colon k\in Z_+\}$ of $\{W^k\colon k\in Z_+\}$ and $\{V_k\colon k\in Z_+\}$ which is a linear contraction semigroup on $L_1(\lambda\times m\times\mu,\mathfrak{X})$ as well as on $L_\infty(\lambda\times m\times\mu,\mathfrak{X})$. (4.6) can be deduced from (4.5) and from the existence of an L_p-function which dominates all the functions

$$\left|\left|\left|\frac{1}{n}\sum_{k=1}^{n}(\tau_k\xi)(i,x,y)\right|\right|\right|,$$

where $\xi(i,x,y)=\delta_1(i)f(x,y)$. To show (4.7), we suppose that μ is finite. If so, the convergence in the mean of order 2 implies convergence in the mean of order 1 for all functions in the dense subset $L_2(m\times\mu,\mathfrak{X})$ of $L_1(m\times\mu,\mathfrak{X})$. Now let $f\in L_1(m\times\mu,\mathfrak{X})$ and write

$$E_a=\left\{(x,y)\colon \frac{1}{a}\leqslant |||f(x,y)|||\leqslant a\right\},$$

$$f_a(x,y)=(f\cdot\chi_{E_a})(x,y),\qquad a=1,2,\ldots$$

Then for each $a\geqslant 1$,

$$\iint\limits_{X\times Y}|||f_a(x,y)|||^2dm(x)d\mu(y)=\iint\limits_{E_a}|||f(x,y)|||^2dm(x)d\mu(y)$$
$$\leqslant a^2\cdot m\times\mu(E_a)<\infty,$$

that is, the functions f_a belong to $L_2(m\times\mu,\mathfrak{X})$ and possess the following properties:

(i) $|||f_a(x,y)|||\leqslant|||f_{a+1}(x,y)|||\leqslant|||f(x,y)|||,\ a\geqslant 1,$

(ii) $\lim\limits_{a\to\infty}|||f_a(x,y)-f(x,y)|||=0\ \ m\times\mu$-a.e.

From (i) and (ii) follows

(iii) $\lim\limits_{a\to\infty}\|f_a-f\|_{L_1(m\times\mu,\mathfrak{X})}=0.$

If for each $a\geqslant 1$, we write

$$\xi_{(n,a)}(i,x,y)=\frac{1}{n}\sum_{k=1}^{n}(\tau_k\xi_a)(i,x,y), \tag{4.8}$$

where $\xi_a(i,x,y)=\delta_1(i)f_a(x,y)$ which is in $L_1(\lambda\times m\times\mu,\mathfrak{X})$, we have

$$\xi_{(n,a)}(1,x,y)=\frac{1}{n}\sum_{k=1}^{n}u_k\cdot\{gU_1^*f_a\}^{*k}(x,y). \tag{4.9}$$

In view of (4.5), (4.8) and (4.9), there exists a function $\xi_a^*\in L_1(\lambda\times m\times\mu,\mathfrak{X})$ such that

$$\lim_{n\to\infty}\xi_{(n,a)}(1,x,y)=\xi_a^*(1,x,y) \tag{4.10}$$

strongly in $\mathfrak{X}$ almost everywhere on $X \times Y$. Noticing that for each $a \geqslant 1$,

$$\|(\xi_{(n,a)})_1\|_{L_\infty(m\times\mu,\mathfrak{X})} \leqslant a,$$
$$\|(\xi_a^*)_1\|_{L_\infty(m\times\mu,\mathfrak{X})} \leqslant a,$$

it ensues from (4.10) and Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \|(\xi_{(n,a)})_1 - (\xi_a^*)_1\|_{L_1(m\times\mu,\mathfrak{X})} = 0. \tag{4.11}$$

However, since

$$\|(\xi_a^*)_1 - (\xi_b^*)_1\|_{L_1(m\times\mu,\mathfrak{X})} \leqslant \|(\xi_a^*)_1 - (\xi_{(n,a)})_1\|_{L_1(m\times\mu,\mathfrak{X})} + \\ + \|(\xi_{(n,b)})_1 - (\xi_b^*)_1\|_{L_1(m\times\mu,\mathfrak{X})} + \|f_a - f_b\|_{L_1(m\times\mu,\mathfrak{X})},$$

(4.11) and (iii) show that $\{(\xi_a^*)_1\}$ is a Cauchy sequence, so that there is a function $f_* \in L_1(m \times \mu, \mathfrak{X})$ with

$$\lim_{a\to\infty} \|f_* - (\xi_a^*)_1\|_{L_1(m\times\mu,\mathfrak{X})} = 0. \tag{4.12}$$

Now

$$\left\|f_* - \frac{1}{n}\sum_{k=1}^{n} u_k\cdot\{gU_1^*f\}^{*k}\right\|_{L_1(m\times\mu,\mathfrak{X})} \\ \leqslant \|f_* - (\xi_a^*)_1\|_{L_1(m\times\mu,\mathfrak{X})} + \|(\xi_a^*)_1 - (\xi_{(n,a)})_1\|_{L_1(m\times\mu,\mathfrak{X})} + \|f_a - f\|_{L_1(m\times\mu,\mathfrak{X})},$$

and therefore, letting first n and then a tend to infinity in the above inequality, the combination of (4.11), (4.12) and (iii) establishes the fact that

$$\lim_{n\to\infty} \left\|f_* - \frac{1}{n}\sum_{k=1}^{n} u_k\cdot\{gU_1^*f\}^{*k}\right\|_{L_1(m\times\mu,\mathfrak{X})} = 0. \tag{4.13}$$

From what (4.5) and (4.13) say, we infer that

$$f^*(x, y) = f_*(x, y) \quad m\times\mu\text{-a.e.} \tag{4.14}$$

and hence, inserting (4.14) into (4.13), we get (4.7). Now (4.1) follows immediately from (4.5) by Fubini's theorem. Similarly to (4.6), (4.2) can be deduced from (4.1), (4.6) and the existence of a dominating function. Finally, we proceed to the proof of (4.3). If the measure μ is finite and if we take $f(x, y) = h(y)$ for $h \in L_1(\mu, \mathfrak{X})$, then

$$\int_Y \left|\left|\left| f_x^*(y) - \frac{1}{n}\sum_{k=1}^{n} u_k\cdot\{gU_1^*f\}_x^{*k}(y)\right|\right|\right| d\mu(y) \\ \leqslant \int_Y |||f_x^*(y)|||\, d\mu(y) + \frac{1}{n}\sum_{k=1}^{n}\int_Y |||(T_kf)_x(y)|||\, d\mu(y)$$

$$= \int_Y |||f_x^*(y)|||\,d\mu(y) + \frac{1}{n}\sum_{k=1}^{n} \int_Y |||h(\psi_{(k,x)}y)|||\,d\mu(y)$$
$$= \|f_x^*\|_{L_1(\mu,\mathfrak{X})} + \|h\|_{L_1(\mu,\mathfrak{X})},$$

which is plainly in $L_1(m)$. Consequently, using this and applying Lebesgue's dominated convergence theorem together with (4.4) and (4.7), we have (4.3) for almost all $x \in X$ and complete the proof of Theorem 4.

The method of proof of Theorem 4 used is reminiscent of that in the author's previous paper [23], and especially, the proof of the mean convergence of order 1 is given for the first time in this paper.

Applications of Theorem 4 yield several variant forms including the known results as the special cases of it. For example, the following corollary as a generalization of Gładysz's theorem ([13], Satz 5) is obtained by taking $U_{(1,x)}$ in Theorem 4 to be the identity for all x.

COROLLARY 1. *Let $\mathfrak{X}$ be reflexive and let ψ be a measure preserving transformation of Y. If $f \in L_p(\mu, \mathfrak{X})$, $1 \leqslant p < \infty$, and g is a complex valued $\mathfrak{F}$-measurable function with $|g(y)| \leqslant 1$, then there is a function $f^* \in L_p(\mu, \mathfrak{X})$ such that the strong limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} u_k g(y) \ldots g(\psi^{k-1}y) f(\psi^k y) = f^*(y)$$

holds with the exception of a μ-null set. Moreover, the function f^* is the limit in the mean of order p with $1 < p < \infty$, and if μ is finite then f^* is also the limit in the mean of order 1.

If $g(x, y) = 1$ for all (x, y), Theorem 4 entails

COROLLARY 2 (cf. Yoshimoto [23], Theorem 7). *Let $\mathfrak{X}$ be reflexive and let $\{U_{(n,x)}\colon (n, x) \in Z_+ \times X\}$ be a strongly $\mathfrak{B}$-measurable contraction quasi semigroup on $L_1(\mu, \mathfrak{X})$ as well as on $L_\infty(\mu, \mathfrak{X})$, associated with φ. Then for any $f \in L_p(m \times \mu, \mathfrak{X})$ with $1 \leqslant p < \infty$, there is a function $f^* \in L_p(m \times \mu, \mathfrak{X})$ such that excepting an m-null set,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} u_k \big(U_{(k,x)}(T_k f)_x\big)(y) = f_x^*(y)$$

strongly in $\mathfrak{X}$ almost everywhere on Y. Except on a set of m-measure zero, f_x^* is the limit in the $L_p(\mu, \mathfrak{X})$-norm with $1 < p < \infty$, and if μ is finite and $f(x, y) \equiv h(y)$ for $h \in L_1(\mu, \mathfrak{X})$, f_x^* is the limit in the $L_1(\mu, \mathfrak{X})$-norm.

Next we shall be concerned with generalizations of the above consideration of weights with one parameter to that of multi-parameter ones.

As in the introduction of this section, we consider sequences

$\{w_k^{(j)}\colon k \geqslant 1\}$, $1 \leqslant j \leqslant N$, of non-negative numbers for which

$$\sum_{k=1}^{\infty} w_k^{(j)} = 1, \quad 1 \leqslant j \leqslant N,$$

and the corresponding sequences $\{u_k^{(j)}\colon k \geqslant 0\}$, $\{\beta_k^{(j)}\colon k \geqslant 1\}$, $1 \leqslant j \leqslant N$. For each j, let λ_j be the measure defined on $\mathscr{A}$ with the corresponding sequence $\{w_k^{(j)}\colon k \geqslant 1\}$ and the linear contraction operator $W_{(j)}$ on $L_1(\lambda_j)$ (as well as on $L_\infty(\lambda_j)$) be defined as before. Set

$$\begin{aligned} a(k_1, \ldots, k_N) &= u_{k_1}^{(1)} \times \ldots \times u_{k_N}^{(N)}, \\ b(k_1, \ldots, k_N) &= \exp\{i\beta(k_1 + \ldots + k_N)\} \ (\beta \text{ real}), \\ c(k_1, \ldots, k_N) &= a(k_1, \ldots, k_N)\cdot b(k_1, \ldots, k_N). \end{aligned}$$

The following theorem is an N-parameter extension of the Beck and Schwartz's random ergodic theorem ([5], Theorem 2, p. 1049). (Cf. Yoshimoto [23], Corollary 2, p. 152, to which Theorem 4 reduces for the special case that $g \equiv 1$, $U_{(0,x)} =$ identity and $U_{(k,x)} = U_x U_{\varphi x} \ldots U_{\varphi^{k-1}x}$, $k \geqslant 1$.)

THEOREM 5. *Let $\mathfrak{X}$ be reflexive and let there be given a strongly $\mathfrak{F}$-measurable quasi semigroup $\{U_{(k,y)}\colon (k, y) \in Z_+^N \times Y\}$ associated with a discrete semiflow $\{\psi_k\colon k \in Z_+^N\}$ on Y, such that every $U_{(k,y)}$ belongs to the Banach space $B(\mathfrak{X})$ of bounded linear operators acting on $\mathfrak{X}$. Suppose that $|||U_{(k,y)}||| \leqslant 1$ for all $(k, y) \in Z_+^N \times Y$. Then for any $f \in L_p(\mu, \mathfrak{X})$, $1 \leqslant p < \infty$, the functions*

$$\frac{1}{n^N} \sum_{k_1=0}^{n-1} \cdots \sum_{k_N=0}^{n-1} b(k_1, \ldots, k_N)\, U_{(k_1,\ldots,k_N,y)}\big(f(\psi_{(k_1,\ldots,k_N)}y)\big)$$

are convergent (as $n \to \infty$) strongly in $\mathfrak{X}$ almost everywhere on Y, as well as in the norm of $L_p(\mu, \mathfrak{X})$ with $1 < p < \infty$. Furthermore, if μ is finite, the limit also exists in the norm of $L_1(\mu, \mathfrak{X})$.

Proof. Let us define, for $f \in L_1(\mu, \mathfrak{X})$,

$$(U^*_{(k_1,\ldots,k_N)} f)(y) = b(k_1, \ldots, k_N)\, U_{(k_1,\ldots,k_N,y)}\big(f(\psi_{(k_1,\ldots,k_N)}y)\big).$$

It is quite clear that $\{U_k^*\colon k \in Z_+^N\}$ is a linear contraction semigroup on $L_1(\mu, \mathfrak{X})$ as well as on $L_\infty(\mu, \mathfrak{X})$. In addition, we observe that for all $f \in L_p(\mu, \mathfrak{X})$ with $1 \leqslant p < \infty$, the set where

$$\sup_{n \geqslant 1} \left|\left|\left| \frac{1}{n^N} \sum_{k_1=0}^{n-1} \cdots \sum_{k_N=0}^{n-1} (U^*_{(k_1,\ldots,k_N)} f)(y) \right|\right|\right| > a > 0$$

has measure which tends to zero as $a \to \infty$. In fact, this results from an ergodic lemma ([10], Lemma 16, p. 166) of Dunford and Schwartz. Therefore, we may get the convergence in the mean of order p with $1 < p < \infty$ and the convergence almost everywhere by the same way as that used by Dunford and Schwartz [10] with the aid of Banach's convergence theorem to be stated below. The proof of the $L_1(\mu, \mathfrak{X})$-mean convergence follows exactly the same line as that in the proof of Theorem 4. The proof of Theorem 5 is herewith completed.

Banach's convergence theorem (which will also be cited later) can be stated as follows.

THEOREM 6. *Let $A_1 \supset A_2 \supset \ldots$ be denumerable sets and for each $a \in A_1$, let U_a be a continuous linear map from a B-space $\mathfrak{G}$ into the space $L_p(\mu, \mathfrak{X})$. Suppose that for each $g \in \mathfrak{G}$, we have*

(i) $\sup_{a \in A_1} |||(U_a g)(y)||| < \infty$ μ-a.e.,

and that for each g in a set dense in $\mathfrak{G}$,

(ii) $\lim_{p\to\infty} \sup_{a,b \in A_p} |||(U_a g)(y) - (U_b g)(y)||| = 0$ μ-a.e.

Then the equation (ii) *is valid for every $g \in \mathfrak{G}$.*

Note. The original form of this theorem stated by Banach for the special case of real valued functions can easily be extended to the vector valued case. In particular, the above form stated by Dunford and Schwartz [10] for the complex valued functions is also verbatim valid for the vector valued case. For the proof, the reader is referred to Dunford and Schwartz [11].

For the case $N = 1$, Theorem 5 is extendable with a more general system of weights, $\{c(k)\}$ instead of $\{b(k)\}$.

Let

$$Y_r^* = Y \times \ldots \times Y, \quad \mathfrak{F}_r^* = \mathfrak{F} \times \ldots \times \mathfrak{F}, \quad \mu_r^* = \mu \times \ldots \times \mu \quad (r \text{ times})$$
$$\Psi^* = \Psi \times \Psi \times \ldots, \qquad \Sigma^* = \Sigma \times \Sigma \times \ldots,$$

where Ψ is the set of all measure preserving transformations on Y and Σ is the σ-algebra of all subsets of Ψ. Let P^* be a probability measure defined on Σ^* by the requirement that the coordinate sequence $\{\eta_n(\psi^*)\}$ ($\eta_n(\psi^*) = \psi_n$ if $\psi^* = \{\psi_n\}$) should be a (strongly) stationary process. This requirement is, of course, equivalent to the fact that the measure P^* is preserved by the one-sided shift transformation σ on Ψ^*.

Consider the skew product S of σ and $\{\xi_{\psi^*}\colon \psi^* \in \Psi^*\}$ obtained by the direct product $\xi_{\psi^*} = \psi_1 \times \ldots \times \psi_r$, where $\psi^* = (\psi_1, \psi_2, \ldots)$.

THEOREM 7. *Let $\mathfrak{X}$ be reflexive and let there be defined on Y_r^* a strongly $\mathfrak{F}_r^*$-measurable function U with values in $B(\mathfrak{X})$. Suppose that $|||U(y_r^*)||| \leq 1$ for all $y_r^* \in Y_r^*$ and that the family $\{\xi_{\psi^*}\colon \psi^* \in \Psi^*\}$ is $\Sigma^* \times \mathfrak{F}_r^*$-measurable. Then for all $f \in L_p(\mu_r^*, \mathfrak{X})$, $1 \leq p < \infty$, there is a set E^* of P^*-measure zero such that for any $\psi^* \in \Psi^* - E^*$, there exists a function $f_{\psi^*}^* \in L_p(\mu_r^*, \mathfrak{X})$*

such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} c(k)\,U(\xi_{\psi^*}y_r^*)\ldots U(\xi_{\sigma^k\psi^*}\ldots\xi_{\psi^*}y_r^*)f(\xi_{\sigma^{k+1}\psi^*}\ldots\xi_{\psi^*}y_r^*) = f^*_{\psi^*}(y_r^*)$$

strongly in $\mathfrak{X}$, for all y_r^* of Y_r^* with the exception of a μ_r^*-null set. Excepting the set E^*, the function $f^*_{\psi^*}$ is the limit in the norm of $L_p(\mu_r^*, \mathfrak{X})$ with $1 < p < \infty$, and if μ is finite, this function is also the limit in the norm of $L_1(\mu_r^*, \mathfrak{X})$.

Proof. If for all $\omega \in \Psi^* \times Y_r^*$, we write $U_{(0,\omega)} =$ identity and

$$U_{(k,\omega)} = U\big(\pi(S\omega)\big)\,U\big(\pi(S^2\omega)\big)\ldots U\big(\pi(S^k\omega)\big), \quad k \geqslant 1,$$

where π denotes the projection $\Psi^* \times Y_r^* \to Y_r^*$, we have a strongly $\Sigma^* \times \mathfrak{F}_r^*$-measurable quasi semigroup $\{U_{(n,\psi^*,y_r^*)}: (n, \psi^*, y_r^*) \in Z_+ \times \Psi^* \times Y_r^*\}$ in $B(\mathfrak{X})$ associated with S, such that

$$|||U_{(n,\psi^*,y_r^*)}||| \leqslant 1 \quad \text{for all } (n, \psi^*, y_r^*) \in Z_+ \times \Psi^* \times Y_r^*.$$

Let us denote the induced contraction semigroup of $\{U_{(n,\psi^*,y_r^*)}\}$ by $\{U_n^*: n \in Z_+\}$ (which exists by Theorem 1) and consider the contraction semigroup $\{U_n^{**}: n \in Z_+\}$ obtained by setting $U_n^{**} = b(n)U_n^*$ for $n \in Z_+$. Then, using the function $F = f\circ\pi$ for $f \in L_p(\mu_r^*, \mathfrak{X})$, all the conclusions of the theorem can be deduced from Theorem 1 of the author's previous paper [23] (or from Theorem 4 of the present paper).

Theorem 7 is another extension of Beck and Schwartz's theorem [5].

For notational convenience, we denote

$$(4.15)\quad \big(D(n_1, \ldots, n_N)f\big)_x(\cdot) = \frac{1}{n_1\ldots n_N}\sum_{k_1=0}^{n_1-1}\cdots\sum_{k_N=0}^{n_N-1} c(k_1, \ldots, k_N)\big(U_{(k_1,\ldots,k_N,x)}(T_{(k_1,\ldots,k_N)}f)_x\big)(\cdot)$$

for $n_1 \geqslant 1, \ldots, n_N \geqslant 1$ and $f \in L_p(m\times\mu)$.

THEOREM 8. *Let $\{U_{(n,x)}: (n, x) \in Z_+^N \times X\}$ be a strongly $\mathfrak{B}$-measurable contraction quasi semigroup on $L_1(\mu)$ associated with $\{\varphi_n: n \in Z_+^N\}$, such that*

$$\|U_{(n,x)}h\|_{L_\infty(\mu)} \leqslant \|h\|_{L_\infty(\mu)} \quad \textit{for} \quad h \in L_1(\mu)\cap L_\infty(\mu).$$

(i) *Let $p > 1$ and $f \in L_p(m\times\mu)$. Then, excepting a set of m-measure zero, the functions* (4.15) *are convergent (as $n_1, \ldots, n_N \to \infty$ independently) almost everywhere on Y, as well as in the norm of $L_p(\mu)$.*

(ii) *Let $p = 1$ and $f \in L_1(m\times\mu)$. Suppose that μ is finite and that f satisfies the following*

$$(4.16)\quad \iint_{X\times Y} |f(x, y)|\log^+|f(x, y)|\,dm(x)\,d\mu(y) < \infty,$$

where the symbol $\log^+ a$ *is defined for* $a > 0$ *to be the larger of* $\log a$ *and* 0. *Then, except for an m-null set, the functions* (4.15) *converge to a function in* $L_1(\mu)$ *almost everywhere on* Y *as* $n_1, \ldots, n_N \to \infty$ *independently, and this limit also exists in the norm of* $L_1(\mu)$.

Proof. We denote the induced contraction semigroup of the given quasi semigroup by $\{U_n^*: n \in Z_+^N\}$ (which exists by Theorem 1) and define *the semigroup* $\{U_n^{**}: n \in Z_+^N\}$ by $U_n^{**} = b(n) U_n^*$ for $n \in Z_+^N$. If we take $\{W_n^*: n \in Z_+^N\}$ to be the direct product of $\{W_{(1)}^k: k \in Z_+\}, \ldots, \{W_{(N)}^k: k \in Z_+\}$, we have

$$(W^*_{(k_1,\ldots,k_N)} \delta_{(1,\ldots,1)})(1, \ldots, 1) = a(k_1, \ldots, k_N)$$

for all $(k_1, \ldots, k_N) \in Z_+^N$, where

$$\delta_{(1,\ldots,1)}(i_1, \ldots, i_N) = \delta_1(i_1) \times \ldots \times \delta_1(i_N).$$

Let us now consider the direct product $\{\tau_n: n \in Z_+^N\}$ of $\{W_n^*: n \in Z_+^N\}$ and $\{U_n^{**}: n \in Z_+^N\}$; then $\{\tau_n: n \in Z_+^N\}$ is norm-contracting on $L_1(\lambda_1 \times \ldots \times \lambda_N \times m \times \mu)$ as well as on $L_\infty(\lambda_1 \times \ldots \times \lambda_N \times m \times \mu)$. For each $n_1, \ldots, n_N \geqslant 1$ and each $g \in L_p(\lambda_1 \times \ldots \times \lambda_N \times m \times \mu)$, consider the average

$$\frac{1}{n_1 \ldots n_N} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_N=0}^{n_N-1} \tau_{(k_1,\ldots,k_N)} g. \tag{4.17}$$

Applying Dunford and Schwartz's theorem ([10], Theorem 9, p. 146) to $\{\tau_n: n \in Z_+^N\}$, we have, for $g \in L_p(\lambda_1 \times \ldots \times \lambda_N \times m \times \mu)$ with $1 < p < \infty$, the almost everywhere convergence and the L_p-mean convergence of the functions (4.17) as $n_1, \ldots, n_N \to \infty$ independently. Furthermore, the functions (4.17) are, for $n_1 \geqslant 1, \ldots, n_N \geqslant 1$, all dominated by a function in $L_p(\lambda_1 \times \ldots \times \lambda_N \times m \times \mu)$. Therefore, using the above facts with the function $\delta_{(1,\ldots,1)} f$ $\big(f \in L_p(m \times \mu)\big)$, we get the first assertion (i) by Fubini's theorem.

To prove the assertion (ii), we make use of the following

LEMMA 7. *Let* U *be a bounded linear operator on* $L_1(m \times \mu)$ *whose* $L_\infty(m \times \mu)$ *norm is also finite. Then there is a positive linear operator* Q *on* $L_1(m \times \mu)$ *whose* $L_1(m \times \mu)$ *and* $L_\infty(m \times \mu)$ *norms do not exceed those of* U *and which is such that*

$$|(U^n f)(\cdot)| \leqslant Q^n\big(|f(\cdot)|\big), \quad n \geqslant 1,\ f \in L_1(m \times \mu).$$

For this result, see Dunford and Schwartz [10], Lemma 4, p. 140.

LEMMA 8. *Let* U *be a linear contraction operator on* $L_1(m \times \mu)$ *as well as on* $L_\infty(m \times \mu)$. *For each* $f \in L_p(m \times \mu)$, *let*

$$f^*(x, y) = \sup_{n \geqslant 1} \left| \frac{1}{n} \sum_{k=0}^{n-1} (U^k f)(x, y) \right|.$$

Then, if $1 < p < \infty$, *we have*

$$\iint\limits_{X\times Y} \big(f^*(x, y)\big)^p dm(x)\, d\mu(y) \leqslant \frac{2^p p}{p-1} \iint\limits_{X\times Y} |f(x, y)|^p dm(x)\, d\mu(y),$$

while if $p = 1$ *(and* $m\times\mu(X\times Y) < \infty$*) we have*

$$\iint\limits_{X\times Y} f^*(x, y)\, dm(x)\, d\mu(y)$$
$$= 2\Big[m\times\mu(X\times Y) + \iint\limits_{X\times Y} |f(x, y)| \log^+ |f(x, y)|\, dm(x)\, d\mu(y)\Big].$$

For this result, see also Dunford and Schwartz [10], Theorem 8, p. 145.

Let

$$V_{k_i}^{(i)} = U^*_{(0,\ldots,0,k_i,0,\ldots,0)}, \qquad k_i \in Z_+, \qquad 1 \leqslant i \leqslant N.$$

Then

$$U^*_{(k_1,\ldots,k_N)} f = V_{k_N}^{(N)} \ldots V_{k_1}^{(1)} f, \qquad (k_1, \ldots, k_N) \in Z_+^N.$$

By Lemma 7 there are positive linear operators $Q^{(i)}$ $(1 \leqslant i \leqslant N)$ on $L_1(m\times\mu)$ whose $L_1(m\times\mu)$ and $L_\infty(m\times\mu)$ norms do not exceed one and which are such that

$$|(V_k^{(i)} f)(\cdot)| \leqslant Q_k^{(i)}\big(|f(\cdot)|\big), \qquad k \in Z_+, \qquad 1 \leqslant i \leqslant N,$$

where $Q_k^{(i)}$ is the kth iterations of $Q^{(i)}$. Moreover, by Lemma 8, there are functions $f_1, \ldots, f_N$ in $L_1(m\times\mu)$ with

$$A(Q^{(1)}, n_1)|f| \leqslant f_1, \qquad n_1 \geqslant 1,$$
$$A(Q^{(2)}, n_2) A(Q^{(1)}, n_1)|f| \leqslant f_2, \qquad n_1, n_2 \geqslant 1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A(Q^{(N)}, n_N) \ldots A(Q^{(1)}, n_1)|f| \leqslant f_N, \qquad n_1, \ldots, n_N \geqslant 1,$$

where

$$A(Q^{(i)}, n_i)|f| = \frac{1}{n_i} \sum_{k=0}^{n_i-1} Q_k^{(i)} |f|, \qquad 1 \leqslant i \leqslant N.$$

Thus

$$\Big|\frac{1}{n_1 \ldots n_N} \sum_{k_1=0}^{n_1-1} \ldots \sum_{k_N=0}^{n_N-1} a(k_1, \ldots, k_N)\, U^{**}_{(k_1,\ldots,k_N)} f\Big|$$
$$\leqslant \frac{1}{n_1 \ldots n_N} \sum_{k_1=0}^{n_1-1} \ldots \sum_{k_N=0}^{n_N-1} |U^*_{(k_1,\ldots,k_N)} f|$$
$$= \frac{1}{n_1 \ldots n_N} \sum_{k_1=0}^{n_1-1} \ldots \sum_{k_N=0}^{n_N-1} |V_{k_N}^{(N)} \ldots V_{k_1}^{(1)} f|$$
$$\leqslant A(Q^{(N)}, n_N) \ldots A(Q^{(1)}, n_1)|f| \leqslant f_N, \qquad n_1, \ldots, n_N \geqslant 1,$$

which shows that the multiple sequence

$$\frac{1}{n_1 \dots n_N} \sum_{k_1=0}^{n_1-1} \dots \sum_{k_N=0}^{n_N-1} (\tau_{(k_1,\dots,k_N)} \delta_{(1,\dots,1)} \cdot f)_{(1,\dots,1)}(\cdot,\cdot)$$

is dominated by a function in $L_1(m \times \mu)$. Now the assertion (ii) follows from a combination of Banach's convergence Theorem 7, Lebesgue's dominated convergence theorem and Fubini's theorem. The proof of Theorem 8 has hereby been completed.

Application of Theorem 8 and the result ([10], Lemma 16, p. 166) of Dunford and Schwartz gives the following result without the assumption (4.16).

THEOREM 9. *Let $\{U_{(k,x)}: (k, x) \in Z_+^N \times X\}$ be as in Theorem 8. Then for every $f \in L_p(m \times \mu)$, $1 \leqslant p < \infty$, there exists a function $f^* \in L_p(m \times \mu)$ such that neglecting an m-null set,*

$$\lim_{n\to\infty} \big(D(n, \dots, n)f\big)_x(y) = f_x^*(y) \ \ \mu\text{-a.e.}$$

The function f_x^* is the limit in the norm of $L_p(\mu)$ with $1 < p < \infty$, and if μ is finite and $f(x, y) \equiv h(y)$ for $h \in L_1(\mu)$ then f_x^* is also the $L_1(\mu)$-limit.

For the proof of the $L_1(\mu)$-mean convergence in Theorem 9, see that of Theorem 4.

Theorem 8 and Theorem 9 allow, for example, the consideration of the limiting behaviors of the weighted averages

$$\frac{1}{n_1 \dots n_N} \sum_{k_1=0}^{n_1-1} \dots \sum_{k_N=0}^{n_N-1} c(k_1, \dots, k_N)\, U_{(k_1,\dots,k_N)} f,$$

$$\frac{1}{n^N} \sum_{k_1=0}^{n-1} \dots \sum_{k_N=0}^{n-1} c(k_1, \dots, k_N)\, U_{(k_1,\dots,k_N)} f$$

for a semigroup $\{U_k: k \in Z_+^N\}$ of linear contraction operators on L_1 as well as on L_∞.

A similar argument applies to the following extension of the so-called "non-commuting" ergodic theorem ([10], Theorem 9, p. 146) due to Dunford and Schwartz.

THEOREM 10 (Yoshimoto [24]). *Let $U_{(i)}$, $1 \leqslant i \leqslant N$, be linear operators on $L_1(m)$ with $\|U_{(i)}\|_{L_1(m)} \leqslant 1$ and $\|U_{(i)}\|_{L_\infty(m)} \leqslant 1$, $1 \leqslant i \leqslant N$. Then for every $f \in L_p(m)$ with $p > 1$, the multiple sequence*

$$\frac{1}{n_1 \dots n_N} \sum_{k_1=0}^{n_1-1} \dots \sum_{k_N=0}^{n_N-1} c(k_1, \dots, k_N)(U_{(1)}^{k_1} \dots U_{(N)}^{k_N} f)(x) \tag{4.18}$$

is convergent (as $n_1, \dots, n_N \to \infty$ independently) almost everywhere on X, as well as in the norm of $L_p(m)$. This sequence (4.18) is dominated by a func-

tion in $L_p(m)$. Furthermore, if m is finite then for every f in $L_1(m)$ belonging to the Zygmund class (see (4.16)), *the sequence* (4.18) *converges almost everywhere on X, as well as in the norm of $L_1(m)$.*

THEOREM 11. *Let $\{U_{(n,x)}: (n, x) \in Z_+ \times X\}$ be a strongly $\mathfrak{B}$-measurable contraction quasi semigroup on $L_1(\mu)$ associated with φ. Let $\{h_k\}$ be a sequence of non-negative $\mathfrak{B}\times\mathfrak{F}$-measurable functions such that except for an m-null set,*

$$\left|\left(U_{(1,x)}(T_1\xi)_x\right)(y)\right| \leqslant (h_{k+1})_x(y) \quad \mu\text{-a.e.}$$

whenever $|\xi(x, y)| \leqslant h_k(x, y)$ $m\times\mu$-a.e. and $\xi \in L_1(m\times\mu)$. Then for all $f \in L_1(m\times\mu)$, there is a set E of m-measure zero such that for any $x \in X - E$,

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} \left(U_{(k,x)}(T_k f)_x\right)(y)}{\sum_{k=0}^{n-1} (h_k)_x(y)}$$

exists and is finite almost everywhere on $\left\{y: \sum_{k=0}^{\infty} (h_k)_x(y) > 0\right\}$.

Moreover, if every $U_{(n,x)}$ is positive then for any $f \in L_1(m\times\mu)$ and $h \in L_1(m\times\mu)$, $h \geqslant 0$,

$$\text{(ii)} \qquad \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} u_k\left(U_{(k,x)}(T_k f)_x\right)(y)}{\sum_{k=0}^{n-1} u_k\left(U_{(k,x)}(T_k h)_x\right)(y)}$$

exists and is finite almost everywhere on the set where $h > 0$.

The assertion of Theorem 11 is deducible from Chacon [8], Theorem, p. 90 (cf. Akcoglu [1]) and Baxter [4], Theorem 1, p. 278, *via* the induced contraction semigroup.

§ 5. Continuous random ergodic theorems

Continuous random ergodic theorems have received an operator-theoretical treatment little up to now. The real step in this direction was taken by Kin ([16], [17]) who obtained some results for a more general class which is considerably larger than that of quasi semigroups consisting of measure preserving transformations.

The difficulty of generalizing these results for measure preserving quasi semigroups at the operator-theoretic level consists in the measurability of contraction quasi semigroups; however, it is surmountable by means of the measurable versions of such quasi semigroups (see Theorem 2 and Theorem 3).

In this section, we generalize and extend the continuous random ergodic theorems for measure preserving transformations to the operator-theoretic level. And the substance of the investigation presented here is to survey the behaviors at infinity and the local behaviors of random operator averages. Two measures m and μ will be finite throughout this section.

THEOREM 12. *Let $\mathfrak{X}$ be reflexive and let $\{U_{(t,x)}: (t,x)\in R_+\times X\}$ be a strongly $\mathfrak{B}$-measurable, m-a.e. strongly continuous contraction quasi semi-group on $L_1(\mu,\mathfrak{X})$ associated with $\{\varphi_t: t\in R_+\}$, such that*

$$\|U_{(t,x)}\xi\|_{L_\infty(\mu,\mathfrak{X})} \leqslant \|\xi\|_{L_\infty(\mu,\mathfrak{X})} \quad \text{for} \quad \xi\in L_1(\mu,\mathfrak{X})\cap L_\infty(\mu,\mathfrak{X}).$$

Then for any $f\in L_1(m\times\mu,\mathfrak{X})$ and any β, there exist a set E of m-measure zero and a function $f^*\in L_1(m\times\mu,\mathfrak{X})$ such that for any $x\in X-E$,

$$\lim_{\Lambda\to\infty}\frac{1}{\Lambda}\int_0^\Lambda e^{i\beta t}[U_{(t,x)}(T_tf)_x](y)\,dt = f^*_x(y), \tag{5.1}$$

$$\lim_{\Lambda\to 0+}\frac{1}{\Lambda}\int_0^\Lambda e^{i\beta t}[U_{(t,x)}(T_tf)_x](y)\,dt = f_x(y) \tag{5.2}$$

strongly in $\mathfrak{X}$ almost everywhere on Y. Moreover, if $1<p<\infty$, let $f\in L_p(m\times\mu,\mathfrak{X})$, and if $p=1$ let $f(x,y)\equiv h(y)$ for $h\in L_1(\mu,\mathfrak{X})$. Then for any p with $1\leqslant p<\infty$,

$$\lim_{\Lambda\to\infty}\left\| f^*_x-\frac{1}{\Lambda}\int_0^\Lambda e^{i\beta t}[U_{(t,x)}(T_tf)_x]\,dt\right\|_{L_p(\mu,\mathfrak{X})} = 0 \quad m\text{-a.e.} \tag{5.3}$$

Proof. Taken up with Theorem 2, there exists on $L_1(m\times\mu,\mathfrak{X})$ as well as on $L_\infty(m\times\mu,\mathfrak{X})$ the induced contraction semigroup $\{U^*_t: t\in R_+\}$ of $\{U_{(t,x)}: (t,x)\in R_+\times X\}$. Put $U^{**}_t = e^{i\beta t}U^*_t$ for all $t\in R_+$.

To prove the theorem, we require the continuous version of Chacon's ergodic theorem [7] and a vector-valued generalization of Krengel's local ergodic theorem [18], subject to the additional condition that

$$\|U^{**}_t\|_{L_\infty(m\times\mu,\mathfrak{X})}\leqslant 1.$$

LEMMA 9. *For every $f\in L_p(m\times\mu,\mathfrak{X})$ with $1\leqslant p<\infty$, the averages*

$$\frac{1}{\Lambda}\int_0^\Lambda [U^{**}_tf](x,y)\,dt \tag{5.4}$$

approach a limit strongly in $\mathfrak{X}$ almost everywhere on $X\times Y$ as $\Lambda\to\infty$. The limit also exists in the norm of $L_p(m\times\mu,\mathfrak{X})$ with $1\leqslant p<\infty$. Especially, *if $1 < p < \infty$, then there exists a function $f^* \in L_p(m \times \mu, \mathfrak{X})$ such that*

$$\left|\left|\left|\frac{1}{\Lambda}\int_0^\Lambda [U_t^{**}f](x,y)\,dt\right|\right|\right| \leqslant |||f^*(x,y)|||, \qquad \Lambda > 0 \tag{5.5}$$

for almost all $(x, y) \in X \times Y$.

Proof. By Fubini's theorem, there is an $m \times \mu$-null set Δ^* for which the average (5.4) exists for all (x, y) not in Δ^*.

Now the proof will be based upon the following identity, which will serve to reduce the present lemma to the discrete case discussed in Chacon [7]. For $\Lambda \geqslant 1$ and $f \in L_p(m \times \mu, \mathfrak{X})$ with $p \geqslant 1$,

$$\frac{1}{\Lambda}\int_0^\Lambda [U_t^{**}f]\,dt = \frac{[\Lambda]}{\Lambda}\left\{\frac{1}{[\Lambda]}\sum_{k=0}^{[\Lambda]-1} U_k^{**}\left(\int_0^1 [U_t^{**}f]\,dt\right) + \right. \tag{5.6}$$

$$\left. + \frac{[\Lambda]+1}{[\Lambda]}\,\frac{1}{[\Lambda]+1}\left(\sum_{k=0}^{[\Lambda]} U_k^{**} - \sum_{k=0}^{[\Lambda]-1} U_k^{**}\right)\left(\int_0^r [U_t^{**}f]\,dt\right)\right\},$$

where $\Lambda = [\Lambda]+r$, $0 \leqslant r < 1$. While, it follows from Fubini's theorem, using Riesz convexity theorem, that

$$\begin{aligned}
\left\|\int_0^1 [U_t^{**}f]\,dt\right\|_{L_p(m\times\mu,\mathfrak{X})}^p &= \iint_{X\times Y}\left|\left|\left|\int_0^1 [U_t^{**}f](x,y)\,dt\right|\right|\right|^p dm(x)\,d\mu(y) \\
&\leqslant \iint_{X\times Y}\int_0^1 |||[U_t^{**}f](x,y)|||^p\,dt\,dm(x)\,d\mu(y) \\
&= \int_0^1\left\{\iint_{X\times Y} |||[U_t^{**}f](x,y)|||^p\,dm(x)\,d\mu(y)\right\}dt \\
&\leqslant \|f\|_{L_p(m\times\mu,\mathfrak{X})}^p .
\end{aligned}$$

A fortiori,

$$\left\|\int_0^r [U_t^{**}f]\,dt\right\|_{L_p(m\times\mu,\mathfrak{X})}^p \leqslant \|f\|_{L_p(m\times\mu,\mathfrak{X})}^p .$$

Thus it is seen that the functions

$$\int_0^1 [U_t^{**}f]\,dt, \qquad \int_0^r [U_t^{**}f]\,dt$$

are in $L_p(m \times \mu, \mathfrak{X})$. Therefore, applying Chacon's ergodic theorem ([7], Theorem 1, p. 171) to the right-hand side of (5.6), we have the convergence almost everywhere of the averages (5.4) as $\Lambda \to \infty$. That if $1 < p < \infty$, then there exists a dominated function f^* in $L_p(m\times\mu, \mathfrak{X})$ which is independent of the parameter Λ and which satisfies (5.5) is inferable from the dominated theorem of Chacon ([7], Theorem 1), using the identity (5.6). The convergence in the mean of order p (>1) ensues from the convergence almost everywhere and the existence of a dominating function. The proof of the convergence in the mean of order 1 follows exactly the same line as that in the proof of Theorem 4. Hence the proof of Lemma 9 is accomplished.

LEMMA 10. *Let for* $f\in L_1(m\times\mu, \mathfrak{X})$ *and* $u>0$,

$$e^*(u) = \left\{(x, y)\colon \sup_{\Lambda>0} \left|\left|\left|\frac{1}{\Lambda}\int_0^{\Lambda} [U_t^{**}f](x, y)\,dt\right|\right|\right| > u\right\}. \tag{5.7}$$

Then

$$m\times\mu\big(e^*(u)\big) \leqslant \frac{2}{u}\cdot\|f\|_{L_1(m\times\mu,\mathfrak{X})}. \tag{5.8}$$

Proof. Let us denote by D the set of positive dyadic rational numbers. By Fubini's theorem, there is an $m\times\mu$-null set Δ^* for which the average (5.4) exists strongly in $\mathfrak{X}$ for all $(x, y)\in(X\times Y)-\Delta^*$. Since U_t^{**} is strongly continuous in t and $\|U_t^{**}\|_{L_1(m\times\mu,\mathfrak{X})}\leqslant 1$, we have

$$\frac{1}{r}\int_0^r [U_t^{**}f]\,dt = \lim_{n\to\infty}\frac{1}{r2^n}\sum_{k=0}^{r2^n-1} U_{\frac{k}{2^n}}^{**}f, \qquad r\in D,$$

in $L_1(m\times\mu, \mathfrak{X})$. Accordingly, passing to subsequences and finding a common subsequence $\{n_i\}$ by the Cantor's diagonal procedure, one gets

$$\frac{1}{r}\int_0^r [U_t^{**}f](x, y)\,dt = \lim_{i\to\infty}\frac{1}{r2^{n_i}}\sum_{k=0}^{r2^{n_i}-1} (U_{\frac{k}{2^{n_i}}}^{**}f)(x, y), \qquad r\in D,$$

strongly in $\mathfrak{X}$ for all $(x, y)\in(X\times Y)-(\Delta^*\cup\Delta^{**})$, where Δ^{**} is a set of $m\times\mu$-measure zero. If we write

$$f_{n_i}^*(x, y) = \sup_{k\geqslant 1}\left|\left|\left|\frac{1}{k}\sum_{j=0}^{k-1} (U_{\frac{j}{2^{n_i}}}^{**}f)(x, y)\right|\right|\right|,$$

then for any $\varepsilon>0$ and (x, y) not in $(\Delta^*\cup\Delta^{**})$, there exists a number $N(r, \varepsilon, x, y)$ such that for all n with $n\geqslant N(r, \varepsilon, x, y)$,

$$f_n^*(x, y) \geqslant \left|\left|\left|\frac{1}{r}\int_0^r [U_t^{**}f](x, y)\,dt\right|\right|\right| - \varepsilon.$$

So

$$\liminf_{n\to\infty} f_n^*(x, y) \geqslant \left|\left|\left| \frac{1}{r} \int_0^r [U_t^{**}f](x, y)\,dt \right|\right|\right|, \qquad r \in D,$$

on $(X\times Y)-(\Delta^*\cup\Delta^{**})$. But, since for each $(x, y)\in(X\times Y)-(\Delta^*\cup\Delta^{**})$, (5.4) is continuous in $\Lambda > 0$, we obtain

$$\sup_{\Lambda>0} \left|\left|\left| \frac{1}{\Lambda} \int_0^\Lambda [U_t^{**}f](x, y)\,dt \right|\right|\right| = \sup_{r\in D} \left|\left|\left| \frac{1}{r} \int_0^r [U_t^{**}f](x, y)\,dt \right|\right|\right|,$$

and thus

$$\liminf_{n\to\infty} f_n^*(x, y) \geqslant \sup_{\Lambda>0} \left|\left|\left| \frac{1}{\Lambda} \int_0^\Lambda U_t^{**}f](x, y)\,dt \right|\right|\right| \quad m\times\mu\text{-a.e.} \tag{5.9}$$

From (5.7) and (5.9),

$$e^*(u) \subset \bigcup_{n=1}^{\infty} \bigcap_{k\geqslant n}^{\infty} e_k^*(u),$$

where

$$e_k^*(u) = \{(x, y)\colon f_k^*(x, y) > u\}.$$

Hence by Chacon's maximal ergodic lemma ([7], p. 166),

$$m\times\mu\big(e^*(u)\big) \leqslant \liminf_{n\to\infty} m\times\mu\big(e_n^*(u)\big) \leqslant \frac{2}{u}\cdot\|f\|_{L_1(m\times\mu,\mathfrak{X})},$$

which proves (5.8) and finishes the proof.

LEMMA 11. *The set*

$$\mathfrak{A} = \left\{\frac{1}{\Lambda}\int_0^\Lambda [U_t^{**}f]\,dt\colon f\in L_1(m\times\mu, \mathfrak{X}),\ 0<\Lambda<1\right\}$$

fs dense in $L_1(m\times\mu, \mathfrak{X})$ and the local ergodic theorem is valid for all iunctions in $\mathfrak{A}$.

For the proof of Lemma 11, refer to Terrell [22], Lemma 1.2, p. 266. (In fact, this lemma can be proved, with trivial modifications, by the same argument as that used by Terrell in the case $\mathfrak{X} = C$.)

LEMMA 12. *If $f\in L_1(m\times\mu, \mathfrak{X})$, then*

$$\lim_{\Lambda\to 0+} \frac{1}{\Lambda}\int_0^\Lambda [U_t^{**}f](x, y)\,dt = f(x, y) \quad m\times\mu\text{-a.e.} \tag{5.10}$$

strongly in $\mathfrak{X}$.

Proof. Suppose that (5.10) is false. Then there exists a function $f \in L_1(m \times \mu, \mathfrak{X})$ such that

$$m \times \mu \left\{(x, y): \limsup_{\Lambda \to 0+} \left|\left|\left| \frac{1}{\Lambda} \int_0^\Lambda [U_t^{**} f](x, y)\, dt \right|\right|\right| > |||f(x, y)||| \right\} > 0,$$

from which follows that there are two positive numbers ξ, η such that

$$m \times \mu \left\{(x, y): \limsup_{\Lambda \to 0+} \left|\left|\left| \frac{1}{\Lambda} \int_0^\Lambda [U_t^{**} f](x, y)\, dt \right|\right|\right| > \xi + |||f(x, y)||| \right\} = \eta.$$

By virtue of Lemma 11, we can choose a function $g \in \mathfrak{A}$ satisfying

$$\|f - g\|_{L_1(m \times \mu, \mathfrak{X})} < \tfrac{1}{8} \xi \eta,$$

whereupon,

$$m \times \mu \{(x, y): |||f(x, y) - g(x, y)||| \geqslant \tfrac{1}{2} \xi\} < \tfrac{1}{4} \eta.$$

Thus, according to Lemma 10,

$$(5.11) \qquad m \times \mu \left\{(x, y): \limsup_{\Lambda \to 0+} \left|\left|\left| \frac{1}{\Lambda} \int_0^\Lambda [U_t^{**} f(-g)](x, y)\, dt \right|\right|\right| > \tfrac{1}{2} \xi \right\}$$

$$\leqslant m \times \mu \left\{(x, y): \sup_{\Lambda > 0} \left|\left|\left| \frac{1}{\Lambda} \int_0^\Lambda [U_t^{**} (f - g)](x, y)\, dt \right|\right|\right| > \tfrac{1}{4} \xi \right\}$$

$$\leqslant \frac{4}{\xi} \cdot \|f - g\|_{L_1(m \times \mu, \mathfrak{X})} < \tfrac{1}{2} \eta.$$

Again, by Lemma 10, we have

$$(5.12) \qquad m \times \mu \left\{(x, y): \limsup_{\Lambda \to 0+} \left|\left|\left| \frac{1}{\Lambda} \int_0^\Lambda [U_t^{**} (f - g)](x, y)\, dt \right|\right|\right| > \tfrac{1}{2} \xi \right\}$$

$$\geqslant m \times \mu \left\{(x, y): \limsup_{\Lambda \to 0+} \left|\left|\left| \frac{1}{\Lambda} \int_0^\Lambda [U_t^{**} f](x, y)\, dt \right|\right|\right| > \tfrac{1}{2} \xi + |||g(x, y)||| \right\}$$

$$\geqslant m \times \mu \left\{(x, y): \limsup_{\Lambda \to 0+} \left|\left|\left| \frac{1}{\Lambda} \int_0^\Lambda [U_t^{**} f](x, y)\, dt \right|\right|\right| > \xi + |||f(x, y)||| \right\}$$

$$- m \times \mu \{(x, y): |||f(x, y) - g(x, y)||| \geqslant \tfrac{1}{2} \xi\} \geqslant \eta - \tfrac{1}{4} \eta = \tfrac{3}{4} \eta.$$

But (5.11) and (5.12) yield a contradiction, thus proving the lemma.

Now (5.1), (5.2) and (5.3) appearing in Theorem 12 are easily established by Lemma 9 and Lemma 12 after the fashion of the proof of Theorem 4 and terminate the proof of Theorem 12.

THEOREM 13. *Let the measure m be σ-finite and let $\mathfrak{X}$ be reflexive. Let $\{U_{(t,x)} : (t, x) \in R_+^N \times X\}$ be a strongly $\mathfrak{B}$-measurable, m-a.e. strongly continuous quasi semigroup associated with a measurable semiflow $\{\varphi_t : t \in R_+^N\}$ on X, such that every $U_{(t,x)}$ belongs to the B-space $B(\mathfrak{X})$. Suppose that $|||U_{(t,x)}||| \leqslant 1$ for all (t, x).*

(i) *For any $f \in L_p(m, \mathfrak{X})$ with $1 < p < \infty$ and any real β, the averages*

$$\frac{1}{\Lambda_1 \ldots \Lambda_N} \int_0^{\Lambda_1} \ldots \int_0^{\Lambda_N} e^{i\beta(t_1+\ldots+t_N)} \left[U_{(t_1,\ldots,t_N,x)}\big(f(\varphi_{(t_1,\ldots,t_N)}x)\big)\right] dt_1 \times \ldots \times dt_N$$

converge strongly for almost all $x \in X$ as $\Lambda_1, \ldots, \Lambda_N \to \infty$ independently, and the limit also exists in the norm of $L_p(m, \mathfrak{X})$.

(ii) *For any $f \in L_1(m, \mathfrak{X})$ and any real β, the limit*

$$\lim_{\Lambda\to\infty} \frac{1}{\Lambda_N} \int_0^{\Lambda} \ldots \int_0^{\Lambda} e^{i\beta(t_1+\ldots+t_N)} \left[U_{(t_1,\ldots,t_N,x)}\big(f(\varphi_{(t_1,\ldots,t_N)}x)\big)\right] dt_1 \times \ldots \times dt_N$$

exists strongly in $\mathfrak{X}$ almost everywhere on X. Furthermore, if m is finite then the limit also exists in the norm of $L_1(m, \mathfrak{X})$.

Proof. Define the semigroup $\{U_t^* : t \in R_+^N\}$ on $L_1(m, \mathfrak{X})$ by letting

$$(U^*_{(t_1,\ldots,t_N)}f)(x) = e^{i\beta(t_1+\ldots+t_N)}\, U_{(t_1,\ldots,t_N,x)}\big(f(\varphi_{(t_1,\ldots,t_N)}x)\big)$$

for $f \in L_1(m, \mathfrak{X})$. Then on the hypothesis bestowed on the given quasi semigroup, $\{U_t^* : t \in R_+^N\}$ turns out to be strongly continuous in t and norm-contracting on $L_1(m, \mathfrak{X})$ as well as on $L_\infty(m, \mathfrak{X})$. Thus it follows from the Riesz convexity theorem that $\|U_t^*\|_{L_p(m,\mathfrak{X})} \leqslant 1$ for $1 < p < \infty$.

To establish (i), we may note that there exists a function $f^* \in L_p(m, \mathfrak{X})$ such that

$$\left|\left|\left|\frac{1}{\Lambda_1 \ldots \Lambda_N} \int_0^{\Lambda_1} \ldots \int_0^{\Lambda_N} [U^*_{(t_1,\ldots,t_N)}f](x)\, dt_1 \ldots dt_N\right|\right|\right| \leqslant |||f^*(x)|||, \qquad \Lambda_1, \ldots, \Lambda_N > 0$$

almost everywhere on X and then use the same argument as that used by Dunford and Schwartz ([10], Theorem 10, p. 157).

As for (ii), we first observe that for every $f \in L_p(m, \mathfrak{X})$ with $p \geqslant 1$, the set where

$$\sup_{\Lambda>0} \left|\left|\left|\frac{1}{\Lambda^N} \int_0^{\Lambda} \ldots \int_0^{\Lambda} [U^*_{(t_1,\ldots,t_N)}f](x)\, dt_1 \ldots dt_N\right|\right|\right| > a > 0$$

has measure which tends to zero as $a \to \infty$ (cf. Dunford and Schwartz [10], Lemma 11, p. 159). Then after the manner of the proof of Theorem 4 and applying Banach's convergence Theorem 6, we get (ii) and complete the proof of Theorem 13.

This result is a continuous extension of Beck and Schwartz's random ergodic theorem [5] (cf. Theorem 5).

In what follows, we use the notation $[M_\beta(\Lambda_1, \ldots, \Lambda_N)f]_x(\cdot)$ instead of

$$\frac{1}{\Lambda_1 \ldots \Lambda_N} \int_0^{\Lambda_1} \cdots \int_0^{\Lambda_N} e^{i\beta(t_1+\ldots+t_N)} [U_{(t_1,\ldots,t_N,x)}(T_{(t_1,\ldots,t_N)}f)_x](\cdot)\, dt_1 \times \ldots \times dt_N$$

for β, $\Lambda_1 > 0, \ldots, \Lambda_N > 0$ and $f \in L_p(m \times \mu)$.

Following Terrell [22], we shall say that

$$(\Lambda_1, \ldots, \Lambda_N) \underset{\varrho}{\to} (0, \ldots, 0)$$

if Λ_1 (> 0), $\ldots$, Λ_N (> 0) approach zero independently and if for a $\varrho > 0$,

$$(\Lambda_i / \Lambda_j) < \varrho, \quad 1 \leqslant i, j \leqslant N.$$

Applications of Dunford and Schwartz's ergodic theorems [10] and Terrell's local ergodic theorems [22] to the induced contraction semigroups give the following results (cf. Kin [16]).

THEOREM 14. *Let $\{U_{(t,x)}\colon (t, x) \in R_+^N \times X\}$ be a strongly $\mathfrak{B}$-measurable, m-a.e. strongly continuous contraction quasi semigroup on $L_1(\mu)$ associated with $\{\varphi_t\colon t \in R_+^N\}$ such that $\|U_{(t,x)}\|_{L_\infty(\mu)} \leqslant 1$. Then for all $f \in L_p(m \times \mu)$ with $1 \leqslant p < \infty$ and for any β, there exist an m-null set E and a function f^* $\in L_p(m \times \mu)$ such that for any $x \in X - E$,*

$$\lim_{\Lambda \to \infty} [M_\beta(\Lambda, \ldots, \Lambda)f]_x(y) = f_x^*(y),$$

$$\lim_{\Lambda \to 0+} [M_\beta(\Lambda, \ldots, \Lambda)f]_x(y) = f_x(y)$$

hold for almost all $y \in Y$. Moreover, if $1 < p < \infty$, let $f \in L_p(m \times \mu)$, and if $p = 1$ then let $f(x, y) \equiv h(y)$ for $h \in L_1(\mu)$. Then for all p with $1 \leqslant p < \infty$,

$$\lim_{\Lambda \to \infty} \|f_x^* - [M_\beta(\Lambda, \ldots, \Lambda)f]_x\|_{L_p(\mu)} = 0 \ \ m\text{-a.e.}$$

THEOREM 15. *Let $\{U_{(t,x)}\colon (t, x) \in R_+^N \times X\}$ be a strongly $\mathfrak{B}$-measurable, m-a.e. strongly continuous quasi semigroup of positive contractions on $L_1(\mu)$ associated with $\{\varphi_t\colon t \in R_+^N\}$. If $f \in L_1(m \times \mu)$, then, except on a set of m-measure zero,*

$$\lim_{(\Lambda_1, \ldots, \Lambda_N) \underset{\varrho}{\to} (0, \ldots, 0)} [M_0(\Lambda_1, \ldots, \Lambda_N)f]_x(y) = f_x(y)$$

almost everywhere on Y.

THEOREM 16. *On the hypothesis of Theorem* 14, *let* $f \in L_p(m \times \mu)$, $1 \leqslant p < \infty$, *and assume that if* $p = 1$, f *belongs to the Zygmund class* (see (4.16)). *Then, except for an m-null set,*

$$[M_\beta(\Lambda_1, \ldots, \Lambda_N)f]_x(y)$$

approaches a limit almost everywhere on Y *as* $\Lambda_1 \to \infty, \ldots, \Lambda_N \to \infty$ *independently. The limit also exists in the norm of* $L_p(\mu)$.

Proof. Let $\{U_t^*: t \in R_+^N\}$ be the induced contraction semigroup of $\{U_{(t,x)}\}$ and put $U^{**}_{(t_1,\ldots,t_N)} = \exp\big(i\beta(t_1 + \ldots + t_N)\big) U^*_{(t_1,\ldots,t_N)}$. For each $\Lambda_1, \ldots, \Lambda_N > 0$ and each $f \in L_p(m \times \mu)$, we consider the average

$$\frac{1}{\Lambda_1 \ldots \Lambda_N} \int_0^{\Lambda_1} \ldots \int_0^{\Lambda_N} [U^{**}_{(t_1,\ldots,t_N)} f] dt_1 \ldots dt_N. \tag{5.13}$$

Then, for $p > 1$ and $f \in L_p(m \times \mu)$, the average (5.13) converges almost everywhere on $X \times Y$ as $\Lambda_1 \to \infty, \ldots, \Lambda_N \to \infty$ independently. The limit also exists in the norm of $L_p(m \times \mu)$ and the functions (5.13) are, for $\Lambda_1, \ldots, \Lambda_N > 0$, all dominated by a function in $L_p(m \times \mu)$. According to Dunford and Schwartz's theorem ([10], Theorem 7, p. 156), the hypothesis (4.16) guarantees that for $f \in L_1(m \times \mu)$, the function

$$f^* = \sup_{0 < \Lambda_1, \ldots, \Lambda_N < \infty} \left| \frac{1}{\Lambda_1 \ldots \Lambda_N} \int_0^{\Lambda_1} \ldots \int_0^{\Lambda_N} [U^{**}_{(t_1,\ldots,t_N)} f] dt_1 \ldots dt_N \right|,$$

which dominates all the functions (5.13), is in $L_1(m \times \mu)$. Thus, since the set $L_2(m \times \mu)$ is dense in $L_1(m \times \mu)$, we may apply Banach's convergence Theorem 6 to obtain the pointwise convergence of the functions (5.13) for $f \in L_1(m \times \mu)$. Furthermore, the $L_1(m \times \mu)$-mean convergence can be deduced from the existence of a dominating function. Hence we come up to the desired conclusion of Theorem 16 by using the above facts and Fubini's theorem.

THEOREM 17 (cf. Kin [17]). *Let* $\{U_{(t,x)}: (t, x) \in R_+ \times X\}$ *be a strongly* $\mathfrak{B}$*-measurable, m-a.e. strongly continuous quasi semigroup of positive contractions on* $L_1(\mu)$ *associated with* $\{\varphi_t: t \in R_+\}$. *If* $f, g \in L_1(m \times \mu)$ *and* g *is positive, then, except for an m-null set,*

$$\lim_{\Lambda \to \infty} \frac{[M_0(\Lambda)f]_x(y)}{[M_0(\Lambda)g]_x(y)},$$

$$\lim_{\Lambda \to 0+} \frac{[M_0(\Lambda)f]_x(y)}{[M_0(\Lambda)g]_x(y)}$$

exist and are finite almost everywhere on the set where $[M_0(\Lambda)g]_x(y) > 0$ *for all* $\Lambda > 0$.

The proof of Theorem 17 uses the ratio ergodic theorems for semigroups (cf. Akcoglu and Cunsolo [3], Fong and Sucheston [12], Akcoglu and Chacon [2]).

Finally, we note that random ergodic theorems, which are recognized as random extensions of ergodic theorems, have some interesting applications in the theory of inhomogeneous processes. For details, the reader is referred to the works by Beck and Schwartz [5], Révész [20] and Kin [16], [17].

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