

LECTURES ON CONSTITUTIVE EXPRESSIONS

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1. Introduction

A constitutive expression is employed to define the response of a material when it is subjected to one or more fields. If the material possesses symmetry properties, restrictions are imposed on the form of the constitutive expression. This leads to consideration of the problem of determining the general form of scalar-valued polynomial functions $W(\mathbf{B}, \mathbf{C}, \dots)$ and tensor-valued polynomial functions $\mathbf{H}(\mathbf{B}, \mathbf{C}, \dots)$ of the tensors $\mathbf{B}, \mathbf{C}, \dots$ which are invariant under a group Γ .

We indicate the manner in which restrictions are imposed on the form of a constitutive expression by the material symmetry. Let $\mathcal{A}_1, \mathcal{A}_2, \dots$ denote the set of symmetry transformations which carry the material from its initial configuration into a final configuration which is indistinguishable from the original. Let \mathbf{x} be a rectangular Cartesian coordinate system whose orientation relative to the preferred directions in the material is specified. Let $\mathbf{A}_1 \mathbf{x}, \mathbf{A}_2 \mathbf{x}, \dots$ denote the rectangular Cartesian coordinate system into which \mathbf{x} is carried by the symmetry transformations $\mathcal{A}_1, \mathcal{A}_2, \dots$ respectively. The matrices $\mathbf{A}_1, \mathbf{A}_2, \dots$ associated with $\mathcal{A}_1, \mathcal{A}_2, \dots$ form a matrix group Γ which is referred to as the symmetry group of the material. The reference frames $\mathbf{A}_1 \mathbf{x}, \mathbf{A}_2 \mathbf{x}, \dots$ are referred to as the set of equivalent coordinate systems associated with the material under consideration.

Consider the constitutive expression given by

$$(1.1) \quad \mathbf{F} = \mathbf{H}(\mathbf{B}, \mathbf{C}, \dots)$$

where $\mathbf{F}, \mathbf{B}, \mathbf{C}, \dots$ are tensors of orders n, p, q, \dots . The components of \mathbf{F}, \dots when referred to the reference frames \mathbf{x} and $\mathbf{A}\mathbf{x}$ respectively are given by

$$(1.2) \quad F_{i_1 \dots i_n}, \quad (\mathbf{A}\mathbf{F})_{i_1 \dots i_n} = A_{i_1 j_1} \dots A_{i_n j_n} F_{j_1 \dots j_n}, \dots$$

If the expression (1.1) is to describe the response of a material whose symmetry is specified by the group Γ of matrices A_1, A_2, \dots , then (1.1) must have the same form when referred to each of the equivalent coordinate systems $A_1 \mathbf{x}, A_2 \mathbf{x}, \dots$. Thus, we require that

$$(1.3) \quad \mathbf{A}H(\mathbf{B}, \mathbf{C}, \dots) = H(\mathbf{A}\mathbf{B}, \mathbf{A}\mathbf{C}, \dots)$$

must hold for each \mathbf{A} belonging to Γ . The equation (1.3) may be written as

$$(1.4) \quad A_{i_1 j_1} \dots A_{i_n j_n} H_{j_1 \dots j_n}(B_{i_1 \dots i_p}, C_{i_1 \dots i_q}, \dots) \\ = H_{i_1 \dots i_n}(A_{i_1 j_1} \dots A_{i_p j_p} B_{j_1 \dots j_p}, A_{i_1 j_1} \dots A_{i_q j_q} C_{j_1 \dots j_q}, \dots).$$

We say that the tensor-valued function $H(\mathbf{B}, \mathbf{C}, \dots)$ is invariant under Γ if (1.3) holds for all \mathbf{A} in Γ . The problem of concern is to determine the general expression for $H(\mathbf{B}, \mathbf{C}, \dots)$ consistent with the restriction (1.3) imposed by the requirement that the constitutive expression be form-invariant under the group Γ defining the material symmetry. An appropriate solution is given by listing a set of tensor-valued functions $H_j(\mathbf{B}, \mathbf{C}, \dots)$, each of which is invariant under Γ , such that any polynomial tensor-valued function $H(\mathbf{B}, \mathbf{C}, \dots)$ which is invariant under Γ is expressible as a linear combination of the $H_j(\mathbf{B}, \mathbf{C}, \dots)$ with coefficients $\phi_j(\mathbf{B}, \mathbf{C}, \dots)$ which are scalar-valued polynomial functions invariant under Γ .

Special cases of this problem are of interest. The classical theories of crystal physics employ constitutive expressions of the form

$$(1.5) \quad T_{i_1 \dots i_n} = c_{i_1 \dots j_m} E_{j_1 \dots j_m}.$$

The tensors T and E are field tensors such as the stress tensor, the strain tensor, ... The tensor c is referred to as a property tensor or a material tensor. The restrictions of the form (1.3) imposed by material symmetry are satisfied if the property tensor c is invariant under the group Γ defining the material symmetry, i.e., if

$$(1.6) \quad c_{i_1 i_2 \dots j_m} = A_{i_1 p_1} A_{i_2 p_2} \dots A_{j_m q_m} c_{p_1 p_2 \dots q_m}$$

holds for all \mathbf{A} belonging to Γ . If Γ is a finite group which defines the symmetry properties of a crystal, the tensor c satisfying (1.6) is referred to as an anisotropic tensor. In Section 2, we discuss some problems concerning anisotropic tensors. In Section 3, we discuss tensors which are invariant under the full orthogonal group and the proper orthogonal group, i.e. isotropic tensors and rotation tensors respectively. In Section 4, we consider the problem of determining the form of a constitutive expression consistent with the restrictions imposed by the effect of a superposed rotation of the physical system. In Section 5, we discuss the manner in which (1.5) may be solved upon application of Schur's lemma. In Section 6, we consider the problem of generating an integrity basis for scalar-valued functions $W(\mathbf{B}, \mathbf{C}, \dots)$ which are invariant under a group Γ . In Section 7, the concept

of a set of invariants of symmetry type $(n_1 n_2 \dots)$ is considered. In Section 8 it is shown how this notion aids in the determination of an integrity basis for functions of second-order tensors which are invariant under the orthogonal group. In Section 9, we employ methods from the theory of group representations to assist with the generation of integrity bases for functions of an arbitrary number of tensors of any order which are invariant under a given crystallographic group.

2. Anisotropic tensors

Let Γ denote the group of symmetry transformations which defines the symmetry properties of the material under consideration. Then, a property tensor c appearing in a constitutive expression which describes the response of this material is required to satisfy

$$(2.1) \quad c_{i_1 \dots i_n} = A_{i_1 j_1} \dots A_{i_n j_n} C_{j_1 \dots j_n}$$

for all A belonging to Γ . A tensor c which satisfies (2.1) for all A belonging to Γ is said to be invariant under Γ . The number of linearly independent n th-order tensors which are invariant under a finite group Γ comprised of A_1, \dots, A_M is given by

$$(2.2) \quad P_n^\Gamma = \frac{1}{M} \sum_{i=1}^M (\text{tr } A_i)^n.$$

Let c_1, \dots, c_P ($P = P_n^\Gamma$) be a set of P_n^Γ linearly independent n th-order tensors which are invariant under Γ . Then any n th-order tensor c which is invariant under Γ is expressible as a linear combination of the c_1, \dots, c_P , i.e.,

$$(2.3) \quad c = a_1 c_1 + \dots + a_P c_P.$$

If Γ is a crystallographic point group, we refer to the tensors invariant under Γ as the anisotropic tensors associated with the group Γ . A set of P_n^Γ linearly independent n th-order tensors c_1, \dots, c_P which are invariant under Γ will be referred to as a complete set of n th-order anisotropic tensors associated with the group Γ .

A constitutive expression of the form (1.5) appropriate for a given crystal class may then be written as

$$(2.4) \quad T_{i_1 \dots i_P} = (a_1 c_{i_1 \dots i_n}^{(1)} + \dots + a_P c_{i_1 \dots i_n}^{(P)}) E_{i_{P+1} \dots i_n}$$

where the tensors c_1, \dots, c_P in (2.4) are those comprising a complete set of anisotropic tensors associated with the group Γ considered. If any of the physical tensors T, E in (2.4) possess symmetry properties, e.g., if

$$(2.5) \quad T_{i_1 i_2} = T_{i_2 i_1}, \quad E_{i_1 i_2 i_3} = E_{i_1 i_3 i_2}, \dots$$

then a number of the terms in (2.4) will be redundant and these must be eliminated. If n is large, this may be difficult.

The n th-order anisotropic tensors c_1, \dots, c_p are comprised of the linearly independent isomers of a number of tensors u, \dots, v . Let $u_1, \dots, u_N, \dots, v_1, \dots, v_M$ denote the linearly independent isomers of u, \dots, v . We proceed by eliminating the redundant terms in each of the expressions

$$(2.6) \quad \begin{aligned} T_{i_1 \dots i_p}^{(u)} &= (a_1 u_{i_1 \dots i_n}^{(1)} + \dots + a_N u_{i_1 \dots i_n}^{(N)}) E_{i_{p+1} \dots i_n}, \dots, \\ T_{i_1 \dots i_p}^{(v)} &= (b_1 v_{i_1 \dots i_n}^{(1)} + \dots + b_M v_{i_1 \dots i_n}^{(M)}) E_{i_{p+1} \dots i_n}. \end{aligned}$$

If we denote the expressions obtained from (2.6) upon eliminating the redundant terms by $T^*(u), \dots, T^*(v)$, then the appropriate expression for T is given by

$$(2.7) \quad T = T^*(u) + \dots + T^*(v).$$

The number of linearly independent terms in each of the expressions (2.6) is an essential piece of information. If this information is lacking, it may be a difficult matter to determine whether all of the redundant terms have been eliminated. Given this information, we may proceed by generating the appropriate number of linearly independent terms rather than by eliminating the redundant terms and this is frequently advantageous. We proceed by investigating the manner in which the isomers of $u_{i_1 \dots i_n}$ and $T_{i_1 \dots i_p} E_{i_{p+1} \dots i_n}$ transform under interchange of the subscripts $i_1 \dots i_n$.

Let S be the permutation of the numbers $1, 2, \dots, n$ which carries 1 into α , 2 into β , \dots , n into γ . Application of the permutation S to the tensor u yields an isomer Su of u which is defined by

$$(2.8) \quad Su_{i_1 i_2 \dots i_n} = u_{i_\alpha i_\beta \dots i_\gamma}.$$

With this definition, we see that the distinct isomers of a tensor u form the carrier space for a representation of the group S_n of permutations of the numbers $1, 2, \dots, n$. This representation may be decomposed into the direct sum of irreducible representations of S_n given by

$$(2.9) \quad \sum_n \alpha_{n_1 \dots n_p} (n_1 \dots n_p)$$

where $(n_1 \dots n_p)$ denotes an irreducible representation of S_n , the $\alpha_{n_1 \dots n_p}$ are positive integers or zero and \sum_n denotes summation over the irreducible representations of S_n . We note that there is an irreducible representation of S_n corresponding to each partition $n_1 \dots n_p$ of n , i.e., to each set of positive integers $n_1 \geq n_2 \geq \dots \geq n_p$ such that $n_1 + \dots + n_p = n$. Thus, for $n = 4$, the expression (2.9) is given by

$$(2.10) \quad \alpha_4(4) + \alpha_{31}(31) + \alpha_{22}(22) + \alpha_{211}(211) + \alpha_{1111}(1111).$$

A set of tensors which forms the carrier space for an irreducible representation $(n_1 \dots n_p)$ is referred to as a set of tensors of symmetry type $(n_1 \dots n_p)$.

A set of tensors which forms the carrier space for the representation given by (2.9) is referred to as a set of tensors of symmetry type $\sum_n \alpha_{n_1 \dots n_p} (n_1 \dots n_p)$.

The number of tensors comprising a set of tensors of symmetry type $(n_1 \dots n_p)$ is given by $f_{n_1 \dots n_p}$ where $f_{n_1 \dots n_p}$ is the degree of irreducible representation and may be found in the first column of the character tables for the symmetric group S_n ($n = 1, \dots, 10$) given by Littlewood ([5]).

We now give a procedure which may be employed to determine the decomposition of a representation whose carrier space is formed by a tensor \mathbf{u} and its distinct isomers. Suppose that there are N linearly independent tensors which may be formed from the $n!$ isomers of \mathbf{u} . We denote these by $\mathbf{u}_1, \dots, \mathbf{u}_N$. With (2.8), we see that the tensor $S\mathbf{u}_i$ is expressible as a linear combination of the $\mathbf{u}_1, \dots, \mathbf{u}_N$. The tensors $\mathbf{u}_1, \dots, \mathbf{u}_N$ form the carrier space for a representation which we denote by (A) . We have

$$(2.11) \quad S\mathbf{u}_i = \mathbf{u}_j a_{ji}(S) \quad (i, j = 1, \dots, N).$$

The $n!$ quantities

$$(2.12) \quad \chi_A(S) = a_{ii}(S) = \text{tr } a(S)$$

are referred to as the components of the character of the representation (A) . The number of times the irreducible representation $(n_1 \dots n_p)$ occurs in the decomposition of the representation is then given by

$$(2.13) \quad \alpha_{n_1 \dots n_p} = \frac{1}{n!} \sum_s \chi_A(S) \chi_{n_1 \dots n_p}(S)$$

where the summation is over the $n!$ permutations of S_n and where $\chi_{n_1 \dots n_p}(S)$ are the components of the character of the irreducible representation $(n_1 \dots n_p)$. If S and S' are permutations belonging to the same class c of permutations, i.e, if they have the same cycle structure, then

$$(2.14) \quad \begin{aligned} \chi_A(S) &= \chi_A(S') = \chi_A(c), \\ \chi_{n_1 \dots n_p}(S) &= \chi_{n_1 \dots n_p}(S') = \chi_{n_1 \dots n_p}(c). \end{aligned}$$

Thus, (2.13) may be rewritten as

$$(2.15) \quad \alpha_{n_1 \dots n_p} = \frac{1}{n!} \sum_c h_c \chi_A(c) \chi_{n_1 \dots n_p}(c)$$

where the summation is over the classes of S_n and h_c is the number of permutations belonging to the class c . Thus to determine the decomposition of a representation (A) , we need only determine the character $\chi_A(s)$ for one permutation belonging to each class of S_n . The quantities $\chi_{n_1 \dots n_p}(c)$ and h_c are given by Littlewood ([5]) for $n \leq 10$.

For example, consider the representation (A) whose carrier space is

formed by the distinct isomers of $T = T_{i_1 i_2 i_3} = T_{i_1 i_3 i_2}$. We employ the notation

$$\begin{aligned}
 (2.16) \quad T_1 &= T_{i_1 i_2 i_3} = T_{i_1 i_3 i_2}, \\
 T_2 &= T_{i_2 i_3 i_1} = T_{i_2 i_1 i_3}, \\
 T_3 &= T_{i_3 i_1 i_2} = T_{i_3 i_2 i_1}.
 \end{aligned}$$

The tensors ST_i and the characters $\chi_A(S)$ may be determined from (2.8), (2.12) and (2.16). We list these quantities in tabular form below. With (2.13) or (2.15), we see immediately that the decomposition of the representation (A) is given by (3)+(21).

We are also concerned with determining the decomposition of representations whose carrier spaces are formed by the isomers of tensors which are the outer products of vectors, second-order tensors, ... If the isomers of $T_{i_1 i_2 i_3}$ and $E_{i_1 i_2}$ for carrier spaces for the representations (A) and (B) respectively, then the isomers of $T_{i_1 i_2 i_3} E_{i_4 i_5}$ form the carrier space for a representation of S_5 which is denoted by $(A) \cdot (B)$ and which is referred to as the direct product of (A) and (B) . The decomposition of the direct product of irreducible representations into the sum of irreducible representations has been considered by Murnaghan ([8], [9]). Let (A) and (B) be reducible representations of S_m and S_n whose decompositions are given respectively by

$$(2.17) \quad \sum_m \alpha_{m_1 \dots m_p} (m_1 \dots m_p), \quad \sum_n \beta_{n_1 \dots n_q} (n_1 \dots n_q).$$

Table 1

S	ST_1	ST_2	ST_3	$\chi_A(S)$	$\chi_3(S)$	$\chi_{21}(S)$	$\chi_{111}(S)$
e	T_1	T_2	T_3	3	1	2	1
(123)	T_2	T_3	T_1	0	1	-1	1
(132)	T_3	T_1	T_2	0	1	-1	1
(12)	T_2	T_1	T_3	1	1	0	-1
(13)	T_3	T_2	T_1	1	1	0	-1
(23)	T_1	T_3	T_2	1	1	0	-1

Then the decomposition of $(A) \cdot (B)$ is given by

$$(2.18) \quad \sum_m \sum_n \alpha_{m_1 \dots m_p} \beta_{n_1 \dots n_q} (m_1 \dots m_p) \cdot (n_1 \dots n_q)$$

where the decomposition of $(m_1 \dots m_p) \cdot (n_1 \dots n_q)$ may be found in the work of Murnaghan ([8], [9]).

If the isomers of tensor $T_{i_1 i_2 i_3}$ forms the carrier space for a representation (A) , then the isomers of the tensor $T_{i_1 i_2 i_3} T_{i_4 i_5 i_6}$ form the carrier space for a reducible representation of S_6 which is denoted by $(A) \otimes (A)$ and is referred to as the symmetrized square of (A) . The decomposition of such

representations has been considered by Murnaghan ([10]) who lists most of the results required for our purposes.

Suppose that the N linearly independent isomers of the n th order property tensor $\mathbf{u} = u_{i_1 \dots i_n}$ form a set of tensors of symmetry type $\sum \alpha_{n_1 \dots n_p} (n_1 \dots n_p)$. Further suppose that the isomers of the tensor $T_{i_1 \dots i_p} E_{i_{p+1} \dots i_n}$ form a set of tensors of symmetry type $\sum \beta_{n_1 \dots n_p} (n_1 \dots n_p)$. Then the number of linearly independent terms in the expression

$$(2.19) \quad T_{i_1 \dots i_p} = (a_1 u_{i_1 \dots i_n}^{(1)} + \dots + a_N u_{i_1 \dots i_n}^{(N)}) E_{i_{p+1} \dots i_n}$$

is given by

$$(2.20) \quad \sum_n \alpha_{n_1 \dots n_p} \beta_{n_1 \dots n_p}$$

Some examples are required to clarify the procedures. These will be given below.

Let us employ the notation

$$(2.21) \quad \begin{aligned} e_1 &= \delta_{ii}, & \dots, & e_3 = \delta_{3i}, \\ e_{11} &= \delta_{1i} \delta_{1j}, & e_{12} &= \delta_{1i} \delta_{2j}, \dots, \\ e_{111} &= \delta_{1i} \delta_{1j} \delta_{1k}, & e_{112} &= \delta_{1i} \delta_{1j} \delta_{2k}, \dots, \\ \sum e_{11} &= e_{11} + e_{22} + e_{33}, & \sum e_{1111} &= e_{1111} + e_{2222} + e_{3333}. \end{aligned}$$

More generally, $\sum e_{112\dots3}$ denotes the sum of the three tensors obtained upon cyclic permutation of the subscripts on $e_{112\dots3}$. Complete sets of tensors of degrees 2, 4, 6, 8 associated with the hexoctahedral crystal class (highest symmetry cubic class) are given by

$$(2.22) \quad \begin{aligned} 2) \quad & \sum e_{11} = \delta_{ij}; & 1; & (2). \\ 4) \quad & \sum e_{1111}; & 1; & (4). \\ & \sum(e_{1122} + e_{2211}); & 3; & (4)+(2). \\ 6) \quad & \sum e_{111111}; & (1); & (6). \\ & \sum(e_{111122} + e_{111133}); & 15; & (6)+(5 \ 1)+(4 \ 2). \\ & \sum(e_{112233} + e_{113322}); & 15; & (6)+(4 \ 2)+(2 \ 2). \\ 8) \quad & \sum e_{11111111}; & 1; & (8). \\ & \sum(e_{11111122} + e_{11111133}); & 28; & (8)+(7 \ 1)+(6 \ 2). \\ & \sum(e_{11112222} + e_{11113333}); & 35; & (8)+(6 \ 2)+(4 \ 4). \\ & \sum(e_{11112233} + e_{11113322}); & 210; & (8)+(7 \ 1)+2(6 \ 2)+ \\ & & & +(5 \ 3)+(5 \ 2 \ 1)+(4 \ 4)+(4 \ 2 \ 2). \end{aligned}$$

The notation is as follows. $\sum(e_{1122} + e_{2211})$ denotes an invariant tensor

(invariant under the hexoctahedral group). The number 3 following denotes the number of linearly independent isomers of this tensor. The notation (4)+(22) indicates that these three isomers form a set of tensors of symmetry type (4)+(22).

We consider the problem of determining the form of the constitutive expression

$$(2.23) \quad T_{ij} = c_{ijkmnp} E_{km} F_{np}$$

appropriate for a hexoctahedral crystal where $T_{ij} = T_{ji}$, $E_{km} = E_{mk}$ and $F_{np} = -F_{pn}$, i.e., T , E and F are of symmetry types (2), (2) and (11) respectively. With the aid of tables given by Murnaghan ([8], [9]), we see that the isomers of $T_{ij} E_{km} F_{np}$ form a set of tensors of symmetry type

$$(2.24) \quad (33)+(51)+(42)+2(41)+2(321)+(3111)+(2211).$$

Since the isomers of $\sum e_{111111}$, $\sum(e_{111122} + e_{111133})$ and $\sum(e_{112233} + e_{113322})$ are sets of tensors of symmetry types (6), (6)+(51)+(42) and (6)+(42)+(222) respectively, we see from (2.19) and (2.20) that we obtain 0, 2 and 1 linearly independent terms in (2.23) when we replace c_{ijkmnp} by a linear combination of the 1, 15 and 15 isomers respectively of the tensors $\sum e_{111111}, \dots, \sum(e_{112233} + e_{113322})$. The appropriate expression is then readily obtained. For example, one of the three terms is given by substituting $\sum(e_{111212} + e_{111313})$ for c_{ijkmnp} in (2.23) so as to yield

$$(2.25) \quad T_{ij}^{(1)} = \delta_{1i} \delta_{1j} (E_{12} F_{12} + E_{13} F_{13}) + \delta_{2i} \delta_{2j} (E_{23} F_{23} + E_{21} F_{21}) + \delta_{3i} \delta_{3j} (E_{31} F_{31} + E_{32} F_{32}).$$

Further details, examples and extensive tables concerning the procedure described above are given by Smith ([13]).

3. Isotropic tensors and rotation tensors

A tensor c which satisfies the equations

$$(3.1) \quad c_{i_1 \dots i_n} = A_{i_1 j_1} \dots A_{i_n j_n} c_{j_1 \dots j_n}$$

for all orthogonal $A = \|A_{ij}\|$, i.e., for all A such that $AA^T = A^T A = I$, is referred to as an isotropic tensor. If the tensor c satisfies (3.1) for all proper orthogonal A , i.e., orthogonal A 's such that $\det A = 1$, then c is referred to as a *rotation* tensor. It is known that every three dimensional n th-order (n is even) isotropic tensor is expressible as a linear combination of the

$$(3.2) \quad p_n = \frac{n!}{(n/2)! 2^{n/2}}$$

distinct isomers of

$$(3.3) \quad \delta_{i_1 i_2} \delta_{i_3 i_4} \cdots \delta_{i_{n-1} i_n}$$

where δ_{ij} denotes the Kronecker delta. Isotropic tensors are even ordered tensors and are also rotation tensors. There are no odd-ordered isotropic tensors. Every three-dimensional n th-order (n is odd) rotation tensor is expressible as a linear combination of the

$$(3.4) \quad q_n = \binom{n}{3} p_{n-3}$$

distinct isomers of

$$(3.5) \quad \varepsilon_{i_1 i_2 i_3} \delta_{i_4 i_5} \cdots \delta_{i_{n-1} i_n}$$

where ε_{ijk} denotes the alternating tensor.

The representation of the general three dimensional n th-order isotropic tensor as a linear combination of the p_n distinct isomers of $\delta_{i_1 i_2} \cdots \delta_{i_{n-1} i_n}$ is objectionable since in general (for $n \geq 8$) these p_n tensors are not linearly independent. The q_n isomers of the tensor (3.5) are also not linearly independent in general (for $n \geq 5$). Let P_n and Q_n denote the number of linearly independent three dimensional n th-order isotropic and odd rotation tensors respectively. The quantities P_n and Q_n may be computed upon employing group-theoretic considerations. We have, with (3.2) and (3.4),

$$(3.6) \quad \begin{aligned} p_2 = P_2 = 1; \quad p_4 = P_4 = 3; \quad p_6 = P_6 = 15; \\ p_8 = 105, \quad P_8 = 91; \quad p_{10} = 945; \quad P_{10} = 603; \dots \\ q_3 = Q_3 = 1; \quad q_5 = 10, \quad Q_5 = 6; \quad q_7 = 105, \quad Q_7 = 36; \\ q_9 = 1260, \quad Q_9 = 232; \quad Q_{11} = 17325, \quad q_{11} = 1585. \end{aligned}$$

The problem of concern is to list a set P_n linearly independent n th-order isotropic tensors for $n = 8, 10, \dots$. Similarly we must be able to list sets of linearly independent n th-order rotation tensors for $n = 5, 7, \dots$. We restrict consideration to the case of isotropic tensors. A detailed discussion of the problem is given by Smith ([12]). We observe that the set of 3 isomers of the isotropic tensor $\delta_{i_1 i_2} \delta_{i_3 i_4}$ forms a set of three tensors of symmetry type (4)+(22). There are 1 and 2 tensors respectively in sets of tensors of type (4) and (22). We denote this information by writing

$$(3.7) \quad \delta_{i_1 i_2} \delta_{i_3 i_4}, \dots; \quad 3; \quad (4)+(22); \quad 1+2.$$

Similarly, we have

$$(3.8) \quad \begin{aligned} \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6}, \dots; \quad 15; \quad (6)+(42)+(222); \quad 1+9+5. \\ \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6} \delta_{i_7 i_8}, \dots; \quad 105; \quad (8)+(62)+(44)+(422)+(2222); \\ 1+20+14+56+14. \\ \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6} \delta_{i_7 i_8} \delta_{i_9 i_{10}}, \dots; \quad 945; \quad (10)+(82)+(64)+(622)+ \\ +(442)+(4222)+(22222); \quad 1+35+90+225+252+300+42. \end{aligned}$$

For the case $n = 8$, there are 91 linearly independent tensors which comprise sets of symmetry type $(8) + (62) + (44) + (422)$. There are 14 tensors forming the set of type (2222) and these correspond to identities relating the isotropic tensors. For the case $n = 10$, there are 603 linearly independent isotropic tensors belonging to the sets (10) , (82) , (64) , (622) , (442) and $342 = 945 - 603$ null tensors corresponding to the sets (4222) , (22222) .

We may proceed by setting up a correspondence between isotropic tensors of order n and the standard tableaux associated with irreducible representations of the symmetric group S_n . Let n_1, n_2, \dots, n_r be positive integers such that

$$(3.9) \quad n_1 + n_2 + \dots + n_r = n, \quad n_1 \geq n_2 \geq \dots \geq n_r.$$

Corresponding to each partition $n_1 n_2 \dots n_r$ of n is a *frame* which consists of r rows of squares of lengths n_1, n_2, \dots, n_r where $n_1 \geq n_2 \geq \dots \geq n_r$. A *tableau* is obtained from a frame by inserting the numbers $1, 2, \dots, n$ in any fashion into the n squares. A *standard tableau* is one in which the integers increase from left to right in any given row and from top to bottom in any given column. Thus the standard tableaux corresponding to the frames 6, 42 and 222 are given by

$$(3.10) \quad \begin{array}{l} 1\ 2\ 3\ 4\ 5\ 6; \\ 1\ 2\ 3\ 4, \quad 1\ 2\ 3\ 5, \quad 1\ 2\ 3\ 6, \quad 1\ 2\ 4\ 5, \quad 1\ 2\ 4\ 6, \\ 5\ 6 \quad 4\ 6 \quad 4\ 5 \quad 3\ 6 \quad 3\ 5 \\ 1\ 2\ 5\ 6, \quad 1\ 3\ 4\ 5, \quad 1\ 3\ 4\ 6, \quad 1\ 3\ 5\ 6; \\ 3\ 4 \quad 2\ 6 \quad 2\ 5 \quad 2\ 4 \\ 1\ 2, \quad 1\ 2, \quad 1\ 3, \quad 1\ 3, \quad 1\ 4. \\ 3\ 4 \quad 3\ 5 \quad 2\ 4 \quad 2\ 5 \quad 2\ 5 \\ 5\ 6 \quad 4\ 6 \quad 5\ 6 \quad 4\ 6 \quad 3\ 6 \end{array}$$

Corresponding to each of these standard tableaux we may list an isotropic tensor. Thus, we have

$$(3.11) \quad \begin{array}{l} \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6}; \\ \delta_{i_2 i_6}^{i_1 i_5} \delta_{i_3 i_4}, \quad \dots, \quad \delta_{i_3 i_4}^{i_1 i_2} \delta_{i_5 i_6}; \\ \delta_{i_2 i_4 i_6}^{i_1 i_3 i_5}, \quad \dots, \quad \delta_{i_4 i_5 i_6}^{i_1 i_2 i_3} \end{array}$$

where

$$(3.12) \quad \delta_{i_2 i_6}^{i_1 i_5} = \begin{vmatrix} \delta_{i_1 i_2} & \delta_{i_1 i_6} \\ \delta_{i_5 i_2} & \delta_{i_5 i_6} \end{vmatrix}, \quad \delta_{i_2 i_4 i_6}^{i_1 i_3 i_5} = \begin{vmatrix} \delta_{i_1 i_2} & \delta_{i_1 i_4} & \delta_{i_1 i_6} \\ \delta_{i_3 i_2} & \delta_{i_3 i_4} & \delta_{i_3 i_6} \\ \delta_{i_5 i_2} & \delta_{i_5 i_4} & \delta_{i_5 i_6} \end{vmatrix}.$$

The set of tensors (3.11) gives 15 linearly independent tensors of order six. Proceeding in the same fashion for $n = 8$, we would set up the 91 standard tableaux corresponding to the partitions 8, 6 2, 4 4, 4 2 2 of 8 and list a tensor corresponding to each tableau in the same fashion as above. For $n = 10, 12, \dots$, the procedure is the same. For a more complete discussion (see [12]).

4. Application. Rotation of the physical system

Let us assume that the stress σ_{ij} in a body is a function of the deformation gradients, i.e.,

$$(4.1) \quad \sigma_{ij} = \sigma_{ij}(F_{pA}) \quad \text{or} \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}(F)$$

where $F = \|F_{pA}\| = \|\partial x_p / \partial X_A\|$ denotes the deformation gradient matrix. If we subject the body to a rigid body rotation, the stress when referred to a set of coordinate axes fixed in the body will remain unaltered. This imposes restrictions on the functional form of (4.1). Thus $\boldsymbol{\sigma}(F)$ must satisfy

$$(4.2) \quad \boldsymbol{\sigma}(QF) = Q\boldsymbol{\sigma}(F)Q^T$$

for all proper orthogonal $Q = \|Q_{ij}\|$, i.e., for all Q such that

$$(4.3) \quad QQ^T = Q^T Q = I, \quad \det Q = 1.$$

This is a special case of the problem discussed in Section 1. Thus, we may consider σ_{ij} to be a symmetric second-order tensor-valued function of three vectors F_{i1}, F_{i2} and F_{i3} which is invariant under the group of all proper orthogonal transformations. This problem has been considered by Green and Rivlin ([1]) who have shown that

$$(4.4) \quad \boldsymbol{\sigma}(F) = F\boldsymbol{\psi}(F^T F)F^T$$

where $\boldsymbol{\psi}$ is an arbitrary symmetric second-order tensor-valued function of the finite strain tensor $C = F^T F$. Let us consider the special case where $\boldsymbol{\sigma}(F)$ is expanded as a polynomial in the quantities F_{i1}, F_{i2}, F_{i3} ($i = 1, 2, 3$). We have

$$(4.5) \quad \sigma_{ij}(F_{kA}) = c_{ijk} F_{k1} + d_{ikj} F_{k2} + e_{ijk} F_{k3} + c_{ijkl} F_{k1} F_{l1} + d_{ijkl} F_{k1} F_{l2} + \dots$$

Upon substituting (4.5) into (4.2), we see that the c_{ijk}, \dots must satisfy

$$(4.6) \quad c_{ijk} = Q_{ip} Q_{jq} Q_{kr} c_{pqr}, \quad \dots, \quad d_{ijkl} = Q_{ip} Q_{jq} Q_{kr} Q_{ls} d_{pqrs}, \quad \dots$$

for all proper orthogonal $Q = \|Q_{ip}\|$. Thus, the $c_{ijk}, \dots, d_{ijkl}, \dots$ are expressible as

$$(4.7) \quad c_{ijk} = a_0 \epsilon_{ijk}, \quad \dots, \quad d_{ijkl} = d_0 \delta_{ij} \delta_{kl} + d_1 \delta_{ik} \delta_{jl} + d_2 \delta_{il} \delta_{jk}, \quad \dots$$

Upon substituting (4.7) into (4.5) and observing that σ_{ij} is symmetric, we obtain

$$(4.8) \quad \sigma_{ij}(F_{kA}) = (c_0 \delta_{ij} \delta_{kl} + c_1 \delta_{ik} \delta_{jl} + c_2 \delta_{il} \delta_{jk}) F_{k1} F_{l1} + \\ + (d_0 \delta_{ij} \delta_{kl} + d_1 \delta_{ik} \delta_{jl} + d_2 \delta_{il} \delta_{jk}) F_{k1} F_{l2} + \dots$$

We now employ the polar factorization which states that a nonsingular matrix F is expressible as the product of an orthogonal matrix R and a positive definite symmetric matrix U , i.e., $F = RU$ or

$$(4.9) \quad F_{i1} = R_{ip} U_{p1}, \quad \dots, \quad F_{i3} = R_{ip} U_{p3}.$$

Upon substituting (4.9) into (4.8), we obtain

$$(4.10) \quad \sigma_{ij}(F_{kA}) = R_{im} R_{jn} \sigma_{mn}(U_{pA})$$

where

$$(4.11) \quad \sigma_{mn}(U_{pA}) = (c_0 \delta_{mn} \delta_{pq} + c_1 \delta_{mp} \delta_{nq} + c_2 \delta_{mq} \delta_{np}) U_{p1} U_{q1} + \\ + (d_0 \delta_{mn} \delta_{pq} + d_1 \delta_{mp} \delta_{nq} + d_2 \delta_{mq} \delta_{np}) U_{p1} U_{q2} + \dots$$

Note that the coefficients of $U_{p1} U_{q1}, \dots$ in (4.11) are exactly the same as the coefficients of the $F_{k1} F_{l1}, \dots$ in (4.8). Further observe that the tensors appearing as coefficients of the $U_{p1} U_{q1}, \dots$ in (4.11) are isotropic tensors. Consequently the expression for $\sigma(U)$ given by (4.11) will satisfy the relation $\sigma(QU) = Q\sigma(U)Q^T$ for all proper orthogonal Q .

It is observed that if (4.2) holds for all proper orthogonal Q , then (4.2) clearly holds for $Q = R^T$ where R is the orthogonal matrix appearing in the polar decomposition $F = RU$ of F . Upon setting $Q = R^T$ and $QF = R^T RU = U$ in (4.2), we see that if $\sigma(F)$ satisfies (4.2), it is expressible in the form

$$(4.12) \quad \sigma(F) = \sigma(RU) = R\sigma(U)R^T.$$

The argument is then made (see [7], [17]) that $\sigma(F)$ is expressible in the form (4.12) where $\sigma(U)$ is an arbitrary symmetric matrix-valued function of U . If we expand $\sigma(U)$ as a polynomial in U , we obtain

$$(4.13) \quad \sigma_{ij}(F_{kA}) = R_{im} R_{jn} \sigma_{mb}(U_{pA})$$

where

$$(4.14) \quad \sigma_{mm}(U_{pA}) = \bar{c}_{mnp} U_{p1} + \bar{d}_{mnp} U_{p2} + \bar{e}_{mnp} U_{p3} + \\ + \bar{c}_{mnpq} U_{p1} U_{q1} + \bar{d}_{mnpq} U_{p1} U_{q2} + \dots$$

and where the \bar{c}_{mnp}, \dots are arbitrary apart from the condition that $\bar{c}_{mnp} = \bar{c}_{nmp}, \dots$. Comparison of (4.11) and (4.14) show that the two expressions differ substantially.

Consider the special case where F is positive definite symmetric. Then

$F = U$ and $R = I$. Upon setting $F = U$ in (4.2), we see that $\sigma(U)$ satisfies $\sigma(QU) = Q\sigma(U)Q^T$ and hence $\sigma(U)$ is not an arbitrary function. The result (4.12) with $\sigma(U)$ arbitrary proceeds on the basis that $\sigma(U)$ is an arbitrary function of U . Thus there appears to be a conflict between the results given in [1] and [7], [17]. We believe that the expression (4.4) due to Green and Rivlin ([1]) is the appropriate result.

5. Application of Schur's lemma

Let us consider the problem of determining the general form of

$$(5.1) \quad t_{ij} = c_{ijk\dots q} e_{klm} e_{npq}, \quad t_{ij} = t_{ji}, \quad e_{ijk} = e_{ikj}$$

which is invariant under the three-dimensional orthogonal group O_3 . The general eighth order tensor which is invariant under O_3 is expressible as

$$(5.2) \quad c_{i_1\dots i_8} = \alpha_1 \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6} \delta_{i_7 i_8} + \alpha_2 \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_7} \delta_{i_6 i_8} + \dots$$

where the right hand side of (5.2) denotes a linear combination of the 105 distinct isomers of $\delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6} \delta_{i_7 i_8}$. Only 91 of these isomers are linearly independent. Explicit expressions for the 91 linearly independent 8th order tensors invariant under O_3 are given by Kearsley and Fong ([2]). If we employ these results and substitute the general expression for $c_{i_1\dots i_8}$ in (5.1), we obtain 91 terms. However, only 15 of these terms are linearly independent and one must solve a tedious algebraic problem in order to obtain the appropriate expression. It is preferable to proceed as follows. We set

$$(5.3) \quad \begin{aligned} t_{ij} &= t_{ij}^{(1)} + t_{ij}^{(2)}, & t_{ij}^{(1)} &= \frac{1}{3} t_{kk} \delta_{ij}, \\ e_{ijk} &= e_{ijk}^{(1)} + \dots + e_{ijk}^{(4)}, \\ 15e_{ijk}^{(1)} &= e_i \delta_{jk} + e_j \delta_{ki} + e_k \delta_{ij}, & e_i &= e_{ipp} + e_{pip} + e_{ppi}, \\ 6e_{ijk}^{(2)} &= 2c_i \delta_{jk} - c_j \delta_{ki} - c_k \delta_{ij}, & c_i &= e_{ipp} - e_{ppi}, \\ 3e_{ijk}^{(3)} &= 2e_{ijk} - e_{jki} - e_{kij} - 3e_{ijk}^{(2)}, \\ 6e_{ijk}^{(4)} &= e_{ijk} + e_{jik} + e_{kji} + e_{ikj} + e_{jki} + e_{kij} - 6e_{ijk}^{(1)}. \end{aligned}$$

We may then write (5.1) as

$$(5.4) \quad t_{ij}^{(1)} + t_{ij}^{(2)} = c_{ijk\dots q} (e_{klm}^{(1)} + \dots + e_{klm}^{(4)}) (e_{npq}^{(1)} + \dots + e_{npq}^{(4)})$$

and consider the 20 separate problems

$$(5.5) \quad t_{ij}^{(\alpha)} = c_{ijk\dots q} e_{klm}^{(\beta)} e_{npq}^{(\gamma)} \quad (\alpha = 1, 2; \beta, \gamma = 1, \dots, 4; \beta \leq \gamma).$$

We may compute the number of linearly independent terms in the expressions obtained from (5.5) by setting $(\alpha, \beta, \gamma) = (1, 1, 1), \dots, (2, 4, 4)$. We find that there is just one linearly independent term in 15 cases and none in the

remaining 5 cases. Thus, we have reduced the complicated algebraic problem of determining the general form of (5.1) which is invariant under O_3 to 15 trivial problems. The validity of this procedure is based on the fact that the independent components of the six tensors $t_{ij}^{(1)}, \dots, e_{ijk}^{(4)}$ form carrier spaces for irreducible representations of the group O_3 . We now show how a variant of this procedure may be employed to establish the general form of constitutive expressions which are invariant under any given crystallographic group.

We now consider the problem of determining the form of

$$(5.6) \quad t_{i_1 \dots i_n} = c_{i_1 \dots j_m} e_{j_1 \dots j_m}$$

which is invariant under the group $\Gamma = (A_1, \dots, A_M)$. Let T and E denote the column matrices whose elements are the n and m independent components of the tensors t and e referred to the reference frame x . Let T_k and E_k denote the column matrices whose elements are the independent components of the tensors t and e when referred to the reference frame $A_k x$. The elements of T_k and E_k are to be arranged in the same order as are the elements of T and E . We may determine an $n \times n$ matrix $S(A_k)$ which relates T_k and T . Thus

$$(5.7) \quad T_k = S(A_k) T.$$

The set of M $n \times n$ matrices $S(A_1), \dots, S(A_M)$ forms a matrix representation of degree n of the group Γ and defines the transformation properties of T under the group $\{A_1, \dots, A_M\}$. We may also determine a set of N $m \times m$ matrices which relate the E_k and E . Thus

$$(5.8) \quad E_k = R(A_k) E.$$

The $R(A_k)$ ($k = 1, \dots, M$) form a matrix representation of degree m of the group Γ .

The equation (5.6) may be rewritten as

$$(5.9) \quad T = CE.$$

The requirement of invariance under $\{A_1, \dots, A_M\}$ becomes

$$(5.10) \quad S(A_k) T = CR(A_k) E$$

where C is an $n \times m$ matrix and where (5.10) holds for $k = 1, \dots, M$. We see from (5.10) that the matrix C is subject to the restrictions that

$$(5.11) \quad S(A_k) C = CR(A_k)$$

must hold for $k = 1, \dots, M$. Let

$$(5.12) \quad Z = QT, \quad X = PE$$

where Q and P are constant non-singular $n \times n$ and $m \times m$ matrices respectively. With (5.7), (5.8) and (5.12) we see that the transformation properties of

Z and X under the group $\{A_1, \dots, A_M\}$ are defined by the matrix representations

$$(5.13) \quad QS(A_k)Q^{-1}, \quad PR(A_k)P^{-1}$$

respectively. The matrix representations (5.13) are said to be equivalent to the representations $S(A_k)$ and $R(A_k)$. There are only a finite number r of inequivalent irreducible representations associated with a given crystallographic group. We denote these representations by $\Gamma_1, \dots, \Gamma_r$ and we assume that the matrices

$$(5.14) \quad \Gamma_i(A_1), \dots, \Gamma_i(A_M) \quad (i = 1, \dots, r)$$

defining the representations $\Gamma_1, \dots, \Gamma_r$ are given. The representation Γ_i is said to be of degree p_i if the matrices $\Gamma_i(A_k)$ are $p_i \times p_i$ matrices. We denote the degree Γ_i by p_i .

With (5.12), the equation (5.9) becomes

$$(5.15) \quad Z = DX, \quad D = QCP^{-1}.$$

With (5.13), we see that the restrictions on the form of D corresponding to (5.11) are given by

$$(5.16) \quad QS(A_k)Q^{-1}D = DPR(A_k)P^{-1}$$

where (5.16) must hold for $k = 1, \dots, M$. We observe that it is always possible to choose the matrices Q and P so that the matrix representations $QS(A_k)Q^{-1}$ and $PR(A_k)P^{-1}$ are decomposed into the direct sum of irreducible representations of Γ . Thus, with the appropriate choice of Q and P , we have

$$(5.17) \quad \begin{aligned} QS(A_k)Q^{-1} &= n_1 \Gamma_1(A_k) \dot{+} \dots \dot{+} n_r \Gamma_r(A_k), \\ PR(A_k)P^{-1} &= m_1 \Gamma_1(A_k) \dot{+} \dots \dot{+} m_r \Gamma_r(A_k) \end{aligned}$$

where the right hand side of (5.17)₁ denotes a block diagonal matrix which contains n_1 matrices $\Gamma_1(A_k), \dots, n_r$ matrices $\Gamma_r(A_k)$ along the diagonal. For example,

$$(5.18) \quad 2\Gamma_1(A_k) \dot{+} \Gamma_2(A_k) = \left\| \begin{array}{ccc} \Gamma_1(A_k) & 0 & 0 \\ 0 & \Gamma_1(A_k) & 0 \\ 0 & 0 & \Gamma_2(A_k) \end{array} \right\|.$$

Let us employ the notation

$$(5.19) \quad Z = \left\| \begin{array}{c} z_1 \\ z_2 \\ \vdots \\ z_r \end{array} \right\|, \quad z_j = \left\| \begin{array}{c} z_{j1} \\ z_{j2} \\ \vdots \\ z_{jn_j} \end{array} \right\|, \quad X = \left\| \begin{array}{c} X_1 \\ X_2 \\ \vdots \\ X_r \end{array} \right\|, \quad X_k = \left\| \begin{array}{c} X_{k1} \\ X_{k2} \\ \vdots \\ X_{km_k} \end{array} \right\|$$

where the z_{ji} are $p_j \times 1$ matrices, the X_{ki} are $p_k \times 1$ matrices and where

$$(5.20) \quad p_1 n_1 + \dots + p_r n_r = n, \quad p_1 m_1 + \dots + p_r m_r = m.$$

With the notation (5.19), the equation (5.15)₁ may be written as

$$(5.21) \quad \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_r \end{pmatrix} = \begin{pmatrix} D^{11} & D^{12} & \dots & D^{1r} \\ D^{21} & D^{22} & \dots & D^{2r} \\ \vdots & \vdots & \dots & \vdots \\ D^{r1} & D^{r2} & \dots & D^{rr} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{pmatrix}$$

where the matrices D^{ij} are of the form

$$(5.22) \quad D^{ij} = \begin{pmatrix} D_{11}^{ij} & \dots & D_{1m_j}^{ij} \\ \vdots & & \vdots \\ D_{n_i 1}^{ij} & \dots & D_{n_i m_j}^{ij} \end{pmatrix}.$$

The matrices $D_{11}^{ij}, D_{12}^{ij}, \dots$ are $p_i \times p_j$ matrices and the matrix D^{ij} is a $p_i n_i \times p_j m_j$ matrix. With (5.17) and (5.21), we see that the restrictions imposed on the matrix D by (5.16) may be written as

$$(5.23) \quad \begin{pmatrix} G_1(A_k) & \cdot & \cdot & \cdot \\ \cdot & G_2(A_k) & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & G_r(A_k) \end{pmatrix} \begin{pmatrix} D^{11} & D^{12} & \dots & D^{1r} \\ D^{21} & D^{22} & \dots & D^{2r} \\ \vdots & \vdots & \dots & \vdots \\ D^{r1} & D^{r2} & \dots & D^{rr} \end{pmatrix} \\ = \begin{pmatrix} D^{11} & D^{12} & \dots & D^{1r} \\ D^{21} & D^{22} & \dots & D^{2r} \\ \vdots & \vdots & \dots & \vdots \\ D^{r1} & D^{r2} & \dots & D^{rr} \end{pmatrix} \begin{pmatrix} P_1(A_k) & \cdot & \cdot & \cdot \\ \cdot & P_2(A_k) & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & P_r(A_k) \end{pmatrix}$$

where

$$(5.24) \quad G_i(A_k) = \begin{pmatrix} \Gamma_i(A_k) & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \Gamma_i(A_k) \end{pmatrix}, \quad P_i(A_k) = \begin{pmatrix} \Gamma_i(A_k) & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \Gamma_i(A_k) \end{pmatrix}.$$

In (5.24), $G_i(A_k)$ and $P_i(A_k)$ are block diagonal matrices where $\Gamma_i(A_k)$ appears on the diagonal n_i and m_i times respectively. With (5.23), we obtain r^2 sets of equations

which must hold for $k = 1, \dots, M$. With (5.22), (5.24) and (5.25), we have

$$(5.26) \quad \left\| \begin{matrix} \Gamma_i(A_k) & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \Gamma_i(A_k) \end{matrix} \right\| \left\| \begin{matrix} D_{11}^{ij} & \dots & D_{1m_j}^{ij} \\ \cdot & \cdot & \cdot \\ D_{n_i 1}^{ij} & \dots & D_{n_i m_j}^{ij} \end{matrix} \right\| \\ = \left\| \begin{matrix} D_{11}^{ij} & \dots & D_{1m_j}^{ij} \\ \cdot & \cdot & \cdot \\ D_{n_i 1}^{ij} & \dots & D_{n_i m_j}^{ij} \end{matrix} \right\| \left\| \begin{matrix} \Gamma_j(A_k) & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \Gamma_j(A_k) \end{matrix} \right\|.$$

This yields the equations

$$(5.27) \quad \Gamma_i(A_k) D_{\alpha\beta}^{ij} = D_{\alpha\beta}^{ij} \Gamma_j(A_k) \quad (\alpha = 1, \dots, n_i; \beta = 1, \dots, m_j)$$

where (5.27) holds for $k = 1, \dots, M$. Schur's Lemma [6] tells that $D_{\alpha\beta}^{ij}$ is the zero matrix if $i \neq j$ and that $D_{\alpha\beta}^{ij}$ is a scalar multiple of the $p_i \times p_i$ identity matrix if $i = j$. Thus, we see that the matrix D appearing in (5.21) is of the form

$$(5.28) \quad \left\| \begin{matrix} D^{11} & \cdot & \cdot \\ \cdot & D^{22} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & D^r \end{matrix} \right\|$$

where

$$(5.29) \quad D^{ii} = \left\| \begin{matrix} D_{11}^{ii} & \dots & D_{1m_i}^{ii} \\ \vdots & & \vdots \\ D_{n_i 1}^{ii} & \dots & D_{n_i m_i}^{ii} \end{matrix} \right\|, \quad D_{\alpha\beta}^{ii} = \lambda_{\alpha\beta}^{ii} I$$

and where I is the $p_i \times p_i$ identity matrix. With (5.28) and (5.29), we may immediately list the general form of the matrix D appearing in (5.15) and (5.21).

Thus the determination of the form of the constitutive expression $T = CE$ which is invariant under Γ is trivial once we have decomposed the set of n components of T and the set of m components of E into $n_1 + \dots + n_r$ and $m_1 + \dots + m_r$ sets of quantities which form the carrier spaces for the irreducible representations of Γ appearing in the decomposition of the representations $S(A_k)$ and $R(A_k)$. We consider the case where Γ is the crystallographic group C_{3v} . The set of matrices defining the material symmetry are given by

(5.30)

$$A_1, A_2, A_3 = \left\| \begin{matrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{matrix} \right\|, \quad \left\| \begin{matrix} -1/2 & \sqrt{3}/2 & \cdot \\ -\sqrt{3}/2 & -1/2 & \cdot \\ \cdot & \cdot & 1 \end{matrix} \right\|, \quad \left\| \begin{matrix} -1/2 & -\sqrt{3}/2 & \cdot \\ \sqrt{3}/2 & -1/2 & \cdot \\ \cdot & \cdot & 1 \end{matrix} \right\|, \\ A_4, A_5, A_6 = \left\| \begin{matrix} -1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{matrix} \right\|, \quad \left\| \begin{matrix} 1/2 & -\sqrt{3}/2 & \cdot \\ -\sqrt{3}/2 & -1/2 & \cdot \\ \cdot & \cdot & 1 \end{matrix} \right\|, \quad \left\| \begin{matrix} 1/2 & \sqrt{3}/2 & \cdot \\ \sqrt{3}/2 & -1/2 & \cdot \\ \cdot & \cdot & 1 \end{matrix} \right\|.$$

There are three inequivalent irreducible representations associated with the group C_{3v} . These are denoted by $\Gamma_1(A_k), \dots, \Gamma_3(A_k)$ and are listed below.

$$\begin{aligned}
 &\Gamma_1(A_1), \dots, \Gamma_1(A_6) = 1, 1, 1, 1, 1, 1, \\
 &\Gamma_2(A_1), \dots, \Gamma_2(A_6) = 1, 1, 1, -1, -1, -1, \\
 (5.31) \quad &\Gamma_3(A_1), \dots, \Gamma_3(A_3) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{vmatrix}, \begin{vmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{vmatrix}, \\
 &\Gamma_3(A_4), \dots, \Gamma_3(A_6) = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{vmatrix}, \begin{vmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{vmatrix}.
 \end{aligned}$$

Suppose that

$$\begin{aligned}
 &1) \quad a, b, c, \dots \\
 &2) \quad \alpha, \beta, \gamma, \dots \\
 (5.32) \quad &3) \quad \begin{vmatrix} y_1 \\ y_2 \end{vmatrix}, \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}, \dots
 \end{aligned}$$

are quantities whose transformation properties under C_{3v} are defined by the matrix representations $\Gamma_1, \dots, \Gamma_3$ respectively. Typical products of the quantities appearing in (5.32) taken two at a time are

$$(5.33) \quad ab, \alpha\alpha, ay_1, ay_2, \alpha\beta, \alpha y_1, \alpha y_2, y_1 z_1, y_1 z_2, y_2 z_1, y_2 z_2.$$

With (5.32), we see that the 11 quantities (5.33) may be split into sets

$$\begin{aligned}
 &1) \quad ab, \alpha\beta, y_1 z_1 + y_2 z_2, \\
 &2) \quad \alpha\alpha, y_1 z_2 - y_2 z_1, \\
 (5.34) \quad &3) \quad \begin{vmatrix} ay_1 \\ ay_2 \end{vmatrix}, \begin{vmatrix} \alpha y_2 \\ -\alpha y_1 \end{vmatrix}, \begin{vmatrix} y_1 z_2 + y_2 z_1 \\ y_1 z_1 - y_2 z_2 \end{vmatrix},
 \end{aligned}$$

whose transformation properties are defined by the representations $\Gamma_1, \dots, \Gamma_3$. With (5.34), we may determine the decomposition of the components $E_i E_j$ ($i, j = 1, \dots, m$), $E_i E_j E_k$ ($i, j, k = 1, \dots, m$), ... into sets X_{1i}, \dots, X_{ri} once we have determined the decomposition of the components E_i ($i = 1, \dots, m$) into such sets.

For example, the transformation properties of a vector v under C_{3v} are defined by the matrices (5.30). With (5.30) and (5.31), it is apparent that the transformation properties of

$$(5.35) \quad v_3 \quad \text{and} \quad \begin{vmatrix} v_1 \\ v_2 \end{vmatrix}$$

are defined by the representations Γ_1 and Γ_3 respectively. With (5.34), we see that the six components $v_i v_j (i, j = 1, 2, 3; i \leq j)$ may be split into the sets

$$(5.36) \quad \begin{aligned} &1) \quad v_3^2, \quad v_1^2 + v_2^2, \\ &3) \quad \begin{vmatrix} v_3 v_1 \\ v_3 v_2 \end{vmatrix}, \quad \begin{vmatrix} 2v_1 v_2 \\ v_1^2 - v_2^2 \end{vmatrix} \end{aligned}$$

whose transformation properties are defined by Γ_1 and Γ_3 respectively. Consider now the constitutive equation

$$(5.37) \quad u_i = c_{ijk} v_j v_k$$

which is required to be invariant under C_{3v} . We set

$$(5.38) \quad \begin{aligned} T_{11} = u_3, \quad T_{31} = \begin{vmatrix} u_1 \\ u_2 \end{vmatrix}, \quad E_{11} = v_3^2, \quad E_{12} = v_1^2 + v_2^2, \\ E_{31} = \begin{vmatrix} v_3 v_1 \\ v_3 v_2 \end{vmatrix}, \quad E_{32} = \begin{vmatrix} 2v_1 v_2 \\ v_1^2 - v_2^2 \end{vmatrix}. \end{aligned}$$

We then have

$$(5.39) \quad \begin{vmatrix} T_{11} \\ T_{31} \end{vmatrix} = \begin{vmatrix} D_{11}^{11} & D_{12}^{11} & 0 & 0 \\ 0 & 0 & D_{11}^{33} & D_{12}^{33} \end{vmatrix} \begin{vmatrix} E_{11} \\ E_{12} \\ E_{31} \\ E_{32} \end{vmatrix}$$

where the $D_{\alpha\beta}^{11}$ and $D_{\alpha\beta}^{33}$ are multiples of 1×1 and 2×2 identity matrices respectively. The equation (5.39) may be written as

$$(5.40) \quad \begin{vmatrix} u_3 \\ u_1 \\ u_2 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & c_4 & 0 \\ 0 & 0 & 0 & c_3 & 0 & c_4 \end{vmatrix} \begin{vmatrix} v_3^2 \\ v_1^2 + v_2^2 \\ v_3 v_1 \\ v_3 v_2 \\ 2v_1 v_2 \\ v_1^2 - v_2^2 \end{vmatrix}.$$

This section closely follows Smith and Kiral ([15]). Some further results on the subject are given by Smith ([14]).

6. The generation of integrity bases

We consider the problem of determining the elements of the integrity basis for polynomial functions of N absolute vectors y_1, y_2, \dots, y_N which are invariant under the group Γ associated with the crystal class D_2 . This crystal class is characterized by the presence of three mutually orthogonal two-fold

axes of rotation. If we take the x_1 , x_2 and x_3 axes to lie along these two-fold axes of rotation, the symmetry transformations are the rotations through 180 degrees about each of the coordinate axes. The set of matrices defining the symmetry properties of the material are then given by

$$(6.1) \quad A_1, A_2, A_3, A_4 = \begin{vmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{vmatrix}, \begin{vmatrix} 1 & . & . \\ . & -1 & . \\ . & . & -1 \end{vmatrix}, \begin{vmatrix} -1 & . & . \\ . & 1 & . \\ . & . & -1 \end{vmatrix}, \begin{vmatrix} -1 & . & . \\ . & -1 & . \\ . & . & 1 \end{vmatrix}.$$

Let y_i and $(A_k y)_i$ denote the components of the absolute vector y when referred to the reference frames x and $A_k x$ respectively. These components are related by the equations

$$(6.2) \quad (A_k y)_i = T_{ij}^{(k)} y_j$$

where the matrices $T_k = T(A_k) = \|T_{ij}^{(k)}\|$ are given by

$$(6.3) \quad T_1, T_2, T_3, T_4 = \begin{vmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{vmatrix}, \begin{vmatrix} 1 & . & . \\ . & -1 & . \\ . & . & -1 \end{vmatrix}, \begin{vmatrix} -1 & . & . \\ . & 1 & . \\ . & . & -1 \end{vmatrix}, \begin{vmatrix} -1 & . & . \\ . & -1 & . \\ . & . & 1 \end{vmatrix}.$$

The restrictions

$$(6.6) \quad P(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m) = P(\alpha_1, \dots, \alpha_n, -\beta_1, \dots, -\beta_m).$$

may be written more explicitly as

$$(6.5) \quad \begin{aligned} P(y_1^{(1)}, y_2^{(1)}, y_3^{(1)}, \dots, y_1^{(N)}, y_2^{(N)}, y_3^{(N)}) \\ &= P(y_1^{(1)}, -y_2^{(1)}, -y_3^{(1)}, \dots, y_1^{(N)}, -y_2^{(N)}, -y_3^{(N)}) \\ &= P(-y_1^{(1)}, y_2^{(1)}, -y_3^{(1)}, \dots, -y_1^{(N)}, y_2^{(N)}, -y_3^{(N)}) \\ &= P(-y_1^{(1)}, -y_2^{(1)}, y_3^{(1)}, \dots, -y_1^{(N)}, -y_2^{(N)}, y_3^{(N)}). \end{aligned}$$

In order to determine the integrity basis, we employ the following obvious theorem.

THEOREM 1. *Let P be a polynomial function of the real quantities $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m$ which satisfies the relation*

$$(6.6) \quad P(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m) = P(\alpha_1, \dots, \alpha_n, -\beta_1, \dots, -\beta_m).$$

Then P is expressible as a polynomial in the quantities

$$(6.7) \quad \alpha_i \quad (i = 1, \dots, n) \quad \text{and} \quad \beta_j \beta_k \quad (j, k = 1, \dots, m).$$

With (6.5) and theorem 1, we then readily see that an integrity basis for

polynomial functions of y_1, \dots, y_N which are invariant under the group Γ defined by (6.1) is formed by the quantities

$$(6.8) \quad y_1^{(i)} y_1^{(j)}, \quad y_2^{(i)} y_2^{(j)}, \quad y_3^{(i)} y_3^{(j)}, \quad y_1^{(i)} y_2^{(j)} y_3^{(k)} \quad (i, j, k = 1, 2, \dots, N).$$

There are a number of theorems such as Theorem 1 above which enable us in principle to determine the integrity basis for a wide variety of problems. However, the integrity bases which are obtained upon application of such theorems will generally contain a number of redundant terms which must be eliminated. This may prove to be a matter of some difficulty. In order to avoid such difficulties, we may employ an iterative procedure. We outline below the application of this procedure to the generation of the multilinear elements of the integrity basis for functions of N vectors y_1, \dots, y_N which are invariant under the group (6.1).

We first determine the number P_n of linearly independent invariants which are multilinear in n vectors y_1, \dots, y_n . The transformation properties of the 3^n quantities $y_i^{(1)} y_j^{(2)} \dots y_k^{(n)}$ ($i, j, k = 1, 2, 3$) under change of reference frames are described by the Kronecker n th power $T_k^{[n]}$ of the matrices T_k . The number P_n is obtained by taking the mean value over the group Γ of the trace of the matrices $T_k^{[n]}$. Since the trace of the Kronecker n th power of a matrix T_k is equal to the n th power of $\text{tr } T_k$, we have

$$(6.9) \quad P_n = \frac{1}{4} \sum_{k=1}^4 (\text{tr } T_k)^n.$$

From (6.3), we see that

$$(6.10) \quad (\text{tr } T_1, \text{tr } T_2, \text{tr } T_3, \text{tr } T_4) = (3, -1, -1, -1).$$

With (6.9) and (6.10), we then have

$$(6.11) \quad P_1 = 0, \quad P_2 = 3, \quad P_3 = 6, \quad P_4 = 21, \dots$$

We now proceed to generate the multilinear elements of the integrity basis. From (6.11), we see that there are no invariants of degree 1 in y_1 and three linearly independent invariants of degree 1, 1 in y_1, y_2 . With (6.5), we readily see that

$$(6.12) \quad y_1^{(1)} y_1^{(2)}, \quad y_2^{(1)} y_2^{(2)}, \quad y_3^{(1)} y_3^{(2)}$$

are invariant under Γ . They are obviously linearly independent. The multilinear elements of the integrity basis of degree two are then comprised of the

$\binom{N}{2}$ sets of invariants obtained by replacing y_1, y_2 in the set of invariants

(6.12) by all possible sets of two different vectors chosen from y_1, \dots, y_N . From (6.11), we see there are six linearly independent invariants of degree 1, 1, 1 in y_1, y_2, y_3 . With (6.5), we readily see that these are given by

$$(6.13) \quad y_i^{(1)} y_j^{(2)} y_k^{(3)} \quad (ijk = 123, 132, 213, 231, 312, 321).$$

The multilinear elements of the integrity basis of degree three are then comprised of the $\binom{N}{3}$ sets of invariants obtained by replacing y_1, y_2, y_3 in (6.13) by all possible sets of three different vectors chosen from y_1, y_2, \dots, y_N . From (6.11), we see that there are 21 linearly independent invariants of degree 1, 1, 1, 1, in y_1, y_2, y_3, y_4 . However, we readily verify that there are also 21 linearly independent invariants of this degree which may be obtained as products of invariants of the form (6.12). Hence there are no invariants of degree 1, 1, 1, 1 in y_1, y_2, y_3, y_4 which are required as elements of the integrity basis.

This iteration process must be terminated at some stage and it is necessary to determine by one means or another an upper bound on the degree of the elements of the integrity basis. For the case under consideration, we may employ a result which says that since the group Γ is comprised of four transformations, the elements of the integrity basis must be of degree four or less. This enables us to state that the typical multilinear elements of the integrity basis for polynomial functions of N vectors invariant under the group (6.1) are given by (6.12) and (6.13). The determination of the non-linear elements of the integrity basis may be carried out in a similar fashion.

In the iterative procedure described above, we know the number P_n of linearly independent invariants of degree n . We determine by inspection the invariants of degree n which may be obtained as products of invariants of degree less than n . Suppose there are R_n such invariants. They are not necessarily all linearly independent. Suppose then that Q_n of these R_n invariants are linearly independent. We must determine these Q_n invariants and then determine $P_n - Q_n$ additional invariants I_1, I_2, \dots such that the I_1, I_2, \dots together with the Q_n invariants which are products of lower order invariants form a set of P_n linearly independent invariants. The I_1, I_2, \dots are then elements of the integrity basis. This can be a very formidable problem when P_n and Q_n are large. In Sections 7 and 9, we discuss methods which essentially reduce this problem to a number of smaller problems which may usually be solved much more readily.

7. Invariants of symmetry type $(n_1 n_2 \dots n_r)$

Let I_1, \dots, I_n be a set of linearly independent invariants which are multilinear in the quantities A_1, \dots, A_M . We choose as an example the invariants (6.13). Let

$$(7.1) \quad \begin{aligned} I_1 &= y_1^{(1)} y_2^{(2)} y_3^{(3)}, & I_2 &= y_1^{(1)} y_2^{(3)} y_3^{(2)}, & I_3 &= y_1^{(2)} y_2^{(3)} y_3^{(1)}, \\ I_4 &= y_1^{(2)} y_2^{(1)} y_3^{(3)}, & I_5 &= y_1^{(3)} y_2^{(1)} y_3^{(2)}, & I_6 &= y_1^{(3)} y_2^{(2)} y_3^{(1)}. \end{aligned}$$

Let s be that permutation of the numbers $1, \dots, M$ which carries 1 into i_1, \dots, M into i_M . The permutation s applied to the subscripts on the tensors A_1, \dots, A_M transforms the invariant $I_j(A_1, \dots, A_M)$ into the invariant

$$(7.2) \quad sI_j(A_1, \dots, A_M) = I_j(A_{i_1}, \dots, A_{i_M}).$$

We assume that the space spanned by I_1, \dots, I_n is invariant under the group S_M of all $M!$ permutations of $1, \dots, M$, i.e., the invariants $sI_j(A_1, \dots, A_M)$ are expressible as linear combinations of I_1, \dots, I_n . Thus,

$$(7.3) \quad sI_j = I_k D_{kj}(s).$$

For example, consider the transformation properties of the quantities (7.1) under permutation of the vectors y_1, y_2, y_3 among themselves. The symmetric group S_3 is comprised of six permutations

$$(7.4) \quad e, (12), (13), (23), (123), (132).$$

In table 2 below, we list the quantities $sI_j(y_1, y_2, y_3)$ for $j = 1, \dots, 6$ and for all permutations s belonging to the set (7.4).

The matrices $D(s)$ appearing in (7.3) are then seen to be given by

$$(7.5) \quad \begin{aligned} D(e) &= \begin{vmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{vmatrix}, & D(12) &= \begin{vmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{vmatrix}, \\ D(13) &= \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}, & D(23) &= \begin{vmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{vmatrix}, \\ D(123) &= \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{vmatrix}, & D(132) &= \begin{vmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \end{aligned}$$

The matrices $D(s)$ given by (7.5) which describe the transformation properties of the invariants (7.1) under the permutations of S_3 are said to form a matrix representation of degree 6 of the symmetric group S_3 . Thus, to every element

Table 2

	I_1	I_2	I_3	I_4	I_5	I_6
e	I_1	I_2	I_3	I_4	I_5	I_6
(12)	I_4	I_3	I_2	I_1	I_6	I_5
(13)	I_6	I_5	I_4	I_3	I_2	I_1
(23)	I_2	I_1	I_6	I_5	I_4	I_3
(123)	I_3	I_4	I_5	I_6	I_1	I_2
(132)	I_5	I_6	I_1	I_2	I_3	I_4

s of S_3 there corresponds a matrix $D(s)$ such that to the product $u = ts$ of two permutations corresponds the matrix

$$(7.6) \quad D(u) = D(t)D(s).$$

For example,

$$(7.7) \quad (13)(23) = (132)$$

and we see from (7.5) that

$$(7.8) \quad D(13)D(23) = D(132).$$

The invariants (7.1) are said to form the carrier space of the representation

$$(7.9) \quad \Gamma = \{D(e), D(12), \dots, D(132)\}.$$

Consider now the invariants defined by

$$(7.10) \quad \begin{aligned} J_1 &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \\ J_2 &= I_1 + I_4 - I_5 - I_6, & J_3 &= I_3 + I_6 - I_1 - I_2, \\ J_4 &= I_1 + I_6 - I_3 - I_4, & J_5 &= I_4 + I_5 - I_1 - I_2, \\ J_6 &= I_1 + I_3 + I_5 - I_2 - I_4 - I_6. \end{aligned}$$

From Table 2, we readily obtain the transformation properties of the invariants J_1, \dots, J_6 under the permutations of y_1, y_2 and y_3 . We list the quantities sJ_i in Table 3.

Table 3

	J_1	J_2	J_3	J_4	J_5	J_6
e	J_1	J_2	J_3	J_4	J_5	J_6
(12)	J_1	J_2	$-J_2 - J_3$	J_5	J_4	$-J_6$
(13)	J_1	J_3	J_2	J_4	$-J_4 - J_5$	$-J_6$
(23)	J_1	$-J_2 - J_3$	J_3	$-J_4 - J_5$	J_5	$-J_6$
(123)	J_1	J_3	$-J_2 - J_3$	$-J_4 - J_5$	J_4	J_6
(132)	J_1	$-J_2 - J_3$	J_2	J_5	$-J_4 - J_5$	J_6

We thus have

$$(7.11) \quad sJ_p = J_q H_{qp}(s)$$

where the matrices $H(s)$ form a matrix representation of S_3 which is said to be equivalent to the representation $D(s)$. From Table 3, we see that the matrices $H(s)$ are all of the form

$$(7.12) \quad H(s) = \begin{pmatrix} K(s) & & & & & & \\ & L(s) & & & & & \\ & & M(s) & & & & \\ & & & N(s) & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix}$$

where K, L, M and N are $1 \times 1, 2 \times 2, 2 \times 2$ and 1×1 matrices respectively and where all of the non-zero components of H appear in the matrices K, L, M and N . The sets of matrices $K(s), \dots, N(s)$ are listed in Table 4.

Table 4

s	e	(12)	(13)	(23)	(123)	(132)
$K(s)$	1	1	1	1	1	1
$L(s)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$
$M(s)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$
$N(s)$	1	-1	-1	-1	1	1

The sets of matrices $K(s), \dots, N(s)$ also form representations of the symmetric group S_3 . The invariants $J_1, (J_2, J_3), (J_4, J_5)$ and J_6 form the carrier spaces for the representations

$$(7.13) \quad \Gamma_1 = \{K(s)\}, \quad \Gamma_2 = \{L(s)\}, \quad \Gamma_3 = \{M(s)\}, \quad \Gamma_4 = \{N(s)\}$$

respectively. We say that the representation $\Gamma = \{D(s)\}$ has been decomposed into the direct sum of the representations $\Gamma_1, \dots, \Gamma_4$. If a matrix representation can be decomposed in this fashion, it is said to be reducible. If not, it is said to be irreducible. Each of the representations (7.13) are irreducible representations of the symmetric group S_3 . The quantity

$$(7.14) \quad \text{char} \{K(s)\} = [\text{tr } K(e), \text{tr } K(12), \text{tr } K(13), \text{tr } K(23), \text{tr } K(123), \text{tr } K(132)]$$

is referred to as the character of the representation $\Gamma_1 = \{K(s)\}$. There are only three inequivalent irreducible representations associated with the symmetric group S_3 . These are denoted by

$$(7.15) \quad (3), \quad (21), \quad (111).$$

The characters of these irreducible representations are given ([5]) by

$$(7.16) \quad \begin{aligned} \text{char}(3) &= [1, 1, 1, 1, 1, 1], \\ \text{char}(21) &= [2, 0, 0, 0, -1, -1], \\ \text{char}(111) &= [1, -1, -1, -1, 1, 1]. \end{aligned}$$

We see from table 4 that

$$(7.17) \quad \begin{aligned} \text{char} \{K(s)\} &= \text{char}(3), \\ \text{char} \{L(s)\} &= \text{char} \{M(s)\} = \text{char}(21), \\ \text{char} \{N(s)\} &= \text{char}(111). \end{aligned}$$

This reflects the fact that the character of any irreducible representation of S_3 must equal either $\text{char}(3)$, $\text{char}(21)$ or $\text{char}(111)$.

The invariant J_1 defined by (7.10) forms the carrier space for the irreducible representation $\{K(S)\}$ for which $\text{char} \{K(s)\}$ is equal to $\text{char}(3)$. We then refer to J_1 as a set of invariants of symmetry type (3). The invariants J_2 and J_3 defined by (7.10) form the carrier space for the representation $\{L(s)\}$. Since $\text{char} \{L(s)\} = \text{char}(21)$, we refer to J_2 and J_3 as a set of invariants of symmetry type (21). Similarly the invariants J_4 , J_5 and J_6 defined by (7.10) are referred to as sets of invariants of symmetry types (21) and (111) respectively.

The number of inequivalent irreducible representations of the symmetric group S_M is equal to the number of partitions of M , i.e., the number of solutions in positive integers of the equation

$$(7.18) \quad n_1 + n_2 + \dots + n_r = M, \quad n_1 \geq n_2 \geq \dots \geq n_r.$$

For example, the partitions of 4 are given by 4, 31, 22, 211, 1111 and the inequivalent irreducible representations of S_4 are denoted by

$$(7.19) \quad (4), \quad (31), \quad (22), \quad (211), \quad (1111).$$

Similarly, the partitions of 5 are given by 5, 41, 32, 311, 221, 2111, 11111 and the inequivalent irreducible representations of S_5 are denoted by

$$(7.20) \quad (5), \quad (41), \quad (32), \quad (311), \quad (221), \quad (2111), \quad (11111).$$

The characters of the various irreducible representations of S_M may be found in the literature for $M \leq 15$ (see [5] for $M = 1, \dots, 8$). If a set of invariants J_1, \dots, J_p forms the carrier space for an irreducible representation Γ of S_M

for which $\text{char } \Gamma = \text{char}(n_1 n_2 \dots n_r)$, we say that the invariants J_1, \dots, J_p form a set of invariants of symmetry type $(n_1 n_2 \dots n_r)$.

In the next section, we give an example to show how we may employ the notion of a set of invariants of symmetry type $(n_1 n_2 \dots n_r)$ to ease the burden of computation involved in determining the multilinear elements of an integrity basis.

8. Integrity basis for N symmetric second-order traceless tensors A_1, \dots, A_N - the proper orthogonal group

We outline the computation yielding the multilinear elements of this integrity basis. We borrow from the discussion of Spencer and Rivlin [16] the following results.

(i) Every multilinear element of the integrity basis involves at most six tensors and is of the form

$$(8.1) \quad \text{tr } A_i A_j \dots A_k.$$

(ii) The trace of a matrix product formed from symmetric 3×3 matrices is unaltered by cyclic permutation of the factors in the product and is also unaltered if the order of the factors in the product is reversed.

For example, we have

$$(8.2) \quad \begin{aligned} \text{tr } A_1 A_2 A_3 &= \text{tr } A_2 A_3 A_1 = \text{tr } A_3 A_1 A_2 \\ &= \text{tr } A_3 A_2 A_1 = \text{tr } A_1 A_3 A_2 = \text{tr } A_2 A_1 A_3. \end{aligned}$$

We may readily compute (see [11]) the number $p_{(n_1 n_2 \dots)}$ of sets of linearly independent invariants of symmetry type $(n_1 n_2 \dots n_r)$. These quantities are listed in Table 5.

Table 5

$(n_1 n_2 \dots)$	(2)	(3)	(4)	(22)	(5)	(41)	(32)	(221)	(11111)	(6)	(42)	(321)	(3111)	(222)
$p_{(n_1 n_2 \dots)}$	1	1	1	2	1	1	1	1	1	2	3	1	1	2
$q_{(n_1 n_2 \dots)}$	0	0	1	1	1	1	1	0	0	2	3	1	0	2
$N_{(n_1 n_2 \dots)}$	1	1	1	2	1	4	5	5	1	1	9	16	10	5

In Table 5, $N_{(n_1 n_2 \dots)}$ denotes the number of invariants comprising a set of invariants of symmetry type $(n_1 n_2 \dots)$ and $q_{(n_1 n_2 \dots)}$ denotes the number of sets of invariants which may be obtained as products of lower order invariants. The computation then proceeds as follows.

(i) *Invariants linear in A_1 .* Since $\text{tr } A_1 = 0$, there are no linearly independent invariants of degree one in A_1 .

(ii) *Invariants multilinear in A_1, A_2 .* There is only a single linearly independent invariant of this degree which is given by

$$(8.3) \quad \text{tr } A_1 A_2.$$

We see that

$$(8.4) \quad e \text{ tr } A_1 A_2 = \text{tr } A_1 A_2, \quad (1\ 2) \text{tr } A_1 A_2 = \text{tr } A_2 A_1 = \text{tr } A_1 A_2.$$

Thus, $\text{tr } A_1 A_2$ forms the carrier space for a matrix representation $D(s)$ of degree one which is given by

$$(8.5) \quad D(e) = \|1\|, \quad D(1\ 2) = \|1\|.$$

From (8.5) and the character tables for S_2 given in [5], we see that $\text{char } \{D(s)\} = [1, 1] = \text{char}(2)$. Hence the invariant $\text{tr } A_1 A_2$ forms a set of invariants of symmetry type (2).

(iii) *Invariants multilinear in A_1, A_2, A_3 .* There is only a single linearly independent invariant of this degree which is given by

$$(8.6) \quad \text{tr } A_1 A_2 A_3.$$

With (8.2), we see that

$$(8.7) \quad \begin{aligned} e \text{ tr } A_1 A_2 A_3 &= \text{tr } A_1 A_2 A_3, \\ (1\ 2) \text{tr } A_1 A_2 A_3 &= \text{tr } A_2 A_1 A_3 = \text{tr } A_1 A_2 A_3, \dots, \\ (1\ 3\ 2) \text{tr } A_1 A_2 A_3 &= \text{tr } A_3 A_1 A_2 = \text{tr } A_1 A_2 A_3. \end{aligned}$$

Thus, $\text{tr } A_1 A_2 A_3$ is readily seen to form the carrier space for a representation $\{E(s)\}$ of S_3 such that $\text{char } \{E(s)\} = \text{char}(3)$. Hence, $\text{tr } A_1 A_2 A_3$ forms a set of invariants of symmetry type (3).

(iv) *Invariants multilinear in A_1, A_2, A_3, A_4 .* From Table 5, we see that there are five linearly independent invariants of this degree which form one set of invariants of symmetry type (4) and two sets of invariants of symmetry type (22). We may obtain three invariants as products of invariants of the form (8.3). These are given by

$$(8.8) \quad \text{tr } A_1 A_2 \text{ tr } A_3 A_4, \quad \text{tr } A_1 A_3 \text{ tr } A_2 A_4, \quad \text{tr } A_1 A_4 \text{ tr } A_2 A_3.$$

We may upon investigating the manner in which the invariants (8.8) behave under permutations of the tensors A_1, \dots, A_4 establish that the invariants (8.8) may be split into a set of invariants of symmetry type (4) and a set of invariants of symmetry type (22). Thus, $q_{(4)} = q_{(22)} = 1$. We then see that we require one set of invariants of symmetry type (22) as elements of the integrity basis (since $p_{(22)} = 2$ and $q_{(22)} = 1$). This set is given by

$$(8.9) \quad \begin{aligned} I_1(A_1, A_2, A_3, A_4) &= Y \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{tr } A_1 A_3 A_2 A_4, \\ I_2(A_1, A_2, A_3, A_4) &= Y \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \text{tr } A_1 A_2 A_3 A_4, \quad (22) \end{aligned}$$

where $Y(\dots)$ denotes a Young symmetry operator (see [11] for a discussion of the properties of $Y(\dots)$ and further references). For example, we have

$$(8.10) \quad Y\left(\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}\right) = \{e+(12)\}\{e+(34)\}\{e-(13)\}\{e-(24)\}.$$

(v) *Invariants multilinear in A_1, \dots, A_5 .* There are 10 invariants of this degree which may be obtained as products of the invariants (8.3) and (8.6). These are given by

$$(8.11) \quad \text{tr } A_1 A_2 \text{ tr } A_3 A_4 A_5, \quad \text{tr } A_1 A_3 \text{ tr } A_2 A_4 A_5, \quad \dots, \quad \text{tr } A_4 A_5 \text{ tr } A_1 A_2 A_3.$$

The ten invariants (8.11) form the carrier space for a representation which is denoted by $(2) \cdot (3)$ and is referred to as the direct product of the representations (2) and (3). The decomposition of representations $(n_1 n_2 \dots) \cdot (m_1 m_2 \dots)$ has been considered by Murnaghan ([8]). We see from tables given in [8] that

$$(8.12) \quad (2) \cdot (3) = (5) + (41) + (32)$$

and hence

$$(8.13) \quad q_{(5)} = q_{(41)} = q_{(32)} = 1.$$

Then from table 5 we see that one set of invariants of symmetry types (11111) and (221) are required as elements of the integrity basis. These will be given by

$$(8.14) \quad Y\left(\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}\right) \text{tr } A_1 A_2 A_3 A_4 A_5$$

and

$$(8.15) \quad Y\left(\begin{matrix} 1 & 2 \\ 3 & 4 \\ 5 \end{matrix}\right) \text{tr } A_1 A_2 A_3 A_4 A_5, \dots$$

(vi) *Invariants multilinear in A_1, \dots, A_6 .* There are $15 + 30 + 10 = 55$ invariants of this degree which may be obtained as products of lower order invariants. These are given by

- (8.16) 1) $\text{tr } A_1 A_2 \text{ tr } A_3 A_4 \text{ tr } A_5 A_6, \dots, 15$ invariants, $(6) + (42) + (222)$;
- 2) $\text{tr } A_1 A_2 I_j(A_3, A_4, A_5, A_6)$ ($j = 1, 2$), $\dots, 30$ invariants, $(42) + (321) + (222)$;
- 3) $\text{tr } A_1 A_2 A_3 \text{ tr } A_4 A_5 A_6, \dots, 10$ invariants, $(6) + (42)$.

In (8.16), we have listed on the right the irreducible representations into which the representations for which the invariants (8.16)₁, ..., (8.16)₃ form the carrier spaces may be decomposed. We then see from (8.16) that

$$(8.17) \quad q_{(6)} = q_{(222)} = 2, \quad q_{(42)} = 3, \quad q_{(321)} = 1.$$

From Table 5, we see that

$$(8.18) \quad p_{(6)} = p_{(222)} = 2, \quad p_{(42)} = 3, \quad p_{(321)} = 1, \quad p_{(3111)} = 1.$$

With (8.17) and (8.18), we see that there must be a single set of 10 invariants of symmetry type (3 1 1 1) present in the integrity basis. This is given by

$$(8.19) \quad Y \begin{pmatrix} 1 & 2 & 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} \text{tr } A_1 A_2 A_3 A_4 A_5 A_6, \dots$$

We thus see that the typical multilinear elements of the integrity basis are given by the invariants (8.3), (8.6), (8.9), (8.14), (8.15) and (8.19). The non-linear elements of the integrity basis are readily obtained once the multilinear elements have been determined. Details of the procedure involved in obtaining the non-linear elements from the multilinear are given by Smith [11].

9. Integrity bases for anisotropic materials

The decomposition procedure employed in Section 5 may be used effectively when we seek to determine the general form of the polynomial expression

$$(9.1) \quad \phi = \phi(B_{i_1 \dots i_p}, C_{i_1 \dots i_q}, \dots)$$

which is invariant under a crystallographic group Γ . We note that the problem of determining the form of the function $T_{i_1 \dots i_n} = \phi_{i_1 \dots i_n}(B_{i_1 \dots i_p}, C_{i_1 \dots i_q}, \dots)$ which is invariant under Γ may be reduced to the problem of determining the form of a scalar-valued function $\phi^*(B_{i_1 \dots i_p}, C_{i_1 \dots i_q}, \dots, T_{i_1 \dots i_n})$ which is invariant under Γ where ϕ^* is linear in $T_{i_1 \dots i_n}$. There is consequently no loss in generality in restricting consideration to the case (9.1). Let $S(A_k)$, $R(A_k)$, ... denote the matrix representations of Γ which define the transformation properties under Γ of the column matrices B , C , ... whose entries are the independent components $B_1, \dots, B_p, C_1, \dots, C_q, \dots$ of the tensors $B_{i_1 \dots i_p}, C_{i_1 \dots i_q}, \dots$. We may then express (9.1) as

$$(9.2) \quad \phi = \bar{\phi}(B, C \dots)$$

where the polynomial function $\bar{\phi}$ is subject to the restriction that

$$(9.3) \quad \bar{\phi}(B, C, \dots) = \bar{\phi}(S(A_k)B, R(A_k)C, \dots)$$

for all A_k belonging to Γ . We may choose matrices Q, P so that

$$\begin{aligned}
 QS(A_k)Q^{-1} &= n_1 \Gamma_1(A_k) \dot{+} \dots \dot{+} n_r \Gamma_r(A_k), \\
 PR(A_k)P^{-1} &= m_1 \Gamma_1(A_k) \dot{+} \dots \dot{+} m_r \Gamma_r(A_k), \\
 QB &= \beta_{11} \dot{+} \dots \dot{+} \beta_{1n_1} \dot{+} \dots \dot{+} \beta_{r1} \dot{+} \dots \dot{+} \beta_{rn_r}, \\
 PC &= \beta_{1,n_1+1} \dot{+} \dots \dot{+} \beta_{1,n_1+m_1} \dot{+} \dots \dot{+} \beta_{r,n_r+1} \dot{+} \dots \dot{+} \beta_{r,n_r+m_r}, \dots
 \end{aligned}
 \tag{9.4}$$

where the transformation properties under Γ of the $\beta_{1i}, \dots, \beta_{ri}$ are defined by the irreducible representations $\Gamma_1, \dots, \Gamma_r$, respectively. With (9.4), we see that (9.2) is expressible in the form

$$\phi = \phi^*(\beta_{1i}, \dots, \beta_{rj}) \quad (i = 1, 2, \dots; j = 1, 2, \dots)
 \tag{9.5}$$

where the function ϕ^* is subject to the restrictions that

$$\phi^*(\beta_{1i}, \dots, \beta_{rj}) = \phi^*(\Gamma_1(A_k)\beta_{1i}, \dots, \Gamma_r(A_k)\beta_{rj})
 \tag{9.6}$$

must hold for all A_k belonging to Γ . We observe that the problem of determining the general form of $\phi(B, C)$ and of $\psi(D, E, F)$, when translated into the form (9.6), will differ only in the number of quantities $\beta_{1i}, \dots, \beta_{rj}$ appearing as arguments of the functions ϕ^* and ψ^* . Thus, if we solve (9.6) for the case where $i = 1, \dots, k_1, \dots, j = 1, \dots, k_r$, where k_1, \dots, k_r are arbitrary, we have then solved the most general problem which may arise. Since we are considering the case where $\phi^*(\dots)$ in (9.6) is a polynomial function, we must determine an integrity basis for functions of $\beta_{1i}, \dots, \beta_{rj}$ which are invariant under Γ . An integrity basis for functions of $\beta_{1i}, \dots, \beta_{rj}$ invariant under Γ is comprised of a set of polynomial functions I_1, I_2, \dots of the $\beta_{1i}, \dots, \beta_{rj}$, each of which is invariant under Γ such that any polynomial function of the $\beta_{1i}, \dots, \beta_{rj}$ which is invariant under Γ is expressible as a polynomial in the elements I_1, I_2, \dots of the integrity basis.

We consider the problem of determining an integrity basis for functions of an arbitrary number of tensors of any order which are invariant under the crystallographic group D_2 . The crystal class D_2 possesses three mutually perpendicular two fold axes of rotation. The matrices defining the symmetry group Γ are given by

$$A_1, \dots, A_4 = \left\| \begin{matrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{matrix} \right\|, \left\| \begin{matrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{matrix} \right\|, \left\| \begin{matrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{matrix} \right\|, \left\| \begin{matrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{matrix} \right\|.
 \tag{9.7}$$

The irreducible representations $\Gamma_1, \dots, \Gamma_4$ associated with the group (9.7) are all of degree 1 and are listed below. The quantities which form the carrier spaces for the irreducible representations $\Gamma_1, \dots, \Gamma_4$ are denoted by $a_1, a_2, \dots, d_1, d_2, \dots$ respectively.

Table 6
Irreducible representations of D_2

	A ₁	A ₂	A ₃	A ₄	
$\Gamma_1(A_2)$	1	1	1	1	a_1, a_2, \dots, a_p
$\Gamma_2(A_2)$	1	1	-1	-1	b_1, b_2, \dots, b_q
$\Gamma_3(A_2)$	1	-1	1	-1	c_1, c_2, \dots, c_r
$\Gamma_4(A_2)$	1	-1	-1	1	d_1, d_2, \dots, d_s

We readily see that every scalar-valued polynomial function $V(a_i, b_j, c_k, d_m)$ of the quantities a_i, \dots, d_m which is invariant under D_2 is expressible as a polynomial in the quantities

$$(9.8) \quad \begin{aligned} &1) \ a_i, \\ &2) \ b_{j_1} b_{j_2}, c_{k_1} c_{k_2}, d_{m_1} d_{m_2}, \\ &3) \ b_j c_k d_m \end{aligned}$$

where $i = 1, \dots, p$; $j, j_1, j_2 = 1, \dots, q$; $k, k_1, k_2 = 1, \dots, r$; $m, m_1, m_2 = 1, \dots, s$.

Every polynomial function $V_2(a_i, \dots, d_m)$ whose transformation properties under D_2 are defined by Γ_2 is expressible as a linear combination of the quantities

$$(9.9) \quad \begin{aligned} &1) \ b_j \ (j = 1, \dots, q), \\ &2) \ c_k d_m \ (k = 1, \dots, r; m = 1, \dots, s) \end{aligned}$$

with coefficients which are polynomials in the quantities (9.8).

Every polynomial function $V_3(a_i, \dots, d_m)$ whose transformation properties under \bar{D}_2 are defined by Γ_3 is expressible as a linear combination of the quantities

$$(9.10) \quad 1) \ c_k \ (k = 1, \dots, r), \quad 2) \ b_j d_m \ (j = 1, \dots, q; m = 1, \dots, s)$$

with coefficients which are polynomials in the quantities (9.8).

Every polynomial function $V_4(a_i, \dots, d_m)$ whose transformation properties under D_2 are defined by Γ_4 is expressible as a linear combination of the quantities

$$(9.11) \quad 1) \ d_m \ (m = 1, \dots, s), \quad 2) \ b_j c_k \ (j = 1, \dots, q; k = 1, \dots, r)$$

with coefficients which are polynomials in the quantities (9.8).

We consider the problem of determining the form of a symmetric second-order tensor-valued function $\sigma(E)$ of a symmetric second-order tensor E which is invariant under D_2 . We employ the notation

$$(9.12) \quad \begin{aligned} (\sigma_1, \dots, \sigma_6) &= (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}), \\ (E_1, \dots, E_6) &= (E_{11}, E_{22}, E_{33}, E_{23}, E_{31}, E_{12}). \end{aligned}$$

We then have

$$(9.13) \quad (A_x \sigma)_i = T_{ij}(A_x) \sigma_j, \quad (A_x E)_i = T_{ij}(A_x) E_j$$

where

$$(9.14) \quad \begin{aligned} T(A_1) &= \text{diag}(1, 1, 1, 1, 1, 1), & T(A_2) &= \text{diag}(1, 1, 1, 1, -1, -1), \\ T(A_3) &= \text{diag}(1, 1, 1, -1, 1, -1), & T(A_4) &= \text{diag}(1, 1, 1, -1, -1, 1). \end{aligned}$$

With (9.14), we readily see that the quantities listed under 1, ..., 4 below form carrier spaces for the irreducible representations $\Gamma_1, \dots, \Gamma_4$ of D_2 .

$$(9.15) \quad 1) \ \sigma_1, \sigma_2, \sigma_3, E_1, E_2, E_3, \quad 2) \ \sigma_4, E_4, \quad 3) \ \sigma_5, E_5, \quad 4) \ \sigma_6, E_6.$$

With (9.8), ..., (9.11) and (9.15), we see that

$$(9.16) \quad \begin{aligned} \sigma_1 &= \phi_1, & \sigma_4 &= \phi_4 E_4 + \phi_5 E_5 E_6, \\ \sigma_2 &= \phi_2, & \sigma_5 &= \phi_6 E_5 + \phi_7 E_4 E_6, \\ \sigma_3 &= \phi_3, & \sigma_6 &= \phi_8 E_6 + \phi_9 E_4 E_5 \end{aligned}$$

where ϕ_i are polynomials in the quantities

$$(9.17) \quad E_1, E_2, E_3, E_4^2, E_5^2, E_6^2, E_4 E_5 E_6.$$

Let us now consider the problem of determining the form of the constitutive expressions which are invariant under the group C_{2v} which is defined by the matrices

$$(9.18) \quad A_1, \dots, A_4 = \begin{vmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix}, \begin{vmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{vmatrix}, \begin{vmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{vmatrix}, \begin{vmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{vmatrix}.$$

There are four inequivalent irreducible representations $\Gamma_1, \dots, \Gamma_4$ associated with the group C_{2v} and they are the same as those listed above for the group D_2 . The results (9.8), ..., (9.11) are again applicable and no further effort is required. The same statement is also applicable for the crystallographic group C_{2h} . A more detailed discussion of this procedure and complete results for most of the crystal classes are given by Kiral and Smith ([3]), and Kiral, Smith and Smith ([4]).

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