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Definable quantifiers in second order arithmetic and elementary extensions of ω-models

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CONTENTS

§ 0. Introduction and terminology

In the present paper we study definable quantifiers in models of second order arithmetic in connection with elementary extensions of countable ω-models.

The idea of generalizing the notion of a quantifier goes back as far as Mostowski's paper [21]. About a decade later several authors observed that definable quantifiers gave a particulary elegant constructions of elementary extensions of countable models. These constructions fitted best the models of second order arithmetic A_2 (a theory widely studied since Mostowski's paper [22]) and of ZF set theory. In particular, Keisler proved in [13] (see also [14]) a theorem saying that every countable ω-model of A_2 has an elementary extension, which is also an ω-model. Mostowski and Suzuki proved in [26] that every countable β-model has an elementary extension, which is an ω-model but not a β-model. Since the early seventies several papers have been written, to mention only Mostowski's [23], [24], [25], Marek's [19], Krivine and McAloon's [17], Hutchinson's [10] or Dubiel's [3] and [4] (for a more complete bibliography see [1]). In particular, Mrs. Dubiel gave in [4] a characterization of definable regular quantifiers in models of set theory ZFC and asked for a similar result for models of second order arithmetic A_2. In the same paper she asked whether there exists an ω-model of A_2 with many nonequivalent definable regular quantifiers. In the present paper we answer the second problem in the positive. However, the results of Sections 4 and 5 show that, contrary to the case of ZFC, the existence of different definable quantifiers strongly depends on the particular model of A_2. Also the methods of defining quantifiers are so different that at the present moment we cannot see any way of giving them a full characterization. In this paper we also consider the problem whether every elementary extension can be obtained by means of a definable quantifier. We answer negatively in case where only regular quantifiers are considered. We prove a surprising result that some generic extensions can be obtained by this method.

In all our considerations we include the schema of choice in the axiomatization of A_2. Very little is known about the existence of definable quantifiers in models of second order arithmetic without choice (see [1] or [6]).

Our main tool in proving results of Sections 3 and 4 is the so-called accurate interpretation of set theory in an extension of A_2. Apart from being applied in proofs of other theorems, the notion of an accurate interpretation itself is worth studying. Now we give our motivation for this.

Let us imagine living beings who know only natural and real numbers (which will be identified with sets of natural numbers), and therefore then use only the language of second order arithmetic. Our paper tries to answer the question of how they can imagine the "external world". After a formalization this leads to the notion of interpretation. Since their knowledge is limited to numbers, we can expect that they know better than we do, how numbers really behave. That is why we consider extensions of A_2 (we still believe that real numbers, whatever they are, have properties described by axioms of A_2). As we usually do in mathematical logic, we identify the world with set theory.

It is natural to restrict ourselves to interpretations whose numbers are isomorphic to numbers of the given extension of A_2 and such that the isomorphism can be defined in this extension – our beings will certainly want their reals to be precisely the arithmetical part of the external world which they try to imagine.

We know of the existence of such interpretations. For example, a classical result (Kreisel [16], Zbierski [33]) says that the theory ZFC^- can be interpreted in A_2 with all the above requirements satisfied. The failure of power set axiom is necessary – we prove that ZFC^-+"$P(\omega)$ is a set" cannot be interpreted in any consistent extension of A_2 under our assumption of preservation of reals. Therefore our beings cannot imagine any world in which there exist powers of infinite sets. It is natural to ask if they can understand the notion of uncountability. The present paper gives a positive answer to this question. The notion of uncountability formalized with the help of accurate interpretation allows one to define various quantifiers in some models of A_2.

The paper is organized as follows. In the first section we recall the notion of a definable quantifier and intruduce a more general notion of a pseudoquantifier. We give a brief exposition of the work of Keisler, Mostowski, Krivine and McAloon concerning applications of the notion of a definable quantifier to elementary extensions of countable models. The second section contains applications of the methods of the first section to the study of countable models of set theory. We recall Mrs. Dubiel's characterization of definable quantifiers in ZFC set theory and then we prove that there exist elementary extensions which are not generated by a definable quantifier. Our next aim is to carry over the results of Section 2 to models of a certain extension of A_2. In order to do this we introduce in Section 3 the notion of an accurate interpretation of set theory in an extension of A_2 and investigate properties of such interpretations. Further, in Section 4, we generalize the results of Section 3 and prove some of the main theorems of the paper. We show a model of A_2 and its elementary extension that cannot be obtained by using a definable regular quantifier. We also show a model of A_2, which has infinitely many nonequivalent definable regular quantifiers, thus solving a problem posed in [4]. In Section 5 we recall our result from [5] that there exists a countable β-model of A_2 with the property that every Cohen generic extension of it is elementary. We show that this forcing extension can be obtained by the method of definable quantifiers. We also give a characterization of elementary extensions of this model in terms of generic extensions.

Throughout the paper we use standard notation. By second-order arithmetic A_2 we mean a theory formulated in first-order language with two unary predicates N and S, partial operations $+, \cdot$ and a binary relation $\in$. $N(x)$ stands for "x is a natural number" and $S(x)$ for "x is a set of natural numbers". The axioms include sentences saying that each object is either a natural number or a set of them (not both at once!) and if $x \in y$ then x is a number and y is a set. We usually use $n, m, k, l, \ldots$ as variables ranging over numbers and $x, y, z, \ldots$ as variables for sets (or seldom for arbitrary objects), thus avoiding the use of predicates N and S in formulas. As further axioms we take the usual Peano axioms with induction written as one statement about sets of numbers, the axiom of extensionality and two schemes of axioms: comprehension $\exists x \forall n [n \in x \equiv \varphi]$ and choice $\forall n \exists n \varphi(n, x) \to \exists x \forall n \varphi(n, (x)_n)$ where $(x_n) = \{m\colon J(n, m) \in x\}$, J being the pairing function: $J(n, m) = 2^n(2m+1)-1$ for natural numbers.

By ZFC^- we mean the theory formulated in the language of set theory whose axioms are those of ZF without the power set axiom, but with the following schema of choice:

$$\forall x \in a \exists y \varphi(x, y) \to \exists f [\mathrm{Func}(f) \;\&\; \mathrm{dom}(f) = a \;\&\; \forall x \in a \varphi(x, f(x))].$$

By a cardinal number in ZFC^- we mean an infinite intitial ordinal, i.e. an aleph. We also informally use definable classes, identifying them with their definitions. For more information about A_2 and ZFC^- we refer the reader to [1].

In the paper we assume familiarity with forcing technique, as described e.g. in [12]. We also use forcing with a proper class of conditions and in this case we recommend the paper [32].

The sign ■ marks the end of a proof.

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§ 1. Quantifiers and elementary extensions

In this section we shall make a few general observations concerning elementary extensions of models in connection with the notion of a definable quantifier.

We consider a first-order language $\mathscr{L}$ and a structure M for $\mathscr{L}$. By $\mathscr{L}_M$ we shall denote the language $\mathscr{L}$ with additional constants for all elements of M. We usually make no distinction between elements of M and the constants denoting them. By Def(M) we denote the family of first-order definable (possibly with parameters from M) subsets of M, i.e. subsets of M definable in M by means of formulas of $\mathscr{L}_M$.

Let us suppose that we are given a family of sets $q \subseteq \mathrm{Def}(M)$. We enlarge the language $\mathscr{L}_M$ by adding a new symbol q and accept a new clause in defining formulas of the new language, denoted later as $\mathscr{L}_M(q)$:

if φ is a formula of $\mathscr{L}_M$ and x a variable then $qx\varphi$ is a formula of $\mathscr{L}_M(q)$.

Next we define the satisfaction relation for formulas of $\mathscr{L}_M(q)$ in the model (M, q) in the natural way, the only nonevident case being

$$(M, q) \models qx\varphi(x, x_1, \ldots, x_n)[a_1, \ldots, a_n] \quad \text{iff} \quad \{a \in M\colon M \models \varphi[a, a_1, \ldots, a_n]\} \in q.$$

Definition 1.1. The family q will be called a *pseudoquantifier in the model* M iff the model (M, q) satisfies the following axioms:

(1) $\forall x[\varphi(x) \to \psi(x)] \to (qx\varphi(x) \to qx\psi(x))$,

(2) $qx[\varphi(x) \vee \psi(x)] \to (qx\varphi(x) \vee qx\psi(x))$,

(3) $qx[x = x]$,

(4) $\neg\exists y qx[x = y]$.

A pseudoquantifier q is *regular* iff the model (M, q) satisfies also the following axioms:

(5) $qx\exists y\varphi \to \exists y qx\varphi \vee qy\exists x\varphi$.

Example 1.2. Let N be an elementary extension of the model M. We define q to be the family of those sets $X \in \mathrm{Def}(M)$, $X = \{a \in M\colon M \models \varphi[a, a_1, \ldots, a_n]\}$, for which there exists an element $b \in N - M$ such that $N \models \varphi[b, a_1, \ldots, a_n]$. It is easy to see that q is a regular pseudoquantifier. It will be denoted by $q_{M,N}$.

We are interested in the following problem: given a countable model M, we want to characterize those sets F of formulas of $\mathscr{L}_M$ with one free variable which have the property that there exists an elementary extension N

of M such that $\varphi \in F$ iff there exists $b \in N - M$ s.t. $N \models \varphi[b]$. Our problem is thus asking for a characterization of all pseudoquantifiers $q_{M,N}$.

An important strengthening of the notion of a pseudoquantifier is the notion of a definable quantifier. We extend the language $\mathscr{L}_M$ by adding a new symbol Q and accept a new clause in the definition of formulas of the new language, denoted by $\mathscr{L}_M(Q)$:

if φ is a formula $\mathscr{L}_M(Q)$ and x a variable then $Qx\varphi$ is a formula of $\mathscr{L}_M(Q)$.

The corresponding clause in the definition of satisfaction is the following

$$(M, q) \models Qx\,\varphi(x, x_1, \ldots, x_n)[a_1, \ldots, a_n] \quad \text{iff} \quad \{a \in M: (M, q) \models \varphi[a, a_1, \ldots, a_n]\} \in q,$$

where q, as before, is a subfamily of $\mathrm{Def}(M)$. Following [4] we introduce the following

DEFINITOǸ 1.3. A pseudoquantifier q is a *definable quantifier in the model* M iff the model (M, q) satisfies axioms (1)–(4) of Definition 1.1 for all formulas of the language $\mathscr{L}_M(Q)$. A definable quantifier q is *regular* iff the model (M, q) satisfies the schema (5) as well.

Now we present another definition of a definable quantifier. Following [17] we introduce

DEFINITION 1.4. By an explicit generalized quantifier in a model M we mean a map Q^0 which to every pair $\langle \varphi, x \rangle$, φ being a formula of $\mathscr{L}_M$ and x a variable, corresponds a formula of $\mathscr{L}_M$, denoted by $Q^0 x\varphi$, in such a way that the following conditions hold:

(1) The variable x is not free in $Q^0 x\varphi$,

(2) if y is a variable different from x then y is a free variable in φ iff it is free in $Q^0 x\varphi$,

(3) if a variable y does not occur in φ and a formula ψ is obtained from φ by replacing each free occurrence of x by y, then

$$M \models Q^0 x\varphi \equiv Q^0 y\psi,$$

(4) if a variable y does not occur in φ and a formula ψ is obtained from φ by replacing each free occurrence of a variable z by y, then

$$M \models y = z \rightarrow (Q^0 x\varphi \equiv Q^0 x\psi),$$

(5) in the model M all axioms analogous to (1)–(4) of Definition 1.1 with Q^0 in place of q are satisfied.

We can easily verify that if Q^0 is an explicit definable quantifier in M then the family $q \subseteq \mathrm{Def}(M)$ consisting of sets of form $\{a \in M:$

$M \models \varphi[a]\}$ for φ such that $M \models Q^0 x \varphi(x)$ is a definable quantifier in M. Moreover, we have

$$M \models Q^0 x \varphi[a_1, \ldots, a_n] \quad \text{iff} \quad (M, q) \models Qx \varphi[a_1, \ldots, a_n],$$

for any formula φ of the language $\mathscr{L}_M$. In particular, it follows that for every formula φ of $\mathscr{L}_M(Q)$ there exists a formula φ^0 of $\mathscr{L}_M$ such that

$$(M, q) \models \varphi[a_1, \ldots, a_n] \quad \text{iff} \quad M \models \varphi^0[a_1, \ldots, a_n].$$

The formula φ^0 is defined inductively (on the length of φ) in the most natural way.

We establish the converse for the case in which the model M has a definable pairing function $j: M^2 \to M$. Let us assume that a family $q \subseteq \mathrm{Def}(M)$ of subsets of M is a definable quantifier in M. We observe that in order to define a mapping Q^0 with properties (1)–(5) it is enough to show that for any formula $\varphi(x, x_1, \ldots, x_n)$ of $\mathscr{L}_M$ the set

$$\{\langle a_1, \ldots, a_n \rangle \in M^n: (M, q) \models Qx \varphi(x, x_1, \ldots, x_n)[a_1, \ldots, a_n]\}$$

is M-definable. Thus, if we choose a definition of this set carefully (so that its variables satisfy conditions (1) and (2)), it can serve as $Q^0 x \varphi$. The satisfaction of conditions (1)–(5) will follow immediately from the definition of Q^0, the fact that q is a definable quantifier in M and the obvious equivalence

$$(M, q) \models Qx \varphi[a_1, \ldots, a_n] \equiv M \models Q^0 x \varphi[a_1, \ldots, a_n].$$

Since there is a pairing function, we may restrict consideration to formulas $\varphi(x, y)$ with only two variables. In order to show that, given such a formula $\varphi(x, y)$, the set

$$X = \{y \in M: (M, q) \models Qx \varphi(x, y)\}$$

is M-definable, observe that

$$(M, q) \models Qx \varphi(x, y) \vee \neg Qx \varphi(x, y),$$

whence

$$(M, q) \models QyQx \varphi(x,) \vee Qy \neg Qx \varphi(x, y).$$

Thus either X or its complement belongs to q and so X is M-definable.

From now on we shall not distinguish between the concept of definable quantifier and explicit definable quantifier. We shall write $Qx \varphi$ instead of $Q^0 x \varphi$, thus identifying a formula of the language $\mathscr{L}_M(Q)$ with the appropriate formula of the language $\mathscr{L}_M$.

It is worth observing that when defining a quantifier, it is enough to determine the value $Qx \varphi(x)$ of the mapping Q for formulas φ without constants. Namely, if a formula $\varphi(x, x_1, \ldots, x_n)$ of $\mathscr{L}_M$ is equivalent to

$\psi(x, x_1, \ldots, x_n, a_1, \ldots, a_m)$ for some ψ of $\mathscr{L}$ and $a_1, \ldots, a_m \in M$, then from condition (4) of Definition 1.4 we easily derive the equivalence

$$M \models Qx\varphi(x, x_1, \ldots, x_n)[b_1, \ldots, b_n] \quad \text{iff}$$
$$M \models Qx\psi(x, x_1, \ldots, x_n, y_1, \ldots, y_m)[b_1, \ldots, b_n, a_1, \ldots, a_m].$$

Now we turn our attention to connections between pseudoquantifiers and elementary extensions of models.

Definition 1.5. A formula $\varphi(x)$ of the language $\mathscr{L}_M$ with one free variable x is called a *countable-like formula* for a pseudoquantifier $q \subseteq \mathrm{Def}(M)$ iff for any formula $\psi(x, y)$ of the language $\mathscr{L}_M$ the following implication holds in the model (M, q):

$$qy\exists x[\varphi(x) \,\&\, \psi(x, y)] \rightarrow \exists xqy\psi(x, y).$$

Otherwise the formula $\varphi(x)$ is called uncountable-like.

We observe that the definition of a countable-like formula remains equivalent if we replace the above implication by a similar one:

$$qy\exists x[\varphi(x) \,\&\, \psi(x, y)] \rightarrow \exists xqy[\varphi(x) \,\&\, \psi(x, y)].$$

We also observe that if $(M, q) \models qx\varphi(x)$ then the formula $\varphi(x)$ is uncountable-like. On the other hand, if $(M, q) \models \neg qx\varphi(x)$ and the pseudoquantifier q is regular then $\varphi(x)$ is countable-like.

We should make clear that the analogous notions are introduced in [17] for definable quantifiers and that the proofs of the above remarks carry over from [17].

The following proposition is the pseudoquantifier version of Theorem 3.6 of [4].

Proposition 1.6. *If q is a pseudoquantifier in M then uncountable-like formulas define in M a regular pseudoquantifier.*

Proof. Let $\bar{q} \subseteq \mathrm{Def}(M)$ consist of sets definable in M by means of formulas of $\mathscr{L}_M$ uncountable-like for q. It is easy to verify that $\bar{q}$ is a pseudoquantifier. We show that it is regular. Let $\varphi(x, y)$ be a formula of $\mathscr{L}_M$ and assume that $(M, \bar{q}) \models qx\exists y\varphi$. It means that the formula $\exists y\varphi(x, y)$ is uncountable-like for q, so there exists a formula $\psi(x, z)$ of $\mathscr{L}_M$ such that

$$(M, q) \models qz\exists x[\exists y\varphi(x, y) \,\&\, \psi(x, z)] \,\&\, \neg\exists xqz\psi(x, z).$$

We define $\vartheta(y, z) \equiv \exists x[\varphi(x, y) \,\&\, \psi(x, z)]$ and observe that

$$(M, q) \models qz\exists y[\exists x\varphi(x, y) \,\&\, \vartheta(y, z)].$$

There are two cases.

(a) $(M, q) \models \neg\exists yqz\vartheta(y, z)$. Then the formula $\exists x\varphi(x, y)$ is uncountable-like for q and so $(M, \bar{q}) \models qy\exists x\varphi(x, y)$.

(b) $(M, q) \models \exists y q z \vartheta(y, z)$. We take $a \in M$ such that $(M, q) \models qz \vartheta(a, z)$, i.e.

$$(M, q) \models qz \exists x [\varphi(x, a) \,\&\, \psi(x, z)].$$

Since $(M, q) \models \neg \exists x qz \psi(x, z)$, the formula $\varphi(x, a)$ is uncountable-like for q. Therefore $(M, \bar{q}) \models qx \varphi(x, a)$ and so $(M, \bar{q}) \models \exists y qx \varphi(x, y)$. ∎

We do not know if uncountable-like formulas for a definable quantifier form a regular definable quantifier or not (see also [4]).

The following theorem, though stated for quantifiers rather than pseudoquantifiers, was formulated and proved by several authors, see e.g. Keisler's [13] or Mostowski's [25]. In fact, their proofs use only the fact that q is a pseudoquantifier. We include the proof for the completeness of reasoning.

THEOREM 1.7. *Let M be a countable model and $q \subseteq \mathrm{Def}(M)$ a pseudoquantifier in M. Then for any formula $\varphi(x)$ of the language $\mathscr{L}_M$, uncountable-like q, there exists a model N such that*

(a) $M \prec N$,

(b) *there exists $b \in N - M$ such that $N \models \varphi[b]$,*

(c) *if a formula $\vartheta(x)$ is countable-like for q and $a \in N$ is such that $N \models \vartheta[a]$ then $a \in M$.*

Sketch of the proof. We find a formula $\psi(x, y)$ such that

$$(M, q) \models qx \exists y [\varphi(y) \,\&\, \psi(x, y)] \,\&\, \neg \exists y qx \psi(x, y).$$

Now we extend the language $\mathscr{L}_M$ by adding a new constant c and inductively we define a theory T in the new language with the following properties:

(1) $\mathrm{Th}((M, a)_{a \in M}) \subseteq T$,

(2) T is complete,

(3) the sentence $\exists y [\varphi(y) \,\&\, \psi(c, y)]$ belongs to T,

(4) if a formula $\vartheta(x)$ is countable-like for q and a sentence of form $\exists x [\vartheta(x) \,\&\, \chi(x)]$ belongs to T then there exists $a \in M$ such that $M \models \vartheta[a]$ and $\chi(a) \in T$,

(5) if Φ is a finite subset of T, a variable x does not occur in Φ and $\Phi(x)$ is obtained from Φ by replacing the constant c by the variable x then $(M, q) \models qx \bigwedge \Phi(x)$.

In particular, condition (5) implies that the theory T is consistent. By Henkin–Orey theorem (see [14], p. 54) there exists a model N of T such that if $\vartheta(x)$ is a countable-like formula for q in M then ϑ is satisfied both in M and N by exactly the same elements. Since $\mathrm{Th}((M, a)_{a \in M}) \subseteq T$, the model N can be taken so that it is an elementary extension of M. From the condition (3) it follows that $N \models \exists y [\varphi(y) \,\&\, \psi(c, y)]$. Let $b \in N$ be such that

$N \models \varphi[b]$ and $N \models \psi(c, b)$. Let us suppose that $b \in M$. Then, by the completeness of T, the sentence $\psi(c, b)$ belongs to T and so (by condition (5)) $(M, q) \models qx\psi(x, b)$. This, however, contradicts $(M, q) \models \neg \exists y qx \psi(x, y)$. Thus $b \in N - M$. ∎

We observe now that if q is a definable quantifier in M then it generates a definable quantifier in any elementary extension N of M. Namely, we define $Qx\varphi$ for formulas φ of the language $\mathscr{L}$ in the same way in N as in M. Then $Qx\varphi$ is a formula of $\mathscr{L}_M$ which is a sublanguage of $\mathscr{L}_N$. The considerations from page 10 convince us that Q is a definable quantifier. It is also easy to see that a formula $\vartheta(x)$ of the language $\mathscr{L}_M$ is countable-like for q in the model M iff it is countable-like for the corresponding definable quantifier in the model N. This observation enables us to strengthen Theorem 1.7 to the following theorem of Krivine and McAloon [17]:

THEOREM 1.8. *Let M be a countable model and $q \subseteq$ Def(M) a definable quantifier in M. Then there exists a model N with the properties*:

(1) $M \prec N$,

(2) *if a formula $\varphi(x)$ is countable-like for q in M and $N \models \varphi(a)$ for some $a \in N$ then $a \in M$*,

(3) *if a formula $\varphi(x)$ is uncountable-like for q in M then there exists $b \in N - M$ such that $N \models \varphi[b]$.*

Sketch of the proof. We enumerate all uncountable-like formulas of $\mathscr{L}_M$ and construct a sequence $M = M_0 \prec M_1 \prec \ldots \prec M_n \prec \ldots$ of elementary extensions of M by iterating Theorem 1.7 so that:

(a) all extensions preserve all countable-like formulas,

(b) the extension M_{n+1} of M_n extends the nth uncountable-like formula.

Finally the union $N = \bigcup_{n \in \omega} M_n$ is the required model. ∎

DEFINITION 1.9 (see [17]). An extension N of a model M is called a *complete end extension* (for a quantifier q) iff it satisfies conditions (1)–(3) from Theorem 1.8.

Now we present a direct method of obtaining complete end extensions of countable models, which is a reformulation of the forcing method from [17].

We extend the language $\mathscr{L}_M$ by adding a sequence of new constants c_n, $n \in \omega$. Next we inductively construct a theory T in the new language satisfying the following conditions:

(1) $\mathrm{Th}((M, a)_{a \in M}) \subseteq T$,

(2) T is complete,

(3) if $M \models Qx\varphi$ then there exists $i \in \omega$ such that $\varphi(c_i) \in T$,

(4) if a formula $\varphi(x)$ is countable-like and a sentence of form $\exists x[\varphi(x) \,\&\, \psi(x)]$ belongs to T then there exists $a \in M$ such that $M \models \varphi[a]$ and $\psi(a) \in T$,

(5) if Φ is a finite subset of T such that constants c_i for $i > n$ do not occur in Φ, $x_0, \ldots, x_n$ are variables which do not occur in Φ and $\Phi(x_0, \ldots, x_n)$ is obtained from Φ by replacing constants $c_0, \ldots, c_n$ by respectively $x_0, \ldots, x_n$ then $M \models Qx_0 \ldots Qx_n \bigwedge\!\!\bigwedge \Phi(x_0, \ldots, x_n)$.

Now the application of Henkin–Orey's theorem gives us a required model N.

Definition 1.10. Let M be a countable model and $q \subseteq \mathrm{Def}(M)$ a definable quantifier in M. A complete end extension N of M will be called a *q-extension* iff there exists a sequence $a_0, a_1, \ldots, a_n, \ldots,$ of elements of N such that the theory T of those sentences of the language $\mathscr{L}_M$ with additional constants c_i, $i \in \omega$, which are true in N (when interpreting c_i as a_i) satisfies conditions (1)–(5) from the alternate proof of Theorem 1.8.

We shall illustrate this definition with an example of a q-extension which closely corresponds to the first proof of Theorem 1.8. Let us consider the following enumerations of sets of formulas of $\mathscr{L}_M$: $\langle \varphi_n;\, n \in \omega \rangle$ is an enumeration of all formulas φ such that $M \models Qx\,\varphi(x)$, $\langle \psi_n;\, n \in \omega \rangle$ is an enumeration of all uncountable-like formulas ψ in the model M and ϑ_n is such that

$$M \models Qx\,\exists y\,[\psi_n(y) \,\&\, \vartheta_n(x, y)] \,\&\, \neg \exists y\, Qx\, \vartheta_n(x, y).$$

We construct an elementary chain $\langle M_n;\, n \in \omega \rangle$ such that $M_0 = M$ and for each n there exists $c_n \in M_{n+1}$ such that

(i) M_{n+1} preserves all countable-like formulas of the model M_n (and hence of M),

(ii) $M_{n+1} \models \varphi_n(c_n) \,\&\, \exists y\,[\psi_n(y) \,\&\, \vartheta_n(c_n, y)]$,

(iii) if $\Phi(x)$ is a formula of $\mathscr{L}_{M_n}$ such that $M_{n+1} \models \Phi(c_n)$ then $M_n \models Qx\,\Phi(x)$.

One can easily see that the union $N = \bigcup_{n \in \omega} M_n$ is a complete end extension of M. Moreover $N \models \varphi_n(c_n)$, and so in order to show that N is a q-extension of M it suffices to verify condition (5). Let $\Phi(x_0, \ldots, x_n)$ be a formula of $\mathscr{L}_M$ and $N \models \Phi(c_0, \ldots, c_n)$. Then $M_{n+1} \models \Phi(c_0, \ldots, c_n)$, whence $M_n \models Qx_n\,\Phi(c_0, \ldots, c_{n-1}, x_n)$. By induction we get

$$M \models Qx_0 \ldots Qx_n\, \Phi(x_0, \ldots, x_n).$$

We would like to ask whether every complete end extension for a definable quantifier q is necessarily a q-extension.

We would also like to remark that both proofs of Theorem 1.8 strongly employ the fact that q is a quantifier. In the first proof we used the equivalent notion of the explicit definable quantifier and in the other we considered formulas of the form $Qx_0 \ldots Qx_n \varphi$. Thus we pose the problem whether for any pseudoquantifier $q \subseteq \mathrm{Def}(M)$ there exists a complete end extension N of M (with respect to the pseudoquantifier q).

In the subsequent sections we shall use the following strengthening of Theorem 1.8, also due to Krivine and McAloon.

THEOREM 1.11. *If M is a countable model and $q \subseteq \mathrm{Def}(M)$ a definable quantifier then there exists a complete end extension N of M such that a subset of M is definable both in M and in N iff it is definable in M by means of a countable-like formula.*

For a proof of this theorem we refer the reader to [17].

In connection with Theorem 1.8 we recall the following definition introduced in [9].

DEFINITION 1.12. Two definable quantifiers $q_1 \subseteq \mathrm{Def}(M)$ and $q_2 \subseteq \mathrm{Def}(M)$ *are equivalent in the model M* iff they have the same countable-like formulas.

In other words, two definable quantifiers are equivalent iff complete end extensions of the model M obtained by means of these quantifiers extend the same formulas of the language $\mathscr{L}_M$. We should mention that this notion of the equivalence differs from that of [4].

We shall be concerned with the problem of the existence of nonequivalent definable quantifiers in models of second order arithmetic A_2. We are interested only in countably additive quantifiers (i.e. such that the formula $N(x)$ is countable-like for them), because they generate elementary extensions of ω-models, which are ω-models as well. We recall that there exist models of A_2 in which all countably additive quantifiers are equivalent (see [4] or [9]). On the other hand, in [9], we constructed a model of A_2 in which the quantifiers "there exist uncountably many" and "there exist arbitrarily large well-orderings" are nonequivalent. In the present paper we shall show that there exists a model of second order arithmetic A_2 in which there are infinitely many nonequivalent countably additive regular definable quantifiers.

§ 2. Elementary extensions of countable models of set theory

In her paper [4] Mrs. Dubiel gave a characterization of definable regular quantifiers in models of ZFC. In this section we repeat her argument for the notion of pseudoquantifier in order to obtain a similar characterization of pseudoquantifiers, and consequently, of elementary extensions of models of ZFC or a certain extension of ZFC^-. Throughout this section $\mathscr{L}$ will be the language of set theory.

Let us consider an extension of ZFC^- obtained by assuming that some formula $\varphi(x, y)$ defines a well-ordering of the universe, similar to the class On of the ordinal numbers. By ST we denote either the theory ZFC or the above defined extension of ZFC^-. We observe that in the theory ST there exists a definable function $\varrho: V \to On$ such that for any α the class $\{x: \varrho(x) < \alpha\}$ is a set. In ZFC we can namely put $\varrho(x) = rk(x)$ and in the other case we put $\varrho(x) =$ order type of $\{y: \varphi(y, x)\}$.

We begin with a trivial observation.

PROPOSITION 2.1. *If N is an elementary extension of a model M of* ST *then there exists a regular pseudoquantifier $q \subseteq \mathrm{Def}(M)$ such that N is a complete end extension of M with respect to q.*

For a proof take the pseudoquantifier $q_{M,N}$ described in Example 1.2. ■

Our present aim will be to characterize regular pseudoquantifiers in models of ST. The following definition is inspired by the definition of a regular class of cardinals (see [13]).

DEFINITION 2.2. A subset I of the set of cardinals of a model M of ST will be called a *regular segment* iff it has the following properties:

(1) $x \in I \;\&\; y \in M \;\&\; M \models \mathrm{Card}[y] \;\&\; x \in_M y \to y \in I$,

(2) if $x \in M$ and $|\bigcup^M x|^M \in I$ then either $|x|^M \in I$ or there exists $y \in M$ such that $x \in_M y$ and $|y|^M \in I$.

We introduce useful notation. Assume that M is a model of ST and $\varphi(x)$ a formula of the language $\mathscr{L}_M$ with one free variable x. Then, if there exists $a \in M$ such that $M \models \forall x[x \in a \equiv \varphi(x)]$, by $|\varphi|^M$ we denote the cardinal of a in the model M. If, however, $\varphi(x)$ defines a proper class in M, we write $|\varphi|^M = On^M$. Thus we consider On^M as a cardinal number of the model M and assume that On^M belongs to every regular segment $I \subseteq \mathrm{Card}^M$.

Now suppose that we are given a regular segment $I \subseteq \mathrm{Card}^M$. We define a family of sets $q_I \subseteq \mathrm{Def}(M)$ so that a set $X = \{a \in M: M \models \varphi[a]\}$ belongs to q_I iff $|\varphi|^M \in I$. It is obvious that this definition does not depend on the choice of φ.

The following two lemmas are the "pseudoquantifier versions" of Theorems 2.3 and 2.8 from [4].

LEMMA 2.3. *If $I \subseteq \mathrm{Card}^M$ is a regular segment then the family q_I is a regular pseudoquantifier in the model M.*

Proof. It is easy to see that q_I is a pseudoquantifier. We show that it is regular. Let $\varphi(x, y)$ be such that $(M, q_I) \models qx\exists y\varphi(x, y)$.

Suppose, first, that the formula $\exists y\varphi(x, y)$ defines in M a proper class.

Then obviously either for some y the class $\{x \in M \colon M \models \varphi[x, y]\}$ is proper or there exists a proper class of those y for which $M \models \exists x \varphi(x, y)$. Therefore we get

$$(M, q_I) \models \exists y q x \varphi(x, y) \vee q y \exists x \varphi(x, y).$$

Now suppose that there exists $a \in M$ such that

$$M \models \forall x [x \in a \equiv \exists y \varphi(x, y)].$$

Then $|a|^M \in I$. We find in M an element b such that

$$M \models \forall z [z \in b \equiv \exists y \forall x [x \in z \equiv \varphi(x, y)]]$$

and b consists of nonempty sets (in the sense of M). It is easy to see that $a = \bigcup^M b$. By the regularity of I either $|b|^M \in I$ or there exists $c \in M$ such that $c \in_M b$ and $|c|^M \in I$. We consider first the case when $|b|^M \in I$. By the definition of b and the axiom of choice in M we find $d \in M$ such that $|d|^M = |b|^M$ and

$$M \models \forall y [y \in d \equiv \exists x \varphi(x, y)].$$

Now $|d|^M \in I$, so $(M, q_I) \models q y [y \in d]$ and hence $(M, q_I) \models q y \exists x \varphi(x, y)$. In the second case we take $c \in M$ such that $c \in_M b$ and $|c|^M \in I$. Then we find $y \in M$ such that $M \models \forall x [x \in c \equiv \varphi(x, y)]$ and notice that then

$$(M, q_I) \models \exists y q x \varphi(x, y). \quad \blacksquare$$

LEMMA 2.4. *Let $q \subseteq \mathrm{Def}(M)$ be a regular pseudoquantifier in a model M of* ST. *Then there exists a regular segment $I \subseteq \mathrm{Card}^M$ such that $q = q_I$.*

Proof. We show first that if a formula φ defines a proper class of M then $(M, q) \models q x \varphi(x)$. Suppose, to the contrary, that $(M, q) \models \neg q x \varphi(x)$. Since q is regular, the formula φ is countable-like for q. We consider the formula $\psi(x, y) \equiv \varrho(x) < \varrho(y)$. Then

$$M \models \forall x \exists y [\varphi(y) \mathbin{\&} \varrho(x) < \varrho(y)],$$

and so

$$(M, q) \models q x \exists y [\varphi(y) \mathbin{\&} \psi(x, y)].$$

It follows that $(M, q) \models \exists y q x \psi(x, y)$, i.e. there exists $a \in M$ such that $(M, q) \models q x [x \in a]$. Since $\varphi(x)$ defines a proper class, there exists $f \in M$ such that $M \models$ "f is a 1-1 function & $\mathrm{dom}(f) = a \mathbin{\&} \forall x \in a \, \varphi(f(x))$", and hence $(M, q) \models q x \exists y [f(x) = y]$. By the regularity of q, either

$$(M, q) \models \exists y q x [f(x) = y] \quad \text{or} \quad (M, q) \models q y \exists x [f(x) = y].$$

Since f is one-to-one, the first case is impossible; hence $(M, q) \models q y \varphi(y)$, which proves the claim.

Now we notice that if for no $a \in M$ we have

$$(M, q) \models qx[x \in a] \quad \text{then} \quad q = q_I \quad \text{for} \quad I = \{On^M\}.$$

The segment I is then regular.

Finally, suppose that there exists $a \in M$ such that $(M, q) \models qx[x \in a]$. We define:

$$I = \{x \in \mathrm{Card}^M : (M, q) \models qy[y \in x]\} \cup \{On^M\}.$$

An argument similar to the proof of the claim shows that if $(M, q) \models qx[x \in a]$ and $|a|^M \leqslant |b|^M$ then $(M, q) \models qx[x \in b]$. This implies that I satisfies condition 1 of Definition 2.2 and that $q = q_I$. It remains to show that I is regular, so let $x \in M$ be such that $|\bigcup^M x|^M \in I$. It means that $(M, q) \models qt[t \in \bigcup^M x]$, i.e.

$$(M, q) \models qt \exists y [t \in y \; \& \; y \in x].$$

By regularity of q we get either

$$(M, q) \models \exists y qt [t \in y \; \& \; y \in x]$$

or

$$(M, q) \models qy \exists t [t \in y \; \& \; y \in x].$$

In the first case we obtain $y \in M$ such that $y \in_M x$ and $(M, q) \models qt[t \in y]$, i.e. $|y|^M \in I$. In the second case $(M, q) \models qy[y \in x]$, i.e. $|x|^M \in I$. ∎

COROLLARY 3.5. *If N is an elementary extension of a model M of* ST *then there exists a regular segment $I \subseteq \mathrm{Card}^M$ such that N is a complete end extension of M with respect to the pseudoquantifier q_I.*

Now we recall Mrs. Dubiel's characterization of regular definable quantifiers in models of ST (see [4]).

Let us suppose that Q is a regular definable quantifier in a model M of ST. Then the corresponding family $q \subseteq \mathrm{Def}(M)$ is a regular pseudoquantifier in M, and so it is of the form q_I for some regular segment I. The segment I is definable in M:

$$x \in I \quad \text{iff} \quad M \models \mathrm{Card}(x) \;\&\; Qy[y \in x];$$

thus it has a least element $\varkappa$. Then $\varkappa$ is a regular cardinal in M and it follows easily that the quantifier Q is defined by:

$$M \models Qx\varphi \quad \text{iff} \quad M \models \text{"there exist at least } \varkappa \, x \text{ such that } \varphi(x)\text{"};$$

or, in case when q does not contain sets of M:

$M \models Qx\varphi$ iff $M \models$ "there exists a proper class of x such that $\varphi(x)$".

Following [4] these quantifiers will be denoted by $Q_\aleph$ and Q_V, respectively.

A similar situation occurs when M is a well-founded model. Then every regular pseudoquantifier $q \subseteq \mathrm{Def}(M)$ is in fact a regular definable quantifier. For a proof we observe that the regular segment I such that $q = q_I$ contains a least element.

THEOREM 2.6. *If M is a countable model of* ST *and $I \subseteq \mathrm{Card}^M$ is a regular segment then there exists a complete end extension of the model M, with respect to the pseudoquantifier q_I.*

Proof. Let us consider the model (M, q_I) and let $\varphi(x)$ be a formula of the language $\mathscr{L}_M$ such that $(M, q_I) \models qx\varphi(x)$. By applying Theorem 1.7 we obtain a model M_1 such that

(a) $M \prec M_1$,

(b) there exists $b \in M_1 - M$ such that $M_1 \models \varphi[b]$,

(c) if $(M, q_I) \models \neg qx\psi(x)$ and $M_1 \models \psi[x]$, then $x \in M$.

(We recall that q_r is a regular pseudoquantifier, so ψ is countable-like iff $(M, q_I) \models \neg qx\psi(x)$).

Now we define $I_1 \subseteq \mathrm{Card}^{M_1}$ as follows:

$$I_1 = \{x \in \mathrm{Card}^{M_1} : \exists y \in I[M_1 \models y \leqslant x]\} \cup \{On^{M_1}\}.$$

We show that I_1 is a regular segment in M_1. There are two cases:

(1) I_1 has a least element $\varkappa$. Then $\varkappa \in I$ and is the least element of I. Thus it is a regular cardinal in M and hence in M_1. Therefore $I_1 = \{x \in \mathrm{Card}^{M_1} : \varkappa \leqslant x\}$ is a regular segment.

(2) I_1 has no least element. Since in ST one can show that $|\bigcup x| \leqslant \max(|x|, \sup\{|y| : y \in x\})$ for x such that $|\bigcup x| \geqslant \omega$, then in order to show that I_1 is regular it suffices to prove that if $x \in M_1$ and $z \in \mathrm{Card}^{M_1}$ are such that $M_1 \models \forall y \in x\, \mathrm{Card}(y)$ and $M_1 \models z = \sup x$, then $z \in I_1$ implies that there exists $y \in M_1$ such that $y \in_{M_1} x$ and $y \in I_1$. Now, if x and z are as above then there exists $t \in I_1$ such that $t \in_{M_1} z$. Hence there exists $y \in M_1$ such that $M_1 \models y \in x \;\&\; t \leqslant y$. But then $y \in_{M_1} x$ and $y \in I_1$, which proves the regularity of I_1.

In particular, it follows that formulas countable-like for q_I in M are countable-like for q_{I_1} in M_1. Similarly, formulas uncountable-like remain uncountable-like. This enables us to initiate the proof of Theorem 1.8.

We enumerate all formulas which are uncountable-like for q in M and construct a sequence of elementary extensions of the model M. We start with

$M_0 = M$ and M_1 constructed so far. The extension M_{n+1} of M_n is obtained in the same way as M_1 from M_0. We assume that the set

$$I_n = \{x \in M_n : \exists y \in I [M_n \models y \leqslant x]\} \cup \{On^{M_n}\}$$

is a regular segment in M_n and extend the model M_n by means of pseudo-quantifier q_{I_n}, extending the nth uncountable-like formula. Then we define a regular segment $I_{n+1} \subseteq \mathrm{Card}^{M_{n+1}}$ by

$$\begin{aligned} I_{n+1} &= \{x \in \mathrm{Card}^{M_{n+1}} : \exists y \in I_n [M_{n+1} \models y \leqslant x]\} \cup \{On^{M_{n+1}}\} \\ &= \{x \in \mathrm{Card}^{M_{n+1}} : \exists y \in I [M_{n+1} \models y \leqslant x]\} \cup \{On^{M_{n+1}}\}. \end{aligned}$$

Now we take $N = \bigcup_{n \in \omega} M_n$ as the required complete end extension. ∎

A proof of a theorem similar to Theorem 2.6 can be found in [15].

As an example of an application of Theorem 2.6 we prove the following

THEOREM 2.7. *There exists a countable model M of* ST *and an elementary extension N of M which is not a complete end extension of M for any definable regular quantifier $q \subseteq \mathrm{Def}(M)$.*

Proof. We already know that if q is a regular definable quantifier in a model M of ST then the regular segment $I \subseteq \mathrm{Card}^M$ such that $q = q_I$ contains the least element. Thus it is enough to show that there exists a countable model M of ST which has a nondefinable regular segment $I \subseteq \mathrm{Card}^M$.

We take a countable well-founded model M_0 of ST with no biggest cardinal (in Section 4 we show that there exist well-founded models of ZFC^- with definable well-ordering of the universe and with no biggest cardinal, which are not models of ZFC) and we consider the definable quantifier Q_V in M_0.

By applying Theorem 1.11 we obtain a model M of ST which extends M_0 and such that the ordinal numbers of M_0 are not definable in M. In particular, the standard (i.e. well-founded) part of M is exactly M_0. Now we define $I \subseteq \mathrm{Card}^M$ to be the set of new cardinal numbers, i.e. cardinals which belong to $M - M_0$. Then I is nondefinable in M and, since a supremum of a set of standard ordinals is obviously standard, I is a regular segment. ∎

Theorem 2.7 shows that it is worth studying the following problem: given a countable model M for some first-order language and an elementary extension N of M, determine whether there exists a definable regular quantifier $q \subseteq \mathrm{Def}(M)$ such that the model N is a complete end extension of M with respect to the quantifier q. If such a quantifier does exist then it is interesting to ask whether N is a q-extension of M. We return to this problem in Section 5.

Now we shall be interested in elementary extensions of ω-models of second-order arithmetic A_2 and we shall aim at proving a theorem similar to 2.7 for models of A_2.

§ 3. Interpretations of set theory in extensions of A_2

Before carrying over the results of Section 2 from set theory to second-order arithmetic we shall study the problem of interpreting set theory in extensions of A_2. The idea behind these investigations is to describe a way of developing in arithmetic some fragments of the theory of cardinal numbers so that it should become possible to apply the techniques of the last section to the study of elementary extensions of models of A_2.

The most important notion of this section is that of interpretation of one theory in another, as described in most text-books on mathematical logic. We shall be interested in interpretations of set theory in arithmetic and vice versa. Throughout this section we denote by $\mathscr{L}$ the language of second-order arithmetic and by $\mathscr{L}^*$ the language of set theory.

There exists a natural interpretation of the second order arithmetic in set theory. Namely, we first choose a definable set Ω which is an isomorphic copy of the set of natural numbers with the property that no subset of Ω is an element of it. We fix one such set Ω. Then we interpret numbers as elements of Ω and sets as subsets of Ω. Given a formula of the language $\mathscr{L}$, we denote by $\bar{\varphi}$ the natural interpretation of φ to the language $\mathscr{L}^*$. In order not to proliferate notation, we shall not distinguish between Ω and the set of natural numbers ω. It is obvious that if φ is an axiom of A_2 then $\mathrm{ZFC}^- \vdash \bar{\varphi}$.

Now suppose that we are given three formulas $I = \langle V, \varepsilon, \approx \rangle$ of the language $\mathscr{L}$ such that V has one free variable and ε and $\approx$ both have two free variables. We shall denote them as $x \varepsilon y$ and $x \approx y$. Now we define inductively the interpretation φ^I of a set-theoretical formula φ in the language $\mathscr{L}$:

$$\begin{aligned}
(x \in y)^I &= (V(x) \mathbin{\&} V(y) \mathbin{\&} x \varepsilon y),\\
(x = y)^I &= (V(x) \mathbin{\&} V(y) \mathbin{\&} x \approx y),\\
(\neg \varphi)^I &= \neg \varphi^I,\\
(\varphi \vee \psi)^I &= \varphi^I \vee \psi^I,\\
(\exists x \varphi)^I &= \exists x [V(x) \mathbin{\&} \varphi^I].
\end{aligned}$$

We say that I is an interpretation of a set theory S in an arithmetic $T \supseteq A_2$ iff for every formula φ of the language $\mathscr{L}^*$, if $S \vdash \varphi$ then $T \vdash \varphi^I$.

DEFINITION 3.1. An interpretation I of a set theory $S \supseteq \mathrm{ZF}^-$ in an arithmetic $T \supseteq A_2$ is *accurate* iff there exists a formula $\Xi(x, y)$ of the language $\mathscr{L}$ such that the following properties are provable in T:

(1) $\Xi(x, y) \to (x \in \omega \vee x \subseteq \omega)^I$,

(2) $\forall x [(x \in \omega \vee x \subseteq \omega)^I \to \exists! y \, \Xi(x, y)]$,

(3) $\forall x \forall y[\Xi(x, y) \to ((x \in \omega)^I \equiv N(y))]$,

(4) $\forall y \exists x \Xi(x, y)$,

(5) $\Xi(x_1, y_1) \,\&\, \Xi(x_2, y_2) \to (x_1 \approx x_2 \equiv y_1 = y_2)$,

(6) $\Xi(x_1, y_1) \,\&\, \Xi(x_2, y_2) \to (x_1 \varepsilon x_2 \equiv y_1 \in y_2)$.

From now on we shall write $\|x\| = y$ instead of $\Xi(x, y)$. Thus, informally speaking, in the theory T one can prove that $\|\cdot\|$ defines an isomorphism between $(\omega \cup P(\omega))^I$ and the universe of the theory T.

A classical result (cf. [16] or [33]) shows that there exists an accurate interpretation of ZFC$^-$ in A_2. Under this interpretation, $A_2 \vdash (\mathrm{V} = \mathrm{HC})^I$, where $\mathrm{V} = \mathrm{HC}$ is a set-theoretical formula which says that every set is countable. It is natural to raise the problem whether there exist extensions $T \supseteq A_2$ and accurate interpretations I of ZFC$^-$ in T such that $T \vdash (\mathrm{V} \neq \mathrm{HC})^I$.

Now assume that $T \supseteq A_2$ is a consistent extension of A_2 and I an accurate interpretation of ZFC$^-$ in T. We begin with the following

LEMMA 3.2. *For any formula $\varphi(x_1, \ldots, x_n)$ of the language $\mathscr{L}$ the following is provable in T:*

$$\forall x_1, \ldots, x_n \forall y_1, \ldots, y_n [\bigwedge_{1 \leqslant i \leqslant n} \|x_i\| = y_i \to (\varphi(y_1, \ldots, y_n) \equiv \bar{\varphi}^I(x_1, \ldots, x_n))].$$

Proof. By easy induction on the lenght of φ, imitating the proof that isomorphic models are elementarily equivalent. ∎

COROLLARY 3.3. *If φ is a sentence of the language $\mathscr{L}^*$ then $T \vdash \overline{\varphi^I}^I \equiv \varphi^I$.*

Proof. By direct application of Lemma 3.2. ∎

As a consequence of the above corollary we obtain the following theorem, which shows that the existence of certain uncountable sets cannot be consistent within accurate interpretations.

THEOREM 3.4. *If T is a consistent extension of A_2 and I an accurate interpretation of* ZFC$^-$ *in T then $T \vdash (P(\omega)$ is not a set$)^I$.*

Proof. Let us suppose, to the contrary, that T does not prove $(P(\omega)$ is not a set$)^I$. Then the theory $T' = T \cup \{(P(\omega)$ is a set$)^I\}$ is consistent. We consider a theory S defined as $S = \{\varphi\colon T' \vdash \varphi^I\}$. Then S is a consistent set theory such that ZFC$^- \supseteq S$ and $S \vdash (P(\omega)$ is a set$)$.

We obtain a contradiction with Tarski's theorem ([30], see also [2], p. 97) by showing that in the theory S it is possible to formalize the definition of truth. In order to do this we prove in S that there exist a definable set M and definable binary relations i, e on M such that, for any set-theoretical sentence φ, $S \vdash \varphi \equiv \varphi^{(M)}$, where $\varphi^{(M)}$ is obtained from φ by replacing $\in$ and $=$ by respectively e and i, and then by restricting quantifiers to M. That will be enough because then we can define $\mathrm{Tr}(\ulcorner\varphi\urcorner) \equiv \langle M, i, e\rangle \models \ulcorner\varphi\urcorner$ and since M is a set, it is provable in S that $\varphi^{(M)} \equiv \langle M, i, e\rangle \models \ulcorner\varphi\urcorner$ for any sentence φ with Gödel number $\ulcorner\varphi\urcorner$. The formula Tr will give us the required formalization of the definition of truth.

The set M and relations $i, e \subseteq M^2$ can be defined in the following way:

$$M = \{x \in \omega \cup P(\omega)\colon \bar{V}(x)\},$$

$$i = \{\langle x, y\rangle \in M^2\colon \overline{(x \approx y)}\} \quad \text{and}$$

$$e = \{\langle x, y\rangle \in M^2\colon \overline{(x \in y)}\}.$$

Obviously, M is a set and the definitions of M, i, e do not require any parameters. By Corollary 3.3, $S \vdash \varphi \equiv \varphi^{(M)}$. Namely, $S \vdash \overline{\varphi^I} \equiv \varphi$ and $S \vdash \overline{\varphi^I} \equiv \varphi^{(M)}$. ■

Theorem 3.4 shows that we cannot define accurate interpretations for a set theory with the power set axiom. However, it is reasonable to ask whether there exist interpretations of ZFC^- with uncountable sets. It turns out that we can give a positive answer for certain extensions of A_2. For A_2 itself the question is open.

THEOREM 3.5. *If I is an accurate interpretation of* ZFC^- *in A_2 and if $A_2 \vdash \mathrm{WO}^I$ then $A_2 \nvdash (\mathrm{V} \neq \mathrm{HC})^I$, where* WO *is a sentence of $\mathscr{L}^*$ saying that every set can be well-ordered.*

Proof. It is enough to show that in the set theory $\mathrm{ZFC}^- + \mathrm{WO} + \mathrm{V} \neq \mathrm{HC}$ it is provable that A_2 is consistent. For a proof observe that WO $+ \mathrm{V} \neq \mathrm{HC}$ implies that there exist uncountable ordinal numbers, in particular ω_1. We consider the constructible universe L. The set $P(\varphi) \cap L_{\omega_1}$ is then a model of A_2. ■

Theorem 3.5 suggests the problem whether WO is provable in ZFC^-. In case of positive answer Theorem 3.5 shows that accurate interpretations with $\mathrm{V} \neq \mathrm{HC}$ can be defined only in proper extensions of A_2. In case of negative answer we can raise another problem, namely whether in A_2 itself one can define an accurate interpretation of $\mathrm{ZFC}^- + \mathrm{V} \neq \mathrm{HC}$.

Now we turn our attention to constructing accurate interpretations of $\mathrm{ZFC}^- + \mathrm{V} \neq \mathrm{HC}$ in proper extensions of A_2. Suppose that we are given three formulas $\Phi(x)$, $J(x, y, z)$ and $E(x, y)$ of the language $\mathscr{L}$. We consider a theory $T \supseteq A_2$ such that the following axioms are provable in T:

(1) $\exists x \exists y [x \neq y \mathbin{\&} \Phi(x) \mathbin{\&} \Phi(y)]$,

(2) $J(x, y, z) \rightarrow \Phi(x) \mathbin{\&} \Phi(y) \mathbin{\&} \Phi(z)$,

(3) $\forall x, y [\Phi(x) \mathbin{\&} \Phi(y) \rightarrow \exists! z J(x, y, z)]$,

(4) $\forall z [\Phi(z) \rightarrow \exists x, y J(x, y, z)]$,

(5) $J(x, y, z) \& J(x', y', z') \rightarrow (z = z' \equiv x = x' \& y = y')$,

(6) $E(x, y) \rightarrow \Phi(x) \& S(y)$,

(7) $\exists z \forall x [\Phi(x) \rightarrow (E(x, z) \equiv \varphi(x))]$ for any formula φ of the language $\mathscr{L}$ in which the variable z does not occur,

(8) $\forall x [\Phi(x) \rightarrow \exists y \psi(x, y)]$

$$\rightarrow \exists z \forall x \big[\Phi(x) \rightarrow \exists y \big[\psi(x, y) \& \\ \& \forall t \big[\Phi(t) \rightarrow \big(E(t, y) \equiv \exists u [J(x, t, u) \& E(u, z)]\big)\big]\big]\big],$$

for any formula $\psi(x, y)$ of the language $\mathscr{L}$ in which the variable z does not occur.

Later we shall formulate another axiom (9) and assume that it is also provable in T.

Since the formulas Φ, J and E will be used for defining an interpretation I, we assume that they do not contain any parameters. However, in future we shall use interpretations to describe definable quantifiers in some models of A_2. Then we shall extend the notion of interpretation, by allowing parameters from the model. Then we shall speak of interpretations in a model rather than of interpretations in a theory. It should be said here that all results which we shall prove for interpretations in theories extending A_2 remain valid for interpretations in models of A_2.

Even a brief look at axioms (1)–(8) shows that J defines a pairing function for the class defined by Φ. Therefore we shall write $J(x, y) = z$ instead of $J(x, y, z)$. From the existence of pairing function and axiom (1) it follows that Φ is infinite. Without loss of generality we can assume that Φ contains all natural numbers and the pairing function J extends the arithmetical pairing function. Axiom (6) shows that E gives a coding of subclasses of Φ by sets of numbers (i.e. reals) and, the schema (7) being a sort of comprehension schema over the class Φ, shows that this coding is in a sense complete. Finally, the schema (8) is a suitable form of choice.

A real x will be called a code of a class $\{t\colon \Phi(t) \& E(t, x)\}$, denoted sometimes by C_x. We shall often identify a class with its code. Because of the existence of a pairing function a real x can also code binary relations over Φ: $\{\langle u, v\rangle\colon \Phi(u) \& \Phi(v) \& E(J(u, v), x)\}$, which is equal to $\{\langle u, v\rangle\colon J(u, v) \in C_x\}$. We shall not distinguish between these two notions of coding and it should be clear from the context which one we mean.

Suppose we are given a real x. Then for each y such that $\Phi(y)$ there exists by comprehension schema (7) a real z such that

$$\forall t \big[\Phi(t) \rightarrow \big(E(t, z) \equiv E(J(y, t), x)\big)\big],$$

i.e. a real z coding the $\{t\colon \Phi(t) \& E(J(y, t), x)\}$. Since we are interested in subclasses of Φ rather than in codes for them any code z for this class will be denoted by $(x)_y$. By using this notation we can express the schema of choice (8) in a more readable way as follows

$$\forall x[\Phi(x) \to \exists y \varphi(x, y)] \to \exists z \forall x[\Phi(x) \to \exists y[y = (z)_x \,\&\, \varphi(x, y)]].$$

We see that in the theory T reals became second-order objects over the class Φ. We can thus follow the construction of an interpretation of ZFC^- in a second-order theory. We shall sketch which notions should be successively introduced and which properties of them should be proved. Proofs are standard and can be found for two special cases in [18] or [33].

These notions and properties are:

(i) A real x codes a binary relation, which is a partial ordering.

(ii) A real y codes the domain of a partial ordering coded by x.

(iii) We must prove that such a code y exists. The proof uses schema (7).

(iv) A real x codes a linear ordering.

(v) A real x codes a well-ordering. We write here that x is a linear ordering and if a real y codes a nonempty subset of the domain of x then y has the least element (in the sense of the ordering x).

(vi) We prove that x is a well-ordering iff it is a linear ordering and every countable nonempty subset of the domain of x has a least element. Notice that a countable subset of Φ is easily coded by a single real. The proof of the equivalence uses schema (8).

(vii) We introduce the notion of isomorphism of binary relations and prove the usual theorem about comparing well-orderings.

(viii) We define the notion of a codable family of codable subclasses of Φ.

(ix) We define the notion of the least upper bound of a codable family of well-orderings and prove that for any such family there exists a well-ordering which is the least upper bound.

(x) We introduce the notion of a well-founded partial ordering and prove the usual principle of transfinite induction.

(xi) We prove a theorem permitting definitions by transfinite induction.

(xii) We define the notion of rank for elements of a well-founded partial ordering. Instead of using ordinal numbers as in set theory we use well-orderings. We prove that ranks exist and are unique up to isomorphism. Observe that we do not distinguish between an ordering and its domain.

This will be our custom from now on. We also recall that we identify a codable subclass of Φ with one of its codes.

Now we are ready to define an interpretation of a set theory in the theory T. The universe of the interpretation will consist of (code for) so called well-founded trees.

DEFINITION 3.6. A well-founded partial ordering of a subclass of the class Φ will be called a *well-founded tree* iff it has the following properties:

(1) it has exactly one maximal element,

(2) the successors of any element of it form a finite well-ordered set.

In particular, it follows from the definition that every node of a well-founded tree has exactly one immediate successor. Also, if x is a node of a well-founded tree and y is a node such that $y < x$, then there exists exactly one immediate predecessor z of x such that $y \leqslant z < x$. The immediate predecessors of the maximal node of a tree will be called *almost maximal nodes*.

For any well-founded tree x and a node y of it we introduce the notion of a restricted tree, denoted by $x \upharpoonright y$, consisting of y itself and all predecessors of y. Now we define a relation $\approx$ between trees.

Given two well-founded trees x and x' we define a relation $y \sim y'$ between nodes of x and nodes of x' by induction on the rank of modes. We put $y \sim y'$ iff for every immediate predecessor z of y there exists an immediate predecessor z' of y' such that $z \sim z'$ and for every immediate predecessor z' of y' there exists an immediate predecessor z of y such that $z \sim z'$. Then we say that trees x and x' *are equivalent*, denoted by $x \approx x'$, iff the relation $\sim$ holds between the maximal nodes of x and x'.

It is easy to prove by induction that if y is a node of a tree x and y' is a node of a tree x' then the relation $\sim$ defined between the trees $x \upharpoonright y$ and $x' \upharpoonright y'$ coincides with that defined between x and x'. In particular, we get the equivalence $y \sim y' \equiv x \upharpoonright y \approx x' \upharpoonright y'$.

Finally, we define a relation ε between trees by saying that $x \varepsilon x'$ iff there exists an almost maximal node y of the tree x' such that $x \approx x' \upharpoonright y$.

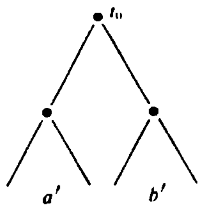
Now, after having denoted by $V(x)$ a formula saying that x is a well-founded tree, we can show that $I = \langle V, \approx, \varepsilon \rangle$ is an accurate interpretation of ZFC^- in T.

We can easily prove by induction that the relation $\approx$ is an equivalence relation between trees and, moreover, that it is a congruence with respect to ε. Thus equality axioms hold in the interpretation and it is now enough to prove the interpretations of axioms of ZFC^- in T.

(a) *Extensionality*. Assume that a and b are well-founded trees such that $\forall c[V(c) \to (c \varepsilon a \equiv c \varepsilon b)]$. In order to show that $a \approx b$, we take any almost maximal node t of a. Then $a \upharpoonright t \varepsilon a$, and so $a \upharpoonright t \varepsilon b$. Thus there exists an almost maximal node u of b such that $a \upharpoonright t \approx b \upharpoonright u$ and thence $t \sim u$. Similarly, for any almost maximal node u of b there exists an almost maximal node t of a such that $t \sim u$ and so $a \approx b$.

(b) *Pair*. Suppose that a and b are well-founded trees. For every element x of the class Φ we define the class $A_x = \{J(x, t): \Phi(t)\}$. The classes A_x are then pairwise disjoint subclasses of Φ. The function $t \to J(x, t)$ defines a one-to-one mapping from Φ onto A_x. Now we take any two classes A_x and

A_y and using the above mappings we copy the trees a and b in A_x and A_y, respectively. We denote these copies by a' and b'. Finally, we take an element t_0 of Φ which belongs to neither of these copies and construct a tree c according to the following picture

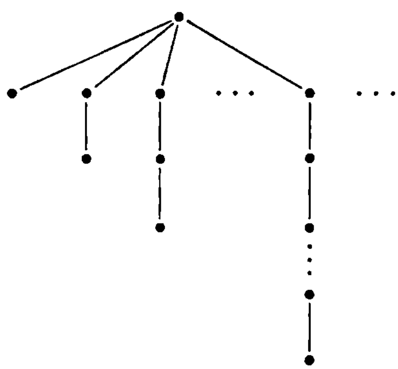


The tree c obviously satisfies the formula $c = \{a, b\}$.

(c) *Empty set.* A tree having only one node plays the role of the empty set.

(d) *Union.* Given a well-founded tree, we construct its union by simply deleting its almost maximal nodes.

(e) *Infinity.* A well-founded tree representing an infinite set can be pictured as follows:



(f) *Regularity.* We first observe that by the definition of the relation $\sim$, if $a \approx b$ then ranks of a and b are equal (the rank of a well-founded tree being the rank of its maximal node). It follows that if $a \varepsilon b$ then the rank of a is strictly smaller than that of b. Thus, given a well-founded tree, we find a minimal ε-element of it by taking a subtree $a \restriction y$ for an almost maximal node y of the smallest rank.

(g) *Replacement and Choice schema.* Suppose we are given a well-founded tree a and a set-theoretical formula $\varphi(x, y)$. Next assume that

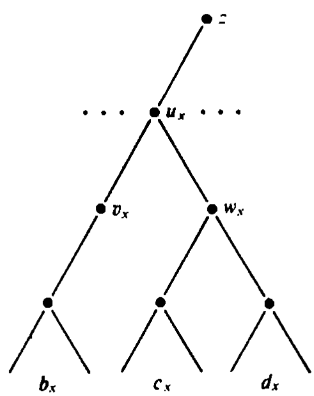
$$\forall x [x \varepsilon a \rightarrow \exists y [V(y) \,\&\, \varphi^I(x, y)]].$$

We want to find a well-founded tree f which is a function with domain a in the interpretation and such that

$$\forall x[x \varepsilon a \rightarrow \varphi^I(x, f(x))],$$

$f(x)$ denoting a tree which is the value of f for the tree x.

The assumption on φ shows that for every almost maximal node x of the tree a there exists a tree y such that $\varphi^I(a \upharpoonright x, y)$. By choice schema (8) there exists a code z selecting for each such node x a tree y_x such that $\varphi^I(a \upharpoonright x, y_x)$. We again use the classes A_x defined in (b). By using the pairing function J we split each class A_x into disjoint sections $A_{x,t}$, for t in the class Φ. Then in three of those sections we define trees b_x, c_x, d_x so that b_x and c_x were isomorphic with $a \upharpoonright x$ and d_x were isomorphic with y_x. We observe that domains of b_x, c_x and d_x are disjoint and that infinitely many elements of A_x are not used. We select three of them, denoted by u_x, v_x, w_x, and finally we select one element z of the class Φ, not used yet. Now we construct a tree g according to the following picture:



x ranges over almost maximal nodes of the tree a

The tree g obviously codes a binary relation with domain a such that if trees r and s are in this relation then $\varphi^I(r, s)$.

Assume that φ has the property that for every x in the tree a there exists exactly one y (up to $\approx$) such that $\varphi^I(x, y)$. Then it is easy to see that for any almost maximal nodes x_1, x_2 of a, if $a \upharpoonright x_1 \approx a \upharpoonright x_2$ then $y_{x_1} \approx y_{x_2}$. It follows that for any such x_1, x_2, if $b_{x_1} \approx b_{x_2}$ and $c_{x_1} \approx c_{x_2}$ then $d_{x_1} \approx d_{x_2}$, i.e. the tree g is a function. This shows that the interpretation I satisfies the replacement schema.

In order to verify the choice schema we formulate another assumption (9) about T, mentioned already on page 24. It is the following:

(9) there exists a real w which codes a well-ordering of the class Φ.

In particular, the axiom (9) is satisfied if there exists a definable well-ordering of the class Φ, i.e. a formula $\leqslant$ which defines a linear ordering of Φ and which satisfies the schema

$$(9') \quad \exists x[\Phi(x) \mathbin{\&} \varphi(x)] \to \exists x[\Phi(x) \mathbin{\&} \varphi(x) \mathbin{\&} \forall y[\Phi(y) \mathbin{\&} \varphi(y) \to x \leqslant y]],$$

for any formula φ of the language $\mathscr{L}$.

We return to the proof of the choice schema. We well-order the class of almost maximal nodes of the tree a and then by induction on that well-ordering we define a tree f by removing some subtrees from g. Namely if for some x and x' bigger than x we have $a \restriction x \approx a \restriction x'$ then we remove $g \restriction u_{x'}$ from g. One can easily see that the tree f is the required function.

Thus we have proved that I is in fact an interpretation of ZFC^- in the theory T. Now we show that it is accurate; and later we shall establish a necessary and sufficient condition for $(\mathrm{V} \neq \mathrm{HC})^I$ to be provable in T.

LEMMA 3.7. *The interpretation I is accurate.*

Proof. We can easily define a formula $v(n, x)$ of the language $\mathscr{L}$ so that the following were provable in T:

$$(1) \qquad v(0, x) \equiv (x \text{ is the empty set})^I,$$

$$(2) \qquad v(n+1, x) \equiv \exists y[v(n, y) \mathbin{\&} (x = y \cup \{y\})^I].$$

Then the formula v defines an isomorphism between the set of natural numbers of T and the natural numbers of the interpretation. Then for any tree x such that x is a set of natural numbers in the interpretation we put

$$\|x\| = \{n : \exists y[y \varepsilon x \mathbin{\&} v(n, y)]\}.$$

Then $\|\cdot\|$ is an isomorphism between $P(\omega)$ of the interpretation and the reals of the theory T.

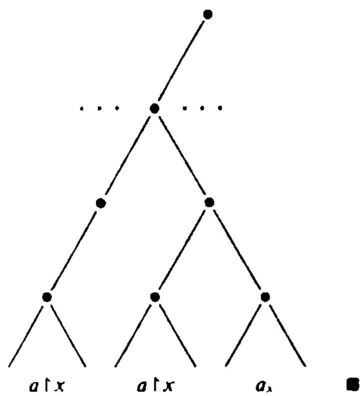
Now, depending on the definition of Ω, we define the required isomorphism between $\Omega \cup P(\Omega)$ of the interpretation and the universe of the theory T. It should be observed that the definition of Ω does not require parameters; and so neither does the definition of the isomorphism. ∎

PROPOSITION 3.8. *There exists a tree b in the interpretation with the property $\forall x[x \varepsilon b \equiv \Phi(\|x\|)]$.*

Proof. By the choice schema (8), for each real x such that $\Phi(x)$ we choose a code for a tree a_x such that $\|a_x\| = x$. Now we easily construct a tree b which collects all trees a_x. ∎

LEMMA 3.9. *In the interpretation I every set is equipotential with a subset of the set denoted by the tree b from Proposition* 38.

Proof. Let a be any well-founded tree. By induction on a well-ordering of the class Φ we can construct a tree a' such that $a \approx a'$ and for any almost maximal nodes x_1, x_2 of the tree a', if $a' \upharpoonright x_1 \approx a' \upharpoonright x_2$ then $x_1 = x_2$. Thus we assume that a itself has this property. We consider the class A of almost maximal nodes of the tree a. Then of course $\forall x [A(x) \rightarrow \Phi(x)]$. For every x from the class A we take a tree a_x such that $\|a_x\| = x$. Then we construct a tree, which is a one-to-one mapping from a into b in the interpretation, according to the following picture:



A class defined by a formula $\varphi(x)$ is called *countable* iff

$$\exists x \forall y [\varphi(y) \rightarrow \exists n [y = (x)_n]].$$

LEMMA 3.10. *A well-founded tree a is countable in the interpretation iff there exists a countable class φ of its almost maximal nodes such that, for any tree $d \varepsilon a$, there exists a node x in φ such that $d \approx a \upharpoonright x$.*

Proof. Let a be a well-founded tree. We consider trees c_n such that $\|c_n\| = n$ and a tree c which collects all c_n. Then obviously c is countable in the interpretation. If a is countable in I then there exists a well-founded tree f which in I is a function from c onto a. Then for each n we can choose an almost maximal node $(x)_n$ so that the pair $\langle c_n, a \upharpoonright (x)_n \rangle$ is an element of f in the sense of ε. The class collecting all $(x)_n$ is the required class φ.

On the other hand, given a class φ with the properties formulated in the lemma one can easily construct a tree-function from c onto a. ∎

COROLLARY 3.11. *The interpretation I satisfies V = HC iff the class Φ is countable.*

Proof. If the class Φ is countable then the tree b from Proposition 3.8 is countable in I and the corollary follows from Lemma 3.9. On the other hand, if Φ is uncountable then Lemma 3.10 shows that the tree b is uncountable, thus giving V $\neq$ HC. ∎

We close this section with two examples of such interpretations.

EXAMPLE 3.12. We take $\Phi(x) \equiv N(x)$. As a pairing function J we take the usual arithmetical pairing function and as $E(x, y)$ we take the membership relation $x \in y$. Then obviously axioms (1)-(9') are provable in A_2, and so we get an accurate interpretation of ZFC$^-$ in A_2. By Corollary 3.11 this interpretation satisfies V = HC. This is exactly the interpretation described in [33].

EXAMPLE 3.13. This is the example announced in [7]. We extend first the arithmetic A_2 by adding an axiom saying that there exist uncountably many constructible reals. Then we define $\Phi(x)$ by saying that x is an infinite constructible set of natural numbers. Obviously, there exists a definable well-ordering of the class Φ, namely the ordering of construction. It is also easy to define a pairing function for the class Φ. We next fix an enumeration of finite sets of natural numbers, $nr(s)$ denoting the number of the set s. Now for each infinite set x of natural numbers we put $S(x) = \{J(n, nr(x \cap n)) : N(n)\}$. Then for x from the class Φ the sets $S(x)$ form a family of almost disjoint sets of numbers. We introduce the coding relation E as follows:

$$E(x, y) \equiv (S(x) \cap y \textit{ is an infinite set}) \ \& \ \Phi(x).$$

We consider the theory $T \supseteq A_2$ obtained from A_2 by adding schemes (7) and (8) as well as the axiom mentioned at the beginning. Since Φ is uncountable in T, we get an accurate interpretation of ZFC$^-$ + V $\neq$ HC in T. Since the class Φ is well-ordered so that the initial segments of Φ are countable, the interpretation satisfies axioms: ω_1 exists and every set is of power at most ω_1. We need only show that the theory T is consistent. But it follows immediately from the fact the theory ZFC + $2^\omega > \omega_1$ + Martin's Axiom is consistent and that in this theory one can prove that every set of reals of power $\leqslant \omega_1$ can be coded by a single real in the way described in T. For the proofs of these facts see [20] and [29]. Thus the consistency of T follows from the consistency of ZFC. In fact it suffices to assume the consistency of ZFC$^-$ + ω_1 exists. Namely, in this theory we can repeat the usual Solovay–Tennenbaum's proof with the only difference that we use iteration of class-many notions of forcing. On the other hand, the consistency of T implies of course the consistency of ZFC$^-$ + ω_1 exists.

Our last example shows that there exist accurate interpretations of ZFC$^-$ + V $\neq$ HC in consistent extensions of A_2. It seems natural to ask whether there exist such interpretations of ZFC$^-$ with many uncountable cardinals. One should also observe that in the interpretations of kind described in this section there always exists a greatest cardinal number. We can thus ask whether there exists an interpretation of ZFC$^-$ with no maximal cardinal. In the next section we shall answer the second question affirmatively, thus also giving a positive answer to the first question.

§ 4. Definable quantifiers in models of A_2

In this section we shall be interested in the notion of a definable quantifier in models of second order arithmetic A_2.

We recall that a model M of A_2 is called an *ω-model* iff the natural numbers of M are of order type ω. Then we can identify them with the elements of ω and so the model M itself can be identified with a subset of $P(\omega)$. A β-model of A_2 is an ω-model with an absolute notion of well-ordering. This means that if $x \in M$ is a well-ordering of the natural numbers in the model M then it is a well-ordering of the natural numbers also outside M.

A definable quantifier Q in a model $M \models A_2$ is said to be *countably additive* iff

$$M \models Qx \exists n \varphi(n, x) \rightarrow \exists n Qx \varphi(n, x)$$

for any formula φ of the language of the model M. In other words, the formula $N(x)$ is countable-like for Q. It follows that if Q is countably additive in an ω-model M then a complete end extension of M with respect to Q is also an ω-model. This remark explains our particular interest in countably additive quantifiers.

Our first problem will be to find a β-model M with a definable quantifier Q such that every complete end extension of M with respect to Q is a β-model with the same height. This will also yield an example of two nonequivalent quantifiers in a model of A_2. Finally, we shall describe a model of A_2 with infinitely many nonequivalent, countably additive, definable regular quantifiers, thus solving one of the problems posed in [4].

Let us recall that Example 3.13 showed a β-model M of A_2 with a definable, uncountable, well-ordered class Φ such that every M-definable subclass of Φ was coded by a single real by means of a binary relation E. Let $\mathrm{WO}(x)$ be a formula which says that x is a code of a well-ordering of a subclass of Φ. For two such codes x and y, let $x \prec y$ means that the well-ordering coded by x is similar to a proper initial segment of the ordering coded by y. Now we define a quantifier Q in the model M as follows:

$$Qx\varphi(x) \equiv \forall x[\mathrm{WO}(x) \rightarrow \exists y[\mathrm{WO}(y) \,\&\, x \prec y \,\&\, \varphi(y)]],$$

i.e. the formula φ is satisfied by some codes of arbitrarily large well-orderings of the class Φ. We can easily prove that Q is a definable quantifier and that the class Φ is countable-like for Q. This follows from the fact that a codable family of well-orderings has a codable least uper bound.

Let us now suppose that a model N is a complete end extension of the model M with respect to the quantifier Q. We show that N is a β-model of the same height as M. Since the class Φ is uncountable and well-ordered in the model M, every well-ordering of the set of natural numbers of M is similar to an initial segment of the class Φ. The model N, as an elementary extension of M, has the same property. Now it is enough to observe that the class Φ is preserved in the extension and so the model N has no new order types of well-orderings of the set of natural numbers, thus being a β-model of the same height.

Now we observe that this extension preserves an uncountable class, so it is not a complete end extension with respect to the quantifier „there exist uncountably many". This shows that in the model M there are at least two nonequivalent countably additive quantifiers, giving in this way another proof of the theorem proved in [9]. Just as in [9] one of the quantifiers is not regular, namely the quantifiers Q. In order to see this let us consider a countable subset b of the class Φ and a well-ordering of the set b in the type ω. Let c be a code of this ordering. Now let us identify reals with permutations of the set of natural numbers. Then for any such permutation π we consider the corresponding permuted well-ordering of the set b. Let $\varphi(\pi, y)$ be a formula which says that y is a code of the well-ordering obtained from that coded by c (c being a parameter in φ) by applying the permutation π. Then obviously $M \models \forall \pi \exists y \varphi(\pi, y)$, and so $M \models Q\pi\exists y \varphi(\pi, y)$. Suppose that Q is regular. Then either $M \models \exists y Q\pi\varphi(\pi, y)$ or $M \models Qy\exists\pi\varphi(\pi,y)$. In the first case we obtain one code for many well-orderings, which is impossible. In the other case we obtain arbitrarily large well-orderings, all of them of type ω, which is also absurd.

Before showing that there exist models of A_2 with many nonequivalent regular quantifiers we shall generalize the notion of an accurate interpretation of ZFC$^-$, developed in Section 3. Our aim will be to construct such an interpretation of the theory ZFC$^-$ + "there is no maximal cardinal number". We shall formulate conditions for a consistent extension $T \supseteq A_2$ which guarantee that such an interpretation exists in T.

Let us first suppose that we are given a formula $\Psi(x, y)$ of the language $\mathscr{L}$, denoted later by $x \leqslant y$. Of course, $x < y$ denotes $\Psi(x, y)$ & $x \neq y$. Our first demand is that T proves.

(1) the formula $\leqslant$ defines a linear ordering of the whole universe.

We shall want the formula $\leqslant$ to define a well-ordering of the universe; so we assume that T proves all instances of the schema

(2) $$\exists x \varphi(x) \to \exists x [\varphi(x) \,\&\, \forall y [\varphi(y) \to x \leqslant y]],$$

for all formulas $\varphi(x)$ of the language $\mathscr{L}$ such that no collision of variables occurs.

We next assume that T proves the following from of collection scheme:

(3) $$\forall x \exists y \varphi(x, y) \to \forall a \exists b \forall x < a \exists y < b \varphi(x, y)$$

for all formulas $\varphi(x, y)$ of the language $\mathscr{L}$.

Finally, we suppose that we are given a formula $E(x, y)$ of the language $\mathscr{L}'$ such that T proves the following from of comprehension scheme for E:

$$\exists a \forall x[\varphi(x) \rightarrow x < a] \rightarrow \exists b \forall y[E(y, b) \equiv \varphi(y)] \tag{4}$$

for any formula $\varphi(x)$ of the language $\mathscr{L}$ for which no collision of variables occurs.

When we consider interpretations of a set theory in a model of the theory T, we shall accept parameters in the formulas Ψ and E (see page 24 for a similar convention).

In the sequel a class φ will be called *bounded* iff

$$\exists a \forall x[\varphi(x) \rightarrow x < a].$$

The comprehension scheme (4) shows that every bounded class is codable by a single real by means of the relation E. By modification of the relation E we can also assume the converse:

$$\forall b \exists a \forall x[E(x, b) \rightarrow x < a].$$

A real b will be called a *code of a bounded class* φ iff $\forall x[E(x, b) \equiv \varphi(x)]$ and this equivalence holds for no real $b' < b$. Obviously, it follows from (2) that every bounded class has a code. The class of codes of bounded classes will be denoted by C. If b is a code then $E(x, b)$ will be also denoted as $x e b$. The codes obviously satisfy the following extensionality axiom:

$$C(a) \;\&\; C(b) \;\&\; \forall x[x e a \equiv x e b] \rightarrow a = b.$$

The class $\{x\colon x e b\}$ for a code b will sometimes be denote by C_b.

For our further investigations it is important to describe a coding of binary relations. By J we denote a definable pairing function for the universe of the theory T, extending the usual arithmetical pairing function. Let K and L be the reverse functions: $K(J(x, y)) = x$, $L(J(x, y)) = y$. By a double application of schema (3) we easily get

$$\forall a \exists b \forall x < a \forall y < b[J(x, y) < b].$$

Now let $\varphi(x, y)$ be a formula of the language $\mathscr{L}$ such that

$$\exists a \forall x \forall y[\varphi(x, y) \rightarrow x < a \;\&\; y < a]$$

(in such a case we shall say that φ defines a bounded binary relation). Then consider the formula $\psi(z) \equiv \varphi(K(z), L(z))$. From the above it follows that ψ defines a bounded class. A code for ψ will be called a *code of the binary relation* defined by φ. Suppose, conversely, that we are given a bounded class ψ. Then if we define a binary relation $\varphi(x, y) \equiv \psi(J(x, y))$, we easily get by

(3) that φ is a bounded relation. Thus every code of a bounded class can be regarded as a code of a bounded binary relation.

An inductive argument shows that for any a there exists b such that $a < b$ and the initial segment determined by b (denoted by $O(b)$) is closed under J, K and L Therefore the function $J \upharpoonright O(b)$ is a pairing function for $O(b)$. The least such b will be called the pairing closure of a. For any code b and any x we shall denote by $(b)_x$ the code for the class $\{y\colon J(x, y) \in b\}$.

Now we are ready to construct an accurate interpretation of ZFC$^-$ in the theory T. The line of the construction follows that described in Section 3. We consider the class of codes of well-founded trees with domains being bounded classes and the membership and equality relations defined in the same way as in Section 3. It follows from the scheme (2) that we may select exactly one code from any equivalence class of the relation $\approx$, namely by choosing the code least in the ordering $\leqslant$. In this way we obtain an interpretation with absolute equality.

The proof that this is an interpretation of ZFC$^-$ follows closely that of Section 3. We shall indicate two minor differences. In the proof of pairing axiom and of the choice scheme we split the class Φ (which contained the domains of all trees under consideration) in many classes by using the pairing function. In our present case we first find a common bound a of the domain of all trees under consideration and next split the initial segment determined by the pairing closure of a. Then we can follow our previous proof. The other remark is that instead of using the scheme of choice (8) (see page 24) we use the collection scheme (3) together with the well-ordering scheme (2). The details of the proof will be omitted.

An analogue of the proof of Lemma 3.7 shows that this interpretation is accurate. Moreover the isomorphism establishing the accuracy can be extended to one level higher up. Namely, given a well-founded tree x which is a set of reals in the interpretation, we show that the class $\{\|y\|\colon y \varepsilon x\}$ is bounded: for any almost maximal node t of the tree x we consider the real $\|x \upharpoonright t\|$ and by applying the collection scheme we find a bound for the class of them which is the required bound. The code for this class will be taken as $\|x\|$. The extended function $\|\cdot\|$ is then an isomorphism between the class of natural numbers, reals numbers and sets of real numbers of the interpretation and the class of bounded classes of reals (i.e. speaking precisely, the codes of bounded classes of reals) of the theory T.

It is also worth observing that in the interpretation the continuum satisfies schemes (1)-(4). This allows us to define in the interpretation the notion of a well-founded tree of reals and extend the function $\|\cdot\|$ to all levels by putting $\|x\| = \{\|y\|\colon y \varepsilon x\}$. A routine proof shows that in the interpretation the notion of well-founded tree in the sense just described coincides with the usual set-theoretical notion. Moreover, every set is the value $\|x\|$ for some well-founded tree x. Let $\tau(u)$ denote the tree x such that $\|x\| = u$. Now we can define in the interpretation a well-ordering of the universe by putting $x \leqslant y$ iff $\tau(x) \leqslant \tau(y)$, where $\leqslant$ on the right hand side denotes the well-ordering of the continuum generated by that of the theory T.

Now we shall introduce the notion of cardinal number in the theory T and show that it corresponds closely to the notion of cardinal numbers in the above interpretation.

We say that a real a is a *cardinal number* iff for no real $b < a$ there exists a code c for a one-to-one function from $O(b)$ onto $O(a)$. If b is a code for a bounded class then the power of b, denoted by $|b|$, is the unique cardinal number a such that there exists a code c for a one-to-one function from $O(a)$ onto the class coded by b. It is obvious that if a cardinal number a is the power of b then a is unique. On the other hand, the class coded by b is well-ordered by the relation $\leqslant$ and can be embedded into the initial segment $O(d)$ for d being an upper bound for the class C_b. Therefore the class C_b is isomorphic with an initial segment of $O(d)$ and hence can be mapped in a one-to-one way onto some initial segment $O(a)$ of $O(d)$. The least a with this property is the power of b.

Now let a be a code for a bounded class. Then we define $c(a)$ to be the cardinal number in the interpretation of the tree $\tau(a)$, i.e. the tree x such that $\|x\| = a$. Let us consider a and b for bounded classes and let c be the code for a one-to-one function from C_a into C_b. Let us take well-founded trees x and y such that $\|x\| = a$ and $\|y\| = b$. Then with the help of c we construct a tree z such that in the interpretation z maps the tree x into y in a one-to-one way. Namely, we let z be a tree consisting of pairs of trees $\langle x \upharpoonright u, y \upharpoonright v \rangle$ where u and v are almost maximal nodes of the trees x and y, respectively, such that $\|y \upharpoonright v\|$is the value of the function C_c for $\|x \upharpoonright u\|$. It then follows that $c(a) \leqslant c(b)$. It is easy to see that if the code c codes a function from C_a onto C_b then the corresponding tree z is a function from x onto y. Thus if $|a| = |b|$ then $c(a) = c(b)$. On the other hand, if a tree z is a one-to-one mapping from a tree x into a tree y then the class $\{J(\|u\|, \|v\|): \langle u, v \rangle \varepsilon z\}$ is a one-to-one mapping from $\|x\|$ into $\|y\|$. It follows that if $c(a) \leqslant c(b)$ then $|a| \leqslant |b|$ and, just as before, if $c(a) = c(b)$ then $|a| = |b|$. Finally, in order to show that c is onto, let a well-founded tree x be a cardinal number in the interpretation. We may assume that the tree x has the property that for different almost maximal nodes t and u the subtrees $x \upharpoonright t$ and $x \upharpoonright u$ are not equivalent. Then we consider the class of almost maximal nodes of the tree x and its code b. It is easy to see that $c(b) = x$. This shows that the function c establishes an isomorphism between the cardinals of T and the cardinals of the interpretation.

We can also introduce in the natural way the notion of a regular cardinal in the theory T. The isomorphism $\|\cdot\|$ extended to the „third type" allows us to prove that the power of b is regular iff the cardinal number $c(b)$ is regular in the interpretation. We shall omit the details of a routine proof.

Since a function which establishes the equivalence of bounded classes is a "third type" object, the existence of the isomorphism $\|\cdot\|$ shows that all properties of cardinal numbers which can be expressed by means of the notion of equivalence and which can be proved in ZFC^- also hold in the theory T. In particular, all laws of cardinal arithmetic which do not use exponentiation are provable in T.

The notion of a cardinal number in the theory T allows us to carry over the results of Section 2 to the case of models of the theory T.

We begin by introducing an analogous notion of a regular segment of the class of cardinals. Let us consider a bounded class with a code b and assume that the elements of C_b are themselves codes for bounded classes. We define $\bigcup b$ to be the code for the class $\{x : \exists y[x \varepsilon y \;\&\; y \varepsilon b]\}$, which is obviously bounded.

DEFINITION 4.1. A subset I of the set of cardinals of a model M of T will be called a *regular segment* iff it has the following properties:

(1) if $x \in I$ and $y > x$ is a cardinal in M then $y \in I$,
(2) if $b \in M$ is a code for a bounded class of codes and $|\bigcup b| \in I$ then either $|b| \in I$ or there exists $y \in M$ such that $y \varepsilon_M b$ and $|y| \in I$.

As in Section 2 we define a family of sets $q_I \subseteq \mathrm{Def}(M)$ so that a set $X = \{x \in M : M \models \varphi[x]\}$ belongs to q_I iff either the formula φ defines in M an unbounded class or φ defines a bounded class with the code of cardinality belonging to I.

We can observe that the fact that a bounded class is embeddable in an unbounded one follows in T from the collection schema (3) and the well-ordering schema (2). Now, by letting the function ϱ to be the identity, we can repeat word-by-word the proofs of Lemmas 2.3 and 2.4. Thus we see that q_I is a regular pseudoquantifier in M and, conversely, every regular pseudoquantifier q in M is of the form q_I for some regular segment I of the class of cardinals of M. The proof of Theorem 2.6 is also easily adaptable to our situation. It only requires the observation that the formula

$$\left|\bigcup x\right| \leqslant \max(|x|, \sup\{|y| : y \in x\}),$$

which was used in the proof, is already provable in ZFC^- and so it holds in the model M.

We thus see that the only definable regular quantifiers Q in models of the theory T are of one of the following forms:

$Q_a x\,\varphi(x) \equiv$ "the formula φ defines an unbounded class or it defines a bounded class with code of power $\geqslant a$",

for a regular cardinal a, or

$Q_v x\,\varphi(x) \equiv$ "the formula φ defines an unbounded class".

It is also clear that if the relation $\leqslant$ well-orders a model M of T then Q_a and Q_b are the only regular pseudoquantifiers in M.

In order to carry over Theorem 2.7 we make a new assumption about the theory Z. Namely, we assume that T proves the axiom

(5) there does not exist a maximal cardinal number.

We then have

THEOREM 4.2. *There exists a countable model M of the theory T and an elementary extension N of M which is not a complete end extension of M for any definable regular quantifier $q \subseteq$ Def(M).*

Proof. We follow the proof of Theorem 2.7. We assume thet there exists a countable model M_0 of T such that the relation $\leqslant$ is a well-ordering of the universe of M. The existence of such a model will be shown later. We apply Theorem 1.11 to obtain a model M which extends M_0 and such that M_0 is exactly the initial well-ordered segment of M. The assumption that in M_0 there is no maximal cardinal number shows that the set of cardinal numbers of M_0 is not definable in M. We define I to be the set of cardinal numbers of M which do not belong to M_0. The regularity of the segment I follows from the fact that the least upper bound of a family of well-orderings is itself a well-ordering. The conclusion now follows exactly in the same way as in Theorem 2.7. ∎

We also observe that a model of the theory T is an example of a model of A_2 with infinitely many definable regular quantifiers. All except one are countably additive. The exception is the quantifiers Q_a where a denotes the cardinal of countable sets.

We shall now show that the theory T is consistent. To this end we apply the following theorem due to Harrington [10]:

THEOREM 4.3. *Let $\mathfrak{M}$ be a countable transitive model of* ZFC *and let $\alpha \in \mathfrak{M}$ be a regular cardinal in $\mathfrak{M}$ such that $\mathfrak{M} \models \alpha^{<\alpha} = \alpha$. Then there exists a ccc generic extension $\mathfrak{N}$ of $\mathfrak{M}$ such that the following conditions hold in $\mathfrak{N}$:*

(1) $2^{\omega} = \alpha$,

(2) *there exists a Σ^1_3 well-ordering of $P(\omega)$ of length α,*

(3) *every set of reals of power $< \alpha$ is Σ^1_3.* ∎

The continuum of the model $\mathfrak{N}$ is a model of the theory T, satisfying axioms (1)-(4). The well-ordering $\leqslant$ exists by (2) of the theorem and as a coding relation E we can take a universal predicate for Σ^1_3 sets. In order to obtain a model for the axiom (5) we use Harrington's theorem with α strongly inaccessible. It is clear that the relation $\leqslant$ is a well-ordering of the continuum of $\mathfrak{N}$ in the absolute sense.

It is worth checking how strong the theory T is when we include all the axioms (1)-(5). We know that we can interpret ZFC$^-$ + "there is no maximal cardinal" in T. In turn, it is easy to see that the constructible universe of that theory is a model of ZFC. Therefore the theory T is at least as strong as ZFC. On the other hand, a closer inspection of the proof of Theorem 4.3 shows that it works as a proof of the following

THEOREM 4.4. *Let $\mathfrak{M}$ be a countable transitive model of* ZFC. *Then there exists a ccc generic extension $\mathfrak{N}$ of the model $\mathfrak{M}$ via a proper class of forcing conditions such that the following hold in $\mathfrak{N}$:*

(1) ZFC$^-$ + "*$P(\omega)$ is a proper class*",

(2) *there is a Σ^1_3 well-ordering of the continuum such that the initial segments of it are sets,*

(3) *all sets of reals are Σ^1_3.* ∎

Obviously the continuum of the model $\mathfrak{N}$ is a model of the theory T satisfying all five axioms (1)-(5). As before, the relation $\leqslant$, which well-orders the continuum of $\mathfrak{N}$ is a well-ordering in the absolute sense. This shows that T is equiconsistent with ZFC.

Theorem 4.3 can be strengthened so that the well-ordering of the continuum is light-face Σ^1_3 (personal communication of L. Harrington). It follows that there exists a theory T as above with a definable (without parameters) accurate interpretation of the theory ZFC$^-$ with no maximal cardinal number.

We close this section with some remarks concerning the notion of an accurate interpretation of ZFC$^-$ in some consistent extension of A_2.

We can ask whether there exists a consistent extension $T \supseteq A_2$ such that for any model $\mathfrak{M}$ of ZFC and $M = (P(\omega))^{\mathfrak{M}}$ the following equivalence is true: $M \models T$ iff $\mathfrak{M} \models 2^\omega > \omega_1$. A result of Platek [27] shows that it is impossible. We shall sketch a proof of this result (cf. [8]) in the case of countable transitive models of ZFC.

Suppose that such a theory T exists. Let us take a countable transitive model $\mathfrak{M}$ of ZFC$+2^\omega > \omega_1$. Then $P(\omega) \cap \mathfrak{M} \models T$. We find a generic mapping f which collapses the cardinal of $P(\omega) \cap \mathfrak{M}$ onto $\omega_1^{\mathfrak{M}}$ such that $P(\omega) \cap \mathfrak{M} = P(\omega) \cap \mathfrak{M}[f]$. Then of course $\mathfrak{M}[f] \models 2^\omega = \omega_1$ and $P(\omega)$ $\mathfrak{M}[f] \models T$, a contradiction.

We can analyse the reason for this failure. It is easy to see that though the continuum of the model $\mathfrak{M}$ is big, this cannot be formulated in the continuum because there may exist functions which establish the equivalence of $P(\omega)$ and ω_1. On the other hand, the same continuum is small in $\mathfrak{M}[f]$, but again this cannot be formulated in the continuum since the function f exists "too high" to be seen from inside the continuum. This suggests another problem, namely, whether we can formulate some conditions in the language of second order arithmetic which would convince us that the continuum of the model is big. We proposed here an accurate interpretation as a solution to this problem. If a model M of A_2 allows an accurate interpretation of $\mathrm{ZFC}^- + \mathrm{V} \neq \mathrm{HC}$ then there exists a model $\mathfrak{M}$ of ZFC^- such that M is the continuum of $\mathfrak{M}$ and $\mathfrak{M}$ contains uncountable sets but the continuum of $\mathfrak{M}$ is a proper class.

Thus, in a sense, the existence of an accurate interpretation formalizes that $\mathfrak{M}$ satisfies the negation of the continuum hypothesis, so its continuum is big. We would like to ask here whether the converse problem also has a positive solution. The problem is thus the following: does there exist an extension $T \supseteq A_2$ such that, for any model $M \models A_2$, $M \models T$ iff there exists a model $\mathfrak{M} \models \mathrm{ZFC}^- +$ "ω_1 exists" $+$ "$P(\omega)$ is a proper class" such that M is the continuum of $\mathfrak{M}$?

For completeness let us mention that though there does not exist a theory $T \supseteq A_2$ such that for a model $\mathfrak{M} \models \mathrm{ZFC}$, $P(\omega) \cap \mathfrak{M} \models T$ implies $\mathfrak{M} \models 2^{\omega} > \omega_1$, there are sentences φ_n of the language of A_2 such that $P(\omega) \cap \mathfrak{M} \models \varphi_n$ implies $\mathfrak{M} \models 2^{\omega} \leqslant \omega_{n+1}$ but not $\mathfrak{M} \models 2^{\omega} \leqslant \omega_n$ (see [8]).

§ 5. Elementary generic extensions

In this section we recall our theorem (see [5]) that there exists a model M of A_2 with a Cohen generic extension which is an elementary extension. As a special case of the question posed at the end of Section 2 we can ask whether there exists a regular definable quantifiers $q \subseteq \mathrm{Def}(M)$ such that this elementary extension is a q-extension. We shall show that the answer is positive.

Throughout this section $\mathfrak{M}$ will be a countable transitive model of $\mathrm{ZFC} + \mathrm{V} = \mathrm{L}$. We shall deal with a model $\mathfrak{M}[G]$ obtained from $\mathfrak{M}$ by adding $\alpha \geqslant \omega_1^M$ Cohen reals.

Thus let $\alpha \in On \cap \mathfrak{M}$ be such that $\mathfrak{M} \models \alpha \geqslant \omega_1$. Let Q be the following notion of forcing:

$$p \in Q \text{ iff } p\colon a \to 2 \text{ for some finite } a \subseteq \alpha \times \omega,\ p \leqslant q \text{ iff } p \supseteq q.$$

For any $A \in \mathfrak{M}$ such that $A \subseteq \alpha$ and any M-generic filter $G \subseteq Q$ we define

$$Q_A = \{p \in Q\colon \mathrm{dom}(p) \subseteq A \times \omega\} \quad \text{and} \quad G_A = G \cap Q_A.$$

By product lemma we get $Q \cong Q_A \times Q_{\alpha - A}$ and hence

$$\mathfrak{M}[G] = \mathfrak{M}[G_A][G_{\alpha - A}].$$

By P we shall denote the usual Cohen notion of forcing:

$p \in P$ iff $p\colon a \to 2$ for some finite $a \subseteq \omega$;
$p \leqslant q$ iff $p \supseteq q$.

We easily observe that if $\mathfrak{M} \models |A| \leqslant \omega$ then the notions of forcing Q_A and P are equivalent, i.e. the corresponding complete Boolean algebras are isomorphic.

We next recall that Q is a ccc notion of forcing.

A proof of the following important lemma can be found in [28]:

LEMMA 5.1. *Let G be an $\mathfrak{M}$-generic filter in Q and $a \subseteq \omega$ be an element of $\mathfrak{M}[G]$. Then*

(1) *There exists $A \subseteq \alpha$, $A \in \mathfrak{M}$ such that $\mathfrak{M} \models |A| \leqslant \omega$ and $a \in \mathfrak{M}[G_A]$. In other words, there exists a real $b \subseteq \omega$, Cohen generic over $\mathfrak{M}$, such that $a \in \mathfrak{M}[b]$.*

(2) *There exists an $\mathfrak{M}[a]$-generic filter $H \subseteq Q$ such that $\mathfrak{M}[G] = \mathfrak{M}[a][H]$.* ∎

Before proving the next theorem we adopt a useful convention. Namely, we shal not distinguish between a Cohen generic real $a \subseteq \omega$ and the $\mathfrak{M}$-generic filter $G \subseteq P$ such that $a = \{n\colon \bigcup G(n) = 0\}$. Thus we shall often write that the real a is itself an $\mathfrak{M}$-generic filter in P.

THEOREM 5.2. *Let $G \subseteq Q$ be on $\mathfrak{M}$-generic filter and a real $a \subseteq \omega$ be Cohen generic over $\mathfrak{M}[G]$. Then $P(\omega) \cap \mathfrak{M}[G] \prec P(\omega) \cap \mathfrak{M}[G][a]$.*

Proof. It is enough to show that for any set-theoretical formula $\varphi(x_1, \ldots, x_n)$, if $a_1, \ldots, a_n \in \mathfrak{M}[G]$ are such that $a_i \subseteq \omega$ and $\mathfrak{M}[G] \models \varphi[a_1, \ldots, a_n]$, then $\mathfrak{M}[G][a] \models \varphi[a_1, \ldots, a_n]$.

Assume therefore that $a_1, \ldots, a_n \in P(\omega) \cap \mathfrak{M}[G]$ and $\mathfrak{M}[G] \models \varphi[a_1, \ldots, a_n]$. By Lemma 5.1 there exists $A \subseteq \alpha$, $A \in \mathfrak{M}$, such that $\mathfrak{M} \models |A| \leqslant \omega$ and $a_1, \ldots, a_n \in \mathfrak{M}[G_A]$. Thence $\mathfrak{M}[G_A][G_{\alpha-A}] \models \varphi[a_1, \ldots, a_n]$ and so, by the homogeneity of forcing $Q_{\alpha-A}$, we obtain $\mathfrak{M}[G_A] \models 1 \Vdash_{Q_{\alpha-A}} \varphi(\hat{a}_1, \ldots, \hat{a}_n)$. We next observe that

$$\mathfrak{M}[G][a] = \mathfrak{M}[G_A][G_{\alpha-A}][a] = \mathfrak{M}[G_A][G_{\alpha-A} \times a].$$

Since the notions of forcing $Q_{\alpha-A}$ and $Q_{\alpha-A} \times P$ are equivalent, there exists an $\mathfrak{M}[G_A]$-generic filter $H \subseteq Q_{\alpha-A}$ such that $\mathfrak{M}[G_A][G_{\alpha-A} \times a] = \mathfrak{M}[G_A][H]$. Therefore, since $\mathfrak{M}[G_A][H] \models \varphi[a_1, \ldots, a_n]$, we obtain $\mathfrak{M}[G][a] \models \varphi[a_1, \ldots, a_n]$. ∎

Our present aim will be to prove that the model $N = P(\omega) \cap \mathfrak{M}[G][a]$ of A_2 is a q-extension of the model $M = P(\omega) \cap \mathfrak{M}[G]$ for some regular definable quantifier $q \subseteq \mathrm{Def}(M)$. First we show that N is a complete end extension of the model M with respect to some regular definable quantifier q. The main step in the proof is the following

THEOREM 5.3. *Suppose a set theoretical formula $\varphi(x, p_1, \ldots, p_n)$ defines in the model $\mathfrak{M}[G]$ a subset of $P(\omega)$, the parameters $p_1, \ldots, p_n$ being either elements of $P(\omega)$ or elements of the model $\mathfrak{M}$. Then the set defined by φ contains a perfect subset in $\mathfrak{M}[G]$ iff it contains an element nonconstructible from the parameters.*

Proof. We follow the proof from [28]. Since the proof of one of the next theorems will be based on the same notation, we give the details.

Let us suppose that the set defined by the formula φ contains an element nonconstructible from the parameters $p_1, \ldots, p_n$. By Lemma 5.1 the model $\mathfrak{M}[G]$ is a generic extension of $\mathfrak{M}[p_1, \ldots, p_n]$ via the same notion of forcing Q, and so without loss of generality we can assume that all parameters $p_1, \ldots, p_n$ belong to $\mathfrak{M}$ (by replacing $\mathfrak{M}$ with $\mathfrak{M}[p_1, \ldots, p_n]$). Thus assume that all parameters in φ are in $\mathfrak{M}$ and φ is satisfied by a nonconstructible real.

Let $K = \{x \in \mathfrak{M}[G]: \mathfrak{M}[G] \models \varphi[x]\}$ and let $a \subseteq \omega$ be a nonconstructible element of K. By Lemma 5.1 there exists an $\mathfrak{M}$-generic Cohen real $u \in \mathfrak{M}[G]$ such that $a \in \mathfrak{M}[u]$. Observe that the fact $a \in \mathfrak{M}[u]$ can be expressed in the model $\mathfrak{M}[G]$ because the model $\mathfrak{M}$ is definable in $\mathfrak{M}[G]$ (as L or $L[p_1, \ldots, p_n]$ in the sense of $\mathfrak{M}[G]$).

We find an ordinal $\beta \in On \cap \mathfrak{M}$ such that $a = F_\beta(u)$, F being Gödel's constructibility function. Then

$$\mathfrak{M}[G] \models F_\beta(u) \subseteq \omega \;\&\; F_\beta(u) \notin \mathfrak{M} \;\&\; \varphi(F_\beta(u)).$$

By Lemma 5.1 we find an $\mathfrak{M}[u]$-generic filter $H \subseteq Q$ such that $\mathfrak{M}[G] = \mathfrak{M}[u][H]$. Then

$$\mathfrak{M}[u][H] \models F_\beta(u) \subseteq \omega \;\&\; F_\beta(u) \notin \mathfrak{M} \;\&\; \varphi(F_\beta(u)).$$

By the homogeneity of the notion of forcing Q we get

$$\mathfrak{M}[u] \models I_Q \Vdash_Q F_{\hat\beta}(\hat u) \subseteq \hat\omega \;\&\; F_{\hat\beta}(\hat u) \notin \mathfrak{M} \;\&\; \varphi(F_{\hat\beta}(\hat u)).$$

Let g be the cononical name for the generic real in the forcing P. Then

$$\mathfrak{M}[u] \models \exists x[x = j_G(g) \;\&\; I \Vdash_Q F_{\hat\beta}(\hat x) \subseteq \hat\omega \;\&\; F_{\hat\beta}(\hat x) \notin \mathfrak{M} \;\&\; \varphi(F_{\hat\beta}(\hat x))].$$

There exists a condition $p_0 \in P$ such that u extends p_0 and

$$p_0 \Vdash \exists x[x = g \;\&\; I \Vdash_Q F_{\hat\beta}(\hat x) \subseteq \hat\omega \;\&\; F_{\hat\beta}(\hat x) \notin \mathfrak{M} \;\&\; \varphi(F_{\hat\beta}(\hat x))].$$

Let us assume now that $u' \in \mathfrak{M}[G]$ is another Cohen generic real over $\mathfrak{M}$ extending p_0. Then

$$\mathfrak{M}[u'] \models I \Vdash_Q F_{\hat\beta}(\hat u') \subseteq \hat\omega \;\&\; F_{\hat\beta}(\hat u') \notin \mathfrak{M} \;\&\; \varphi(F_{\hat\beta}(\hat u')).$$

By Lemma 5.1 there exists an $\mathfrak{M}[u']$-generic filter $H \subseteq Q$ such that $\mathfrak{M}[u'][H'] = \mathfrak{M}[G]$ and so

$$\mathfrak{M}[G] \models F_\beta(u') \subseteq \omega \;\&\; F_\beta(u') \notin \mathfrak{M} \;\&\; \varphi(F_\beta(u')).$$

We next assume that we are given a pair $\langle u_1, u_2\rangle$ of reals, which is $\mathfrak{M}$-generic over $\boldsymbol{P}^2$ and such that both u_1, u_2 extend p_0. Then we claim that $F_\beta(u_1) \neq F_\beta(u_2)$.

Indeed $F_\beta(u_1) \in \mathfrak{M}[u_1]$ and, $F_\beta(u_2) \in \mathfrak{M}[u_2]$. Since $\mathfrak{M}[u_1] \cap \mathfrak{M}[u_2] = \mathfrak{M}$ and $F_\beta(u_i) \notin \mathfrak{M}$, then $F_\beta(u_1) \neq F_\beta(u_2)$.

Now we consider the set $S = \bigcup_{n\in\omega} 2^n$. We claim that in the model $\mathfrak{M}[G]$ there exists a function $f\colon S \to \boldsymbol{P}$ which has the following properties:

(1) $f(0) = p_0$,

(2) f is one-to-one,

(3) $s_1 \subseteq s_2 \to f(s_1) \subseteq f(s_2)$,

(4) if s_1 is incompatible with s_2 (i.e. $s_1 \cup s_2 \notin S$) then $f(s_1)$ is incompatible with $f(s_2)$,

(5) for each open dense subset $D \subseteq \boldsymbol{P}$, $D \in \mathfrak{M}$, there exists $n \in \omega$ such that $f(s) \in D$ for all s of length $\geqslant n$,

(6) for each open dense subset $D \subseteq \boldsymbol{P}^2$, $D \in \mathfrak{M}$, there exists $n \in \omega$ such that $\langle f(s_1), f(s_2)\rangle \in D$ for all $s_1, s_2 \in S$ of length $\geqslant n$ and such that $s_1 \neq s_2$.

In order to show that such a function f exists in $\mathfrak{M}[G]$ we consider in $\mathfrak{M}$ the following set of conditions $\boldsymbol{R}$:

$r \in \boldsymbol{R}$ iff there exists $n \in \omega$ such that $r\colon \bigcup_{k\leqslant n} 2^k \to \boldsymbol{P}$ and r satisfies conditions (1)–(4) of the above conditions imposed on f,
$r \leqslant r'$ iff $r \supseteq r'$.

We notice that the set $\boldsymbol{R}$ is countable in the model $\mathfrak{M}$, so it is equivalent to the notion of forcing $\boldsymbol{P}$. Thus in the model $\mathfrak{M}[G]$ there exist $\mathfrak{M}$-generic filters over $\boldsymbol{R}$. Let $F \subseteq \boldsymbol{R}$ be such a filter and $f = \bigcup F$. We show that f has the required properties. A standard argument shows that f is a function and $f\colon S \to \boldsymbol{P}$. It is trivial to verify that f has properties (1)–(4).

In order to prove that f has property (5), we take any open dense set $D \subseteq \boldsymbol{P}$, $D \in \mathfrak{M}$. Define

$$E = \{r \in \boldsymbol{R}\colon \exists n \in \omega \forall s \in S[lh(s) = n \to s \in \mathrm{dom}(r)\ \&\ r(s) \in D]\}.$$

Then it is easy to see that $E \in \mathfrak{M}$ and E is an open dense subset of $\boldsymbol{R}$; so there exists a condition r such that $r \in E \cap F$.

Then $r \subseteq f$ and, moreover, there exists $n \in \omega$ such that $r(s) \in D$ (and hence $f(s) \in D$) for $s \in S$ of length n. Since D is open, $f(s) \in D$ also for s of length bigger than n.

The condition (6) can be verified similarly.

Now we observe that for any function $h\colon \omega \to 2$, $h \in \mathfrak{M}[G]$ the real $\gamma(h) = \bigcup_{n\in\omega} f(h \upharpoonright n)$ belongs to $\mathfrak{M}[G]$ and is Cohen generic over $\mathfrak{M}$. Moreover, $\gamma(h)$ extends p_0. Also for different $h_1, h_2\colon \omega \to 2$, the pair $\langle \gamma(h_1), \gamma(h_2)\rangle$ is $\mathfrak{M}$-generic over P^2.

In particular, for $h\colon \omega \to 2$, $h \in \mathfrak{M}[G]$, we have

$$\mathfrak{M}[G] \models F_\beta(\gamma(h)) \subseteq \omega \;\&\; F_\beta(\gamma(h)) \notin \mathfrak{M} \;\&\; \varphi\big(F_\beta(\gamma(h))\big)$$

and for different $h_1, h_2 \in \mathfrak{M}[G]$, $F_\beta(\gamma(h_1)) \neq F_\beta(\gamma(h_2))$.

We define a function i by the formula

$$i(h) = F_\beta(\gamma(h)).$$

The above considerations prove that $i\colon 2^\omega \to K$ in the model $\mathfrak{M}[G]$ and that i is one-to-one.

It is enough to show that i is continuous. Then by the compactness of the Cantor set 2^ω, i is a homeomorphism and so K contains a perfect set.

Thus let $h_0\colon \omega \to 2$, $h_0 \in \mathfrak{M}[G]$ and $n \in \omega$. We are looking for $m \in \omega$ such that for any $h\colon \omega \to 2$, $h \in \mathfrak{M}[G]$ such that $h \upharpoonright m = h_0 \upharpoonright m$, we also have $i(h) \upharpoonright n = i(h_0) \upharpoonright n$. We consider the set

$$D = \{q \in P\colon q \text{ is incompatible with } p_0 \text{ or } q \leqslant p_0 \text{ and } \exists s \in 2^n [q \Vdash \hat{s} \subseteq \chi(F_{\hat\beta}(g))]\},$$

where $\chi(x)$ denotes the characteristic function of the set x.

Since $p_0 \Vdash \mathit{1} \Vdash_Q F_{\hat\beta}(\hat g) \subseteq \hat\omega$ and the sentence $F_\beta(x) \subseteq \omega$ is absolute, we have $p_0 \Vdash F_{\hat\beta}(g) \subseteq \hat\omega$. A standard argument shows that p_0 has an extension deciding the first n values of $\chi(F_{\hat\beta}(g))$. Thus D is dense in P. It is obvious that D is open and $D \in \mathfrak{M}$.

By the property (5) of the function f there exists $m \in \omega$ such that $f(t) \in D$ for $t \in S$ of length $\geqslant m$. In particular $f(h_0 \upharpoonright m) \in D$. Since $f(h_0 \upharpoonright m)$ extends p_0, there exists $s_0 \in 2^n$ such that $f(h_0 \upharpoonright m)$ forces $\hat s_0 \subseteq \chi(F_{\hat\beta}(g))$ and hence $s_0 \subseteq \chi(F_\beta(\gamma(h_0)))$.

Now we take any $h\colon \omega \to 2$, $h \in \mathfrak{M}[G]$ such that $h \upharpoonright m = h_0 \upharpoonright m$. Then $f(h \upharpoonright m) = f(h_0 \upharpoonright m)$, and so

$$f(h \upharpoonright m) \Vdash \hat s_0 \subseteq \chi(F_{\hat\beta}(g)).$$

Therefore $s_0 \subseteq \chi(F_\beta(\gamma(h_0)))$, and so

$$F_\beta(\gamma(h)) \upharpoonright n = F_\beta(\gamma(h_0)) \upharpoonright n.$$

Finally, $i(h) \upharpoonright n = i(h_0) \upharpoonright n$, which proves that the set K contains a perfect subset.

For the proof of the converse implication let us assume that there exists in the model $\mathfrak{M}[G]$ a perfect set such that all elements of it are constructible from the same real a. In other words, there exists a function $p\colon S \to S$ and a real a such that

(1) p is a one-to-one function preserving the ordering and incompatibility (i.e. the range of p is a perfect subset of the tree S),

(2) for every function $h\colon \omega \to 2$, $h \in \mathfrak{M}[G]$ the real $p[h]$ defined as $p[h] = \bigcup_{n\in\omega} p(h \upharpoonright n)$ is constructible from a.

We observe that any function $h\colon \omega \to 2$ is constructible from p and $p[h]$. Therefore in the model $\mathfrak{M}[G]$ every function $h\colon \omega \to 2$ is constructible from two reals: p and a, which is impossible. ∎

Before proceeding we shall make a remark on the above theorem. Namely as a byproduct of the proof we obtain the following property of Cohen's model $\mathfrak{M}[a]$:

Proposition 5.4. *Let a be a Cohen generic real over the model $\mathfrak{M}$. If a formula $\varphi(x, p_1, \ldots, p_n)$ defines a subset of $P(\omega)$ and moreover*

(1) $\mathfrak{M}[p_1, \ldots, p_n] \neq \mathfrak{M}[a]$,

(2) *there exists a real $b \notin \mathfrak{M}$ such that $\mathfrak{M}[b] \neq \mathfrak{M}[a]$ and $\mathfrak{M}[a] \models \varphi[b, p_1, \ldots, p_n]$,*

then the set defined by φ contains a perfect subset.

For the proof we follow the line of Theorem 5.3 making use of the following lemma instead of Lemma 5.1:

Lemma 5.5. *If a is a Cohen generic real over $\mathfrak{M}$ and $x \in \mathfrak{M}[a]$ is a real such that $\mathfrak{M}[x] \neq \mathfrak{M}[a]$ then there exists a Cohen generic real b over the model $\mathfrak{M}[x]$ such that $\mathfrak{M}[a] = \mathfrak{M}[x][b]$.*

We omit the details of the proofs. ∎

We return to our proof. In the model $M = P(\omega) \cap \mathfrak{M}[G]$ we define a quantifier Q as follows

$$Qx\,\varphi(x, x_1, \ldots, x_n) \equiv \text{there exists a perfect set of reals } x \text{ satisfying the formula } \varphi,$$

which can be written in the language $\mathscr{L}$ as

$$Qx\,\varphi(x, x_1, \ldots, x_n) \equiv \text{there exists a perfect tree } p\colon S \to S \text{ such that for any } h\colon \omega \to 2 \text{ we have } \varphi(p[h], x_1, \ldots, x_n).$$

Theorem 5.3 shows that the quantifier Q has the following equivalent forms:

$$Qx\,\varphi(x, x_1, \ldots, x_n) \equiv \text{there exists a real } x \text{ nonconstructible from } x_1, \ldots, x_n \text{ such that } \varphi(x, x_1, \ldots, x_n),$$

$Qx\,\varphi(x, x_1, \ldots, x_n) \equiv$ for every real y there exists a real x nonconstructible from y such that $\varphi(x, x_1, \ldots, x_n)$.

PROPOSITION 5.6. *Q is a regular definable quantifier in the model $M = P(\omega) \cap \mathfrak{M}[G]$.*

Proof. It is obvious that Q is a definable quantifier. We prove that it is regular. We shall use the first of the above equivalent forms of Q. Assume that $M \models Qx\exists y\,\varphi(x, y, x_1, \ldots, x_n)$. There exists a real $z \in M$, nonconstructible from $x_1, \ldots, x_n$, such that $M \models \exists y\,\varphi(z, y, x_1, \ldots, x_n)$. Take $y \in M$ such that $M \models \varphi(z, y, x_1, \ldots, x_n)$. There are two cases.

(1) y is constructible from $x_1, \ldots, x_n$. Then z is nonconstructible from $y, x_1, \ldots, x_n$, so $M \models Qx\,\varphi(x, y, x_1, \ldots, x_n)$. Therefore $M \models \exists y\,Qx\,\varphi(x, y, x_1, \ldots, x_n)$.

(2) y is nonconstructible from $x_1, \ldots, x_n$. Then $M \models \exists x\,\varphi(x, y, x_1, \ldots, x_n)$ and so $M \models Qy\exists x\,\varphi(x, x_1, \ldots, x_n)$. ∎

PROPOSITION 5.7. *The model $N = P(\omega) \cap \mathfrak{M}[G][a]$ is a complete end extension of the model $M = P(\omega) \cap \mathfrak{M}[G]$ with respect to the quantifier Q.*

Proof. We already know that N is an elementary extension of the model M.

Now let a formula $\varphi(x)$ of the language $\mathscr{L}_M$ be countable-like in M for the quantifier Q. Then $M \models \neg\, Qx\,\varphi(x)$, i.e. there exists a real $a \in M$ such that the formula φ is satisfied in M only by elements constructible from a. Thus φ has the same property in the model N. Let $x \in N$ be such that $N \models \varphi[x]$. Then x is constructible in N from a and since the heights of M and N are equal, x must belong to M.

On the other hand, if $M \models Qx\,\varphi(x)$ (which, by regularity of Q, means that φ is uncountable-like for Q in M) then there exists a perfect tree $p\colon S \to S$ such that all $p[h]$ for $h\colon \omega \to 2$ satisfy the formula $\varphi(x)$ in M. The same holds in N and it suffices to take $h\colon \omega \to 2$ from $N - M$ to obtain an element $p[h]$ of $N - M$ which satisfies the formula φ in the model N. ∎

THEOREM 5.8. *The model N is a Q-extension of the model M.*

Proof. We have to find a sequence $c_0, c_1, \ldots, c_n, \ldots$ of elements of the model N such that the following conditions hold:

(1) if a formula $\varphi(x)$ of the language $\mathscr{L}_M$ is such that $M \models Qx\,\varphi(x)$ then there exists $i \in \omega$ such that $N \models \varphi[c_i]$,

(2) if a formula $\varphi(x_0, \ldots, x_n)$ of the language $\mathscr{L}_M$ is such that $N \models \varphi[c_0, \ldots, c_n]$, then $M \models Qx_0 \ldots Qx_n\,\varphi(x_0, \ldots, x_n)$.

We enumerate all formulas $\varphi(x)$ of the language $\mathscr{L}_M$ such that $M \models Qx\,\varphi(x)$; let $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$ be such an enumeration. Now we define a sequence $b_0, b_1, \ldots, b_n, \ldots$ of Cohen generic reals over $\mathfrak{M}[G]$ and a sequence $\beta_0, \beta_1, \ldots, \beta_n, \ldots$ of ordinals so that the following conditions hold:

(i) $b_n \in \mathfrak{M}[G][a]$ for $n \in \omega$,
(ii) b_n is $\mathfrak{M}[G][b_0, \ldots, b_{n-1}]$-generic,
(iii) $\mathfrak{M}[G][a] \models \varphi_n[F_{\beta_n}(b_n)]$.

Suppose that b_i and β_i are already defined for $i < n$ and that $\mathfrak{M}[G][b_0, \ldots, b_{n-1}] \neq \mathfrak{M}[G][a]$. We consider the formula φ_n. Since $M \models Qx\varphi_n(x)$, the formula φ_n is satisfied in M by some element nonconstructible from the parameters occurring in φ_n. From the proof of Theorem 5.3 it follows that there exists in the model $\mathfrak{M}[G][a]$ a function $f: S \to \boldsymbol{P}$ and an ordinal β_n such that for any function $h: \omega \to 2$, $h \in \mathfrak{M}[G][a]$, the real $\gamma(h) = \bigcup_{n \in \omega} f(h \restriction n)$ is $\mathfrak{M}[G][b_0, \ldots, b_{n-1}]$-generic and that $\mathfrak{M}[G][a] \models \varphi_n\big(F_{\beta_n}(\gamma(h))\big)$. Namely, such a function is added by Cohen forcing over the model $\mathfrak{M}[G][b_0, \ldots, b_{n-1}]$ and since we have assumed that $\mathfrak{M}[G][b_0, \ldots, b_{n-1}] \neq \mathfrak{M}[G][a]$, there exists in $\mathfrak{M}[G][a]$ a Cohen generic real over $\mathfrak{M}[G][b_0, \ldots, b_{n-1}]$. We take any real of the form $\gamma(h)$ as our b_n. We next observe that in the model $\mathfrak{M}[G][a]$ the pair $\langle b_n, \gamma(h_1)\rangle$ is $\mathfrak{M}[G][b_0, \ldots, b_{n-1}]$-generic over $\boldsymbol{P}^2$, for any $h_1 \neq h$, so that $\mathfrak{M}[G][b_0, \ldots, b_{n-1}][b_n] \neq \mathfrak{M}[G][a]$. The construction is thus finished.

We put $c_n = F_{\beta_n}(b_n)$. Then $\mathfrak{M}[G][a] \models \varphi_n[c_n]$, and so the condition (1) is fulfilled.

For the proof of (2) we take a formula $\varphi(x_0, \ldots, x_n)$ such that $N \models \varphi[c_0, \ldots, c_n]$. Since b_n is $\mathfrak{M}[G][b_0, \ldots, b_{n-1}]$-generic and $c_0, \ldots, c_{n-1}$ are constructible from $b_0, \ldots, b_{n-1}$, respectively, the real c_n is nonconstructible from $c_0, \ldots, c_{n-1}$. The model N is an elementary extension of M, so by applying Theorem 5.3 to the model M we get $N \models Qx_n\varphi(c_0, \ldots, c_{n-1}, x_n)$. By repeating this reasoning n times we finally get $N \models Qx_0 \ldots Qx_n\varphi(x_0, \ldots, x_n)$, and hence $M \models Qx_0 \ldots Qx_n\varphi(x_0, \ldots, x_n)$. ■

We close this section with a study of the form of complete end extensions of our model M with respect to the quantifier Q. We shall prove that such a complete end extension is the continuum of a generic extension of the model $\mathfrak{M}$ over the set of conditions $\boldsymbol{Q}$, for $\alpha = \omega_1^{\mathfrak{M}}$.

Thus we now assume that $\boldsymbol{C}$ is the set of conditions defined as $\boldsymbol{Q}$ for the particular choice of α, namely for $\alpha = \omega_1^{\mathfrak{M}}$. We also fix the quantifier Q.

We first observe that a filter $H \subseteq \boldsymbol{C}$ is $\mathfrak{M}$-generic iff H intersects every maximal antichain $X \subseteq \boldsymbol{C}$ which belongs to $\mathfrak{M}$. Then we notice that every maximal antichain $X \subseteq \boldsymbol{C}$, $X \in \mathfrak{M}$, is hereditarily countable in $\mathfrak{M}$. Thus we shall consider the model $\mathfrak{M}_0 = \mathrm{HC}^{\mathfrak{M}}$. Of course $\mathfrak{M}_0 \models \mathrm{ZFC}^- + \mathrm{V} = \mathrm{L} + \mathrm{V}$

= HC. The forcing C is a definable class in $\mathfrak{M}_0$ and it is a coherent notion of forcing in $\mathfrak{M}_0$ in the sense of [32].

We recall that a filter H in a coherent notion of forcing $C \subseteq \mathfrak{M}_0$ is $\mathfrak{M}_0$-generic iff it intersects every $\mathfrak{M}_0$-definable dense class of conditions. Since this is equivalent to intersecting every maximal antichain in C (which must belong to $\mathfrak{M}_0$), we see that a filter $H \subseteq C$ is $\mathfrak{M}_0$-generic iff it is $\mathfrak{M}$-generic. Moreover, if $H \subseteq C$ is $\mathfrak{M}$-generic and $x \in \mathfrak{M}[H]$ is hereditarily countable in $\mathfrak{M}[H]$ then $x \in \mathfrak{M}_0[H]$. Thus it suffices to show that if a model N is a complete end extension of the model M then there exists an $\mathfrak{M}_0$-generic filter $H \subseteq C$ such that $N = P(\omega) \cap \mathfrak{M}_0[H]$.

We shall need the following important property of the model $\mathfrak{M}_0[H]$. For $\xi \in On \cap \mathfrak{M}_0 = \omega_1^{\mathfrak{M}}$ we put $C_\xi = \{p \in C\colon \operatorname{dom}(p) \subseteq \xi \times \omega\}$. Then by putting $H_\xi = H \cap C_\xi$ we get $H_\xi \in \mathfrak{M}_0[H]$ and, moreover, $\mathfrak{M}_0[H]$ is the least model of ZFC$^-$ which contains $\mathfrak{M}_0$ and all the H_ξ for $\xi \in On \cap \mathfrak{M}_0$.

Now let N be a complete end extension of the model M. We can easily observe that N is a β-model of the same height as M. Namely we first notice that the formula "x is constructible" is countable-like for Q in M and hence absolute between M and N. Now it is enough to see that if $x \in N$ is a well-ordering then there exists a constructible real $a \in M$ such that the ordering x is isomorphic to the ordering of reals constructible earlier than a. This shows that there exists a countable transitive model $\mathfrak{N} \models \text{ZFC}^- + \text{V} = \text{HC}$ such that $N = P(\omega) \cap \mathfrak{N}$, $\mathfrak{M}_0[G] \prec \mathfrak{N}$, and the height of $\mathfrak{N}$ is equal to that of $\mathfrak{M}_0$. It is easy to see that the model $\mathfrak{N}$ satisfies the following axioms:

(A1) $\forall x \exists y [x \in L[y]$ & y is Cohen generic over $L]$,

(A2) $\forall x \exists y [y$ is Cohen generic over $L[x]]$.

It is obvious that our aim will be achieved once we prove the following

THEOREM 5.9. *Let $\mathfrak{N}$ be a countable transitive model of* $\text{ZFC}^- + \text{V} = \text{HC} + (\text{A1}) + (\text{A2})$ *such that $On \cap \mathfrak{N} = On \cap \mathfrak{M}_0$ where $\mathfrak{M}_0 \models \text{ZFC}^- + \text{V} = \text{HC} + \text{V} = \text{L}$. Then there exists an $\mathfrak{M}_0$-generic filter $H \subseteq C$ such that $\mathfrak{N} = \mathfrak{M}_0[H]$.*

Proof. We repeat the reasoning from [31].

We define in $\mathfrak{N}$ the following class of conditions:

$p \in R$ iff p is an L-generic (i.e. $\mathfrak{M}_0$-generic) filter in C_α for some $\alpha \in On \cap \mathfrak{N}$,

$p \leqslant q$ iff $p \supseteq q$.

Let $H' \subseteq R$ be an $\mathfrak{N}$-generic filter. We put $H = \bigcup H'$. It is obvious that $H \subseteq C$. We shall verify that H is an $\mathfrak{M}_0$-generic filter in C and that $\mathfrak{N} = \mathfrak{M}_0[H]$.

It is easy to see that H is a filter in C. In order to show that it is $\mathfrak{M}_0$-generic, we take any $\mathfrak{M}_0$-definable class $D \subseteq C$, dense in C. We define $E = \{r \in R\colon r \cap D \neq 0\}$. Clearly, E is a subclass of R definable in $\mathfrak{N}$, since $\mathfrak{M}_0$ is definable in $\mathfrak{N}$ as $L^{\mathfrak{N}}$. We show E is dense in R.

Let $p \in \boldsymbol{R}$ and take $\alpha \in On \cap \mathfrak{M}$ such that $p \subseteq \boldsymbol{C}_\alpha$ and p is $\mathfrak{M}_0$-generic in $\boldsymbol{C}_\alpha$. We consider the set $D_\alpha = \{q \upharpoonright \alpha \times \omega \colon q \in D\}$, which obviously belongs to $\mathfrak{M}_0$. We can easily see that D_α is dense in $\boldsymbol{C}_\alpha$ and so $p \cap D_\alpha \neq 0$. Let $q \in p \cap D_\alpha$. We find a condition $q' \in D$ such that $q = q' \upharpoonright \alpha \times \omega$. If $q = q'$ then $q \in p \cap D$ and so $p \in E$. Thus assume that $q \neq q'$. Let β be such that $q' \in \boldsymbol{C}_\beta$. We can assume that $\beta > \alpha$. By (A2) there exists a Cohen generic real y over $L[p]$. Since Cohen's forcing $\boldsymbol{P}$ is equivalent to $\boldsymbol{C}_\beta - \boldsymbol{C}_\alpha$, there exists $r \supseteq p$ which is L-generic over $\boldsymbol{C}_\beta$. We can assume that $q' \in r$. In order to see this, we must observe that if we change values of all conditions is a generic filter on a fixed finite set, the resulting filter is also generic. Now we get $q' \in D$ and then $r \cap D \neq 0$, i.e. $r \in E$. Thus we have shown that E is dense.

Now let $r \in E \cap H'$. Then $r \cap D \neq 0$; thus let us take $p \in r \cap D$. Then $p \in r \in H'$ and so $p \in H$, whence $H \cap D \neq 0$. This proves that H is $\mathfrak{M}_0$-generic.

It remains to show that $\mathfrak{N} = \mathfrak{M}_0[H]$. We first show that $\mathfrak{M}_0[H] \subseteq \mathfrak{N}$ by proving that $H_\alpha \in \mathfrak{N}$ for each $\alpha \in On \cap \mathfrak{M}_0$.

By genericity of the filter H', for each such α there exists an L-generic filter $p \subseteq \boldsymbol{C}_\alpha$ such that $p \in H'$. It is enough to prove that $p = H_\alpha$.

It is clear that $p \subseteq H_\alpha$. Thus let $q \in H_\alpha$. We take $p' \in H'$ such that $q \in p'$. We can assume that p' is an L-generic filter in $\boldsymbol{C}_\beta$ for some $\beta > \alpha$. Since p and p' are both in H', $p \subseteq p'$ and so $p = p' \cap \boldsymbol{C}_\alpha$. It follows that $q \in p$.

Now it remains to that $\mathfrak{N} \subseteq \mathfrak{M}_0[H]$. It suffices to show that

$$\forall x \in \mathfrak{N} \exists \alpha \in On \cap \mathfrak{M}_0 [x \in \mathfrak{M}_0[H_\alpha]].$$

Let $x \in \mathfrak{N}$. We consider the class $D_x \subseteq \boldsymbol{R}$: $D_x = \{p \in \boldsymbol{R} \colon x \in L[p]\}$. It is obviously $\mathfrak{N}$-definable. We show that it is dense in $\boldsymbol{R}$.

Let $q \in \boldsymbol{R}$. Then q is L-generic in $\boldsymbol{C}_\alpha$ for some $\alpha \in On \cap \mathfrak{M}_0$. We take $y \in \mathfrak{N}$, Cohen generic over $\mathfrak{M}_0$, such that q, $x \in L[y]$ in $\mathfrak{N}$ (we use (A1) here) and consider models $\mathfrak{M}_0[q] \subseteq \mathfrak{M}_0[y]$. If $\mathfrak{M}_0[q] = \mathfrak{M}_0[y]$ then $x \in L[q]$ in $\mathfrak{N}$ and hence $q \in D_x$. Thus assume that $\mathfrak{M}_0[q] \neq \mathfrak{M}_0[y]$. By Lemma 5.5 there exists a Cohen generic real z over $\mathfrak{M}_0[q]$ such that $\mathfrak{M}_0[y] = \mathfrak{M}_0[q][z]$. Now it is easy to find an $\mathfrak{M}_0$-generic filter $p \subseteq \boldsymbol{C}_{\alpha+1}$ such that $q \subseteq p$ and $\mathfrak{M}_0[y] = \mathfrak{M}_0[p]$. Then obviously $p \in D_x$.

Now we take $p \in H' \cap D_x$. Let $\alpha \in \mathfrak{M}_0$ be such that p is an $\mathfrak{M}_0$-generic filter in $\boldsymbol{C}_\alpha$. Then $p = H_\alpha$ and so $x \in \mathfrak{M}_0[H_\alpha]$. ∎

It follows from the above theorem that if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are countable transitive models of $\mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC} + (\mathrm{A1}) + (\mathrm{A2})$ and $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$, then $\mathfrak{M}_1 \prec \mathfrak{M}_2$. We do not know whether for these models there exists an $\mathfrak{M}_1$-generic filter $G \subseteq \boldsymbol{C}$ such that $\mathfrak{M}_1[G] = \mathfrak{M}_2$.

Added in proof. Then problem posed after Theorem 3.5 of whether WO is provable in ZFC^- has been answered negatively by Z. Szczepaniak.

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