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MICROLOCAL PROPERTIES OF A CLASS OF PSEUDODIFFERENTIAL OPERATORS WITH DOUBLE INVOLUTIVE CHARACTERISTICS

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1. This paper deals with the microlocal properties of a class of second order pseudodifferential operators (ψ.d.o.) with double involutive characteristics and elliptic subprincipal symbol. Assuming the principal symbol to be nonnegative a necessary and sufficient condition for hypoellipticity with loss of the smoothness of solutions equal to 1 has been established by Boutet de Monvel [1] and Hörmander [7]. In contrast with them we shall suppose that the real part of the subprincipal symbol of the operator is negative, while its imaginary part vanishes at some points (but not identically). Then necessary and sufficient conditions for the operators under consideration to be (micro) locally hypoelliptic and solvable are obtained. The results are similar to the main theorems for principal type ψ.d.o. [3], [4].

Throughout the paper the notions of wave front set of a distribution, homogeneous canonical transformation, microlocalized Sobolev space $H^s_{\mathrm{mce}}(\varrho^0)$ as well as of quasihomogeneous wave front set and anisotropic Sobolev space are supposed to be well known [4], [9], [10], [12].

To begin with, several definitions are presented.

DEFINITION 1 (classical, local). A ψ.d.o. $P(x, D)$ is called *locally solvable* at a point $x_0 \in X$ if there exists a neighbourhood $\omega \ni x_0$ such that for each function $f \in C_0^\infty(X)$ one can find a distribution $u \in \mathscr{D}'(X)$ satisfying $P(x, D)u = f$ in ω.

DEFINITION 2 (Trèves). The operator $P(x, D)$ is called *microlocally solvable* at a point $\varrho^0 \in T^*(X)\setminus 0$ if one can find an integer $N \in \boldsymbol{Z}$ and a conic neighbourhood Γ of ϱ^0 such that for each $f \in H^N_{\mathrm{loc}}(X)$ there exists a distribution $u \in \mathscr{D}'(X)$ satisfying the relation $\Gamma \cap \mathrm{WF}(Pu - f) = \emptyset$.

The operator P is *microlocally hypoelliptic* at the point ϱ^0 if and only if $\varrho^0 \notin \mathrm{WF}(Pu) \Rightarrow \varrho^0 \notin \mathrm{WF}(u)$. The standard hypoellipticity is given by the equality sing supp Pu = sing supp u.

2. We shall consider classical second order ψ.d.o. with the following principal symbol:

$$(1) \qquad p_2^0(z, \zeta) = \sum_{i,j=1}^{n-1} a_{ij}(x, y; \xi, \eta)\xi_i \xi_j, \qquad \operatorname{ord}_\zeta a_{ij} = 0,$$

$$(a_{ij}(z, \zeta))_{i,j=1}^{n-1} > 0, \qquad a_{ij}(z, \zeta) = a_{ji}(z, \zeta),$$

$$z = (x, y) \in \boldsymbol{R}^{n-1} \times \boldsymbol{R}^1, \qquad \zeta = (\xi, \eta) \in \boldsymbol{R}^{n-1} \times \boldsymbol{R}^1.$$

The *characteristic set* of P is given by the formula

$$\operatorname{Char} p_2^0 = \{(z, \zeta)\colon\ \xi = 0,\ \eta \neq 0\}$$

and the *subprincipal symbol* p_1' of P is defined as follows:

$$p_1'(z, \zeta) = p_1(z, \zeta) + \frac{i}{2}\sum_{j=1}^{n} \frac{\partial^2 p_2^0}{\partial z_j \partial \zeta_j}(z, \zeta), \qquad \operatorname{ord}_\zeta p_1 = 1.$$

Write $\Sigma_{n-1}^* = \{(x, y; \lambda, \tilde{\eta})\colon \lambda = (dx_1, \ldots, dx_{n-1}),\ y = y_0 = \text{const},\ \tilde{\eta} = \tilde{\eta}^0 = \text{const} \neq 0\}$. Then the principal symbol p_2^0 induces the following quadratic form on $\Sigma_{n-1}^*|_{\operatorname{Char} p_2^0}$:

$$(2) \qquad \sum_{i,j=1}^{n-1} a_{ij}(x, y; 0, \tilde{\eta})\eta_i \eta_j, \qquad a_{ij}(z; 0, \tilde{\eta}) = a_{ij}(z; 0, \operatorname{sgn} \tilde{\eta}),\ \tilde{\eta} \in \boldsymbol{R}^1 \setminus 0.$$

The Hamiltonian vector field H^M of the form (2) on Σ_{n-1}^* is

$$2\sum_{i,j=1}^{n-1} a_{ij}(x, y; 0, \tilde{\eta})\eta_i \frac{\partial}{\partial x_j} - \sum_{i,j,k=1}^{n-1} \frac{\partial a_{ik}}{\partial x_j}\eta_i \eta_k \frac{\partial}{\partial \eta_j}.$$

Consider now the function

$$(3) \qquad I_P(\varrho, \eta) = \sum_{j,k=1}^{n-1} a_{jk}(\varrho)\eta_j \eta_k + p_1'(\varrho), \qquad \varrho \in \operatorname{Char} p_2^0,$$
$$\eta = (\eta_1, \ldots, \eta_{n-1}) \in \boldsymbol{R}^{n-1} \setminus 0.$$

Obviously $I_P \in C^\infty(\Sigma^*)$. As the zero bicharacteristics of $\operatorname{Re} I_P$ will play an important role further, we shall give an explicit formula for them.

Thus assume that the asymptotic development of the full symbol p of P is given by

$$(4) \qquad p \sim \sum_{i,j=1}^{n-1} a_{ij}(x, y; \xi, \eta)\xi_i \xi_j + p_1(z, \zeta) + p_0(z, \zeta) + \ldots,$$
$$\operatorname{ord}_\zeta p_{-j} = j \geqslant 0,\ j \in \boldsymbol{Z},$$

and $p_1'|_{\operatorname{Char} p_2^0} = p_1'|_{\xi=0} = p_1(z; 0, \eta) = |\eta|\, p_1(z; 0, \operatorname{sgn} \eta)$.

Put $c_1 = \operatorname{Re} p_1$, $d_1 = \operatorname{Im} p_1$ and suppose that $c(z; 0, \eta) < 0$. Then the nondegenerate Hamiltonian vector field of $\operatorname{Re} I_P$ on Σ^*_{n-1} is

$$(5)\quad 2\sum_{i,j=1}^{n-1} a_{ij}(x, y; 0, \eta)\eta_i \frac{\partial}{\partial x_j} - \sum_{i,j,k=1}^{n-1} \frac{\partial a_{ik}}{\partial x_j}(x, y; 0, \eta)\eta_i\eta_k \frac{\partial}{\partial \eta_j} - \sum_{j=1}^{n-1} \frac{\partial c_1}{\partial x_j}(x, y; 0, \eta)\frac{\partial}{\partial \eta_j}, \qquad \eta \in \boldsymbol{R}^1 \setminus 0$$

((y, η) are, of course, parameters). We recall that an integral curve γ of the vector field (5) is called a *zero bicharacteristic* iff $\operatorname{Re} I_P|_\gamma \equiv 0$.

We now formulate the first result of this paper.

THEOREM 1. *Consider a ψ.d.o. $P(z, D)$ with principal symbol* (1) *and let* $\operatorname{Re} p_1'|_{\operatorname{Char} p_2^0} < 0$. *Suppose that* $\operatorname{Im} p_1'|_{\operatorname{Char} p_2^0}$ *conserves its sign in a neighbourhood of each zero bicharacteristic γ of* $\operatorname{Re} I_P$ *and* $\operatorname{Im} p_1'|_{\operatorname{Char} p_2^0}$ *does not vanish identically on any open interval of γ. Then $P(z, D)$ is microlocally hypoelliptic and*

$$\varrho^0 \in \operatorname{Char} p_2^0,\ Pu \in H^s_{\mathrm{mce}}(\varrho^0) \quad \Rightarrow \quad u \in H^{s+1/2}_{\mathrm{mce}}(\varrho^0)$$

for each real s. Both P and P^* are locally solvable.

No result on hypoellipticity holds when $\operatorname{Im} p_1'|_{\operatorname{Char} p_2^0}$ changes its sign on a single bicharacteristic of $\operatorname{Re} I_P$.

THEOREM 1'. *Let a ψ.d.o. $P(z, D)$ have the principal symbol* (1) *in a conic neighbourhood of $\varrho^0 \in \operatorname{Char} p_2^0$,* $\operatorname{Re} p_1'(\varrho^0) < 0$, $\operatorname{Im} p_1'(\varrho^0) = 0$. *Assume that one can find a covector $\eta^0 \in \boldsymbol{R}^{n-1} \setminus 0$ with the properties*: $\operatorname{Re} I_P(\varrho^0, \eta^0) = 0$ *and* $\operatorname{Im} p_1'|_{\operatorname{Char} p_2^0}$ *changes its sign at ϱ^0 along the zero bicharacteristic γ of* $\operatorname{Re} I_P$ *passing through (ϱ^0, η^0). Then $P(z, D)$ is not microhypoelliptic at ϱ^0. Moreover, there exists a distribution $u \in \mathscr{E}'(X)$ such that $\varrho^0 \notin \operatorname{WF}(Pu)$, $\operatorname{WF}(u) = \{t\varrho^0 : t > 0\}$ in some cone about ϱ^0.*

Recall that $\operatorname{Im} p_1'$ *changes its sign* along a curve γ at a point ϱ^0 if $\operatorname{Im} p_1'|_{\gamma_+} \leqslant 0$ ($\geqslant 0$), $\operatorname{Im} p_1'|_{\gamma_-} \geqslant 0$ ($\leqslant 0$) and $\operatorname{Im} p_1'$ does not vanish identically on any open interval of γ_+ (γ_-). Here γ_+ (γ_-) is the part of the curve γ positively (negatively) oriented with respect to (ϱ^0, η^0).

In some special cases a theorem on local nonsolvability is true.

THEOREM 2. *Assume that $P(z, D)$ is a ψ.d.o. with principal symbol* (1), $\varrho^0 \in \operatorname{Char} p_2^0$, $\operatorname{Re} p_1'(\varrho^0) < 0$, $\operatorname{Im} p_1'(\varrho^0) = 0$. *Suppose that there exists a covector $\eta^0 \in \boldsymbol{R}^{n-1} \setminus 0$ for which* $\operatorname{Re} I_P(\varrho^0, \eta^0) = 0$ *and* $\operatorname{Im} p_1'|_{\operatorname{Char} p_2^0}$ *has a simple zero at ϱ^0 along the zero bicharacteristic of* $\operatorname{Re} I_P$ *passing through (ϱ^0, η^0). Then P is microlocally nonsolvable at ϱ^0. Moreover, for each conic neighbourhood*

$\Gamma \ni \varrho^0$ and each pair of numbers $s' < s$ one can find a function $u \in H^{s'}_{\mathrm{loc}}(X)$ such that $\mathrm{WF}(Pu) \cap \Gamma = \emptyset$,

$$\mathrm{WF}(u) \cap \Gamma = \mathrm{WF}_s(u) \cap \Gamma = \{t\varrho^0 \colon t > 0\} \quad (\varrho^0 \notin \mathrm{WF}_s(u) \Leftrightarrow \varrho^0 \in H^s_{\mathrm{mce}}(\varrho^0)).$$

To illustrate our theorems a simple example is presented.

EXAMPLE 1. Consider in $\boldsymbol{R}^n$ the Schrödinger operator with a small perturbation of the first derivative with respect to the time variable D_y:

$$(6) \qquad P(z, D) = \sum_{j=1}^{n-1} D_j^2 + \big(-1 + i\varphi(x)\big) D_y,$$

$$\varphi(x) \in C^\infty(\boldsymbol{R}^{n-1}; \boldsymbol{R}^1),\ z = (x_1, \ldots, x_{n-1}, y),\ D_j = \frac{1}{i} \frac{\partial}{\partial x_j}.$$

I. Let $\varphi \equiv 0$. The operator P turns out to be microlocally hypoelliptic at points of the type $(z_0; \xi^0, \eta^0)$, $\xi^0 \neq 0$ and $(z_0; 0, \tilde{\eta}^0)$, $\tilde{\eta}^0 < 0$. In the latter case the microlocal loss of smoothness of solutions of the equation $Pu = f$ is equal to 1 [1]. Boutet de Monvel has shown that a result on the propagation of singularities is valid in a conic neighbourhood of the point $(z_0; 0, \tilde{\eta}^0)$, $\tilde{\eta}^0 > 0$ [2].

II. Let $\varphi \not\equiv 0$. Obviously $\operatorname{Re} I_P = \sum_{j=1}^{n-1} \xi_j^2 - \eta$, $\eta > 0$, and therefore the zero bicharacteristics of $\operatorname{Re} I_P$ can be written as follows:

$$l\colon \quad \begin{matrix} x_j = 2\xi_j^0 t + x_{j0} \\ \xi_j = \xi_j^0 \end{matrix}, \quad \sum_{j=1}^{n-1} \xi_j^{02} = \tilde{\eta}^0, \quad y = y_0.$$

1) Assume first that the function φ does not change its sign and if $\varphi(x_0) = 0$ then φ does not vanish identically on any interval of straight lines $l \subset \boldsymbol{R}^{n-1}_x$ passing through the point x_0. We claim that P is microlocally hypoelliptic. More precisely,

$$P(z, D) u \in H^s_{\mathrm{mce}}(\varrho^0), \quad \varrho^0 \notin \mathrm{Char}\ p_2^0 \quad \Rightarrow \quad u \in H^{s+2}_{\mathrm{mce}}(\varrho^0);$$

$$P(z, D) u \in H^s_{\mathrm{mce}}(\varrho^0), \quad \varrho^0 = (z_0; 0, \tilde{\eta}^0 < 0) \in \mathrm{Char}\ p_2^0 \quad \Rightarrow \quad u \in H^{s+1}_{\mathrm{mce}}(\varrho^0);$$

$$P(z, D) u \in H^s_{\mathrm{mce}}(\varrho^0), \quad \varrho^0 = (z_0; 0, \tilde{\eta}^0 > 0) \in \mathrm{Char}\ p_2^0 \quad \Rightarrow \quad u \in H^{s+1/2}_{\mathrm{mce}}(\varrho^0).$$

2) Suppose now that φ changes its sign along a line l passing through x_0. For the sake of simplicity we can assume that φ changes its sign at the point x_0. The additional assumption that φ is monotonically increasing (decreasing) along l in a small neighbourhood of x_0 implies the microlocal nonsolvability of P at the point $\varrho^0 = (x_0, y_0; 0, \eta^0 > 0)$. Moreover, for each conic neighbourhood $\Gamma \ni \varrho^0$ and each pair of real numbers $s' < s$ one can find a distribution $u \in H^{s'}_{\mathrm{loc}}(\boldsymbol{R}^n)$ such that $\mathrm{WF}(Pu) \cap \Gamma = \emptyset$, $\mathrm{WF}(u) \cap \Gamma = \mathrm{WF}_s(u) \cap \Gamma = \{t\varrho^0 \colon t > 0\}$.

3) Let $\varphi|_{\tilde{l}} \equiv 0$, where $\tilde{l}$ is an open interval lying on l. Suppose that $\partial\varphi/\partial l_j|_{\tilde{l}} \equiv 0$, where $l, l_1, \ldots, l_{n-2}$ form an orthogonal basis in $\boldsymbol{R}_x^{n-1}$. Then we claim that the operator P is nonhypoelliptic at the points of $\tilde{l}$. Thus the condition on $\operatorname{Im} p_1'|_{\operatorname{Char} p_2^0}$ in Theorem 1 not to vanish identically on any open interval of γ is essential.

An application of the machinery of nonhomogeneous wave front sets enables us to obtain more precise positive results concerning operators considered in Theorem 1. Thus let the weight $M = (1, \ldots, 1, 2)$ and denote by $\mathrm{WF}_M(u)$ the M-quasihomogeneous wave front set of the distribution u [9]. The symbol $H^{s,M}(\boldsymbol{R}^n)$ stands for the anisotropic Sobolev space with weight $[\zeta]_M \approx (|\xi|^2 + |\eta|)^{1/2}$. The definition of the microlocalized Sobolev space $H^{s,M}_{\mathrm{mcc}}(\varrho^0)$ is standard and we omit the details.

PROPOSITION. *Consider an M-quasihomogeneous ψ.d.o. P with principal symbol*

$$p_{2,M}^0(z, \zeta) = \sum_{i,j=1}^{n-1} a_{ij}(z, \eta)\xi_i \xi_j + c_1(z, \eta) + i d_1(z, \eta),$$

where $\operatorname{ord}_\eta a_{ij} = 0$, $\operatorname{ord}_\eta c_1 = \operatorname{ord}_\eta d_1 = 1$, $a_{ij} = a_{ji}$ and $(a_{ij})_{i,j=1}^{n-1} > 0$. Suppose that $c_1(z, \eta) < 0$ and assume that a point $\varrho^0 = (z_0; \xi^0, \eta^0 > 0)$ satisfies the following two conditions:

$$p_{2,M}^0(\varrho^0) = 0, \qquad -i\{\bar{p}_{2,M}^0, p_{2,M}^0\}_M(\varrho^0) > 0.\text{(}^1\text{)}$$

Then P is M-microhypoelliptic at ϱ^0 with anisotropic loss of smoothness of solutions equal to $1/2$, i.e. $\varrho^0 \notin \mathrm{WF}_M(Pu) \Rightarrow \varrho^0 \notin \mathrm{WF}_M(u)$ and $Pu \in H^{s,M}_{\mathrm{mcc}}(\varrho^0) \Rightarrow u \in H^{s+3/2,M}_{\mathrm{mcc}}(\varrho^0)$.

EXAMPLE 2. Let

$$P(z, D) = \sum_{j=1}^{n-1} D_{x_j}^2 + (-1 + i\langle a, x\rangle) D_y, \qquad a \in \boldsymbol{R}^{n-1}\setminus 0,$$

$a_j > 0$, $1 \leqslant j \leqslant n-1$. Consider the point $\varrho^0 = (z_0, \zeta^0)$, $\zeta^0 = (\xi^0, \tilde{\eta}^0)$, $|\xi^0|^2 = \tilde{\eta}^0$, $z_0 = (0, y_0)$, $\xi^0 = (\xi_1^0, \ldots, \xi_{n-1}^0)$, $\xi_j^0 > 0$, $1 \leqslant j \leqslant n-1$. The proposition above states that $\varrho^0 \notin \mathrm{WF}_M(Pu) \Rightarrow \varrho^0 \notin \mathrm{WF}_M(u)$. On the other hand, the operator P is neither hypoelliptic nor locally solvable at the point $(0, y_0; 0, \tilde{\eta}^0 > 0)$. In other words, $\varrho^0 = (z_0; 0, \tilde{\eta}^0 > 0) \notin \mathrm{WF}(Pu)$ does not imply that $(\varphi u)\hat{}$, $\varphi \in C_0^\infty$, is rapidly decreasing about the direction $(0, \tilde{\eta}^0 > 0)$. Assuming $\varrho^0 = (z_0; \xi^0, \tilde{\eta}^0 > 0) \notin \mathrm{WF}_M(u)$ we have that $(\varphi u)\hat{}(\zeta)$ is rapidly decreasing in the intersection of two paraboloids with the axis $(0, \tilde{\eta}^0)$ which contain ϱ^0 $(c_0 \eta \leqslant |\xi|^2 \leqslant c_1 \eta;\ \xi_j > 0,\ 1 \leqslant j \leqslant n-1)$.

(¹) $\{f, g\}_M(z, \zeta) = \sum_{j=1}^{n-1} \left(\frac{\partial f}{\partial \xi_j}(z, \zeta) \frac{\partial g}{\partial x_j}(z, \zeta) - \frac{\partial f}{\partial x_j}(z, \zeta) \frac{\partial g}{\partial \xi_j}(z, \zeta) \right).$

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1. The idea of the proof of Theorem 1 is contained in Boutet de Monvel [2] (see also [10]), hence only a short sketch will be given. We will show that in each quasiconic neighbourhood Γ_1: $\eta \geqslant c_0|\xi|^2$, $c_0 > 0$, the operator P can be considered as a quasihomogeneous classical ψ.d.o. belonging to the class $S_{\mathrm{cl}}^{2,M}(\Gamma_1)$, $M = (1, \ldots, 1, 2)$. To simplify the notation we shall suppose that $a(z, \zeta) \in S_{\mathrm{cl}}^{m}$, $m \in \boldsymbol{R}^1$, i.e. $\mathrm{ord}_\zeta a = m$. Applying the Taylor formula we get

$$a(z, \zeta) = \sum_{|\alpha| < N} \frac{\partial^\alpha a}{\partial \xi^\alpha}(z; 0, \eta)\frac{\xi^\alpha}{\alpha!} + \sum_{|\alpha| = N} \frac{\xi^\alpha}{\alpha!}\int_0^1 (1-t)^{N-1}\frac{\partial^\alpha a}{\partial \xi^\alpha}(z; t\xi, \eta)\, dt.$$

Thus the remainder R_N is estimated in the following way:

$$\begin{aligned} |R_N| &\leqslant C|\xi|^N(1+t|\xi|+|\eta|)^{m-N} \\ &\leqslant C' \begin{cases} |\eta|^{N/2}(1+|\eta|^{1/2}+|\eta|)^{m-N}, & m-N \geqslant 0, \\ |\eta|^{N/2}(1+|\eta|)^{m-N}, & m-N < 0, \end{cases} \end{aligned}$$

$$\forall (z, \zeta) \in K \times \Gamma_1, \ K \Subset \Omega.$$

Having in mind that $1+|\eta| \sim 1+|\eta|+|\xi|^2$ in Γ_1 we conclude that

$$|R_N(z, \zeta)| \leqslant \mathrm{const}\,(1+|\eta|+|\xi|^2)^{m-N/2} \approx (1+[\zeta]_M)^{2m-N},$$

and moreover

$$|D_\xi^\alpha D_\eta^\beta R_N(z, \zeta)| \leqslant C_{\alpha\beta}(1+[\zeta]_M)^{2m-N-|\alpha|-2|\beta|}, \quad \forall (z, \zeta) \in K \times \Gamma_1.$$

Obviously, the functions $(\partial^\alpha a/\partial\xi^\alpha)(x, y; 0, \eta)\xi^\alpha$, $|\alpha| < N$, are M-quasihomogeneous of order $2m-|\alpha|$ in Γ_1 and according to the definition of the class $S_{1,0}^{2m,M}(\Gamma_1)$ we obtain $a \in S_{1,0}^{2m,M}(\Gamma_1)$ [12]. The same procedure applied to the symbols $a_{ij}(z, \zeta)$, $p_1(z, \zeta)$, $p_{-k}(z, \zeta)$, $k \geqslant 0$, $k \in \boldsymbol{Z}$, enables us to conclude that $P(z, D)$ is a classical second order M-quasihomogeneous ψ.d.o. in Γ_1 with the following principal symbol:

$$(7) \quad p_{2,M}^0(z, \zeta) = \sum_{i,j=1}^{n-1} a_{ij}(z; 0, \eta)\xi_i\xi_j + c_1(z; 0, \eta) + id_1(z; 0, \eta),$$

$$c_1(z; 0, \eta) < 0.$$

Because of the positivity of the matrix $(a_{ij})_{i,j=1}^{n-1}|_{\mathrm{Char}\, p_2^0}$ the symbol $p_{2,M}^0$ is either quasielliptic or of quasiprincipal type (i.e. $p_{2,M}^0(\varrho^0) = 0$, $\varrho^0 = (z_0, \zeta^0) \in \Gamma_1 \Rightarrow \mathrm{grad}_\xi\, p_{2,M}^0(\varrho^0) \neq 0$). An interesting result due to Segala, Theorem 5.5 from [12], and the main assumption of Theorem 1 show that P is M-microhypoelliptic in Γ_1 and

$$p_{2,M}^0(\varrho^0) \neq 0, \ Pu \in H_{\mathrm{mce}}^{s,M}(\varrho^0) \quad \Rightarrow \quad u \in H_{\mathrm{mce}}^{s+2,M}(\varrho^0),$$

while

$$p^0_{2,M}(\varrho^0) = 0, \;\; Pu \in H^{s,M}_{\mathrm{mcc}}(\varrho^0) \;\; \Rightarrow \;\; u \in H^{s+1,M}_{\mathrm{mcc}}(\varrho^0).$$

2. Consider now the operator P with symbol (4) in the paraboloid $\Gamma_2 = \{\zeta = (\xi, \eta)\colon |\xi|^2 \geqslant c\eta \geqslant 0, c \gg 1\}$. It is evident that

$$|p| \geqslant \mathrm{const}\,|\xi|^2 - \mathrm{const}\,(|\xi| + |\eta|) \geqslant c'\,|\xi|^2 - \mathrm{const}\,|\eta|,$$

as $|\xi| \geqslant 1$. So $|p| \geqslant \mathrm{const}\,(|\xi| + |\eta|^{1/2})^2$ and consequently $|p| \sim [\zeta]^2_M$ in Γ_2.

The inequality $|\xi| \leqslant [\zeta]_M$, $\forall \zeta \in \Gamma_2$, implies that

$$|D^{\alpha_1}_\xi D^{\alpha_2}_\eta p| \leqslant C_\alpha (|\xi| + |\eta|^{1/2})^{-|\alpha_1| - \alpha_2} |p|, \quad \alpha = (\alpha_1, \alpha_2).$$

In fact,

$$|D^{\alpha_1}_\xi p| \leqslant c\,|\xi| \leqslant c\,|p|\,[\zeta]^{-1}_M, \quad |\alpha| = 1, \text{ and}$$

$$|D^{\alpha_2}_\eta p| \leqslant C \frac{|\xi|^2}{|\xi| + |\eta|} \leqslant C\,|\xi|, \quad \alpha_2 = 1.$$

Suppose now that $|\alpha| \geqslant 2$, $|\alpha| = |\beta| + 2$. We have

$$|D^\alpha_\zeta p| = |D^\beta_\zeta (D^2_\zeta p)| \leqslant C_\alpha (|\xi| + |\eta|)^{-|\beta|}$$
$$\leqslant C'_\alpha [\zeta]^{-|\beta| - 2}_M |p| \approx [\zeta]^{-|\alpha|}_M |p| \quad (|\xi| \sim [\zeta]_M \text{ in } \Gamma_2).$$

To estimate in Γ_2 the derivatives of $D^\alpha_\zeta(1/p)$ we use the identities $p \cdot p^{-1} \equiv 1$ and $D^\alpha_\zeta(p \cdot p^{-1}) \equiv 0$ when $|\alpha| \geqslant 1$. We get inductively $1/p \in S^{-2,M}_{1/2,0}(\Omega \times \Gamma_2)$ and therefore the operator P turns out to be M-microhypoelliptic in Γ_2, i.e.

$$\varrho^0 = (z_0, \zeta^0) \in \Omega \times \Gamma_2, \;\; \varrho^0 \notin \mathrm{WF}_M(Pu) \;\; \Rightarrow \;\; \varrho^0 \notin \mathrm{WF}_M(u).$$

Moreover,

$$Pu \in H^{s,M}_{\mathrm{mce}}(\varrho^0) \;\; \Rightarrow \;\; u \in H^{s+2,M}_{\mathrm{mce}}(\varrho^0), \quad \forall s \in \mathbf{R}^1.$$

To complete the proof the following result will be useful.

Proposition 1. *Assume that* $\mathrm{WF}_M(u) \subset \{(x, y, \zeta)\colon \zeta = (\xi, \eta) \in \mathbf{R}^n, \eta > 0\}$. *Then* $\mathrm{WF}(u) = \{(x, y; 0, \eta)\colon \exists \xi \in \mathbf{R}^{n-1}\colon (x, y; \xi, \eta) \in \mathrm{WF}_M(u)\}$ [2].

Suppose now that the operator P with symbol (4) satisfies the condition $Pu \in C^\infty$, $u \in \mathscr{D}'$. Combining the considerations above and Proposition 1 we come to the conclusion that $\varrho^0 = (z_0; 0, \bar{\eta}^0 > 0) \notin \mathrm{WF}(u)$. Standard arguments from the theory of ψ.d.o. show that $\mathrm{WF}(Pu) = \mathrm{WF}(u)$. The desired microlocal smoothness of solutions of the equation $Pu = f$ is a simple consequence of Theorem 4.3 from [10]. We omit the details.

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The proof of Theorem 1′ is based on some results obtained in [2] and on the main theorem of Moyer–Hörmander concerning the microlocal nonsolvability of principal type ψ.d.o. [8].

We shall denote by $\hat{\hat{f}}$ the partial Fourier transform with respect to y of the tempered distribution $f(x, y) \in \mathcal{S}'(\mathbf{R}^{n-1} \times \mathbf{R}^1_y)$ and by $\mathcal{S}'_+$, $\mathcal{S}'_{++}$ the following two spaces of distributions:

$$\mathcal{S}'_+ = \{f \in \mathcal{S}': \operatorname{supp} \hat{f}(\xi, \eta) \subset \{\zeta = (\xi, \eta): \eta \geqslant \varepsilon(|\xi|+1), \varepsilon = \text{const} > 0\},$$

$$\mathcal{S}'_{++} = \{f \in \mathcal{S}': \operatorname{supp} \hat{f}(\xi, \eta) \subset \{\zeta: \eta \geqslant \varepsilon(|\xi|^2+1), \varepsilon > 0\}\}.$$

Boutet de Monvel introduced in [2] a special integral operator U giving a link between the classical and M-homogeneous wave front sets of distributions. This is the definition of U:

$$T = Uf = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iy\eta^2} \hat{\hat{f}}(x, \eta)\, d\eta. \tag{8}$$

The properties of U we are interested in are formulated below.

(a) $\|Uf\|_{M,t} \sim \|f\|_{t-1/2}$, $\forall f \in \mathcal{S}'_+$, where $\|f\|_t$ is the usual Sobolev norm and $\|f\|^2_{M,t} = \int\int (1+|\xi|+|\eta|^{1/2})^{2t} |f(\xi, \eta)|^2 \, d\xi\, d\eta$.

(b) U is an isomorphism from $\mathcal{S}'_+$ onto $\mathcal{S}'_{++}$; $f \in \mathcal{S}'_+ \cap \mathcal{S} \Leftrightarrow Uf \in \mathcal{S}$;

$$Ux_j = x_j U, \quad U\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j} U, \quad i\frac{\partial}{\partial y} U = U\left(\frac{\partial^2}{\partial y^2}\right),$$
$$z = (x, y), \qquad 1 \leqslant j \leqslant n-1.$$

(c) Suppose that $P'(z, D)$ is an M-quasihomogeneous ψ.d.o. with principal symbol p'^0, $M = (1, \ldots, 1, 2)$ and $f \in \mathcal{S}'_+$. Then one can find a classical homogeneous ψ.d.o. $P(z, D)$ with principal symbol p^0 such that $p^0(x, y; \xi, \eta) = p'^0(x, 0; \xi, \eta^2)$, $\forall \eta > 0$, and

$$P' Uf = UPf, \tag{9}$$

(d) Assume that a distribution $f \in \mathcal{S}'_+$ coincides outside some compact set $K \Subset \mathbf{R}^n$ with a function $g \in \mathcal{S}$. Then $\mathrm{WF}_M(Uf)$ is the image of $\mathrm{WF}(f)$ under the mapping χ, χ: $(x, y; \xi, \eta) \to (x, 0; \xi, \eta^2)$.

Let $\varrho^0 = (0, 0; 0, 1)$, $\zeta^0 = (0, 1)$. The principal symbol $p^0_{2,M}$ of the operator P in the paraboloid Γ_1 is given by (7). Therefore the zero quasibicharacteristic γ of $\operatorname{Re} p^0_{2,M}$ passing through the point $(0, 0; \eta^0, 1)$, $\sum_{i,j=1}^{n-1} a_{ij}(\varrho^0)\eta^0_i \eta^0_j + c_1(\varrho^0) = 0$, $\eta^0 = (\eta^0_1, \ldots, \eta^0_{n-1})$, has the equations

$$\gamma: \begin{cases} \dot{x}_j = 2\sum_{i=1}^{n-1} a_{ij}(x, 0; \zeta^0)\xi_i, \quad 1 \leqslant j \leqslant n-1, \\ \dot{\xi}_j = -\sum_{i,k=1}^{n-1} \dfrac{\partial a_{ik}}{\partial x_j}(x, 0; \zeta^0)\xi_i \xi_k - \dfrac{\partial c_1}{\partial x_j}(x, 0; \zeta^0), \\ x(0) = 0, \ \xi(0) = \eta^0; \quad y \equiv 0, \ \eta \equiv 1. \end{cases}$$

We can suppose that $d_1|_\gamma = d_1(x(t), 0; \zeta^0)$ changes its sign from + to − at the point ϱ^0.

To apply (9) the classical homogeneous principal type ψ.d.o. Q with principal symbol

$$q_2^0(x, \xi, \eta) = \sum_{i,j=1}^{n-1} a_{ij}(x, 0; 0, 1)\xi_i \xi_j + \eta^2 (c_1(x, 0; 0, 1) + id_1(x, 0; 0, 1))$$

will be investigated.

The zero bicharacteristic of $\operatorname{Re} q_2^0(x, \zeta)$ $(x = x(t), \xi = \xi(t))$ passing through the characteristic point $\lambda^0 = (0, 0; \eta^0, 1) \in \operatorname{Char} q_2^0$ has the same equations as γ. A theorem of Godin from [5] and the cited result of Moyer–Hörmander enable us to conclude that there exists a distribution $f \in \mathscr{E}'$ such that $Qf \in C_0^\infty$, $\operatorname{WF}(f) = \{t\lambda^0 \colon t > 0\}$. Repeating the same arguments as in [2], p. 13–14, we get Theorem 1'.

4

1. When proving Theorem 2 the following necessary condition for microlocal solvability of linear ψ.d.o. will be used.

Proposition 2 [8]. *Assume that T is a compact cone in $T^*(X)\backslash 0$ with base $\omega_T \Subset X$ and a ψ.d.o. $P(z, D)$ is microlocally solvable in T. Suppose that $\omega \subset X$ is an open set with $\omega \Supset \omega_T$, and $N \in \mathbf{Z}$ is the number occurring in Definition 2. Then one can find an integer ν, a ψ.d.o. A of order ν with full symbol identically vanishing in some conic neighbourhood of T and a constant $C > 0$ such that the following inequality holds:*

$$(9) \qquad \|u\|_{-N} \leqslant C(\|P^* u\|_\nu + \|u\|_{-N-n} + \|Au\|_0), \qquad u \in C_0^\infty(\omega)$$

(P^* stands for the L_2 adjoint operator of P).

Thus Theorem 2 will be established at the point $\varrho^0 = (z_0, \zeta^0)$ $(\in T)$ if the estimate (9) is violated for arbitrary N, ν, $\omega \ni z_0$ and A, where $\operatorname{ord} A = \nu$ and $\sigma(A) \equiv 0$ about ϱ^0.

The proof of Theorem 2 can be reduced to showing that the inequality

$$(10) \qquad \|v\|_0 \leqslant \tilde{C}(\|Qv\|_0 + \|Q_1 v\|_0), \qquad \forall v \in C_0^\infty(\omega'),$$

breaks down, where $\varrho^0 = (x_0, y_0; 0, \eta^0 > 0)$, $z_0 = (x_0, y_0) \in \omega'$, Q is a classical ψ.d.o. of order $\nu + N + 2$ with principal symbol

$$q^0_{\nu+N+2} = f^2(z)|\zeta|^{\nu+N} \sum_{i,j=1}^{n-1} a_{ij}(z, \zeta)\xi_i \xi_j, \qquad f \equiv 1 \text{ about } \varrho^0,\ f \in C_0^\infty(\omega'),$$

$$q'_{\nu+N+1}|_{\operatorname{Char} p_2^0} = |\zeta|^{\nu+N} \bar{p}_1|_{\operatorname{Char} p_2^0}, \qquad \operatorname{ord}_\zeta Q_1 = \nu + N$$

and the full symbol $\sigma(Q_1) \equiv 0$ in a cone Γ about ϱ^0.

For the sake of simplicity we assume that $z_0 = 0$. Let us fix ω', $\tilde{C} > 0$

and the operator Q_1 from (10). The estimate (10) fails to hold for a function $v(z)$ of the following type:

$$\varphi(z)\, e^{i\lambda^2 y\tilde{\eta}^0 + i\lambda h(z)},$$

$\lambda > 0$ a parameter, $\varphi(z) \in C_0^\infty(\omega')$, $h(z) \in C^\infty(\omega')$ and $\operatorname{Im} h(z) \sim c_0 |z|^2$, $z \to 0$, $c_0 > 0$. It can easily be shown that for each classical symbol $p_m(z, \zeta)$, $\operatorname{ord}_\zeta p_m = m$, and for each pair of integers M, M' there exists a number $N_1 \in \boldsymbol{Z}_+ \setminus 0$ such that all the derivatives of order $\leqslant M$ of the function

$$\lambda^{M'} \{p_m(z, D)\big(\varphi(z)\, e^{i\lambda^2 y\tilde{\eta}^0 + i\lambda h(z)}\big) - e^{i\lambda^2 y\tilde{\eta}^0} \sum_{|\alpha| \leqslant N_1} p_m^{(\alpha)}(z, \lambda^2 \zeta^0)\, D_z^\alpha\big(\varphi(z)\, e^{i\lambda h(z)}\big)\}$$

are bounded in the space of the variable z ($\forall \lambda \geqslant 1$; $p_m^{(\alpha)}(z, \zeta) = \partial^\alpha p_m / \partial \zeta^\alpha$, $\zeta^0 = (0, \tilde{\eta}^0)$).

Obviously $\sigma(Q_1)|_\Gamma = 0 \Rightarrow Q_1\big(\varphi(z)\, e^{i\lambda^2 y\tilde{\eta}^0 + i\lambda h(z)}\big) = O(\lambda^{-\infty})$. We shall construct the phase function $h(z)$ and the amplitudes $\varphi_j(z) \in C_0^\infty(\omega')$ in such a way that

$$Q(z, D)\Big(\sum_{j=0}^{L} \varphi_j(z)\, \lambda^{-j} e^{i\lambda^2 y\tilde{\eta}^0 + i\lambda h(z)}\Big) = O(\lambda^{-R}),$$

$\lambda \to \infty$ and $R \gg 1$; $\varphi_0(0) = 1$, $\varphi_j(0) = 0$, $j \geqslant 1$.

To find $h(z)$ the following nonlinear Cauchy problem will be considered:

$$\sum_{i,j=1}^{n-1} a_{ij}(z, \zeta^0) \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} + \bar{p}_1(z, \zeta^0) = 0, \tag{11}$$

$$\operatorname{grad}_x h(0) = \eta^0 \in \boldsymbol{R}^{n-1} \setminus 0$$

and η^0 is defined in Theorem 2. As we are looking for an approximate solution modulo $O(|z|^\sigma)$, $z \to 0$, $\sigma \gg 1$, of (11) we can assume that the coefficients a_{ij}, $1 \leqslant i, j \leqslant n-1$ and $p_1(z, \zeta^0)$ are analytic functions in ω'.

Without loss of generality we can suppose $\operatorname{grad}_x h(0) = (\eta_1^0, 0, \ldots, 0) = \eta^0$ since the assumption of Theorem 2 is invariant under rotations in $\boldsymbol{R}_x^{n-1}$. Therefore

$$\begin{aligned} 0 \neq & \Big\{ \sum_{j,k=1}^{n-1} a_{jk}(z, \zeta^0)\, \eta_j \eta_k + c_1(z, \zeta^0),\, d_1(z, \zeta^0) \Big\}_M (\varrho^0, \eta^0) \\ & = 2 \sum_{k=1}^{n-1} a_{1k}(\varrho^0) \frac{\partial d_1}{\partial x_k}(\varrho^0)\, \eta_1^0. \end{aligned} \tag{12}$$

According to the Cauchy–Kovalevska theorem there exists a unique analytic solution of the Cauchy problem for (11) with initial data

$$h|_{x_1 = 0} = \sum_{j,l=1}^{n} \alpha_{jl} z_j z_l |_{x_1 = 0}$$

in a neighbourhood of the origin.

Lemma 1.6.4 from [6] and the inequality (12) imply the existence of a symmetric matrix (α_{jl}), $\mathrm{Im}(\alpha_{jl}) > 0$, such that

$$h(z) = \frac{\partial h(0)}{\partial z_1} z_1 + \sum_{j,l=1}^{n} \alpha_{jl} z_j z_l + O(|z|^3), \qquad z \to 0.$$

The observations that $z_j = x_j$, $1 \leqslant j \leqslant n-1$, and $\|v\|_0^2 = \lambda^{-n/2}[c|\varphi_0(0)|^2 + o(1)]$, $c = \text{const} > 0$, complete the proof of Theorem 2.

The local nonsolvability of the operator $P(z, D)$ ((6), case II.2)) is shown in [11].

2. The proof of the Proposition is based on the well-known identity

$$\|Pu\|_0^2 = \|P^* u\|_0^2 + (C_3(z, D)u, u), \qquad \forall u \in C_0^\infty(K), \tag{13}$$

where $C_3(z, D) = [P^*, P]\ (z, D)$.

The principal symbol $c_{3,M}^0(z, \zeta)$ of the M-quasihomogeneous operator $C_3(z, D)$ is equal to

$$c_{3,M}^0(z, \zeta) = -i\{\bar{p}_{2,M}^0, p_{2,M}^0\}_M(z, \zeta) = 2\eta \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} a_{jk}(z, 1)\xi_k \frac{\partial d_1}{\partial x_j}(z, 1).$$

$c_{3,M}^0(z, \zeta)$ is M-quasihomogeneous of order 3, of course, and $p_{2,M}^0(\varrho^0) = 0$, $\tilde{\eta}^0 > 0 \Rightarrow \xi^0 \neq 0$. Assume that $c_{3,M}^0 > 0$ in an M-quasiconic neighbourhood Γ of the point ϱ^0 and denote by $\psi(z, \zeta)$ an M-quasihomogeneous function of order 0 with respect to ζ having the properties: $\psi|_{\Gamma_1} \equiv 1$, $\Gamma_1 \ni \varrho^0$, $\Gamma_1 \Subset \Gamma$, $\psi \in C_0^\infty(\Gamma)$. Putting $\psi(z, D)u \in C_0^\infty(K)$ in (13) we get

$$\|P(\psi u)\|_0^2 \geqslant \mathrm{Re}\big(C_3(z, D)(\psi(z, D)u), \psi(z, D)u\big), \qquad \forall u \in C_0^\infty(K).$$

Let $\hat{c}_{3,M}^0 > 0$ be a 3rd order M-quasihomogeneous symbol such that $\hat{c}_{3,M}^0 = c_{3,M}^0$ in a quasiconic neighbourhood of $\operatorname{supp}\psi$. Then $\big(c_{3,M}^0(z, D) - \hat{c}_{3,M}^0(z, D)\big)\circ\psi$ is an infinitely smoothing operator. Gårding's inequality [9], [12] asserts that

$$\|P(\psi u)\|_0^2 \geqslant c_0 \|\psi u\|_{3/2,M}^2 + c_1 \|\psi u\|_{1,M}^2, \qquad \forall u \in C_0^\infty(K), \tag{14}$$

$c_0 = \text{const} > 0$, $c_1 = \text{const}$.

The desired M-microlocal hypoellipticity can easily be deduced from (14). In fact, consider the ψ.d.o. $q_\varepsilon(z, D)$ with symbol

$$q_\varepsilon(z, \zeta) = \varphi(z)(1 + [\zeta]_M^2)^{s/2}(1 + \varepsilon^2 [\zeta]_M^2)^{-1/2},$$

$$\varphi \in C_0^\infty(\mathbf{R}^n), \quad \varphi \equiv 1 \text{ about the point } z_0.$$

Certainly, $\mathrm{ord}_{\zeta,M}\, q_\varepsilon = s-1$. A simple calculation shows that $q_\varepsilon \in S_{1,0}^{s-1,M}$, and moreover $\{q_\varepsilon(z, \zeta)\}_{0<\varepsilon\leqslant 1}$ is a bounded subset of $S_{1,0}^{s,M}$. On the other hand, the estimate (14) remains true for each $u \in H_{\mathrm{mce}}^{2,M}(\varrho^0)$. To verify the last statement, (14) is rewritten as follows:

$$\|\psi Pu\|_0^2 + \|\psi_1 u\|_{1,M}^2 \geqslant c_0 \|\psi u\|_{3/2,M}^2 + c_1 \|\psi u\|_{1,M}^2, \tag{15}$$

$\forall u \in C_0^\infty(K)$, where $\psi_1(z, D)$ is a 0th order M-quasihomogeneous ψ.d.o. with principal symbol $(1+[\zeta]_M^2)^{-1/2}\{\psi, p_{2,M}^0\}_M$, $\operatorname{supp}\psi_{1,M}^0 \Subset \Gamma$. Then $u \in H_{\mathrm{mce}}^{2,M}(\varrho^0) \Rightarrow Pu \in H_{\mathrm{mce}}^{0,M}(\varrho^0) \Rightarrow \psi Pu \in H^0(\boldsymbol{R}^n)$, etc.

Assume now $u \in H_{\mathrm{mce}}^{s+1,M}(\varrho^0)$ and $Pu \in H_{\mathrm{mce}}^{s,M}(\varrho^0)$. Then we can find a small enough conic neighbourhood $\Gamma_2 \ni \varrho^0$, $\Gamma_2 \Subset \Gamma_1$, for which $q_\varepsilon(z, D)\psi_2(z, D)u \in H_{\mathrm{mce}}^{2,M}(\varrho^0)$, $\operatorname{supp}\psi_2 \Subset \Gamma_2$, $\psi_2 \equiv 1$ about ϱ^0. Thus

$$\|\psi P(q_\varepsilon \psi_2 u)\|_0^2 + \|\psi_1 q_\varepsilon \psi_2 u\|_{1,M}^2 \geqslant c_0 \|\psi q_\varepsilon \psi_2 u\|_{3/2,M}^2 + c_1 \|\psi q_\varepsilon \psi_2 u\|_{1,M}^2.$$

Because of the boundedness of q_ε in $S_{1,0}^{s,M}$ we have $\|\psi_1 q_\varepsilon \psi_2 u\|_{1,M} \leqslant \text{const}$, the constant not depending on ε. The identity

$$\psi P(q_\varepsilon \psi_2) = \psi q_\varepsilon \psi_2 P + \psi [P, q_\varepsilon \psi_2]$$

implies that $\|\psi [P, q_\varepsilon \psi_2] u\|_0 \leqslant \text{const}$ since the commutator $[P, q_\varepsilon \psi_2]$ is concentrated in Γ_2 and the set $\{[P, q_\varepsilon \psi_2]\}_{0<\varepsilon\leqslant 1}$ is bounded in $S_{1,0}^{s+1,M}$; $\|\psi q_\varepsilon \psi_2 Pu\|_0 \leqslant \text{const}$. Thus $\|q_\varepsilon \psi \psi_2 u\|_{3/2,M} \leqslant \text{const}$, $\forall \varepsilon \in (0, 1] \Rightarrow \|q_0 \psi \psi_2 u\|_{3/2,M} \leqslant \text{const} \Rightarrow u \in H_{\mathrm{mce}}^{s+3/2,M}(\varrho^0)$. The desired M-microhypoellipticity is shown.

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