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ON THE MEAN VALUE OF THE REMAINDER TERM OF THE PRIME NUMBER FORMULA

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In the present work we shall investigate the function

$$\Delta(x) := \psi(x) - x := \sum_{p^m \leqslant x} \log p - x. \tag{1.1}$$

This can be expressed by the non-trivial zeros ϱ of the Riemann zeta function as follows:

$$\Delta(x) = -\sum_{|\gamma| \leqslant x} \frac{x^\varrho}{\varrho} + O(\log^2 x) \tag{1.2}$$

where we shall always write $\varrho = \beta + i\gamma$. Phragmén already proved in the 19th century that

$$\Delta(x) = \Omega(x^{\beta_0 - \varepsilon}) \tag{1.3}$$

if $\zeta(\varrho_0) = 0$, but this result was completely ineffective. The problem of finding explicit Ω-type theorems was formulated by Littlewood in 1937 [6]. Somewhat more generally one can raise the following problems. Let us suppose $\zeta(\varrho_0) = 0$ $(\beta_0 \geqslant 1/2, \gamma_0 > 0)$ and let $Y > c(\varrho_0)$, where $c(\varrho_0)$ is an effective constant depending on ϱ_0. The question is for which functions $f_i(x, \varrho_0) \gg x^{\beta_0 - \varepsilon}$ and $A(Y)$ we can assert:

Problem 1. $|\Delta(x)| \geqslant f_1(x, \varrho_0)$ for some $x \in [Y, A(Y)]$;

Problem 2. $\max_{x \leqslant Y} |\Delta(x)| \geqslant f_2(Y, \varrho_0)$;

Problem 3. $D(Y) := \frac{1}{Y}\int_1^Y |\Delta(x)|\, dx \geqslant f_3(Y, \varrho_0)$.

Problems 1 and 2 were solved in 1950 by Turán [8], who made use of his power sum method. He showed (in a slightly modified formulation)

$$(1.4) \qquad \max_{x \leqslant Y} |\Delta(x)| \geqslant Y^{\beta_0} \exp\left(-c_1 \frac{\log Y}{\log \log Y} \log \log \log Y \right)$$

for $Y > c_2 \exp(\exp(|\varrho_0|))$, where $c_\nu > 0$ always denotes an explicitly calculable constant. His lower bound was proved by S. Knapowski for $D(Y)$ too. (The result is implicitly contained in [4].)

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The present author has succeeded in solving Problem 1 with the function

$$(2.1) \qquad f_1(x, \varrho_0, \varepsilon) = (1-\varepsilon) \frac{x^{\beta_0}}{|\varrho_0|}$$

[7], which gives the expected oscillation "caused by a particular zero ϱ_0". It has even been shown that for $Y > c(\varrho_0, \varepsilon)$, $I = [Y, Y^{4 \cdot 10^4 \varepsilon^{-2} \log \gamma_0}]$ and $|\varrho_0| > 400\varepsilon^{-2}$

$$(2.2) \qquad \min_{x \in I} \frac{\Delta(x)}{f_1(x, \varrho_0, \varepsilon)} \leqslant -1 < 1 \leqslant \max_{x \in I} \frac{\Delta(x)}{f_1(x, \varrho_0, \varepsilon)}.$$

If we now investigate only the problem of lower estimation of $|\Delta(x)|$, a slight improvement and a more elegant formulation of the above result are given by

Theorem 1. *For* $Y > c_3 \gamma_0^{40}$ *there exists an*

$$(2.3) \qquad x \in [Y, Y^{6 \log \gamma_0 + 60}] := I^*$$

such that

$$(2.4) \qquad |\Delta(x)| > \frac{x^{\beta_0}}{|\varrho_0 + 4|}.$$

It is a slight imperfection of the above result that the lower estimation $(1-\varepsilon) x^{\beta_0}/|\varrho_0|$ is reached only for $|\varrho_0| > c\varepsilon^{-1/2}$, an assumption of type $Y > c(\varrho_0, \varepsilon)$ being insufficient for this purpose. But with a slight additional effort we can show also

Theorem 1′. *For* $Y > \max\left(c_4 (\gamma_0/\varepsilon)^{14}, \exp\left((c_5/\varepsilon\gamma_0)^2\right)\right)$ *there exists an* $x \in I^*$ (see (2.3)) *such that*

$$(2.5) \qquad |\Delta(x)| > (1-\varepsilon) \frac{x^{\beta_0}}{|\varrho_0|}.$$

We are not able to prove as good estimations for Problems 2 and 3. But a quite satisfactory lower bound is furnished by the following theorem, which even gives a good localization for large values of $\Delta(x)$.

THEOREM 2. *If ϱ_0 is a zeta-zero with multiplicity ν, then for $Y > e^{|\gamma_0|+4}$, $\mathscr{I} = [Y/(100 \log Y), Y]$ we have*

$$\max_{x\in\mathscr{I}} |\Delta(x)| \geqslant \frac{1}{Y}\int\limits_{x\in\mathscr{I}} |\Delta(x)|\,dx > \frac{Y^{\beta_0}|\zeta^{(\nu)}(\varrho_0)|}{6(\nu-1)!\,|\varrho_0|^3} - c_6. \tag{2.6}$$

In particular,

$$\max_{x\leqslant Y} |\Delta(x)| \geqslant D(Y) > \frac{Y^{\beta_0}|\zeta^{(\nu)}(\varrho_0)|}{6(\nu-1)!\,|\varrho_0|^3} - c_6. \tag{2.7}$$

If we take $\varrho_0 = 1/2 + i\cdot 14.13\ldots$ (and consider the value of c_6), this implies

COROLLARY 1. *For $Y > 2$ we have*

$$\max_{x\leqslant Y} |\Delta(x)| \geqslant D(Y) > \sqrt{Y}/22\,000. \tag{2.8}$$

Any improvement of this inequality by a factor greater than 22000 should already imply the falsity of the Riemann hypothesis, since Cramér [3] proved in 1922 that on the Riemann hypothesis

$$D(Y) < \sqrt{Y} \quad (Y > c_7). \tag{2.9}$$

If the Riemann hypothesis is true, then (1.2) gives trivially

$$\Delta(x) = O(\sqrt{x}\log^2 x) \tag{2.10}$$

but for estimates from below we know only that

$$\Delta(x) = \Omega(\sqrt{x}\log\log\log x), \tag{2.11}$$

which was proved by Littlewood [5] in 1914. According to a conjecture of Montgomery

$$\overline{\lim_{x\to\infty}} \frac{|\Delta(x)|}{\sqrt{x}(\log\log\log x)^2} = \pm\frac{1}{2\pi}, \tag{2.12}$$

which would fill the gap between (2.10) and (2.11).

Now for the average value of $|\Delta(x)|$ we know the precise order of magnitude, if we assume the Riemann hypothesis.

COROLLARY 2. *For $Y > c_7$ we have on the Riemann hypothesis*

$$\sqrt{Y}/22000 < D(Y) < \sqrt{Y}. \tag{2.13}$$

Let

$$\theta = \sup_{\zeta(\varrho)=0} \operatorname{Re} \varrho \tag{2.14}$$

and with the terminology of Ingham we shall say that θ is attained if there is a zeta zero ϱ_0 on the line $\sigma = \theta$, i.e.,

$$\varrho_0 = \theta + i\gamma_0. \tag{2.15}$$

COROLLARY 3. *If θ is attained, then*

$$c_1(\varrho_0) Y^\theta < D(Y) < c_2(\theta) Y^\theta. \tag{2.16}$$

The lower bound is naturally a special case of Theorem 1. The upper bound follows in the case of $\theta = 1/2$ from the theorem of Cramér (see (2.9)), while for $\theta > 1/2$ we have by density theorems even

$$|\Delta(x)| < c_2(\theta) x^\theta \tag{2.17}$$

since

$$\sum_{\beta \geqslant (\theta+1/2)/2} \frac{1}{|\varrho|} < c(\theta). \tag{2.18}$$

Since the usual way of obtaining Ω-type theorems or lower estimations for $D(Y)$ is through some weighted mean value estimates of $\Delta(x)$, one cannot expect lower estimations, which should hold for a positive proportion of all x's. Therefore it is surprising that, without assuming anything on the linear independence or dependence of the imaginary parts of the zeta-zeros, only with a relatively natural assumption, one can show the following

COROLLARY 4. *If θ is attained, then*

$$\frac{1}{Y} \left|\{x \leqslant Y; |\Delta(x)| > c_3(\varrho_0) Y^{\beta_0}\}\right| > c_4(\varrho_0). \tag{2.19}$$

If $\theta > 1/2$, this is a trivial consequence of (2.16) and (2.17); if $\theta = 1/2$, then besides Corollary 1 we need the result of Cramér in the original form

$$\frac{1}{Y} \int_1^Y \Delta^2(x)\, dx = O(Y) \tag{2.20}$$

(of which (2.9) is only a consequence).

Further it is interesting to note that by (2.16) and (2.17) one can formulate

COROLLARY 5. *If θ is attained and $\theta > 1/2$, then*

$$\max_{x \leqslant Y} |\Delta(x)| \leqslant c_5(\varrho_0) D(Y). \tag{2.21}$$

Finally we note that it is possible to prove a relatively good lower bound for $D(Y)$ without any factor of $|\zeta^{(\nu)}(\varrho_0)|$ type. Namely, for $Y > e^{|\varrho_0|}$ the present author has shown

$$D(Y) \geqslant \frac{1}{Y} \int_{Y\exp(-c_8 \log_2^2 Y)}^{Y} |\Delta(x)|\, dx > Y^{\beta_0} \exp(-c_9 \log_2^2 Y) \tag{2.22}$$

where $\log_2 Y = \log\log Y$. This result has important applications. Making use of (2.22) the author has shown that $\pi(x) - \mathrm{li}\, x$ changes sign in every interval of the form

$$[Y \exp(-c_{10} \log_2^3 Y),\ Y] \quad \text{for} \quad Y > Y_0, \tag{2.23}$$

where Y_0 is an ineffective constant. Further, the author has proved with the aid of (2.22) that the number of sign changes of $\pi(x) - \mathrm{li}\, x$ in the interval $[2, Y]$ is

$$V_1(Y) > c_{11} \frac{\log Y}{\log_2^3 Y} \quad \text{for} \quad Y > c_{12}. \tag{2.24}$$

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We sketch the proof of Theorem 1. A crucial role is played by the continuous form of the power sum theorem of Cassels [2], according to which for arbitrary complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $d > 0$

$$\max_{d \leqslant t \leqslant (2n-1)d} \frac{\left|\sum_{i=1}^{n} e^{\alpha_i t}\right|}{|e^{\alpha_1 t}|} \geqslant 1. \tag{3.1}$$

Let

$$a \in \left[\frac{\log Y}{10}, \frac{6(\log \gamma_0 + 10)\log Y}{22}\right], \tag{3.2}$$

$$A := \max_{e^{10a} \leqslant x \leqslant e^{22a}} \frac{|\Delta(x)|}{\left(\dfrac{x^{\beta_0}}{|\varrho_0 + 3|}\right)}, \tag{3.3}$$

$$H(s) := \frac{\zeta'}{\zeta}(s) + \frac{s}{s-1} = \int_1^\infty \Delta(x) \frac{d}{dx}(x^{-s})\, dx. \tag{3.4}$$

It is easy to show that

$$(3.5)\qquad U := \frac{1}{2\pi i}\int\limits_{(3)} H(s+\varrho_0)\, e^{as^2+15as}\, ds$$

$$= \frac{1}{2\sqrt{\pi a}}\int\limits_1^\infty \frac{\Delta(x)}{x^{1+\varrho_0}}\left(-\varrho_0+\frac{15a-\log x}{2a}\right)\exp\left(-\frac{(15a-\log x)^2}{4a}\right)dx.$$

Using (3.3) and $\Delta(x) = O(x)$ one can obtain from the right-hand side of (3.5)

$$(3.6)\qquad |U| \leqslant A + O(e^{-(5/4)a}|\varrho_0|).$$

On the other hand moving the path of integration on the left-hand side of (3.5) onto the line $\sigma = -2$ and using Jensen's inequality, one can show that

$$(3.7)\qquad U = \sum_{|\gamma-\gamma_0|\leqslant 3} e^{\{(\varrho-\varrho_0)^2+15(\varrho-\varrho_0)\}a} + O\left(e^{-(5/4)a}\log|\varrho_0|\right).$$

Since a result of Backlund [1] implies

$$(3.8)\qquad \sum_{|\gamma-\gamma_0|\leqslant 3} 1 < \tfrac{15}{11}(\log\gamma_0+10),$$

estimating the power sum in (3.7) by Cassels' theorem, one can derive (2.3) and (2.4) by easy calculation.

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We shall sketch the proof of the following weakened form of Theorem 2:

THEOREM 2′. *If ϱ_0 is a simple zeta-zero, then*

$$(4.1)\qquad \int\limits_1^Y |\Delta(x)|\,dx > \frac{c_{13}|\zeta'(\varrho_0)|}{|\varrho_0|^4}\, Y^{1+\beta_0} - c_{14}Y^{5/4}.$$

Let

$$(4.2)\qquad \lambda = \log Y,$$

$$(4.3)\qquad G(s) := -(s-2)\zeta'(s-1)-(s-1)\zeta(s-1),$$

$$(4.4)\qquad H(s) := \frac{G(s)}{(s-1)(s-2)\zeta(s-1)} = \int\limits_1^\infty \frac{\Delta(x)}{x^s}\, ds,$$

$$h(s) := \frac{(s-2)\zeta(s-1)}{(s-1-\varrho_0)(s+1)^4}, \tag{4.5}$$

$$w(u) = \frac{1}{2\pi i} \int_{(3)} e^{us} h(s)\, ds. \tag{4.6}$$

Our starting formula in this case is

$$U^* := \frac{1}{2\pi i} \int_{(3)} h(s) H(s) e^{\lambda s} ds = \int_1^\infty \Delta(x) w(\lambda - \log x)\, dx. \tag{4.7}$$

From the left-hand side of (4.7) we obtain

$$U^* = -\zeta'(\varrho_0)\left(1 - \frac{1}{\varrho_0}\right)(\varrho_0 + 2)^{-4} e^{\lambda(1+\varrho_0)} + O(e^{5\lambda/4}). \tag{4.8}$$

On the other hand, one can easily show

$$|w(u)| < c_0 \quad \text{for} \quad u \geqslant 0, \tag{4.9}$$

$$w(u) = 0 \quad \text{for} \quad u \leqslant 0. \tag{4.10}$$

Now (4.7)–(4.10) give Theorem 2′.

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