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GENERALIZED SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR ORDINARY LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER IN THE COLOMBEAU ALGEBRA

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Abstract. It is shown that from the fact that a homogeneous problem has a unique trivial solution it follows that a non-homogeneous problem has a solution in the Colombeau algebra.

1. Introduction. We consider the following problem

$$x''(t) + p(t)x'(t) + q(t)x(t) = r(t), \tag{1.0}$$

$$x(a) = d_1, \quad x(b) = d_2, \quad a, b \in \mathbb{R},\ a < b,\ d_1, d_2 \in \overline{\mathbb{R}}, \tag{1.1}$$

where p, q and r are elements of the Colombeau algebra $\mathcal{G}(\mathbb{R})$; d_1, d_2 are known elements of the Colombeau algebra $\overline{\mathbb{R}}$ of generalized real numbers; $x(a), x(b)$ are understood as the values of the generalized function x at the points a and b, respectively (see [2], [3]). The elements p, q and r are given. The multiplication, the derivative, the sum and the equality are meant in the Colombeau algebra sense. We prove theorems on existence and uniqueness of solutions of the problem (1.0)–(1.1).

In the papers [3], [10] some linear differential equations with coefficients from the Coloumbeau algebra were examined. Certain problems from the quantum field theory lead to such equations. However, these equations cannot be considered in the theory of distributions, due to difficulties in defining multiplication of distributions.

2. Notation. Let $\mathcal{D}(\mathbb{R})$ be the set of all C^∞ functions $\mathbb{R} \to \mathbb{R}$ with compact support. For $q = 1, 2, \ldots$ we denote by $\mathcal{A}_q$ the set of all functions $\Phi \in \mathcal{D}(\mathbb{R})$ such that

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$$\int_{-\infty}^{\infty}\Phi(t)\,dt=1,\qquad \int_{-\infty}^{\infty}t^k\Phi(t)\,dt=0,\quad 1\le k\le q. \tag{2.1}$$

Next, $\mathcal{E}[\mathbb{R}]$ is the set of all functions $R:\mathcal{A}_1\times\mathbb{R}\to\mathbb{R}$ such that $R(\Phi,t)\in C^\infty$ for every fixed $\Phi\in\mathcal{A}_1$.

If $R\in\mathcal{E}[\mathbb{R}]$, then $D_kR(\Phi,t)=\frac{d^k}{dt^k}(R(\Phi,t))$.

For given $\Phi\in\mathcal{D}(\mathbb{R})$ and $\varepsilon>0$, we define Φ_ε by

$$\Phi_\varepsilon(t)=\frac{1}{\varepsilon}\Phi\left(\frac{t}{\varepsilon}\right). \tag{2.2}$$

An element R of $\mathcal{E}[\mathbb{R}]$ is *moderate* if for every compact set K of $\mathbb{R}$ and every k there is $N\in\mathbb{N}$ such that the following condition holds: for every $\Phi\in\mathcal{A}_N$ there are $c>0$ and $\eta>0$ such that

$$\sup_{t\in K}|D_kR(\Phi_\varepsilon,t)|\le c\varepsilon^{-N}\qquad \text{if } 0<\varepsilon<\eta. \tag{2.3}$$

We denote by $\mathcal{E}_{\mathcal{M}}[\mathbb{R}]$ the set of all moderate elements of $\mathcal{E}[\mathbb{R}]$.

By Γ we denote the set of all increasing functions α from $\mathbb{N}$ into $\mathbb{R}^+$ such that $\alpha(q)$ tends to ∞ as $q\to\infty$.

We define an ideal $\mathcal{N}[\mathbb{R}]$ in $\mathcal{E}_{\mathcal{M}}[\mathbb{R}]$ as follows: $R\in\mathcal{N}[\mathbb{R}]$ if for every compact set K of $\mathbb{R}$ and every k there are $N\in\mathbb{N}$ and $\alpha\in\Gamma$ such that the following condition holds: for every $q\ge N$ and $\Phi\in\mathcal{A}_q$ there are $c>0$ and $\eta>0$ such that

$$\sup_{t\in K}|D_kR(\Phi_\varepsilon,t)|\le c\varepsilon^{\alpha(q)-N}\qquad \text{if } 0<\varepsilon<\eta. \tag{2.4}$$

The algebra $\mathcal{G}(\mathbb{R})$ (the *Colombeau algebra*) is defined as the quotient algebra of $\mathcal{E}_{\mathcal{M}}[\mathbb{R}]$ with respect to $\mathcal{N}[\mathbb{R}]$ (see [3]).

We denote by $\mathcal{E}_0$ the set of all functions from $\mathcal{A}_1$ into $\mathbb{R}$. Next, we denote by $\mathcal{E}_{\mathcal{M}}$ the set of all the so-called *moderate elements of* $\mathcal{E}_0$ defined by

$$\mathcal{E}_{\mathcal{M}}=\{R\in\mathcal{E}_0: \text{there is } N\in\mathbb{N} \text{ such that for every } \Phi\in\mathcal{A}_N \text{ there are } c>0,\ \eta>0 \text{ such } |R(\Phi_\varepsilon)|\le c\varepsilon^{-N} \text{ if } 0<\varepsilon<\eta\}. \tag{2.5}$$

Further, we define an ideal $\mathcal{T}$ of $\mathcal{E}_{\mathcal{M}}$ by

$$\mathcal{T}=\{R\in\mathcal{E}_0: \text{there are } N\in\mathbb{N} \text{ and } \alpha\in\Gamma \text{ such that for every } q\ge N \text{ and } \Phi\in\mathcal{A}_q \text{ there are } c>0 \text{ and } \eta>0 \text{ such that } |R(\Phi_\varepsilon)|\le c\varepsilon^{\alpha(q)-N} \text{ if } 0<\varepsilon<\eta\}. \tag{2.6}$$

We define an algebra $\overline{\mathbb{R}}$ by setting

$$\overline{\mathbb{R}}=\mathcal{E}_{\mathcal{M}}/\mathcal{T}\qquad\text{(see [3])}.$$

If $R\in\mathcal{E}_{\mathcal{M}}[\mathbb{R}]$ is a representative of $G\in\mathcal{G}[\mathbb{R}]$, then for a fixed t the map $Y:\Phi\to R(\Phi,t)\in\mathbb{R}$ is defined on $\mathcal{A}_1$ and $Y\in\mathcal{E}_{\mathcal{M}}$. The class of Y in $\mathbb{R}$ depends only on G and t. This class is denoted by $G(t)$ and is called the *value of the generalized function G at the point t* (see [3]).

We say that $G\in\mathcal{G}(\mathbb{R})$ is a *constant generalized function* on $\mathbb{R}$ if it admits a representative $R(\Phi,t)$ which is independent of $t\in\mathbb{R}$. With any $Z\in\overline{\mathbb{R}}$ we associate a constant generalized function which admits $R(\Phi,t) = Z(\Phi)$ as its representative, provided that we denote by Z a representative of Z (see [3]).

We put

$$\|x\|_1 = \max_{t\in[a,b]} |x(t)| + \max_{t\in[a,b]} |x'(t)|$$

$$\|x\| = \max_{t\in[a,b]} |x(t)|, \qquad \text{where } x \in \mathbb{C}^\infty.$$

Throughout the paper K denotes a compact set in $\mathbb{R}$.

We denote by $R_p(\Phi,t)$, $R_{x_0}(\Phi)$ and $R_{x(t_0)}(\Phi)$ representatives of elements p, x_0 and $x(t_0)$.

We say that $x \in \mathcal{G}(\mathbb{R})$ is a *solution* of the equation (1.0) if x satisfies this equation in $\mathcal{G}(\mathbb{R})$.

The definition of generalized functions on an open interval $(A,B) \subset \mathbb{R}$ is almost the same as the definition on the whole $\mathbb{R}$ (see [3]). In this paper we shall prove theorems on generalized solutions of linear differential equations of second order on $\mathbb{R}$. It is not difficult to observe that these theorems are also true in the case when the generalized functions p, q and r are considered on an interval (A,B) such that $-\infty < A < a < b < B < \infty$.

3. Main results. First, we shall introduce

HYPOTHESIS H.

(3.0) $p, q, r \in \mathcal{G}(\mathbb{R})$,

(3.1) the elements $p, q \in \mathcal{G}(\mathbb{R})$ admit representatives $R_p(\Phi,t)$ and $R_q(\Phi,t)$ with the following properties: for every K there is $N \in \mathbb{N}$ such that for every $\Phi \in \mathcal{A}_N$ there are constants $c > 0$ and $\eta > 0$ such that

$$\sup_{t\in K}\Big|\int_{t_0}^{t} |R_p(\Phi_\varepsilon,s)|\,ds\Big| \le c, \quad \sup_{t\in K}\Big|\int_{t_0}^{t} |R_q(\Phi_\varepsilon,s)|\,ds\Big| \le c \quad \text{if } 0<\varepsilon<\eta,$$

(3.2) the element $p \in \mathcal{G}(\mathbb{R})$ admits a representative $R_p(\Phi,t)$ with the following property: there is $N \in \mathbb{N}$ such that for every $\Phi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$ and $\gamma > 0$ such that

$$I_1(p,\Phi_\varepsilon) = \int_a^b |R_p(\Phi_\varepsilon,t)|\,dt \le \frac{4}{b-a} - \gamma \quad \text{if } 0<\varepsilon<\varepsilon_0,$$

(3.3) the elements $p, q \in \mathcal{G}(\mathbb{R})$ admit representatives $R_p(\Phi,t)$ and $R_q(\Phi,t)$ with the following property: there is $N \in \mathbb{N}$ such that for every $\Phi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$ and $\gamma > 0$ such that

$$I_2(p,q,\Phi_\varepsilon) = \int_a^b |R_p(\Phi_\varepsilon,t)|\,dt + \int_a^b |R_q(\Phi_\varepsilon,t)|\,dt \le \frac{4}{b-a+4} - \gamma \quad \text{if } 0<\varepsilon<\varepsilon_0.$$

Now we shall give theorems on the existence and uniqueness of the solution of the problem (1.0)–(1.1). Apart from the problem (1.0)–(1.1), we shall examine the homogeneous problem of the form

$$\begin{cases} x''(t) + p(t)x'(t) + q(t)x(t) = 0, \\ x(a) = x(b) = 0, \quad a < b, \; a, b \in \mathbb{R}. \end{cases} \tag{3.4}$$

THEOREM 3.1. *Assume that the conditions* (3.0)–(3.1) *hold and zero is the unique solution of the problem* (3.4) *in* $\mathcal{G}(\mathbb{R})$. *Then the problem* (1.0)–(1.1) *has a unique solution* x *in* $\mathcal{G}(\mathbb{R})$.

Remark 3.1. Let δ denote the generalized function (the delta Dirac's generalized function) which admits as a representative the functions $R_\delta(\Phi, t) = \Phi(-t)$, where $\Phi \in \mathcal{A}_1$. Then δ has the property (3.1).

Remark 3.2. It is not difficult to show that the problem

$$x'(t) = 2\delta'(t)\delta(t)x(t), \tag{3.5}$$

$$x(-1) = 1 \tag{3.6}$$

has no solution in $\mathcal{G}(\mathbb{R})$ (see [10]).

If p, q and r have properties (3.0)–(3.1), then the problem

$$x''(t) + p(t)x'(t) + q(t)x(t) = r(t), \tag{3.7}$$

$$x(t_0) = r_1, \quad x'(t_0) = r_2, \quad t_0 \in \mathbb{R}, \; r_1, r_2 \in \overline{\mathbb{R}}, \tag{3.8}$$

has a unique solution $x \in \mathcal{G}(\mathbb{R})$ (see [10]). Moreover, every solution x of the equation (3.7) has a representation

$$x = c_1\varphi_0 + c_2\psi + Q, \tag{3.9}$$

where φ_0 and ψ are solutions of the problems:

$$\begin{cases} \varphi_0''(t) + p(t){\varphi'}_0(t) + q(t)\varphi_0(t) = 0, \\ \varphi_0(t_0) = 1, \quad {\varphi'}_0(t_0) = 0, \end{cases} \tag{3.10}$$

$$\begin{cases} \psi''(t) + p(t)\psi'(t) + q(t)\psi(t) = 0, \\ \psi(t_0) = 0, \quad \psi'(t_0) = 1, \end{cases} \tag{3.11}$$

Q is a particular solution of the equation (3.7) and c_1 and c_2 are generalized constant functions on $\mathbb{R}$. The solution x is the class of solutions of the problems:

$$x''(t) + R_p(\Phi, t)x'(t) + R_q(\Phi, t)x(t) = R_r(\Phi, t), \tag{3.12}$$

$$x(t_0) = R_{r_1}(\Phi), \quad x'(t_0) = R_{r_2}(\Phi), \quad \Phi \in \mathcal{A}_1 \quad \text{(see [10])}. \tag{3.13}$$

Remark 3.3. If $p \in C^\infty$ and if

$$I(p) = \int_a^b |p(t)|\, dt < \frac{4}{b-a}, \tag{3.14}$$

then p has the property (3.2). If $p \in L^1_{\mathrm{loc}}(\mathbb{R})$ (i.e. $p \in L^1(I)$ for every compact interval $I \subset \mathbb{R}$) and if p satisfies (3.14), then we put

$$R_p(\Phi_\varepsilon, t) = \int_{-\infty}^{\infty} p(t+\varepsilon u)\Phi(u)\,du = (p * \Phi_\varepsilon)(t) \tag{3.15}$$

where $\Phi \in \mathcal{A}_1$ (see [3]). Taking into account that

$$\begin{aligned} |R_p(\Phi_\varepsilon, t)| - \Big|\int_{-\infty}^{\infty} p(t)\Phi(u)du\Big| &\le \Big|R_p(\Phi_\varepsilon, t) - \int_{-\infty}^{\infty} p(t)\Phi(u)\,du\Big| \\ &\le \int_{-\infty}^{\infty} |p(t+\varepsilon u) - p(t)||\Phi(u)|\,du, \end{aligned} \tag{3.16}$$

we conclude that p has the properties (3.1)–(3.2).

Remark 3.4. Let $\widetilde{\delta}$ denote the generalized function defined by

$$R_{\tilde{\delta}}(\Phi, t) = \frac{\Phi(-t)}{\int_{-\infty}^{\infty} |\Phi(-t)|\,dt}, \quad \Phi \in \mathcal{A}_1. \tag{3.17}$$

Moreover, let $a = -1$ and $b = 1$. Then $\widetilde{\delta}$ has the properties (3.1)–(3.2).

Remark 3.5. Let $p, q \in L^1_{\mathrm{loc}}(\mathbb{R})$ and let

$$\widetilde{I}(p,q) = \int_a^b |p(t)|\,dt + \int_a^b |q(t)|\,dt < \frac{4}{b-a+4}. \tag{3.18}$$

Then p and q have the properties (3.1) and (3.3).

Theorem 3.2. *Assume* (3.1)–(3.2). *Then the problem*

$$x''(t) + p(t)x(t) = 0, \tag{3.19}$$

$$x(a) = x(b) = 0, \qquad a < b,\ a, b \in \mathbb{R}, \tag{3.20}$$

has only the trivial solution in $\mathcal{G}(\mathbb{R})$.

Theorem 3.3. *Assume the conditions* (3.1) *and* (3.3). *Then the problem* (3.4) *has only the trivial solution in* $\mathcal{G}(\mathbb{R})$.

Remark 3.6. In the papers [6], [9], [11], [12] under various assumptions the situation when linear differential equations of second order (with additional conditions) have only a trivial solution is considered.

4. Proofs.

In order to prove Theorem 3.1, we shall prove three lemmas.

Lemma 4.1. *Suppose that all assumptions of Theorem* 3.1 *are satisfied and let* $R_{\varphi_0}(\Phi_\varepsilon, t)$ *and* $R_\Psi(\Phi_\varepsilon, t)$ *be solutions of the problems*:

$$\varphi_0''(t) + R_p(\Phi_\varepsilon, t)\varphi_0'(t) + R_q(\Phi_\varepsilon, t)\varphi_0(t) = 0, \tag{4.1}$$

$$\varphi_0(a) = 1, \quad \varphi_0'(a) = 0, \quad \Phi \in \mathcal{A}_1, \tag{4.2}$$

and

$$\psi''(t) + R_p(\Phi_\varepsilon, t)\psi'(t) + R_q(\Phi_\varepsilon, t)\psi(t) = 0, \tag{4.3}$$

$$\psi(a) = 0, \quad \psi'(a) = 1, \quad \Phi \in \mathcal{A}_1, \tag{4.4}$$

respectively. Then there is $N \in \mathbb{N}$ *such that for every* $q \geq N$ *and* $\Phi \in \mathcal{A}_q$ *there is* $\eta > 0$ *such that*

$$R_\Psi(\Phi_\varepsilon, b) \neq 0 \quad \text{if } 0 < \varepsilon < \eta. \tag{4.5}$$

Proof. Suppose (4.5) is not true. Then for every $N \in \mathbb{N}$ there exist $q \geq N$ and $\Phi \in \mathcal{A}_q$ such that for every $\eta > 0$ there exists $\varepsilon > 0$ with the following property:

$$R_\Psi(\Phi_\varepsilon, a) = 0, \quad R_\Psi(\Phi_\varepsilon, b) = 0, \quad R_{\Psi'}(\Phi_\varepsilon, a) = 1. \tag{4.6}$$

Let $R_{\overline{\Psi}} : \mathcal{A}_1 \times \mathbb{R} \to \mathbb{R}$ be the mapping defined by

$$R_{\overline{\Psi}}(\Phi, t) = \begin{cases} R_\Psi(\Phi, t) & \text{if } \Phi = \Phi_\varepsilon,\ R_\Psi(\Phi_\varepsilon, a) = R_\Psi(\Phi_\varepsilon, b) = 0, \\ & R_{\Psi'}(\Phi_\varepsilon, a) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Then $R_{\overline{\Psi}}(\Phi, t) \in \mathcal{E}_{\mathcal{M}}[\mathbb{R}]$ (because $R_\Psi(\Phi, t) \in \mathcal{E}_{\mathcal{M}}[\mathbb{R}]$) and $R_{\overline{\Psi}} \notin \mathcal{N}[\mathbb{R}]$). Defining $\overline{\Psi}$ as the class of $R_{\overline{\Psi}}(\Phi, t)$ in $\mathcal{G}(\mathbb{R})$, we conclude that $\overline{\Psi}$ is a non-trivial solution of the problem (3.4), which is impossible. Thus (4.5) holds.

LEMMA 4.2. *Let all assumptions of Lemma* 4.1 *be satisfied. Then there is a number* $q \in \mathbb{N}$ *such that for every* $\Phi \in \mathcal{A}_q$ *there are* $c > 0,\ \eta > 0$ *such that*

$$|R_\Psi(\Phi_\varepsilon, b)|^{-1} \leq c\varepsilon^{-q} \quad \text{if } 0 < \varepsilon < \eta. \tag{4.8}$$

Proof. Suppose the lemma is false. Then for every $q \in \mathbb{N}$ there exists $\Phi \in \mathcal{A}_q$ such that for every $c, \eta > 0$ there exists $\varepsilon > 0$ with the following property:

$$|R_\Psi(\Phi_\varepsilon, b)| < \frac{1}{c}\varepsilon^q, \quad 0 < \varepsilon < \eta. \tag{4.9}$$

We choose a sequence $\Phi_1, \ldots, \Phi_m, \ldots$ such that $\Phi_m \in \mathcal{A}_m$ and for every $c, \eta > 0$ there exists $\varepsilon > 0$ with the following property:

$$|R_\Psi(\Phi_\varepsilon, b)| < \frac{1}{c}\varepsilon^m, \quad 0 < \varepsilon < \eta. \tag{4.10}$$

Without loss of a generality we can assume that

$$\Phi_m \in \mathcal{A}_{m+r_m} - \mathcal{A}_{m+1+r_m}, \tag{4.11}$$

where $r_m \geq 0$ and r_m is the smallest number such that (4.10) holds (for fixed m). We put

$$C = \{\Phi_1, \ldots, \Phi_m, \ldots\}, \quad \Phi_m = \Phi_{m,i}, \tag{4.12}$$

where i denotes the number of different elements of the set $\{\Phi_1, \ldots, \Phi_m\}$. We consider the family E of functions defined by

$$E = \{\Phi_{m,i_\varepsilon} :\ \Phi_{m,i} \in C,\ \varepsilon > 0;\ m, i = 1, 2, \ldots\}. \tag{4.13}$$

Evidently, $\Phi_{m,i_\varepsilon} \in \mathcal{A}_{m+r_m} - \mathcal{A}_{m+1+r_m}$. Thus, for $c > 0$ and $0 < \eta < 1$ there exists $\beta > 0$ such that

$$|R_\Psi(\Phi_{m,i_\beta}, b)| < \frac{1}{c}\beta^i, \quad \beta < \eta. \tag{4.14}$$

Now we define a mapping $\lambda : \mathcal{A}_1 \to \mathbb{R}$ as follows:

$$\lambda(\Phi) = \begin{cases} 1 & \text{if } \Phi = \Phi_{m,i_\beta} \text{ for some } m, i \text{ and } 0 < \beta < 1, \\ & \text{and } \Phi_{m,i_\beta} \text{ satisfies (4.14) for some } c > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

We claim that

$$\lambda(\Phi)R_\Psi(\Phi, b) \in \mathcal{T}. \tag{4.16}$$

Suppose (4.16) is not true. Then for every $N \in \mathbb{N}$ and $\alpha \in \Gamma$ there exist $q \geq N$ and $\Phi \in \mathcal{A}_q$ such that for every $c > 0$ and $\eta > 0$ there exists $\varepsilon > 0$ with

$$|R_\Psi(\Phi_\varepsilon, b)\lambda(\Phi_\varepsilon)| > c\varepsilon^{\alpha(q)-N}. \tag{4.17}$$

Let N_1 be a number such that (4.5) and (4.17) hold and let $\alpha(q) = i$, where $\Phi_q = \Phi_{q,i}$. Then

$$\lambda(\Phi_\varepsilon) = 1 \quad \text{(by (4.17))} \tag{4.18}$$

and

$$\Phi_\varepsilon(t) = \overline{\Phi}_{m,i\beta}(t) \quad \text{for all } t \in \mathbb{R}, \tag{4.19}$$

where $\overline{\Phi}_{m,i} \in C$. Hence

$$\Phi(t) = \overline{\Phi}_{m,i\gamma}(t) \quad \text{and} \quad \overline{\Phi}_{m,i\gamma} \in E, \tag{4.20}$$

where $\gamma > 0$ and

$$\Phi_{\overline{\varepsilon}}(t) = \overline{\Phi}_{m,i\gamma\cdot\overline{\varepsilon}}(t), \quad \overline{\Phi}_{m,i\gamma\cdot\overline{\varepsilon}}(t) \in E \quad \text{if } \overline{\varepsilon} > 0. \tag{4.21}$$

Taking into account (4.14) and (4.17), we get

$$\frac{1}{\overline{c}}\overline{\varepsilon}^i\gamma^i > c\overline{\varepsilon}^{i-N_1}, \tag{4.22}$$

where $\overline{c} > 1$ and $0 < \overline{\varepsilon} < \eta$. Let $c = 1$ and let $\overline{\varepsilon}$ be sufficiently small. Then

$$\overline{\varepsilon}^{N_1}\gamma^i > \overline{c} > 1, \tag{4.23}$$

which is impossible. Thus (4.16) holds. Let $x = \lambda \cdot \psi$, where λ denotes the class of $\lambda(\Phi)$ in $\overline{\mathbb{R}}$. Then x is a solution of the problem (3.4) and $x \neq 0$ (because $x'(a) = \lambda\psi'(a) = \lambda \neq 0$), which contradicts the assumptions of Lemma 4.2.

LEMMA 4.3. *Let all assumptions of Lemma* 4.1 *be satisfied and let* $d \in \overline{\mathbb{R}}$. *Then the equation*

$$c \cdot \psi(b) = d \tag{4.24}$$

has a unique solution in $\overline{\mathbb{R}}$.

P r o o f. The uniqueness of solution of (4.24) follows from the assumptions of Lemma 4.1. Now we shall prove existence of a solution of (4.24). To this purpose, we define a mapping $R_c : \mathcal{A}_1 \to \mathbb{R}$ as follows:

$$R_c(\Phi) = \begin{cases} \dfrac{R_d(\Phi)}{R_\Psi(\Phi, b)} & \text{if } R_\Psi(\Phi, b) \neq 0, \\ 1 & \text{otherwise.} \end{cases} \tag{4.25}$$

Applying Lemmas 4.1–4.2, we deduce that there is $N \in \mathbb{N}$ such that for every $q \geq N$ and $\Phi \in \mathcal{A}_q$ there is $\eta > 0$ such that $R_c(\Phi_\varepsilon)R_\Psi(\Phi_\varepsilon b) = R_d(\Phi_\varepsilon)$ if $0 < \varepsilon < \eta$. Hence $R_c(\Phi) \in \mathcal{E}_\mathcal{M}$, which implies our assertion.

R e m a r k 4.1. Let c be defined as the class of all elements $R_c(\Phi)$ in $\overline{\mathbb{R}}$ defined by (4.25). Then we shall write $c = \psi^{-1}(b)d$.

R e m a r k 4.2. It is known that $(|t|)(0)$ is a non-zero element of $\overline{\mathbb{R}}$ and it is not invertible (see [2], p. 147). Thus $\overline{\mathbb{R}}$ is not a field.

P r o o f o f T h e o r e m 3.1. The uniqueness of solutions of the problem (1.0)–(1.1) follows from the assumptions of Theorem 3.1. It is sufficient to show the existence of a solution. If x is a solution of the equation (3.7), then by applying (3.9) we get

$$r_1 = x(a) = c_1 + Q(a) \tag{4.26}$$

and

$$r_2 = x(b) = c_1\varphi_0(b) + c_2\psi(b) + Q(b). \tag{4.27}$$

Hence, by Lemma 4.3 we conclude that

$$c_1 = r_1 - Q(a) \tag{4.28}$$

and

$$c_2 = \psi^{-1}(b)(r_2 - Q(b) - \varphi_0(b)(r_1 - Q(a))), \tag{4.29}$$

which was to be proved.

P r o o f o f T h e o r e m s 3.2–3.3. If x is a solution of the problem (3.4), then

$$R_{x''}(\Phi, t) + R_p(\Phi, t)R_{x'}(\Phi, t) + R_q(\Phi, t)R_x(\Phi, t) = \overline{\eta}(\Phi, t), \tag{4.30}$$

$$R_{x(a)}(\Phi) = \eta_1(\Phi), \quad R_{x(b)}(\Phi) = \eta_2(\Phi), \tag{4.31}$$

where

$$\Phi \in \mathcal{A}_1, \quad \overline{\eta} \in \mathcal{N}[\mathbb{R}] \quad \text{and} \quad \eta_1, \eta_2 \in \mathcal{T}. \tag{4.32}$$

Hence

$$\begin{aligned} R_x(\Phi, t) = \int_a^b G(t, s)(-R_p(\Phi, s)R_{x'}(\Phi, s) - R_q(\Phi, s)R_x(\Phi, s) + \overline{\eta}(\Phi, s))\, ds \\ + \eta_1(\Phi) + \frac{\eta_2(\Phi) - \eta_1(\Phi)}{b - a}(t - a), \end{aligned} \tag{4.33}$$

where

$$G(t,s) = \begin{cases} \dfrac{(t-b)(s-a)}{b-a} & \text{if } a \le s \le t \le b, \\ \dfrac{(a-t)(b-s)}{b-a} & \text{if } a \le t \le s \le b. \end{cases} \tag{4.34}$$

Taking into account that

$$\sup_{t,s\in[a,b]} |G(t,s)| = \frac{b-a}{4} \tag{4.35}$$

we have

$$\|R_x(\Phi_\varepsilon,t)\| \le \frac{b-a}{4}\Big(I_2(p,q,\Phi_\varepsilon)\|R_x(\Phi_\varepsilon,t)\|_1 + \int_a^b |\overline{\eta}(\Phi_\varepsilon,s)|\,ds\Big) + 2|\eta_1(\Phi_\varepsilon)| + |\eta_2(\Phi_\varepsilon)| \tag{4.36}$$

and

$$\|R_{x'}(\Phi_\varepsilon,t)\| \le I_2(p,q,\Phi_\varepsilon)\|R_x(\Phi_\varepsilon,t)\|_1 + \int_a^b |\overline{\eta}(\Phi_\varepsilon,s)|\,ds + \frac{|\eta_2(\Phi_\varepsilon)| + |\eta_1(\Phi_\varepsilon)|}{b-a}, \tag{4.37}$$

where ε is sufficiently small. Relations (4.36)–(4.37) yield

$$\|R_x(\Phi_\varepsilon,t)\|_1 \le \frac{b-a+4}{4} I_2(p,q,\Phi_\varepsilon)\|R_x(\Phi_\varepsilon,t)\|_1 + \eta_3(\Phi_\varepsilon), \tag{4.38}$$

where

$$\eta_3(\Phi_\varepsilon) = \frac{b-a+4}{4}\int_a^b |\overline{\eta}(\Phi_\varepsilon,s)|\,ds + |\eta_2(\Phi_\varepsilon)| + 2|\eta_1(\Phi_\varepsilon)| + \frac{(|\eta_2(\Phi_\varepsilon)| + |\eta_1(\Phi_\varepsilon)|)}{b-a} \quad \text{(see [6])}. \tag{4.39}$$

On the other hand, $R_x(\Phi_\varepsilon,t)$ is a solution of the problem

$$x''(t) + R_p(\Phi_\varepsilon,t)x'(t) + R_q(\Phi_\varepsilon,t)x(t) = \overline{\eta}(\Phi_\varepsilon,t), \tag{4.40}$$

$$x(t_0) = R_x(\Phi_\varepsilon,t_0), \quad x'(t_0) = R_{x'}(\Phi_\varepsilon,t_0), \quad a < t_0 < b \tag{4.41}$$

and

$$R_x(\Phi,t_0), R_{x'}(\Phi,t_0) \in \mathcal{T} \quad \text{(by (4.38), (4.39), (4.41) and [10])}. \tag{4.42}$$

Thus

$$R_x(\Phi,t) \in \mathcal{N}[\mathbb{R}]. \tag{4.43}$$

This proves Theorem 3.3.

The proof of Theorem 3.2 is similar to that of Theorem 3.3. We start with the equalities

$$R_{x''}(\Phi,t) + R_p(\Phi,t)R_x(\Phi,t) = \overline{\eta}(\Phi,t), \tag{4.44}$$

$$R_{x(a)}(\Phi) = \eta_1(\Phi), \quad R_{x(b)}(\Phi) = \eta_2(\Phi), \tag{4.45}$$

where $\Phi \in \mathcal{A}_1$ and x is a solution of the problem (3.17)–(3.18). Applying (4.33) and (4.36), we can see that

$$\|R_x(\Phi_\varepsilon, t)\| \le \frac{b-a}{4} I_1(p, \Phi_\varepsilon)\|R_x(\Phi_\varepsilon, t)\| + \widetilde{\eta}_3(\Phi_\varepsilon), \tag{4.46}$$

where

$$\widetilde{\eta}_3(\Phi_\varepsilon) = \frac{b-a}{4}\int\limits_a^b |\overline{\eta}(\Phi_\varepsilon, s)|\, ds + 2|\eta_1(\Phi_\varepsilon)| + |\eta_2(\Phi_\varepsilon)| \tag{4.47}$$

and ε is sufficiently small number.

As in the proof of Theorem 3.3, we conclude that $R_x(\Phi, t)$ has property (4.43), which completes the proof of Theorem 3.2.

5. Final remarks

COROLLARY 5.1. *Let assumptions* (3.0), (3.1) *and* (3.3) *be satisfied and let* $R_x(\Phi_\varepsilon, t)$ *be a solution of the problem*

$$x''(t) + R_p(\Phi_\varepsilon, t)x'(t) + R_q(\Phi_\varepsilon, t)x(t) = R_r(\Phi_\varepsilon, t), \tag{5.0}$$

$$x(a) = R_{d_1}(\Phi_\varepsilon), \quad x(b) = R_{d_2}(\Phi_\varepsilon), \quad \Phi \in \mathcal{A}_N \tag{5.1}$$

(*for small* $\varepsilon > 0$). *Then there exists a unique solution* x *of the problem* (1.0)–(1.1) *and* x *is the class of elements* $R_x(\Phi, t)$ *in* $\mathcal{G}(\mathbb{R})$.

Proof. It is easy to see that $R_x(\Phi_\varepsilon, t)$ satisfies the equality

$$\begin{aligned} R_x(\Phi_\varepsilon, t) = \int\limits_a^b G(t,s)(&-R_p(\Phi_\varepsilon, s)R_{x'}(\Phi_\varepsilon, s) - R_q(\Phi_\varepsilon, s)R_x(\Phi_\varepsilon, s) \\ &+ R_r(\Phi_\varepsilon, s))\, ds + R_{d_1}(\Phi_\varepsilon) + \frac{R_{d_2}(\Phi_\varepsilon) - R_{d_1}(\Phi_\varepsilon)}{b-a}(t-a), \end{aligned} \tag{5.2}$$

where G is defined by (4.34). Using (4.35), we obtain

$$\|R_x(\Phi_\varepsilon, t)\|_1 \le \frac{b-a+4}{4} I_2(p, q, \Phi_\varepsilon)\|R_x(\Phi_\varepsilon, t)\|_1 + \beta(\Phi_\varepsilon), \tag{5.3}$$

where $|\beta(\Phi_\varepsilon)| \le c\varepsilon^{-N}$ (for small $\varepsilon > 0$). Hence

$$R_x(\Phi, t_0), R_{x'}(\Phi, t_0) \in \mathcal{E}_{\mathcal{M}} \tag{5.5}$$

where t_0 is a fixed point in the interval (a, b). Denote by r_1 and r_2 the classes of $R_x(\Phi, t_0)$ and $R_{x'}(\Phi, t_0)$ respectively in $\overline{\mathbb{R}}$. Then we see that the problem (3.7)–(3.8) has exactly one solution x in $\mathcal{G}(\mathbb{R})$ and x is the class of $R_x(\Phi, t)$ in $\mathcal{G}(\mathbb{R})$ ($R_x(\Phi, t)$ is a solution of the problem (3.12)–(3.13) too). On the other hand,

$$R_x(\Phi, a) = r_{d_1}(\Phi), \quad R_x(\Phi, b) = R_{d_2}(\Phi), \tag{5.6}$$

which completes the proof of Corollary 5.1.

By similar arguments we get

COROLLARY 5.2. *Let assumptions* (3.0)–(3.2) *be satisfied and let* $R_x(\Phi_\varepsilon, t)$ *be a solution of the problem*

$$x''(t) + R_p(\Phi_\varepsilon, t)x(t) = R_r(\Phi_\varepsilon, t), \tag{5.7}$$

$$x(a) = R_{d_1}(\Phi_\varepsilon), \quad x(b) = R_{d_2}(\Phi_\varepsilon) \tag{5.8}$$

(*for* $\Phi \in \mathcal{A}_N$ *and small* $\varepsilon > 0$). *Then there exists a unique solution* x *of the problem*

$$x''(t) + p(t)x(t) = r(t), \tag{5.9}$$

$$x(a) = d_1, \quad x(b) = d_2, \quad a < b,\ d_1, d_2 \in \overline{\mathbb{R}}, \tag{5.10}$$

and x *is the class of the element* $R_x(\Phi, t)$ *in* $\mathcal{G}(\mathbb{R})$.

R e m a r k 5.1. It is known that the algebra $\mathcal{E}_f(\mathbb{R})$ of all piecewise continuous functions on $\mathbb{R}$ is not a subalgebra of $\mathcal{G}(\mathbb{R})$ (see [3]). On the other hand, if $g_1, g_2 \in C^\infty$, then the classical product and the product in $\mathcal{G}(\mathbb{R})$ give rise to the same elements of $\mathcal{G}(\mathbb{R})$ (see [4]).

COROLLARY 5.3. *Let* $p, q, r \in C^\infty$. *Moreover*, *let* $r_1, r_2 \in \mathbb{R}$. *Then the classical and generalized solutions* (*i.e. solutions in the Colombeau algebra*) *of the problem* (3.7)–(3.8) *give rise to the same elements of* $\mathcal{G}(\mathbb{R})$ (see [4], [10]).

R e m a r k 5.2. If necessary, we denote the product in $\mathcal{G}(\mathbb{R})$ by $\odot$, to avoid confusion with the classical product. To repair the consistency problem for multiplication we give the definition introduced by J. F. Colombeau in [3].

An element u of $\mathcal{G}(\mathbb{R})$ is said *to admit* a member $W \in \mathcal{D}'(\mathbb{R})$ as *associated distribution* if it has a representative $R_u(\Phi, t)$ with the following property: for every $\Psi \in \mathcal{D}(\mathbb{R})$ there is $N \in \mathbb{N}$ such that for every $\Phi \in \mathcal{A}_N$ we have

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} R_u(\Phi_\varepsilon, t)\Psi(t)\, dt = W(\Psi). \tag{5.11}$$

If u admits an associated distribution W, then this associated distribution is unique (see [3], p. 64).

COROLLARY 5.4. *Assume that* $p, q, r \in L^1_{\mathrm{loc}}(\mathbb{R})$, $d_1, d_2 \in \mathbb{R}$,

$$\widetilde{I}(p, q) < \frac{4}{b - a + 4}, \tag{5.13}$$

x *is a solution of* (1.0)–(1.1) *in the Carathéodory sense*, *and* $\overline{x} \in \mathcal{G}(\mathbb{R})$ *is a solution of the problem*

$$x''(t) + p(t) \odot x'(t) + q(t) \odot x(t) = r(t),$$
$$x(a) = d_1, \quad x(b) = d_2.$$

Then $\overline{x}$ *admits an associated distribution which equals* x.

P r o o f. This follows from the fact that $p * \Phi_\varepsilon \to p$, $q * \Phi_\varepsilon \to q$, $r * \Phi_\varepsilon \to r$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ (see [1]) and the continuous dependence of x on the coefficients p, q and r.

COROLLARY 5.5. *Assume that* $p, r \in L^1_{\mathrm{loc}}(\mathbb{R})$, $d_1, d_2 \in \mathbb{R}$,

$$I(p) < \frac{4}{b-a},$$

x is a solution of the problem (5.9)–(5.10) *in the Carathéodory sense*, *and* $\overline{x} \in \mathcal{G}(\mathbb{R})$ *is a solution of the problem*

$$x''(t) + p(t) \odot x(t) = r(t),$$
$$x(a) = d_1, \quad x(b) = d_2.$$

Then $\overline{x}$ admits an associated distribution which equals x.

R e m a r k 5.3. Continuous solutions of ordinary differential equations can be considered in another way (for example, [5], [7]. [8], [11], [13], [14]).

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