POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCCXLI

ŁUKASZ STETTNER

Ergodic control of Markov processes with mixed observation structure

WARSZAWA 1995

Łukasz Stettner
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-950 Warszawa, Poland
E-mail: stettner@impan.impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by

P R I N T E D I N P O L A N D

ISSN 0012-3862

CONTENTS

1. Introduction 5
2. Preliminary results and assumptions 7
3. Approximation of the invariant measure 14
4. Construction of nearly optimal control functions 24
4.1. Approximation of admissible control functions 24
4.2. State space discretization 25
4.3. Comments on further discretizations 31
5. Nearly optimal control values 31
6. An example 35
References 36

1991 *Mathematics Subject Classification*: Primary 93E20; Secondary 93E11, 93E25.
Received 28.12.1993; revised version 16.8.1994.

1. Introduction

On a probability space (Ω, F, P) consider a controlled discrete time Markov process (x_i) with values in a state space E that is a closed, not necessarily bounded subset of $\mathbb{R}^d$, and transition operator $P^{a_i}(x_i, dz)$ in a generic period i, where a_i is a control that takes values in a convex compact set $U \subset \mathbb{R}^\kappa$, $\kappa \geq 1$. Assume that we have a partial observation y_i of the state process x_i only. Denote by Y^i the σ-field generated by all admissible observations gathered up to time i, i.e. $Y^i = \sigma\{y_j : j \leq i\}$. Then it seems natural to assume that the admissible control a_i, in a generic period i, is a U-valued, Y^i-adapted random variable. The control of partially observable models defined above has been the subject of intensive studies for the case of finite or discounted cost functionals (see e.g. [3], [7], [13] and references therein).

In this paper we are interested in optimization of the ergodic cost, i.e. average cost per unit time functional. Problems of this type are very hard to study and only few positive results are obtained so far (see [11], [16] as well as other references below). To guarantee nice ergodic properties of the so-called filtering process (defined below), in what follows we shall assume that in a certain domain $\Gamma \subset E$, which is a compact set, the process (x_i) is completely observed. Outside of Γ, the observation is partial with known density r of its distribution, that is, for the observation process y_i and $Y^i = \sigma\{y_j : j \leq i\}$, $Y^0 = \{\emptyset, \Omega\}$, we have

(A1) $$P\{y_{i+1} \in A \mid x_0, \ldots, x_{i+1}, Y^i\} = \chi_{A\cap\Gamma}(x_{i+1}) + \chi_{\Gamma^c}(x_{i+1}) \int_{A\cap\Gamma^c} r(x_{i+1}, y)\, dy$$

for $i = 0, 1, \ldots$, and any $A \in \mathcal{B}(E)$, the σ-field of Borel subsets of E; we have denoted by Γ^c the set $E \setminus \Gamma$.

An observation of the form (A1) arises in the situation when we are monitoring the system through a "window", i.e. a bounded domain Γ in which due to the accuracy of the measurements we have complete observation of the state and signals that enter our detectors outside of Γ, or measurements outside of the range of Γ are noisy. In particular, (A1) is satisfied in a singular observation model considered in Section 6. Notice that if our observation depends continuously on the value of the state process, we have to admit unbounded values of r, since close to the boundary of Γ the observation measure generated by r approaches the Dirac delta measure.

Although we are interested in the case when the state space E is uncountable, one can easily adapt all results formulated in the paper to the case of E countable replacing integrals by suitable summations.

Define the filtering process (π_i) corresponding to (x_i) with initial law μ and observation (y_i), as a measure-valued process on the space $P(E)$ of probability measures on E, endowed with the topology of weak convergence, as follows:

$$\pi_0^\mu(A) = \mu(A), \quad \pi_i^\mu(A) = P_\mu\{x_i \in A \mid Y^i\}. \tag{1}$$

In order to obtain our results, since we are interested in control over an infinite horizon, and will apply weak convergence techniques, we shall assume that the controls a_i are of the form

$$a_i = u(\pi_i^\mu) \tag{2}$$

with $u : P(E) \to U$ a continuous function. In what follows we shall denote by $\mathcal{A}$ the class of continuous functions from $P(E)$ into U.

Being interested in ergodic stochastic control, for a given continuous bounded function $c : E \times U \to \mathbb{R}^+ = [0, \infty)$ we consider as objective function to be minimized the functional

$$J_\mu(u) = \limsup_{n\to\infty} n^{-1} E_\mu\Big\{\sum_{i=0}^{n-1} c(x_i, u(\pi_i^{\mu,u}))\Big\}, \tag{3}$$

where we point out the dependence of our filtering process (π_i^μ) on the control function $u \in \mathcal{A}$.

By suitable discretizations of the state space and the space control parameters we construct a nearly optimal control function $u \in \mathcal{A}$ for the cost functional $J_\mu(u)$.

Since generally the true filtering process $(\pi_i^{\mu,u})$ cannot be computed to make the procedure feasible we show that for the controls a_i of the form (2), where u is a nearly optimal control function and π_i^μ is replaced by a computable approximating filtering process, the cost functional $J_\mu((a_i))$ does not exceed the optimal value of $J_\mu(u)$ over $u \in \mathcal{A}$ plus a given small $\varepsilon > 0$.

Notice that since our main tool is weak convergence techniques, similarly to the case of other stochastic control problems (see e.g. [9]) we are not able to compute any approximation errors. However, the approach presented in the paper seems to be the only method that allows us, provided we use a sufficiently fine discretization, to construct nearly optimal control values.

Optimal ergodic control of a partially observed Markov chain with bounded cost function c has been the subject of several papers ([2], [5], [8], [11], [12], [16], [17], see also [1] and references therein). However, only in [11] and [12] a general state space was considered. The approach of [11] and [12] was based on a regeneration property of the controlled filtering process and additive noise observation structure which assure the existence of a unique invariant measure for the controlled filtering process. Below, thanks to a particular observation structure we obtain the existence of a unique invariant measure for the controlled filtering process (see Lemma 3) without restrictive regeneration assumptions of [11] and [12]. Moreover, we allow the observation density $r(x,y)$ to be of a general form and the observation measure generated by r to converge, close to the boundary of Γ, to the Dirac delta measure.

The general scheme of the paper is similar to that of [11]. Namely, in Section 3 we prove a fundamental theorem on approximations of the invariant measure, which is applied in Section 4 to construct a nearly optimal control function. In Section 5 we construct an approximating filtering process π_i^m such that for m sufficiently large, the control which is the value of the nearly optimal control function at π_i^m is nearly optimal. A class of examples for which all assumptions of the paper are satisfied is constructed in Section 6. Since we do not have a regeneration property and allow the observation noise to be of a general nature, we cannot apply the results of [11]. Furthermore, we assume the continuity of the observation structure, which requires the unboundedness of the observation density $r(x,y)$, for x, y close to Γ, and which in turn creates additional difficulties in the proofs.

2. Preliminary results and assumptions

We start with an adaptation of Lemmas 1 and 2 of [16] to the observation structure given by (A1).

LEMMA 1. *Assume the filtering process π_i^μ is controlled with control a_i adapted to Y^i in a generic period i. Then, under* (A1), *for $i=0,1,2,\ldots$ and $A\in\mathcal{B}(E)$, we have P_μ-a.e.*

$$\begin{aligned}(4)\qquad \pi_{i+1}^\mu(A) &= \chi_{A\cap\Gamma}(y_{i+1}) + \chi_{\Gamma^c}(y_{i+1}) \\ &\quad\times \int_{A\cap\Gamma^c} r(z,y_{i+1})\,P^{a_i}(\pi_i^\mu,dz)\Big(\int_{\Gamma^c} r(z,y_{i+1})\,P^{a_i}(\pi_i^\mu,dz)\Big)^{-1} \\ &=: M^{a_i}(y_{i+1},\pi_i^\mu)(A) =: \chi_{A\cap\Gamma}(y_{i+1}) + \chi_{\Gamma^c}(y_{i+1})N^{a_i}(y_{i+1},\pi_i^\mu)(A),\end{aligned}$$

where we implicitly defined the operators M and N, and where $P^a(\nu,dz)=\int_E P^a(z_1,dz)\,\nu(dz_1)$ for $a\in U$ and $\nu\in P(E)$.

Proof. We follow the proof of Lemma 1 of [16]. For $A,B\in\mathcal{B}(E)$ and ψ_i a Y^i-measurable bounded random variable we have

$$\begin{aligned}&\int_\Omega M^{a_i}(y_{i+1},\pi_i^\mu)(B)\chi_A(y_{i+1})\psi_i\,dP_\mu \\ &= \int_\Omega M^{a_i}(x_{i+1},\pi_i^\mu)(B)\chi_{A\cap\Gamma}(x_{i+1})\psi_i\,dP_\mu\end{aligned}$$

$$+\int_{\Omega} \chi_{\Gamma^c}(x_{i+1}) \int_{A\cap\Gamma^c} M^{a_i}(y,\pi_i^\mu)(B) r(x_{i+1},y)\,dy\,\psi_i\,dP_\mu$$

$$=\int_{\Omega} \chi_{B\cap\Gamma}(x_{i+1})\chi_{A\cap\Gamma}(x_{i+1})\psi_i\,dP_\mu$$

$$+\int_{\Omega}\int_{A\cap\Gamma^c} M^{a_i}(y,\pi_i^\mu)(B)\int_{\Gamma^c} r(z,y)\,P^{a_i}(\pi_i^\mu,dz)\,dy\,\psi_i\,dP_\mu$$

$$=\int_{\Omega} \chi_{B\cap\Gamma}(x_{i+1})\chi_{A}(x_{i+1})\psi_i\,dP_\mu$$

$$+\int_{\Omega}\int_{A\cap\Gamma^c}\int_{B\cap\Gamma^c} r(z,y)P^{a_i}(\pi_i^\mu,dz)\,dy\,\psi_i\,dP_\mu$$

$$=\int_{\Omega} \chi_{B\cap\Gamma}(x_{i+1})\chi_{A}(y_{i+1})\psi_i\,dP_\mu+\int_{\Omega} \chi_{B\cap\Gamma^c}(x_{i+1})\chi_{A\cap\Gamma^c}(y_{i+1})\psi_i\,dP_\mu$$

$$=\int_{\Omega} \chi_{B}(x_{i+1})\chi_{A}(y_{i+1})\,\psi_i\,dP_\mu$$

and by the definition of conditional expectation we obtain (4). ■

LEMMA 2. *Assume* (A1) *holds, and* $a_i = u(\pi_i^\mu)$ *for* $i = 0,1,2,\ldots,$ *where* $u : P(E) \to U$ *is a fixed Borel measurable function. Then* $(\pi_i^{\mu,u})$ *is a* Y^i *Markov process with transition operator*

$$\Pi^u(\nu,F)=\int_{\Gamma} F(\delta_z)\,P^{u(\nu)}(\nu,dz)+\int_{\Gamma^c}\int_{\Gamma^c} r(z,y)F(M^u(y,\nu))P^{u(\nu)}(\nu,dz)\,dy \tag{5}$$

where $\nu\in P(E)$, F *is a bounded Borel measurable function on* $P(E)$, *and by* $M^u(y,\nu)$ *we denote* $M^{u(\nu)}(y,\nu)$.

Proof. Similarly to the proof of Lemma 2 of [16] we have

$$E_\mu\{F(\pi_{i+1}^{\mu,u})\mid Y^i\}=E_\mu\Big\{\chi_\Gamma(x_{i+1})F(M^u(x_{i+1},\pi_i^{\mu,u}))+\chi_{\Gamma^c}(x_{i+1})\int_{\Gamma^c} r(x_{i+1},y)F(M^u(y,\pi_i^{\mu,u}))\,dy\,\Big|\,Y^i\Big\}$$

$$=\int_{\Gamma} F(\delta_z)\,P^{u(\pi_i^{\mu,u})}(\pi_i^{\mu,u},dz)+\int_{\Gamma^c}\int_{\Gamma^c} r(z,y)F(M^u(y,\pi_i^{\mu,u}))\,dy\,P^{u(\pi_i^{\mu,u})}(\pi_i^{\mu,u},dz),$$

from which (5) follows. ■

The operator Π^u is said to have the *Feller property* if it maps continuous bounded functions $F : P(E) \to \mathbb{R}$ into the same. Our next result gives precisely this property for Π^u, under the following assumptions:

(A2) for fixed $a \in U$, $P^a(x,\cdot)$ is Feller,

(A3) if $U \ni a_m \to a$, then $P^{a_m}(x,\cdot) \Rightarrow P^a(x,\cdot)$ uniformly in x from compact subsets of E, where $\Rightarrow$ stands for weak convergence in $P(E)$,

(A4) $r(z,y)$ is continuous for $z, y \in \Gamma^c$, and is bounded on the set $\Gamma^c_\delta = \{(z,y) \in \Gamma^c \times \Gamma^c\colon\ \varrho_E(z,\Gamma) \ge \delta\} \cup \{(z,y) \in \Gamma^c \times \Gamma^c\colon\ \varrho_E(z,y) \ge \delta\}$, for $\delta > 0$, with ϱ_E standing for a metric on E compatible with the topology. Moreover, if $\Gamma^c \ni y_m \to y \in \Gamma$ and $B(y,\delta) = \{z \in \Gamma^c : \varrho_E(z,y) \le \delta\}$ for $\delta > 0$, then

$$\inf_{a\in U}\inf_{x\in K}\int\limits_{B(y,\delta)} r(z,y_m)\,P^a(x,dz) \to \infty$$

as $m \to \infty$, for any compact subset $K \subset E$,

(A5) if $\Gamma^c \ni z_m \to z$, then $R(z_m,\cdot) \Rightarrow R(z,\cdot)$ as $m \to \infty$ with

$$R(z,A) := \begin{cases} \int_{A\cap\Gamma^c} r(z,y)\,dy & \text{for } z \in \Gamma^c,\\ \chi_A(z) & \text{for } z \in \Gamma,\end{cases}$$

for $A \in \mathcal{B}(E)$,

(A6) for $x \in E$ and $a \in U$ we have $P^a(x,\partial\Gamma) = 0$ with $\partial\Gamma$ being the boundary of Γ.

Let us notice that by (A5) our observation measure $R(x,\cdot)$ depends continuously on the state x, and therefore the observation density $r(x,y)$ has to converge to infinity as y becomes close to x with x close to the boundary of Γ.

In what follows we shall assume that (A1) is satisfied and $u_i = u(\pi_i^{\mu,u})$ for some $u \in \mathcal{A}$.

PROPOSITION 1. *Under* (A1)–(A4), (A6), *for* $u \in \mathcal{A}$ *the mapping*

$$\overline{\Gamma^c} \times P(E) \ni (y,\nu) \mapsto N^u(y,\nu) \quad \textit{is continuous}, \tag{6}$$

where $\overline{\Gamma^c}$ *is the closure of* Γ^c, *and* $N^u(y,\nu)(A) = \chi_A(y)$ *for* $y \in \partial\Gamma$. *Moreover, assuming additionally* (A5), *for* $F \in C(P(E))$, *the space of continuous bounded functions on* $P(E)$, *and* $u \in \mathcal{A}$ *we have* $\Pi^u(\nu,F) \in C(P(E))$.

P r o o f. Notice first that by (A2) and (A3) the mapping

$$P(E) \ni \nu \mapsto P^{u(\nu)}(\nu,\cdot) \in P(E) \tag{7}$$

is continuous. In fact, for $\varphi \in C(E)$ and $\nu_n \Rightarrow \nu$, we have

$$|P^{u(\nu_n)}(\nu_n,\varphi)-P^{u(\nu)}(\nu,\varphi)| \le \Big|\int\limits_K (P^{u(\nu_n)}(z,\varphi)-P^{u(\nu)}(z,\varphi))\,\nu_n(dz)\Big| + 2\|\varphi\|\varepsilon + \Big|\int\limits_{\mathbb{R}^d} P^{u(\nu)}(z,\varphi)\,(\nu_n(dz)-\nu(dz))\Big| = \mathrm{I}_n+\mathrm{II}_n+\mathrm{III}_n,$$

where K is a compact subset of E such that $\nu_n(K),\nu(K)\ge 1-\varepsilon$ and $\|\varphi\| = \sup_{z\in E}|\varphi(z)|$. By (A3), $\mathrm{I}_n\to 0$, by (A2), $\mathrm{III}_n\to 0$ as $n\to\infty$, and because of the tightness of ν_n, ε can be chosen arbitrarily small. Thus (7) holds.

To show (6) assume first that $\Gamma^c\ni y_n\to y\in\Gamma^c$ and $P(E)\ni\nu_n\Rightarrow\nu$.

It is sufficient to prove that for $\varphi\in C(E)$,

$$\int\limits_{\Gamma^c} r(z,y_n)\varphi(z)\,P^{u(\nu_n)}(\nu_n,dz)\to\int\limits_{\Gamma^c} r(z,y)\varphi(z)\,P^{u(\nu)}(\nu,dz)$$

as $n\to\infty$.

Letting $r(z,y)=0$ for $z\in\Gamma$ and $y\in E$, we have

$$\Big|\int\limits_{\Gamma^c} r(z,y_n)\varphi(z)\,P^{u(\nu_n)}(\nu_n,dz)-\int\limits_{\Gamma^c} r(z,y)\varphi(z)\,P^{u(\nu)}(\nu,dz)\Big| \tag{8}$$
$$\le\Big|\int\limits_{\Gamma^c\cap K}(r(z,y_n)-r(z,y))\varphi(z)\,P^{u(\nu_n)}(\nu_n,dz)\Big| + \varepsilon\|\varphi\|(\|r\|_{y_n}+\|r\|_y) + \Big|\int\limits_E \chi_{\Gamma^c}(z)\varphi(z)r(z,y)\,(P^{u(\nu_n)}(\nu_n,dz)-P^{u(\nu)}(\nu,dz))\Big| = \mathrm{I}_n+\mathrm{II}_n+\mathrm{III}_n,$$

where by the tightness of $P^{u(\nu_n)}(\nu_n,\cdot)$ (which follows from (7)) we have chosen a compact set $K\subset E$ such that

$$P^{u(\nu_n)}(\nu_n,K)\ge 1-\varepsilon$$

and we denote by $\|r\|_y$ the supremum of $r(z,y)$ over $z\in\Gamma^c$. By the continuity of r (see (A4)) and the boundedness of $\|r\|_{y_n}$ (since $y_n\to y\notin\partial\Gamma$), I_n converges to 0. Since by (7), (A6) and Theorem 1.2.1(v) of [4], III_n goes to 0 as $n\to\infty$, we obtain (6), for $y\in\Gamma^c$. By (A4), if $\Gamma^c\ni y_n\to y\in\Gamma$ and $\nu_n\Rightarrow\nu$ we have $N^u(y_n,\nu_n)(\varphi)\to\varphi(y)$. Therefore $N^u(y,\nu)$ is continuous for $y\in\Gamma^c$ and $\nu\in P(E)$.

It remains to show the Feller property of Π^u.

Let $F\in C(P(E))$ and $P(E)\ni\nu_n\Rightarrow\nu$. By (7) and (A6),

$$\int\limits_\Gamma F(\delta_z)\,(P^{u(\nu_n)}(\nu_n,dz)-P^{u(\nu)}(\nu,dz))\to 0\quad\text{as } n\to\infty.$$

Therefore we have to show the continuity in ν of the second part of $\Pi^u(\nu,F)$, i.e.

the convergence

$$\Big|\int_{\Gamma^c}\int_{\Gamma^c} r(z,y)F(M^u(y,\nu_n))\,dy\,P^{u(\nu_n)}(\nu_n,dz) - \int_{\Gamma^c}\int_{\Gamma^c} r(z,y)F(M^u(y,\nu))\,dy\,P^{u(\nu)}(\nu,dz)\Big| \to 0 \quad \text{as } n\to\infty.$$

For any $\varepsilon > 0$ one can find by (7) a compact set $K \subset E$ such that

$$P^{u(\nu_n)}(\nu_n, K) \ge 1-\varepsilon \quad \text{for } n = 1,2,\ldots$$

Given K by (A5) there is a compact set $L \subset E$ such that

$$\int_{L\cap\Gamma^c} r(z,y)\,dy \ge 1-\varepsilon \quad \text{for } z \in K\cap\Gamma^c.$$

Therefore

$$\begin{aligned}
&\Big|\int_{\Gamma^c}\int_{\Gamma^c} r(z,y)F(M^u(y,\nu_n))\,dy\,P^{u(\nu_n)}(\nu_n,dz) - \int_{\Gamma^c}\int_{\Gamma^c} r(z,y)F(M^u(y,\nu))\,dy\,P^{u(\nu)}(\nu,dz)\Big| \\
&\le \int_{\Gamma^c}\int_{\Gamma^c} r(z,y)|F(M^u(y,\nu_n)) - F(M^u(y,\nu))|\,dy\,P^{u(\nu_n)}(\nu_n,dz) \\
&\quad + \Big|\int_{\Gamma^c}\int_{\Gamma^c} r(z,y)F(M^u(y,\nu))\,dy\,(P^{u(\nu_n)}(\nu_n,dz) - P^{u(\nu)}(\nu,dz))\Big| \\
&\le \int_{\Gamma^c\cap K}\int_{\Gamma^c\cap L} r(z,y)|F(N^u(y,\nu_n)) - F(N^u(y,\nu))|\,dy\,P^{u(\nu_n)}(\nu_n,dz) + 4\|F\|\varepsilon \\
&\quad + \Big|\int_E \chi_{\Gamma^c}(z)\int_{\Gamma^c} r(z,y)F(N^u(y,\nu))\,dy\,(P^{u(\nu_n)}(\nu_n,dz) - P^{u(\nu)}(\nu,dz))\Big| \\
&= \mathrm{I}_n + \mathrm{II}_n + \mathrm{III}_n.
\end{aligned}$$

By (6), $\mathrm{I}_n \to 0$, and from (6), (7) and (A5), (A6) also $\mathrm{III}_n \to 0$. Finally, $\Pi^u(\nu,F)$ is continuous in ν. ∎

Remark 1. If instead of (A4) we assume

(A4′) $r(x,y)$ is continuous and bounded on Γ^c,

then the mapping (6) is continuous for $y \notin \partial\Gamma$. This is the case when at the boundary $\partial\Gamma$ we have an abrupt change in the precision of observation and the observation structure is therefore discontinuous. Assuming additionally (instead of (A5))

(A5′) if $z_m \to z \in \Gamma^c$, then $R(z_m, \cdot) \Rightarrow R(z, \cdot)$ as $m \to \infty$, and for any compact set $K \subset E$ the family $\{R(z, \cdot) : z \in K \cap \Gamma^c\}$ is tight,

we obtain the Feller property for the transition operator $\Pi^u(\nu, F)$. Although as we pointed out in the introduction we are interested in the case of unbounded $r(x, y)$, it is worth noticing here that all results below hold for the case when (A4) and (A5) are replaced by (A4′) and (A5′) respectively.

Under a Markovian control of the form (2) the cost functional (3) can be rewritten as

$$J_\mu(u) = \limsup_{n\to\infty} n^{-1} E_\mu \Big\{ \sum_{i=0}^{n-1} \int_E c(z, u(\pi_i^{\mu,u}))\, \pi_i^{\mu,u}(dz) \Big\}, \tag{9}$$

and the partially observed ergodic control problem is then reduced to the ergodic control problem for a completely observed Feller Markov process $(\pi_i^{\mu,u})$. To solve the latter problem we need some ergodic properties of $(\pi_i^{\mu,u})$. For this purpose we assume

(A7) there is a compact set $\Gamma_1 \subset \Gamma$ such that

(i) for $a \in U$ and $x \in E$,

$$P^a(x, \partial\Gamma_1) = 0, \tag{10}$$

(ii) for $\mu \in P(E)$ and $u \in \mathcal{A}$,

$$E_\mu^u T_{\Gamma_1} < \infty, \tag{11}$$

where $T_{\Gamma_1} = \inf\{s \ge 0 : x_s \in \Gamma_1\}$ and E_μ^u stands for the conditional expectation of the filtering process starting from μ with control $u \in \mathcal{A}$,

(iii) for $\tau = T_{\Gamma^c} + T_{\Gamma_1} \circ \Theta_{T_{\Gamma^c}}$, where Θ_t stands for the Markov shift operator corresponding to the state process,

$$\sup_{x\in\Gamma_1} \sup_{u\in\mathcal{A}} E_x^u \tau^2 < \infty, \tag{12}$$

(iv) if we let $\tau_1 = \tau$, $\tau_{n+1} = \tau_n + \tau \circ \Theta_{\tau_n}$, then the embedded Markov chain $(\pi_{\tau_n}^{\mu,u}) = (x_{\tau_n})$ has a unique invariant measure η^u and the strong law of large numbers holds for (x_{τ_n}).

In other words, τ is the first time of return to Γ_1 after hitting Γ^c.

R e m a r k 2. To guarantee (12) we usually impose Lyapunov type conditions. In particular, (12) is satisfied when

$$\inf_{a\in U} \inf_{x\in E} P^a(x, \Gamma_1) > 0 \quad \text{and} \quad \inf_{a\in U} \inf_{x\in E} P^a(x, \Gamma^c) > 0.$$

R e m a r k 3. Since Γ_1 is compact, if $P_x^u\{x_\tau \in \cdot\}$ is Feller for $x \in \Gamma_1$ and $u \in \mathcal{A}$, then there is an invariant measure η^u for (x_{τ_n}). However, it is not necessarily unique.

R e m a r k 4. It follows from Theorem 6.2 of Chapter 5 of [6] and its proof that the law of large numbers is satisfied when (x_{τ_n}) is uniformly ergodic, or more generally when there are no invariant sets of η^u measure zero.

LEMMA 3. *Under* (A1), (A7), *for any* $u \in \mathcal{A}$ *there is a unique invariant measure* Φ^u *for* $(\pi_i^{\mu,u})$, *and it is of the form*

$$\Phi^u(F) = \int_{\Gamma_1} E_x^u\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{\delta_x,u})\Big\}\,\eta^u(dx)\,\Big(\int_{\Gamma_1} E_x^u\{\tau\}\,\eta^u(dx)\Big)^{-1} \tag{13}$$

for any $F \in b\mathcal{B}(P(E))$, *the family of bounded Borel functions on* $P(E)$. *Moreover*, *for* $F \in b\mathcal{B}(P(E))$ *and* $\mu \in P(E)$ *we have*

$$\lim_{n\to\infty} n^{-1} E_\mu\Big\{\sum_{i=0}^{n-1} F(\pi_i^{\mu,u})\Big\} = \Phi^u(F). \tag{14}$$

P r o o f. Since η^u is invariant for $(\pi_{\tau_n}^{\mu,u})$, we have

$$\int_{\Gamma_1} E_x^u\Big\{\sum_{i=0}^{\tau-1} \Pi^u F(\pi_i^{\delta_x,u})\Big\}\,\eta^u(dx) = \int_{\Gamma_1} E_x^u\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{\delta_x,u})\Big\}\,\eta^u(dx),$$

where $\Pi^u F(\nu) = \Pi^u(\nu, F)$ and clearly Φ^u is an invariant measure for $(\pi_i^{\mu,u})$. It remains to show (14) from which the uniqueness of Φ^u follows.

For $g \in b\mathcal{B}(P(E))$ by the law of large numbers for martingales we have

$$\lim_{n\to\infty} n^{-1}\Big(\sum_{i=0}^{\tau_n-1} g(\pi_i^{\delta_x,u}) - \sum_{i=0}^{n-1} E_{x_{\tau_i}}\Big[\sum_{j=0}^{\tau-1} g(\pi_j^{\delta_{x_{\tau_i}},u})\Big]\Big) = 0 \qquad P_x\text{-a.e.}$$

for $x \in \Gamma_1$.

Now, by the law of large numbers for (x_{τ_n}),

$$\lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} E_{x_{\tau_i}}\Big[\sum_{j=0}^{\tau-1} g(\pi_j^{\delta_{x_{\tau_i}},u})\Big] = \int_{\Gamma_1} E_x\Big\{\sum_{j=0}^{\tau-1} g(\pi_j^{\delta_x,u})\Big\}\,\eta^u(dx) \qquad P_x\text{-a.e.}$$

for $x \in \Gamma_1$.

Letting $g \equiv F$ or $g \equiv 1$ in the above two identities we obtain

$$\lim_{n\to\infty} \tau_n^{-1}\sum_{i=0}^{\tau_n-1} F(\pi_i^{\delta_x,u}) = \Phi^u(F) \qquad P_x\text{-a.e.} \tag{15}$$

for $x \in \Gamma_1$. Since it is straightforward to verify that

$$\lim_{n\to\infty} \tau_n^{-1}\sum_{i=0}^{\tau_n-1} F(\pi_i^{\delta_x,u}) = \lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} F(\pi_i^{\delta_x,u}) \qquad P_x\text{-a.e.}$$

for $x \in \Gamma_1$, from (15) and (11) we obtain (14). ∎

Remark 5. If $u : P(E) \to U$ is Borel measurable, and η^u is an invariant measure for (x_{τ_n}) and

$$\sup_{x\in\Gamma_1} E^u_x \tau < \infty,$$

then Φ^u defined by (13) is an invariant measure for $(\pi_i^{\mu,u})$.

By Lemma 3 we easily obtain

COROLLARY 1. *Under the assumptions of Lemma* 3,

$$J_\mu(u) = \int_{P(E)} \int_E c(z, u(\nu))\, \nu(dz)\, \Phi^u(d\nu) \tag{16}$$

for $u \in \mathcal{A}$.

3. Approximation of the invariant measure

By (16) it is clear that the approximation of the cost functional $J_\mu(u)$ can be now reduced to the approximation of the invariant measure Φ^u.

Below, we formulate our fundamental approximation result which will be used in Section 4 to construct nearly optimal control functions. We approximate (x_i) by (x_i^m), which is a controlled Markov process with transition operator $P_m^{a_i}$, control function u_m and the density of observation r_m. We assume

(D1) for $U \ni a_m \to a$, we have $P_m^{a_m}(x,\cdot) \Rightarrow P^a(x,\cdot)$, uniformly in x from compact subsets of E,

(D2) $u_m : P(E) \to U$ are Borel measurable and $u_m(\nu) \to u(\nu)$ with $u \in \mathcal{A}$ as $m \to \infty$, uniformly in ν from compact subsets of $P(E)$,

(D3) (i) $r_m \in b\mathcal{B}(\Gamma^c \times \Gamma^c)$, $r_m(z,y) \to r(z,y)$ uniformly in (z,y) from compact subsets of $\Gamma^c \times \Gamma^c$ as $m \to \infty$, and $r_m(z,y)$ are uniformly (in m) bounded on Γ^c_δ for $\delta > 0$,

(ii) for any compact $K \subset \Gamma^c$,

$$\sup_{z\in K} \int_{\Gamma^c} |r(z,y) - r_m(z,y)|\, dy \to 0 \quad \text{as } m \to \infty,$$

(iii) if $\Gamma^c \ni y_m \to y \in \Gamma$, then for $\delta > 0$ and any compact set $K \subset E$,

$$\inf_{a\in U} \inf_{x\in K} \int_{B(y,\delta)} r_m(z, y_m)\, P^a_m(x, dz) \to \infty \quad \text{as } m \to \infty,$$

with $B(y,\delta) = \{z \in \Gamma^c : \varrho_E(z,y) < \delta\}$,

(D4) for any compact set $K \subset E$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$P^a_m(z, \Gamma(\delta)) < \varepsilon \quad \text{for } a \in U,\ z \in K,\ m = 1, 2, \ldots,$$

with $\Gamma(\delta) = \{z \in \Gamma^c : \varrho_E(z, \Gamma) < \delta\}$, (D5) for the compact set Γ_1 of the assumption (A7) and the Markov times T_{Γ_1}, τ, τ_n, defined in (A7), considered now as Markov times with respect to (x_i^m), we have

(i) $E_\mu^{u_m,m} T_{\Gamma_1} < \infty$ for $\mu \in P(E)$,

(ii) $\sup_{x\in\Gamma_1} \sup_m E_x^{u_m,m} \tau^2 < \infty$ with $E_x^{u_m,m}$ standing for conditional expectation of (x_i^m) starting from x, and controlled in a generic period i by $u(\pi_i^{m,\delta_x,u_m})$, where π_i^{m,δ_x,u_m} is the filtering process corresponding to (x_i^m),

(iii) for any u_m of (D2) there exists an invariant measure $\eta_m^{u_m}$ for $(x_{\tau_n}^{u_m})$, the embedded Markov chain in (x_i^m) controlled in a generic period i by $u_m(\pi_i^{m,\delta_x,u_m})$.

Under the above assumptions, provided the observation structure of x_i^m is of the form (A1) with r replaced by r_m, and in a generic period i the control $u(\pi_i^{m,\mu,u})$ is applied, by Remark 5 there is an invariant measure Φ^{u_m} for π_i^{m,μ,u_m} of the form

$$\Phi_m^{u_m}(F) = \int_{\Gamma_1} E_x^{u_m,m}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{m,\delta_x,u_m})\Big\}\,\eta_m^{u_m}(dx) \times \Big(\int_{\Gamma_1} E_x^{u_m,m}\{\tau\}\,\eta_m^{u_m}(dx)\Big)^{-1} \tag{17}$$

for any $F \in b\mathcal{B}(P(E))$.

The main result of this section can be formulated as follows:

THEOREM 1. *Let* (A1)–(A7) *and* (D1)–(D5) *be satisfied. Then*

$$\Phi_m^{u_m} \Rightarrow \Phi^u \quad \textit{weakly on } P(P(E)) \textit{ as } m\to\infty.$$

Proof. We divide the proof into a sequence of lemmas and corollaries.

LEMMA 4. *Under* (D1) *and* (D2),

$$P_m^{u_m(\nu)}(\nu,\cdot) \Rightarrow P^{u(\nu)}(\nu,\cdot) \quad \textit{as } m\to\infty,$$

uniformly in ν from compact subsets of $P(E)$.

Proof. Denote by ϱ_U a metric on U. By (D1) for $\varphi\in C(E)$ for each compact set $K\subset E$ and $\varepsilon>0$ one can find $\delta>0$ and a positive integer N such that for $m>N$,

$$\text{if } \varrho_U(a,a') < \delta \text{ for } a,a'\in U \text{ then } \sup_{x\in K} |P_m^a(x,\varphi) - P^{a'}(x,\varphi)| < \varepsilon. \tag{18}$$

Let $H\subset P(E)$ be a compact set. By tightness of H for each $\varepsilon>0$ there is a compact set $K\subset E$ such that $\nu(K)\ge 1-\varepsilon\|\varphi\|^{-1}$ for $\nu\in H$.

Given $\varepsilon>0$ and a compact set $K\subset E$ with the above property we choose $\delta>0$ for which (18) holds.

By (D2),

$$\exists_{N'}\forall_{m\geq N'}\quad \sup_{\nu\in H}\varrho_U(u_m(\nu),u(\nu))<\delta.$$

Therefore

$$\begin{aligned}&|P_m^{u_m(\nu)}(\nu,\varphi)-P^{u(\nu)}(\nu,\varphi)|\\&\quad\leq 2\varepsilon+\sup_{x\in K}|P_m^{u_m(\nu)}(x,\varphi)-P^{u(\nu)}(x,\varphi)|\leq 3\varepsilon\quad \text{for } m\geq\max\{N,N'\},\end{aligned}$$

from which the assertion of Lemma 4 follows. ∎

There are two important consequences of Lemma 4.

Corollary 2. *Let* (D1), (D2) *and* (A2)–(A3) *be satisfied. For* $f\in C(E)$ *and set* $A\in\mathcal{B}(E)$ *such that*

$$\sup_{x\in E}\sup_{a\in U}P^a(x,\partial A)=0 \tag{19}$$

we have

$$P_m^{u_m(\nu)}(\nu,f\chi_A)\to P^{u(\nu)}(\nu,f\chi_A) \tag{20}$$

as $m\to\infty$, *uniformly in* ν *from compact subsets of* $P(E)$.

Proof. Assume (20) does not hold, i.e. for $\nu_m\Rightarrow\nu$ we have

$$|P_m^{u_m(\nu_m)}(\nu_m,f\chi_A)-P^{u(\nu_m)}(\nu_m,f\chi_A)|\geq\delta>0. \tag{21}$$

By (A2) and (A3), $P^{u(\nu_m)}(\nu_m,\cdot)\Rightarrow P^{u(\nu)}(\nu,\cdot)$. Therefore for any $f_1\in C(E)$, by Lemma 4,

$$\begin{aligned}&|P_m^{u_m(\nu_m)}(\nu_m,f_1)-P^{u(\nu)}(\nu,f_1)|\\&\quad\leq|P_m^{u_m(\nu_m)}(\nu_m,f_1)-P^{u(\nu_m)}(\nu_m,f_1)|+|P^{u(\nu_m)}(\nu_m,f_1)-P^{u(\nu)}(\nu,f_1)|\to 0\end{aligned}$$

and $P_m^{u_m(\nu_m)}(\nu_m,\cdot)\Rightarrow P^{u(\nu)}(\nu,\cdot)$. By Theorem 1.2.1(v) of [4],

$$P_m^{u_m(\nu_m)}(\nu_m,f\chi_A)\to P^{u(\nu)}(\nu,f\chi_A)$$

and

$$P^{u(\nu_m)}(\nu_m,f\chi_A)\to P^{u(\nu)}(\nu,f\chi_A)\quad \text{as } m\to\infty,$$

a contradiction to (21). Thus (20) is satisfied. ∎

Using the tightness arguments we obtain the following

Corollary 3. *Under* (D1), (D2), (A2), (A3), *for each compact subset* H *of* $P(E)$ *and* $\varepsilon>0$, *there is a positive integer* m_0 *and a compact set* $K\subset E$ *such that for* $\nu\in H$ *and* $m\geq m_0$,

$$P_m^{u_m(\nu)}(\nu,K)\geq 1-\varepsilon,\qquad P^{u(\nu)}(\nu,K)\geq 1-\varepsilon. \tag{22}$$

Proof. By (A2) and (A3) the family of measures $\{P^{u(\nu)}(\nu,\cdot):\nu\in H\}$ is compact in $P(E)$. Therefore there is a compact set $\overline{K}\subset E$ such that

$$P^{u(\nu)}(\nu,\overline{K})\geq 1-\varepsilon/2\quad \text{for } \nu\in H.$$

Let

$$\varphi(x)=\begin{cases}1-\inf_{z\in\bar K}\varrho_E(x,z) & \text{if } \inf_{z\in\bar K}\varrho_E(x,z)\le 1,\\ 0 & \text{elsewhere,}\end{cases}$$

where ϱ_E denotes a metric compatible with the topology of E. Clearly $\varphi\in C(E)$ and by Lemma 4 for $m\ge m_0$,

$$\sup_{\nu\in H}|P_m^{u_m(\nu)}(\nu,\varphi)-P^{u(\nu)}(\nu,\varphi)|\le\varepsilon/2.$$

Thus for $K=\{x\in E:\inf_{z\in\bar K}\varrho_E(x,z)\le 1\}$, $m\ge m_0$ and $\nu\in H$ we have

$$\begin{aligned}P_m^{u_m(\nu)}(\nu,K)&\ge P_m^{u_m(\nu)}(\nu,\varphi)\ge P^{u(\nu)}(\nu,\varphi)-\varepsilon/2\\&\ge P^{u(\nu)}(\nu,\overline{K})-\varepsilon/2\ge 1-\varepsilon,\end{aligned}$$

which was our claim. ∎

Before we formulate our next lemma, by analogy to (4) we define the measures $N_m^u(y,\nu)(A)=\chi_A(y)$ for $y\in\Gamma$,

$$N_m^u(y,\nu)(A) := \int_{A\cap\Gamma^c} r_m(z,y)\,P_m^{u(\nu)}(\nu,dz)\Big\{\int_{\Gamma^c} r_m(z,y)\,P_m^{u(\nu)}(\nu,dz)\Big\}^{-1}\quad\text{for } y\in\Gamma^c$$

and

$$M_m^u(y,\nu)(A):=\chi_{A\cap\Gamma}(y)+\chi_{\Gamma^c}(y)N_m^u(y,\nu)(A).$$

LEMMA 5. *Assume* (A2)–(A6) *and* (D1)–(D3). *Then*

$$N_m^{u_m}(y,\nu)\Rightarrow N^u(y,\nu)\quad\text{as } m\to 0$$

uniformly on compact subsets of $\overline{\Gamma^c}\times P(E)$.

Proof. It suffices to show that for $\overline{\Gamma^c}\ni y_m\to y$ and $P(E)\ni\nu_m\Rightarrow\nu$ we have

$$|N_m^{u_m(\nu_m)}(y_m,\nu_m)(\varphi)-N^{u(\nu_m)}(y_m,\nu_m)(\varphi)|\to 0$$

for any $\varphi\in C(E)$. Since by Proposition 1,

$$N^{u(\nu_m)}(y_m,\nu_m)(\varphi)\to N^{u(\nu)}(y,\nu)(\varphi)$$

it remains to show that

$$N_m^{u_m(\nu_m)}(y_m,\nu_m)(\varphi)\to N^{u(\nu)}(y,\nu)(\varphi).$$

We consider two cases: $y\in\Gamma^c$ and $y\in\partial\Gamma^c$. If $y\in\Gamma^c$ it is sufficient to prove the convergence of the numerator of $N_m^{u_m(\nu_m)}(y_m,\nu_m)(\varphi)$ to the numerator of $N^{u(\nu)}(y,\nu)(\varphi)$.

We have

$$\Big|\int\limits_{\Gamma^c} r_m(z,y_m)\varphi(z)\,P_m^{u_m(\nu_m)}(\nu_m,dz)-\int\limits_{\Gamma^c} r(z,y)\varphi(z)\,P^{u(\nu)}(\nu,dz)\Big|$$
$$\le \int\limits_{\Gamma^c}|r_m(z,y_m)-r(z,y)|\,|\varphi(z)|\,P_m^{u_m(\nu_m)}(\nu_m,dz)$$
$$+\Big|\int\limits_{\Gamma^c} r(z,y)\varphi(z)\,(P_m^{u_m(\nu_m)}(\nu_m,dz)-P^{u(\nu)}(\nu,dz))\Big|$$
$$=\mathrm{I}_m+\mathrm{II}_m.$$

Since $y\in\Gamma^c$, by (A4) and (D3)(i), there exist positive constants $m_0(y)$ and $M(y)$ such that for $m>m_0(y)$ and $z\in\Gamma^c$,

$$\max\{r(x,y),r_m(x,y_m)\}<M(y).$$

Now, by the tightness of $\{\nu_m:m=1,2,\ldots\}$ and (D4), for given $\varepsilon>0$ one can find $\delta>0$ such that

$$\mathrm{I}_m\le 4\varepsilon\|\varphi\|M(y)+\int\limits_{\Gamma^c\cup(\Gamma(\delta))^c}|r_m(z,y_m)-r(z,y)|\,|\varphi(z)|\,P_m^{u_m(\nu_m)}(\nu_m,dz).$$

Therefore by Corollary 3, there exists a positive integer m_1 and a compact set $K\subset E$ such that for $m>\max\{m_0,m_1\}$,

$$\mathrm{I}_m\le 6\varepsilon\|\varphi\|M(y)\sup_{z\in\Gamma^c\cap(\Gamma(\delta))^c}|r_m(z,y_m)-r(z,y)|\,\|\varphi\|$$

and consequently by (D3)(i), $\mathrm{I}_m\to 6\varepsilon\|\varphi\|M(y)$ as $m\to\infty$. By Corollary 2 and (A4) also $\mathrm{II}_m\to 0$, which completes the proof in the case when $y\in\Gamma^c$.

Now, if $y\in\partial\Gamma^c$, then by (D3)(iii),

$$N_m^{u_m(\nu_m)}(y_m,\nu_m)(\varphi)\to\varphi(y)=N^{u(\nu)}(y,\nu)(\varphi).$$

Consequently, $N_m^{u_m(\nu_m)}(y_m,\nu_m)\Rightarrow N^{u(\nu)}(y,\nu)$ and the proof of Lemma 5 is complete. ■

For $F\in b\mathcal{B}(P(E))$ and $\nu\in P(E)$ let

$$\Pi_m^u(\nu,F)=\int\limits_{\Gamma}F(\delta_z)\,P_m^{u(\nu)}(\nu,dz)$$
$$+\int\limits_{\Gamma^c}\int\limits_{\Gamma^c} r_m(z,y)F(M_m^u(y,\nu))\,dy\,P_m^{u(\nu)}(\nu,dz).$$

Given a set $A\in\mathcal{B}(E)$ denote by $\widetilde{A}$ the set of all measures δ_x with $x\in A$.

Lemma 6. *Assume* (A2)–(A7), (D1)–(D4), *and suppose that* $b\mathcal{B}(P(E))\ni F_m\to F\in C(P(E))$ *uniformly on compact subsets of* $P(E)$, *and the* F_m *are uniformly bounded. Let* $A=\Gamma$ *or* $A=\Gamma_1$. *Then*

$$\Pi_m^{u_m}(\nu,F_m)\to\Pi^u(\nu,F)\tag{23}$$

and

$$\Pi_m^{u_m}(\nu, F_m\chi_{\tilde{A}}) \to \Pi^u(\nu, F\chi_{\tilde{A}}) \tag{24}$$

as $m \to \infty$, uniformly in ν from compact subsets of $P(E)$.

Proof. Let H be a compact subset of $P(E)$. We have

$$\begin{aligned}
&|\Pi_m^{u_m}(\nu, F_m) - \Pi^u(\nu, F)| \\
&\quad \le \int_\Gamma |F_m(\delta_z) - F(\delta_z)|\, P_m^{u_m(\nu)}(\nu, dz) \\
&\qquad + \Big| \int_\Gamma F(\delta_z)\, (P_m^{u_m(\nu)}(\nu, dz) - P^{u(\nu)}(\nu, dz))\Big| \\
&\qquad + \Big| \int_{\Gamma^c} \int_{\Gamma^c} (r_m(z,y) - r(z,y)) F_m(M_m^{u_m}(y,\nu))\, dy\, P_m^{u_m(\nu)}(\nu, dz)\Big| \\
&\qquad + \Big| \int_{\Gamma^c} \int_{\Gamma^c} r(z,y)(F_m(M_m^{u_m}(y,\nu)) - F(M^u(y,\nu)))\, dy\, P_m^{u_m(\nu)}(\nu, dz)\Big| \\
&\qquad + \Big| \int_{\Gamma^c} \int_{\Gamma^c} r(z,y) F(M^u(y,\nu))\, dy\, (P_m^{u_m(\nu)}(\nu, dz) - P^{u(\nu)}(\nu, dz))\Big| \\
&\quad = \mathrm{I}_m + \mathrm{II}_m + \mathrm{III}_m + \mathrm{IV}_m + \mathrm{V}_m.
\end{aligned}$$

Since $\sup_{z\in\Gamma} |F_m(\delta_z) - F(\delta_z)| \to 0$, $\mathrm{I}_m \to 0$ as $m \to \infty$. By Corollary 2 and (A6) also $\mathrm{II}_m \to 0$ as $m \to \infty$ uniformly in $\nu \in H$. Given $\varepsilon > 0$, by Corollary 3 we can find m_0 and a compact set $K \subset E$ such that (22) holds for $m \ge m_0$.

By compactness of H there is a compact set $K_1 \subset E$ such that $\nu(K_1) \ge 1 - \varepsilon$ for $\nu \in H$. Choosing now $\delta > 0$ such that

$$P_m^a(z, \Gamma(\delta)) < \varepsilon \quad \text{for } a \in U,\ z \in K_1,\ m = 1, 2, \ldots,$$

which we can do by (D4), we obtain

$$\begin{aligned}
\mathrm{III}_m &\le \int_{\Gamma^c\cap K} \int_{\Gamma^c} |r_m(z,y) - r(z,y)|\, \|F_m\|\, dy\, P_m^{u_m(\nu)}(\nu, dz) + 2\varepsilon\|F_m\| \\
&\le \sup_{z_1\in K_1} \int_{\Gamma^c\cap K} \int_{\Gamma^c} |r_m(z,y) - r(z,y)|\, \|F_m\|\, dy\, P_m^{u_m(\nu)}(z_1, dz) \\
&\qquad + 2\varepsilon\|F_m\| + 2\varepsilon\|F_m\| \\
&\le \sup_{z_1\in K_1} \int_{\{\Gamma^c\setminus\Gamma(\delta)\}\cap K} \int_{\Gamma^c} |r_m(z,y) - r(z,y)|\, \|F_m\|\, dy\, P_m^{u_m(\nu)}(z_1, dz) + 6\varepsilon\|F_m\| \\
&\le \sup_{z\in\{\Gamma^c\setminus\Gamma(\delta)\}\cap K} \|F_m\| \int_{\Gamma^c} |r_m(z,y) - r(z,y)|\, dy + 6\varepsilon\|F_m\|.
\end{aligned}$$

Letting $m\to\infty$, since $\|F_m\|\le C$, by (D3)(ii) we obtain $\limsup_{m\to\infty}\mathrm{III}_m\le 6\varepsilon C$, uniformly in $\nu\in H$.

Furthermore, by (A5) there is a compact set $L\subset E$ such that

$$\sup_{z\in K} R(z,L^c)<\varepsilon$$

and

$$\mathrm{IV}_m\le 4\varepsilon C+\int\limits_{\Gamma^c\cap K}\int\limits_{\Gamma^c\cap L} r(z,y)|F_m(N_m^{u_m}(y,\nu))-F(N^u(y,\nu))|\,dy\,P_m^{u_m(\nu)}(\nu,dz).$$

Therefore, by Lemma 5, uniformly in $\nu\in H$, $\limsup_{m\to\infty}\mathrm{IV}_m\le 4\varepsilon C$. Finally, by Lemma 4 and (A5), $\mathrm{V}_m\to 0$ uniformly in $\nu\in H$.

Summarizing, $\mathrm{I}_m+\mathrm{II}_m+\mathrm{III}_m+\mathrm{IV}_m+\mathrm{V}_m\to 0$ as $m\to\infty$ uniformly in $\nu\in H$ and consequently (23) holds.

The proof of (24) is almost immediate since

$$\begin{aligned}\Pi_m^{u_m}(\nu,F_m\chi_{\tilde A}) &= \int\limits_{A\cap\Gamma} F_m(\delta_z)\,P_m^{u_m(\nu)}(\nu,dz)\\ &\to \int\limits_{A\cap\Gamma} F(\delta_z)P^{u(\nu)}(\nu,dz)=\Pi^u(\nu,F\chi_{\tilde A})\quad \text{as } m\to\infty\end{aligned}$$

by Corollary 2.

The proof of Lemma 6 is complete. ∎

In the next lemma we extend (24) to the probabilities of more complex events.

LEMMA 7. *Assume* (A1)–(A7) *and* (D1)–(D4). *Let* $b\mathcal{B}(P(E))\ni F_m\to F\in C(P(E))$ *uniformly on compact subsets of* $P(E)$ *and* F_m *be uniformly bounded. Then for each* $i=1,2,\dots,$

$$\text{(25)}\quad \begin{aligned}&E_x^{u_m,m}\{\chi_{B_1}(\pi_1^{m,\delta_x,u_m})\dots\chi_{B_i}(\pi_i^{m,\delta_x,u_m})F_m(\pi_i^{m,\delta_x,u_m})\}\\ &\qquad\to E_x^u\{\chi_{B_1}(\pi_1^{\delta_x,u})\dots\chi_{B_i}(\pi_i^{\delta_x,u})F(\pi_i^{\delta_x,u})\}\end{aligned}$$

as $m\to\infty$, *uniformly in* $x\in\Gamma_1$ *with* $B_k\in\{\widetilde\Gamma_1,\widetilde\Gamma_1^c,\widetilde\Gamma,\widetilde\Gamma^c\}$ *for* $k=1,\dots,i$, *where* $\widetilde{\ }$ *denotes the operator defined before Lemma* 6.

Proof. We use induction on i. For $i=1$, (25) holds by (24). Assume that (25) holds for i. For $i+1$ we have

$$\begin{aligned}&E_x^{u_m,m}\{\chi_{B_1}(\pi_1^{m,\delta_x,u_m})\dots\chi_{B_{i+1}}(\pi_{i+1}^{m,\delta_x,u_m})F_m(\pi_{i+1}^{m,\delta_x,u_m})\}\\ &\quad=E_x^{u_m,m}\{\chi_{B_1}(\pi_1^{m,\delta_x,u_m})\dots\chi_{B_i}(\pi_i^{m,\delta_x,u_m})\Pi_m^{u_m}(\pi_i^{m,\delta_x,u_m},\chi_{B_{i+1}}F_m)\}\\ &\quad\to E_x^u\{\chi_{B_1}(\pi_1^{\delta_x,u})\dots\chi_{B_i}(\pi_i^{\delta_x,u})\Pi^u(\pi_i^{\delta_x,u},\chi_{B_{i+1}}F)\}\\ &\quad=E_x^u\{\chi_{B_1}(\pi_1^{\delta_x,u})\dots\chi_{B_i}(\pi_i^{\delta_x,u})\chi_{B_{i+1}}(\pi_{i+1}^{\delta_x,u})F(\pi_{i+1}^{\delta_x,u})\}\end{aligned}$$

as $m\to\infty$, uniformly in $x\in\Gamma_1$, by induction hypothesis and (24), which is the desired conclusion. ∎

With formula (25) we can now obtain

COROLLARY 4. *Under* (A1)–(A7) *and* (D1)–(D5), *for* $f \in C(\Gamma_1)$ *and* $F \in C(P(E))$ *we have*

$$E_x^{u_m,m}\{f(x_\tau^m)\} \to E_x^u\{f(x_\tau)\} \tag{26}$$

and

$$E_x^{u_m,m}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{m,\delta_x,u_m})\Big\} \to E_x^u\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{\delta_x,u})\Big\} \tag{27}$$

uniformly in $x \in \Gamma_1$, *as* $m \to \infty$, *with* τ *standing on the left hand side of* (26)–(27) *for the Markov time defined in* (A7) *with respect to* (x_i^m), *while on the right hand side for the Markov time corresponding to* (x_i).

P r o o f. Clearly we have

$$f(x_\tau) = \sum_{i=2}^{\infty}\sum_{j=1}^{i-1} \chi_{\tilde{\Gamma}}(\pi_1)\dots\chi_{\tilde{\Gamma}}(\pi_{j-1})\chi_{\tilde{\Gamma}^c}(\pi_j) \times \chi_{\tilde{\Gamma}_1^c}(\pi_{j+1})\dots\chi_{\tilde{\Gamma}_1^c}(\pi_{i-1})\chi_{\tilde{\Gamma}_1}(\pi_i)\pi_i(f) \tag{28}$$

and

$$\sum_{i=0}^{\tau-1} F(\pi_i) = \sum_{i=0}^{\infty}\chi_{\tilde{\Gamma}}(\pi_0)\dots\chi_{\tilde{\Gamma}}(\pi_i)\Big[F(\pi_i) + \chi_{\tilde{\Gamma}^c}(\pi_{i+1})F(\pi_{i+1}) + \chi_{\tilde{\Gamma}^c}(\pi_{i+1})\sum_{k=i+2}^{\infty}\chi_{\tilde{\Gamma}_1^c}(\pi_{i+2})\dots\chi_{\tilde{\Gamma}_1^c}(\pi_k)F(\pi_k)\Big] \tag{29}$$

and the analogous representations hold for $f(x_\tau^m)$ and $\sum_{i=0}^{\tau-1} F(\pi_i^m)$.

By (A7)(iii) and (D4)(ii) to prove (26) and (27) it is sufficient to show that for each $i, j, k = 1, 2, \dots$ with $j < i < k-1$,

$$\begin{aligned} E_x^{u_m,m}\{&\chi_{\tilde{\Gamma}}(\pi_1^{m,\delta_x,u_m})\dots\chi_{\tilde{\Gamma}}(\pi_{j-1}^{m,\delta_x,u_m})\chi_{\tilde{\Gamma}^c}(\pi_j^{m,\delta_x,u_m})\chi_{\tilde{\Gamma}_1^c}(\pi_{j+1}^{m,\delta_x,u_m})\dots \\ &\dots\chi_{\tilde{\Gamma}_1^c}(\pi_{i-1}^{m,\delta_x,u_m})\chi_{\tilde{\Gamma}_1}(\pi_i^{m,\delta_x,u_m})\pi_i^{m,\delta_x,u_m}(f)\} \\ \to E_x^u\{&\chi_{\tilde{\Gamma}}(\pi_1^{\delta_x,u})\dots\chi_{\tilde{\Gamma}}(\pi_{j-1}^{\delta_x,u})\chi_{\tilde{\Gamma}^c}(\pi_j^{\delta_x,u})\chi_{\tilde{\Gamma}_1^c}(\pi_{j+1}^{\delta_x,u})\dots \\ &\dots\chi_{\tilde{\Gamma}_1^c}(\pi_{i-1}^{\delta_x,u})\chi_{\tilde{\Gamma}_1}(\pi_i^{\delta_x,u})\pi_i^{\delta_x,u}(f)\} \end{aligned} \tag{30}$$

and

$$\begin{aligned} E_x^{u_m,m}\{&\chi_{\tilde{\Gamma}}(\pi_1^{m,\delta_x,u_m})\dots\chi_{\tilde{\Gamma}}(\pi_i^{m,\delta_x,u_m})[F(\pi_i^{m,\delta_x,u_m}) \\ &+ \chi_{\tilde{\Gamma}^c}(\pi_{i+1}^{m,\delta_x,u_m})F(\pi_{i+1}^{m,\delta_x,u_m}) \\ &+ \chi_{\tilde{\Gamma}^c}(\pi_{i+1}^{m,\delta_x,u_m})\chi_{\tilde{\Gamma}_1^c}(\pi_{i+2}^{m,\delta_x,u_m})\dots\chi_{\tilde{\Gamma}_1^c}(\pi_k^{m,\delta_x,u_m})F(\pi_k^{m,\delta_x,u_m})]\} \end{aligned} \tag{31}$$

$$\to E_x^u\{\chi_{\tilde\Gamma}(\pi_1^{\delta_x,u})\ldots\chi_{\tilde\Gamma}(\pi_i^{\delta_x,u})[F(\pi_i^{\delta_x,u})+\chi_{\tilde\Gamma^c}(\pi_{i+1}^{\delta_x,u})F(\pi_{i+1}^{\delta_x,u})$$
$$+\chi_{\tilde\Gamma^c}(\pi_{i+1}^{\delta_x,u})\chi_{\tilde\Gamma_1^c}(\pi_{i+2}^{\delta_x,u})\ldots\chi_{\tilde\Gamma_1^c}(\pi_k^{\delta_x,u})F(\pi_k^{\delta_x,u})]\}$$

as $m\to\infty$, uniformly in $x\in\Gamma_1$.

Applying Lemma 7 we obtain (30) and (31), which completes the proof. ∎

The representations (28)–(29) are crucial in the proof of the following

LEMMA 8. *Assume* (A1)–(A7). *For* $f\in C(\Gamma_1)$, $F\in C(P(E))$ *and* $u\in\mathcal{A}$ *the mappings*

$$\Gamma_1\ni x\mapsto E_x^u\{f(x_\tau)\},\tag{32}$$

$$\Gamma_1\ni x\mapsto E_x^u\Big\{\sum_{i=0}^{\tau-1}F(\pi_i^{\delta_x,u})\Big\}\tag{33}$$

are continuous and bounded.

Proof. By (28), (29) and (A7)(iii) it suffices to show the continuity of the mappings

$$\Gamma_1\ni x\mapsto E_x^u\{\chi_{\tilde\Gamma}(\pi_1^{\delta_x,u})\ldots\chi_{\tilde\Gamma}(\pi_{j-1}^{\delta_x,u})\chi_{\tilde\Gamma^c}(\pi_j^{\delta_x,u})\chi_{\tilde\Gamma_1^c}(\pi_{j+1}^{\delta_x,u})\ldots$$
$$\ldots\chi_{\tilde\Gamma_1^c}(\pi_{i-1}^{\delta_x,u})\chi_{\tilde\Gamma_1}(\pi_i^{\delta_x,u})\pi_i^{\delta_x,u}(f)\}$$

and

$$\Gamma_1\ni x\mapsto E_x^u\{\chi_{\tilde\Gamma}(\pi_1^{\delta_x,u})\ldots\chi_{\tilde\Gamma}(\pi_i^{\delta_x,u})[F(\pi_i^{\delta_x,u})+\chi_{\tilde\Gamma^c}(\pi_{i+1}^{\delta_x,u})F(\pi_{i+1}^{\delta_x,u})$$
$$+\chi_{\tilde\Gamma^c}(\pi_{i+1}^{\delta_x,u})\chi_{\tilde\Gamma_1^c}(\pi_{i+2}^{\delta_x,u})\ldots\chi_{\tilde\Gamma_1^c}(\pi_k^{\delta_x,u})F(\pi_k^{\delta_x,u})]\}$$

for $i,j,k=1,2,\ldots$ with $j<i<k-1$.

For this purpose it is sufficient to show by induction the following hypothesis:

(34) for $F\in C(P(E))$, $i=1,2,\ldots$, the mapping
$$\Gamma_1\ni x\mapsto E_x^u\{\chi_{B_1}(\pi_1^{\delta_x,u})\ldots\chi_{B_i}(\pi_i^{\delta_x,u})F(\pi_i^{\delta_x,u})\}$$
with $B_k\in\{\widetilde\Gamma,\widetilde\Gamma^c,\widetilde\Gamma_1,\widetilde\Gamma_1^c\}$ for $k=1,\ldots,i$ is continuous.

The step $i=1$ follows immediately from the Feller property of Π^u and (A6), (A7)(i). Assuming (34) to be true for i we have

$$E_x^u\{\chi_{B_1}(\pi_1^{\delta_x,u})\ldots\chi_{B_{i+1}}(\pi_{i+1}^{\delta_x,u})F(\pi_{i+1}^{\delta_x,u})\}$$
$$=E_x^u\{\chi_{B_1}(\pi_1^{\delta_x,u})\ldots\chi_{B_i}(\pi_i^{\delta_x,u})\Pi^u(\pi_i^{\delta_x,u},\chi_{B_{i+1}}F)\}.$$

By continuity of the mapping $\nu\mapsto\Pi^u(\nu,\chi_{B_{i+1}}F)$ and the induction hypothesis, we obtain (34) for $i+1$. Therefore (34) is satisfied, from which we obtain the continuity of the mappings (32) and (33). ∎

In (D5)(iii) we assume that the embedded Markov chains $(x_{\tau_n}^m)$ corresponding to the control functions u_m have invariant measures $\eta_m^{u_m}$. Since Γ_1 is compact, the measures are tight, and one can choose a convergent subsequence. In the next lemma we identify the limit.

LEMMA 9. *Under* (A1)–(A7) *and* (D1)–(D4) *we have*

(35) $$\eta_m^{u_m} \Rightarrow \eta^u \quad \textit{as } m \to \infty.$$

P r o o f. Assume $\eta_m^{u_m} \Rightarrow \overline{\eta}^u$. Then by Lemma 8 for $f \in C(\Gamma_1)$ we have

$$\Big| \int_{\Gamma_1} E_x^u\{f(x_\tau)\}\,\overline{\eta}^u(dx) - \int_{\Gamma_1} f(x)\,\overline{\eta}^u(dx)\Big|$$
$$\le \Big| \int_{\Gamma_1} E_x^u\{f(x_\tau)\}\,(\overline{\eta}^u(dx) - \eta_m^{u_m}(dx))\Big|$$
$$+ \Big| \int_{\Gamma_1} (E_x^u\{f(x_\tau)\} - E_x^{u_m,m}\{f(x_\tau^m)\})\,\eta_m^{u_m}(dx)\Big|$$
$$+ \Big| \int_{\Gamma_1} f(x)\,(\eta_m^{u_m}(dx) - \overline{\eta}^u(dx))\Big| \to 0$$

as $m \to \infty$ by (26). Therefore $\overline{\eta}^u$ is invariant for (x_{τ_n}) and by (A7)(iv), $\overline{\eta}^u = \eta^u$. ■

We are now in a position to complete the proof of Theorem 1.

By (13) and (17) it suffices to show that

(36) $$\int_{\Gamma_1} E_x^{u_m}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{m,\delta_x,u_m})\Big\}\,\eta_m^{u_m}(dx) \to \int_{\Gamma_1} E_x^{u}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{\delta_x,u})\Big\}\,\eta^u(dx) \quad \text{as } m \to \infty.$$

We have

$$\Big| \int_{\Gamma_1} E_x^{u_m}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{m,\delta_x,u_m})\Big\}\,\eta_m^{u_m}(dx) - \int_{\Gamma_1} E_x^{u}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{\delta_x,u})\Big\}\,\eta^u(dx)\Big|$$
$$\le \sup_{x\in\Gamma_1} \Big|E_x^{u_m}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{m,\delta_x,u_m})\Big\} - E_x^{u}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{\delta_x,u})\Big\}\Big|$$
$$+ \int_{\Gamma_1} E_x^{u}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{\delta_x,u})\Big\}\,(\eta_m^{u_m}(dx) - \eta^u(dx))$$

and by (27), the continuity of the mapping (33), and (35) we obtain (36). In consequence of (36), $\Phi_m^{u_m} \Rightarrow \Phi^u$ as $m \to \infty$. The proof of Theorem 1 is finished. ■

4. Construction of nearly optimal control functions

In this section using Theorem 1 we construct a nearly optimal control function, that is, a continuous function $u : P(E) \to U$ for which the value of the cost functional $J_\mu(u)$ of (3) is nearly optimal. The section is divided into three parts in which we approximate the controls and the state space and give some remarks and comments on further approximations.

4.1. Approximation of admissible control functions. The class $\mathcal{A}$ of admissible control functions is too large for approximation purposes. Therefore for given positive L, positive integer n and a sequence $\varphi_1, \varphi_2, \ldots$ dense in $C_0(\mathbb{R}^d)$, the space of continuous functions vanishing at infinity, we define the class

$$\mathcal{A}(L,n) = \{u \in \mathcal{A} : u(\nu) = \overline{u}(\nu(\varphi_1), \ldots, \nu(\varphi_n)), \quad \text{where } \overline{u} : \mathbb{R}^n \to U \text{ is Lipschitz with Lipschitz constant } L\}.$$

Assume

(A8) $\quad c \in C(E \times U)$

and impose the following assumption on the original state process (x_i):

(A9) $\quad$ for any compact set $K \subset E$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{z \in K} \sup_{a \in U} P^a(z, \Gamma(\delta)) < \varepsilon$$

with $\Gamma(\delta)$ defined in (D4).

We have

PROPOSITION 2. *Assume* (A1)–(A9). *Then for* $\mu \in P(E)$,

$$\lim_{L \to \infty,\, n \to \infty} \inf_{u \in \mathcal{A}(L,n)} J_\mu(u) = \inf_{u \in \mathcal{A}} J_\mu(u). \tag{37}$$

Proof. By the Stone–Weierstrass theorem (see Thm. 9.28 of [10] and Appendix of [15]), each $u \in \mathcal{A}$ can be approximated uniformly on compact subsets of $P(\mathbb{R}^d)$ by functions $u_{L,n} \in \mathcal{A}(L,n)$ with L, n sufficiently large.

Let $\Phi^{u_{L,n}}$ and Φ^u be invariant measures corresponding to the controlled filtering processes $(\pi_i^{u_{L,n}})$ and (π_i^u) controlled by the functions $u_{L,n}$ and u respectively. Then by Theorem 1,

$$\Phi^{u_{L,n}} \Rightarrow \Phi^u \quad \text{as } L, n \to \infty$$

and the family $\{\Phi^u, \Phi^{u_{L,n}} : L > 0,\ n > 0\}$ is tight. Therefore given $\varepsilon > 0$, there is a compact set $H \subset P(E)$ such that

$$\Phi^{u_{L,n}}(H) \geq 1 - \varepsilon \quad \text{for } L > 0,\ n > 0.$$

By (A8),

$$\int_E c(x, u_{L,n}(\nu))\, \nu(dx) \to \int_E c(x, u(\nu))\, \nu(dx)$$

as $L, n \to \infty$, uniformly in $\nu \in H$.

Using Corollary 1, we obtain

$$
\begin{aligned}
&|J_\mu(u) - J_\mu(u_{L,n})| \\
&\qquad \le \Big| \int\limits_{P(E)} \int\limits_E c(z, u(\nu))\, \nu(dz)\, (\Phi^u(d\nu) - \Phi^{u_{L,n}}(d\nu)) \Big| \\
&\qquad\quad + \int\limits_H \Big| \int\limits_E c(z, u(\nu))\, \nu(dz) - \int\limits_E c(z, u_{L,n}(\nu))\, \nu(dz) \Big| \Phi^{u_{L,n}}(d\nu) + 2\|c\|\varepsilon.
\end{aligned}
$$

Letting $L, n \to \infty$, since ε can be chosen arbitrarily small we have

$$\lim_{L,n\to\infty} J_\mu(u_{L,n}) = J_\mu(u)$$

and (37) follows. ■

Remark 6. As one can see from the proof of the last proposition, for $\mathcal{A} \ni u_m \to u \in \mathcal{A}$ uniformly on compact subsets of $P(E)$, we have $J_\mu(u_m) \to J_\mu(u)$.

4.2. State space discretization. The filtering process $(\pi_i^{\mu,u})$ corresponding to the controlled state process (x_i) is infinite-dimensional. Since we are looking for the function u of $\pi_i^{\mu,u}$ that minimizes the cost functional $J_\mu(u)$ of (3) it is important to get the corresponding filtering process with values in a finite-dimensional space, and one way to achieve this is to discretize (x_i). Therefore we partition E into a finite number of disjoint Borel sets B_k^m, $k = 1, \ldots, k_m$, such that

(i) $\bigcup_{k=1}^{k_m} B_k^m = E$, $\quad \bigcup_{k=1}^{k_r} B_k^m = \Gamma_1$,

$\bigcup_{k=1}^{k_p} B_k^m = \Gamma \quad$ with $r \le p < m$,

(ii) B_k^m have nonempty interiors, and the closures of B_k^m for $k < k_m$ are compact,

(iii) $\sup_{k<k_m} \operatorname{diam}(B_k^m) \to 0$ as $m \to \infty$ with $\operatorname{diam}(B)$ standing for the diameter of the set B,

(iv) $B_{k_m}^m \supset B_{k_{m+1}}^{m+1}$, and $\bigcap_{m=1}^\infty B_{k_m}^m = \emptyset$,

(v) for $m = 1, 2, \ldots$ and $k = 1, \ldots, k_m$, there are indices $r_1, \ldots, r_{i(k)}$ such that

$$B_k^m = \bigcup_{p=1}^{i(k)} B_{r_p}^{m+1}.$$

For each fixed m we choose a set $Z^m = \{z_k^m : k = 1, \ldots, k_m\}$ of selectors with the following properties:

$$z_k^m \in \operatorname{int}(B_k^m), \quad \{z_k^m : k = 1, \ldots, k_m\} \subset \{z_k^{m+1} : k = 1, \ldots, k_{m+1}\}, \tag{38}$$
$$z_{k_m}^m \to \infty \quad \text{for } m \to \infty.$$

We also require that

(B1) the partition (B_k^m) and selectors Z^m are such that for $a \in U$ and $k, p = 1, \ldots, k_m$,

$$P^a(z_p^m, \partial B_k^m) = 0.$$

The above assumption means that we choose the partition (B_k^m) of E and selectors Z^m in such a way that the sets B_k^m are $P^a(z_p^m, \cdot)$ continuous.

Let

$$r_m(x,y) = \Big(\int\limits_{B_k^m \cap \Gamma^c} dz\Big)^{-1}\Big[\int\limits_{B_k^m \cap \Gamma^c} r(z_j^m, z)\,dz + \frac{1}{k_m - k_p - 1}\int\limits_{B_{k_m}^m \cap \Gamma^c} r(z_j^m, z)\,dz\Big] \tag{39}$$

for $x \in B_j^m$ and $y \in B_k^m$ with $k_p < k < k_m$, $k_p < j$, and

$$r_m(x,y) = 0 \quad \text{for } y \in B_{k_m}^m \text{ and } y \notin \Gamma^c,$$

and

$$c_m(x,a) = c(z_j^m, a) \tag{40}$$

for $x \in B_j^m$ and $a \in U$.

Clearly, for fixed $x \in E$, $r_m(x,y)$ is a density and almost immediately we have

LEMMA 10. *Under* (A4) *and* (A5), r_m *defined by* (39) *satisfies* (D3). *Moreover*, *under* (A8),

$$c_m(x,a) \to c(x,a) \quad \textit{as } m \to \infty$$

uniformly in x from compact subsets of E and in $a \in U$. Furthermore, (A9) *implies* (D4). ∎

Consider now a simplex S^m in $\mathbb{R}^{k_m}$ given by

$$S^m = \Big\{(s_1, \ldots, s_{k_m}) : 0 \le s_k \le 1,\ k = 1, \ldots, k_m,\ \sum_{k=1}^{k_m} s_k = 1\Big\}$$

and define a class $\mathcal{A}_m(L,n)$ of control functions on S^m by

$$\mathcal{A}_m(L,n) = \Big\{u \in C(S^m, U) : u(s) = \overline{u}\Big(\sum_{k=1}^{k_m} \varphi_1(z_k^m)s_k, \ldots, \sum_{k=1}^{k_m} \varphi_n(z_k^m)s_k\Big),$$
$$\text{for } \overline{u} : \mathbb{R}^n \to U \text{ Lipschitz with Lipschitz constant } L\Big\}.$$

Following [11] define the mappings

$$\begin{aligned}
\mathcal{L}_m &: P(E) \ni \nu \mapsto \sum_{k=1}^{k_m} \nu(B_k^m)\delta_{z_k^m} \in P(E),\\
\overline{\mathcal{L}}_m &: C(P(E)U) \ni u \mapsto \overline{\mathcal{L}}_m u \quad \text{ with } \overline{\mathcal{L}}_m u(\nu) = u(\mathcal{L}_m \nu),\\
\widetilde{\mathcal{L}}_m &: \mathcal{A} \ni u \mapsto \widetilde{\mathcal{L}}_m u \in C(S^m, U) \quad \text{ with } \widetilde{\mathcal{L}}_m u(s) = u\Big(\sum_{k=1}^{k_m} s_k \delta_{z_k^m}\Big),
\end{aligned}$$

and recall Lemma 3.1 of [11]:

LEMMA 11. (i) $\mathcal{L}_m\nu \Rightarrow \nu$ *as* $m \to \infty$, *uniformly on compact subsets of* $P(E)$,

(ii) *for* $u \in \mathcal{A}$, $\overline{\mathcal{L}}_m u(\nu) \to u(\nu)$ *as* $m \to \infty$, *uniformly on compact subsets of* $P(E)$,

(iii) *if* $\mathcal{B}(P(E), U) \ni u_m \to u \in C(P(E), U)$, *uniformly on* $P(E)$, *then* $\overline{\mathcal{L}}_m u_m(\nu) \to u(\nu)$, *uniformly on compact subsets of* $P(E)$ *as* $m \to \infty$, *where* $\mathcal{B}(P(E), U)$ *stands for the set of all bounded Borel functions from* $P(E)$ *into* U. ∎

Let $E_m = \{1, \ldots, k_m\}$, $\Gamma_1^m = \{1, \ldots, k_r\}$, $\Gamma^m = \{1, \ldots, k_p\}$. We identify z_k^m with k for $k = 1, \ldots, k_m$ and consider two approximating transition kernels

$$\overline{P}_m^a(x, \cdot) = P^a(z_k^m, \cdot) \quad \text{ for } x \in B_k^m \tag{41}$$

and

$$P_m^a(k, j) = P^a(z_k^m, B_j^m). \tag{42}$$

By analogy to Section 3.1 of [11], for fixed $u \in \mathcal{A}$ consider two partially observed systems:

I. The unobserved process $\overline{x}_i^m$ evolves in E according to the transition operator $\overline{P}_m^{a_i}(x, \cdot)$ in a generic period i, with initial law μ and observation $\overline{y}_i^m \in Z_m$ such that

$$\begin{aligned}
&P\{\overline{y}_{i+1}^m \in A \mid \overline{x}_0^m, \ldots, \overline{x}_{i+1}^m, \overline{y}_1^m, \ldots, \overline{y}_i^m\}\\
&\qquad = \chi_{A\cap\Gamma}(\overline{x}_{i+1}^m) + \chi_{\Gamma^c}(\overline{x}_{i+1}^m) \int_{A\cap\Gamma^c} r_m(\overline{x}_{i+1}^m, y)\, dy
\end{aligned} \tag{43}$$

and with control $a_i = \overline{\mathcal{L}}_m u(\overline{\pi}_i^m)$, where $\overline{\pi}_i^m$ stands for the filtering process corresponding to $\overline{x}_i^m$ with the observation $\overline{y}_i^m$.

II. The unobserved process x_i^m evolves in E_m according to the transition matrix $P_m^{a_i}(k, p)$ in a generic period i, with initial law $(\mu(B_1^m), \ldots, \mu(B_{k_m}^m))$ and observation $y_i \in E_m$ satisfying

$$P\{y_{i+1}^m = k \mid x_0^m, \ldots, x_{i+1}^m = j,\ y_1^m, \ldots, y_i^m\} = \chi_{k\cap\Gamma_m}(j) + \chi_{\Gamma_m^c}(j) r_m^{j,k} \tag{44}$$

with

$$r_m^{j,k} = \int_{B_k^m\cap\Gamma^c} r(z_j^m, y)\, dy + \frac{1}{k_m - k_p - 1} \int_{B_{k_m}^m\cap\Gamma^c} r(z_j^m, y)\, dy \quad \text{ for } k < k_m,$$

$r_m^{j,k_m} = 0$, and with control $a_i = \widetilde{\mathcal{L}}_m u(\pi_i^m)$ where π_i^m is the filtering process corresponding to x_i^m and (y_i^m).

Define, analogously to (4),

$$\overline{M}_m^a(y,\nu)(A) = \chi_{A\cap\Gamma}(y) + \chi_{\Gamma^c}(y)\overline{N}_m^a(y,\nu)(A),$$
$$\overline{N}_m^a(y,\nu)(A) = \int\limits_{A\cap\Gamma^c} r_m(z,y)\,\overline{P}_m^a(\nu,dz)\Big(\int\limits_{\Gamma^c} r_m(z,y)\overline{P}_m^a(\nu,dz)\Big)^{-1}$$

for $y \in E$ and $\nu \in P(E)$, and

$$M_m^a(y,s)(k) = \chi_{k\cap\Gamma_m}(y) + \chi_{\Gamma_m^c}(y)N_m^a(y,s)(k),$$
$$N_m^a(y,s)(k) = r_m^{k,y}P_m^a(s,k)\Big(\sum_{j=1}^{k_m} r_m^{j,y}P_m^a(s,j)\Big)^{-1}$$

for $y \in E_m$ and $s \in S^m$ with $P_m^a(s,k) = \sum_{j=1}^{k_m} P_m^a(j,k)s_j$.

We have

LEMMA 12. *The filtering processes* $(\overline{\pi}_i^m)$ *and* (π_i^m) *have the following representations*:

$$\overline{\pi}_{i+1}^m(A) = \overline{M}_m^{\tilde{\mathcal{L}}_m u}(\overline{y}_{i+1}^m,\overline{\pi}_i^m)(A) \quad \textit{for } A\in\mathcal{B}(E), P\textit{-a.e.},$$
$$\pi_{i+1}^m(k) = M_m^{\tilde{\mathcal{L}}_m u}(y_{i+1}^m,\pi_i^m)(k) \quad \textit{for } k\in E_m, P\textit{-a.e.},$$
$$\overline{\pi}_0^m(A) = \mu(A), \qquad \pi_0^m(k) = \mu(B_k^m).$$

The processes $(\overline{\pi}_i^m)$, (π_i^m) *are Markov with respect to the* σ*-fields* $\overline{Y}_m^i = \sigma\{\overline{y}_1^m,\ldots,\overline{y}_i^m\}$ *and* $Y_m^i = \sigma\{y_1^m,\ldots,y_i^m\}$ *respectively*, *with transition operators*

$$\overline{\Pi}_m^{\tilde{\mathcal{L}}_m u}(\nu,F) = \int\limits_{\Gamma} F(\delta_z)\,\overline{P}_m^{\tilde{\mathcal{L}}_m u(\nu)}(\nu,dz) + \int\limits_{\Gamma^c}\int\limits_{\Gamma^c} r_m(z,y)F(\overline{M}_m^{\tilde{\mathcal{L}}_m u(\nu)}(y,\nu))\,dy\,\overline{P}_m^{\tilde{\mathcal{L}}_m u(\nu)}(\nu,dz) \tag{45}$$

and

$$\Pi_m^{\tilde{\mathcal{L}}_m u}(s,f) = \sum_{k=1}^{k_p} f(\delta_k)P_m^{\tilde{\mathcal{L}}_m u(s)}(s,k) + \sum_{k=k_p+1}^{k_m}\sum_{j=k_p+1}^{k_m} r_m^{j,k} f(M_m^{\tilde{\mathcal{L}}_m u(s)}(k,s))P_m^{\tilde{\mathcal{L}}_m u(s)}(s,j) \tag{46}$$

for F *and* f *being bounded Borel measurable functions on* $P(E)$ *and* S^m *respectively*, *with* δ_k *denoting the element of* S^m *with the* k*-th coordinate* 1 *and zeros elsewhere. Moreover*, *under* (A3) *and* (B1) *and* $u\in\mathcal{A}$, *the operator* $\Pi_m^{\tilde{\mathcal{L}}_m u}$ *is Feller*, *i.e. transforms the space of continuous functions on* S *into itself.*

Proof. The first part of the lemma follows by considerations analogous to those of Lemmas 1 and 2. The Feller property of $\Pi_m^{\tilde{\mathcal{L}}_m u}$ is a consequence of (B1) and the proof of Proposition 1. ∎

The filtering process (π_i^m) plays a fundamental role in the construction of nearly optimal control functions. As we shall see below we will minimize the cost functional

$$J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) := \limsup_{n\to\infty} n^{-1} E_{\overline{\mu}}\Big\{\sum_{i=0}^{n-1} c^m(x_i^m, \widetilde{\mathcal{L}}_m u(\pi_i^m))\Big\} \tag{47}$$

over $u \in \mathcal{A}(L,n)$ with $c^m(j,a) = c(z_j^m, a)$ and $\overline{\mu} = (\mu(B_1^m),\ldots,\mu(B_{k_m}^m))$.

Since

$$J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) = \limsup_{n\to\infty} n^{-1} E_{\overline{\mu}}\Big\{\sum_{i=0}^{n-1}\sum_{j=1}^{k_m} c^m(j, \widetilde{\mathcal{L}}_m u(\pi_i^m))\pi_i^m(j)\Big\}, \tag{48}$$

the calculation of the cost functional leads to the study of the ergodic properties of (π_i^m).

In what follows, we shall assume

(B2) (i) $E_j^{u,m} T_{\Gamma_1^m} < \infty$ for any $u \in C(S^m, U)$ and $j = 1,\ldots,k_m$,

(ii) for $\tau = T_{(\Gamma^m)^c} + T_{\Gamma_1^m} \circ \Theta_{T_{(\Gamma^m)^c}}$,

$$\sup_m \sup_{u\in C(S^m,U)} \sup_{j=1,\ldots,k_r} E_j^{u,m}\{\tau^2\} < \infty,$$

(iii) for any $u \in C(S^m,U)$ there is a unique invariant measure η_m^u for $x_{\tau_n}^m$, where $\tau_1 = \tau$ and $\tau_{n+1} = \tau_n + \tau\circ\Theta_{\tau_n}$,

where for given $u \in C(S^m,U)$, x_i^m stands for the unobserved process on E_m with transition matrix $P^{u(\pi_i^m)}(k,p)$, observation y_i^m of the form (44), and filtering process π_i^m, and $E_j^{u,m}$ denotes the measure generated by x_i^m with initial state $x_0^m = j$.

The assumption (B2)(i), (iii) says that the discretization should preserve the ergodic properties of the original model. Only the assumption (B2)(ii) seems to be stronger, since a uniform (in m) bound for the second moment of τ is required.

By (B2), the second part of Remark 3 and Lemma 3 for $u \in C(S^m,U)$ there is a unique invariant measure $\widehat{\Phi}_m^u$ for (π_i^m) and has the form

$$\widehat{\Phi}_m^u(f) = \sum_{j=1}^{k_r} E_j^{u,m}\Big\{\sum_{i=0}^{\tau-1} f(\pi_i^m)\Big\}\eta_m^u(j)\cdot\Big(\sum_{j=1}^{k_r} E_j^{u,m}\{\tau\}\eta_m^u(j)\Big)^{-1} \tag{49}$$

for $f \in b\mathcal{B}(S^m)$, the set of bounded Borel functions on S^m.

Consequently, analogously to Corollary 1, by Lemma 3 we obtain

$$J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) = \int_{S^m} \sum_{j=1}^{k_m} c^m(j, \tilde{\mathcal{L}}_m u(s)) s_j\, \widehat{\Phi}_m^{\tilde{\mathcal{L}}_m u}(ds). \tag{50}$$

We are now in a position to formulate the main result of this section.

PROPOSITION 3. *Assume* (A1)–(A9) *and* (B1)–(B2). *Then*

$$\sup_{u\in\mathcal{A}(L,n)} |J^m_{\overline{\mu}}(\widetilde{\mathcal{L}}_m u) - J_\mu(u)| \to 0 \quad \text{as } m\to\infty, \tag{51}$$

uniformly in $\mu \in P(E)$ *with* $\overline{\mu} = (\mu(B^m_1),\ldots,\mu(B^m_{k_m}))$.

Proof. Assume, contrary to (51), that

$$|J^m_{\overline{\mu}_m}(\widetilde{\mathcal{L}}_m u_m) - J_{\mu_m}(u_m)| \ge \delta > 0 \tag{52}$$

for $m = 1,2,\ldots$ and $u_m \in \mathcal{A}(L,n)$. By the compactness of $\mathcal{A}(L,n)$ we can assume that $u_m(\nu) \to u(\nu)$ with $u \in \mathcal{A}(L,n)$, uniformly on $P(E)$. Therefore by Lemma 11(iii),

$$\overline{\mathcal{L}}_m u_m(\nu) \to u(\nu) \quad \text{as } m\to\infty, \tag{53}$$

uniformly on compact subsets of $P(E)$.

Now, notice that the Markov times T_{Γ_1}, T_{Γ^c}, τ, τ_n for the process $(\overline{x}^m_i)$ with initial law μ coincide with the respective Markov times $T_{\Gamma^m_1}$, $T_{(\Gamma^m)^c}$, τ, τ_n for the process (x^m_i) with initial law $\overline{\mu} = (\mu(B^m_1),\ldots,\mu(B^m_{k_m}))$. Moreover, one can check that under (B2) for $B \in \mathcal{B}(\Gamma_1)$,

$$\overline{\eta}^{\bar{\mathcal{L}}_m u_m}(B) := \sum_{j=1}^{k_r} E^{\bar{\mathcal{L}}_m u_m, m}_{z^m_j}\{\overline{x}^m_\tau \in B\}\eta^{\tilde{\mathcal{L}}_m u_m}_m(j) \tag{54}$$

is an invariant measure for $\overline{x}^m_{\tau_n}$.

Therefore for $F \in b\mathcal{B}(P(E))$,

$$\Phi^{\bar{\mathcal{L}}_m u_m}_m(F) := \int_{\Gamma_1} E^{\bar{\mathcal{L}}_m u_m, m}_x\Big\{\sum_{i=0}^{\tau-1} F(\overline{\pi}^m_i)\Big\}\,\overline{\eta}^{\bar{\mathcal{L}}_m u_m}_m(dx) \times \Big(\int_{\Gamma_1} E^{\bar{\mathcal{L}}_m u_m, m}_x\{\tau\}\,\overline{\eta}^{\bar{\mathcal{L}}_m u_m}_m(dx)\Big)^{-1} \tag{55}$$

is an invariant measure for $\overline{\pi}^m_i$ with control $a_i = \overline{\mathcal{L}}_m u_m(\overline{\pi}_i)$, and

$$J^m_{\bar{\mu}_m}(\widetilde{\mathcal{L}}_m u_m) = \int_{P(E)}\int_E c_m(x, \overline{\mathcal{L}}_m u_m(\nu))\,\nu(dx)\,\Phi^{\bar{\mathcal{L}}_m u_m}_m(d\nu). \tag{56}$$

We show that $\Phi^{\bar{\mathcal{L}}_m u_m}_m \Rightarrow \Phi^u$ weakly on $P(P(E))$. For this purpose we use Theorem 1, with $P^a_m(x,\cdot) = \overline{P}^a_m(x,\cdot)$ defined in (40), r_m given in (39), and $u_m = \overline{\mathcal{L}}_m u_m$. By (A3), clearly (D1) is satisfied. From (53) we have (D2). Then (D3) and (D4) follow from Lemma 10 and by (B2) and (54) we obtain (D5). Therefore

$$\Phi^{\bar{\mathcal{L}}_m u_m}_m \Rightarrow \Phi^u \quad \text{as } m\to\infty$$

and similarly to the proof of Proposition 2, using Lemma 10 we obtain

$$J^m_{\bar{\mu}_m}(\widetilde{\mathcal{L}}_m u) \to J_\mu(u) \quad \text{as } m\to\infty,$$

which together with Remark 6 contradicts (52). Thus (51) holds. ∎

We have the following consequences of Proposition 3:

Corollary 5. *Under the assumptions of Proposition* 3,

(i) *for any initial law* $\overline{\mu} \in S$ *and* $\mu \in P(E)$,

$$\inf_{u\in\mathcal{A}_m(L,n)} J^m_{\overline{\mu}}(u) \to \inf_{u\in\mathcal{A}(L,n)} J_\mu(u) \quad \text{as } m\to\infty, \tag{57}$$

(ii) *if* $\widetilde{\mathcal{L}}_m u_m$ *is* ε*-optimal for* $J^m_{\overline{\mu}}$ *over* $\mathcal{A}_m(L,n)$ *for* m *such that*

$$\sup_{u\in\mathcal{A}(L,n)} |J^m_{\overline{\mu}}(\widetilde{\mathcal{L}}_m u) - J_\mu(u)| < \varepsilon,$$

then u_m *is* 3ε*-optimal for* J_μ *over* $\mathcal{A}(L,n)$.

Proof. Since $\widetilde{\mathcal{L}}_m\mathcal{A}(L,n) = \mathcal{A}_m(L,n)$, (i) immediately follows from (51). To show (ii) notice that

$$J_\mu(u_m) \le J^m_{\overline{\mu}}(\widetilde{\mathcal{L}}_m u_m) + \varepsilon \le \inf_{u\in\mathcal{A}(L,n)} J^m_{\overline{\mu}}(\widetilde{\mathcal{L}}_m u) + 2\varepsilon \le \inf_{u\in\mathcal{A}(L,n)} J_\mu(u) + 3\varepsilon. \ \blacksquare$$

In view of Corollary 5 the problem of construction of a nearly optimal control function u is reduced to minimization of $J^m_{\overline{\mu}}$ with respect to $\widetilde{\mathcal{L}}_m u$, where $u \in \mathcal{A}(L,n)$. By (48) the latter problem can be viewed as a finite-dimensional, complete observation control problem of the filtering process (π^m_i) with values in the simplex S^m and long run average cost functional of the form (48).

4.3. Comments on further discretizations. The problem formulated at the end of the preceding section cannot be solved explicitly, since the new state space S^m and the set of admissible controls $\mathcal{A}_m(L,n)$ are infinite. Therefore we discretize S^m in a way similar to Section 4.2. Moreover, we discretize the set U, and consider a discrete-valued control of the approximating Markov chain, embedded in S^m. For our minimization problem no dynamic programming arguments seem to be directly applicable, since we have a particular class of admissible controls. This difficulty can be overcome by adapting stochastic techniques of global optimization (see [14], [18]), e.g. a suitable version of the simulated annealing algorithm. For more details concerning computer implementation of that part of the paper see [13].

5. Nearly optimal control values

For a given control function u, the filtering process $(\pi^{\mu,u}_i)$ has values in an infinite-dimensional space and therefore practically $u(\pi^{\mu,u}_i)$ cannot be computed. Therefore it seems reasonable to approximate $(\pi^{\mu,u}_i)$ by a filtering process π^m_i with values in a simplex S^m. Since the only available information is a sequence (y_i), which is a noisy observation of (x_i), we apply (y_i) to the formula of Lemma 12 for π^m_i, and construct a computable approximation of π^m_i; the latter, when applied to the nearly optimal control function u of the cost functional $J_{\bar{\mu}}$, will yield nearly optimal control values. In this section we justify the above construction.

Assume the state process (x_i) has initial law μ and is controlled by $a_i = u(\pi_i^{(\nu)})$, $u \in \mathcal{A}$, with $\pi_i^{(\nu)}$ defined as follows:

$$\pi_0^{(\nu)} = \nu, \qquad \pi_{i+1}^{(\nu)} = M^{u(\pi_i^{(\nu)})}(y_{i+1}^{(\nu)}, \pi_i^{(\nu)}), \tag{58}$$

where $(y_i^{(\nu)})$ denotes the observation of (x_i) satisfying (A1). Clearly, the filtering process $(\pi_i^{(\mu,u)})$ corresponding to (x_i) with observation $(y_i^{(\nu)})$ is given by

$$\pi_0^{(\mu,\nu)} = \mu, \qquad \pi_{i+1}^{(\mu,\nu)} = M^{u(\pi_i^{(\nu)})}(y_{i+1}^{(\nu)}, \pi_i^{(\mu,\nu)}). \tag{59}$$

Consider the state space discretization introduced in Section 4.2. For $y \in \mathbb{R}^d$ let

$$\overline{w}_m y = z_k^m \quad \text{if } y \in B_k^m$$

and

$$w_m y = k \quad \text{if } y \in B_k^m, \text{ for } k = 1, \ldots, k_m.$$

Assume now that the state process (x_i) with initial law μ is controlled by $a_i = \overline{\mathcal{L}}_m u(\overline{\pi}_i^{m(\nu)}) = \widetilde{\mathcal{L}}_m u(\pi_i^{m(\nu)})$, $u \in \mathcal{A}$, with $\overline{\pi}_i^{m(\nu)}$, $\pi_i^{m(\nu)}$ given by the recursive formulae

$$\begin{aligned} &\overline{\pi}_0^{m(\nu)} = \nu, \\ &\overline{\pi}_{i+1}^{m(\nu)} = M_m^{\bar{\mathcal{L}}_m u(\bar{\pi}_i^{m(\nu)})}(\overline{w}_m y_{i+1}^{(\nu)}, \overline{\pi}_i^{m(\nu)}), \end{aligned} \tag{60}$$

$$\begin{aligned} &\pi_0^{m(\bar{\nu})} = \overline{\nu} = (\nu(B_1^m), \ldots, \nu(B_{k_m}^m)), \\ &\pi_{i+1}^{m(\bar{\nu})} = M_m^{\tilde{\mathcal{L}}_m u(\pi_i^{m(\bar{\nu})})}(w_m y_{i+1}^{(\nu)}, \pi_i^{m(\bar{\nu})}), \end{aligned} \tag{61}$$

where $(y_i^{(\nu)})$ denotes again the observation of (x_i) satisfying (A1). The filtering process $\pi_i^{m(\mu,\nu)}$ corresponding to (x_i) with observation $(y_i^{(\nu)})$ this time has the form

$$\begin{aligned} &\pi_0^{m(\mu,\nu)} = \mu, \\ &\pi_{i+1}^{m(\mu,\nu)} = M^{\bar{\mathcal{L}}_m u(\bar{\pi}_i^{m(\nu)})}(y_{i+1}^{(\nu)}, \pi_i^{m(\mu,\nu)}) = M^{\tilde{\mathcal{L}}_m u(\pi_i^{m(\bar{\nu})})}(y_{i+1}^{(\nu)}, \pi_i^{m(\mu,\nu)}). \end{aligned} \tag{62}$$

It appears that the pairs $(\pi_i^{(\nu)}, \pi_i^{(\mu,\nu)})$, $(\overline{\pi}_i^{m(\nu)}, \pi_i^{m(\mu,\nu)})$ and $(\pi_i^{m(\bar{\nu})}, \pi_i^{m(\mu,\nu)})$ form Markov processes. Denote their transition operators by T^u, $\overline{T}_m^u$ and T_m^u respectively.

Let $E_{\nu,\mu,m}^{\bar{\mathcal{L}}_m u}$ and $E_{\bar{\nu},\mu,m}^{\tilde{\mathcal{L}}_m u}$ be the conditional expectations of the pairs $(\overline{\pi}_i^{m(\nu)}, \pi_i^{m(\mu,\nu)})$ and $(\pi_i^{m(\bar{\nu})}, \pi_i^{m(\mu,\nu)})$ respectively.

Assume

(B3) for any $u \in \mathcal{A}$,

(i) $E_{\nu,\mu,m}^{\bar{\mathcal{L}}_m u} T_{\Gamma_1} < \infty$ for $\nu, \mu \notin \Gamma_1$,

(ii) $\sup_{x\in\Gamma_1}\sup_m E^{\tilde{\mathcal{L}}_m u}_{\bar{w}_m x,x,m}\tau^2<\infty$, where $\tau=T_{\Gamma^c}+T_{\Gamma_1}\circ\Theta_{T_{\Gamma^c}}$,

(iii) there is a unique invariant measure η^u_m for (x_{τ_n}), where $\tau_1=\tau$, $\tau_{n+1}=\tau_n+\tau\circ\Theta_\tau$, and the strong law of large numbers holds for (x_{τ_n}).

The Markov times T_{Γ_1}, T_{Γ^c}, τ above are all with respect to (x_i) with control $a_i=\overline{\mathcal{L}}_m u(\overline{\pi}_i^{m(\nu)})=\widetilde{\mathcal{L}}_m u(\pi_i^{m(\bar{\nu})})$.

The assumption (B3) is in particular satisfied when the original state process is uniformly stable no matter what control is applied.

Theorem 2. *Under* (A1)–(A9), (B1) *and* (B3), *for* $\mu\in P(E)$,

$$J_\mu((\widetilde{\mathcal{L}}_m u(\pi_i^{m(\nu)})))\to J_\mu(u)\quad \text{as } m\to\infty. \tag{63}$$

Proof. The proof is rather long and technical. It follows the steps of Theorem 1. Therefore we only sketch some preliminary steps which show that we can indeed apply the considerations of Theorem 1.

Notice first that the measures

$$\widehat{\Psi}^u_m(F_1):=\int_{\Gamma_1} E^{\tilde{\mathcal{L}}_m u}_{w_m x,x,m}\Big\{\sum_{i=0}^{\tau-1}F_1(\pi_i^{m(\delta_{w_n x})},\pi_i^{m(\delta_{w_m x},\delta_x)})\Big\}\,\eta^u_m(dx)\times\Big(\int_{\Gamma_1}E^{\tilde{\mathcal{L}}_m u}_{w_m x,x,m}\{\tau\}\,\eta^u_m(dx)\Big)^{-1} \tag{64}$$

and

$$\Psi^u_m(F_2):=\int_{\Gamma_1} E^{\bar{\mathcal{L}}_m u}_{\bar{w}_m x,x,m}\Big\{\sum_{i=0}^{\tau-1}F_2(\overline{\pi}_i^{m(\delta_{\bar{w}_n x})},\pi_i^{m(\delta_{\bar{w}_m x},\delta_x)})\Big\}\,\eta^u_m(dx)\times\Big(\int_{\Gamma_1}E^{\bar{\mathcal{L}}_m u}_{w_m x,x,m}\{\tau\}\,\eta^u_m(dx)\Big)^{-1} \tag{65}$$

defined for $F_1\in b\mathcal{B}(S^m\times P(E))$ and $F_2\in b\mathcal{B}(P(E)\times P(E))$ are invariant for the operators T^u_m and $\overline{T}^u_m$ respectively.

Since

$$\begin{aligned} J_\mu((\widetilde{\mathcal{L}}_m u(\pi_i^{m(\nu)})))&=\int_{S^m}\int_{P(E)}\int_E c(x,\widetilde{\mathcal{L}}_m u(s))\,\nu(dx),\\ \widehat{\Psi}^u_m(ds,d\nu)&=\int_{P(E)}\int_{P(E)}\int_E c(x,\overline{\mathcal{L}}_m u(\nu_1))\,\nu_2(dx)\,\Psi^u_m(d\nu_1,d\nu_2),\end{aligned} \tag{66}$$

by Lemma 11 and (A8) it remains to show that

$$\Psi^u_m\Rightarrow\Psi^u, \tag{67}$$

where for $F\in b\mathcal{B}(P(E)\times P(E))$,

$$\Psi^u(F) = \int_{P(E)\times P(E)} F(\nu,\nu)\,\Phi^u(d\nu). \tag{68}$$

By (13) and (65) to obtain (67) it is sufficient to prove that

$$\eta_m^u \Rightarrow \eta^u \tag{69}$$

and

$$E^{\bar{\mathcal{L}}_m u}_{\bar{w}_m x,x,m}\Big\{\sum_{i=0}^{\tau-1} F(\overline{\pi}_i^{m(\delta_{\bar{w}_m x})}, \pi_i^{m(\delta_{\bar{w}_m},\delta_x)})\Big\} \to E_x^u\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{\delta_x}, \pi_i^{\delta_x})\Big\} \tag{70}$$

uniformly in $x \in \Gamma_1$ as $m \to \infty$.

The proof of (69), (70) consists of analogs of Lemmas 5, 6, 7, 9. In particular, we show an analog of Lemma 6 which says that if $F_m \in b\mathcal{B}(P(E)\times P(E))$ converges uniformly on compact subsets of $P(E)\times P(E)$ to $F \in C(P(E)\times P(E))$ and $A = \Gamma$ or Γ_1 then

$$\overline{T}_m^u(\nu,\mu,F_m\chi_{\tilde{A}\times\tilde{A}}) \to T^u(\nu,\mu,F\chi_{\tilde{A}\times\tilde{A}}) \quad \text{as } m\to\infty,$$

uniformly on compact subsets of $P(E)\times P(E)$. Then by induction (cf. Lemma 7 and Corollary 4) we obtain (70), and almost immediately (69) (cf. Lemma 9). The details are left to the reader. ∎

We are now in a position to conclude our approximation.

PROPOSITION 4. *Assume* (A1)–(A9) *and* (B1)–(B3)*. For given $\varepsilon > 0$ one can choose L and n so large that*

$$\inf_{u\in\mathcal{A}(L,n)} J_\mu(u) \le \inf_{u\in\mathcal{A}} J_\mu(u) + \varepsilon, \tag{71}$$

and then $m_0(L,n)$ such that for $m \ge m_0(L,n)$,

$$\sup_{u\in\mathcal{A}(L,n)} |J_{\bar{\mu}}^m((\widetilde{\mathcal{L}}_m u) - J_\mu(u)| < \varepsilon. \tag{72}$$

If $\widehat{u}_m$ is ε-optimal for $J_{\bar{\mu}}^m$ over $\mathcal{A}_m(L,n)$ and m is so large that

$$|J_\mu((\widetilde{\mathcal{L}}_m u_m(\pi_i^{mu(\nu)}))) - J_\mu(u_m)| < \varepsilon, \tag{73}$$

then

$$J_\mu((\widehat{u}_m(\pi_i^{mu(\nu)}))) \le \inf_{u\in\mathcal{A}} J_\mu(u) + 5\varepsilon. \tag{74}$$

Proof. The possibility of choosing L and n for which (71) holds follows from Proposition 2. Then by Proposition 3 we justify the choice of $m_0(L,n)$ such that for $m \ge m_0(L,n)$, (72) is satisfied.

Denote by u_m an element of $\mathcal{A}(L,n)$ such that $\widehat{u}_m = \widetilde{\mathcal{L}}_m u_m$. By (73) and Corollary 5(ii) we have

$$J_\mu((\widehat{u}_m(\pi_i^{mu(\nu)}))) \le J_\mu(u_m) + \varepsilon \le \inf_{u\in\mathcal{A}(L,n)} J(u) + 4\varepsilon$$

and immediately from (71) we obtain (74), which completes the proof. ∎

We can summarize our approximation as follows. To obtain nearly optimal control values it suffices to find, for a sufficiently large m, an ε-optimal control function $\widehat{u}_m$ of $J_{\bar{\mu}}^m$, and then to construct an approximating filtering process $\pi_i^{m(\nu)}$, using the observation values (y_i). The controls $a_i = \widehat{u}_m(\pi_i^{m(\nu)})$ yield nearly optimal control values for our original cost functional J_μ.

6. An example

The majority of the assumptions imposed in the paper are standard, and do not seem to be restrictive. There is a group of ergodicity assumptions (A7)(iii), (iv), (B2), (B3) which require the models to be "uniformly" stable for each admissible control.

Below we present one simple model for which all assumptions are satisfied.

Let

$$x_{i+1} = Ax_i + f(x_i, a_i) + B(x_i)w_i, \tag{75}$$

where the w_i are independent $N(0,1)$ vectors on $\mathbb{R}^d$, A is an asymptotically stable matrix, the functions f and B are continuous in their arguments and bounded, with B having furthermore a bounded inverse (matrix).

Assume Γ is the Cartesian product of the intervals $[-b_k, b_k]$, $k = 1, \ldots, d$.

Let the kth coordinate y_{n+1}^k of y_{n+1} be equal to the kth coordinate x_{n+1}^k of x_{n+1} when $x_{n+1}^l \in [-b_l, b_l]$, for each $l = 1, \ldots, d$. Otherwise, y_{n+1}^k equals

$$\begin{aligned} \max\{x_{n+1}^k + g(x_{n+1})v_{n+1}^k, x_{n+1}^k\} \quad & \text{for } x_{n+1}^k \geq b_k, \\ \min\{\max\{x_{n+1}^k + g(x_{n+1})v_{n+1}^k, -b_k\}, b_k\} \quad & \text{for } -b_k \leq x_{n+1}^k \leq b_k, \\ \min\{x_{n+1}^k + g(x_{n+1})v_{n+1}^k, x_{n+1}^k\} \quad & \text{for } x_{n+1}^k \leq -b_k, \end{aligned}$$

where v_n^k, for $k = 1, \ldots, d$ and $n = 1, 2, \ldots$, are i.i.d. $N(0,1)$, independent of (w_n), and g is a continuous bounded function with support equal to the complement of the Cartesian product of the intervals $[-b_k, b_k]$, $k = 1, 2, \ldots$

Notice that by the form of observation structure defined above, if x_n is in Γ^c, the observation y_n is also in Γ^c.

Under the above assumptions, (A1)–(A7) and (A9) hold, and by a suitable choice of a partition we can also guarantee (B1)–(B3) to be satisfied.

Acknowledgements. The author wishes to thank Prof. W. Runggaldier from Padova for his comments improving readability of the paper and for finding a gap in the example.

References

[1] A. Arapostatis, V. S. Borkar, E. Fernandez-Gaucherand, M. K. Ghosh, and S. Marcus, *Discrete-time controlled Markov processes with average cost criterion, a survey*, SIAM J. Control Optim. 31 (1993), 282–344.

[2] A. Arapostatis, E. Fernandez-Gaucherand and S. Marcus, *Analysis of an adaptive partially observed controlled Markov chain*, IEEE Trans. Automat. Control 38 (1993), 987-992.

[3] D. P. Bertsekas and S. E. Schreve, *Stochastic Optimal Control: The Discrete Time Case*, Academic Press, New York, 1978.

[4] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.

[5] G. Di Masi and Ł. Stettner, *On adaptive control of a partially observed Markov chain*, Applicationes Math. 22 (1994), 165–180.

[6] J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953.

[7] O. Hernandez–Lerma, *Adaptive Markov Control Processes*, Springer, New York, 1989.

[8] M. Kurano, *On the existence of an optimal stationary J-policy in non-discounted Markovian decision processes with incomplete state information*, Bull. Math. Statist. 17 (1977), 77–81.

[9] H. J. Kushner and P. G. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, Springer, 1993.

[10] H. L. Royden, *Real Analysis*, MacMillan, New York, 1968.

[11] W. Runggaldier and Ł. Stettner, *Nearly optimal controls for stochastic ergodic problems with partial observation*, SIAM J. Control Optim. 31 (1993), 180–218.

[12] —, —, *Partially observable control problems with compulsory shifts of the state*, IIASA Working Paper 92–34, May 1992.

[13] —, —, *Approximations of Discrete Time Partially Observed Control Problems*, Appl. Math. Monographs CNR, in preparation.

[14] F. Schoen, *Stochastic techniques for global optimization: a survey of recent advances*, J. Global Optimization 1 (1991), 207–228.

[15] Ł. Stettner, *On invariant measures of filtering processes*, in: Proc. 4-th Bad Honnef Conf. on Stochastic Differential Systems, Lecture Notes in Control and Inform. Sci. 126, Springer, 1989, 279–292.

[16] —, *Ergodic control of partially observed Markov processes with equivalent transition probabilities*, Applicationes Math. 22 (1993), 25–38.

[17] K. Wakuta, *Semi-Markov decision processes with incomplete state observation—average cost criterion*, J. Oper. Res. Soc. Japan 24 (1981), 95–108.

[18] R. Zieliński und P. Neumann, *Stochastische Verfahren zur Suche nach dem Minimum einer Funktion*, Akademie-Verlag, Berlin, 1983.