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Invariant extension of Haar measure

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CONTENTS

§ 1. Introduction

In 1950, Kakutani and Oxtoby [10] proved that Haar measure on the circle group can be extended to an invariant, countable additive measure for which the corresponding $\mathfrak{L}_2$-space has Hilbert space dimension $2^{\mathfrak{c}}$. The same result holds for any compact metric group (see Hewitt and Ross [3], § 16).

On the other hand, using the result of Kakutani and Kodaira [11], in 1959 Hulanicki [8] proved that discontinuous characters of the circle group can be made simultaneously measurable under an invariant extension of Haar measure. This result was extended by Itzkowitz [9] for connected compact infinite Abelian groups and by Hewitt and Ross [5] for arbitrary locally compact Abelian groups by showing that $2^{\mathrm{card}(G)}$ characters of any infinite locally compact Abelian group G can be made simultaneously measurable under an invariant extension of Haar measure.

In this paper, we generalize the theorem of Kakutani and Oxtoby for arbitrary locally compact Hausdorff group.

In § 2, we give an extension of a given finite measure space to a very much larger measure space so that the extended measure has the same transformation law for a certain class of transformations as the original measure. The proof is based on the method of Kakutani and Oxtoby, but we use the terminology of Hewitt and Ross ([3], § 16). The essential difference is in the elimination of the original version's topological feature. This makes possible the use of purely set-theoretic and measure-theoretic devices.

In § 3, using the results of § 2, we extend the Haar measure to a very much larger measure space on an arbitrary locally compact nondiscrete Hausdorff group.

In § 4, we prove that the Lebesgue measure on the n-dimensional Euclidean space can be extended to a measure for which the corresponding $\mathfrak{L}_2$-space has Hilbert space dimension $2^{\mathfrak{c}}$ and this measure is transformed by differentiable coordinate transformations as the Lebesgue measure.

We shall use the axiom of choice, but we do not use the continuum hypothesis.

All notations and terms not explained as they occur are as in the monograph of Hewitt and Ross [3] and [4].

If A is a set we shall denote it's *cardinal number* by $\mathrm{card}(A)$ and it's *complement* by A'. For a function f let $\mathrm{dmn} f$ denote it's *domain* and let the *multiplicity* $N(f, y)$ denote the number of elements (possibly ∞) of $f^{-1}\{y\}$.

For a topological space X the *weight* of X is the least cardinal number of an open basis. The *character* of a topological space X is the least cardinal number $\mathfrak{n}$ for which there is an open basis at x with cardinal number not greater then $\mathfrak{n}$ whenever $x \in X$.

Let μ be a measure on a set X and let $\mathcal{M}$ denote the σ-algebra which is the domain of μ. The *weight* of the measure μ (or the weight of the measure space $(X, \mathcal{M}, \mu)$) is the least cardinal number of a basis of $(X, \mathcal{M}, \mu)$. A *subspace* of $(X, \mathcal{M}, \mu)$ is a measure space $(M, \mathcal{M}_M, \mu_M)$ where $M \in \mathcal{M}$, $\mathcal{M}_M = \{A\colon A \in \mathcal{M},\ A \subset M\}$ and μ_M is the restriction of μ onto $\mathcal{M}_M$. The *character* of μ is the least cardinal number $\mathfrak{n}$ for which $(M, \mathcal{M}_M, \mu_M)$ has a base with cardinal number not greater then $\mathfrak{n}$ whenever $M \in \mathcal{M}$ and $\mu(M) < \infty$. By a *Radon measure* we mean a measure μ, over a locally compact Hausdorff space X, with the following three properties:

If V is an open subset of X, then V is μ measurable and $\mu(V) = \sup\{\mu(K)\colon K \text{ is compact, } K \subset V\}$.

If K is a compact subset of X, then $\mu(K) < \infty$.

If A is a μ measurable subset of X, then

$$\mu(A) = \inf\{\mu(V)\colon V \text{ is open, } A \subset V\}.$$

μ measurable functions are considered to be defined μ almost everywhere. We say that f is a *μ integrable function* if and only if f is a μ measurable extended real valued function whose integral is defined (possible $-\infty$ or ∞).

We remark that if $(X, \mathcal{M}, \mu)$ is a finite measure space then the weight of μ and the Hilbert space dimension of the corresponding $\mathfrak{L}_2$-space are equal, whenever one of them is infinite. (See Hewitt and Ross [3], (16.12).)

§ 2. Covariant extension of measures

2.1. Definition. A measure μ (the measure space $(X, \mathcal{M}, \mu)$) is called *J covariant measure* if and only if there is a set T of mappings τ such that each $\tau \in T$ maps a μ measurable set into X, and for every $\tau \in T$ an extended real valued nonnegative μ measurable function $J(\tau)$ is defined satisfying the following property:

(1) The function $N(\tau|A, y)$ is μ measurable and

$$\int_A J(\tau)(x)\,d\mu(x) = \int_X N(\tau|A, y)\,d\mu(y)$$

whenever $A \in \mathcal{M}$ and $\tau \in T = \operatorname{dmn} J$.

If τ is one-to-one and $J(\tau)$ is μ almost equal to the characteristic function of $\operatorname{dmn}\tau$ whenever $\tau \in T$ then the measure μ (or the measure space $(X, \mathcal{M}, \mu)$) is called *T invariant*.

We remark that if every element τ of T is one-to-one then condition (1) is equivalent to

(2) $\tau(A) \in \mathscr{M}$ and $\mu(\tau(A)) = \int_A J(\tau)(x)\,d\mu(x)$ whenever $A \in \mathscr{M}$ and $\tau \in T$.

Hence if every element τ of T is one-to-one then μ is T invariant if and only if

(3) $\tau(A) \in \mathscr{M}$ and $\mu(\tau(A)) = \mu(A)$ whenever $A \in \mathscr{M}$, $\tau \in T$ and $A \subset \operatorname{dmn} \tau$.

2.2. Examples. 1. Let $X = R^n$, μ the Lebesgue measure on R^n, and let T be the group of all translations of R^n. Then μ is T invariant. In this case, $J(\tau)(x) = 1$ for every $x \in R^n$ whenever $\tau \in T$.

2. Let $X = R^n$, μ the Lebesgue measure on R^n, and let T be the set of all linear mappings of R^n into itself. If $J(\tau)(x) = |\det \tau|$ whenever $x \in R^n$ and $\tau \in T$, then μ is J covariant.

3. Let $X = R^n$, μ the Lebesgue measure on R^n, and let T be the set of all continuously differentiable functions τ which map an open subset of R^n into R^n. Let $J(\tau)(x)$ be equal to the absolute value of the Jacobian of τ at x if $x \in \operatorname{dmn} \tau$ and 0 if $x \notin \operatorname{dmn} \tau$. Then μ is J covariant.

2.3. Remark. If $(X, \mathscr{M}, \mu)$ is a J covariant measure space then we have for every nonnegative μ integrable function f the following formulae concerning transformation of integrals:

(1) For every $\tau \in \operatorname{dmn} J$ the function $y \mapsto \sum_{x \in \tau^{-1}\{y\}} f(x)$ is a μ measurable function and

$$\int_X f(x) \cdot J(\tau)(x)\,d\mu x = \int_X \Big(\sum_{x \in \tau^{-1}\{y\}} f(x) \Big)\,d\mu y.$$

This conclusion is obvious in case f is the characteristic function of a μ measurable subset of X. Hence, using approximation with linear combinations of characteristic functions, we get it for nonnegative μ measurable functions. For arbitrary μ integrable functions we can use the definition of the integral.

2.4. Theorem. *Let* $(X, \mathscr{M}, \mu)$ *a* J *covariant complete measure space,* $0 < \mu(X) < \infty$, *and suppose that every element of* $T = \operatorname{dmn} J$ *is one-to-one. Suppose that there exists an infinite cardinal number* $\mathfrak{n}$ *and a subset* $\mathscr{F}$ *of* 2^X *with the following properties:*

(1) $\operatorname{card}(X) = 2^{\mathfrak{n}}$;

(2) $\operatorname{card}(T) \leqslant 2^{\mathfrak{n}}$;

(3) $\operatorname{card}(\mathscr{F}) \leqslant 2^{\mathfrak{n}}$ *and for every* $A \in \mathscr{M}$, $\mu(A) > 0$ *there exists a* $F \in \mathscr{F}$ *for which* $\operatorname{card}(F) = 2^{\mathfrak{n}}$ *and* $F \subset A$.

Then there exists a J covariant measure space $(X, \mathcal{N}, \nu)$ with weight $2^{2^{\mathfrak{n}}}$ which is an extension of $(X, \mathcal{M}, \mu)$.

Proof. The construction will be carried out in five steps.

I. As the first step we shall give a J covariant extension λ of μ such that every set $A \subset X$ for which $\operatorname{card}(A) < 2^{\mathfrak{n}}$ is λ measurable and has λ measure 0.

Let $\mathcal{P}$ be the family of subsets A of X for which $\operatorname{card}(A) < 2^{\mathfrak{n}}$. By (3) if a μ measurable set is in $\mathcal{P}$ then it has measure 0. Using the inequality of König, we have that

$$\operatorname{card}\Big(\bigcup_{i=1}^{\infty} A_i\Big) < (2^{\mathfrak{n}})^{\aleph_0} = 2^{\mathfrak{n} \times \aleph_0} = 2^{\mathfrak{n}}$$

whenever $A_1, A_2, \ldots \in \mathcal{P}$.

Let $\mathcal{L}$ be the family of all subsets L of X such that for some $M \in \mathcal{M}$ the symmetric difference $L \triangle M$ is in $\mathcal{P}$. It is easy to verify that $\mathcal{L}$ is a σ-algebra and that if we put $\lambda(L) = \mu(M)$ then $\lambda(L)$ defined unambiguously on $\mathcal{L}$ and $(X, \mathcal{L}, \lambda)$ is a complete measure space. Finally, we show that λ is a J covariant extension of μ. Since every element τ of T is a one-to-one function, we have to prove that $\tau(L) \in \mathcal{L}$ and

$$\int_L J(\tau)(x)\, d\lambda x = \lambda\big(\tau(L)\big)$$

whenever $L \in \mathcal{L}$ and $\tau \in T$. Let us choose an $M \in \mathcal{M}$ for which $M \triangle L \in \mathcal{P}$. Hence $\tau(M) \triangle \tau(L) \in \mathcal{P}$, because τ is a one-to-one function, and so $\tau(L) \in \mathcal{L}$, $\lambda\big(\tau(L)\big) = \mu\big(\tau(M)\big)$. Consequently,

$$\begin{aligned}\int_L J(\tau)(x)\, d\lambda x &= \int_M J(\tau)(x)\, d\lambda x \\ &= \int_X \xi_M(x) \cdot J(\tau)(x)\, d\lambda x \overset{(1)}{=} \int_X \xi_M(x) \cdot J(\tau)(x)\, d\mu x \\ &= \mu\big(\tau(M)\big) = \lambda\big(\tau(L)\big),\end{aligned}$$

where ξ_M is the characteristic function of the set M. Equality (1) above can be proved by using approximation with linear combinations of characteristic functions.

II. We prove that there exists a family $\{X_\gamma\}_{\gamma \in \Gamma}$ of subsets X_γ of X with the following properties:

(4) $\operatorname{card}(\Gamma) = 2^{\mathfrak{n}}$.

(5) The sets X_γ, $\gamma \in \Gamma$ are pairwise disjoint.

(6) If $L \in \mathcal{L}$ and $\lambda(L) > 0$ then $L \cap X_\gamma \neq \emptyset$ whenever $\gamma \in \Gamma$.

(7) For every subset Γ_0 of Γ and for every $\tau \in T$

$$\operatorname{card}\Big(\tau\big(\bigcup_{\gamma \in \Gamma_0} X_\gamma\big) \triangle \Big(\tau(X) \cap \big(\bigcup_{\gamma \in \Gamma_0} X_\gamma\big)\Big)\Big) < 2^{\mathfrak{n}}.$$

Let Ω be the smallest ordinal having cardinal number 2^{n}. Let A be an arbitrary set with cardinal number 2^{n}. We may and do suppose that every element of $\mathscr{F}$ has cardinal number 2^{n}. Since $\operatorname{card}(A\times\mathscr{F}) = 2^{\text{n}}$, there exists an $\mathscr{L}\mapsto(a_\alpha, F_\alpha)$ one-to-one mapping of the set of ordinals $\{\alpha\colon 0\leqslant\alpha<\Omega\}$ onto $A\times\mathscr{F}$. The transfinite sequence $F_0, \ldots, F_\alpha, \ldots, 0\leqslant\alpha<\Omega$ contains every element F of $\mathscr{F}$ 2^{n} times. Let us choose a mapping $\alpha\mapsto\tau_\alpha$ of the set $\{\alpha\colon 0\leqslant\alpha<\Omega\}$ onto the set $\{1_X\}\cup T$for which $\tau_0 = 1_X$ where 1_X is the identical mapping of X onto itself. For each $x\in X$ and each ordinal $\alpha<\Omega$ let $C_\alpha(x)$ denote the set of points of X that can be written as

$$\tau_{\beta_1}^{\varepsilon_1}\circ\ldots\circ\tau_{\beta_n}^{\varepsilon_n}(x)$$

where $0\leqslant\beta_k\leqslant\alpha$ and ε_k is 1 or -1, $k = 1, 2, \ldots, n$ and $n = 1, 2, 3,$ Clearly, we have $x\in C_\alpha(x)$ and for $x\in X$ and $0\leqslant\beta\leqslant\alpha<\Omega$ we have $\tau_\beta(C_\alpha(x)) = \tau_\beta(X)\cap C_\alpha(x)$. We also have

$$\operatorname{card}(C_\alpha(x))\leqslant\max\{\operatorname{card}(\alpha), \aleph_0\} < 2^{\text{n}}.$$

We shall show that there exists a transfinite double sequence

$$\{x_\beta^\alpha\colon 0\leqslant\beta\leqslant\alpha<\Omega\}$$

of elements of X such that:

$x_\beta^\alpha\in F_\alpha$ for $0\leqslant\beta\leqslant\alpha<\Omega$;

the sets $\{C_\alpha(x_\beta^\alpha)\colon 0\leqslant\beta\leqslant\alpha<\Omega\}$ are pairwise disjoint.

If we agree that $(\gamma, \delta) < (\alpha, \beta)$ whenever $\gamma<\alpha$ or whenever $\gamma=\alpha$ and $\delta<\beta$ (lexicographic ordering), then $\{(\alpha, \beta)\colon 0\leqslant\beta\leqslant\alpha<\Omega\}$ is a well ordered set. We shall define the sequence $\{x_\beta^\alpha\colon 0\leqslant\beta\leqslant\alpha<\Omega\}$ by transfinite induction. Let x_0^0 be an arbitrary point of F_0. Suppose that $0\leqslant\beta\leqslant\alpha<\Omega$ and that x_δ^γ has already been defined for all pairs $(\gamma, \delta) < (\alpha, \beta)$, $0\leqslant\delta\leqslant\gamma$. Consider the union $D(\alpha, \beta)$ of the sets $C_\alpha(x_\delta^\gamma)$ as (γ, δ) runs over all pairs $(\gamma, \delta) < (\alpha, \beta)$. Then

$$\operatorname{card}(D(\alpha, \beta))\leqslant(\operatorname{card}(\alpha))^2\cdot\max\{\operatorname{card}(\alpha), \aleph_0\} < 2^{\text{n}}.$$

Since $\operatorname{card}(F_\alpha) = 2^{\text{n}}$, the set $F_\alpha\cap D(\alpha, \beta)'$ is nonvoid. Let x_β^α be an arbitrary point of $F_\alpha\cap D(\alpha, \beta)'$. Then $C_\alpha(x_\beta^\alpha)$ is disjoint from every $C_\gamma(x_\delta^\gamma)$, $(\gamma, \delta) < (\alpha, \beta)$. Otherwise we would have

$$\tau_{\beta_1}^{\varepsilon_1}\circ\ldots\circ\tau_{\beta_n}^{\varepsilon_n}(x_\beta^\alpha) = \tau_{\delta_1}^{\eta_1}\circ\ldots\circ\tau_{\delta_m}^{\eta_m}(x_\delta^\gamma)$$

where $\beta_k\leqslant\alpha$, $\delta_j\leqslant\gamma\leqslant\alpha$, ε_k is 1 or -1, and η_j is 1 or -1, $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Hence

$$x_\beta^\alpha = \tau_{\beta_n}^{-\varepsilon_n}\circ\quad\circ\tau_{\beta_1}^{-\varepsilon_1}\circ\tau_{\delta_1}^{\eta_1}\circ\ldots\circ\tau_{\delta_m}^{\eta_m}(x_\delta^\gamma)\in D(\alpha, \beta)$$

and this contradicts to the choice of x_β^α.

Now let

$$\Gamma = \{\nu\colon \nu \text{ is an ordinal and } 0 \leqslant \nu < \Omega\}$$

and let

$$X_\nu = \bigcup \{C_\alpha(x_\nu^\alpha)\colon \nu \leqslant \alpha < \Omega\}, \qquad \nu \in \Gamma$$

Properties (4) and (5) are obvious. If $L \in \mathscr{L}$ and $\lambda(L) > 0$ then there exists a $M \in \mathscr{M}$ for which $\operatorname{card}(L \triangle M) < 2^{\aleph}$ and $\mu(M) = \lambda(L) > 0$. Hence there exists a $F \in \mathscr{F}$ such that $F \subset M$. Let $L = B \cup (M \cap A')$, where $\operatorname{card}(A) < 2^{\aleph}$ and $\operatorname{card}(B) < 2^{\aleph}$. Since $x_\nu^\alpha \in F$ and $x_\nu^\alpha \in C_\alpha(x_\nu^\alpha) \subset X_\nu$ whenever $\nu \leqslant \alpha < \Omega$ and $F_\alpha = F$, we have that $F \cap X_\nu$ has at least $2^{\aleph}$ elements. Hence M and $B \cup (M \cap A')$ contain also $2^{\aleph}$ elements from X_ν which proves (6).

To prove (7) let $\Gamma_0 \subset \Gamma$ and $\tau \in T$. Suppose that $0 \leqslant \gamma < \Omega$ and $\tau_\gamma = \tau$. Using that

$$\tau_\gamma\big(C_\alpha(x_\nu^\alpha)\big) = \tau_\gamma(X) \cap C_\alpha(x_\nu^\alpha) \qquad \text{if} \qquad \gamma \leqslant \alpha < \Omega$$

and

$$\bigcup_{\nu \in \Gamma_0} X_\nu = \bigcup \{C_\alpha(x_\nu^\alpha)\colon \nu \in \Gamma_0,\ \nu \leqslant \alpha < \Omega\},$$

we have that

$$(8) \quad \tau_\gamma\Big(\bigcup_{\nu \in \Gamma_0} X_\nu\Big) \triangle \Big(\tau_\gamma(X) \cap \Big(\bigcup_{\nu \in \Gamma_0} X_\nu\Big)\Big) \subset \bigcup \{C_\alpha(x_\nu^\alpha) \cup \tau_\gamma\big(C_\alpha(x_\nu^\alpha)\big)\colon \nu \in \Gamma_0,\ \nu \leqslant \alpha < \gamma\}.$$

Because

$$\operatorname{card}\Big(C_\alpha(x_\nu^\alpha) \cup \tau_\gamma\big(C_\alpha(x_\nu^\alpha)\big)\Big) \leqslant \max\{\operatorname{card}(\alpha), \aleph_0\}$$

the right side of (8) has cardinal number less that $2^{\aleph}$.

III. We shall prove that there exists a family $\{E_\vartheta\}_{\vartheta \in \Theta}$ of distinct subsets of X such that

(9) $\operatorname{card}(\Theta) = 2^{2^{\aleph}}$;

(10) if $\tau \in T$ then with the notations $E^{-1} = E'$, $E^1 = E$

$$\operatorname{card}\Big(\tau\Big(\bigcap_{n=1}^{\infty} E_{\vartheta_n}^{\varepsilon_n}\Big) \triangle \Big(\tau(X) \cap \Big(\bigcap_{n=1}^{\infty} E_{\vartheta_n}^{\varepsilon_n}\Big)\Big)\Big) < 2^{\aleph}$$

for every sequence $\{\vartheta_n\}_{n=1}^{\infty}$ of (not necessarily distinct) elements from Θ and for every sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ where ε_n is 1 or -1, $n = 1, 2,$

(11) $\Big(\bigcap_{n=1}^{\infty} E_{\vartheta_n}^{\varepsilon_n}\Big) \cap L \neq \emptyset$ whenever $L \in \mathscr{L}$, $\lambda(L) > 0$;

$\{\vartheta_n\}_{n=1}^{\infty}$ is a sequence of distinct elements from Θ and $\{\varepsilon_n\}_{n=1}^{\infty}$ is a sequence where ε_n is 1 or -1, $n = 1, 2,$

The proof is based on the following lemma which is a special case of a result of Tarski ([12], Hilfsatz 3.16):

There exists a family $\{\Gamma_\vartheta\}_{\vartheta\in\Theta}$ *of distinct subsets of* Γ *for which* $\operatorname{card}(\Theta) = 2^{2^\mathfrak{n}}$ *and the set* $\bigcap_{n=1}^{\infty} \Gamma_{\vartheta_n}^{\varepsilon_n}$ *is nonvoid for every sequence* $\{\vartheta_n\}_{n=1}^{\infty}$ *of distinct elements from* Θ *and every sequence* $\{\varepsilon_n\}_{n=1}^{\infty}$ *with* $\varepsilon_n = 1$ *or* -1, $n = 1, 2, \ldots$

Now let $E_\vartheta = \bigcup \{X_\nu\colon \nu\in\Gamma_\nu\}$. (9) is trivial. (10) follows from (7) and from the fact that the family of all subsets A of X for which

$$\operatorname{card}\big(\tau(A)\,\triangle\,(\tau(X)\cap A)\big) < 2^\mathfrak{n}$$

is a σ-algebra. To prove (11) we notice that

$$E_\vartheta^\varepsilon \supset \bigcup\{X_\nu\colon \nu\in\Gamma_\vartheta^\varepsilon\} \quad \text{if } \varepsilon \text{ is } 1 \text{ or } -1,$$

and hence, if $\nu_0 \in \bigcap_{n=1}^{\infty} \Gamma_{\vartheta_n}^{\varepsilon_n}$, then

$$\bigcap_{n=1}^{\infty} E_{\vartheta_n}^{\varepsilon_n} \supset \bigcap_{n=1}^{\infty} \big(\bigcup\{X_\nu\colon \nu\in\Gamma_{\vartheta_n}^{\varepsilon_n}\}\big) \supset X_{\nu_0}.$$

IV. Consider the family $\mathscr{E}$ of all subsets E of X having the form

$$E = \bigcup\nolimits_{\{\varepsilon_1,\ldots,\varepsilon_n\}} \Big(\big(\bigcap_{k=1}^{n} E_{\vartheta_k}^{\varepsilon_k}\big)\cap L_{\varepsilon_1,\ldots,\varepsilon_n}\Big) \tag{12}$$

where n is a positive integer, $\{\vartheta_1, \ldots, \vartheta_n\}$ is a finite subset of Θ, and for each sequence $\{\varepsilon_1, \ldots, \varepsilon_n\}$ with $\varepsilon_k = 1$ or -1, $k = 1, 2, \ldots, n$, $L_{\varepsilon_1,\ldots,\varepsilon_n}$ is an element of $\mathscr{L}$. Here $\bigcup_{\{\varepsilon_1,\ldots,\varepsilon_n\}}$ means the union over the 2^n distinct sequences $\{\varepsilon_1, \ldots, \varepsilon_n\}$ where $\varepsilon_k = 1$ or -1. Similarly, $\sum_{\{\varepsilon_1,\ldots,\varepsilon_n\}}$ will denote the sum over all of the distinct sequences $\{\varepsilon_1, \ldots, \varepsilon_n\}$ (n is fixed).

We shall prove that the family $\mathscr{E}$ is an algebra of subsets of X, $\mathscr{L} \subset \mathscr{E}$ and if

$$\nu(E) = \frac{1}{2^n} \sum\nolimits_{\{\varepsilon_1,\ldots,\varepsilon_n\}} \lambda(L_{\varepsilon_1,\ldots,\varepsilon_n}) \tag{13}$$

whenever E has the form (12), then ν is well defined and finitely additive on $\mathscr{E}$ and $\nu(L) = \lambda(L)$ for all $L\in\mathscr{L}$. Finally, ν can be extended to a countable additive measure (which we shall denote also by ν) defined on the σ-algebra $\mathscr{N}$ generated by $\mathscr{E}$.

First we observe that if E has the form (12) and $\{\vartheta_{n+1}, \ldots, \vartheta_m\}$ is a finite subset of Θ disjoint from $\{\vartheta_1, \ldots, \vartheta_n\}$, then

$$E = \bigcup\nolimits_{\{\varepsilon_1,\ldots,\varepsilon_m\}} \Big(\big(\bigcap_{j=1}^{m} E_{\vartheta_j}^{\varepsilon_j}\big)\cap L_{\varepsilon_1,\ldots,\varepsilon_m}\Big)$$

where $L_{\varepsilon_1,\ldots,\varepsilon_m}$ is defined to be $L_{\varepsilon_1,\ldots,\varepsilon_n}$ for all sequences $\{\varepsilon_1, \ldots, \varepsilon_m\}$ where $\varepsilon_j = 1$ or -1. Moreover, we have

$$\frac{1}{2^n}\sum_{\{\varepsilon_1,\ldots,\varepsilon_n\}}\lambda(L_{\varepsilon_1,\ldots,\varepsilon_n}) = \frac{1}{2^m}\sum_{\{\varepsilon_1,\ldots,\varepsilon_m\}}\lambda(L_{\varepsilon_1,\ldots,\varepsilon_m}).$$

These observations imply that whenever in the proof we have two sets E and F in $\mathscr{E}$, we may suppose that E has the form (12) and F has the form

$$F = \bigcup_{\{\varepsilon_1,\ldots,\varepsilon_n\}}\Big(\big(\bigcap_{k=1}^{n} E_{\vartheta_n}^{\varepsilon_n}\big)\cap K_{\varepsilon_1,\ldots,\varepsilon_n}\Big) \tag{14}$$

where $K_{\varepsilon_1,\ldots,\varepsilon_n} \in \mathscr{L}$ if $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is a sequence of 1's and -1's. That is, the sets $\{\vartheta_1, \ldots, \vartheta_n\}$ in (12) and (14) are the same.

It is easy to see that $\mathscr{E}$ is an algebra and $\mathscr{L} \subset \mathscr{E}$. Suppose that E has the forms (12) and (14), i.e., $E = F$. In this case,

$$\big(\bigcap_{k=1}^{n} E_{\vartheta_k}^{\varepsilon_k}\big)\cap L_{\varepsilon_1,\ldots,\varepsilon_n} = \big(\bigcap_{k=1}^{n} E_{\vartheta_k}^{\varepsilon_k}\big)\cap K_{\varepsilon_1,\ldots,\varepsilon_n},$$

and hence

$$(L_{\varepsilon_1,\ldots,\varepsilon_n} \triangle K_{\varepsilon_1,\ldots,\varepsilon_n}) \subset \big(\bigcap_{k=1}^{n} E_{\vartheta_k}^{\varepsilon_k}\big)'. \tag{15}$$

Since the intersection of $\bigcap_{k=1}^{n} E_{\vartheta_k}^{\varepsilon_k}$ with any λ measurable set having positive λ measure is nonvoid, the set on the left side of (15) has λ measure 0. Hence

$$\lambda(L_{\varepsilon_1,\ldots,\varepsilon_n}) = \lambda(K_{\varepsilon_1,\ldots,\varepsilon_n})$$

and ν is well defined on $\mathscr{E}$. Clearly, $\nu(L) = \lambda(L)$ if $L \in \mathscr{L}$

We now show that ν is finitely additive on $\mathscr{E}$. Suppose that E and F are disjoint sets having the forms (12) and (14). Then

$$\big(\bigcap_{k=1}^{n} E_{\vartheta_k}^{\varepsilon_k}\big)\cap L_{\varepsilon_1,\ldots,\varepsilon_n}\cap K_{\varepsilon_1,\ldots,\varepsilon_n} = \emptyset$$

so that

$$L_{\varepsilon_1,\ldots,\varepsilon_n}\cap K_{\varepsilon_1,\ldots,\varepsilon_n} \subset \big(\bigcap_{k=1}^{n} E_{\vartheta_k}^{\varepsilon_k}\big)',$$

and hence the set on the left side has λ measure 0. This proves that

$$\lambda(L_{\varepsilon_1,\ldots,\varepsilon_n}\cup K_{\varepsilon_1,\ldots,\varepsilon_n}) = \lambda(L_{\varepsilon_1,\ldots,\varepsilon_n}) + \lambda(K_{\varepsilon_1,\ldots,\varepsilon_n}),$$

and hence $\nu(E\cup F) = \nu(E) + \nu(F)$.

To show that ν can be extended to a countable additive measure on $\mathscr{N}$ it sufficies to prove that if $\{E_p\}_{p=1}^{\infty}$ is a decreasing sequence of sets in $\mathscr{E}$ and

$\nu(E_p) \geqslant \alpha > 0$ for some positive number α and $p = 1, 2, \ldots$, then $\bigcap_{p=1}^{\infty} E_p \neq \emptyset$. (See Theorem 13.A, Halmos [2]; the above property implies immediately that ν is a measure on the algebra $\mathscr{E}$ in Halmos's sense.) We may suppose that we have a sequence $\{\vartheta_n\}_{n=1}^{\infty}$ of distinct elements of Θ, a strictly increasing sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers, and elements $L_{\varepsilon_1,\ldots,\varepsilon_{n_p}}$ in $\mathscr{L}$ for each sequence $\{\varepsilon_1, \ldots, \varepsilon_{n_p}\}$ such that

$$E_p = \bigcup_{\{\varepsilon_1,\ldots,\varepsilon_{n_p}\}} \Big(\big(\bigcap_{k=1}^{n_p} E_{\vartheta_k}^{\varepsilon_k}\big) \cap L_{\varepsilon_1,\ldots,\varepsilon_{n_p}}\Big). \tag{16}$$

Replacing $L_{\varepsilon_1,\ldots,\varepsilon_{n_p}}$ by $\bigcap_{q=1}^{p} L_{\varepsilon_1,\ldots,\varepsilon_{n_q}}$, we may suppose that

$$L_{\varepsilon_1,\ldots,\varepsilon_{n_{p+1}}} \subset L_{\varepsilon_1,\ldots,\varepsilon_{n_p}} \tag{17}$$

for $p = 1, 2, \ldots$ and for every sequence $\{\varepsilon_1, \ldots, \varepsilon_{n_{p+1}}\}$.

For each p, since

$$\nu(E_p) = \frac{1}{2^{n_p}} \sum_{\{\varepsilon_1,\ldots,\varepsilon_{n_p}\}} \lambda(L_{\varepsilon_1,\ldots,\varepsilon_{n_p}}) \geqslant \alpha > 0,$$

there exists at least one combination $\{\varepsilon_1, \ldots, \varepsilon_{n_p}\}$ such that $\lambda(L_{\varepsilon_1,\ldots,\varepsilon_{n_p}}) \geqslant \alpha$.

By appealing to relation (17) it is then not difficult to see that there exists a particular sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ with $\varepsilon_n = 1$ or -1, $n = 1, 2, \ldots$, such that

$$\lambda(L_{\varepsilon_1,\ldots,\varepsilon_{n_p}}) \geqslant \alpha$$

for $p = 1, 2, \ldots$ It follows, again by (17), that for this particular sequence $\{\varepsilon_n\}_{n=1}^{\infty}$

$$\lambda\big(\bigcap_{p=1}^{\infty} L_{\varepsilon_1,\ldots,\varepsilon_{n_p}}\big) \geqslant \alpha,$$

and hence

$$\bigcap_{p=1}^{\infty} E_p \supset \bigcap_{p=1}^{\infty} \Big(\big(\bigcap_{k=1}^{n_p} E_{\vartheta_k}^{\varepsilon_k}\big) \cap L_{\varepsilon_1,\ldots,\varepsilon_{n_p}}\Big) = \big(\bigcap_{k=1}^{\infty} E_{\vartheta_k}^{\varepsilon_k}\big) \cap \big(\bigcap_{p=1}^{\infty} L_{\varepsilon_1,\ldots,\varepsilon_{n_p}}\big) \neq \emptyset.$$

This proves that ν can be extended to a measure on $\mathscr{N}$, also denoted by ν.

V. We now complete the proof by showing that $(X, \mathscr{N}, \nu)$ is J covariant and has weight $2^{2^{\aleph}}$. Indeed, $E_\vartheta \in \mathscr{E} \subset \mathscr{N}$ for every $\vartheta \in \Theta$ and

$$\nu(E_{\vartheta_1} \triangle E_{\vartheta_2}) = \tfrac{1}{2} \cdot \mu(X)$$

whenever $\vartheta_1 \neq \vartheta_2$. Hence $(X, \mathscr{N}, \nu)$ has weight $2^{2^{\aleph}}$.

To show that $(X, \mathscr{N}, \nu)$ is J covariant we have to prove that

$$\tau(N) \in \mathscr{N} \text{ and } \int_N J(\tau)\, d\nu = \nu(\tau(N)) \tag{18}$$

whenever $\tau \in T$ and $N \in \mathscr{N}$ Suppose first that $N = E$ for some $E \in \mathscr{E}$, that is, N has the form (12). Then

$$\tau(E) = \bigcup_{(\varepsilon_1,\ldots,\varepsilon_n)} \Big(\tau\big(\bigcap_{k=1}^{n} E_{g_k}^{\varepsilon_k}\big) \cap \tau(L_{\varepsilon_1,\ldots,\varepsilon_n})\Big).$$

If we let

$$E_0 = \bigcup_{(\varepsilon_1,\ldots,\varepsilon_n)} \Big(\big(\bigcap_{k=1}^{n} E_{g_k}^{\varepsilon_k}\big) \cap \tau(L_{\varepsilon_1,\ldots,\varepsilon_n})\Big)$$

we find that

$$\tau(E) \triangle E_0 \subset \bigcup_{(\varepsilon_1,\ldots,\varepsilon_n)} \Big(\tau\big(\bigcap_{k=1}^{n} E_{g_k}^{\varepsilon_k}\big) \triangle \big(\tau(X) \cap \big(\bigcap_{k=1}^{n} E_{g_k}^{\varepsilon_k}\big)\big)\Big),$$

and hence $\operatorname{card}\big(\tau(E) \triangle E_0\big) < 2^{\mathfrak{n}}$. This proves that $\tau(E) \triangle E_0 \in \mathscr{L} \subset \mathscr{E}$; hence $\tau(E) \in \mathscr{E}$ and

$$\begin{aligned} \nu\big(\tau(E)\big) = \nu(E_0) &= \frac{1}{2^n} \sum_{(\varepsilon_1,\ldots,\varepsilon_n)} \lambda\big(\tau(L_{\varepsilon_1,\ldots,\varepsilon_n})\big) \\ &= \frac{1}{2^n} \sum_{(\varepsilon_1,\ldots,\varepsilon_n)} \int_{L_{\varepsilon_1,\ldots,\varepsilon_n}} J(\tau)\, d\lambda. \end{aligned}$$

If we prove that

$$\int_{\left(\bigcap_{k=1}^{n} E_{g_k}^{\varepsilon_k}\right) \cap L_{\varepsilon_1,\ldots,\varepsilon_n}} f\, d\nu = \frac{1}{2^n} \int_{L_{\varepsilon_1,\ldots,\varepsilon_n}} f\, d\lambda \tag{19}$$

whenever f is a nonnegative λ measurable function, then we have that

$$\nu\big(\tau(E)\big) = \int_E J(\tau)\, d\nu.$$

But (19) is clear if f is the characteristic function of a λ measurable set. Hence we obtain (19) by approximation by linear combinations of characteristic functions.

Now let $\mathscr{B}$ consist of those sets $N \in \mathscr{N}$ which satisfy (18). Since $\mathscr{E} \subset \mathscr{B}$, Theorem 6.B of Halmos [2] shows that $\mathscr{B} = \mathscr{N}$ if $\mathscr{B}$ is a monotone class; that is, if $\mathscr{B}$ is closed under the formation of unions of increasing sequences of sets and of intersection of decreasing sequences of sets. It is easy to verify that $\mathscr{B}$ is a monotone class.

2.5. Remark. The method used in the first step of the proof of Theorem 2.4 may be applied if $(X, \mathscr{M}, \mu)$ is a not complete measure space and $\mathscr{P}$ is the family of all subsets of X which are contained in a μ measurable set of measure 0. This shows that in Theorem 2.4 the measure space $(X, \mathscr{M}, \mu)$ needs not to be complete, but we may add to the statement of Theorem 2.4 the completeness of $(X, \mathscr{N}, \nu)$.

§ 3. An invariant extension of Haar measure

In this section we give an invariant extension of left Haar measure to a very much larger measure space on an arbitrary locally compact nondiscrete Hausdorff group. The facts used in the proof are summed up in Theorems 3.1 and 3.2.

3.1. THEOREM. [Haar] *Let G be a locally compact Hausdorff group. Then there exists a measure λ on G for which*

(1) *λ is a complete Radon measure on G;*

(2) *if L is a λ measurable subset of G and $z \in G$ then zL is a λ measurable subset of G and $\lambda(zL) = \lambda(L)$;*

(3) *if V is a nonvoid open subset of G then $\lambda(V) > 0$;*

(4) *a subset L of G is λ measurable if and only if $L \cap C$ is λ measurable whenever C is a compact subset of G.*

There exists one and only one continuous homomorphism Δ of G into the multiplicative group of positive real numbers such that

(5) *if L is a λ measurable subset of G and $z \in G$ then Lz and L^{-1} are λ measurable and*

$$\lambda(Lz) = \Delta(z) \cdot \lambda(L);$$

$$\lambda(L^{-1}) = \int_L \Delta(z^{-1})\, d\lambda z.$$

The measure λ is essentially unique: If λ^* is another measure with properties (1)–(4) then the domains of λ and λ^* are the same and there exists a positive real number c for which $\lambda(L) = c \cdot \lambda^*(L)$ whenever L is an element of the common domain.

The function Δ is called *modular function* of G. Any measure λ satisfying (1)–(4) is called a *left Haar measure* on G.

3.2. THEOREM. *Let G be a locally compact nondiscrete Hausdorff group and let $(G, \mathscr{L}, \lambda)$ be a left Haar measure space on G. Let $\mathfrak{n}$ denote the character of the topological space G. Then $\mathfrak{n}$ is an infinite cardinal number. If X is a subset of G which has a σ-compact closure X^- then* $\operatorname{card}(X) \leqslant 2^{\mathfrak{n}}$ *and X as a subspace has weight not greater then $\mathfrak{n}$. If X is a λ measurable σ-finite subset of G with $\lambda(X) > 0$ then* $\operatorname{card}(X) = 2^{\mathfrak{n}}$. *If $0 < \lambda(X) < \infty$ then the subspace $(X, \mathscr{L}_X, \lambda_X)$ has weight $\mathfrak{n}$, and hence the character of the measure λ is equal to $\mathfrak{n}$.*

Proof. $\mathfrak{n}$ is infinite, because G is nondiscrete. Let V be an open symmetric neighbourhood of the identity of G which has a compact closure, and let $H = \bigcup_{n=1}^{\infty} V^n$. Then H is an open, closed, compactly generated

σ-compact subgroup of G. As a consequence of a theorem of Čech and Pospíšil, $\operatorname{card}(H) = 2^{\mathfrak{n}}$ and the weight of H is $\mathfrak{n}$. (See Hewitt and Ross [4], 28.58.) If X is a subset of G with σ-compact closure X^- then X can be covered by countable many cosets xH where $x \in G$. Since every coset xH has weight $\mathfrak{n}$ and cardinal number $2^{\mathfrak{n}}$, we have that $\operatorname{card}(X) \leqslant 2^{\mathfrak{n}}$ and X as a subspace has weight not greater than $\mathfrak{n}$.

Now suppose that X is λ measurable, σ-finite and $\lambda(X) > 0$. Let $X = \bigcup_{i=0}^{\infty} X_i$, where $X_0, X_1, \ldots$ are λ measurable sets with finite λ measure and $\lambda(X_0) > 0$. We may suppose that X_0 is a Borel set. By a theorem of Steinhaus the set $X_0(X_0^{-1})$ contains an open neighbourhood U of the identity (see Hewitt and Ross [3], (20.17)). Since H is σ-compact, it can be covered by countable many sets xU.

Hence

$$2^{\mathfrak{n}} = \operatorname{card}(H) \leqslant \aleph_0 \cdot \operatorname{card}(U) = \operatorname{card}(U)$$
$$\leqslant \operatorname{card}\left(X_0(X_0^{-1})\right) \leqslant \left(\operatorname{card}(X_0)\right)^2 = \operatorname{card}(X_0) \leqslant \operatorname{card}(X).$$

To prove the other inequality let us choose open sets $U_0, U_1, U_2, \ldots$ for which $X_i \subset U_i$ and $\lambda(U_i) < \infty$ $(i = 0, 1, \ldots)$. Since $U_i \cap xH$ is void or has positive λ measure for $x \in G$, $(i = 1, 2, \ldots)$, we have that U_i intersects only countable many cosets xH. This proves that $\operatorname{card}(U_i) \leqslant 2^{\mathfrak{n}}$ and hence $\operatorname{card}(X) \leqslant 2^{\mathfrak{n}}$.

To show the last statement let X be a λ measurable subset of G with finite positive measure. First suppose that X is compact. Then X as a subspace has a topological base $\mathscr{C}$ with $\operatorname{card}(\mathscr{C}) \leqslant 2^{\mathfrak{n}}$. Let

$$\mathscr{B} = \left\{ \bigcup_{i=1}^{n} C_i \colon n \text{ is a natural number, } C_1, \ldots, C_n \in \mathscr{C} \right\}.$$

Given that λ_X is a Radon measure on X, if $\varepsilon > 0$ and $M \in \mathscr{L}$, $M \subset X$ there exist subsets C and U of X such that C is compact, U is open in X, $C \subset M \subset U$ and $\lambda(U \cap C') < \varepsilon$. Choosing a $B \in \mathscr{B}$ for which $C \subset B \subset U$, we have that $\lambda(M \triangle B) < \varepsilon$, and hence $\mathscr{B}$ is a base of $(X, \mathscr{L}_X, \lambda_X)$. Clearly, $\operatorname{card}(\mathscr{B}) \leqslant \mathfrak{n}$.

If X is not compact then let us choose compact sets X_n and measure theoretic bases $\mathscr{B}_n$ in X_n $(n = 1, 2, \ldots)$, for which $X_n \subset X$, $\lambda(X \cap X_n') < \frac{1}{n}$. It is clear that $\mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}_n$ is a base in $(X, \mathscr{L}_X, \lambda_X)$ and $\operatorname{card}(\mathscr{B}) \leqslant \mathfrak{n}$.

Now let $\mathscr{B}$ be an arbitrary base of $(X, \mathscr{L}_X, \lambda_X)$. We shall prove that $\operatorname{card}(\mathscr{B}) \geqslant \mathfrak{n}$. We may suppose that X is compact. If $B \in \mathscr{B}$ and n is a natural number then let

$$V_{B,n} = \{x \colon x \in G \quad \text{and} \quad \lambda(xB \cap B) > \lambda(B) - 1/n\}$$

and let

$$\mathscr{V} = \{V_{B,n}\colon\ B \in \mathscr{B},\ n = 1, 2, \ldots\}.$$

We shall prove that $\mathscr{V}$ is an open base at e, the identity of G. By Hewitt and Ross [3], (20.17) the function $x \mapsto \lambda(xA \cap A)$ is continuous on G whenever $A \in \mathscr{L}$ and A^- is compact. This proves that every element of $\mathscr{V}$ is an open neighbourhood of e. Now let U be an arbitrary neighbourhood of e and let V be an open neighbourhood of e for which $VV^{-1} \subset U$. Since X is compact and it is covered by the sets Vy, $y \in G$, there exists a $y_0 \in G$ for which $\lambda(X \cap Vy_0) > 0$. Let $A = X \cap Vy_0$ and let n be a natural number such that $\frac{1}{n} < \lambda(A)$. Let us choose a set $B \in \mathscr{B}$ for which $\lambda(A \triangle B) < \frac{1}{8n}$. Using the inclusion

$$xB \cap B \subset (xA \cap A) \cup (A \triangle B) \cup x(A \triangle B),$$

we have

$$\lambda(xB \cap B) \leqslant \lambda(xA \cap A) + 1/4n$$

and, by

$$A \subset B \cup (A \triangle B),$$

we have

$$\lambda(A) \leqslant \lambda(B) + 1/8n.$$

Hence

$$\lambda(A) - \lambda(xA \cap A) \leqslant \lambda(B) - \lambda(xB \cap B) + 1/2n$$

for $x \in G$. This shows that

$$\begin{aligned} V_{B,2n} = \{x\colon\ \lambda(B) - \lambda(xB \cap B) < 1/2n\} \\ &\subset \{x\colon\ \lambda(A) - \lambda(xA \cap A) < 1/n\} \\ &\subset \{x\colon\ \lambda(xA \cap A) > 0\} \subset \{x\colon\ xA \cap A \neq \emptyset\} \\ &= AA^{-1} \subset Vy_0 y_0^{-1} V^{-1} = VV^{-1} \subset U. \end{aligned}$$

This proves that $\mathscr{V}$ is an open base at e. Clearly,

$$\operatorname{card}(\mathscr{V}) \leqslant \operatorname{card}(\mathscr{B}) \cdot \aleph_0$$

By Hevitt and Ross [3], (11.44) we have that the inequality $\operatorname{card}(\mathscr{B}) < \aleph_0$ is impossible, because G is nondiscrete and so $\lambda\{e\} = 0$. Hence $\operatorname{card}(\mathscr{B}) \geqslant \mathfrak{n}$.

3.3 THEOREM. *Let G be a locally compact nondiscrete Hausdorff group and λ a left Haar measure on G. Let $\mathscr{L}$ denote the family of the λ measurable* *sets, let Δ denote the modular function of G, and let* $\mathfrak{n}$ *denote the character of the topological space G. Then the measure space $(G, \mathscr{L}, \lambda)$ (which has by Theorem 3.2 character* $\mathfrak{n}$*) can be extended to a complete measure space $(G, \mathscr{K}, \varkappa)$ such that $\varkappa$ has character $2^{2^{\mathfrak{n}}}$ and if $z \in G$, $K \in \mathscr{K}$ then*

(1) *$zK \in \mathscr{K}$ and $\varkappa(zK) = \varkappa(K)$;*

(2) *$Kz \in \mathscr{K}$ and $\varkappa(Kz) = \Delta(z) \cdot \varkappa(K)$;*

(3) *$K^{-1} \in \mathscr{K}$ and $\varkappa(K^{-1}) = \int_K \frac{1}{\Delta(z)} d\varkappa z$.*

Proof. The construction will be carried out in four steps.

I. First we shall give a suitable dissection of G. Let X be an open symmetric neighbourhood of the identity e of G with compact closure. Let H denote the open and closed subgroup of G generated by X. Since H is σ-compact, there exist elements $y_1, y_2, \ldots \in H$ such that $\bigcup_{n=1}^{\infty} y_n X = H$. Let $Y_1 = y_1 X$ and $Y_n = y_n X \cap (\bigcup_{i=1}^{n-1} Y_i)'$ for $n > 1$. Then the sets $Y_1, Y_2, Y_3,$ are disjoint λ measurable sets with finite λ measure, $\bigcup_{n=1}^{\infty} Y_n = H$ and $Y_n \subset y_n X$.

Let Y be a subset of G containing one and only one element from each coset xH, $x \in G$. Let $\{x_\gamma\colon \gamma \in \Gamma\}$ be the set of all elements yy_n, where $y \in Y$ and $n = 1, 2,$ Let $X_\gamma = yY_n$ if $x_\gamma = yy_n$. Then $\{X_\gamma\colon \gamma \in \Gamma\}$ is a disjoint family of subsets of G, each X_γ is λ measurable and has finite λ measure, and $x_\gamma^{-1} X_\gamma \subset X$ for every $\gamma \in \Gamma$.

II. We prove that for a subset A of G the following conditions are equivalent:

(4) A^- is σ-compact.

(5) A is contained in a λ measurable subset of G with σ-finite λ measure.

(6) $A \cap X_\gamma \neq \emptyset$ is satisfied at most countable many $\gamma \in \Gamma$.

It is clear that (4) implies (5). If (5) is satisfied then $A = \bigcup_{n=1}^{\infty} A_n$ where each A_n is contained in a λ measurable subset B_n of G with finite measure. Clearly we may suppose that B_n is open. Given that H is open, every coset xH, $x \in G$ is open, and hence if the set $xH \cap B_n$ is nonvoid then it has positive λ measure. Hence B_n intersects only countable many cosets. This proves (6). (6) implies (4), because A is a subset of the closed set $\bigcup \{xH\colon xH \cap A \neq \emptyset\}$.

The equivalence of (4), (5), (6) proves that if $A \in \mathscr{L}$ is not σ-finite then $A \cap X_\gamma \neq \emptyset$ is valid for more than countable many γ's ($\gamma \in \Gamma$), and if $A \in \mathscr{L}$ is σ-finite then

$$\lambda(A) = \sum_{\gamma \in \Gamma} \lambda(A \cap X_\gamma) = \sum_{\gamma \in \Gamma} \lambda\big(x_\gamma^{-1}(A \cap X_\gamma)\big).$$

III. Now we give a "local" extension of λ using Theorem 2.4. Let $(X, \mathscr{M}, \mu)$ denote the subspace $(X, \mathscr{L}_X, \lambda_X)$. Clearly, $\operatorname{card}(X) \doteq 2^{\mathfrak{n}}$. For every $x, y \in G$ let

$$V_{x,y} = X \cap xXy,$$

$$\tau_{x,y}\colon V_{x^{-1},y^{-1}} \to V_{x,y}, \ \tau_{x,y}(v) = xvy,$$

$$\tau^*_{x,y}\colon V_{y,x} \to V_{x,y}, \ \tau^*_{x,y}(v) = xv^{-1}y.$$

Let $T = \{\tau_{x,y}, \tau^*_{x,y}\colon x, y \in G\}$. We shall prove that $\operatorname{card}(T) \leqslant 2^{\mathfrak{n}}$. Since $V_{x,y}$ is an open subset of the locally compact topological space X with weight $\mathfrak{n}$ (topologized as the subspace of G), there exist at most $2^{\mathfrak{n}}$ different $V_{x,y}$. A continuous function mapping $V_{x,y}$ into X is uniquely determined by its values on a dense subset of $V_{x,y}$. Since $V_{x,y}$ has a base of cardinal number not greater than $\mathfrak{n}$, $V_{x,y}$ has a dense subset of cardinal number not greater than $\mathfrak{n}$. Hence there exist at most $(2^{\mathfrak{n}})^{\mathfrak{n}} = 2^{\mathfrak{n}\times\mathfrak{n}} = 2^{\mathfrak{n}}$ different continuous functions mapping $V_{x,y}$ into X. Given that $\tau_{x,y}$ and $\tau^*_{x,y}$ are continuous functions, $\operatorname{card}(T) \leqslant 2^{\mathfrak{n}}$.

Let J be defined in the following way:

$$J(\tau_{x,y})(v) = \begin{cases} 0 & \text{if } v \in X \cap (V_{x^{-1},y^{-1}})'; \\ \Delta(y) & \text{if } v \in V_{x^{-1},y^{-1}} \end{cases} \tag{7}$$

$$J(\tau^*_{x,y})(v) = \begin{cases} 0 & \text{if } v \in X \cap (V_{y,x})'; \\ \Delta(v^{-1}y) & \text{if } v \in V_{y,x}. \end{cases} \tag{8}$$

Then by 3.1 we have

$$\int_A J(\tau_{x,y})(v)\, d\mu v = \mu(\tau_{x,y}(A)) \tag{9}$$

and

$$\int_A J(\tau^*_{x,y})(v)\, d\mu v = \mu(\tau^*_{x,y}(A)) \tag{10}$$

for all $A \in \mathscr{M}$ and $x, y \in G$. This proves that $J(\tau)$ is uniquely defined on T since if $\tau \in T$ is represented as $\tau_{x,y}$ or $\tau^*_{x,y}$ in different ways, then by (9), (10) $J(\tau_{x,y})$ or $J(\tau^*_{x,y})$ should be the same since they are positive and continuous on the open set $\operatorname{dmn}\tau = V$ and 0 on $X \cap V'$. This shows that J is uniquely defined on T and $J(\tau)$ is a nonnegative μ measurable function whenever $\tau \in T$; moreover, the measure space $(X, \mathscr{M}, \mu)$ is J covariant. Now let

$$\mathscr{F} = \{F\colon F \text{ is compact}, \ F \subset X, \ \mu(F) > 0\}.$$

If $A \in \mathscr{M}$ and $\mu(A) > 0$ then by $\mu(A) \leqslant \mu(X) < \infty$ there exists an $F \in \mathscr{F}$ for which $F \subset A$. Since for each $F \in \mathscr{F}$ the set $X \cap F'$ is an open subset of the topological space X with weight $\mathfrak{n}$, we have $\operatorname{card}(\mathscr{F}) \leqslant 2^{\mathfrak{n}}$.

Applying Theorem 2.4 and Remark 2.5, we get a J covariant complete extension $(X, \mathscr{N}, \nu)$ of $(X, \mathscr{M}, \mu)$ with weight $2^{2^{\mathfrak{n}}}$

IV. Finally we extend the measure ν to a measure $\varkappa$. Let $\mathscr{K} = \{K\colon K \subset G,\ x_\gamma^{-1}(K \cap X_\gamma) \in \mathscr{N}$ if $\gamma \in \Gamma\}$ and

$$\varkappa(K) = \begin{cases} \infty & \text{if } K \in \mathscr{K} \text{ and } K^- \text{ is not } \sigma\text{-compact;} \\ \sum\limits_{\gamma \in \Gamma} \nu(x_\gamma^{-1}(K \cap X_\gamma)) & \text{if } K \in \mathscr{K} \text{ and } K^- \text{ is } \sigma\text{-compact.} \end{cases}$$

It is easy to see that $(G, \mathscr{K}, \varkappa)$ is a complete measure space and that it is an extension of $(G, \mathscr{L}, \lambda)$.

To show that $\varkappa$ satisfies condition (1) suppose that $z \in G$ and $K \in \mathscr{K}$ Let $\gamma, \delta \in \Gamma$, $x = x_\gamma^{-1} z x_\delta$ and $y = e$. If

$$N = x_\delta^{-1}(K \cap z^{-1} X_\gamma \cap X_\delta)$$

then $N \in \mathscr{N}$, because $K \in \mathscr{K}$, $z^{-1} X_\gamma \in \mathscr{L} \subset \mathscr{K}$ and $K \cap z^{-1} X_\gamma \in \mathscr{K}$ Moreover,

$$\begin{aligned} N &= x_\delta^{-1}(K \cap z^{-1} X_\gamma \cap X_\delta) \\ &= x_\delta^{-1} K \cap x_\delta^{-1} z^{-1} x_\gamma (x_\gamma^{-1} X_\gamma) \cap x_\delta^{-1} X_\delta \\ &\subset x_\delta^{-1} z^{-1} x_\gamma X \cap X = x^{-1} X y^{-1} \cap X = V_{x^{-1}, y^{-1}}. \end{aligned}$$

By the properties of ν

$$\begin{aligned} xNy &= x_\gamma^{-1} z x_\delta (x_\delta^{-1}(K \cap z^{-1} X_\gamma \cap X_\delta)) \\ &= x_\gamma^{-1}(zK \cap X_\gamma \cap z X_\delta) \in \mathscr{N} \text{ and } \nu(xNy) = \nu(N). \end{aligned}$$

Since X_δ^- is σ-compact, the closure of zX_δ is also σ-compact, therefore the set

$$\Delta_\gamma = \{\delta\colon \delta \in \Gamma,\ X_\gamma \cap z X_\delta \neq \emptyset\}$$

is countable. Hence

$$\begin{aligned} x_\gamma^{-1}(zK \cap X_\gamma) &= x_\gamma^{-1}\Big(\bigcup_{\delta \in \Delta_\gamma} (zK \cap X_\gamma \cap z X_\delta)\Big) \\ &= \bigcup_{\delta \in \Delta_\gamma} x_\gamma^{-1}(zK \cap X_\gamma \cap z X_\delta) \in \mathscr{N} \end{aligned}$$

for $\gamma \in \Gamma$ This shows that $zK \in \mathscr{K}$ Moreover,

$$\nu(x_\gamma^{-1}(zK \cap X_\gamma)) = \sum_{\delta \in \Delta_\gamma} \nu(x_\delta^{-1}(K \cap z^{-1} X_\gamma \cap X_\delta)) = \varkappa(K \cap z^{-1} X_\gamma)$$

whenever $\gamma \in \Gamma$ Now if K^- is not σ-compact then zK^- is not σ-compact and $\varkappa(zK) = \infty = \varkappa(K)$. If K^- is σ-compact then zK^- is also σ-compact, and hence

$$zK \cap X_\gamma \neq \emptyset \quad \text{and} \quad z^{-1}(zK \cap X_\gamma) = K \cap z^{-1} X_\gamma \neq \emptyset$$

for countable many γ's ($\gamma \in \Gamma$). This proves that

$$\varkappa(zK) = \sum_{\gamma \in \Gamma} \nu\big(x_\gamma^{-1}(zK \cap X_\gamma)\big) = \sum_{\gamma \in \Gamma} \varkappa(K \cap z^{-1} X_\gamma) = \varkappa(K).$$

Now we show that $\varkappa$ satisfies condition (2). Let $z \in G$, $K \in \mathscr{K}$, $\gamma, \delta \in \Gamma$, $x = x_\gamma^{-1} x_\delta$ and $y = z$. Similarly to the proof of (1), if

$$N = x_\delta^{-1}(K \cap X_\gamma z^{-1} \cap X_\delta)$$

then $N \in \mathscr{N}$ and $N \subset V_{x^{-1}, y^{-1}}$. Hence using the properties of ν we have that

$$xNy = x_\gamma^{-1}(Kz \cap X_\gamma \cap X_\delta z)$$

and

$$\nu(xNy) = \Delta(z) \cdot \nu(N).$$

Given that the set

$$\Delta_\gamma = \{\delta\colon \delta \in \Gamma,\ X_\gamma \cap X_\delta z \neq \emptyset\}$$

is countable, we have that

$$x_\gamma^{-1}(Kz \cap X_\gamma) \in \mathscr{N} \quad \text{if } \gamma \in \Gamma$$

Hence $Kz \in \mathscr{K}$ and

$$\nu\big(x_\gamma^{-1}(Kz \cap X_\gamma)\big) = \Delta(z) \cdot \varkappa(K \cap X_\gamma z^{-1})$$

for $\gamma \in \Gamma$ From this we get (2) similarly to the proof of (1).

To prove (3) we remark that

$$\int_{zK} f(v)\, d\varkappa v = \int_K f(z^{-1} v)\, d\varkappa v \tag{11}$$

whenever $z \in G$, $K \in \mathscr{K}$ and $f\colon G \to [0, \infty]$ is a $\varkappa$ measurable function. This statement is obvious from (1) if f is the characteristic function of a $\varkappa$ measurable set. For nonnegative $\varkappa$ measurable functions it can be proved using approximation by linear combinations of characteristic functions.

Now let $K \in \mathscr{K}$, $\gamma, \delta \in \Gamma$, $y = x_\delta^{-1}$ and $x = x_\gamma^{-1}$. If

$$N = x_\delta^{-1}(K \cap X_\gamma^{-1} \cap X_\delta),$$

then $N \in \mathscr{N}$ and $N \subset V_{y,x}$. Hence by the properties of ν we have

$$xN^{-1} y = x_\gamma^{-1}(K^{-1} \cap X_\gamma \cap X_\delta^{-1}) \in \mathscr{N},$$

$$\nu(xN^{-1} y) = \int_N \Delta(v^{-1} y)\, d\nu v = \Delta(x_\delta^{-1}) \cdot \int_N \Delta(v^{-1})\, d\nu v.$$

Given that the set

$$\Delta_\gamma = \{\delta:\ \delta \in \Gamma,\ X_\gamma \cap X_\delta^{-1} \neq \emptyset\}$$

is countable, we have that

$$x_\gamma^{-1}(K^{-1} \cap X_\gamma) \in \mathcal{N} \quad \text{whenever } \gamma \in \Gamma$$

Hence $K^{-1} \in \mathcal{K}$ Moreover,

$$\begin{aligned} \nu\big(x_\gamma^{-1}(K^{-1} \cap X_\gamma)\big) &= \sum_{\delta \in \Delta_\gamma} \nu\big(x_\gamma^{-1}(K^{-1} \cap X_\gamma \cap X_\delta^{-1})\big) \\ &= \sum_{\delta \in \Delta_\gamma} \Delta(x_\delta^{-1}) \cdot \int_{x_\delta^{-1}(K \cap X_\gamma^{-1} \cap X_\delta)} \Delta(v^{-1})\, d\nu v \\ &= \sum_{\delta \in \Delta_\gamma} \Delta(x_\delta^{-1}) \cdot \int_{(K \cap X_\gamma^{-1} \cap X_\delta)} \Delta(x_\delta v^{-1})\, d\varkappa v \\ &= \sum_{\delta \in \Delta_\gamma} \int_{(K \cap X_\gamma^{-1} \cap X_\delta)} \Delta(v^{-1})\, d\varkappa v = \int_{K \cap X_\gamma^{-1}} \Delta(v^{-1})\, d\varkappa v. \end{aligned}$$

If K^- is not σ-compact then $(K^{-1})^-$ is not σ-compact too and the everywhere positive function $1/\Delta$ has integral ∞ over K. Hence in this case (3) is satisfied. If K^- is σ-compact then $K \cap X_\gamma^{-1} \neq \emptyset$ holds for countable many γ's ($\gamma \in \Gamma$), and hence

$$\varkappa(K^{-1}) = \sum_{\gamma \in \Gamma} \varkappa\big(x_\gamma^{-1}(K^{-1} \cap X_\gamma)\big) = \sum_{\gamma \in \Gamma} \int_{K \cap X_\gamma^{-1}} (v^{-1})\, d\varkappa v = \int_K \Delta(v^{-1})\, d\varkappa v. \ \square$$

§ 4. Covariant extension of Lebesgue measure

In this section we prove that the Lebesgue measure on the n-dimensional Euclidean space can be extended to a measure with character 2^c and this measure is transformed by coordinate transformations like the Lebesgue measure.

If f is a function defined on a subset of R^n and $x \in R^n$ we shall use the notion of the *approximative limit of f at x*, the *approximative upper limit of f at x*, the *differential of f at x*, and the *approximative differential of f at x* denote by $ap \lim_{z \to x} f(z)$, $ap \limsup_{z \to x} f(z)$, $Df(x)$ and $ap Df(x)$, respectively. The definition and the detailed discussion of these notions can be found in Federer [1], 2.8, 2.9, 2.10, 3.1 and 3.2.

The following theorem concerning transformations of integrals is a reformulation of Federer [1], 3.2.1 and 3.2.3.

4.1. THEOREM. *Let $(R^n, \mathcal{L}, \lambda)$ be the Lebesgue measure space. Let T be the* *family of all functions τ mapping a Lebesgue measurable subset of R^n into R^n such that*

$$ap \limsup_{x \to a} \frac{|\tau(x) - \tau(a)|}{|x-a|} < \infty \quad \text{if} \quad a \in \operatorname{dmn} \tau$$

where $|\ |$ is the usual norm on R^n. If $\tau \in T$ then let $J(\tau)(x)$ be the absolute value of the determinant of $ap\, D\tau(x)$ provided that τ is approximative differentiable at x and is 0 elsewhere. Then $(R^n, \mathscr{L}, \lambda)$ is J covariant.

Using the general theorem 2.4, we shall prove the following

4.2. THEOREM. *With the notation of preceding theorem, there exists a J covariant complete extension $(R^n, \mathscr{K}, \varkappa)$ of $(R^n, \mathscr{L}, \lambda)$ with character 2^c.*

Proof. Let

$$X = \{x\colon\ x = (x_1, \ldots, x_n) \in R^n, \quad 0 \leqslant x_i < 1 \ (i = 1, \ldots, n)\},$$
$$\mathscr{M} = \{M\colon\ M \in \mathscr{L},\ M \subset X\},$$

and let μ be the restriction of λ to $\mathscr{M}$. Let T_0 denote the set of all Lipschitzian functions τ (see Federer [1], 2.2.7) mapping a Borel subset of X into X. Clearly, $T_0 \subset T$. Let J_0 denote the restriction of J to T_0 and

$$\mathscr{F} = \{F\colon\ F \subset X,\ F \text{ is compact and } \lambda(F) > 0\}.$$

Applying Theorem 2.4 and Remark 2.5 to the J_0 covariant measure space $(X, \mathscr{M}, \mu)$, we get a J_0 covariant complete extension $(X, \mathscr{N}, \nu)$ of $(X, \mathscr{M}, \mu)$ with character 2^c.

Denote by Z^n the set of all points of R^n with integer coordinates. Let

$$\mathscr{K} = \{K\colon\ K \subset R^n \quad \text{and} \quad (K \cap (X+z)) - z \in \mathscr{N} \quad \text{if} \quad z \in Z^n\};$$
$$\varkappa(K) = \sum_{z \in Z^n} \nu\big((K \cap (X+z)) - z\big) \quad \text{if} \quad K \in \mathscr{K}$$

It is easy to see that $(R^n, \mathscr{K}, \varkappa)$ is a complete measure space with character 2^c and that $\mathscr{L} \subset \mathscr{K}$, $\mathscr{N} \subset \mathscr{K}$, $\varkappa|\mathscr{L} = \lambda$ and $\varkappa|\mathscr{N} = \nu$. We shall prove that $(R^n, \mathscr{K}, \varkappa)$ is J covariant.

First suppose that $\tau \in T$, τ is one-to-one Lipschitzian function and $\operatorname{dmn} \tau$ is a Borel set. Let A be the domain of τ and let

$$A_{z_1, z_2} = A \cap (X + z_1) \cap \tau^{-1}(X + z_2)$$

whenever $z_1, z_2 \in Z^n$. Then the sets A_{z_1, z_2} are disjoint Borel sets with union A. Let

$$\tau_{z_1, z_2}(x) = \tau(x + z_1) - z_2 \quad \text{if} \quad x \in A_{z_1, z_2} \text{ and } z_1, z_2 \in Z^n.$$

Then $\tau_{z_1,z_2} \in T_0$. Hence if $K \in \mathscr{K}$ then

$$\begin{aligned}(\tau(K)\cap(X+z_2))-z_2 &= \tau(K\cap\tau^{-1}(X+z_2))-z_2 \\ &= \tau(K\cap(\bigcup_{z_1\in Z^n} A_{z_1,z_2}))-z_2 = \bigcup_{z_1\in Z^n}(\tau(K\cap A_{z_1,z_2})-z_2) \\ &= \bigcup_{z_1\in Z^n} \tau_{z_1,z_2}((K\cap A_{z_1,z_2})-z_1)\in\mathcal{N},\end{aligned}$$

because $(K\cap A_{z_1,z_2})-z_1 \in \mathcal{N}$ whenever $z_1 \in Z^n$. This proves that $\tau(K)\in\mathscr{K}$

Using the definition of the approximative differential, it is not hard to prove that the function $J(\tau|B)$ is λ almost equal with the product of $J(\tau)$ and the characteristic function of B if $\tau\in T$ and $B\in\mathscr{L}$ From this fact, it follows that

$$\begin{aligned}\varkappa(\tau(K)) &= \sum_{z_2\in Z^n} \nu((\tau(K)\cap(X+z_2))-z_2) \\ &= \sum_{z_2\in Z^n}\sum_{z_1\in Z^n} \nu(\tau_{z_1,z_2}((K\cap A_{z_1,z_2})-z_2)) \\ &= \sum_{z_2\in Z^n}\sum_{z_1\in Z^n}\int_{K\cap A_{z_1,z_2}} J(\tau)(x+z_1)\,d\nu x\end{aligned}$$

if $K\in\mathscr{K}$ Using approximation with linear combinations of characteristic functions, it is not hard to prove that

$$\int_N f(x)\,d\nu x = \int_{N+z} f(x-z)\,d\varkappa x$$

whenever $z\in Z^n$, $N\in\mathcal{N}$ and f is a nonnegative λ measurable function. Hence

$$\varkappa(\tau(K)) = \sum_{z_2\in Z^n}\sum_{z_1\in Z^n}\int_{K\cap A_{z_1,z_2}} J(\tau)(x)\,d\varkappa x = \int_K J(\tau)(x)\,d\varkappa x.$$

Now suppose that $\tau\colon R^n\to R^n$ is a Lipschitzian function. By Theorems 3.1.6 and 3.2.2 of Federer [1], there exist disjoint Borel sets $A_0, A_1,$ for which $\bigcup_{i=0}^{\infty} A_i = R^n$, $\tau|A_i$ is one-to-one for $i = 1, 2, \ldots$, and $J(\tau)(x) = 0$ for λ almost all $x\in A_0$; that is, $\tau(A_0)\in\mathscr{L}$ and $\lambda(\tau(A_0)) = 0$. Now, if $K\in\mathscr{K}$ then $\tau(K\cap A_0)\in\mathscr{K}$, $\varkappa(\tau(K\cap A_0)) = 0$, and $\tau(K\cap A_i)\in\mathscr{K}$ Thus the function

$$N(\tau|K, y) = \sum_{i=0}^{\infty} N(\tau|K\cap A_i, y)$$

is $\varkappa$ measurable, and

$$\begin{aligned}\int_{R^n} N(\tau|K, y)\,d\varkappa y &= \sum_{i=0}^{\infty} \int_{R^n} N(\tau|K\cap A_i, y)\,d\varkappa y \\ &= \sum_{i=0}^{\infty} \int_{K\cap A_i} J(\tau|A_i)(x)\,d\varkappa x = \sum_{i=0}^{\infty} \int_{K\cap A_i} J(\tau)(x)\,d\varkappa x \\ &= \int_K J(\tau)(x)\,d\varkappa x.\end{aligned}$$

Next suppose that τ is an arbitrary element of T, and denote by A the domain of τ. By Theorem 3.1.8 of Federer [1], there exist disjoint λ measurable sets $A_1, A_2,$ for which $\bigcup_{i=1}^{\infty} A_i = A$ and $\tau|A_i$ is a Lipschitzian function ($i = 1, 2, \ldots$). By Kirszbraun's theorem (see Federer [1], 2.10.43), there exists a Lipschitzian extension τ_i: $R^n \to R^n$ of $\tau|A_i$ ($i = 1, 2, \ldots$). If ξ_{A_i} denotes the characteristic function of A_i, then the functions $\xi_{A_i}\cdot J(\tau_i)$, $J(\tau|A_i)$ and $\xi_{A_i}\cdot J(\tau)$ are λ almost equal. Hence

$$N(\tau|K, y) = \sum_{i=1}^{\infty} N(\tau|A_i\cap K, y) = \sum_{i=1}^{\infty} N(\tau_i|A_i\cap K, y)$$

is a $\varkappa$ measurable function and

$$\begin{aligned}\int_{R^n} N(\tau|K, y)\,d\varkappa y &= \sum_{i=1}^{\infty} \int_{R^n} N(\tau|A_i\cap K, y)\,d\varkappa y \\ &= \sum_{i=1}^{\infty} \int_{R^n} N(\tau_i|A_i\cap K, y)\,d\varkappa y = \sum_{i=1}^{\infty} \int_{A_i\cap K} J(\tau_i)(x)\,d\varkappa x \\ &= \sum_{i=1}^{\infty} \int_{A_i\cap K} J(\tau)(x)\,d\varkappa x = \int_K J(\tau)(x)\,d\varkappa x. \quad \square\end{aligned}$$

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