POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCCLIII

WIESŁAW KRÓLIKOWSKI

On Fueter–Hurwitz regular mappings

WARSZAWA 1996

Wiesław Królikowski
Institute of Mathematics
Polish Academy of Sciences
Łódź Branch
Narutowicza 56
PL-90-136 Łódź, Poland

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by

PRINTED IN POLAND

ISSN 0012-3862

CONTENTS

1991 *Mathematics Subject Classification*: 15A63, 15A66, 30G35, 32A30, 32K15, 53C10–53C40.
Key words and phrases: generalized Cauchy–Riemann equations, Fueter–Hurwitz equation, hypercomplex regular functions, quaternionic analysis, quaternionic manifolds.
Received 2.3.1994; revised version 17.3.1995.

Abstract

The paper deals with a special class of mappings from the so-called hypercomplex analysis.

In the first part, several aspects of Fueter regular and biregular functions are considered. Various formulations of the Fueter derivative, some associated Chain Rules and a classification of quaternionic functions with holomorphic slices are given. Moreover, a pointwise criterion for local biregularity in terms of the Jacobian differential is proved.

In the second part, the Lichnerowicz homotopic invariant is generalized to the case of quaternionic Kähler manifolds and the idea of the construction is applied to many different contexts. The most significant result is the proof that regular mappings in the sense of Fueter appear in a natural way in the Eells theory of harmonic mappings.

The third part is a fragment of the so-called theory of Hurwitz pairs. In this theory an analogue of the classical Cauchy–Riemann equations can be defined. It is called a Fueter–Hurwitz equation. A construction of polynomial solutions of an arbitrary degree to the FH-equation, general solution of the Fourier type and its integral representation are given.

A supercomplex structure in the theory of pseudo-Euclidean Hurwitz pairs is introduced and its algebraic and geometric properties are described. Moreover, a necessary and sufficient condition for the existence of such structures is given.

Finally, a counterpart to the Hurwitz condition in the theory of Clifford algebras is found.

Dedicated to my Son

Introduction

The present paper concentrates on a special class of mappings from the so-called "hypercomplex analysis". The starting point was the problem of finding an analog of a holomorphic mapping from complex analysis in the quaternionic case. It is well known that there are many equivalent conditions for holomorphy in complex analysis. It turned out (the fact discovered by several authors) that the transmission of such conditions to the quaternions gives in each case different classes of functions (see e.g. [58]). The best promising attempt was the definition of "quaternionic holomorphy", proposed in 1935 by Fueter [16], which generalized the Cauchy–Riemann equations. Henceforth these mappings appeared in the literature as "regular mappings in the sense of Fueter". They have many properties "analogous" to those of holomorphic mappings although the proofs are very difficult from the technical point of view due, among other things, to the noncommutativity of quaternions. In many cases the analogy does not exist, for example, there are no simple functions which are regular in the sense of Fueter. In 1979 Sudbery [60] collected, classified and proved most of the fundamental properties of regular mappings. Anyway, up to now, there does not exist a "quaternionic analysis" in the same sense as the complex analysis; nevertheless, in 1970s Imaeda [23] presented an exceptionally beautiful, simple and convincing application of quaternions to electromagnetism.

Chapter I is devoted to some problems connected with regular mappings in the sense of Fueter. First of all we investigate the problem of the existence of biregular mappings that are very important from the point of view of applications. To show the existence and prove many properties of such mappings we used a computer. Here I wish to express my deep thanks to Prof. R. M. Porter from CINVESTAV in Mexico City who spent many hours working on the computer to help me find answers to many of my questions. Finally, we show examples of linear and nonlinear biregular mappings in the sense of Fueter. Moreover, we prove that it is impossible to find non-affine manifolds admitting regular functions in the sense of Fueter as transition functions. We also introduce a new class of functions, i.e. LR-biregular functions, and justify their importance from the viewpoint of applications. Furthermore, as a curious detail, we give "simple" proofs of two important theorems: on the ellipticity of the Fueter operator and a fundamental fact that functions expressed by power series (in the quaternionic variables) are not regular in the sense of Fueter. The proofs reveal how many technical difficulties appear in the 4-dimensional case.

Chapter II is connected with the notion of a "quaternionic manifold". The definition used here omits completely the problem of transition functions and it is connected with the so-called Berger list [4]. Berger proved that the holonomy group of an irreducible

Riemannian manifold which is not a symmetric space, is a subgroup of one of the following groups:

$$\mathrm{SO}(n), \quad U(n/2), \quad \mathrm{Sp}(n/4)\cdot\mathrm{Sp}(1), \quad \mathrm{Sp}(n/4),$$
$$G_2\ (n=7), \quad \mathrm{Spin}(7)\ (n=8), \quad \mathrm{Spin}(9)\ (n=16).$$

The manifolds whose holonomy group is contained in $\mathrm{SO}(n)$ are oriented Riemannian manifolds. One can prove only general theorems about the topology of such manifolds.

The manifolds with holonomy groups in $U(n/2)$ are complex Kähler manifolds, very important in complex analysis.

The next group from the Berger list constitutes a basis of the following definition:

DEFINITION. A $4n$-dimensional Riemannian manifold is called *quaternionic* if its holonomy group is a subgroup of $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ (see e.g. [7, 52]).

In Chapter II we generalize an important homotopy invariant, introduced in 1969 by Lichnerowicz [38], to the case of quaternionic (in the above sense) Kähler manifolds by the use of the concept of the quaternionic Kähler form (see Martinelli [47], 1965). Due to the results of Bonan [8] (and others [18, 25, 47]) who noticed first that this direction of developing quaternionic analysis on quaternionic manifolds is appropriate because one can get some analogs of the notions and theorems of complex analysis, we are able to generalize many other results to the quaternionic case. Let us emphasize that finding a counterpart to the Kähler form on the quaternionic manifold gave rise to our generalizations. In particular, due to the papers by Bonan [8], Martinelli [47], Kraines [25] and others we obtain new results on non-triviality of the homotopy group in some important cases.

However, the most significant result presented in this chapter is the proof that regular mappings in the sense of Fueter appear in a natural way in the Eells theory of harmonic mappings [13, 14]. Using the Eells condition [13] for harmonicity we get new results on harmonic mappings from 4-dimensional conformally flat manifolds. In particular, using properties of regular mappings in the sense of Fueter, we obtain a new characterization of harmonic maps from the 4-dimensional torus.

Some of these problems were discussed at the University "La Sapienza" in Rome. The author wants to express his hearty thanks to Prof. Stefano Marchiafava for numerous discussions and encouragement in this work.

Chapter III is a part of the so-called "theory of Hurwitz pairs" initiated by Ławrynowicz and Rembieliński [41–45] and still developed in many centres (e.g. Toulouse, Tokyo, Mexico City, etc.). The starting point of this theory is the following "Hurwitz problem" [21, 22, 41, 53]:

Let S and V denote real vector spaces of dimensions p and n ($p \le n$), respectively, endowed with the scalar products $\langle\,,\,\rangle_S$ and $\langle\,,\,\rangle_V$. For what values of p and n does a bilinear mapping $f: S\times V \to V$ satisfying the "Hurwitz condition"

$$\|f(s,v)\|_V = \|s\|_S\|v\|_V, \qquad s\in S,\ v\in V,$$

exist? It turns out that in the case of $S = V$ $(p = n)$ the Hurwitz problem has the solution: $n = 1, 2, 4, 8$. The above result is very important. What does it mean? In the case $n = 1$ we get the set of real numbers $\mathbb{R}$ as V, in the case $n = 2$ the complex numbers $\mathbb{C}$, for $n = 4$ we have the quaternions $\mathbb{H}$ and, finally, for $n = 8$ the octonions $\mathbb{O}$. Note that if we take three different spaces S, V and W of dimensions p, n and r, respectively, the problem of the existence of a bilinear mapping $f : S \times V \to W$ with the property $\|f(s, v)\|_W = \|s\|_S \|v\|_V$ is still open!

For any Hurwitz pair (S, V, f) we can consider mappings $F : S \to V$ and impose on them some condition of "regularity" (general holomorphy) of the type

$$D \circ F = 0, \qquad (\star)$$

where D is a differential operator which is a natural generalization of the operators:

$$D_{\mathbb{C}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(= \frac{\partial}{\partial \overline{z}}\right) \quad \text{(Cauchy–Riemann equations from complex analysis),}$$

$$D_{\mathbb{H}} := \frac{1}{4}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right) \quad \text{(Fueter operator in the quaternionic case)}$$

and $\circ$ denotes the Hurwitz bilinear mapping f.

The equation $(\star)$ is called the *Fueter–Hurwitz equation* (F-H equation).

Hurwitz pairs have more and more applications in mathematics and mathematical physics [36, 43, 44]. Therefore we decided to examine the problem of the existence and properties of mappings satisfying the condition $(\star)$. We give a construction of polynomial solutions of an arbitrary degree to the F-H equation. We also show a general solution of the Fourier type and its integral representation within the internal structure of the given Hurwitz pair.

Although the technical proofs are very complicated we show that the analysis of mappings which are regular in the sense of Fueter–Hurwitz has features of complex analysis, and that complex analysis (as the simplest) is a special case of the analysis in the Hurwitz pairs. Using another idea of Ławrynowicz and Rembieliński [42, 44] we show how one can introduce the so-called "supercomplex structure" in the theory of pseudo-Euclidean Hurwitz pairs. We describe algebraic and geometric properties of supercomplex structures and give a necessary and sufficient condition for the existence of such structures. This is one of the most significant results shown in this chapter.

Moreover, we give an effective construction of a complex scalar product in the space V of the given pseudo-Euclidean Hurwitz pair (S, V, f) by the use of supercomplex structure. This result is a counterpart in the theory of Hurwitz pairs to the so-called "lifting theorem" by Fröhlich–McEvett [15].

The final part of Chapter III is devoted to finding an analogue of the Hurwitz condition in the theory of Clifford algebras. This result, although easy enough to prove, is rather unexpected. Thus, the pseudo-Euclidean norm is replaced by the spinor one. This fact lets us hope that the use of "tools" from the theory of Clifford algebras could help in solving the general Hurwitz problem which is still open.

I. Quaternionic regular and biregular functions in the sense of Fueter

I.1. Introduction. A basic question in the development of quaternionic analysis is a suitable generalization of the notion of holomorphy. At the outset it may not be clear which of several conditions, equivalent for holomorphic mappings of complex numbers, can best be generalized to the quaternionic skew field $\mathbb{H}$.

A typical element of $\mathbb{H}$ can be written as $q = x_0 + ix_1 + jx_2 + kx_3$, $x_0, x_1, x_2, x_3 \in \mathbb{R}$, and the quaternionic units satisfy $i^2 = j^2 = k^2 = ijk = -1$. The conjugation of $\mathbb{H}$ is given by

$$q^+ = x_0 - ix_1 - jx_2 - kx_3$$

and the norm $\|q\| = (qq^+)^{1/2} = (q^+q)^{1/2}$ can be used to express the inverse element: for $q \in \mathbb{H}$, $q \neq 0$, we have $q^{-1} = q^+/\|q\|^2$.

Recall some fundamental facts.

Definition. A function $f : \mathbb{H} \to \mathbb{H}$ is called (*left*) *differentiable* at q if the limit

$$\frac{df}{dq} = \lim_{h\to 0} h^{-1}[f(q+h) - f(q)]$$

exists.

It is the most natural definition at first sight but it leads to a very restricted class of functions. Namely, we have

Theorem. *If df/dq exists, then $f(q) = a + qb$, $a, b \in \mathbb{H}$.*

Quaternions do not commute, hence the reasonable generalization of the term $a_n z^n$ from the complex case is

$$a_0 q a_1 q \ldots q a_{n+1}, \quad a_i \in \mathbb{H}, \quad i = 0, 1, \ldots, n+1.$$

But the definition of holomorphy using sums of such terms leads to a quite general class of functions, namely to the real-analytic mappings from $\mathbb{R}^4$ to $\mathbb{R}^4$.

In 1935 R. Fueter [16] proposed a definition of "regularity" for quaternionic functions via an analogue of the Cauchy–Riemann equations. The class of Fueter regular functions seems in many ways to express very well the spirit of complex analysis in the quaternionic context, as many classical results (e.g. Cauchy's integral formula, Morera's theorem, the Laurent expansion, ...) carry over in a more or less natural way [58, 60]. But, because of the non-commutativity of quaternions, many properties of holomorphic functions cannot be generalized to the Fueter regular functions. For instance, the composition of two regular functions is not, in general, regular, so we cannot define a "quaternionic" manifold via Fueter regular transition functions. Nevertheless, this theory is still being developed.

In this chapter we consider several aspects of Fueter regular functions and begin an investigation of biregular functions, that is, invertible regular functions with a regular inverse. Sections I.1 and I.2 contain various formulations of the Fueter derivative and some associated Chain Rules, together with a simple verification that the Fueter differential operator is elliptic. In Section I.3 we classify quaternionic functions with holomorphic slices and show their relation to regular functions. In Section I.4 we study the components of the quaternionic monomial q^n and prove that its Fueter derivative is real. We apply this to show that no non-constant regular function can be represented by a simple quaternionic power series. Section I.5 begins the study of biregular mappings; a pointwise criterion is given for local biregularity in terms of the Jacobian differential, and the existence of non-affine biregular mappings is established. The class of LR-biregular mappings (left regular with a right regular inverse) is introduced and its pointwise criterion is found to be considerably simpler. In Section I.6 we observe that a quaternionic manifold admitting a notion of regular function is necessarily affine and define regular and LR-biregular functions on affine quaternionic manifolds. In Section I.7 we develop a formalism in which a "Leibniz rule" for the Fueter operator can be expressed.

I.2. Fueter derivative and regular functions. Denote by $\mathbb{H}$ the field of quaternions. (Well known facts about quaternions and regular functions which are not cited specifically in this work may be found, for example, in the general references [50, 58, 60].)

Let $q = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$ be a quaternion, where $x_\alpha \in \mathbb{R}$, $\alpha = 0, 1, 2, 3$. Write $\partial_\alpha = \partial/\partial x_\alpha$ and form the following symbolic differential expressions:

$$D^+ = \tfrac{1}{4}(\partial_0 + i\partial_1 + j\partial_2 + k\partial_3), \qquad D = \tfrac{1}{4}(\partial_0 - i\partial_1 - j\partial_2 - k\partial_3).$$

DEFINITION 1.2.1. A function $f = f^0 + if^1 + jf^2 + kf^3 : \Omega \to \mathbb{H}$ is said to be (*left*) *regular* in the domain $\Omega \subseteq \mathbb{H}$ if f is differentiable in the usual sense as a mapping of an open set in $\mathbb{R}^4$ to $\mathbb{R}^4$, and $D^+ \cdot f = 0$ in Ω; it is (*left*) *antiregular* if $D \cdot f = 0$.

Similarly, a function f is *right regular* (resp. *right antiregular*) if $f \cdot D^+ = 0$ (resp. $f \cdot D = 0$). (By notational convention, $g\partial_\alpha = \partial_\alpha g$ for any real function g.) If f is invertible and both f and f^{-1} are regular, then f is said to be *biregular*.

We will occasionally use the "·" as above to stress that quaternionic multiplication is performed. For clarity, some statements about left operators will be accompanied by the symmetric results for the corresponding right-sided operators. From the definition it follows that the linear mapping with matrix $A = (a_{\alpha\beta}) \in \mathbb{R}^{4\times 4} = \{4 \times 4 \text{ real matrices}\}$ is regular if and only if the four *real Fueter* equations

$$\begin{aligned} &\varepsilon_{00}a_{00} + \varepsilon_{11}a_{11} + \varepsilon_{22}a_{22} + \varepsilon_{33}a_{33} = 0,\\ &\varepsilon_{01}a_{01} + \varepsilon_{10}a_{10} + \varepsilon_{23}a_{23} + \varepsilon_{32}a_{32} = 0,\\ &\varepsilon_{02}a_{02} + \varepsilon_{13}a_{13} + \varepsilon_{20}a_{20} + \varepsilon_{31}a_{31} = 0,\\ &\varepsilon_{03}a_{03} + \varepsilon_{12}a_{12} + \varepsilon_{21}a_{21} + \varepsilon_{30}a_{30} = 0, \end{aligned} \tag{1.2.1}$$

are satisfied, where $\varepsilon_{\alpha\beta} = \pm 1$, the signs being given by the first of the following matrices:

$$\begin{pmatrix} + & + & + & + \\ + & - & - & + \\ + & + & - & - \\ + & - & + & - \end{pmatrix}, \begin{pmatrix} + & - & - & - \\ + & + & + & - \\ + & - & + & + \\ + & + & - & + \end{pmatrix}, \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & - & - & + \\ + & + & - & - \end{pmatrix}, \begin{pmatrix} + & - & - & - \\ + & + & - & + \\ + & + & + & - \\ + & - & + & + \end{pmatrix}.$$

The remaining three sets of signs correspond to $Df = 0$, $fD^+ = 0$, $fD = 0$, respectively.

PROPOSITION 1.2.1. *The linear differential operators $f \to D^+f$, Df, fD^+, fD, regarded as acting on smooth functions on domains in $\mathbb{R}^4$, are elliptic.*

P r o o f. Express left multiplication by quaternionic units in real coordinates:

$$4D^+f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \partial_0 f + \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \partial_1 f$$

$$+ \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \partial_2 f + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \partial_3 f.$$

Thus the symbol [11] of the operator $f \to D^+f$ is the matrix of first degree polynomials

$$\xi = (\xi_0, \xi_1, \xi_2, \xi_3) \to \frac{1}{4} \begin{pmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ \xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{pmatrix}.$$

Since the determinant $(\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2)^2$ vanishes only when $\xi = 0$, the operator is elliptic. The remaining cases are treated similarly. ■

Write $\mathcal{F} = \mathcal{F}_{\mathbb{R}} = \{A \in \mathbb{R}^{4\times 4} : A \text{ is regular}\}$. Observe that $A_1, A_2 \in \mathcal{F}$ does not imply $A_1A_2 \in \mathcal{F}$, and that the identity matrix I is not in $\mathcal{F}$. A smooth function f is regular if its real Jacobian differential df lies in $\mathcal{F}$ at every point of its domain.

Complex notation. The quaternion q can be expressed uniquely as $q = u + vj$, where $u = x_0 + ix_1,\ v = x_2 + ix_3 \in \mathbb{R} + i\mathbb{R}$. Note that $jv = \overline{v}j$, so

$$(u_1 + v_1j)(u_2 + v_2j) = (u_1u_2 - v_1\overline{v}_2) + (u_1v_2 + v_1\overline{u}_2)j. \tag{1.2.2}$$

Similarly, we may express $2D^+ = \frac{1}{2}(\partial_0 + i\partial_1) + \frac{1}{2}(\partial_2 + i\partial_3)j = \frac{\partial}{\partial\overline{u}} + \frac{\partial}{\partial\overline{v}}j$, $2D = \frac{\partial}{\partial u} - \frac{\partial}{\partial\overline{v}}j$, and $f(q) = \phi(q) + \psi(q)j$. Then a formal calculation analogous to (1.2.2) yields

$$D^+f = \tfrac{1}{2}(\phi_{\overline{u}} - \overline{\psi_{\overline{v}}}) + \tfrac{1}{2}(\psi_{\overline{u}} + \overline{\phi_{\overline{v}}})j, \qquad Df = \tfrac{1}{2}(\phi_u + \overline{\psi_{\overline{v}}}) + \tfrac{1}{2}(\psi_u - \overline{\phi_{\overline{v}}})j,$$

where the subscripts indicate formal complex derivatives. The following is immediate.

Proposition 1.2.2. *Let $f : \Omega \to \mathbb{H}$ be differentiable. Then $D^+f = 0$ if and only if the following system is satisfied*:

$$\phi_v = -\overline{\psi}_u, \qquad \psi_v = \overline{\phi}_u. \tag{1.2.3}$$

Similarly, $Df = 0$ *if and only if* $\phi_v = \overline{\psi}_{\overline{u}}, \psi_v = -\overline{\phi}_{\overline{u}}$.

We will refer to (1.2.3) as "the complex Fueter equations".

Formula (1.2.2) also suggests (see for example [12]) considering the injective homomorphism $\mu : \mathbb{H} \to \mathbb{C}^{2\times 2} = \{2 \times 2 \text{ complex matrices}\}$ defined by

$$\mu(q) = \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}.$$

Then $\mu(D^+f) = \mu(D^+)\mu(f)$, $\mu(Df) = \mu(D)\mu(f)$, where

$$\mu(D^+) = \frac{1}{2}\begin{pmatrix} \partial/\partial\overline{u} & \partial/\partial\overline{v} \\ -\partial/\partial v & \partial/\partial u \end{pmatrix}, \qquad \mu(D) = \frac{1}{2}\begin{pmatrix} \partial/\partial u & -\partial/\partial\overline{v} \\ \partial/\partial v & \partial/\partial\overline{u} \end{pmatrix}.$$

The following matrix notation will prove more convenient for our purposes. For f as above define

$$J_{\mathbb{C}}(f) = \begin{pmatrix} \phi_u & \phi_{\overline{u}} & \phi_v & \phi_{\overline{v}} \\ \overline{\phi}_u & \overline{\phi}_{\overline{u}} & \overline{\phi}_v & \overline{\phi}_{\overline{v}} \\ \psi_u & \psi_{\overline{u}} & \psi_v & \psi_{\overline{v}} \\ \overline{\psi}_{\overline{u}} & \overline{\psi}_{\overline{u}} & \overline{\psi}_v & \overline{\psi}_{\overline{v}} \end{pmatrix} \in \mathcal{M},$$

where $\mathcal{M} \subseteq \mathbb{C}^{4\times 4}$ is the collection of matrices B with block form $(B_{00}, B_{01}, B_{10}, B_{11})$, each $B_{\alpha\beta}$ of the form $(a, \overline{b}; b, \overline{a})$, $a, b \in \mathbb{C}$. The second and fourth columns of matrices in $\mathcal{M}$ are redundant; this notation permits expressing the Chain Rule as $J_{\mathbb{C}}(f \circ g) = J_{\mathbb{C}}(f) \circ J_{\mathbb{C}}(g)$, $J_{\mathbb{C}}(f^{-1}) = (J_{\mathbb{C}} f)^{-1}$. The complex Fueter equations (1.2.3), say $D^+ f = 0$, are equivalent to $J_{\mathbb{C}}(f) \in \mathcal{F}_{\mathbb{C}}$, where

$$\mathcal{F}_{\mathbb{C}} = \{(b_{\alpha\beta}) \in \mathcal{M} : b_{02} = -b_{30},\ b_{22} = b_{10}\}. \tag{1.2.4}$$

Similarly, $Df = 0$ corresponds to the conditions $b_{02} = \overline{b}_{20}, b_{22} = -\overline{b}_{00}$, while $fD^+ = 0$ corresponds to $b_{12} = -\overline{b}_{20}, b_{22} = \overline{b}_{10}$ and $fD = 0$ corresponds to $b_{12} = b_{30}, b_{22} = -b_{00}$.

I.3. Quaternionic partial derivatives. We describe yet another way of expressing the Fueter equations. Introduce the following "conjugations":

$$\begin{aligned} \overline{q} &:= x_0 - ix_1 - jx_2 - kx_3, \\ \overline{q}^{(1)} &:= x_0 - ix_1 + jx_2 + kx_3, \\ \overline{q}^{(2)} &:= x_0 + ix_1 - jx_2 + kx_3, \\ \overline{q}^{(3)} &:= x_0 + ix_1 + jx_2 - kx_3, \end{aligned} \tag{1.3.1}$$

which give

$$\begin{aligned} x_0 &= \tfrac{1}{4}(\overline{q} + \overline{q}^{(1)} + \overline{q}^{(2)} + \overline{q}^{(3)}), & x_2 &= \tfrac{1}{4}(\overline{q} - \overline{q}^{(1)} + \overline{q}^{(2)} - \overline{q}^{(3)})j, \\ x_1 &= \tfrac{1}{4}(\overline{q} + \overline{q}^{(1)} - \overline{q}^{(2)} - \overline{q}^{(3)})i, & x_3 &= \tfrac{1}{4}(\overline{q} - \overline{q}^{(1)} - \overline{q}^{(2)} + \overline{q}^{(3)})k. \end{aligned}$$

To define symbolic partial derivatives with respect to the $\overline{q}^{(\alpha)}$, consider more generally any four $\mathbb{R}$-linear isomorphisms $\sigma_0, \sigma_1, \sigma_2, \sigma_3 : \mathbb{H} \to \mathbb{H}$ with the property that one may solve for the x_α uniquely in terms of the $\sigma_\alpha(q)$; that is, the expressions $\partial x_\alpha / \partial \sigma_\beta(q)$ are defined uniquely by the condition

$$x_\alpha = \sum_\beta \sigma_\beta(q) \cdot \frac{\partial x_\alpha}{\partial \sigma_\beta(q)} \tag{1.3.2}$$

for $0 \le \alpha \le 3$. For any quaternionic function f define $\partial f / \partial \sigma_\alpha(q)$ by

$$\frac{\partial f}{\partial \sigma_\alpha(q)} = \sum_\beta \frac{\partial x_\beta}{\partial \sigma_\alpha(q)} \frac{\partial f}{\partial x_\beta}. \tag{1.3.3}$$

LEMMA 1.3.1. *Assume f is linear. Then*

$$f(q) = \sum_\alpha \sigma_\alpha(q) \cdot \frac{\partial f}{\partial \sigma_\alpha(q)}.$$

Proof. By (1.7) and (1.3.2),

$$\sum_\alpha \sigma_\alpha(q)\frac{\partial f}{\partial\sigma_\alpha(q)} = \sum_\alpha\sum_\beta \sigma_\alpha(q)\frac{\partial x_\beta}{\partial\sigma_\alpha(q)}\frac{\partial f}{\partial x_\beta} = \sum_\beta x_\beta \frac{\partial f}{\partial x_\beta} = f(q). \ \blacksquare$$

PROPOSITION 1.3.1. *Let $p \to q \to f$ be smooth mappings. Then*

$$\frac{\partial f}{\partial\sigma_\alpha(p)} = \sum_\beta \frac{\partial\sigma_\beta(q)}{\partial\sigma_\alpha(p)}\cdot\frac{\partial f}{\partial\sigma_\beta(q)}.$$

Proof. It suffices to prove the statement assuming both maps are linear. Then by two applications of Lemma 1.3.1,

$$f(q(p)) = \sum_\beta \sigma_\beta(q)\frac{\partial f}{\partial\sigma_\beta(q)} = \sum_\beta\Big(\sum_\alpha \sigma_\alpha(p)\frac{\partial\sigma_\beta(q)}{\partial\sigma_\alpha(p)}\Big)\frac{\partial f}{\partial\sigma_\beta(q)},$$

while by Lemma 1.3.1 again

$$f(q(p)) = \sum_\alpha \sigma_\alpha(p)\frac{\partial f}{\partial\sigma_\alpha(p)}.$$

Comparison of the coefficients of $\sigma_\alpha(q)$ gives the desired result. $\blacksquare$

Returning to the specific conjugations (1.3.1), we see that $\partial x_\alpha/\partial\overline{q} = \partial x_\alpha/\partial\sigma_0(q)$ takes the values $1/4,\ i/4,\ j/4,\ k/4$ for $\alpha = 0,1,2,3$. Therefore according to (1.7) we have defined $\partial f/\partial\overline{q} = \sum_\alpha(\partial x_\alpha/\partial\overline{q})\partial_\alpha f = D^+f$.

If we take instead $\sigma_0(q),\ \sigma_1(q),\ \sigma_2(q),\ \sigma_3(q)$ equal to

$$\begin{aligned} q &:= x_0 + ix_1 + jx_2 + kx_3,\\ q^{(1)} &:= x_0 + ix_1 - jx_2 - kx_3,\\ q^{(2)} &:= x_0 - ix_1 + jx_2 - kx_3,\\ q^{(3)} &:= x_0 - ix_1 - jx_2 + kx_3, \end{aligned} \tag{1.3.4}$$

(thus $q^{(1)} = iqi^{-1}$, etc.) then it will be found that $\partial f/\partial q = D\cdot f$.

It must be stressed that $\partial/\partial\overline{q}, \partial/\partial q$ are not intrinsically determined by the definitions of q and $\overline{q}$, but depend on the complete system of four conjugations used. Further, if the order of multiplication in definitions (1.3.2), (1.7) is reversed, giving

$$x_\alpha = \sum_\beta \frac{\partial x_\alpha}{\partial\sigma_\beta(q)}\sigma_\beta(q), \quad \frac{\partial f(q)}{\partial\sigma_\alpha(q)} = \sum_\beta \frac{\partial f}{\partial x_\beta}\frac{\partial x_\beta}{\partial\sigma_\alpha(q)},$$

then Proposition 1.3.1 will need to be adjusted in an obvious way, and the sets of conjugations (1.3.2), (1.3.4) will cause $\partial f/\partial\overline{q},\ \partial f/\partial q$ to evaluate the right-sided operators $f\cdot D^+,\ f\cdot D$, respectively. From Proposition 1.3.1 one may derive various "Chain Rules" such as the following:

PROPOSITION 1.3.2. *Let $f : \Omega_2 \to \mathbb{H}$ and $g : \Omega_1 \to \Omega_2$ be differentiable mappings with domains in $\mathbb{H}$. Let $\partial/\partial\overline{q}^{(\alpha)}$, $\partial/\partial q^{(\alpha)}$ be determined by the systems* (1.3.1), (1.3.4), *respectively. Then*

$$D^+(f\circ g) = (D^+\overline{g})\cdot((D^+f)\circ g) + \sum_{\alpha=1}^{3}(D^+\overline{g}^{(\alpha)})\cdot\Big(\frac{\partial f}{\partial\overline{q}^{(\alpha)}}\circ g\Big),$$

$$D(f\circ g) = (Dg)\cdot((Df)\circ g) + \sum_{\alpha=1}^{3}(Dg^{(\alpha)})\cdot\Big(\frac{\partial f}{\partial q^{(\alpha)}}\circ g\Big),$$

$$D^+(f\circ g) = \sum_{\alpha=0}^{3}(D^+g^\alpha)\cdot(\partial_\alpha f)\circ g,\qquad D(f\circ g) = \sum_{\alpha=0}^{3}(Dg^\alpha)\cdot(\partial_\alpha f)\circ g,$$

where in the latter two formulas, $g = g^0 + ig^1 + jg^2 + kg^3$.

P r o o f. The first two rules are an application of Proposition 1.3.1 with $\partial f/\partial\overline{q} = D^+f$, $\partial f/\partial q = Df$, respectively. To verify the remaining two, use $\sigma_\alpha(q) = x_\alpha$ and deduce from Proposition 1.3.1 that $\partial_\beta(f\circ g) = \sum_\alpha(\partial_\beta g^\alpha)((\partial_\alpha f)\circ g)$. ■

As an application of the Chain Rules we have the following elementary formulas.

COROLLARY 1.3.1. *Let* $g(q) = aq+b$, $h(q) = qc+d$ *be left and right affine quaternionic mappings*, $a, b, c, d \in \mathbb{H}$. *Then*

$$D^+(f\circ g) = \overline{a}(D^+f)\circ g,$$

$$D(f\circ g) = \Big(a_0Df + ia_1\frac{\partial f}{\partial q^{(1)}} + ja_2\frac{\partial f}{\partial q^{(2)}} + ka_3\frac{\partial f}{\partial q^{(3)}}\Big)\circ g,$$

$$D^+(f\circ h) = (c_0D^+f - ic_1\frac{\partial f}{\partial\overline{q}^{(1)}} - jc_2\frac{\partial f}{\partial\overline{q}^{(2)}} - kc_3\frac{\partial f}{\partial\overline{q}^{(3)}})\circ h,$$

$$D(f\circ h) = c(Df)\circ h,$$

where $a = a_0 + ia_1 + ja_2 + ka_3$, $c = c_0 + ic_1 + jc_2 + kc_3$. *Further*, $D^+(h\circ f) = (D^+f)c$, $D(h\circ f) = (Df)c$.

P r o o f. Observe that $D^+\overline{g} = \overline{a}$, $Dh = c$, $D^+\overline{g}^{(\alpha)} = 0 = Dh^\alpha$ for $\alpha = 1, 2, 3$. Thus the formulas for $D^+(f\circ g)$, $D(f\circ h)$ follow from the first two Chain Rules of Proposition 1.3.2. The formulas for $D(f\circ g)$, $D^+(f\circ h)$ are obtained similarly. Finally, since $\partial_0 h = c$, $\partial_1 h = ic$, etc., the formulas for $D^+(h\circ f)$, $D(h\circ f)$ follow from the latter two Chain Rules of Proposition 1.3.2 (or directly from the definitions of D, D^+). ■

I.4. Functions with holomorphic slices. Since $\mathbb{H}$ is an extension of $\mathbb{C}$ and regularity is a generalization of holomorphy, it is natural to look for relations between these two classes of functions. Fueter [16] showed how to construct a regular function applying a holomorphic function symmetrically with respect to the three pure quaternionic units. Here we take a different point of view and examine the holomorphy of restrictions of quaternionic mappings to complex planes.

For each non-zero pure quaternion $p = ip_1 + jp_2 + kp_3$, define an $\mathbb{R}$-linear mapping $\iota_p : \mathbb{C}\to\mathbb{H}$ by the conditions $\iota_p(1) = 1$ and $\iota_p(i) = p$. The injection ι_p is an isomorphism of $\mathbb{C}$ onto the plane $\mathbb{R} + p\mathbb{R} \subseteq \mathbb{H}$. Let $\varrho_p : \mathbb{H}\to\mathbb{R}+p\mathbb{R}$ denote the orthogonal projection onto this plane with respect to the Euclidean inner product $\langle\vec{x},\vec{y}\rangle = \sum_{\alpha=0}^{3} x_\alpha y_\alpha$, and then define the projection $\pi_p : \mathbb{H}\to\mathbb{C}$ by $\pi_p = \iota_p\circ\varrho_p$. We have $\pi_p\circ\iota_p = \iota_p$ and $\iota_p\circ\pi_p = \varrho_p$.

Now consider any $\mathbb{R}$-linear function $A : \mathbb{H} \to \mathbb{H}$. The *p-slice* of A is defined to be the $\mathbb{R}$-linear function $A_p = \pi_p \circ A \circ \iota_p : \mathbb{C} \to \mathbb{C}$. We will say that A is *holomorphic on slices* if for every pure quaternion p with $\|p\| = 1$, the p-slice A_p is holomorphic. More generally, a differentiable map f of an open set of $\mathbb{H}$ to $\mathbb{H}$ is *holomorphic on slices* if its real differential df is holomorphic on slices at each point of the domain. The definition of *antiholomorphic on slices* is analogous. For present purposes it will be sufficient to consider $\mathbb{R}$-linear functions. Write $\overline{A}q = \overline{A(q)}$.

PROPOSITION 1.4.1. *If A is holomorphic (resp. antiholomorphic) on slices, then $\overline{A}$ and $q \to A(\overline{q})$ are antiholomorphic (resp. holomorphic) on slices.*

Proof. First note that $\iota_p(z) = \overline{\iota_{\overline{p}}}(z)$, so $\iota_p(\pi_p(q)) = \varrho_p(q) = \varrho_{\overline{p}}(\overline{q}) = \overline{\iota_{\overline{p}}\pi_{\overline{p}}(\overline{q})} = \iota_p(\overline{\pi_{\overline{p}}(\overline{q})})$. Since ι_p is injective, $\pi_p(q) = \pi_{\overline{p}}(\overline{q})$. From this fact it follows that $\overline{A}_p(z) = \pi_p(\overline{A\iota_p(z)}) = \pi_{\overline{p}}A\iota_{\overline{p}}(\overline{z})$, $z \in \mathbb{C}$. Therefore $\overline{A}_p$ is antiholomorphic (resp. holomorphic) when $A_{\overline{p}}$ is holomorphic (resp. antiholomorphic). Further, $\pi_p A(\overline{\iota_p(z)}) = \pi_{\overline{p}}(\overline{A\iota_{\overline{p}}(z)}) = \overline{A}_{\overline{p}}(z)$ is an antiholomorphic (resp. holomorphic) function of z when $\overline{A}_{\overline{p}}$ is antiholomorphic (resp. holomorphic). ∎

The matrices of ι_p and π_p with respect to the standard bases of $\mathbb{R}^2 \cong \mathbb{C}$ and $\mathbb{R}^4 \cong \mathbb{H}$ are

$$\iota_p = \begin{pmatrix} 1 & 0 \\ 0 & p_1 \\ 0 & p_2 \\ 0 & p_3 \end{pmatrix}, \qquad \pi_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p_1 & p_2 & p_3 \end{pmatrix}, \tag{1.4.1}$$

respectively. This is obvious for ι_p from the definition. For π_p note that the given matrix sends $1 = (1,0,0,0)^T$ to $1 = (1,0)^T$ and $p = (0,p_1,p_2,p_3)^T$ to $i = (0,1)^T$ (where $\vec{x}^T$ denotes the transpose of the vector $\vec{x}$). Further, if $c = c_0 + ic_1 + jc_2 + kc_3$ is orthogonal to $\mathbb{R} + p\mathbb{R}$, then $0 = \langle c, 1\rangle = c_0$ and $0 = \langle c, p\rangle = c_1p_1 + c_2p_2 + c_3p_3$, from which it follows that the matrix sends c to 0, and thus agrees with π_p on any orthogonal basis of the form $1, p, c, c'$. Identify A with its matrix $A = (a_{\alpha\beta})$; then we find that

$$A_p = \begin{pmatrix} a_{00} & \sum_{\beta=1}^3 p_\beta a_{0\beta} \\ \sum_{\alpha=1}^3 p_\alpha a_{\alpha 0} & \sum_{\alpha=1}^3 \sum_{\beta=1}^3 p_\alpha p_\beta a_{\alpha\beta} \end{pmatrix} \in \mathbb{C}^{2\times 2}.$$

We conclude that A is holomorphic on slices if and only if the (Cauchy–Riemann) equations

$$\sum_{\alpha=1}^3 \sum_{\beta=1}^3 p_\alpha p_\beta a_{\alpha\beta} = a_{00}, \qquad \sum_{\alpha=1}^3 p_\alpha a_{\alpha 0} = -\sum_{\beta=1}^3 p_\beta a_{0\beta}$$

are satisfied whenever $p_1^2 + p_2^2 + p_3^2 = 1$. Choosing various values for the p_α one finds easily that this is equivalent to

$$a_{\alpha\alpha} = a_{00}, \qquad a_{\alpha\beta} = -a_{\beta\alpha},$$

whenever $1 \le \alpha \ne \beta \le 3$. Therefore A is holomorphic on slices precisely when it is of the form

$$A = \begin{pmatrix} a & -b & -c & -d \\ b & a & -e & -f \\ c & e & a & -g \\ d & f & g & a \end{pmatrix}, \tag{1.4.2}$$

i.e., is of the form $A = M + aI$, where $M^T = -M$.

We call A *invertibly holomorphic on slices* when it is invertible and both A and A^{-1} are holomorphic on slices (the term "bi-holomorphic on slices" unfortunately would be ambiguous). The following conclusions may be drawn with the aid of (1.2.1).

THEOREM 1.4.1. (a) *Let $A \in \mathbb{R}^{4\times 4}$ be holomorphic on slices. Then A is regular if and only if it is of the form*

$$A = \begin{pmatrix} 0 & -b & -c & -d \\ b & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix},$$

and it is antiregular if and only if it is of the form

$$A = \begin{pmatrix} 0 & -b & -c & -d \\ b & 0 & -d & c \\ c & d & 0 & -b \\ d & -c & b & 0 \end{pmatrix}.$$

In the antiregular case, if any of b, c, d is non-zero then $A^{-1} = -(b^2+c^2+d^2)^{-1}A$ is again both invertibly holomorphic on slices and antiregular.

(b) *Let A be antiholomorphic on slices. Then A is regular if and only if it is of the form*

$$A = \begin{pmatrix} 0 & b & c & d \\ b & 0 & d & -c \\ c & -d & 0 & b \\ d & c & -b & 0 \end{pmatrix};$$

and if any of b, c, d is non-zero, then $A^{-1} = (b^2+c^2+d^2)^{-1}A^T$ is again antiholomorphic on slices, and is right regular. A is antiregular if and only if it is of the form

$$A = \begin{pmatrix} 0 & b & c & d \\ b & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}.$$

We next classify the functions invertibly holomorphic on slices. Begin with A given by (1.4.2), and suppose that A^{-1} is of the same form. In particular, the minors of the diagonal terms of A, namely, $a(a^2+e^2+f^2+g^2)$, $a(a^2+c^2+d^2+g^2)$, $a(a^2+b^2+d^2+f^2)$, $a(a^2+b^2+c^2+e^2)$ must be equal. If $a \neq 0$ then this implies $e^2+f^2+g^2 = c^2+d^2+g^2 = b^2+d^2+f^2 = b^2+c^2+e^2$, from which $g^2 = b^2$, $f^2 = c^2$, $e^2 = d^2$. Taking into account the remaining minors, we see that

$$g = -b,\ f = c,\ e = -d \quad \text{or else} \quad g = b,\ f = -c,\ e = d.$$

This leads to the following classification:

THEOREM 1.4.2. *Let A be invertibly holomorphic on slices. Then the matrix A is of one of the following forms*:

$$I:\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix},\ II:\begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix},\ III:\begin{pmatrix} 0 & -b & -c & -d \\ b & 0 & -e & -f \\ c & e & 0 & -g \\ d & f & g & 0 \end{pmatrix}.$$

Conversely, any invertible A of these forms is invertibly holomorphic on slices.

As is well known, the matrices of type (I) are the linear mappings defined by left multiplication, $q \to (a+ib+jc+kd)q$, while type (II) corresponds to right multiplication. The invertible matrices of these types form multiplicative groups. However, the full class of functions invertibly holomorphic on slices is not closed under composition. Further, it follows from Theorem 1.4.1 that there exists no function both regular and invertibly holomorphic on slices. By Proposition 1.4.1 the classification for matrices invertibly antiholomorphic on slices is

$$I':\begin{pmatrix} a & b & c & d \\ b & -a & d & -c \\ c & -d & -a & b \\ d & c & -b & -a \end{pmatrix},\ II':\begin{pmatrix} a & b & c & d \\ b & -a & -d & c \\ c & d & -a & -b \\ d & -c & b & -a \end{pmatrix},\ III':\begin{pmatrix} 0 & b & c & d \\ b & 0 & e & f \\ c & -e & 0 & g \\ d & -f & -g & 0 \end{pmatrix}$$

For the next result we recall the operators defined in Section 3 via (1.3.1), (1.3.4).

THEOREM 1.4.3. *Let $A:\mathbb{H}\to\mathbb{H}$ be $\mathbb{R}$-linear. If $\partial A/\partial\overline{q}^{(1)} = \partial A/\partial\overline{q}^{(2)} = \partial A/\partial\overline{q}^{(3)} = 0$, then $\overline{A}$ is a left multiplication mapping. If $\partial\overline{A}/\partial q^{(1)} = \partial\overline{A}/\partial q^{(2)} = \partial\overline{A}/\partial q^{(3)} = 0$, then $\overline{A}$ is a right multiplication mapping.*

P r o o f. By the earlier discussion, $4\partial/\partial\overline{q}^{(1)} = \partial_0 + i\partial_1 - j\partial_2 - k\partial_3, 4\partial/\partial\overline{q}^{(2)} = \partial_0 - i\partial_1 + j\partial_2 - k\partial_3, 4\partial/\partial\overline{q}^{(3)} = \partial_0 - i\partial_1 - j\partial_2 + k\partial_3$. From this a simple calculation shows that $\partial A/\partial\overline{q}^{(\gamma)} = 0$ precisely when $A = (a_{\alpha\beta})$ satisfies (1.2.1) with the signs $\varepsilon_{\alpha\beta}$ given by

$$\begin{pmatrix} + & + & - & \\ + & - & + & - \\ + & + & + & + \\ + & - & - & + \end{pmatrix},\quad \begin{pmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \\ + & + & + & + \end{pmatrix},\quad \begin{pmatrix} + & - & - & + \\ + & + & + & + \\ + & - & + & - \\ + & + & - & - \end{pmatrix},$$

for $\gamma = 1,2,3$, respectively. If these hold simultaneously, then it follows easily that A is of type (II′) shown above. The assertion regarding $\partial\overline{A}/\partial q^{(\gamma)}$ is proved similarly. ∎

The following dual statement may be verified in the same way:

THEOREM 1.4.4. *If $D^+\overline{A}^{(1)} = D^+\overline{A}^{(2)} = D^+\overline{A}^{(3)} = 0$, then A is a left multiplication*; *if $DA^{(1)} = DA^{(2)} = DA^{(3)} = 0$, then A is a right multiplication.*

I.5. Non-regularity of simple quaternionic power series. It is well known that the quaternionic power series which are obtained by summing monomials of the form $a_1qa_2q\ldots qa_m \in \mathbb{H}$, comprise all real-analytic $\mathbb{H}$-valued functions on $\mathbb{H}$ (see e.g. [58]). Special power series representing a regular function in the sense of Fueter were found by Sudbery [60] to obtain a counterpart of Laurent's theorem and the residue theorem. Here we limit the discussion to series of the form $\sum a_nq^n$ or $\sum q^na_n$, which we call *simple power series*. Fueter noted [16] that the Laplacian $\Delta(\sum q^na_n)$ is regular. (The result is known but our method is completely original and it shows how the explicit calculations are difficult in the quaternionic case.)

THEOREM 1.5.1. *Let* $f(q)=\sum_{n=0}^{\infty} a_n q^n$ *or* $f(q)=\sum_{n=0}^{\infty} q^n a_n$, $a_n \in \mathbb{H}$, *be a convergent simple power series in the domain* $\Omega \subset \mathbb{H}$ *defining a regular function* f. *Then* $a_n = 0$ *for* $n \geq 1$.

The proof will be given via a series of lemmas, the main goal being parts (a) and (b) of Lemma 1.5.4. Let q be given by (1.3.4) and express its nth power as

$$q^n = r_n^0 + ir_n^1 + jr_n^2 + kr_n^3, \tag{1.5.1}$$

where the r_n^α are to be regarded as functions of the x_α. Thus $r_0^0 = 1, r_0^1 = r_0^2 = r_0^3 = 0$.

LEMMA 1.5.1. *The functions* r_n^α *are homogeneous polynomials of degree* n *in* x_0, x_1, x_2, x_3 *and for* $n \geq 1$ *they satisfy*

$$\begin{aligned} r_n^0 &= x_0 r_{n-1}^0 - x_1 r_{n-1}^1 - x_2 r_{n-1}^2 - x_3 r_{n-1}^3, \\ r_n^1 &= x_1 r_{n-1}^0 + x_0 r_{n-1}^1 - x_3 r_{n-1}^2 + x_2 r_{n-1}^3, \\ r_n^2 &= x_2 r_{n-1}^0 + x_3 r_{n-1}^1 + x_0 r_{n-1}^2 - x_1 r_{n-1}^3, \\ r_n^3 &= x_3 r_{n-1}^0 - x_2 r_{n-1}^1 + x_1 r_{n-1}^2 + x_0 r_{n-1}^3. \end{aligned} \tag{1.5.2}$$

Proof. This is verified by induction on n using (1.5.1) and the definition of quaternionic multiplication. ∎

In what follows, α, β, γ will denote the distinct indices $1, 2, 3$ in any order.

LEMMA 1.5.2. *For all* $n \geq 0$, $x_\alpha r_n^\beta = x_\beta r_n^\alpha$.

Proof. For $n = 0$ we have $r_0^\alpha = x_\alpha$ and the assertion is trivial. Suppose it holds when n is replaced by $n-1$. Then the last three equations of (1.5.2) reduce to the common form

$$r_n^\alpha = x_\alpha r_{n-1}^0 + x_0 r_{n-1}^\alpha, \tag{1.5.3}$$

from which $x_\alpha r_n^\beta - x_\beta r_n^\alpha = x_0(x_\alpha r_{n-1}^\beta - x_\beta r_{n-1}^\alpha) = 0$. By induction, the assertion holds for all n. ∎

As a consequence of Lemma 1.5.2 we see that (1.5.3) is valid for $n \geq 1$.

LEMMA 1.5.3. *The polynomials* r_n^α *and their partial derivatives are related by*

a) $x_\alpha \partial_0 r_n^\beta = x_\beta \partial_0 r_n^\alpha$,
b) $x_\alpha \partial_\gamma r_n^\beta = x_\beta \partial_\gamma r_n^\alpha$,
c) $r_n^\alpha = x_\alpha \partial_\beta r_n^\beta - x_\beta \partial_\beta r_n^\alpha$.

Proof. Differentiate the formula of Lemma 1.5.2 with respect to x_0, x_γ, x_β, respectively. ∎

LEMMA 1.5.4. *The following relations hold*:

a) $\partial_0 r_n^\alpha = -\partial_\alpha r_n^0$,
b) $\partial_\beta r_n^\alpha = \partial_\alpha r_n^\beta$,
c) $x_\gamma \partial_0 r_n^0 = x_\alpha \partial_\alpha r_n^\gamma + x_\beta \partial_\beta r_n^\gamma + x_\gamma \partial_\gamma r_n^\gamma$.

P r o o f. Statements a), b), c) are easily verified for $n = 0$. Assume inductively that they hold when n is replaced by $n-1$. For a) it suffices to consider $\alpha = 1$. Differentiate (1.5.2), (1.5.3) and obtain, after some rearrangement,

$$\partial_1 r_n^0 + \partial_0 r_n^1 = x_0(\partial_1 r_{n-1}^0 + \partial_0 r_{n-1}^1) + (x_1\partial_0 r_{n-1}^0 - x_1\partial_1 r_{n-1}^1 - x_2\partial_1 r_{n-1}^2 - x_3\partial_1 r_{n-1}^3).$$

The first group of terms in parentheses vanishes by induction hypothesis a), while by hypothesis b) the second is equal to $x_1\partial_0 r_{n-1}^0 - x_1\partial_1 r_{n-1}^1 - x_2\partial_2 r_{n-1}^1 - x_3\partial_3 r_{n-1}^1$, which in turn vanishes by hypothesis c). Thus a) holds. For statement b) we will consider $\alpha = 1, \beta = 2$. By assertion a) of Lemma 1.5.3a, $x_1\partial_2 r_{n-1}^0 = -x_1\partial_0 r_{n-1}^2 = -x_1\partial_0 r_{n-1}^1 = x_2\partial_1 r_{n-1}^0$. Differentiate (1.5.3) and invoke hypothesis b) to obtain

$$\partial_2 r_n^1 - \partial_1 r_n^2 = x_0(\partial_2 r_{n-1}^1 - \partial_1 r_{n-1}^2) + (x_1\partial_2 r_{n-1}^0 - x_2\partial_1 r_{n-1}^0) = 0,$$

and b) is verified. Finally, we prove c). Assume $\alpha = 1,\ \beta = 2,\ \gamma = 3$. From (1.5.2) and (1.5.3),

$$\begin{aligned} &x_3\partial_0 r_n^0 - x_1\partial_1 r_n^3 - x_2\partial_2 r_n^3 - x_3\partial_3 r_n^3 \\ &\quad = x_0(x_3\partial_0 r_{n-1}^0 - x_1\partial_1 r_{n-1}^3 - x_2\partial_2 r_{n-1}^3 - x_3\partial_3 r_{n-1}^3) \\ &\qquad - x_1x_3(\partial_0 r_{n-1}^1 + \partial_1 r_{n-1}^0) - x_2x_3(\partial_0 r_{n-1}^2 + \partial_2 r_{n-1}^0) - x_3^2(\partial_0 r_{n-1}^3 + \partial_3 r_{n-1}^0). \end{aligned}$$

By the inductive hypotheses a) and c) the grouped terms again vanish. This completes the proof. ■

LEMMA 1.5.5. *For all* $n \geq 0$,

$$D^+ q^n = \tfrac{1}{4}(\partial_0 r_n^0 - \partial_1 r_n^1 - \partial_2 r_n^2 - \partial_3 r_n^3).$$

In particular, D^+q^n *is real for all* $q \in \mathbb{H}$.

P r o o f. Apply parts a) and b) of the preceding lemma. ■

LEMMA 1.5.6. a) *The coefficient of* x_0^n *in* r_n^0 *is* 1.
b) *The coefficient of* $x_0^{n-1}x_\alpha$ *in* r_n^α *is* n.

P r o o f. By (1.5.2) and induction, r_n^0 may be written $r_n^0 = x_0(x_0^{n-1} + s_0) - x_1s_1 - x_2s_2 - x_3s_3$ for certain polynomials $s_\alpha(x_0, x_1, x_2, x_3)$, where s_0 is not divisible by x_0^{n-1}. Thus a) follows. The proof of b) is similar. ■

P r o o f o f T h e o r e m 1.5.1. By the last two lemmas, we have

$$\begin{aligned} 4D^+q^n &= \partial_0(x_0^n + \ldots) - \partial_1(nx_0^{n-1}x_1 + \ldots) - \partial_2(nx_0^{n-1}x_2 + \ldots) \\ &\quad - \partial_3(nx_0^{n-1} + \ldots) \\ &= nx_0^{n-1} - nx_0^{n-1} - nx_0^{n-1} - nx_0^{n-1} + \ldots = -2nx_0^{n-1} + \ldots \end{aligned}$$

Therefore $2D^+f = \sum(-na_nx_0^{n-1} + \ldots) = 0$ (whether f is a right or a left simple power series), which implies that $na_n = 0$ for all n. Therefore f is constant. ■

It is to be expected that the quaternionic monomials and their partial derivatives will exhibit many symmetries. Let $\pi_{\alpha\beta}$ be the permutation of $\{x_0, x_1, x_2, x_3\}$ which interchanges x_α, x_β. We state the following without proof.

PROPOSITION 1.5.1. *For all* $n \geq 1$, $r_n^0 \circ \pi_{\alpha\beta} = r_n^0$, $r_n^\gamma \circ \pi_{\alpha\beta} = r_n^\gamma$, *and* $r_n^\alpha \circ \pi_{\alpha\beta} r_n^\beta$.

Proposition 1.5.2. *For all* $n \geq 1$,

$$\partial_0 r_n^0 \circ \pi_{\alpha\beta} = \partial_0 r_n^0, \quad \partial_\alpha r_n^0 \circ \pi_{\alpha\beta} = \partial_\beta r_n^0,$$
$$\partial_\alpha r_n^\alpha \circ \pi_{\alpha\beta} = \partial_\beta r_n^\beta, \quad \partial_\alpha r_n^\beta \circ \pi_{\alpha\beta} = \partial_\beta r_n^\alpha, \quad \textit{and} \quad \partial_\alpha r_n^\gamma \circ \pi_{\alpha\beta} = \partial_\beta r_n^\gamma.$$

I.6. Biregular mappings

Biregular linear mappings. From the real Fueter equations (1.2.1) the set $\mathcal{F}$ of real regular matrices is seen to be a 12-dimensional $\mathbb{R}$-linear subspace of $\mathbb{R}^{4\times 4}$. The collection of biregular linear mappings is thus $\mathcal{F}^* = \mathcal{F}^*_{\mathbb{R}} = \{A \in \mathcal{F} : \det A \neq 0,\ A^{-1} \in \mathcal{F}\}$. If we write $\Phi(A) = A^{-1}$, we see that $\mathcal{F}^* = \mathcal{F} \cap \Phi(F \cap \mathrm{GL}(4,\mathbb{R}))$, which is a real algebraic variety of dimension not less than 8.

Inasmuch as the equations defining $\mathcal{F}^*$ are quite cumbersome, we will use the complex formulation. Define $\mathcal{F}^*_{\mathbb{C}} = \{B \in \mathcal{F}_{\mathbb{C}} : \det B \neq 0,\ B^{-1} \in \mathcal{F}_{\mathbb{C}}\}$. There is a natural one-to-one correspondence $\mathcal{F}^* \to \mathcal{F}^*_{\mathbb{C}}$. By (1.2.4), an invertible $B \in \mathcal{F}_{\mathbb{C}}$ is in $\mathcal{F}^*_{\mathbb{C}}$ when $(B^{-1})_{0,2} = -(B^{-1})_{3,0}$ and $(B^{-1})_{2,2} = (B^{-1})_{1,0}$. Therefore we have the following characterization of biregular matrices.

Theorem 1.6.1. *Let* $B = (b_{\alpha\beta}) \in \mathcal{M}$. *Then* $B \in \mathcal{F}^*_{\mathbb{C}}$ *if and only if* $\det B \neq 0, b_{02} = -b_{30},\ b_{22} = b_{10}$, *and*

$$\det \begin{pmatrix} \overline{b}_{10} & -b_{30} & \overline{b}_{12} \\ \overline{b}_{00} & b_{12} & -\overline{b}_{30} \\ \overline{b}_{20} & b_{32} & \overline{b}_{10} \end{pmatrix} = \det \begin{pmatrix} b_{10} & \overline{b}_{00} & b_{12} \\ b_{20} & \overline{b}_{30} & b_{10} \\ b_{30} & \overline{b}_{20} & b_{32} \end{pmatrix}, \tag{1.6.1a}$$

$$\det \begin{pmatrix} b_{00} & \overline{b}_{10} & \overline{b}_{12} \\ b_{10} & \overline{b}_{00} & -\overline{b}_{30} \\ b_{30} & \overline{b}_{20} & \overline{b}_{10} \end{pmatrix} = -\det \begin{pmatrix} b_{10} & b_{12} & -\overline{b}_{30} \\ b_{20} & b_{10} & \overline{b}_{32} \\ b_{30} & b_{32} & \overline{b}_{10} \end{pmatrix}. \tag{1.6.1b}$$

From this formulation, some simple solutions may be obtained by inspection. For example (1.13) is satisfied when $b_{10} = b_{30} = b_{20} + \overline{b}_{12} = 0$. Allowing the remaining parameters to vary freely, we obtain a few particular cases of biregular maps:

Corollary 1.6.1. *Let* $t_1, t_2, t_3 \in \mathbb{C},\ t_1 t_2 + t_3^2 \neq 0$. *Then*

$$\begin{pmatrix} t_1 & 0 & 0 & -t_3 \\ 0 & \overline{t}_1 & -\overline{t}_3 & 0 \\ t_3 & 0 & 0 & t_2 \\ 0 & \overline{t}_3 & \overline{t}_2 & 0 \end{pmatrix} \in \mathcal{F}^*_{\mathbb{C}}.$$

From this it may be seen that the composition of two biregular mappings is not, in general, biregular.

Existence of non-linear biregular mappings. Thus far, all the biregular mappings considered have been linear. We now give examples of biregular mappings with non-constant differential. Corollary 1.6.1 says that $f = \phi + \psi j$ will be locally biregular wherever it is non-singular when its complex coefficient functions satisfy, for example,

$$\phi_{\overline{u}} = \psi_{\overline{u}} = 0, \quad \phi_v = \psi_v = 0, \quad \phi_{\overline{v}} = -\psi_u.$$

This is solved by $\phi(u,v) = \lambda'(u)\overline{v} + \mu(u), \psi(u,v) = -\lambda(u) + \nu(u)$, where λ, μ are holomorphic and ν is antiholomorphic. Here $t_1 = \lambda''(u)\overline{v} + \mu'(u),\ t_2 = 0,\ t_3 = -\lambda'(u)$

so $t_1t_2 + t_3^2 = [\lambda'(u)]^2$ is non-zero whenever $\lambda'(u) \neq 0$. This provides many non-trivial examples of biregular mappings.

LR-biregular mappings. One may define numerous classes of mappings by requiring f, f^{-1} to have any desired properties related to regularity. We will say that the invertible mapping f is *left-right biregular* (*LR-biregular*) when f is (left) regular and f^{-1} is right regular. Let $\mathcal{F}_{\mathbb{C}}^{\mathrm{LR}} \subseteq \mathcal{M}$ denote the set of matrices corresponding to LR-biregular mappings. By the same reasoning as above, one finds that if $B \in \mathcal{F}_{\mathbb{C}}^{*}$ is invertible, then B^{-1} is right regular if and only if $(B^{-1})_{12} = -\overline{(B^{-1})_{20}}, (B^{-1})_{22} = \overline{(B^{-1})_{10}}$; that is,

$$\det\begin{pmatrix} b_{00} & -b_{30} & \overline{b}_{12} \\ b_{10} & b_{12} & -\overline{b}_{30} \\ b_{20} & b_{32} & \overline{b}_{10} \end{pmatrix} = \det\begin{pmatrix} \overline{b}_{10} & b_{00} & -b_{30} \\ \overline{b}_{20} & b_{30} & b_{32} \\ \overline{b}_{30} & b_{20} & b_{10} \end{pmatrix}, \tag{1.6.2a}$$

$$\det\begin{pmatrix} b_{00} & \overline{b}_{10} & \overline{b}_{12} \\ b_{10} & \overline{b}_{00} & -\overline{b}_{30} \\ b_{30} & \overline{b}_{20} & \overline{b}_{10} \end{pmatrix} = -\det\begin{pmatrix} \overline{b}_{10} & \overline{b}_{12} & -b_{30} \\ \overline{b}_{20} & \overline{b}_{10} & b_{32} \\ \overline{b}_{30} & \overline{b}_{32} & b_{10} \end{pmatrix}. \tag{1.6.2b}$$

Although it might not appear so at first sight, equations (1.14) are considerably simpler than (1.13). After expanding these determinants and some rearrangement, we get the following characterization of LR-biregular matrices.

THEOREM 1.6.2. *Let $B = (b_{\alpha\beta}) \in \mathcal{M}$. Then $B \in \mathcal{F}_{\mathbb{C}}^{\mathrm{LR}}$ if and only if* $\det B \neq 0$, $b_{22} = b_{10}$, $b_{02} = -b_{30}$ *and*

$$\begin{aligned} &b_{12}(b_{00}\overline{b}_{10} - \overline{b}_{12}b_{30}) + b_{10}(b_{00}\overline{b}_{20} + \overline{b}_{12}b_{32}) + b_{20}(\overline{b}_{10}b_{32} + \overline{b}_{20}b_{30}) = 0, \\ &\overline{b}_{00}(b_{00}\overline{b}_{10} - \overline{b}_{12}b_{30}) + \overline{b}_{30}(b_{00}\overline{b}_{20} + \overline{b}_{12}b_{32}) - \overline{b}_{32}(\overline{b}_{10}b_{32} + \overline{b}_{20}b_{30}) = 0. \end{aligned} \tag{1.6.3}$$

COROLLARY 1.6.2. *Let $t_1, t_2, t_3, t_4 \in \mathbb{C}$, $t_1t_2 + t_3^2 \neq 0$. Then*

$$B = \begin{pmatrix} t_1 & t_3t_4 & -t_3 & t_1t_4 \\ \overline{t}_3\overline{t}_4 & \overline{t}_1 & \overline{t}_1\overline{t}_4 & -\overline{t}_3 \\ -\overline{t}_2\overline{t}_4 & \overline{t}_3 & \overline{t}_3\overline{t}_4 & \overline{t}_2 \\ t_3 & -t_2t_4 & t_2 & t_3t_4 \end{pmatrix} \tag{1.6.4}$$

is an element of $\mathcal{F}_{\mathbb{C}}^{\mathrm{LR}}$.

Proof. Since $\det B = -|t_1t_2 + t_3^2|^2(|t_4|^2 + 1)^2$, B is non-singular. It is immediate that the entries $b_{\alpha\beta}$ of B satisfy the remaining conditions of Theorem 1.6.2; indeed all the terms in parentheses in (1.6.4) vanish. ■

It must be stressed that (1.6.4) is by no means the most general LR-biregular linear mapping. Even so, it would be difficult to find explicitly the family $\{f = \phi + \psi j : J_{\mathbb{C}}f$ is of the form (1.6.4)$\}$. We give some examples of non-affine LR-biregular mappigs in this family. Suppose ϕ and $\overline{\psi}$ are holomorphic functions of u and v; that is, $\phi_{\overline{u}} = \psi_u = 0$, $\phi_{\overline{v}} = \psi_v = 0$. Suppose further that $\overline{\psi}_u = -\phi_u$. (As a simple illustration, consider $\phi(u,v) = \lambda(u) + \mu(v), \psi(u,v) = -\overline{u\mu'(v)}$ for λ, μ holomorphic.) Then for $f = \phi + \psi j$ we conclude that $J_{\mathbb{C}}f$ is of the form (1.6.4). The non-singularity condition $\phi_u\overline{\psi}_v + (\overline{\psi}_u)^2 \neq 0$ is easily assured.

One may easily see that the class $\mathcal{F}_{\mathbb{C}}^{\mathrm{LR}}$ is also not preserved under composition.

I.7. Leibniz rule for the Fueter operator. Since the quaternionic product of two regular mappings is in general not regular, regular mappings do not satisfy the Leibniz rule. Here we present some formulas which resemble the classical one. The verification of the Leibniz rule for quaternionic functions of a real variable is straightforward.

PROPOSITION 1.7.1. *If* $f,\ g:(t_1,t_2)\to\mathbb{H}$ *are differentiable quaternionic curves defined in a real interval*, *then*

$$\frac{d}{dt}(f\cdot g)(t)=\frac{df}{dt}(t)\cdot g(t)+f(t)\cdot\frac{dg}{dt}(t). \tag{1.7.1}$$

Take symbols $\widehat{1},\ \widehat{i},\ \widehat{j},\ \widehat{k}$ and define

$$\widehat{\mathbb{H}}=\{\widehat{q}=\widehat{1}q_0+\widehat{i}q_1+\widehat{j}q_2+\widehat{k}q_3 : q_\alpha\in\mathbb{H}\text{ for }\alpha=0,1,2,3\}.$$

Thus $\widehat{\mathbb{H}}\cong\mathbb{H}^4$, endowed with a multiplication scheme satisfying $\widehat{i}2=\widehat{j}^2=\widehat{k}^2=-1$, $\widehat{i}\widehat{j}\widehat{k}=-1$. There is a *canonical linear inclusion* $\iota:\mathbb{H}\to\widehat{\mathbb{H}}$ defined by $\iota(q_0)=\widehat{1}q_0$, and a *canonical linear projection* $\pi:\widehat{\mathbb{H}}\to\mathbb{H}$ satisfying $\pi(\widehat{\varepsilon})=\varepsilon$ for each unit $\widehat{\varepsilon}=\widehat{1},\ \widehat{i},\ \widehat{j},\ \widehat{k}$. These are both ring homomorphisms. It is natural to write q_0 in place of $\widehat{1}q_0$ for $q_0\in\mathbb{H}$. Let us define a quaternion-valued "inner product" $\langle\langle\cdot,\cdot\rangle\rangle:\widehat{\mathbb{H}}\times\widehat{\mathbb{H}}\to\mathbb{H}$ by the requirement that it be additive in each variable and that $\langle\langle\widehat{\varepsilon}p_0,\widehat{\delta}q_0\rangle\rangle=\varepsilon\delta\langle p_0,q_0\rangle$ for any units $\widehat{\varepsilon},\widehat{\delta}$ and any $p_0,q_0\in\mathbb{H}$. (Recall that $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product.) Explicitly this means

$$\begin{aligned}\langle\langle\widehat{p},\widehat{q}\rangle\rangle&=\langle\langle p_0+\widehat{i}p_1+\widehat{j}p_2+\widehat{k}p_3,q_0+\widehat{i}q_1+\widehat{j}q_2+\widehat{k}q_3\rangle\rangle\\&=(\langle p_0,q_0\rangle-\langle p_1,q_1\rangle-\langle p_2,q_2\rangle-\langle p_3,q_3\rangle)\\&\quad+i(\langle p_0,q_1\rangle+\langle p_1,q_0\rangle+\langle p_2,q_3\rangle-\langle p_3,q_2\rangle)\\&\quad+j(\langle p_0,q_2\rangle-\langle p_1,q_3\rangle+\langle p_2,q_0\rangle+\langle p_3,q_1\rangle)\\&\quad+k(\langle p_0,q_3\rangle+\langle p_1,q_2\rangle-\langle p_2,q_1\rangle+\langle p_3,q_0\rangle).\end{aligned}$$

We define the left and right *antilinear inclusions* $\mathbb{H}\to\widehat{\mathbb{H}}$ by

$$\begin{aligned}q_0&\to{}^*q_0=\overline{q}_0+\widehat{i}(i\overline{q}_0)+\widehat{j}(j\overline{q}_0)+\widehat{k}(k\overline{q}_0),\\q_0&\to q_0^*=\overline{q}_0+\widehat{i}(\overline{q}_0i)+\widehat{j}(\overline{q}_0j)+\widehat{k}(\overline{q}_0k),\end{aligned}$$

respectively. Using these, the following rule may be stated for expressing multiplication of $\mathbb{H}$ in terms of $\widehat{\mathbb{H}}$.

PROPOSITION 1.7.2. *Let* $p_0,q_0\in\mathbb{H}$. *Then* $\langle\langle\widehat{1}p_0,{}^*q_0\rangle\rangle=p_0\cdot q_0=\langle\langle\widehat{1}p_0^*,q_0\rangle\rangle$.

Proof. Writing $p_0=y_0+iy_1+jy_2+ky_3,\ q_0=x_0+ix_1+jx_2+kx_3$, one finds that

$$\begin{aligned}\langle\langle p_0,{}^*q_0\rangle\rangle&=\langle p_0,\overline{q}_0\rangle+i\langle p_0,i\overline{q}_0\rangle+j\langle p_0,j\overline{q}_0\rangle+k\langle p_0,\overline{q}_0\rangle\\&=(y_0x_0-y_1x_1-y_2x_2-y_3x_3)\\&\quad+i(y_0x_1+y_1x_0+y_2x_3-y_3x_2)\\&\quad+j(y_0x_2-y_1x_3+y_2x_0+y_2x_1)\\&\quad+k(y_0x_3+y_1x_2-y_2x_1+y_3x_0)\\&=p_0\cdot q_0.\end{aligned}$$

The other equality is similar. ■

PROPOSITION 1.7.3. *Let $p_0 \in \mathbb{H}$ and $\widehat{q} \in \widehat{\mathbb{H}}$. Then*

$$\begin{aligned} \langle\langle \widehat{i}p_0, \widehat{q}\rangle\rangle &= i\langle\langle p_0, \widehat{q}\rangle\rangle, & \langle\langle \widehat{q}, \widehat{i}p_0\rangle\rangle &= \langle\langle \widehat{q}, p_0\rangle\rangle i, \\ \langle\langle \widehat{j}p_0, \widehat{q}\rangle\rangle &= j\langle\langle p_0, \widehat{q}\rangle\rangle, & \langle\langle \widehat{q}, \widehat{j}p_0\rangle\rangle &= \langle\langle \widehat{q}, p_0\rangle\rangle j, \\ \langle\langle \widehat{k}p_0, \widehat{q}\rangle\rangle &= k\langle\langle p_0, \widehat{q}\rangle\rangle, & \langle\langle \widehat{q}, \widehat{k}p_0\rangle\rangle &= \langle\langle \widehat{q}, p_0\rangle\rangle k. \end{aligned}$$

Proof. From Proposition 1.7.2,

$$\begin{aligned} \langle\langle \widehat{i}p_0, \widehat{q}\rangle\rangle &= \langle\langle \widehat{i}p_0, q_0 + iq_1 + jq_2 + kq_3\rangle\rangle \\ &= i\langle p_0, q_0\rangle - \langle p_0, q_1\rangle + k\langle p_0, q_2\rangle - j\langle p_0, q_3\rangle = i\langle\langle p_0, \widehat{q}\rangle\rangle. \end{aligned}$$

The other equalities are similar. ∎

By Proposition 1.7.2 the Leibniz rule (1.7.1) for curves $f, g : (t_1, t_2) \to \mathbb{H}$ can be expressed in the form

$$\frac{d}{dt}(fg) = \left\langle\left\langle \widehat{1}\frac{df}{dt}, {}^*g \right\rangle\right\rangle + \left\langle\left\langle f^*, \widehat{1}\frac{dg}{dt} \right\rangle\right\rangle. \tag{1.7.2}$$

Observe that $\langle\langle dg/dt, f^*\rangle\rangle = \langle\langle f^*, dg/dt\rangle\rangle$ since $dg/dt \in \mathbb{H}$ and $\langle\cdot,\cdot\rangle$ is symmetric. (However, $\langle\langle\cdot,\cdot\rangle\rangle$ is not symmetric.) For $f : \Omega \to \mathbb{H}$, $\Omega \subseteq \mathbb{H}$, define the $\widehat{\mathbb{H}}$-valued operators $\widehat{D}^+$, $\widehat{D}$ by

$$\begin{aligned} \widehat{D}^+ f &= \tfrac{1}{4}(\partial_0 + \widehat{i}\partial_1 + \widehat{j}\partial_2 + \widehat{k}\partial_3)\cdot f, \\ \widehat{D} f &= \tfrac{1}{4}(\partial_0 - \widehat{i}\partial_1 - \widehat{j}\partial_2 - \widehat{k}\partial_3)\cdot f. \end{aligned}$$

The Fueter operator can be expressed as $D^+ = \pi \circ \widehat{D}^+$. These operators satisfy the following Leibniz rules.

THEOREM 1.7.1. *Let $f, g : \Omega \to \mathbb{H}$ be differentiable, $\Omega \subseteq \mathbb{H}$. Then*

$$\begin{aligned} D^+(f\cdot g) &= \langle\langle \widehat{D}^+ f, {}^*g\rangle\rangle + \langle\langle f^*, \widehat{D}^+ g\rangle\rangle, \\ D(f\cdot g) &= \langle\langle \widehat{D} f, {}^*g\rangle\rangle + \langle\langle f^*, \widehat{D} g\rangle\rangle. \end{aligned} \tag{1.7.3}$$

Proof. From the foregoing,

$$\begin{aligned} 4D^+(fg) &= \partial_0(fg) + i\partial_1(fg) + j\partial_2(fg) + k\partial_3(fg) \\ &= (\langle\langle \partial_0 f, {}^*g\rangle\rangle + \langle\langle \partial_0 g, f^*\rangle\rangle) + i(\langle\langle \partial_1 f, {}^*g\rangle\rangle + \langle\langle \partial_1 g, f^*\rangle\rangle) \\ &\quad + j(\langle\langle \partial_2 f, {}^*g\rangle\rangle + \langle\langle \partial_2 g, f^*\rangle\rangle) + k(\langle\langle \partial_3 f, {}^*g\rangle\rangle + \langle\langle \partial_3 g, f^*\rangle\rangle) \\ &= (\langle\langle \partial_0 f, {}^*g\rangle\rangle + \langle\langle \widehat{i}\partial_1 f, {}^*g\rangle\rangle + \langle\langle \widehat{j}\partial_2 f, {}^*g\rangle\rangle + \langle\langle \widehat{k}\partial_3 f, {}^*g\rangle\rangle) \\ &\quad + (\langle\langle \partial_0 g, f^*\rangle\rangle + \langle\langle \widehat{i}\partial_1 g, f^*\rangle\rangle + \langle\langle \widehat{j}\partial_2 g, f^*\rangle\rangle + \langle\langle \widehat{k}\partial_3 g, f^*\rangle\rangle) \\ &= 4(\langle\langle \widehat{D}^+ f, {}^*g\rangle\rangle + \langle\langle \widehat{D}^+ g, f^*\rangle\rangle). \end{aligned}$$

The verification for Df is similar. ∎

A rule for $\mathbb{H}$-valued functions may be expressed as follows.

COROLLARY 1.7.1. *Let $f, g : \Omega \to \mathbb{H}$ be differentiable, $\Omega \subseteq \mathbb{H}$. Then*

$$D^+\cdot(f\cdot g) = \pi[(\widehat{D}^+\cdot f)(\widehat{1}g) + (\widehat{1}f)(\widehat{D}^+\cdot g)].$$

P r o o f. This follows from the relations

$$D^+(fg) = (f_0g + fg_0) + i(f_1g + fg_1) + j(f_2g + fg_2) + k(f_3g + fg_3),$$

$$\begin{aligned}\pi[(\widehat{D}^+ f)(\widehat{1}g)] &= \pi[\widehat{1}f_0 + \widehat{i}f_1 + \widehat{j}f_2 + \widehat{k}f_3)](\widehat{1}g)\\ &= \pi[(\widehat{1}f_0g + \widehat{i}f_1g + \widehat{j}f_2g + \widehat{k}f_3g)]\\ &= f_0g + if_1g + jf_2g + kf_3g = \pi[(\widehat{1}f)(\widehat{D}^+ g)]. \ \blacksquare\end{aligned}$$

Right-sided operators $f \to f\widehat{D}^+, f\widehat{D}$ are defined analogously and satisfy similar formulas.

I.8. Regular functions on manifolds. In this section we show that LR-biregularity is a natural notion to consider on certain quaternionic manifolds.

Let M be a real differentiable manifold of dimension 4. Let $x = (x_0, x_1, x_2, x_3) : U \to \mathbb{R}^4$ be a smooth local coordinate system in an open set U in M. Define $q \in \mathbb{H}$ by (1.2.1). Then a function $F : M \to \mathbb{H}$ is *regular* with respect to this coordinate system if $f = F \circ x^{-1}$ is a regular function of q. The following fact limits the family of manifolds which can admit functions regular in the above sense.

PROPOSITION 1.8.1. *Let $g : \Omega_2 \to \Omega_1$ be a diffeomorphism of domains in H. Suppose $f \circ g$ is regular in Ω_2 for every regular f defined in Ω_1. Then g is a left affine map $g(q) = aq + b,\ a, b \in \mathbb{H}$. Conversely, for any left affine g, $f \circ g$ is regular whenever f is.*

Since left multiplication by a non-real constant never coincides with a right multiplication, the following observation is immediate.

COROLLARY 1.8.1. *Let $X \in \mathbb{R}^{4\times 4}$. Then $XA, AX \in \mathcal{F}$ for all $A \in \mathcal{F}$ if and only if $X = aI$ for some $a \in \mathbb{R}$, where I stands for the identity matrix.*

P r o o f o f P r o p o s i t i o n 1.8.1. According to Proposition 1.3.2,

$$D^+\overline{g}^{(1)}\frac{\partial f}{\partial \overline{q}^1} + D^+\overline{g}^{(2)}\frac{\partial f}{\partial \overline{q}^{(2)}} + D^+\overline{g}^{(3)}\frac{\partial f}{\partial \overline{q}^{(3)}} = 0$$

whenever $D^+f = 0$. For this to hold for all regular f, we must have

$$D^+\overline{g}(1) = D^+\overline{g}^{(2)} = D^+\overline{g}^{(3)} = 0.$$

By Theorem 1.4.4 the differential $dg \in \mathbb{R}^{4\times 4}$ has the form of a left multiplication mapping. It is well known [60] that any quaternionic function whose differential is of this form at each point is left affine. Conversely, by Corollary 1.3.1, pre-composition by such functions preserves regularity. $\blacksquare$

There are, of course, contexts in which mappings more general than affine can preserve regularity in a weaker sense. (For instance, let $g(q) = (aq + b)/(cq + d)^{-1}$ $(a, b, c, d \in \mathbb{H}, a^{-1}b - c^{-1}d \neq 0)$ be a quaternionic Möbius transformation. Then f is regular if and only if $\varrho \cdot (f \circ g)$ is regular [60], where the left factor ϱ is given by $\varrho(q) = (|b - ac^{-1}|^{-2}|cq + d|^{-2} \times (cq + d)^{-1}$. However, the definition we have given for regular F on M implies that if x^α, x^β are compatible local coordinates, and if $f^\alpha = F \circ (x^\alpha)^{-1}, f^\beta = F \circ (x^\beta)^{-1}$, then $f^\beta = f^\alpha \circ g$ where $g = x^\alpha \circ (x^\beta)^{-1}$ must satisfy the condition of Proposition 1.8.1. This may be summarized as follows.

THEOREM 1.8.1. *The only manifolds M modelled locally on $\mathbb{H}$ and admitting a well-defined notion of "regular function" $F : M \to \mathbb{H}$ are the left affine manifolds.*

Examples of affine quaternionic manifolds are discussed, for example, in [57]. Associated with a left affine manifold M is its *canonical bundle* ζ_M. This is the bundle with fiber $\mathbb{H}$ for which the portions $U \times \mathbb{H}$, $V \times \mathbb{H}$ over coordinate neighbourhoods U, V sharing transition function $g(q) = aq + b$ are identified via $(q, p) \approx (g(q), pa^{-1})$. Note that ζ_M is a right $\mathbb{H}$-bundle. There are three other natural $\mathbb{H}$-line bundles $\overline{\zeta}_M$, $\zeta_{M^{-1}}$, $\overline{\zeta}_{M^{-1}}$ obtained by replacing pa^{-1} with $\overline{a}^{-1}p$, ap, $p\overline{a}$, respectively. (Due to the non-commutativity of $\mathbb{H}$, a rule such as pa does not satisfy the cocycle condition for a bundle, nor can we define higher powers of ζ_M.) The *conjugate manifold* $\overline{M}$ is the right affine manifold obtained by replacing each chart of M by its conjugate. One may identify naturally $\overline{\zeta_M} = \overline{\zeta}_M$. Let $C^\infty(M, \mathbb{H})$ denote the collection of smooth $\mathbb{H}$-valued functions on M.

PROPOSITION 1.8.2. *Let M be a left affine manifold. The left Fueter operator $f \to D^+ f$ induces a linear operator $C^\infty(M, \mathbb{H}) \to \Gamma(M, \overline{\zeta}_M)$.*

Proof. Let $F \in C^\infty(M, \mathbb{H})$. For each chart x_α, write $f_\alpha = F \circ x_\alpha$ and $\sigma_\alpha = D^+ f_\alpha$. The coordinate transition functions are $g_{\alpha\beta}(q) = x_\alpha \circ x_\beta^{-1}(q) = a_{\alpha\beta} q + b_{\alpha\beta}$. By Corollary 1.3.1, $D^+ f_\beta = D^+(f_\alpha \circ g_{\alpha\beta}) = \overline{a_{\alpha\beta}}(D^+ f_\alpha \circ g_{\alpha\beta})$, so $\sigma_\alpha \circ g_{\alpha\beta} = \overline{a_{\alpha\beta}}^{-1} \sigma_\beta$. Therefore $D^+ F = (\sigma_\alpha)$ is a section of $\overline{\zeta}_M$. ■

A minor extension of Theorem 1.8.1 shows that in order for regular functions $F : M \to N$ to be defined between two manifolds, M must be left affine and N right affine. Since the natural functions defined on N are the right regular ones, the natural notion of invertibly regular function in this context is that of LR-biregular one. Corollary 1.3.1 gives the following:

THEOREM 1.8.2. *Let M, M' be left affine quaternionic manifolds and N, N' right affine ones. Let $G : M \to M'$ and $H : N \to N'$ be left and right affine mappings, respectively. Let $F : M \to N$ be LR-biregular. Then $H \circ F \circ G$ is LR-biregular.*

Note that similar statements hold when "regular" is replaced by "antiregular" and "right" is exchanged with "left".

II. Fueter regular functions and harmonicity

II.1. Introduction. The class of Fueter regular functions seems in many ways to express very well the spirit of complex analysis in the quaternionic context. Sometimes it is very difficult to obtain a required analogy because of long and hard calculations. In many cases no analogy exists.

It is very interesting that Fueter regular functions appear in a natural way in the Eells theory of harmonic mappings. It turned up, to our surprise, that the Eells condition for harmonicity [13] is, for some kind of 4-dimensional manifolds, equivalent to the existence of an antiregular function in the sense of Fueter. It is the most significant result in our paper. This discovery once again testifies the importance of Fueter regular functions.

In particular, rewriting the Eells condition for harmonicity in quaternionic variables we obtain new results on harmonic mappings from 4-dimensional conformally flat manifolds.

Quaternionic manifolds are defined in many various ways. Because of the non-commutativity of quaternions one cannot expect a definition via transition functions. It seems that the most important trial is done by the group $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$ [7, 52]. We generalized the Lichnerowicz homotopy invariant [38] to the "quaternionic" manifolds applying the old concept of the "quaternionic" Kähler form due to Martinelli [47] and others [4, 8, 25, 52].

In 1969 A. Lichnerowicz [38] found a smooth homotopy invariant, $K(\Phi)$, of maps $\Phi : M \to N$ between a compact special almost Hermitian manifold M and an almost Kähler manifold N. He defined this invariant in terms of the Kähler forms of the manifolds as

$$K(\Phi) = \int_M \langle \omega^M, \Phi^* \omega^N \rangle \, dV_M .$$

He also showed that $K(\Phi)$ can be expressed by means of two partial energies $E'(\Phi)$ and $E''(\Phi)$ naturally associated with the map Φ, if one refers to the almost complex structures of M and N.

It is very much surprising to observe that the idea of the construction of $K(\Phi)$ can be applied to many different contexts. Under suitable general hypothesis a homotopy invariant $K_{\xi,\eta}(\Phi)$ can be considered for smooth maps $\Phi : (M,g) \to (N,h)$ between Riemannian manifolds which admit "canonically" defined p-forms $\xi \in \Lambda^p M$ and $\eta \in \Lambda^p N$ playing the role of the Kähler 2-form in the complex case.

We noticed that in the case of Riemannian manifolds with holonomy group contained in the well known Berger list [4], such forms always exist, and can be used in the definition of the homotopy invariant without any additional hypothesis.

In particular, we show that even in the case of $U(n)$ the main Lichnerowicz invariant is only the first of a series of possible ones. It is also remarkable that in the case of forms ξ and η of maximal degree (equal to the common dimension of the manifolds in question) the invariant $K_{\xi,\eta}(\Phi)$ is expressed by the degree of Φ up to a constant depending on $\mathrm{Vol}(M)$.

As an application of the homotopy invariants introduced in this paper we prove some homotopy results for certain classes of smooth maps $\Phi : M \to N$, where M and N are manifolds equipped with G-structures from the Berger list.

The manifolds considered in this section are assumed to be connected, orientable, compact and without boundary.

II.2. Quaternionic manifolds—foundations. Let us begin with some basic definitions. Denote by $\mathbb{H}$ the field of quaternions. Let V be a real m-dimensional vector space and V^* the space dual to V. Consider the following vector spaces over $\mathbb{H}$:

$$\begin{aligned} V^{\mathbb{H}} &:= V \otimes_{\mathbb{R}} \mathbb{H} \text{ — quaternionification of } V \text{ (a right } \mathbb{H}\text{-vector space)},\\ \overline{V^{\mathbb{H}}} &:= \mathbb{H} \otimes_{\mathbb{R}} V \text{ — quaternionic conjugation of } V^{\mathbb{H}} \text{ (a left } \mathbb{H}\text{-vector space)},\\ (V^{\mathbb{H}})^* &:= \mathbb{H} \otimes_{\mathbb{R}} V^* \text{ — dual of } V^{\mathbb{H}} \text{ (a left } \mathbb{H}\text{-vector space)},\\ (\overline{V^{\mathbb{H}}})^* &:= V^* \otimes_{\mathbb{R}} \mathbb{H} \text{ — dual of } \overline{V^{\mathbb{H}}} \text{ (a right } \mathbb{H}\text{-vector space)}. \end{aligned}$$

We use the fact that $\overline{qv} = \overline{v}\overline{q}$, $v \in V^{\mathbb{H}}$, $q \in \mathbb{H}$.

R e m a r k 2.2.1. $(V^{\mathbb{H}})^*$ can be considered as an $\mathbb{R}$-vector space of $\mathbb{R}$-linear forms on V with values in $\mathbb{H}$. $\overline{V^{\mathbb{H}}}$ can be identified with $V^{\mathbb{H}}$; multiplication by a quaternion q in $\overline{V^{\mathbb{H}}}$ is substituted with multiplication by $\overline{q}$ in $V^{\mathbb{H}}$ ($\overline{q}$ is the conjugation of q; we have $q\overline{v} = \overline{v\overline{q}}$).

Let $(1, i, j, k)$ be a fixed standard basis of $\mathbb{H}$ with the well known properties: $i^2 = j^2 = k^2 = -1$, $ijk = -1$. Consider three involutions of $\mathbb{H}$, denoted by σ_1, σ_2 and σ_3 given by inner automorphisms and corresponding to i, j, k, respectively, which satisfy the following conditions:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathrm{id}, \quad \sigma_1\sigma_2 = \sigma_2\sigma_1 = \sigma_3,$$
$$\sigma_2\sigma_3 = \sigma_3\sigma_2 = \sigma_1, \quad \sigma_3\sigma_1 = \sigma_1\sigma_3 = \sigma_2.$$

Using the quaternionic units, the σ_i can be expressed as

$$\sigma_1(q) = -iqi, \quad \sigma_2(q) = -jqj, \quad \sigma_3(q) = -kqk, \quad q \in \mathbb{H}.$$

Then we set

$$q^1 := \sigma_1(q), \quad q^2 := \sigma_2(q), \quad q^3 := \sigma_3(q), \quad q \in \mathbb{H}.$$

Explicitly, if $q = x_0 + ix_1 + jx_2 + kx_3$, then

$$q^1 = x_0 + ix_1 - jx_2 - kx_3,$$
$$q^2 = x_0 - ix_1 + jx_2 - kx_3,$$
$$q^3 = x_0 - ix_1 - jx_2 + kx_3.$$

Note that

$$iq^1 = qi, \quad q^1 = iq, \quad jq^2 = qj, \quad q^2j = jq, \quad kq^3 = qk, \quad q^3k = kq.$$

We also have

$$iq = q^1i, \quad iq^1 = qi, \quad iq^2 = q^3i, \quad iq^3 = q^2i,$$
$$jq = q^2j, \quad jq^1 = q^3j, \quad jq^2 = qj, \quad jq^3 = q^1j,$$
$$kq = q^3k, \quad kq^1 = q^2k, \quad kq^2 = q^1k, \quad kq^3 = qk.$$

Let $\mathrm{Sp}(m)$ be the group of automorphisms of the right quaternionic vector space $\mathbb{H}^m$ which are unitary with respect to the canonical Hermitian product

$$Q \cdot P = \sum_\alpha q^\alpha \overline{p}^\alpha, \quad Q = (q^1, \ldots, q^m), \ P = (p^1, \ldots, p^m) \in \mathbb{H}^m.$$

Recall that the enhancement $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$ of $\mathrm{Sp}(m)$ is the group of $\mathbb{R}$-linear automorphisms $T_{A,q}$ of $\mathbb{H}^m$ defined by

$$T_{A,q}(Q) := AQ \cdot q, \quad Q \in \mathbb{H}^m,$$

where "·" means quaternionic multiplication by a unitary quaternion $q \in \mathrm{Sp}(1)$ and A is a transformation from $\mathrm{Sp}(m)$.

The group $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$ preserves the Euclidean product $g_0(Q, P) = \mathrm{Re}(Q \cdot P)$ in $\mathbb{H}^m \cong \mathbb{R}^{4m}$, hence it is a subgroup of $\mathrm{SO}(4m)$.

DEFINITION 2.2.1 [7]. An *almost quaternionic manifold* is a $4m$-dimensional real manifold with a $\mathrm{GL}(m;\mathbb{H}) \cdot \mathrm{Sp}(1)$-structure (i.e. a reduction of its tangent principal bundle to the subgroup $\mathrm{GL}(m;\mathbb{H}) \cdot \mathrm{Sp}(1)$ of $\mathrm{GL}(4m;\mathbb{R})$).

Definition 2.2.2 [7]. A Riemannian metric on an almost quaternionic manifold is said to be *quaternionic-Hermitian* if there exists a common reduction of the $\mathrm{GL}(m;\mathbb{H})\cdot\mathrm{Sp}(1)$ and $\mathrm{SO}(4m)$-bundles to $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$.

Let $(1, i_1, i_2, i_3)$ be an arbitrary base of $\mathbb{H}$ (i.e. $i_1^2 = i_2^2 = i_3^2 = -1,\ i_1i_2i_3 = -1$). Then right multiplication by i_1, i_2 and i_3 determines a triple (I_1, I_2, I_3) of complex structures on $\mathbb{R}^{4m} \cong \mathbb{H}^m$ satisfying the following conditions:

$$I_1^2 = I_2^2 = I_3^2 = -\mathrm{Id}, \quad I_1I_2I_3 = -\mathrm{Id},$$

where Id stands for the identity mapping in $\mathbb{R}^{4m}$.

Any two such triples (I_1, I_2, I_3) and (I_1', I_2', I_3') are related by a transformation

$$I_h' = \sum_{k=1}^{3} c_{hk} I_k, \qquad h = 1, 2, 3,$$

with $(c_{hk}) \in \mathrm{SO}(3)$.

Definition 2.2.3 [7]. An *almost quaternionic structure* is defined as a covering $\{U_i\}$ of the manifold with two almost complex structures I_i and J_i such that $I_iJ_i = -J_iI_i$ and the 3-dimensional vector spaces of endomorphisms generated by I_i, J_i and $K_i := I_iJ_i$:

$$\mathrm{End}_{U_i} := \{\alpha I_i + \beta J_i + \gamma K_i : \alpha, \beta, \gamma \in \mathbb{R}\}$$

are the same on all of the manifold.

Then a Riemannian metric g is *quaternionic-Hermitian* if g is Hermitian for each I and J.

Definition 2.2.4. The *standard enhanced quaternionic structure* of $\mathbb{H}^m$ is the 3-dimensional subspace Q_0 of the space $\mathrm{End}_{\mathbb{R}}\,\mathbb{H}^m$ generated by (any) triple (I_1, I_2, I_3) as above. We call (I_1, I_2, I_3) an *admissible hypercomplex base* of Q_0 (we also write $(I_1, I_2, I_3) \in Q_0$).

Let $(I_1, I_2, I_3) \in Q_0$. Consider the 2-forms ω_1, ω_2 and ω_3 defined by

$$\omega_k(Q, P) := g_0(Q, I_kP), \qquad k = 1, 2, 3.$$

Definition 2.2.5 [7, 47]. Define

$$\Omega_0 := \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3.$$

Theorem 2.2.1 [6]. *Ω_0 is a well defined 4-form, independent of $(I_1, I_2, I_3) \in Q_0$ and invariant with respect to the group $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$. Moreover, it is non-degenerate because it satisfies the condition*

$$(\Omega_0)^m = (2m+1)!\,\mathrm{vol}(\mathbb{R}^{4m}).$$

Definition 2.2.6. 1. A Riemannian manifold (M^{4m}, g) with a given $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$-structure is called an *almost-Hermitian-quaternionic manifold*. The 4-form Ω corresponding to Ω_0 is called the *fundamental* 4-form (or Kähler form) of (M^{4m}, g).

2. An almost-quaternionic-Hermitian manifold (M^{4m}, g) is called an *almost-quaternionic-Kähler manifold* if the fundamental 4-form is closed.

3. An almost-quaternionic-Hermitian manifold (M^{4m}, g) is called a *quaternionic-Kähler manifold* if its holonomy group is contained in $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$.

R e m a r k 2.2.2. The most important example of an almost-quaternionic-Kähler manifold is the quaternionic projective space $\mathbb{HP}^m$ with a standard metric (cf. e.g. [8, 47]).

R e m a r k 2.2.3. 1. Since $\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)=\mathrm{SO}(4)$, every 4-dimensional oriented Riemannian manifold is in a natural way an almost-quaternionic-Kähler manifold.

2. If $4m > 8$ then the holonomy group of an almost-quaternionic-Kähler manifold is contained in $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$ (cf. [52]).

Let $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ be a C^∞-mapping between two almost-quaternionic-Hermitian manifolds of dimensions $4m$ and $4n$, respectively. We can regard a quaternionic extension $d^{\mathbb{H}}\Phi$ of the differential $d\Phi$ as a section of the bundle $(\Phi^{-1}T^{\mathbb{H}}N) \otimes_{\mathbb{H}} (T^{\mathbb{H}}M)^*$ over M.

Let $p \in M$. It is well known (see e.g. [8]) that $T_p^{\mathbb{H}}M := T_pM \otimes_{\mathbb{R}} \mathbb{H}$ can be decomposed in the following way:

$$T_p^{\mathbb{H}}M = U_p^{\mathbb{H}} \oplus \tau_1 U_p^{\mathbb{H}} \oplus \tau_2 U_p^{\mathbb{H}} \oplus \tau_3 U_p^{\mathbb{H}},$$

where τ_1, τ_2, τ_3 are the semi-involutions defined on $T_p^{\mathbb{H}}M$ as

$$\tau_1 = \mathrm{id}\otimes\sigma_1, \quad \tau_2 = \mathrm{id}\otimes\sigma_2, \quad \tau_3 = \mathrm{id}\otimes\sigma_3$$

and $\sigma_1, \sigma_2, \sigma_3$ are the involutions of the algebra of quaternions $\mathbb{H}$.

Take $p \in M$ and let U be a small open neighbourhood of p. On U, by Definition 2.3, there are three almost complex structures I, J and $K := IJ$. Using I, J and K we can define $U_p^{\mathbb{H}}$ by

$$U_p^{\mathbb{H}} := \{X + iIX + jJX + kKX : X \in T_pM\}.$$

Then we get

$$\begin{aligned}
\tau_1 U_p^{\mathbb{H}} &= \{X + iIX - jJX - kKX : X \in T_pM\},\\
\tau_2 U_p^{\mathbb{H}} &= \{X - iIX + jJX - kKX : X \in T_pM\},\\
\tau_3 U_p^{\mathbb{H}} &= \{X - iIX - jJX + kKX : X \in T_pM\}.
\end{aligned}$$

R e m a r k 2.2.4. There exist elements $X_1, \ldots, X_m$ of T_pM such that the system $(X_1, \ldots, X_m, IX_1, \ldots, IX_m, JX_1, \ldots, JX_m, KX_1, \ldots, KX_m)$ forms an orthonormal basis for $T_p^{\mathbb{H}}M$ with respect to the metric g.

P r o o f. It is analogous to that in the complex case [24]. ■

Let $(x_0^i, x_1^i, x_2^i, x_3^i)$, $i = 1, \ldots, m$, be local real coordinates at the point p. We can introduce the quaternionic coordinates $(q_1, \ldots, q_m)$ as follows:

$$q_k := x_0^k + ix_1^k + jx_2^k + kx_3^k, \quad k = 1, \ldots, m.$$

If the almost quaternionic structure (I, J, K) is integrable then we can assume that I, J and K are given by

$$\begin{aligned}
&I\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big) = -\frac{\partial}{\partial x^k_{1\,|p}}, \quad J\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big) = -\frac{\partial}{\partial x^k_{2\,|p}}, \quad K\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big) = -\frac{\partial}{\partial x^k_{3\,|p}},\\
&I\Big(\frac{\partial}{\partial x^k_{1\,|p}}\Big) = \frac{\partial}{\partial x^k_{0\,|p}}, \quad J\Big(\frac{\partial}{\partial x^k_{1\,|p}}\Big) = \frac{\partial}{\partial x^k_{3\,|p}}, \quad K\Big(\frac{\partial}{\partial x^k_{1\,|p}}\Big) = -\frac{\partial}{\partial x^k_{2\,|p}},\\
&I\Big(\frac{\partial}{\partial x^k_{2\,|p}}\Big) = -\frac{\partial}{\partial x^k_{3\,|p}}, \quad J\Big(\frac{\partial}{\partial x^k_{2\,|p}}\Big) = \frac{\partial}{\partial x^k_{0\,|p}}, \quad K\Big(\frac{\partial}{\partial x^k_{2\,|p}}\Big) = \frac{\partial}{\partial x^k_{1\,|p}},\\
&I\Big(\frac{\partial}{\partial x^k_{3\,|p}}\Big) = \frac{\partial}{\partial x^k_{2\,|p}}, \quad J\Big(\frac{\partial}{\partial x^k_{3\,|p}}\Big) = -\frac{\partial}{\partial x^k_{1\,|p}}, \quad K\Big(\frac{\partial}{\partial x^k_{3\,|p}}\Big) = \frac{\partial}{\partial x^k_{0\,|p}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial}{\partial q_{k\,|p}} &= \frac{1}{4}\Big(\frac{\partial}{\partial x^k_{0\,|p}} - i\frac{\partial}{\partial x^k_{1\,|p}} - j\frac{\partial}{\partial x^k_{2\,|p}} - k\frac{\partial}{\partial x^k_{3\,|p}}\Big)\\
&= \frac{1}{4}\Big[\frac{\partial}{\partial x^k_{0\,|p}} + iI\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big) + jJ\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big) + kK\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big)\Big] \in U^{\mathbb{H}}_p,\\
\frac{\partial}{\partial q^1_{k\,|p}} &= \frac{1}{4}\Big(\frac{\partial}{\partial x^k_{0\,|p}} - i\frac{\partial}{\partial x^k_{1\,|p}} + j\frac{\partial}{\partial x^k_{2\,|p}} + k\frac{\partial}{\partial x^k_{3\,|p}}\Big)\\
&= \frac{1}{4}\Big[\frac{\partial}{\partial x^k_{0\,|p}} + iI\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big) - jJ\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big) - kK\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big)\Big] \in \tau_1 U^{\mathbb{H}}_p,\\
\frac{\partial}{\partial q^2_{k\,|p}} &= \frac{1}{4}\Big(\frac{\partial}{\partial x^k_{0\,|p}} + i\frac{\partial}{\partial x^k_{1\,|p}} - j\frac{\partial}{\partial x^k_{2\,|p}} + \frac{\partial}{\partial x^k_{3\,|p}}\Big)\\
&= \frac{1}{4}\Big[\frac{\partial}{\partial x^k_{0\,|p}} - iI\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big) + jJ\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big) - kK\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big)\Big] \in \tau_2 U^{\mathbb{H}}_p,\\
\frac{\partial}{\partial q^3_{k\,|p}} &= \frac{1}{4}\Big(\frac{\partial}{\partial x^k_{0\,|p}} + i\frac{\partial}{\partial x^k_{1\,|p}} + j\frac{\partial}{\partial x^k_{2\,|p}} - k\frac{\partial}{\partial x^k_{3\,|p}}\Big)\\
&= \frac{1}{4}\Big[\frac{\partial}{\partial x^k_{0\,|p}} - iI\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big) - jJ\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big) + kK\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big)\Big] \in \tau_3 U^{\mathbb{H}}_p.
\end{aligned}$$

Then the system $\{\partial/\partial q_{1\,|p}, \ldots, \partial/\partial q_{m\,|p}\}$ forms a basis for $U^{\mathbb{H}}_p$ and $\{\partial/\partial q^\alpha_{k\,|p}, \ldots, \partial/\partial q^\alpha_{k\,|p}\}$ form bases for $\tau_\alpha U^{\mathbb{H}}_p$, $\alpha = 1,2,3$.

It is also clear that $\{dq_{1\,|p}, \ldots, dq_{m\,|p}\}$ and $\{dq^\alpha_{1\,|p}, \ldots, dq^\alpha_{m\,|p}\}$ form bases for $(U^{\mathbb{H}}_p)^*$ and $(\tau_\alpha U^{\mathbb{H}}_p)^*$, $\alpha = 1,2,3$, respectively.

II.3. Energies of mappings. Take into account the decompositions:

$$\begin{aligned}
T^{\mathbb{H}}_p M &= U^{\mathbb{H}}_p \oplus \tau_1 U^{\mathbb{H}}_p \oplus \tau_2 U^{\mathbb{H}}_p \oplus \tau_3 U^{\mathbb{H}}_p,\\
T^{\mathbb{H}}_{\Phi(p)} N &= \widehat{U}^{\mathbb{H}}_{\Phi(p)} \oplus \widehat{\tau}_1 \widehat{U}^{\mathbb{H}}_{\Phi(p)} \oplus \widehat{\tau}_2 \widehat{U}^{\mathbb{H}}_{\Phi(p)} \oplus \widehat{\tau}_3 \widehat{U}^{\mathbb{H}}_{\Phi(p)}.
\end{aligned} \tag{2.3.1}$$

Then the quaternionified differential $d^{\mathbb{H}}\Phi : T^{\mathbb{H}}M \to T^{\mathbb{H}}N$ determines various partial differentials by composing the inclusions of $U^{\mathbb{H}}$ and $\tau_i U^{\mathbb{H}}$, $i = 1,2,3$, into $T^{\mathbb{H}}M$ and the projections of $T^{\mathbb{H}}N$ onto $\widehat{U}^{\mathbb{H}}$ and $\widehat{\tau}_i \widehat{U}^{\mathbb{H}}$, $i = 1,2,3$.

Note that the partial differential $\partial\Phi : U^{\mathbb{H}} \to \widehat{U}^{\mathbb{H}}$ is independent of the choice of the systems of semi-involutions (τ_1, τ_2, τ_3) and $(\widehat{\tau}_1, \widehat{\tau}_2, \widehat{\tau}_3)$, defined on $T^{\mathbb{H}}M$ and $T^{\mathbb{H}}N$, respectively.

In the neighbourhood of p and $\Phi(p)$ we choose orthonormal Hermitian bases $(\eta_i^M, \eta_{i'}^M, \eta_{i''}^M, \eta_{i'''}^M;\ i = 1, \ldots, m)$, $(\varepsilon_\alpha^N, \varepsilon_{\alpha'}^N, \varepsilon_{\alpha''}^N, \varepsilon_{\alpha'''}^N;\ \alpha = 1, \ldots, n)$ in $T_p^{\mathbb{H}}M$ and $T_{\Phi(p)}^{\mathbb{H}}N$, respectively. If $(\theta_M^i, \theta_M^{i'}, \theta_M^{i''}, \theta_M^{i'''};\ i = 1, \ldots, m)$ denotes the dual base to η^M, then we get

$$\begin{aligned} d^{\mathbb{H}}\Phi = {}& \varepsilon_\alpha(\Phi_i^\alpha\theta^i + \Phi_{i'}^\alpha\theta^{i'} + \Phi_{i''}^\alpha\theta^{i''} + \Phi_{i'''}^\alpha\theta^{i'''}) \\ & + \varepsilon_{\alpha'}(\Phi_i^{\alpha'}\theta^i + \Phi_{i'}^{\alpha'}\theta^{i'} + \Phi_{i''}^{\alpha'}\theta^{i''} + \Phi_{i'''}^{\alpha'}\theta^{i'''}) \\ & + \varepsilon_{\alpha''}(\Phi_i^{\alpha''}\theta^i + \Phi_{i'}^{\alpha''}\theta^{i'} + \Phi_{i''}^{\alpha''}\theta^{i''} + \Phi_{i'''}^{\alpha''}\theta^{i'''}) \\ & + \varepsilon_{\alpha'''}(\Phi_i^{\alpha'''}\theta^i + \Phi_{i'}^{\alpha'''}\theta^{i'} + \Phi_{i''}^{\alpha'''}\theta^{i''} + \Phi_{i'''}^{\alpha'''}\theta^{i'''}). \end{aligned}$$

DEFINITION 2.3.1. We define the *first partial density* of Φ by

$$e' := \sum_{\alpha,i} \overline{\Phi_i^\alpha}\Phi_i^\alpha,$$

where "$\bar{\cdot}$" denotes quaternionic conjugation.

LEMMA 2.3.1. *The definition of $e'(\Phi)$ is independent of the choice of bases in $T_p^{\mathbb{H}}M$ and $T_{\Phi(p)}^{\mathbb{H}}N$, respectively.*

Proof. We prove the lemma in two steps.

a) Assume that $(\widetilde{\varepsilon}_\beta)$, $\beta = 1, \ldots, n$, is another base of $\widehat{U}_{\Phi(p)}^{\mathbb{H}}$. Then there is a transformation $A = (A_\beta^\alpha) \in \mathrm{Sp}(n)$ such that $\widetilde{\varepsilon}_\beta = \varepsilon_\alpha A_\beta^\alpha$. It is easy to see that in this new base we have $\widetilde{\Phi}_i^\beta = A_\alpha^\beta\Phi_i^\alpha$. Thus we get

$$\overline{\widetilde{\Phi}_i^\beta}\widetilde{\Phi}_i^\beta = \overline{(A_\alpha^\beta\Phi_i^\alpha)}(A_\gamma^\beta\Phi_i^\gamma) = \overline{\Phi_i^\alpha}\,\overline{A_\alpha^\beta}A_\gamma^\beta\Phi_i^\gamma = \overline{\Phi_i^\alpha}\,\overline{A_\alpha^\beta}A_\beta^{T\gamma}\Phi_i^\gamma = \overline{\Phi_i^\alpha}\delta_\alpha^\gamma\Phi_i^\gamma = \overline{\Phi_i^\alpha}\Phi_i^\gamma,$$

as required.

b) Let $(\widetilde{\eta}_j)$, $j = 1, \ldots, m$, be another base of $U_p^{\mathbb{H}}$ and $(\widetilde{\theta}^j)$ a dual base to $(\widetilde{\eta}_j)$. Then we have $\widetilde{\theta}^j = B_i^j\theta^i$, where $B = (B_i^j) \in \mathrm{Sp}(m)$. Thus $\widetilde{\Phi}_i^\beta = \Phi_j^\beta B_i^j$ and hence

$$\begin{aligned} \overline{\widetilde{\Phi}_i^\beta}\widetilde{\Phi}_i^\beta &= \overline{(\Phi_j^\beta B_i^j)}(\Phi_k^\beta B_i^k) = \overline{B_i^k}\,\overline{\Phi_j^\beta}\Phi_k^\beta B_i^k = \overline{\Phi_j^\beta}(\overline{B_k^j}B_i^k)\Phi_k^\beta \\ &= \overline{\Phi_j^\beta}(\overline{B_i^j}B_k^{Ti})\Phi_k^\beta = \overline{\Phi_k^\beta}\delta_k^j\Phi_k^\beta = \overline{\Phi_k^\beta}\Phi_k^\beta \end{aligned}$$

because $\mathrm{Re}(qq') = \mathrm{Re}(q'q)$. ∎

Remark 2.3.1. The definition of $e'(\Phi)$ is independent of the choice of semi-involutions in the decompositions (2.3.1) of $T^{\mathbb{H}}M$ and $T^{\mathbb{H}}N$, respectively.

Proof. Indeed, let us change the system of the semi-involutions from (τ_1, τ_2, τ_3) to $(\sigma_1, \sigma_2, \sigma_3)$. According to this change the base $(\eta_i, \eta_{i'}, \eta_{i''}, \eta_{i'''})$ transforms into $(\widetilde{\eta}_i, \widetilde{\eta}_{i'}, \widetilde{\eta}_{i''}, \widetilde{\eta}_{i'''})$. Note that $\widetilde{\eta}_i = \eta_i$ and $(\widetilde{\eta}_{i'}, \widetilde{\eta}_{i''}, \widetilde{\eta}_{i'''})$ are obtained due to the transformation of $(\eta_{i'}, \eta_{i''}, \eta_{i'''})$ by means of an element from $\mathrm{Sp}(3)$. Then we get $\widetilde{\Phi}_i^\alpha = \Phi_i^\alpha$. We deal analogously with the changes of the system $(\widehat{\tau}_1, \widehat{\tau}_2, \widehat{\tau}_3)$. ∎

COROLLARY 2.3.1. *The quantity $e'(\Phi)$ is well defined.*

Remark 2.3.2. In Lemma 2.3.1 one can take the groups $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ and $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$ instead of the groups $\mathrm{Sp}(n)$ and $\mathrm{Sp}(m)$, respectively.

DEFINITION 2.3.2. We define the *second partial energy density* as follows:

$$e''(\Phi) = \sum_{\alpha,i}(\overline{\Phi_{i'}^\alpha}\Phi_{i'}^\alpha + \overline{\Phi_{i''}^\alpha}\Phi_{i''}^\alpha + \overline{\Phi_{i'''}^\alpha}\Phi_{i'''}^\alpha).$$

LEMMA 2.3.2. *The definition of $e''(\Phi)$ is independent of the choice of bases in $T_p^{\mathbb{H}}M$ and $T_{\Phi(p)}^{\mathbb{H}}N$, respectively, and of the choice of semi-involutions introduced to decompose $T^{\mathbb{H}}M$ and $T^{\mathbb{H}}N$.*

Proof. The proof is similar to those of Lemma 2.3.1 and Remark 2.3.1 and thus it may be omitted. ∎

Remark 2.3.3. The first and second partial energy densities are invariant under the groups $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ and $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$, acting from the left and right, respectively. This means that if $(A,q)\in \mathrm{Sp}(n)\,\mathrm{Sp}(1)$ and $(B,w)\in \mathrm{Sp}(m)\,\mathrm{Sp}(1)$ then $e'(\Phi)$ and $e''(\Phi)$ are invariant under the following transformations:

$$\widetilde{\Phi}_i^\beta = (A,q)\Phi_i^\alpha := A_\alpha^\beta \Phi_i^\alpha q, \quad \widetilde{\Phi}_i^\beta = \Phi_\gamma^\beta (B,w) := w\Phi_j^\beta B_i^j,$$
$$\widetilde{\Phi}_i^\beta = (A,q)[\Phi_j^\alpha(B,w)] = (A,q)[w\Phi_j^\alpha B_i^j] = wA_\alpha^\beta \Phi_j^\alpha B_i^j q,$$
$$\widetilde{\Phi}_i^\beta = [(A,q)\Phi_i^\alpha](B,w) = [A_\alpha^\beta \Phi_i^\alpha q](B,w) = wA_\alpha^\beta \Phi_j^\alpha B_i^j q.$$

DEFINITION 2.3.3 [13]. The *energy density* of Φ is the function

$$e(\Phi) := \tfrac{1}{2}|d\Phi|^2$$

with $|d\Phi|^2 := g^{ij}h_{\alpha\beta}\Phi_i^\alpha\Phi_j^\beta$, where $(\Phi_i^\alpha) := (\partial\Phi^\alpha/\partial x_i)$ is a real local representation of $d\Phi$.

DEFINITION 2.3.4 [13]. The *energy* of Φ is the number

$$E(\Phi) := \int_M e(\Phi)v_g,$$

where v_g is the volume element on M (provided that M is compact).

LEMMA 2.3.3. *We have*

$$\tfrac{1}{2}e(\Phi) = e'(\Phi) + e''(\Phi).$$

Proof. By Remark 2.3.4 (see below) the proof is evident. ∎

Remark 2.3.4. If $\Phi:\Omega\to\mathbb{H}$, $\Omega\subset\mathbb{H}$ open, is differentiable (see Definition 1.1) and $\Phi=\Phi^0+i\Phi^1+j\Phi^2+k\Phi^3$ then

$$\left|\frac{\partial\Phi}{\partial q}\right|^2 + \left|\frac{\partial\Phi}{\partial q^1}\right|^2 + \left|\frac{\partial\Phi}{\partial q^2}\right|^2 + \left|\frac{\partial\Phi}{\partial q^3}\right|^2 = \frac{1}{4}\sum_{i=0}^{3}(\Phi_i^\alpha)^2, \quad \Phi_i^\alpha = \frac{\partial\Phi^\alpha}{\partial x_i}.$$

Proof. The left-hand side of the above formula is equal to 1/16 of

$$\begin{aligned}
&[(\Phi_0^0+\Phi_1^1+\Phi_2^2+\Phi_3^3)^2+(\Phi_0^1-\Phi_1^0-\Phi_2^3+\Phi_3^2)^2\\
&\quad+(\Phi_0^2+\Phi_1^3-\Phi_2^0-\Phi_3^1)^2+(\Phi_0^3-\Phi_1^2+\Phi_2^1-\Phi_3^0)^2]\\
&\quad+[(\Phi_0^0+\Phi_1^1-\Phi_2^2-\Phi_3^3)^2+(\Phi_0^1-\Phi_1^0+\Phi_2^3-\Phi_3^2)^2\\
&\quad+(\Phi_0^2+\Phi_1^3+\Phi_2^0+\Phi_3^1)^2+(\Phi_0^3-\Phi_1^2-\Phi_2^1+\Phi_3^0)^2]\\
&\quad+[(\Phi_0^0-\Phi_1^1+\Phi_2^2-\Phi_3^3)^2+(\Phi_0^1+\Phi_1^0-\Phi_2^3-\Phi_3^2)^2\\
&\quad+(\Phi_0^2-\Phi_1^3-\Phi_2^0+\Phi_3^1)^2+(\Phi_0^3+\Phi_1^2+\Phi_2^1+\Phi_3^0)^2]\\
&\quad+[(\Phi_0^0-\Phi_1^1-\Phi_2^2+\Phi_3^3)^2+(\Phi_0^1+\Phi_1^0+\Phi_2^3+\Phi_3^2)^2
\end{aligned}$$

$$+ (\Phi_0^2 - \Phi_1^3 + \Phi_2^0 - \Phi_3^1)^2 + (\Phi_0^3 + \Phi_1^2 - \Phi_2^1 - \Phi_3^0)^2]$$
$$= 4[(\Phi_0^0)^2 + (\Phi_1^0)^2 + (\Phi_2^0)^2 + (\Phi_3^0)^2 + (\Phi_0^1)^2 + (\Phi_1^1)^2 + (\Phi_2^1)^2 + (\Phi_3^1)^2$$
$$+ (\Phi_0^2)^2 + (\Phi_1^2)^2 + (\Phi_2^2)^2 + (\Phi_3^2)^2 + (\Phi_0^3)^2 + (\Phi_1^3)^2 + (\Phi_2^3)^2 + (\Phi_3^3)^2]. \blacksquare$$

For M compact we set

$$E'(\Phi) := \int_M e'(\Phi), \quad E''(\Phi) := \int_M e''(\Phi).$$

Then we obtain

$$E(\Phi) = E'(\Phi) + E''(\Phi).$$

Our aim was to generalize the following result of Eells and Lemaire [13]:

THEOREM 2.3.1. *Let (M, g) and (N, h) be almost Kähler manifolds and $\Phi : M \to N$ a smooth map. If ω^M and ω^N represent the Kähler forms on M and N, respectively, then*

$$\langle \omega^M, \Phi^* \omega^N \rangle = e'(\Phi) - e''(\Phi),$$

where

$$e'(\Phi) = g^{i\overline{j}} h_{\alpha\overline{\beta}} \Phi_i^\alpha \overline{\Phi_j^\beta}, \quad e''(\Phi) = g^{i\overline{j}} h_{\alpha\overline{\beta}} \Phi_{\overline{j}}^\alpha \overline{\Phi_{\overline{i}}^\beta}$$

and Φ_i^α (resp. $\Phi_{\overline{j}}^\alpha$) is the matrix representation of $\partial\Phi$ (resp. $\overline{\partial}\Phi$) in local frame fields.

In spite of several efforts we have not found a quaternionic generalization of the above theorem.

II.4. Lichnerowicz-type homotopy invariant—quaternionic case. Let us recall the definition of the Lichnerowicz invariant [38].

Take two almost Hermitian manifolds: (M, J, g) and (N, J', h), where J, J' denote almost complex structures and g, h almost Hermitian metrics on M and N, respectively.

Let $\Phi : (M, J, g) \to (N, J', h)$ be a smooth map. One defines (see e.g. [13]) the partial energy densities of Φ as the following squares of complex norms:

$$e'(\Phi) = |\partial\Phi|^2 = g^{i\overline{j}} h_{\alpha\overline{\beta}} \Phi_i^\alpha \overline{\Phi_j^\beta}, \quad e''(\Phi) = |\overline{\partial}\Phi|^2 = g^{i\overline{j}} h_{\alpha\overline{\beta}} \Phi_{\overline{j}}^\alpha \overline{\Phi_{\overline{i}}^\beta},$$

where Φ_i^α (resp. $\Phi_{\overline{j}}^\alpha$) is the matrix representation of $\partial\Phi$ (resp. $\overline{\partial}\Phi$) in the given local coordinate system.

For M compact we set

$$E'(\Phi) := \int_M e'(\Phi)\, dV_g, \quad E''(\Phi) := \int_M e''(\Phi)\, dV_g,$$

where dV_g is the volume element on M.

Obviously, Φ is holomorphic iff $E''(\Phi) = 0$ and antiholomorphic iff $E''(\Phi) = 0$.

Lichnerowicz [38] introduced the following quantity:

$$K(\Phi) := E'(\Phi) - E''(\Phi)$$

and he proved that:

1. If (M, J, g) and (N, J', h) are almost Kähler manifolds then $K(\Phi)$ is a smooth homotopy invariant.

2. If ω^M and ω^N represent the Kähler forms on M and N, respectively, then

$$K(\Phi) = \int_M \langle \omega^M, \Phi^* \omega^N \rangle \, dV_g.$$

Now, let us pose the following problem:

Problem 2.4.1. Find an analogue of the Lichnerowicz invariant on a quaternionic manifold.

Let (M^{4m}, g) and (N^{4n}, h) be two almost-quaternionic-Hermitian manifolds with $\dim_{\mathbb{R}} M^{4m} = 4m$ and $\dim_{\mathbb{R}} N^{4n} = 4n$. Suppose that M^{4m} is compact. Assume that $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a smooth mapping. Define

$$K(\Phi) := \int_M \langle \Omega_M, \Phi^* \Omega_N \rangle \, dV_g,$$

where Ω_M and Ω_N represent the 4-fundamental forms on M and N, respectively (see Sect. II.2).

We will prove the following:

Theorem 2.4.1. *Let (M^{4m}, g) and (N^{4n}, h) be two almost-quaternionic-Kähler manifolds (M^{4m} being compact). Suppose that $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a smooth map. Then $K(\Phi)$ is a smooth homotopy invariant.*

We will need two lemmas.

Lemma 2.4.1. *Let $\Phi_t : M \to N$ ($t \in \mathbb{R}$) be a smooth family of maps between the smooth manifolds M and N, and let ω^N be a closed 4-form on N. Then*

$$\frac{\partial}{\partial t}(\Phi_t^* \omega^N) = d[\Phi_t^*(i_{\partial\Phi/\partial t} \omega^N)],$$

where $i_X \omega^N$ denotes the interior product of the vector X with the 4-form ω^N.

Denote by $\mathcal{C}(TM)$ the set of all C^∞-vector fields on M and by $C^\infty(M)$ the set of all C^∞-functions $M \to \mathbb{R}$. Recall that for any vector field $X \in \mathcal{C}(TM)$ one defines a map $i_X : \Lambda^p T^* M \to \Lambda^{p-1} T^* M$ as follows: Let ω be a p-form on M regarded as p-linear mapping

$$\mathcal{C}(TM) \times \overset{(p)}{\ldots} \times \mathcal{C}(TM) \to C^\infty(M).$$

Then one defines a $(p-1)$-form $i_X \omega$ on M by setting

$$i_X \omega(X_1, \ldots, X_{p-1}) := \omega(X, X_1, \ldots, X_{p-1})$$

for $X_1, \ldots, X_{p-1} \in \mathcal{C}(TM)$.

Proof of Lemma 2.4.1. The above statement is a straightforward generalization of the Homotopy Lemma in [13].

Since $d\omega^N = 0$, on M we have $d(\Phi_t^*\omega^N) = 0$ for all t. Consider the smooth map $\Psi : \mathbb{R} \times M \to N$ defined by $\Psi(t,x) = \Phi_t(x)$. Denoting by $\widetilde{d}$ the exterior differential on $\mathbb{R} \times M$ we obtain $\widetilde{d}(\Psi^*\omega^N) = 0$.

Now we claim that $\Psi^*\omega^N = \Phi_t^*\omega^N + \widetilde{d}t \wedge \Phi_t^*[i_{\partial\Phi_t/\partial t}\omega^N]$. Indeed, for $X, Y, W, Z \in \mathcal{C}(TM)$ we get

$$\begin{aligned}\Psi^*\omega^N(X,Y,W,Z) &= \omega^N(d\Psi \cdot X, d\Psi \cdot Y, d\Psi \cdot W, d\Psi \cdot Z)\\ &= \omega^N(d\Phi_t \cdot X, d\Phi_t \cdot Y, d\Phi_t \cdot W, d\Phi_t \cdot Z),\end{aligned}$$

and $\widetilde{d}t(X) = 0$, whereas for $X, Y, W \in \mathcal{C}(TM)$ we have

$$\begin{aligned}\Psi^*\omega^N(\partial/\partial t, X, Y, W) &= \omega^N(\partial\Psi/\partial t, d\Psi \cdot X, d\Psi \cdot Y, d\Psi \cdot W)\\ &= [i_{\partial\Phi_t/\partial t}\omega^N](d\Phi_t \cdot X, d\Phi_t \cdot Y, d\Phi_t \cdot W)\\ &= \Phi_t^*[i_{\partial\Phi_t/\partial t}\omega^N](X,Y,W)\\ &= [\widetilde{d}t \wedge \Phi_t^*(i_{\partial\Phi_t/\partial t}\omega^N)](\partial/\partial t, X, Y, W)\end{aligned}$$

and $\Phi_t^*\omega^N(\partial/\partial t, X) = 0$.

Finally, we have

$$\begin{aligned}0 = \widetilde{d}(\Psi^*\omega^N) &= \widetilde{d}(\Phi_t^*\omega^N) + \widetilde{d}(\widetilde{d}t \wedge \Phi_t^*[i_{\partial\Phi_t/\partial t}\omega^N])\\ &= \widetilde{d}t \wedge \frac{\partial}{\partial t}(\Phi_t^*\omega^N) + d(\Phi_t^*\omega^N) - \widetilde{d}t \wedge \widetilde{d}(\Phi_t^*[i_{\partial\Phi_t/\partial t}\omega^N])\\ &= \widetilde{d}t \wedge \left\{\frac{\partial}{\partial t}(\Phi_t^*\omega^N) - d(\Phi_t^*[i_{\partial\Phi_t/\partial t}\omega^N])\right\}. \ \blacksquare\end{aligned}$$

LEMMA 2.4.2. *Let Ω be the fundamental 4-form on an almost-quaternionic-Kähler manifold of real dimension $4m$. Then*

$$\star\Omega = \frac{12m}{(2m)!}\Omega^{m-1},$$

where "$\star$" denotes the Hodge operator on forms.

P r o o f o f T h e o r e m 2.4.1. By the definition of the Hodge operator we have

$$K(\Phi) := \int_M \langle \Omega^M, \Phi^*\Omega^N \rangle \, dV_g = \int_M \Omega^M \wedge \star \Phi^*\Omega^N = \int_M \Phi^*\Omega^N \wedge \star\, \Omega^M.$$

Let Φ_0 and Φ_1 be two maps from M to N homotopic through a family Φ_t, $t \in [0,1]$. Since Ω^N is closed, Lemma 2.4.1 yields

$$\begin{aligned}\Phi_1^*\Omega^N - \Phi_0^*\Omega^N &= \int_0^1 \frac{\partial}{\partial t}(\Phi_t^*\Omega^N)\,dt\\ &= \int_0^1 d[\Phi_t(i_{\partial\Phi_t/\partial t}\Omega^N)]\,dt = d\int_0^1 \Phi_t^*(i_{\partial\Phi_t/\partial t}\Omega^N)\,dt \equiv d\alpha,\end{aligned}$$

where $\alpha := \int_0^1 \Phi_t^*(i_{\partial\Phi_t/\partial t}\Omega^N)dt$ is a 3-form on M. Therefore

$$\begin{aligned} K(\Phi_1) - K(\Phi_0) &= \int_M \langle \Omega^M, \Phi_1^*\Omega^N - \Phi_0^*\Omega^N \rangle \, dV_g \\ &= \int_M \Omega^M \wedge \star (\Phi_1^*\Omega^N - \Phi_0^*\Omega^N) \\ &= \int_M (\Phi_1^*\Omega^N - \Phi_0^*\Omega^N) \wedge \star\, \Omega^M = \int_M d\alpha \wedge \star\, \Omega^M. \end{aligned}$$

By Lemma 2.4.2 we have $d(\star\, \Omega^M) = 0$, and so $K(\Phi_1) - K(\Phi_0) = 0$. ∎

Definition 2.4.1. Let (M^{4m}, g) and (N^{4n}, h) be almost-quaternionic-Hermitian manifolds and $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ a smooth map. Φ is called *Q-holomorphic* if for every point $p \in M^{4m}$ and each hypercomplex base $(I_1', I_2', I_3') \in Q_p^M$ there exists a hypercomplex base $(I_1, I_2, I_3) \in Q_{\Phi(p)}^N$, such that

$$I_a(\Phi_*)_p = (\Phi_*)_p I_a', \quad a = 1, 2, 3. \tag{2.4.1}$$

Example 2.4.1. Any 4-dimensional, oriented, Riemannian manifold (M^4, g) can be considered as an almost-quaternionic-Kähler manifold (see Sect. 2). A diffeomorphism $\Phi : (M^4, g) \to (M^4, g)$ is Q-holomorphic iff it preserves the fixed orientation.

Remark 2.4.1. Assume that (N^{4n}, h) is an almost-quaternionic-Hermitian (resp. Kähler) manifold and M^{4m} any smooth, orientable $4m(\leq 4n)$-dimensional manifold. Suppose that $\Phi : M^{4m} \to N^{4n}$ is a smooth immersion and the following condition is satisfied: for every point $p \in M^{4m}$ the space $\Phi_*(T_pM^{4m})$ is a quaternionic subspace of $T_{\Phi(p)}N^{4n}$.

Consider on M^{4m} the Riemannian metric $g := \Phi^*h$. Then there is a unique (natural) quaternionic structure Q^M on M^{4m} such that (M^{4m}, g) is an almost-quaternionic-Hermitian (resp. Kähler) manifold endowed with the fundamental 4-form $\Omega^M := \Phi^*\Omega^N$ and $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a Q-holomorphic map. The manifold (M^{4m}, Φ) is called an *immersed almost quaternionic submanifold* of (N^{4n}, h).

Proposition 2.4.1. *Let (M^{4m}, g) and (N^{4n}, h) be almost-quaternionic-Kähler manifolds. Suppose that $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a Q-holomorphic isometric mapping. Then*

$$K(\Phi) = 12m(2m+1)\,\mathrm{Vol}(M).$$

In particular, Φ *cannot be homotopic to a constant map.*

Proof. Let $p \in M^{4m}$. Choose orthonormal bases of the form

$$(e_1, I_1'e_1, I_2'e_1, I_3'e_1, \ldots, e_m, I_1'e_m, I_2'e_m, I_3'e_m)$$

and

$$(f_1, I_1f_1, I_2f_1, I_3f_1, \ldots, f_n, I_1f_n, I_2f_n, I_3f_n)$$

in T_pM^{4m} and $T_{\Phi(p)}N^{4n}$, respectively, where (I_1', I_2', I_3') is a hypercomplex base of Q_p^M, (I_1, I_2, I_3) is a hypercomplex base of $Q_{\Phi(p)}^N$ and the suitable condition (2.4.1) holds.

Let Φ be Q-holomorphic. It is clear that $\Omega^m = \Phi^*\Omega^N$ and $\langle \Omega^M, \Phi^*\Omega^N \rangle_{|p} = \|\Omega^M\|_p^2$. Notice that the only components of Ω_p^M which are different from zero are those that correspond (up to permutation) to the 4-ples of vectors

$$(e_t, I_a e_t, e_s, I_a e_s), \quad (I_b e_t, I_c e_t, I_b e_s, I_c e_s) \quad \text{for } t, s = 1, \ldots, m,\ t \neq s \tag{2.4.2}$$

and

$$(e_t, I_a e_t, I_b e_s, I_c e_s) \quad \text{for } t, s = 1, \ldots, m \tag{2.4.3}$$

for any cyclic permutation (a, b, c) of $(1, 2, 3)$. It is easy to see that, up to permutation, there are $3m(m-1)$ different components of the type (2.4.2), $3m(m-1)$ different components of the type (2.4.3) with $t \neq s$ and m different components of the type (2.4.3) with $t = s$. By simple calculations we get

$$\Omega(e_t, I_a e_t, e_s, I_a e_s) = \Omega(I_b e_t, I_c e_t, I_b e_s, I_c e_s) = \Omega(e_t, I_a e_t, I_b e_s, I_c e_s) = 2 \quad \text{for } t \neq s$$

and

$$\Omega(e_t, I_a e_t, I_b e_t, I_c e_t) = 6.$$

Since $\|\Omega^M\|_p^2 = 3m(m-1)2^2 + 3m(m-1)2^2 + m6^2 = 12m(2m+1)$, by integrating we get the required formula. ∎

R e m a r k 2.4.2. Proposition 2.4.1, with a slight modification, holds when Φ is a conformal immersion.

Let us recall that 4-dimensional immersed quaternionic submanifolds M^4 of a quaternionic-Kähler manifold (N^{4n}, h), $n > 1$, are totally geodesic and semi-conformally flat. In the case when the scalar curvature of (N^{4n}, h) is positive the only possible types for M^4 are $\mathbb{HP}^1 \equiv S^4$ and $\mathbb{CP}^2$ ([46]).

Corollary 2.4.1. *Any immersed quaternionic submanifold of a quaternionic-Kähler manifold* (N^{4n}, h), $n > 1$, *which is isometric to* $\mathbb{HP}^1$ *defines a non-trivial element in the homotopy group* $\pi_4(N^{4n})$.

R e m a r k 2.4.3. The above fact was very well known in the case when (N^{4n}, h) is the quaternionic projective space $\mathbb{HP}^m$ and $\Phi : \mathbb{HP}^1 \to \mathbb{HP}^m$ is a canonical immersion of the quaternionic projective line (see e.g. [62]).

The following three propositions show the cases when we can compute $K(\Phi)$ more precisely.

Proposition 2.4.2. *Let* (M^{4m}, g) *and* (N^{4n}, h) *be almost-quaternionic-Kähler manifolds* (M^{4m} *being compact*). *Suppose that* $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ *is a smooth map such that the cohomology class* $[\Phi^* \Omega^N]$ *equals* $c[\Omega^M]$ *for some* $c \in \mathbb{R}$. *Then*

$$K(\Phi) = 12m(2m-1)c \operatorname{Vol}(M).$$

P r o o f. By hypothesis $\Phi^* \Omega^N = c\Omega^M + d\beta$, so we have

$$\begin{aligned} K(\Phi) &= \int_M \langle \Omega^M, \Phi^* \Omega^N \rangle dV_g = \int_M \Omega^M \wedge \star \Phi^* \Omega^N = \int_M \Omega^M \wedge \star (c\Omega^M + d\beta) \\ &= c \int_M \Omega^M \wedge \star \Omega^M + \int_M \Omega^M \wedge \star d\beta. \end{aligned}$$

Now, note that

$$\int_M \Omega^M \wedge \star d\beta = \int_M d\beta \wedge \star \Omega^M = \int_M d(\beta \wedge \star \Omega^M) = 0.$$

By Lemma 2.4.2 we have

$$K(\Phi) = c \int_M \Omega^M \wedge \star \Omega^M = c\frac{12m}{(2m)!} \int_M (\Omega^M)^m.$$

By Kraines [25], $(\Omega)^m = (2m+1)!\,\mathrm{vol}(M)$, so we get

$$K(\Phi) = c\frac{12m(2m+1)!}{(2m)!} \int_M \mathrm{vol}(M) = 12m(2m+1)c\,\mathrm{Vol}(M),$$

as required. ■

PROPOSITION 2.4.3. *Let (M^4, g) and (N^4, h) be two connected, compact almost-quaternionic-Kähler manifolds of the same dimension 4 and $\Phi : (M^4, g) \to (N^4, h)$ be a smooth map. Then*

$$K(\Phi) = 36 \deg(\Phi)\,\mathrm{Vol}(N),$$

where $\deg(\Phi)$ denotes the degree of Φ (see e.g. [13, 48]).

Proof. By the definition of $\deg(\Phi)$ we have

$$\int_M \Phi^*(dV_N) = \deg(\Phi)\,\mathrm{Vol}(N).$$

Then, again by the result of Kraines [25]: $\Omega^M = 3!\,\mathrm{vol}(M)$, we get

$$\begin{aligned}\int_M \Phi^*(dV_N) &= \frac{1}{3!} \int_M \Phi^*(\Omega^N) \wedge 1 = \frac{1}{3!} \int_M \Phi^*(\Omega^N) \wedge \left(\frac{1}{6} \star \Omega^M\right) \\ &= \frac{1}{36} \int_M \Phi^*(\Omega^N) \wedge \star \Omega^M = \frac{1}{36} K(\Phi)\end{aligned}$$

as required, because by Lemma 2.4.2 we have $\star\,\Omega^M = 6$. ■

PROPOSITION 2.4.4. *Let (M^{4m}, g) and (N^{4n}, h) be almost-quaternionic-Kähler manifolds (M^{4m} compact). Suppose that $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a smooth map such that $[\Phi^*\Omega^N] = c[\Omega^M]$ for some $c \in \mathbb{R}$ and $\mathrm{rank}(d\Phi) < 4m$. Then $K(\Phi) = 0$.*

Proof. Since $\mathrm{rank}(d\Phi) < 4m$, we have $(\Phi^*\Omega^N) = 0$, so that

$$c^m \int_M (\Omega^M)^m = \int_M (\Phi^*\Omega^N)^m = 0,$$

therefore $c = 0$ and $[\Phi^*\Omega^N] = 0$. The assertion now follows from Proposition 2.4.2. ■

COROLLARY 2.4.2. *Let (M^{4m}, g) and (N^{4n}, h) be almost-quaternionic-Kähler manifolds. Suppose that $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ ($m > n$) is a Q-holomorphic submersion (not necessarily Riemannian). If $\dim H^4(M^{4m}, \mathbb{R}) = 1$ (e.g. $M^{4m} = \mathbb{HP}^m$) then $m = n$ and Φ is a Q-holomorphic diffeomorphism.*

To prove Corollary 2.4.2 we need the following:

Remark 2.4.4. Let $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ be a Q-holomorphic submersion. Then on M^{4m} there exists a metric g' such that (M^{4m}, g') is an almost-quaternionic-Kähler manifold and $\Phi' := \Phi : (M^{4m}, g') \to (N^{4n}, h)$ is a Q-holomorphic Riemannian submersion. The construction of g' is as follows:

Let V and H be the vertical and horizontal distributions defined by Φ on M^{4m} of dimensions $4(m-n)$ and $4n$, respectively: $V := \mathrm{Ker}\,\Phi_*$, $H := V^{\perp}$. Note that for each point $p \in M^{4m}$ the spaces V_p and H_p are quaternionic subspaces of T_pM^{4m}. We define the new metric g' by requiring that the restrictions of g' and g to V_p coincide and that the restriction of g' to H_p is an isometry.

Proof of Corollary 2.4.2. By the above Remark 2.4.4 we can suppose that Φ is a Q-holomorphic Riemannian submersion. Then, analogously to the proof of Proposition 2.4.1, we get $K(\Phi) = 12m(2m+1)\,\mathrm{Vol}(M)$. If $m > n$, Proposition 2.4.4 would give $K(\Phi) = 0$, which is a contradiction. ■

Definition 2.4.2 [19]. Let (N^{4n}, h) be an almost-quaternionic-Hermitian manifold and M^{4m} any smooth $4m$-dimensional manifold. Suppose that $\Phi : M^{4m} \to N^{4n}$ is a smooth immersion. We will say that (M^{4m}, Φ) is an *immersed Lagrangian submanifold* of (N^{4n}, h) if at every point $p \in M^{4m}$ and for any hypercomplex base $(I_1, I_2, I_3) \in Q^N_{\Phi(p)}$ the four subspaces $\Phi_*(T_pM^{4m})$, $I_a\Phi_*(T_pM^{4m})$, $a = 1, 2, 3$, are totally orthogonal in $T_{\Phi(p)}N^{4n}$.

Proposition 2.4.5. *Let (N^{4n}, h) be an almost-quaternionic-Kähler manifold. Suppose that M^{4m} is a compact, oriented, $4m$-dimensional manifold $(m < n)$. Let $\Phi_1 : M^{4m} \to N^{4n}$ and $\Phi_2 : M^{4m} \to N^{4n}$ be two immersions such that (M^{4m}, Φ_1) and (M^{4m}, Φ_2) are almost quaternionic and Lagrangian, respectively. Then Φ_1 and Φ_2 cannot be homotopic.*

Proof. By Remark 2.4.1 we can consider M^{4m} as an almost-quaternionic-Kähler manifold with the Riemannian metric $g = \Phi^*h$ and the almost-quaternionic structure Q_M naturally induced by Φ_1^*. By Definition 4.2 we get $\Phi_2^*(\Omega^N) = 0$ (see [19]). Hence $K(\Phi_2) = 0$. On the other hand, by Remark 2.4.1 and Proposition 2.4.1, $K(\Phi_1) \neq 0$. Thus the statement follows by Theorem 2.4.1. ■

II.5. Lichnerowicz-type homotopy invariant for G-structures

a) *General situation.* We were very much surprised to observe that the idea of the construction of the Lichnerowicz smooth homotopy invariant $K(\Phi)$ can be applied to many different contexts. In the previous section we considered the quaternionic case, but under suitable general hypotheses a homotopy invariant $K_{\xi,\eta}(\Phi)$ can be considered for smooth maps $\Phi : (M, g) \to (N, h)$ between Riemannian manifolds which admit "canonically" defined p-forms $\xi \in \Lambda^pM$ and $\eta \in \Lambda^pN$ playing the role of the Kähler 2-form in the complex case (see Sect. 4).

Indeed, we noticed that in the case of Riemannian manifolds with holonomy group appearing in the well known Berger list [4] such forms always exist and can be used in the definition of the homotopy invariant without any additional hypothesis (Theorem 2.5.2).

Let $G \subseteq \mathrm{SO}(m)$ and $G' \subseteq \mathrm{SO}(n)$ be two Lie groups. Suppose that $\eta_0 \in \Lambda^p\mathbb{R}^m$ and $\xi_0 \in \Lambda^p\mathbb{R}^n$ are given forms of degree p which are invariant by G and G', respectively.

Suppose that (M^m, g) and (N^n, h) are smooth, oriented, Riemannian manifolds with given G-structure and G'-structure, subordinate to their respective $\mathrm{SO}(m)$ and $\mathrm{SO}(n)$-structures.

In the following the manifold M^m will always be assumed to be compact.

Notice that on M^m and N^n there are canonically defined p-forms η and ξ corresponding to η_0 and ξ_0, respectively.

Definition 2.5.1. Let $\Phi : (M^m, g) \to (N^n, h)$ be a smooth map. Define

$$K_{\eta,\xi}(\Phi) := \int_M \langle \eta, \Phi^*\xi\rangle \, dV_M.$$

Theorem 2.5.1. *If ξ is closed and η co-closed then $K_{\eta,\xi}(\Phi)$ is a smooth homotopy invariant, i.e. it is constant on the connected components of $C(M, N)$ (space of smooth maps from M^m to N^n endowed with the usual topology of uniform convergence).*

A key point in proving the result of Lichnerowicz and its present generalization is the following fundamental Homotopy Lemma [13, 39]:

Homotopy Lemma. *Let $\Phi_t : M \to N$ $(t \in \mathbb{R})$ be a smooth family of maps between the manifolds M and N. If ω is a closed p-form on N then*

$$\frac{\partial}{\partial t}(\Phi_t^*\omega) = d[\Phi_t^*(i_{\Psi(t)}\omega)],$$

where $i_{\Psi(t)}\omega$ denotes the interior product of the vector $\Psi(t) := \frac{\partial}{\partial t}\Phi_t$ with the form ω.

Proof of Theorem 2.5.1. If "$\star$" denotes the Hodge operator on forms then we have (see e.g. [40])

$$K_{\eta,\xi}(\Phi) = \int_M \Phi^*\xi \wedge \star\,\eta.$$

Let Φ_0 and Φ_1 be two maps from M to N, homotopic through a smooth family Φ_t, $t \in [0, 1]$. Since, by the assumption, $d(\star\,\eta) = 0$, the Homotopy Lemma yields

$$\Phi_1^*(\xi) - \Phi_0^*(\xi) = \int_{[0,1]} \frac{\partial}{\partial t}(\Phi_t^*\xi)\, dt = \int_{[0,1]} d[\Phi_t^*(i_{\Psi(t)}\xi)]\, dt = d\varrho,$$

where $\varrho := \int_{[0,1]} \Phi_t^*(i_{\Psi(t)}\xi)\, dt$. Therefore

$$\begin{aligned} K_{\eta,\xi}(\Phi_1) - K_{\eta,\xi}(\Phi_0) &= \int_M \langle \eta, \Phi_1^*\xi - \Phi_0^*\xi\rangle\, dV_M = \int_M (\Phi_1^*\xi - \Phi_0^*\xi) \wedge \star\,\eta \\ &= \int_M d\varrho \wedge \star\,\eta = \int_M d(\varrho \wedge \star\,\eta) = 0. \end{aligned}$$

Thus we obtain $K_{\eta,\xi}(\Phi_1) = K_{\eta,\xi}(\Phi_0)$, as required. ■

Notice that in the case when $G \subseteq U(m')$, $G' \subseteq U(n')$, $m = 2m'$, $n = 2n'$ and $\eta = \omega^M$, $\xi = \omega^N$, where ω^M and ω^N denote the Kähler forms on M and N, respectively, Theorem 2.5.1 is nothing but the result of Lichnerowicz [38] (see also [13], p. 48).

Let us return to the general case and assume that G, G, η_0, ξ_0 are as above. The following statement is a special case of Theorem 2.5.1:

THEOREM 2.5.2. *Let* (M^m, g) *and* (N^n, h) *be Riemannian manifolds with holonomy groups* G *and* G', *respectively. Suppose that* η *and* ξ *are p-forms corresponding to* η_0 *and* ξ_0 [40] *which are invariant by the parallel transport. Then* $K_{\eta,\xi}(\Phi)$ *is a smooth homotopy invariant of the smooth maps* $\Phi : (M^m, g) \to (N^n, h)$.

Proof. It is sufficient to remark that since ξ and η are invariant by the parallel transport, they are closed and co-closed, respectively (cf. [40]). ■

b) *Special cases*: *holonomy groups* G_2 *and* Spin(7). Consider G_2- and Spin(7)-structures. We must point out that the existence of *compact* manifolds with holonomy group G_2 or Spin(7) is still an open problem. Nevertheless, if the holonomy group is contained in G_2 or Spin(7) one can easily construct some examples.

Let $(\mathbb{R}^7, g_0)$ be the standard Euclidean 7-dimensional space with the standard orientation and metric g_0. It is well known that by an appropriate identification of $\mathbb{R}^7$ with the imaginary octonions (endowed with octonion multiplication) one can define an alternating vector-cross-product $P : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ with the following properties:

1) $P(X, Y)$ is orthogonal to both vectors X and Y,
2) $\|P(X,Y)\|^2 = \|X\|^2\|Y\|^2 \cdot [g_0(X,Y)]^2$.

(Such a vectorial product is unique up to isometry.)

The Lie group G_2 is characterized as the subgroup of SO(7) which preserves P (see e.g. [18, 19]).

The 3-form $\alpha_0 \in \Lambda^3\mathbb{R}^7$ defined by

$$\alpha_0(X, Y, Z) := g_0(X, P(Y, Z)), \quad X, Y, Z, \in \mathbb{R}^7,$$

is G_2-invariant.

Every 7-dimensional, oriented, Riemannian manifold (M^7, g) with a given G_2-structure becomes naturally equipped with a fundamental 3-form α, corresponding to α_0 (we can also consider the fundamental 4-form $\beta := \star\,\alpha$).

The following proposition is an immediate application of Theorem 2.5.2.

PROPOSITION 2.5.1. *Let* (M^7, g) *and* (N^7, h) *be Riemannian manifolds with holonomy groups contained in* G_2. *Suppose that* M^7 *is compact and* $\Phi : M^7 \to N^7$ *is a smooth map. Then*

$$K_\alpha(\Phi) := \int \langle \alpha^M, \Phi^*\alpha^N \rangle \, dV_M \quad \textit{and} \quad K_\beta := \int_M \langle \beta^M, \Phi^*\beta^N \rangle \, dV_M$$

are smooth homotopy invariants.

Let us recall that a 3-dimensional vector subspace of $\mathbb{R}^7$ is called *special* if it is closed under P. Every special subspace admits an orthonormal base of the form $(X, Y, P(X, Y))$ (see e.g. [18]). Similarly, we say that a 4-dimensional vector subspace V^4 of $\mathbb{R}^7$ is *special* if it is orthogonal to a 3-dimensional special subspace. Correspondingly, if (M^7, g) is a Riemannian manifold with a given G_2-structure one can introduce a notion of *special submanifolds* of dimensions 3 or 4.

PROPOSITION 2.5.2. *Let* (N^7, h) *be a Riemannian manifold with holonomy group contained in* G_2. *Then an immersion* $\Phi : M^3 \to N^7$ (*resp.* $\Phi : M^4 \to N^7$) *of a compact*, *oriented*, *special* 3-*dimensional* (*resp.* 4-*dimensional*) *submanifold of* (N^7, h) *cannot be* *homotopic to a constant map. In particular, if M^3 (resp. M^4) is homeomorphic to the sphere S^3 (resp. S^4) then the map Φ defines a non-trivial element in the homotopy group $\pi_3(N^7)$ (resp. $\pi_4(N^7)$).*

Proof. For simplicity consider only the 3-dimensional case. Observe that for any 3-dimensional, oriented, Riemannian manifold (M^3, g) and for any smooth map $\Phi : M^3 \to N^7$ we can consider the quantity

$$K_{dV_M,\alpha}(\Phi) := \int\limits_M \langle dV_M, \Phi^*\alpha^N \rangle \, dV_M,$$

which is a homotopy invariant by Theorem 2.5.1.

Now, let M^3 be a special submanifold immersed by Φ. Consider on M^3 the Riemannian metric $g := \Phi^*h$. Then $dV_M = \Phi^*\alpha^N$ (possibly after changing the orientation of M^3) and we have $\langle dV_M, \Phi^*\alpha^N \rangle_p = 1$. Hence, $K_{dV_M,\alpha}(\Phi)$ is different from zero, and thus Φ cannot be homotopic to a constant map. ∎

We can obtain similar results for the group Spin(7). Let us recall its definition.

By the appropriate identification of $\mathbb{R}^8$ with the Cayley algebra of octonions one can define an alternating three-linear product

$$P^* : \mathbb{R}^8 \times \mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}^8$$

with the following properties:

1) $P^*(X_1, X_2, X_3)$ is orthogonal to each vector X_1, X_2 and X_3,
2) $\|P^*(X_1, X_2, X_3)\|^2 = \det[g_0(X_i, X_j)]$, $i, j = 1, 2, 3$.

(There are two such products up to isometries of $\mathbb{R}^8$.)

The Lie group Spin(7) is the automorphism group of $\mathbb{R}^8$ which preserves such a fixed product P^*.

The 4-form $\gamma_0 \in \Lambda^4\mathbb{R}^8$ defined by

$$\gamma_0(X, Y, Z, T) = g_0(X, P^*(Y, Z, T))$$

is obviously invariant by the group Spin(7) as is the form $\widetilde{\gamma}_0 := \star\gamma_0$.

Thus every 8-dimensional, oriented, Riemannian manifold (M^8, g) with a given Spin(7)-structure is naturally equipped with a fundamental 4-form γ. In analogy to the case of G_2 we have the following:

PROPOSITION 2.5.3. *Let (M^8, g) and (N^8, h) be Riemannian manifolds with holonomy group* Spin(7). *Suppose that M^8 is compact and $\Phi : M^8 \to N^8$ is a smooth map. Then*

$$K_\gamma(\Phi) := \int\limits_M \langle \gamma^M, \Phi^*\gamma^N \rangle \, dV_M \quad \textit{and} \quad K_{\tilde{\gamma}}(\Phi) := \int\limits_M \langle \widetilde{\gamma}^M, \Phi^*\widetilde{\gamma}^N \rangle \, dV_M$$

are smooth homotopy invariants.

Remark 2.5.1. One can also introduce a notion of *special subspace* of $\mathbb{R}^8$ with respect to P^*, i.e. a 4-dimensional subspace closed under P^*, to find a counterpart of Proposition 2.5.2.

R e m a r k 2.5.2. The hypothesis on (M^7, g) (resp. (M^8, g)) in Proposition 2.5.1 (resp. Proposition 2.5.3) can be weakened by supposing that a G_2-structure (resp. Spin(7)-structure) is given and $d(\star\alpha^M) = 0$ (resp. $d(\star\gamma^M) = 0$).

c) *Generalization of the Lichnerowicz invariant in the complex case.* Assume that $G = U(m)$ and $G' = U(n)$. Recall that the standard Kähler 2-form of $\mathbb{R}^{2m} \cong \mathbb{C}^m$ ([24])

$$\omega_0 = \frac{i}{2}(dz^1 \wedge d\overline{z}^1 + \ldots + dz^m \wedge d\overline{z}^m)$$

is $U(m)$-invariant. Also, the rth exterior powers of ω_0, $(\omega_0)^r := \omega_0 \wedge \overset{(r)}{\ldots} \wedge \omega_0$, are $U(m)$-invariant for $r = 1, \ldots, m$.

Let (M^{2m}, g) and (N^{2n}, h) be two almost Hermitian manifolds equipped with the Kähler forms ω^M and ω^N, respectively. For a smooth map $\Phi : (M^{2m}, g) \to (N^{2n}, h)$ we define

$$K_r(\Phi) := \int_M \langle (\omega^M)^r, (\Phi^*\omega^N)^r \rangle \, dV_M, \qquad r = 1, \ldots, m.$$

THEOREM 2.5.3. *Let (M^{2m}, g) and (N^{2n}, h) be almost Kähler manifolds. Then $K_r(\Phi)$ is a smooth homotopy invariant for $r = 1, \ldots, m$.*

P r o o f. By the hypothesis $d(\omega^M)^r = d(\omega^N)^r = 0$ for any positive integer r. By Lemma 2.5.1 below we also have $d[\star(\omega^M)^r] = 0$; then we apply Theorem 2.5.1. ∎

R e m a r k 2.5.3. If we weaken the hypothesis and assume that (M^{2m}, g) is a special almost Hermitian manifold (i.e. $d(\star\omega^M) = 0$) then for $r = 1$ we obtain nothing but the Lichnerowicz invariant ([13], [38]).

LEMMA 2.5.1. *Let ω be the fundamental 2-form on an almost Kähler manifold of real dimension $2m$. Then*

$$\star(\omega)^r = \frac{r!}{(m-r)!}\omega^{m-r} \quad \textit{for } r = 1, \ldots, m.$$

P r o o f. It is analogous to the proof of Lemma 2.4.2 in the quaternionic case. Let us only recall the well known identity (see e.g. [13])

$$\omega^m = m! \operatorname{vol}(\mathbb{R}^{2m}),$$

where $\operatorname{vol}(\mathbb{R}^{2m})$ is the volume form corresponding to the canonical Euclidean metric of $\mathbb{R}^{2m}$ with the canonical orientation. ∎

Now, let us give some applications of the invariants introduced above, with a view to extension to other cases. Some of them are essentially known (see [13]) but this formulation seems to be important because it does not use the notions of the partial energies.

DEFINITION 2.5.2. Let (N^{2n}, h) be an almost Hermitian manifold with an almost complex structure J and M^{2m} any smooth, orientable, $2m$-dimensional manifold ($m \le n$). Suppose that $\Phi : M^{2m} \to N^{2n}$ is a smooth immersion. We say that (M^{2m}, Φ) is an *immersed almost complex submanifold* of (N^{2n}, h) if for every point $p \in M^{2m}$ the vector space $\Phi_*(T_pM^{2m})$ is a J-invariant subspace of $T_{\Phi(p)}N^{2n}$.

Remark 2.5.4. Notice that the manifold M^{2m} (as above) endowed with the Riemannian metric $g := \Phi^* h$ admits a unique almost complex structure J' such that $J \circ \Phi_* = \Phi_* \circ J'$. With respect to J', (M^{2m}, g) is an almost Hermitian manifold and Φ is holomorphic (cf. [13]).

Proposition 2.5.4. *Let (N^{2n}, h) be an almost Kähler manifold with an almost complex structure J and (M^{2m}, Φ) be a compact immersed almost complex submanifold of (N^{2n}, h). Then the immersion $\Phi : M^{2m} \to N^{2n}$ is not homotopic to a constant map. In particular, if M^{2m} is homeomorphic to the sphere S^2 $(m = 1)$ the map Φ defines a non-trivial element in the homotopy group $\pi_2(N^{2n})$.*

Proof. By Remark 2.5.4 consider M^{2m} as an almost Hermitian manifold. Then $\omega^M = \Phi^* \omega^N$ is the Kähler form on M^{2m}. Hence (M^{2m}, g) is almost Kähler and at every point $p \in M^{2m}$ we get $\langle \omega^M, \Phi^* \omega^N \rangle_p = \|\omega^M\|_p^2 = m$. In particular, $K_1(\Phi) \neq 0$ and the statement follows by Theorem 2.5.3. ■

Definition 2.5.3 [19]. Let (N^{2n}, h) be an almost Hermitian manifold with an almost complex structure J and M^{2m} any smooth $2m$-dimensional manifold. Suppose that $\Phi : M^{2m} \to N^{2n}$ is a smooth immersion. We say that (M^{2m}, Φ) is an *immersed Lagrangian submanifold* of (N^{2n}, h) if at every point $p \in M^{2m}$ the vector subspaces $\Phi_*(T_p M^{2m})$ and $J\Phi_*(T_p M^{2m})$ are totally orthogonal in $T_{\Phi(p)} N^{2n}$.

Proposition 2.5.5. *Let (N^{2n}, h) be an almost Kähler manifold. Suppose that M^{2m} is a compact, oriented, $2m$-dimensional manifold $(m \leq n)$. Let $\Phi_1 : M^{2m} \to N^{2n}$ and $\Phi_2 : M^{2m} \to N^{2n}$ be two immersions such that (M^{2m}, Φ_1) and (M^{2m}, Φ_2) are almost complex and Lagrangian, respectively. Then Φ_1 and Φ_2 cannot be homotopic.*

Proof. By Remark 2.5.4 we can consider M^{2m} as an almost Kähler manifold with respect to the Riemannian metric $g := \Phi^* h$ and the almost complex structure J' naturally induced by Φ_1^*. By Definition 5.3 we get $\Phi_2^*(\omega^N) = 0$ (see [19]). Hence $K_1(\Phi_2) = 0$. On the other hand, by Proposition 2.5.4, we have $K_1(\Phi_1) \neq 0$. Thus the statement follows immediately by Theorem 2.5.3. ■

Proposition 2.5.6. *Let (M^{2m}, g) and (N^{2n}, h) be almost Kähler manifolds. Suppose that $\Phi : M^{2m} \to N^{2n}$ is a smooth map such that the cohomology class $[\Phi^*(\omega^N)^r]$ equals $c_r[(\omega^M)^r]$ for some $r \in \{1, \ldots, \min(m, n)\}$ and some $c_r \in \mathbb{R}$. Then*

$$K_r(\Phi) = c_r \frac{m!r!}{(m-r)!} \operatorname{Vol}(M).$$

In particular, $c_r \neq 0$ if and only if $K_r(\Phi) \neq 0$.

Proof. By hypothesis $\Phi^*(\omega^N)^r = c_r(\omega^M)^r + d\varrho$ for some $\varrho \in \Lambda^{r-1}M$. Then by Lemma 2.5.1 we get

$$\begin{aligned} K_r(\Phi) &= \int_M \Phi^*(\omega^N)^r \wedge \star(\omega^M)^r = c_r \int_M (\omega^M)^r \wedge \star(\omega^M)^r + \int_M d\varrho \wedge \star(\omega^M)^r \\ &= c_r \frac{r!}{(m-r)!} \int_M (\omega^M)^m + \frac{r!}{(m-r)!} \int_M d[\varrho \wedge (\omega^M)^{m-r}] = c_r \frac{m!r!}{(m-r)!} \operatorname{Vol}(M), \end{aligned}$$

as required. ■

PROPOSITION 2.5.7. *Let* (M^{2m}, g) *and* (N^{2m}, h) *be two connected, compact, almost Kähler manifolds of real dimension* $2m$. *Assume that* $\Phi : M^{2m} \to N^{2m}$ *is a smooth map. Then*

$$K_m(\Phi) = (m!)^2 \deg(\Phi) \operatorname{Vol}(M),$$

where $\deg(\Phi)$ *denotes the degree of* Φ (see e.g. [4]).

Proof. By the definition of $\deg(\Phi)$ we have

$$\int_M \Phi^*(dV_N) = \deg(\Phi) \operatorname{Vol}(N).$$

Then by Lemma 2.5.1 we get

$$\int_M \Phi^*(dV_N) = \frac{1}{m!} \int_M \Phi^*(\omega^N)^m = \frac{1}{m!^2} \int_M \Phi^*(\omega^N)^m \wedge \star (\omega^M)^m = \frac{1}{m!^2} K_M(\Phi),$$

as required. ∎

PROPOSITION 2.5.8. *Let* (M^{2m}, g) *and* (N^{2n}, h) *be almost Kähler manifolds. Suppose that* $\Phi : M^{2m} \to N^{2n}$ *is a smooth map such that the cohomology class* $[\Phi^* \omega^N]$ *equals* $c[\omega^M]$ *for some* $c \in \mathbb{R}$. *If* $\operatorname{rank}(d\Phi) < 2m$, *then* $K_r(\Phi) = 0$ *for* $r = 1, \ldots, m$.

II.6. Stress-energy tensor and harmonic maps. Let V be a real vector space of dimension $4n$. A quaternionic structure in V corresponds to the following decomposition (see Sect. II.2):

$$V^{\mathbb{H}} = U^{\mathbb{H}} \oplus \tau_1 U^{\mathbb{H}} \oplus \tau_2 U^{\mathbb{H}} \oplus \tau_3 U^{\mathbb{H}}.$$

Consider the tensor products:

$$V^{\mathbb{H}} \otimes_{\mathbb{H}} (V^{\mathbb{H}})^*, \quad V^{\mathbb{H}} \otimes_{\mathbb{H}} \overline{V^{\mathbb{H}}}, \quad \overline{(V^{\mathbb{H}})^*} \otimes_{\mathbb{H}} (V^{\mathbb{H}})^*, \ldots$$

Each of them can be decomposed into the direct sum of 16 components with respect to the given quaternionic structure in V. In particular, to every real covariant 2-tensor S on V, $S \in V^* \otimes_{\mathbb{R}} V^*$, there corresponds a 2-tensor $S^{\mathbb{H}} \in \overline{(V^{\mathbb{H}})^*} \otimes_{\mathbb{H}} (V^{\mathbb{H}})^*$ which can be decomposed into 4 components, namely real tensors $S_0^{\mathbb{H}}, S_1^{\mathbb{H}}$, $S_2^{\mathbb{H}}$, and $S_3^{\mathbb{H}}$ (of genus $0, 1, 2, 3$, respectively) in the following way:

$$\begin{aligned}
S_0^{\mathbb{H}} &\in \overline{(U^{\mathbb{H}})^*} \otimes_{\mathbb{H}} (U^{\mathbb{H}})^* + (\tau_1 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_1 U^{\mathbb{H}})^* + (\tau_2 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_2 U^{\mathbb{H}})^* + (\tau_3 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_3 U^{\mathbb{H}})^*, \\
S_1^{\mathbb{H}} &\in \overline{(U^{\mathbb{H}})^*} \otimes_{\mathbb{H}} (\tau_1 U^{\mathbb{H}})^* + (\tau_1 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (U^{\mathbb{H}})^* + (\tau_2 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_3 U^{\mathbb{H}})^* + (\tau_3 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_2 U^{\mathbb{H}})^*, \\
S_2^{\mathbb{H}} &\in \overline{(U^{\mathbb{H}})^*} \otimes_{\mathbb{H}} (\tau_2 U^{\mathbb{H}})^* + (\tau_2 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (U^{\mathbb{H}})^* + (\tau_3 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_1 U^{\mathbb{H}})^* + (\tau_1 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_3 U^{\mathbb{H}})^*, \\
S_3^{\mathbb{H}} &\in \overline{(U^{\mathbb{H}})^*} \otimes_{\mathbb{H}} (\tau_3 U^{\mathbb{H}})^* + (\tau_3 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (U^{\mathbb{H}})^* + (\tau_1 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_2 U^{\mathbb{H}})^* + (\tau_2 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_1 U^{\mathbb{H}})^*.
\end{aligned}$$

In each line the tensors are mutally biconjugate.

Assume that

$$\dim_{\mathbb{R}} V = 4.$$

Let $x \in V$ and $(\partial/\partial x^0{}_{|x}, \partial/\partial x^1{}_{|x}, \partial/\partial x^2{}_{|x}, \partial/\partial x^3{}_{|x})$ be a base of the tangent space $T_x V$ (this base is good for every $x \in V$ and because $T_x V \cong V$, we can treat it as a base for V) and let (dx^0, dx^1, dx^2, dx^3) be the dual base. Consider a real symmetric 2-tensor on V:

$$S = dx^i S_{ij} dx^j.$$

The quaternionic decomposition of $S^{\mathbb{H}}$ into pure components looks as follows:

$$\begin{aligned} S^{\mathbb{H}} = &\, d\overline{q} s_{00} dq + d\overline{q^1} s_{11} dq^1 + d\overline{q^2} s_{22} dq^2 + d\overline{q^3} s_{33} dq^3 \\ &+ d\overline{q} s_{01} dq^1 + d\overline{q^1} s_{10} dq + d\overline{q^2} s_{23} dq^3 + d\overline{q^3} s_{32} dq^2 \\ &+ d\overline{q} s_{02} dq^2 + d\overline{q^2} s_{20} dq + d\overline{q^1} s_{13} dq^3 + d\overline{q^3} s_{31} dq^1 \\ &+ d\overline{q} s_{03} dq^3 + d\overline{q^3} s_{30} dq + d\overline{q^1} s_{12} dq^2 + d\overline{q^2} s_{21} dq^1, \end{aligned}$$

where

$$s_{00} = s_{11} = s_{22} = s_{33} \in \mathbb{R},$$

$$\begin{aligned} &s_{10} = (s_{01})^1, \quad s_{23} = \tau_2 s_{01}, \quad s_{32} = \tau_2 \overline{s}_{01}, \quad s_{23} = (s_{32})^1, \\ &s_{20} = (s_{02})^2, \quad s_{13} = \tau_1 s_{02}, \quad s_{31} = \tau_1 \overline{s}_{02}, \quad s_{21} = (s_{12})^3, \\ &s_{30} = (s_{03})^3, \quad s_{12} = \tau_3 s_{03}, \quad s_{21} = \tau_3 \overline{s}_{03}, \quad s_{31} = (s_{13})^2. \end{aligned}$$

Hence

$$\begin{aligned} S^{\mathbb{H}} = &\, s_{00}(d\overline{q} dq + d\overline{q^1} dq^1 + d\overline{q^2} dq^2 + d\overline{q^3} dq^3) \\ &+ d\overline{q} s_{01} dq^1 + d\overline{q^1} \overline{s}_{01} dq + d\overline{q^2} (\tau_2 s_{01}) dq^3 + d\overline{q^3} (\tau_2 \overline{s}_{01}) dq^2 \\ &+ d\overline{q} s_{02} dq^2 + d\overline{q^2} \overline{s}_{02} dq + d\overline{q^1} (\tau_1 s_{02}) dq^3 + d\overline{q^3} (\tau_1 \overline{s}_{02}) dq^1 \\ &+ d\overline{q} s_{03} dq^3 + d\overline{q^3} \overline{s}_{03} dq + d\overline{q^1} (\tau_3 s_{03}) dq^2 + d\overline{q^2} (\tau_3 \overline{s}_{03}) dq^1. \end{aligned}$$

The relationship between the quaternionic components s_{ij} and the real components S_{mn} is the following:

(2.6.1a)
$$\begin{aligned} \tfrac{1}{4} S_{00} &= s_{00} + \mathrm{Re}(s_{01} + s_{02} + s_{03}), \\ \tfrac{1}{4} S_{11} &= s_{00} + \mathrm{Re}(s_{01} - s_{02} - s_{03}), \\ \tfrac{1}{4} S_{22} &= s_{00} + \mathrm{Re}(-s_{01} + s_{02} - s_{03}), \\ \tfrac{1}{4} S_{33} &= s_{00} + \mathrm{Re}(-s_{01} - s_{02} + s_{03}), \end{aligned}$$

(2.6.1b)
$$\begin{aligned} \tfrac{1}{4} S_{01} &= -\mathrm{Re}[(s_{02} + s_{03})i] = -\mathrm{Re}[(\pm s_{01} + s_{02} + s_{03})i], \\ \tfrac{1}{4} S_{02} &= -\mathrm{Re}[(s_{01} + s_{03})j] = -\mathrm{Re}[(s_{01} \pm s_{02} + s_{03})j], \\ \tfrac{1}{4} S_{03} &= -\mathrm{Re}[(s_{01} + s_{02})k] = -\mathrm{Re}[(s_{01} + s_{02} \pm s_{03})k], \end{aligned}$$

(2.6.1c)
$$\begin{aligned} \tfrac{1}{4} S_{12} &= -\mathrm{Re}[(s_{01} - s_{02})k] = -\mathrm{Re}[(s_{01} - s_{02} \pm s_{03})k], \\ \tfrac{1}{4} S_{13} &= -\mathrm{Re}[-(s_{01} - s_{03})j] = -\mathrm{Re}[(-s_{01} \pm s_{02} + s_{03})j], \\ \tfrac{1}{4} S_{23} &= \mathrm{Re}[-(s_{02} - s_{03})i] = \mathrm{Re}[(\pm s_{01} - s_{02} + s_{03})i]. \end{aligned}$$

Writing $\pm s_{0k}$ we indicate that we are free to choose the sign due to the fact that $\mathrm{Re}(s_{01} i) = \mathrm{Re}(s_{02} j) = \mathrm{Re}(s_{03} k) = 0$.

On the other hand, we have

(2.6.1d)
$$\begin{aligned} s_{00} &= \tfrac{1}{16}(S_{00} + S_{11} + S_{22} + S_{33}), \\ s_{01} &= \tfrac{1}{16}(S_{00} + S_{11} - S_{22} - S_{33}) + \tfrac{1}{8}(S_{02} - S_{13})j + \tfrac{1}{8}(S_{03} + S_{12})k, \\ s_{02} &= \tfrac{1}{16}(S_{00} - S_{11} + S_{22} - S_{33}) + \tfrac{1}{8}(S_{01} + S_{23})i + \tfrac{1}{8}(S_{03} - S_{12})k, \\ s_{03} &= \tfrac{1}{16}(S_{00} - S_{11} - S_{22} + S_{33}) + \tfrac{1}{8}(S_{01} - S_{23})i + \tfrac{1}{8}(S_{02} + S_{13})j, \end{aligned}$$

$$\begin{aligned} s_{10} &= (s_{01})^1 = \tfrac{1}{16}(S_{00}+S_{11}-S_{22}-S_{33}) - \tfrac{1}{8}(S_{02}-S_{13})j - \tfrac{1}{8}(S_{03}+S_{12})k,\\ s_{20} &= (s_{02})^2 = \tfrac{1}{16}(S_{00}-S_{11}+S_{22}-S_{33}) - \tfrac{1}{8}(S_{01}+S_{23})i - \tfrac{1}{8}(S_{03}-S_{12})k,\\ s_{30} &= (s_{03})^3 = \tfrac{1}{16}(S_{00}-S_{11}-S_{22}+S_{33}) - \tfrac{1}{8}(S_{01}-S_{23})i - \tfrac{1}{8}(S_{02}+S_{13})j, \end{aligned} \tag{2.6.1e}$$

$$\begin{aligned} s_{21} &= \tfrac{1}{16}(S_{00}-S_{11}-S_{22}+S_{33}) - \tfrac{1}{8}(S_{01}-S_{23})i + \tfrac{1}{8}(S_{02}+S_{13})j,\\ s_{31} &= \tfrac{1}{16}(S_{00}-S_{11}+S_{22}-S_{33}) - \tfrac{1}{8}(S_{01}+S_{23})i + \tfrac{1}{8}(S_{03}-S_{12})k,\\ s_{32} &= \tfrac{1}{16}(S_{00}+S_{11}-S_{22}-S_{33}) - \tfrac{1}{8}(S_{02}-S_{13})j + \tfrac{1}{8}(S_{03}+S_{12})k. \end{aligned} \tag{2.6.1f}$$

Hereafter we will consider a real Riemannian manifold M which is locally conformally flat with $\dim_{\mathbb{R}} M = 4$ (e.g. the sphere $S^4 \approx \mathbb{HP}^1$ or the torus $T^4 \equiv \mathbb{H}/\mathbb{Z}^4$, see e.g. [7]). Then we can assume that in a neighbourhood of every point p of M there exists a system of local coordinates (x^0, x^1, x^2, x^3) such that the metric g is expressed by

$$g = g^0_{\mathbb{R}}[(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2], \tag{2.6.2}$$

where $g^0_{\mathbb{R}}$ is a real positive C^∞-function defined near p. Consider a quaternionic coordinate $q := x^0 + ix^1 + jx^2 + kx^3$ associated with the given system of real coordinates. Then the quaternionic expression of g is

$$g = g^0_{\mathbb{H}}[d\overline{q} \otimes dq + d\overline{q^1} \otimes dq^1 + d\overline{q^2} \otimes dq^2 + d\overline{q^3} \otimes dq^3]. \tag{2.6.3}$$

Comparing the expression for g in real and quaternionic coordinates we get:

Proposition 2.6.1. $g^0_{\mathbb{R}} = 4g^0_{\mathbb{H}}$.

Definition 2.6.1 [13]. Let $\Phi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. The *stress-energy tensor* of Φ is the symmetric 2-tensor on M given by

$$S(\Phi) := e \cdot g - \Phi^* h,$$

where $e = e(\Phi)$ denotes the *energy density* of Φ:

$$e(\Phi) := \tfrac{1}{2}|d\Phi|^2 = \tfrac{1}{2} g^{ij} h_{\alpha\beta} \Phi^\alpha_i \Phi^\beta_j,$$

and $(\Phi^\alpha_i) = (\partial\Phi^\alpha/\partial x^i)$ is a local representation of $d\Phi$.

We will compute the quaternionic components of $S(\Phi)$ in the case when (M, g) is a 4-dimensional Riemannian manifold which is locally conformally flat.

I. Computation of $\Phi^* h$. The real expression of the metric h on N is the following:

$$h = \sum_{\alpha,\beta} h_{\alpha\beta} dy^\alpha \otimes dy^\beta \qquad (h_{\alpha\beta} = h_{\beta\alpha}),$$

where (y^α) is a local system of real coordinates defined in an open neighbourhood of $\Phi(p)$, p a fixed point in M. If (x^j) is a system of local coordinates near p then

$$\Phi^* h = dx^i \Phi^\alpha_i h_{\alpha\beta} \Phi^\beta_j dx^j.$$

In order to pass to the quaternionic coordinates we have to consider the extension of the metric h to the quaternionified tangent bundle of N, $T^{\mathbb{H}}N := TN \otimes_{\mathbb{R}} \mathbb{H}$. If $X^{\mathbb{H}}, Y^{\mathbb{H}} \in T^{\mathbb{H}}_{\Phi(p)}N$ then

$$X^{\mathbb{H}} = (X^1, \ldots, X^n), \qquad Y^{\mathbb{H}} = (Y^1, \ldots, Y^m),$$

where $4n = \dim_{\mathbb{R}} N$ and

$$X^\alpha := X_0^\alpha + iX_1^\alpha + jX_2^\alpha + kX_3^\alpha, \qquad Y^\beta := Y_0^\beta + iY_1^\beta + jY_2^\beta + kY_3^\beta.$$

Then we set

$$\begin{aligned}\langle X^{\mathbb{H}}, Y^{\mathbb{H}}\rangle &= \langle X^{\mathbb{H}}, Y^{\mathbb{H}}\rangle_h \\ &:= \sum_{\alpha,\beta=1}^{n} (X_0^\alpha + iX_1^\alpha + jX_2^\alpha + kX_3^\alpha) h_{\alpha\beta} (Y_0^\beta + iY_1^\beta + jY_2^\beta + kY_3^\beta).\end{aligned}$$

Take into account the following expressions:

$$\begin{aligned}\frac{\partial\Phi^\alpha}{\partial q} &= \frac{1}{4}\Big(\frac{\partial\Phi^\alpha}{\partial x^0} - i\frac{\partial\Phi^\alpha}{\partial x^1} - j\frac{\partial\Phi^\alpha}{\partial x^2} - k\frac{\partial\Phi^\alpha}{\partial x^3}\Big),\\ \frac{\partial\Phi^\alpha}{\partial q^1} &= \frac{1}{4}\Big(\frac{\partial\Phi^\alpha}{\partial x^0} - i\frac{\partial\Phi^\alpha}{\partial x^1} + j\frac{\partial\Phi^\alpha}{\partial x^2} + k\frac{\partial\Phi^\alpha}{\partial x^3}\Big),\\ \frac{\partial\Phi^\alpha}{\partial q^2} &= \frac{1}{4}\Big(\frac{\partial\Phi^\alpha}{\partial x^0} + i\frac{\partial\Phi^\alpha}{\partial x^1} - j\frac{\partial\Phi^\alpha}{\partial x^2} + k\frac{\partial\Phi^\alpha}{\partial x^3}\Big),\\ \frac{\partial\Phi^\alpha}{\partial q^3} &= \frac{1}{4}\Big(\frac{\partial\Phi^\alpha}{\partial x^0} + i\frac{\partial\Phi^\alpha}{\partial x^1} + j\frac{\partial\Phi^\alpha}{\partial x^2} - k\frac{\partial\Phi^\alpha}{\partial x^3}\Big).\end{aligned}$$

Since we can write

$$\begin{aligned}\Phi^* = &\Big(d\overline{q}\frac{\partial\Phi^\alpha}{\partial\overline{q}} + d\overline{q^1}\frac{\partial\Phi^\alpha}{\partial\overline{q^1}} + d\overline{q^2}\frac{\partial\Phi^\alpha}{\partial\overline{q^2}} + d\overline{q^3}\frac{\partial\Phi^\alpha}{\partial\overline{q^3}}\Big) h_{\alpha\beta}\\ &\times \Big(\frac{\partial\Phi^\beta}{\partial q}dq + \frac{\partial\Phi^\beta}{\partial q^1}dq^1 + \frac{\partial\Phi^\beta}{\partial q^2}dq^2 + \frac{\partial\Phi^\beta}{\partial q^3}dq^3\Big),\end{aligned}$$

we get

$$\begin{aligned}\Phi^* h = {}& d\overline{q}\Big\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q}\Big\rangle dq + d\overline{q^1}\Big\langle \frac{\partial\Phi}{\partial\overline{q^1}}, \frac{\partial\Phi}{\partial q^1}\Big\rangle dq^1 + d\overline{q^2}\Big\langle \frac{\partial\Phi}{\partial\overline{q^2}}, \frac{\partial\Phi}{\partial q^2}\Big\rangle dq^2\\ &+ d\overline{q^3}\Big\langle \frac{\partial\Phi}{\partial\overline{q^3}}, \frac{\partial\Phi}{\partial q^3}\Big\rangle dq^3 + d\overline{q}\Big\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q^1}\Big\rangle dq^1 + d\overline{q^1}\Big\langle \frac{\partial\Phi}{\partial\overline{q^1}}, \frac{\partial\Phi}{\partial q}\Big\rangle dq\\ &+ d\overline{q^2}\Big\langle \frac{\partial\Phi}{\partial\overline{q^2}}, \frac{\partial\Phi}{\partial q^3}\Big\rangle dq^3 + d\overline{q^3}\Big\langle \frac{\partial\Phi}{\partial\overline{q^3}}, \frac{\partial\Phi}{\partial q^2}\Big\rangle dq^2 + d\overline{q}\Big\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q^2}\Big\rangle dq^2\\ &+ d\overline{q^2}\Big\langle \frac{\partial\Phi}{\partial\overline{q^2}}, \frac{\partial\Phi}{\partial q}\Big\rangle dq + d\overline{q^1}\Big\langle \frac{\partial\Phi}{\partial\overline{q^1}}, \frac{\partial\Phi}{\partial q^3}\Big\rangle dq^3 + d\overline{q^3}\Big\langle \frac{\partial\Phi}{\partial\overline{q^3}}, \frac{\partial\Phi}{\partial q^1}\Big\rangle dq^1\\ &+ d\overline{q}\Big\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q^3}\Big\rangle dq^3 + d\overline{q^3}\Big\langle \frac{\partial\Phi}{\partial\overline{q^3}}, \frac{\partial\Phi}{\partial q}\Big\rangle dq + d\overline{q^1}\Big\langle \frac{\partial\Phi}{\partial\overline{q^1}}, \frac{\partial\Phi}{\partial q^2}\Big\rangle dq^2\\ &+ d\overline{q^2}\Big\langle \frac{\partial\Phi}{\partial\overline{q^2}}, \frac{\partial\Phi}{\partial q^1}\Big\rangle dq^1.\end{aligned}$$

Note that

$$\begin{aligned}\Big\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q}\Big\rangle &= \Big\langle \frac{\partial\Phi}{\partial\overline{q^1}}, \frac{\partial\Phi}{\partial q^1}\Big\rangle = \Big\langle \frac{\partial\Phi}{\partial\overline{q^2}}, \frac{\partial\Phi}{\partial q^2}\Big\rangle = \Big\langle \frac{\partial\Phi}{\partial\overline{q^3}}, \frac{\partial\Phi}{\partial q^3}\Big\rangle \ (\in \mathbb{R}),\\ \Big\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q^1}\Big\rangle &= \Big\langle \frac{\partial\Phi}{\partial\overline{q^1}}, \frac{\partial\Phi}{\partial q}\Big\rangle^1 = \Big\langle \frac{\partial\Phi}{\partial\overline{q^2}}, \frac{\partial\Phi}{\partial q^3}\Big\rangle^2 = \Big\langle \frac{\partial\Phi}{\partial\overline{q^3}}, \frac{\partial\Phi}{\partial q^2}\Big\rangle^3,\end{aligned}$$

$$\left\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q^1} \right\rangle = \overline{\left\langle \frac{\partial\Phi}{\partial\overline{q^1}}, \frac{\partial\Phi}{\partial q} \right\rangle}, \qquad \left\langle \frac{\partial\Phi}{\partial\overline{q^2}}, \frac{\partial\Phi}{\partial q^3} \right\rangle = \overline{\left\langle \frac{\partial\Phi}{\partial\overline{q^3}}, \frac{\partial\Phi}{\partial q^2} \right\rangle},$$
$$\left\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q^2} \right\rangle = \left\langle \frac{\partial\Phi}{\partial\overline{q^2}}, \frac{\partial\Phi}{\partial q} \right\rangle^2 = \left\langle \frac{\partial\Phi}{\partial\overline{q^1}}, \frac{\partial\Phi}{\partial q^3} \right\rangle = \left\langle \frac{\partial\Phi}{\partial\overline{q^3}}, \frac{\partial\Phi}{\partial q^1} \right\rangle^3,$$
$$\left\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q^2} \right\rangle = \overline{\left\langle \frac{\partial\Phi}{\partial\overline{q^2}}, \frac{\partial\Phi}{\partial q} \right\rangle}, \qquad \left\langle \frac{\partial\Phi}{\partial\overline{q^1}}, \frac{\partial\Phi}{\partial q^3} \right\rangle = \overline{\left\langle \frac{\partial\Phi}{\partial\overline{q^3}}, \frac{\partial\Phi}{\partial q^1} \right\rangle},$$
$$\left\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q^3} \right\rangle = \left\langle \frac{\partial\Phi}{\partial\overline{q^3}}, \frac{\partial\Phi}{\partial q} \right\rangle^3 = \left\langle \frac{\partial\Phi}{\partial\overline{q^1}}, \frac{\partial\Phi}{\partial q^2} \right\rangle^1 = \left\langle \frac{\partial\Phi}{\partial\overline{q^2}}, \frac{\partial\Phi}{\partial q^1} \right\rangle^2,$$
$$\left\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q^3} \right\rangle = \overline{\left\langle \frac{\partial\Phi}{\partial\overline{q^3}}, \frac{\partial\Phi}{\partial q} \right\rangle}, \qquad \left\langle \frac{\partial\Phi}{\partial\overline{q^1}}, \frac{\partial\Phi}{\partial q^2} \right\rangle = \overline{\left\langle \frac{\partial\Phi}{\partial\overline{q^2}}, \frac{\partial\Phi}{\partial q^1} \right\rangle}$$

as in the general case for symmetric 2-tensors. Then the explicit expressions look as follows:

$$\left\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q} \right\rangle = \tfrac{1}{16}(|\Phi_0|^2 + |\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2),$$
$$\left\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q^1} \right\rangle = \tfrac{1}{16}[(\Phi_0^\alpha + i\Phi_1^\alpha + j\Phi_2^\alpha + k\Phi_3^\alpha)h_{\alpha\beta}(\Phi_0^\beta - i\Phi_1^\beta + j\Phi_2^\beta + k\Phi_3^\beta)]$$
$$= \tfrac{1}{16}\{(|\Phi_0|^2 + |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2) + 2[\langle\Phi_0,\Phi_2\rangle - \langle\Phi_1,\Phi_3\rangle]j + 2[\langle\Phi_0,\Phi_3\rangle + \langle\Phi_1,\Phi_2\rangle]k\},$$
$$\left\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q^2} \right\rangle = \tfrac{1}{16}[(\Phi_0^\alpha + i\Phi_1^\alpha + j\Phi_2^\alpha + k\Phi_3^\alpha)h_{\alpha\beta}(\Phi_0^\beta + i\Phi_1^\beta - j\Phi_2^\beta + k\Phi_3^\beta)] \tag{2.6.4}$$
$$= \tfrac{1}{16}\{(|\Phi_0|^2 - |\Phi_1|^2 + |\Phi_2|^2 - |\Phi_3|^2) + 2[\langle\Phi_0,\Phi_1\rangle - \langle\Phi_2,\Phi_3\rangle]i + 2[\langle\Phi_0,\Phi_3\rangle - \langle\Phi_1,\Phi_2\rangle]k\},$$
$$\left\langle \frac{\partial\Phi}{\partial\overline{q}}, \frac{\partial\Phi}{\partial q^3} \right\rangle = \tfrac{1}{16}[(\Phi_0^\alpha + i\Phi_1^\alpha + j\Phi_2^\alpha + k\Phi_3^\alpha)h_{\alpha\beta}(\Phi_0^\beta + i\Phi_1^\beta + j\Phi_2^\beta - k\Phi_3^\beta)]$$
$$= \tfrac{1}{16}\{(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 + |\Phi_3|^2) + 2[\langle\Phi_0,\Phi_1\rangle - \langle\Phi_2,\Phi_3\rangle]i + 2[\langle\Phi_0,\Phi_2\rangle + \langle\Phi_1,\Phi_3\rangle]j\}.$$

Finally, we get

$$\begin{aligned}\Phi^*h = {} & \langle\Phi_{\overline{q}},\Phi_q\rangle(d\overline{q}dq + d\overline{q^1}dq^1 + d\overline{q^2}dq^2 + d\overline{q^3}dq^3) \\ & + [d\overline{q}\langle\Phi_{\overline{q}},\Phi_{q^1}\rangle dq^1 + d\overline{q^1}\langle\Phi_{\overline{q}},\Phi_{q^1}\rangle^1 dq + d\overline{q^2}\langle\Phi_{\overline{q}},\Phi_{q^1}\rangle^2 dq^3 + d\overline{q^3}\langle\Phi_{\overline{q}},\Phi_{q^1}\rangle^3 dq^2] \\ & + [d\overline{q}\langle\Phi_{\overline{q}},\Phi_{q^2}\rangle dq^2 + d\overline{q^2}\langle\Phi_{\overline{q}},\Phi_{q^2}\rangle^2 dq + d\overline{q^1}\langle\Phi_{\overline{q}},\Phi_{q^2}\rangle^1 dq^3 + d\overline{q^3}\langle\Phi_{\overline{q}},\Phi_{q^2}\rangle^3 dq^1] \\ & + [d\overline{q}\langle\Phi_{\overline{q}},\Phi_{q^3}\rangle dq^3 + d\overline{q^3}\langle\Phi_{\overline{q}},\Phi_{q^3}\rangle^3 dq + d\overline{q^1}\langle\Phi_{\overline{q}},\Phi_{q^3}\rangle^1 dq^2 + d\overline{q^2}\langle\Phi_{\overline{q}},\Phi_{q^3}\rangle^2 dq^1].\end{aligned}$$

II. Computation of $e(\Phi)\cdot g$. By the definition we have

$$e(\Phi) = \tfrac{1}{2}|d\Phi|^2 = \tfrac{1}{2}g^{ij}h_{\alpha\beta}\Phi_i^\alpha\Phi_j^\beta = \tfrac{1}{2}g^0_{\mathbb{H}}(|\Phi_0|^2 + |\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2).$$

Then

$$\begin{aligned}e(\Phi)\cdot g &= \tfrac{1}{8}(|\Phi_0|^2 + |\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2)(d\overline{q}dq + d\overline{q^1}dq^1 + d\overline{q^2}dq^2 + d\overline{q^3}dq^3) \\ &= 2\langle\Phi_{\overline{q}},\Phi_q\rangle(d\overline{q}dq + d\overline{q^1}dq^1 + d\overline{q^2}dq^2 + d\overline{q^3}dq^3).\end{aligned}$$

Finally, we can write an explicit expression for the stress-energy tensor of Φ:

$$
\begin{aligned}
S(\Phi) = {} & \langle \Phi_{\overline{q}}, \Phi_q \rangle (d\overline{q}dq + d\overline{q^1}dq^1 + d\overline{q^2}dq^2 + d\overline{q^3}dq^3) \\
& - [d\overline{q}\langle \Phi_{\overline{q}}, \Phi_{q^1} \rangle dq^1 + d\overline{q^1}\langle \Phi_{\overline{q}}, \Phi_{q^1} \rangle^1 dq + d\overline{q^2}\langle \Phi_{\overline{q}}, \Phi_{q^1} \rangle^2 dq^3 \\
& + d\overline{q^3}\langle \Phi_{\overline{q}}, \Phi_{q^1} \rangle^3 dq^2] \\
& - [d\overline{q}\langle \Phi_{\overline{q}}, \Phi_{q^2} \rangle dq^2 + d\overline{q^2}\langle \Phi_{\overline{q}}, \Phi_{q^2} \rangle^2 dq + d\overline{q^1}\langle \Phi_{\overline{q}}, \Phi_{q^2} \rangle^1 dq^3 \\
& + d\overline{q^3}\langle \Phi_{\overline{q}}, \Phi_{q^2} \rangle^3 dq^1] \\
& - [d\overline{q}\langle \Phi_{\overline{q}}, \Phi_{q^3} \rangle dq^3 + d\overline{q^3}\langle \Phi_{\overline{q}}, \Phi_{q^3} \rangle^3 dq + d\overline{q^1}\langle \Phi_{\overline{q}}, \Phi_{q^3} \rangle^1 dq^2 \\
& + d\overline{q^2}\langle \Phi_{\overline{q}}, \Phi_{q^3} \rangle^2 dq^1].
\end{aligned} \tag{2.6.5}
$$

According to the above decomposition we can define four tensors S_0, S_1, S_2, S_3 so that every square bracket [ ] corresponds to one component of the decomposition of $S(\Phi)$ into these tensors: $S(\Phi) = S_0 + S_1 + S_2 + S_3$.

PROPOSITION 2.6.2. (Φ *is conformal*) $\Leftrightarrow$ ($S_1 = S_2 = S_3 = 0$).

Proof. Notice that the equations

$$\langle \Phi_{\overline{q}}, \Phi_{q^1} \rangle = \langle \Phi_{\overline{q}}, \Phi_{q^2} \rangle = \langle \Phi_{\overline{q}}, \Phi_{q^3} \rangle = 0,$$

which express the vanishing of the components in S_1, S_2 and S_3, are equivalent to the conditions:

$$
\begin{gathered}
|\Phi_0|^2 = |\Phi_1|^2 = |\Phi_2|^2 = |\Phi_3|^2, \\
\langle \Phi_0, \Phi_i \rangle = 0, \quad i = 1,2,3, \quad \langle \Phi_i, \Phi_j \rangle = 0, \quad i \neq j, \ i,j \neq 0,
\end{gathered}
$$

which just express the conformality of Φ. ∎

Remark 2.6.1. (Φ is conformal) $\Leftrightarrow$ ($S(\Phi)$ is pure of genus 0).

COROLLARY 2.6.1. *If* Φ *is locally regular, i.e.* $\Phi_{\overline{q}} = 0$, *then* $S(\Phi) = 0$.

COROLLARY 2.6.2. $S(\Phi) = 0$ *if and only if* $\Phi = \text{const}$.

Proof. By (2.6.5), $S(\Phi) = 0$ if and only if the following conditions are satisfied:

$$\langle \Phi_{\overline{q}}, \Phi_q \rangle = \langle \Phi_{\overline{q}}, \Phi_{q^1} \rangle = \langle \Phi_{\overline{q}}, \Phi_{q^2} \rangle = \langle \Phi_{\overline{q}}, \Phi_{q^3} \rangle = 0.$$

By (2.6.4) the above equalities are equivalent to

$$|\Phi_0|^2 = |\Phi_1|^2 = |\Phi_2|^2 = |\Phi_3|^2 = 0 \quad \text{and} \quad \langle \Phi_i, \Phi_k \rangle = 0, \quad i,k = 0,1,2,3.$$

But this is possible if and only if Φ is a constant. ∎

COROLLARY 2.6.3. $\operatorname{Trace} S(\Phi) = 2e(\Phi)$.

Proof. By the definition of S we have

$$
\begin{aligned}
\operatorname{Trace} S(\Phi) &= \operatorname{Trace}[e \cdot g - \Phi^* h] = e \operatorname{Trace} g - \operatorname{Trace} \Phi^* h \\
&= e g^{ij} g_{ij} - g^{ij} (\Phi^* h)_{ij} = 4e - g^{ij} \Phi_i^\alpha \Phi_j^\beta h_{\alpha\beta} = 4e - 2e = 2e. \ ∎
\end{aligned}
$$

Recall that if S is a real 2-tensor on a (real) Riemannian manifold, then one defines the *divergence* of S (see e.g. [13]) in the local coordinates (x^i) by

$$(\operatorname{div}_{\mathbb{R}} S)_i = (\operatorname{div} S)_i := g^{jk}\nabla_{\partial_j} S_{ki}.$$

DEFINITION 2.6.2. A 2-tensor S defined on a 4-dimensional almost quaternionic manifold (M^4, g) with standard enhanced quaternionic structure Q_0 is *Hermitian* if for any $p \in M^4$ we have

$$S(I_\alpha X, I_\alpha Y) = S(X, Y) \quad \text{for } \alpha = 1, 2, 3,$$

where $(I_1, I_2, I_3) \in Q_{0|p}$ and $X, Y \in T_p M^4$.

Remark 2.6.2. ($S(\Phi)$ is pure of genus 0) $\Leftrightarrow$ ($S(\Phi)$ is Hermitian).

Remark 2.6.3. (Φ is conformal) $\Leftrightarrow$ ($S(\Phi)$ is Hermitian).

PROPOSITION 2.6.3. *Let $\Phi : (M^4, g) \to (N^{4n}, h)$ be a smooth mapping between two almost-quaternionic-Hermitian manifolds. Assume that (M^4, g) is locally conformally flat. If Φ is Q-holomorphic then $S(\Phi)$ is Hermitian.*

Proof. By the definition of $S(\Phi)$ it is enough to show that $\Phi^* h$ is Hermitian on (M^4, g). Indeed,

$$\begin{aligned}\Phi^* h(I_\alpha X, I_\alpha Y) &= h(d\Phi(I_\alpha X), d\Phi(I_\alpha Y)) = h(I'_\alpha(d\Phi(X), I'_\alpha(d\Phi(Y)) \\ &= h(d\Phi(X), d\Phi(Y)) = \Phi^* h(X, Y). \ \blacksquare\end{aligned}$$

PROPOSITION 2.6.4. *Let $\Phi : (M^4, g) \to (N^{4n}, h)$ be a smooth mapping between two almost-quaternionic-Hermitian manifolds. Assume that (M^4, g) is locally conformally flat. If Φ is Q-holomorphic then Φ is harmonic if and only if it is homothetic.*

Proof. Note that

$$e(\Phi) = 2g^0_{\mathbb{H}}\langle \Phi_{\overline{q}}, \Phi_q\rangle. \tag{2.6.6}$$

If Φ is Q-holomorphic then Φ is conformal. By Remark 2.6.1 we get

$$S(\Phi) = g^0_{\mathbb{H}}\langle \Phi_{\overline{q}}, \Phi_q\rangle g. \tag{2.6.7}$$

On the other hand, conformality of Φ means that $\Phi^* h = \mu g$ for some continuous and non-negative function μ defined on M. By (2.6.6) and (2.6.7) we obtain $\mu = g^0_{\mathbb{H}}\langle \Phi_{\overline{q}}, \Phi_q\rangle$. Thus

$$\begin{aligned}(\operatorname{div} S(\Phi) = 0) &\Leftrightarrow [\langle d(g^0_{\mathbb{H}}\langle \Phi_{\overline{q}}, \Phi_q\rangle), g\rangle = 0] \Leftrightarrow [d(g^0_{\mathbb{H}}\langle \Phi_{\overline{q}}, \Phi_q\rangle) = 0] \\ &\Leftrightarrow (d\mu = 0) \Leftrightarrow (\mu = \text{const}) \Leftrightarrow (\Phi \text{ is homothetic}).\end{aligned}$$

If Φ is homothetic then $\Phi^* h = \mu_0 g$, where $\mu_0 = \text{const}$. So, we have

$$S(\Phi) = (e - \mu_0) g \quad \text{and} \quad \operatorname{div} S(\Phi) = \langle de, g\rangle.$$

But on the other hand, $e = 0$, and so Φ is harmonic. $\blacksquare$

DEFINITION 2.6.3. We define the *quaternionic divergence* of the quaternionic 2-tensor $s_{\mathbb{H}}$ by

$$(\operatorname{div}_{\mathbb{H}} S^{\mathbb{H}})_\gamma := g^{\alpha\beta}_{\mathbb{H}} \nabla_{\partial_\alpha} s_{\beta\gamma},$$

where α, β, γ stand for q, q^1, q^2, q^3.

LEMMA 2.6.1. *If the metric g is locally conformally flat then*

$$
64\,\mathrm{Re}[(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_q] = (\mathrm{div}_{\mathbb{R}}\, S)_0, \quad 64\,\mathrm{Re}[i(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{1}}] = (\mathrm{div}_{\mathbb{R}}\, S)_1,
$$
$$
64\,\mathrm{Re}[j(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{2}}] = (\mathrm{div}_{\mathbb{R}}\, S)_2, \quad 64\,\mathrm{Re}[k(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{3}}] = (\mathrm{div}_{\mathbb{R}}\, S)_3. \tag{2.6.8}
$$

P r o o f. By the assumption, the metric g has the form (2.6.3). Hence, the only non-zero quaternionic components of g are g^{00}, $g^{\overline{11}}$, $g^{\overline{22}}$, $g^{\overline{33}}$ and they equal $g^0_{\mathbb{H}} \neq 0$, which is real. Then we have

$$
(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_0 = \sum_{\alpha} g^{\alpha\alpha} \nabla_{\partial_\alpha} s_{\alpha 0}, \quad (\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{k}} = \sum_{\alpha} g^{\alpha\alpha} \nabla_{\partial_\alpha} s_{\alpha\overline{k}}, \quad k = 1,2,3.
$$

where $\nabla_{\partial/\partial q} = \nabla_{\frac{1}{4}(\partial/\partial x_0 - i\partial/\partial x_1 - j\partial/\partial x_2 - k\partial/\partial x_3)} := \frac{1}{4}[\nabla_{\partial_0} - i\nabla_{\partial_1} - j\nabla_{\partial_2} - k\nabla_{\partial_3}]$, etc. Hence

$$
\begin{aligned}
(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_0 &= g^0_{\mathbb{H}}[\nabla_{\partial/\partial q} s_{00} + \nabla_{\partial/\partial q^1} s_{\overline{1}0} + \nabla_{\partial/\partial q^2} s_{\overline{2}0} + \nabla_{\partial/\partial q^3} s_{\overline{3}0}],\\
(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{1}} &= g^0_{\mathbb{H}}[\nabla_{\partial/\partial q} s_{0\overline{1}} + \nabla_{\partial/\partial q^1} s_{\overline{11}} + \nabla_{\partial/\partial q^2} s_{\overline{21}} + \nabla_{\partial/\partial q^3} s_{\overline{31}}],\\
(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{2}} &= g^0_{\mathbb{H}}[\nabla_{\partial/\partial q} s_{0\overline{2}} + \nabla_{\partial/\partial q^1} s_{\overline{12}} + \nabla_{\partial/\partial q^2} s_{\overline{22}} + \nabla_{\partial/\partial q^3} s_{\overline{32}}],\\
(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{3}} &= g^0_{\mathbb{H}}[\nabla_{\partial/\partial q} s_{0\overline{3}} + \nabla_{\partial/\partial q^1} s_{\overline{13}} + \nabla_{\partial/\partial q^2} s_{\overline{23}} + \nabla_{\partial/\partial q^3} s_{\overline{33}}].
\end{aligned} \tag{2.6.9}
$$

The real divergence of S looks as follows:

$$
(\mathrm{div}_{\mathbb{R}}\, S)_i = g^0_{\mathbb{R}}[\nabla_{\partial_0} S_{0i} + \nabla_{\partial_1} S_{1i} + \nabla_{\partial_2} S_{2i} + \nabla_{\partial_3}] \quad \text{for } i = 0,1,2,3.
$$

By the relationship (2.6.1) between quaternionic and real components of the tensor S we get the required equalities (2.6.8). Indeed, let us show the first relation: $\mathrm{Re}[(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_0] = (\mathrm{div}_{\mathbb{R}}\, S)_0$. By simple calculations we have

$$
\begin{aligned}
(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_0 &= g^0_{\mathbb{H}}[\nabla_{\partial/\partial q} s_{00} + \nabla_{\partial/\partial q^1} s_{\overline{1}0} + \nabla_{\partial/\partial q^2} s_{\overline{2}0} + \nabla_{\partial/\partial q^2} s_{\overline{3}0}]\\
&= \tfrac{1}{4} g^0_{\mathbb{R}} \tfrac{1}{4}[\nabla_{\partial_0} - i\nabla_{\partial_1} - j\nabla_{\partial_2} - k\nabla_{\partial_3}] \tfrac{1}{16}(S_{00} + S_{11} + S_{22} + S_{33})\\
&\quad + \tfrac{1}{4}[\nabla_{\partial_0} - i\nabla_{\partial_1} - j\nabla_{\partial_2} - k\nabla_{\partial_3}]\\
&\quad \times \big\{\tfrac{1}{16}(S_{00} + S_{11} - S_{22} - S_{33}) - \tfrac{1}{8}(S_{02} - S_{13})j - \tfrac{1}{8}(S_{03} + S_{12})k\big\}\\
&\quad + \tfrac{1}{4}[\nabla_{\partial_0} - i\nabla_{\partial_1} - j\nabla_{\partial_2} - k\nabla_{\partial_3}]\\
&\quad \times \big\{\tfrac{1}{16}(S_{00} - S_{11} + S_{22} - S_{33}) - \tfrac{1}{8}(S_{01} + S_{23})i - \tfrac{1}{8}(S_{03} - S_{12})k\big\}\\
&\quad + \tfrac{1}{4}[\nabla_{\partial_0} - i\nabla_{\partial_1} - j\nabla_{\partial_2} - k\nabla_{\partial_3}]\\
&\quad \times \big\{\tfrac{1}{16}(S_{00} - S_{11} - S_{22} + S_{33}) - \tfrac{1}{8}(S_{01} - S_{23})i - \tfrac{1}{8}(S_{02} + S_{13})j\big\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathrm{Re}\,(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_0 &= \tfrac{1}{256} g^0_{\mathbb{R}} \nabla_{\partial_0}(S_{00} + S_{11} + S_{22} + S_{33})\\
&\quad + \tfrac{1}{256} g^0_{\mathbb{R}} \nabla_{\partial_0}(S_{00} + S_{11} - S_{22} - S_{33})\\
&\quad + \tfrac{1}{128} g^0_{\mathbb{R}}[\nabla_{\partial_2}(S_{02} - S_{13}) + \nabla_{\partial_3}(S_{03} + S_{12})]\\
&\quad + \tfrac{1}{256} g^0_{\mathbb{R}} \nabla_{\partial_0}(S_{00} - S_{11} + S_{22} - S_{33})\\
&\quad + \tfrac{1}{128} g^0_{\mathbb{R}}[\nabla_{\partial_1}(S_{01} + S_{23}) + \nabla_{\partial_3}(S_{03} - S_{12})]\\
&\quad + \tfrac{1}{256} g^0_{\mathbb{R}} \nabla_{\partial_0}(S_{00} - S_{11} - S_{22} + S_{33})\\
&\quad + \tfrac{1}{128} g^0_{\mathbb{R}}[\nabla_{\partial_1}(S_{01} - S_{23}) + \nabla_{\partial_2}(S_{02} + S_{13})]\\
&= \tfrac{1}{64} g^0_{\mathbb{R}}[\nabla_{\partial_0} S_{00} + \nabla_{\partial_1} S_{10} + \nabla_{\partial_2} S_{20} + \nabla_{\partial_3} S_{30}] = \tfrac{1}{64}(\mathrm{div}_{\mathbb{R}}\, S)_0.
\end{aligned}
$$

Now, let us prove the second relation of (2.6.8). Compute

$$
\begin{aligned}
(\operatorname{div}_{\mathbb{H}} S^{\mathbb{H}})_{\overline{1}} &= g^0_{\mathbb{H}}[\nabla_{\partial/\partial q} s_{0\overline{1}} + \nabla_{\partial/\partial q^1} s_{\overline{11}} + \nabla_{\partial/\partial q^2} s_{\overline{21}} + \nabla_{\partial/\partial q^3} s_{\overline{31}}] \\
&= \tfrac14 g^0_{\mathbb{R}}\big[\tfrac14(\nabla_{\partial_0} - i\nabla_{\partial_1} - j\nabla_{\partial_2} - k\nabla_{\partial_3}) \\
&\quad \times \big\{\tfrac{1}{16}(S_{00}+S_{11}-S_{22}-S_{33}) + \tfrac18(S_{02}-S_{13})j + \tfrac18(S_{03}+S_{12})k\big\} \\
&\quad + \tfrac14(\nabla_{\partial_0} - i\nabla_{\partial_1} + j\nabla_{\partial_2} + k\nabla_{\partial_3})\big\{\tfrac{1}{16}(S_{00}+S_{11}+S_{22}+S_{33})\big\} \\
&\quad + \tfrac14(\nabla_{\partial_0} + i\nabla_{\partial_1} - j\nabla_{\partial_2} + k\nabla_{\partial_3}) \\
&\quad \times \big\{\tfrac{1}{16}(S_{00}-S_{11}-S_{22}+S_{33}) - \tfrac18(S_{01}-S_{23})i + \tfrac18(S_{02}-S_{13})j\big\} \\
&\quad + \tfrac14(\nabla_{\partial_0} + i\nabla_{\partial_1} + j\nabla_{\partial_2} - k\nabla_{\partial_3}) \\
&\quad \times \big\{\tfrac{1}{16}(S_{00}-S_{11}+S_{22}-S_{33}) - \tfrac18(S_{01}+S_{23})i + \tfrac18(S_{03}-S_{12})k\big\}\big].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\operatorname{Re}[i(\operatorname{div}_{\mathbb{H}} S^{\mathbb{H}})_{\overline{1}}] \\
&= \operatorname{Re}\big[ig^0_{\mathbb{R}}\big\{\tfrac{i}{256}(-\nabla_{\partial_1})(S_{00}+S_{11}-S_{22}-S_{33}) \\
&\quad + \tfrac{i}{128}\nabla_{\partial_3}(S_{02}-S_{13}) - \tfrac{i}{128}\nabla_{\partial_2}(S_{02}+S_{12}) - \tfrac{i}{256}\nabla_{\partial_1}(S_{00}+S_{11}+S_{22}+S_{33}) \\
&\quad + \tfrac{i}{256}\nabla_{\partial_1}(S_{00}-S_{11}-S_{22}+S_{33}) - \tfrac{i}{128}\nabla_{\partial_0}(S_{01}-S_{23}) - \tfrac{i}{128}\nabla_{\partial_3}(S_{02}+S_{13}) \\
&\quad + \tfrac{i}{256}\nabla_{\partial_1}(S_{00}-S_{11}+S_{22}-S_{33}) - \tfrac{i}{128}\nabla_{\partial_0}(S_{01}+S_{23}) + \tfrac{i}{128}\nabla_{\partial_2}(S_{03}-S_{12})\big\}\big] \\
&= \tfrac{1}{64} g^0_{\mathbb{R}}[\nabla_{\partial_0}S_{01} + \nabla_{\partial_1}S_{11} + \nabla_{\partial_2}S_{21} + \nabla_{\partial_3}S_{31}] = \tfrac{1}{64}(\operatorname{div}_{\mathbb{R}} S)_1.
\end{aligned}
$$

The other relations are proved analogously. ■

Let us quote a very important result of Eells [13]:

THEOREM 2.6.1 [13]. *Suppose that $\Phi : (M,g) \to (N,h)$ is a smooth mapping between two smooth Riemannian manifolds. If Φ is harmonic then $S(\Phi)$ is conservative* (*i.e.* $\operatorname{div} S(\Phi) = 0$). *If Φ is a differentiable submersion almost everywhere and* $\operatorname{div}_{\mathbb{R}} S(\Phi) = 0$, *then Φ is harmonic.*

LEMMA 2.6.2. *Let $\Phi : (M^4,g) \to (N^{4n},h)$ be a smooth map between two smooth Riemannian manifolds. Assume that M is locally conformally flat. If Φ is harmonic then Φ locally satisfies the following equations*:

$$
\begin{aligned}
&\operatorname{Re}[(\operatorname{div}_{\mathbb{H}} S^{\mathbb{H}}(\Phi))_0] = 0, \quad \operatorname{Re}[i(\operatorname{div}_{\mathbb{H}} S^{\mathbb{H}}(\Phi))_{\overline{1}}] = 0, \\
&\operatorname{Re}[j(\operatorname{div}_{\mathbb{H}} S^{\mathbb{H}}(\Phi))_{\overline{2}}] = 0, \quad \operatorname{Re}[k(\operatorname{div}_{\mathbb{H}} S^{\mathbb{H}}(\Phi))_{\overline{3}}] = 0,
\end{aligned}
\tag{2.6.10}
$$

where $S^{\mathbb{H}}(\Phi)$ denotes the quaternionic stress-energy tensor of Φ.

Proof. This follows from the result of Eells (Theorem 2.6.1) and Lemma 2.6.1. ■

PROPOSITION 2.6.5. *Let $\Phi : (M^4,g) \to (N^{4n},h)$ be a smooth map between two smooth Riemannian manifolds. Assume that M^4 is locally conformally flat. If Φ is a differentiable submersion almost everywhere and locally the system* (2.6.10) *is satisfied then Φ is harmonic.*

PROPOSITION 2.6.6. *Under the above assumptions the system* (2.6.10) *is equivalent to the following one*:

$$
(2.6.11)\qquad \begin{aligned}
&\mathrm{Re}\{\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_q\rangle - \nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^1}\rangle - \nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^2}\rangle - \nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^3}\rangle\} = 0,\\
&\mathrm{Re}\{i[\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^1}\rangle - \nabla_{\partial/\partial q^1}\langle \Phi_{\overline{q}}, \Phi_q\rangle + \nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^3}\rangle + \nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^2}\rangle]\} = 0,\\
&\mathrm{Re}\{j[\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^2}\rangle + \nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^3}\rangle - \nabla_{\partial/\partial q^2}\langle \Phi_{\overline{q}}, \Phi_q\rangle + \nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^1}\rangle]\} = 0,\\
&\mathrm{Re}\{k[\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^3}\rangle + \nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^2}\rangle + \nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^1}\rangle - \nabla_{\partial/\partial q^3}\langle \Phi_{\overline{q}}, \Phi_q\rangle]\} = 0.
\end{aligned}
$$

P r o o f. Substituting the quaternionic expression $S^{\mathbb{H}}$ for the stress-energy tensor $S(\Phi)$ into (2.6.9) we get

$$
\begin{aligned}
\frac{(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_0}{g^0_{\mathbb{H}}} &= \nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_q\rangle - \nabla_{\partial/\partial q^1}\langle \Phi_{\overline{q}}, \Phi_{q^1}\rangle^1 - \nabla_{\partial/\partial q^2}\langle \Phi_{\overline{q}}, \Phi_{q^2}\rangle^2 - \nabla_{\partial/\partial q^3}\langle \Phi_{\overline{q}}, \Phi_{q^3}\rangle^3,\\
\frac{(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{1}}}{g^0_{\mathbb{H}}} &= -\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^1}\rangle + \nabla_{\partial/\partial q^1}\langle \Phi_{\overline{q}}, \Phi_q\rangle - \nabla_{\partial/\partial q^2}\langle \Phi_{\overline{q}}, \Phi_{q^3}\rangle^2 - \nabla_{\partial/\partial q^3}\langle \Phi_{\overline{q}}, \Phi_{q^2}\rangle^3,\\
\frac{(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{2}}}{g^0_{\mathbb{H}}} &= -\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^2}\rangle - \nabla_{\partial/\partial q^1}\langle \Phi_{\overline{q}}, \Phi_{q^3}\rangle^1 + \nabla_{\partial/\partial q^2}\langle \Phi_{\overline{q}}, \Phi_q\rangle - \nabla_{\partial/\partial q^3}\langle \Phi_{\overline{q}}, \Phi_{q^1}\rangle^3,\\
\frac{(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{3}}}{g^0_{\mathbb{H}}} &= -\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^3}\rangle - \nabla_{\partial/\partial q^1}\langle \Phi_{\overline{q}}, \Phi_{q^2}\rangle^1 - \nabla_{\partial/\partial q^2}\langle \Phi_{\overline{q}}, \Phi_{q^1}\rangle^2 + \nabla_{\partial/\partial q^3}\langle \Phi_{\overline{q}}, \Phi_q\rangle.
\end{aligned}
$$

Since

$$\nabla_{\partial/\partial q^i}\langle \Phi_{\overline{q}}, \Phi_{q^m}\rangle^i = [\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^m}\rangle]^i$$

for $i, m = 0, 1, 2, 3$, we have

$$
\begin{aligned}
\frac{(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_0}{g^0_{\mathbb{H}}} &= \nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_q\rangle - (\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^1}\rangle)^1 - (\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^2}\rangle)^2 - (\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^3}\rangle)^3,\\
\frac{(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{1}}}{g^0_{\mathbb{H}}} &= -\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^1}\rangle + \nabla_{\partial/\partial q^1}\langle \Phi_{\overline{q}}, \Phi_q\rangle - (\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^3}\rangle)^2 - (\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^2}\rangle)^3,\\
\frac{(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{2}}}{g^0_{\mathbb{H}}} &= -\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^2}\rangle - (\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^3}\rangle)^1 + \nabla_{\partial/\partial q^2}\langle \Phi_{\overline{q}}, \Phi_q\rangle - (\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^1}\rangle)^3,\\
\frac{(\mathrm{div}_{\mathbb{H}}\, S^{\mathbb{H}})_{\overline{3}}}{g^0_{\mathbb{H}}} &= -\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^3}\rangle - (\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^2}\rangle)^1 - (\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^1}\rangle)^2 + \nabla_{\partial/\partial q^3}\langle \Phi_{\overline{q}}, \Phi_q\rangle.
\end{aligned}
$$

Now, note that

$$\mathrm{Re}\,[\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^m}\rangle]^i = \mathrm{Re}\,\nabla_{\partial/\partial q}\langle \Phi_{\overline{q}}, \Phi_{q^m}\rangle, \qquad i, m = 1, 2, 3.$$

So, finally we conclude that the system (2.6.10) is equivalent to (2.6.11), as required. ∎

LEMMA 2.6.3. *Let $\Phi : (M^4, g) \to (N^{4n}, h)$ be a smooth map between two smooth Riemannian manifolds. Assume that M is locally conformally flat. If Φ is harmonic then Φ satisfies the following system of real equations*:

$$
(2.6.12)\qquad \begin{aligned}
&\tfrac{1}{32}\nabla_{\partial_0}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2)\\
&\qquad\qquad + \nabla_{\partial_1}\langle \Phi_0, \Phi_1\rangle + \nabla_{\partial_2}\langle \Phi_0, \Phi_2\rangle + \nabla_{\partial_3}\langle \Phi_0, \Phi_3\rangle = 0,\\
&\tfrac{1}{32}\nabla_{\partial_1}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2)\\
&\qquad\qquad - \nabla_{\partial_0}\langle \Phi_0, \Phi_1\rangle + \nabla_{\partial_2}\langle \Phi_0, \Phi_3\rangle - \nabla_{\partial_3}\langle \Phi_0, \Phi_2\rangle = 0,\\
&\tfrac{1}{32}\nabla_{\partial_2}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2)\\
&\qquad\qquad - \nabla_{\partial_0}\langle \Phi_0, \Phi_2\rangle - \nabla_{\partial_1}\langle \Phi_0, \Phi_3\rangle + \nabla_{\partial_3}\langle \Phi_0, \Phi_1\rangle = 0,
\end{aligned}
$$

(2.6.12) [cont.]
$$\frac{1}{32}\nabla_{\partial_3}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2) \\ -\nabla_{\partial_0}\langle\Phi_0,\Phi_3\rangle + \nabla_{\partial_1}\langle\Phi_0,\Phi_2\rangle - \nabla_{\partial_2}\langle\Phi_0,\Phi_1\rangle = 0,$$

where $\Phi_i := \partial\Phi/\partial x^i$ *and* $\langle\cdot,\cdot\rangle$ *means the real scalar product. Moreover, if* Φ *is a differentiable submersion almost everywhere and the system* (2.6.12) *is satisfied, then* Φ *is harmonic.*

Proof. Note that

$$\begin{aligned}
\operatorname{Re}\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_q\rangle &= \tfrac{1}{64}\nabla_{\partial_0}(|\Phi_0|^2+|\Phi_1|^2+|\Phi_2|^2+|\Phi_3|^2),\\
\operatorname{Re}\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_{q^1}\rangle &= \tfrac{1}{64}\{\nabla_{\partial_0}(|\Phi_0|^2+|\Phi_1|^2-|\Phi_2|^2-|\Phi_3|^2)\\
&\quad+2\nabla_{\partial_2}[\langle\Phi_0,\Phi_2\rangle-\langle\Phi_1,\Phi_3\rangle]+2\nabla_{\partial_3}[\langle\Phi_0,\Phi_3\rangle+\langle\Phi_1,\Phi_2\rangle]\},\\
\operatorname{Re}\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_{q^2}\rangle &= \tfrac{1}{64}\{\nabla_{\partial_0}(|\Phi_0|^2-|\Phi_1|^2+|\Phi_2|^2-|\Phi_3|^2)\\
&\quad+2\nabla_{\partial_1}[\langle\Phi_0,\Phi_1\rangle+\langle\Phi_2,\Phi_3\rangle]+2\nabla_{\partial_3}[\langle\Phi_0,\Phi_3\rangle-\langle\Phi_1,\Phi_2\rangle]\},\\
\operatorname{Re}\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_{q^3}\rangle &= \tfrac{1}{64}\{\nabla_{\partial_0}(|\Phi_0|^2-|\Phi_1|^2-|\Phi_2|^2+|\Phi_3|^2)\\
&\quad+2\nabla_{\partial_1}[\langle\Phi_0,\Phi_1\rangle-\langle\Phi_2,\Phi_3\rangle]+2\nabla_{\partial_2}[\langle\Phi_0,\Phi_2\rangle+\langle\Phi_1,\Phi_3\rangle]\},\\
\operatorname{Re} i\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_{q^1}\rangle &= \tfrac{1}{64}\{\nabla_{\partial_1}(|\Phi_0|^2+|\Phi_1|^2-|\Phi_2|^2-|\Phi_3|^2)\\
&\quad-2\nabla_{\partial_3}[\langle\Phi_0,\Phi_2\rangle-\langle\Phi_1,\Phi_3\rangle]+2\nabla_{\partial_2}[\langle\Phi_0,\Phi_3\rangle+\langle\Phi_1,\Phi_2\rangle]\},\\
\operatorname{Re} i\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_{q^2}\rangle &= \tfrac{1}{64}\{\nabla_{\partial_1}(|\Phi_0|^2-|\Phi_1|^2+|\Phi_2|^2-|\Phi_3|^2)\\
&\quad-2\nabla_{\partial_0}[\langle\Phi_0,\Phi_1\rangle+\langle\Phi_2,\Phi_3\rangle]+2\nabla_{\partial_2}[\langle\Phi_0,\Phi_3\rangle-\langle\Phi_1,\Phi_2\rangle]\},\\
\operatorname{Re} i\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_{q^3}\rangle &= \tfrac{1}{64}\{\nabla_{\partial_1}(|\Phi_0|^2-|\Phi_1|^2-|\Phi_2|^2+|\Phi_3|^2)\\
&\quad-2\nabla_{\partial_0}[\langle\Phi_0,\Phi_1\rangle-\langle\Phi_2,\Phi_3\rangle]-2\nabla_{\partial_3}[\langle\Phi_0,\Phi_2\rangle+\langle\Phi_1,\Phi_3\rangle]\},\\
\operatorname{Re} j\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_{q^1}\rangle &= \tfrac{1}{64}\{\nabla_{\partial_2}(|\Phi_0|^2+|\Phi_1|^2-|\Phi_2|^2-|\Phi_3|^2)\\
&\quad-2\nabla_{\partial_0}[\langle\Phi_0,\Phi_2\rangle-\langle\Phi_1,\Phi_3\rangle]-2\nabla_{\partial_1}[\langle\Phi_0,\Phi_3\rangle+\langle\Phi_1,\Phi_2\rangle]\},\\
\operatorname{Re} j\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_{q^2}\rangle &= \tfrac{1}{64}\{\nabla_{\partial_2}(|\Phi_0|^2-|\Phi_1|^2+|\Phi_2|^2-|\Phi_3|^2)\\
&\quad+2\nabla_{\partial_3}[\langle\Phi_0,\Phi_1\rangle+\langle\Phi_2,\Phi_3\rangle]-2\nabla_{\partial_1}[\langle\Phi_0,\Phi_3\rangle-\langle\Phi_1,\Phi_2\rangle]\},\\
\operatorname{Re} j\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_{q^3}\rangle &= \tfrac{1}{64}\{\nabla_{\partial_2}(|\Phi_0|^2-|\Phi_1|^2-|\Phi_2|^2+|\Phi_3|^2)\\
&\quad+2\nabla_{\partial_3}[\langle\Phi_0,\Phi_1\rangle-\langle\Phi_2,\Phi_3\rangle]-2\nabla_{\partial_0}[\langle\Phi_0,\Phi_2\rangle+\langle\Phi_1,\Phi_3\rangle]\},\\
\operatorname{Re} k\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_{q^1}\rangle &= \tfrac{1}{64}\{\nabla_{\partial_3}(|\Phi_0|^2+|\Phi_1|^2-|\Phi_2|^2-|\Phi_3|^2)\\
&\quad+2\nabla_{\partial_1}[\langle\Phi_0,\Phi_2\rangle-\langle\Phi_1,\Phi_3\rangle]-2\nabla_{\partial_0}[\langle\Phi_0,\Phi_3\rangle+\langle\Phi_1,\Phi_2\rangle]\},\\
\operatorname{Re} k\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_{q^2}\rangle &= \tfrac{1}{64}\{\nabla_{\partial_3}(|\Phi_0|^2-|\Phi_1|^2+|\Phi_2|^2-|\Phi_3|^2)\\
&\quad-2\nabla_{\partial_2}[\langle\Phi_0,\Phi_1\rangle+\langle\Phi_2,\Phi_3\rangle]-2\nabla_{\partial_0}[\langle\Phi_0,\Phi_3\rangle-\langle\Phi_1,\Phi_2\rangle]\},\\
\operatorname{Re} k\nabla_{\partial/\partial q}\langle\Phi_{\overline{q}},\Phi_{q^3}\rangle &= \tfrac{1}{64}\{\nabla_{\partial_3}(|\Phi_0|^2-|\Phi_1|^2-|\Phi_2|^2+|\Phi_3|^2)\\
&\quad-2\nabla_{\partial_2}[\langle\Phi_0,\Phi_1\rangle-\langle\Phi_2,\Phi_3\rangle]+2\nabla_{\partial_1}[\langle\Phi_0,\Phi_2\rangle+\langle\Phi_1,\Phi_3\rangle]\},\\
\operatorname{Re} i\nabla_{\partial/\partial q^1}\langle\Phi_{\overline{q}},\Phi_q\rangle &= \tfrac{1}{64}\nabla_{\partial_1}(|\Phi_0|^2+|\Phi_1|^2+|\Phi_2|^2+|\Phi_3|^2),\\
\operatorname{Re} j\nabla_{\partial/\partial q^2}\langle\Phi_{\overline{q}},\Phi_q\rangle &= \tfrac{1}{64}\nabla_{\partial_2}(|\Phi_0|^2+|\Phi_1|^2+|\Phi_2|^2+|\Phi_3|^2),\\
\operatorname{Re} k\nabla_{\partial/\partial q^3}\langle\Phi_{\overline{q}},\Phi_q\rangle &= \tfrac{1}{64}\nabla_{\partial_3}(|\Phi_0|^2+|\Phi_1|^2+|\Phi_2|^2+|\Phi_3|^2).
\end{aligned}$$

Now, substituting the above expressions into (2.6.11) we get immediately (2.6.12), as required. ∎

Definition 2.6.4. For the maps as in Lemma 2.6.3 we define the vector field $\vec{A} = \vec{A}(\Phi) = (A_0(\Phi), A_1(\Phi), A_2(\Phi), A_3(\Phi))$ by

$$\begin{aligned}
A_0 &= \tfrac{1}{2}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2) - \langle \Phi_0, \Phi_1\rangle - \langle \Phi_0, \Phi_2\rangle - \langle \Phi_0, \Phi_3\rangle,\\
A_1 &= \tfrac{1}{2}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2) + \langle \Phi_0, \Phi_1\rangle + \langle \Phi_0, \Phi_2\rangle - \langle \Phi_0, \Phi_3\rangle,\\
A_2 &= \tfrac{1}{2}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2) - \langle \Phi_0, \Phi_1\rangle + \langle \Phi_0, \Phi_2\rangle + \langle \Phi_0, \Phi_3\rangle,\\
A_3 &= \tfrac{1}{2}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2) + \langle \Phi_0, \Phi_1\rangle - \langle \Phi_0, \Phi_2\rangle + \langle \Phi_0, \Phi_3\rangle.
\end{aligned}$$

Definition 2.6.5. Let (M^4, g) be a 4-dimensional smooth Riemannian manifold. A vector field $\vec{A}$ defined on M^4 is called *solenoidal* if in any system of local (real) coordinates (x^i), $i = 0,1,2,3$, the vector field $\vec{A} = (A^0, A^1, A^2, A^3)$ satisfies the following condition:

$$\operatorname{Div} \vec{A}(x^0, x^1, x^2, x^3) := \nabla_{\partial_0} A^0 + \nabla_{\partial_1} A^1 + \nabla_{\partial_2} A^2 + \nabla_{\partial_3} A^3 = 0.$$

Corollary 2.6.4. *Every harmonic map $\Phi : (M^4, g) \to (N^{4n}, h)$ between smooth Riemannian manifolds, where (M^4, g) is locally conformally flat, generates a solenoidal vector field defined on M^4.*

Proof. This follows from Lemma 2.6.3. If Φ is harmonic then it satisfies the system (2.6.12) but this implies that $\operatorname{Div} \vec{A}(\Phi) = 0$. ∎

Proposition 2.6.6. *If $\Phi : (M^4, g) \to (N^{4n}, h)$ is a smooth map between smooth Riemannian manifolds and (M^4, g) is locally conformally flat, then the following conditions are equivalent:*

1) $\operatorname{div}_{\mathbb{R}} S(\Phi) = 0$,

2)

$$\begin{aligned}
\operatorname{Re}\{(s_{00} + s_{01} + s_{02} + s_{03})\nabla_{\partial/\partial q}\} &= 0,\\
\operatorname{Re}\{(s_{00} + s_{01} - s_{02} - s_{03})i\nabla_{\partial/\partial q}\} &= 0,\\
\operatorname{Re}\{(s_{00} - s_{01} + s_{02} - s_{03})j\nabla_{\partial/\partial q}\} &= 0,\\
\operatorname{Re}\{(s_{00} - s_{01} - s_{02} + s_{03})k\nabla_{\partial/\partial q}\} &= 0,
\end{aligned}$$

where $S(\Phi)$ denotes the stress-energy tensor of Φ, s_{0i}, $i = 0,1,2,3$, are the quaternionic components on $S(\Phi)$ and the operator $\nabla_{\partial/\partial q}$ acts ob the right.

Proof. By (2.6.1), where we choose the sign "+" for s_{0k}, $k = 1,2,3$, we find that the condition $\operatorname{div}_{\mathbb{R}} S = 0$ is equivalent to the following system of equations:

$$\begin{aligned}
&\nabla_{\partial_0}[s_{00} + \operatorname{Re}(s_{01} + s_{02} + s_{03})] + \nabla_{\partial_1}[-\operatorname{Re}(s_{01} + s_{02} + s_{03})i]\\
&\qquad + \nabla_{\partial_2}[-\operatorname{Re}(s_{01} + s_{02} + s_{03})j] + \nabla_{\partial_3}[-\operatorname{Re}(s_{01} + s_{02} + s_{03})k] = 0,\\
&\nabla_{\partial_0}[-\operatorname{Re}(s_{01} + s_{02} + s_{03})i] + \nabla_{\partial_1}[s_{00} + \operatorname{Re}(s_{01} - s_{02} - s_{03})]\\
&\qquad + \nabla_{\partial_2}[-\operatorname{Re}(s_{01} - s_{02} + s_{03})k] + \nabla_{\partial_3}[-\operatorname{Re}(-s_{01} + s_{02} + s_{03})j] = 0,\\
&\nabla_{\partial_0}[-\operatorname{Re}(s_{01} + s_{02} + s_{03})j] + \nabla_{\partial_1}[-\operatorname{Re}(s_{01} - s_{02} + s_{03})k]\\
&\qquad + \nabla_{\partial_2}[s_{00} + \operatorname{Re}(-s_{01} + s_{02} - s_{03})] + \nabla_{\partial_3}[-\operatorname{Re}(s_{01} - s_{02} + s_{03})i] = 0,\\
&\nabla_{\partial_0}[-\operatorname{Re}(s_{01} + s_{02} + s_{03})k] + \nabla_{\partial_1}[-\operatorname{Re}(-s_{01} + s_{02} + s_{03})j]\\
&\qquad + \nabla_{\partial_2}[-\operatorname{Re}(s_{01} - s_{02} + s_{03})i] + \nabla_{\partial_3}[s_{00} + \operatorname{Re}(-s_{01} - s_{02} + s_{03})] = 0.
\end{aligned} \tag{2.6.13}$$

Since

$$\operatorname{Re} s_{01}i = \operatorname{Re} s_{02}j = \operatorname{Re} s_{03}k = 0, \qquad \operatorname{Re} s_{00}i = \operatorname{Re} s_{00}j = \operatorname{Re} s_{00}k = 0,$$

we have

$$\begin{aligned} -\operatorname{Re}[(s_{00}+s_{01}-s_{02}-s_{03})i] &= \operatorname{Re}[(s_{00}+s_{01}+s_{02}+s_{03})i],\\ -\operatorname{Re}[(s_{00}-s_{01}+s_{02}-s_{03})j] &= \operatorname{Re}[(s_{00}+s_{01}+s_{02}+s_{03})j],\\ -\operatorname{Re}[(s_{00}-s_{01}-s_{02}+s_{03})k] &= \operatorname{Re}[(s_{00}+s_{01}+s_{02}+s_{03})k]. \end{aligned}$$

Using the above relations we can rewrite the equations (2.6.13) in the following form:

$$\begin{aligned} &\nabla_{\partial_0}\operatorname{Re}(s_{00}+s_{01}+s_{02}+s_{03}) - \nabla_{\partial_1}\operatorname{Re}(s_{00}+s_{01}+s_{02}+s_{03})i\\ &\qquad -\nabla_{\partial_2}\operatorname{Re}(s_{00}+s_{01}+s_{02}+s_{03})j - \nabla_{\partial_3}\operatorname{Re}(s_{00}+s_{01}+s_{02}+s_{03})k = 0,\\ &\nabla_{\partial_0}\operatorname{Re}(s_{00}+s_{01}-s_{02}-s_{03})i + \nabla_{\partial_1}\operatorname{Re}(s_{00}+s_{01}-s_{02}-s_{03})\\ &\qquad -\nabla_{\partial_2}\operatorname{Re}(s_{00}+s_{01}-s_{02}-s_{03})k + \nabla_{\partial_3}\operatorname{Re}(s_{00}+s_{01}-s_{02}-s_{03})j = 0,\\ &\nabla_{\partial_0}\operatorname{Re}(s_{00}-s_{01}+s_{02}-s_{03})j + \nabla_{\partial_1}\operatorname{Re}(s_{00}-s_{01}+s_{02}-s_{03})k\\ &\qquad +\nabla_{\partial_2}\operatorname{Re}(s_{00}-s_{01}+s_{02}-s_{03}) - \nabla_{\partial_3}\operatorname{Re}(s_{00}-s_{01}+s_{02}-s_{03})i = 0,\\ &\nabla_{\partial_0}\operatorname{Re}(s_{00}-s_{01}-s_{02}+s_{03})k - \nabla_{\partial_1}\operatorname{Re}(s_{00}-s_{01}-s_{02}+s_{03})j\\ &\qquad +\nabla_{\partial_2}\operatorname{Re}(s_{00}-s_{01}-s_{02}+s_{03})i + \nabla_{\partial_3}\operatorname{Re}(s_{00}-s_{01}-s_{02}+s_{03}) = 0. \end{aligned} \tag{2.6.14}$$

Now, note that for any quaternionic function f we have

$$\frac{\partial}{\partial x^i}\operatorname{Re} f = \operatorname{Re}\frac{\partial f}{\partial x^i}, \quad \nabla_{\partial_i}\operatorname{Re} f = \operatorname{Re}\nabla_{\partial_i} f, \quad i = 0,1,2,3.$$

Thus the system (2.6.14) is equivalent to

$$\begin{aligned} \operatorname{Re}\{(s_{00}+s_{01}+s_{02}+s_{03})(\nabla_{\partial_0} - i\nabla_{\partial_1} - j\nabla_{\partial_2} - k\nabla_{\partial_3})\} &= 0,\\ \operatorname{Re}\{(s_{00}+s_{01}-s_{02}-s_{03})i(\nabla_{\partial_0} - i\nabla_{\partial_1} - j\nabla_{\partial_2} - k\nabla_{\partial_3})\} &= 0,\\ \operatorname{Re}\{(s_{00}-s_{01}+s_{02}-s_{03})j(\nabla_{\partial_0} - i\nabla_{\partial_1} - j\nabla_{\partial_2} - k\nabla_{\partial_3})\} &= 0,\\ \operatorname{Re}\{(s_{00}-s_{01}-s_{02}+s_{03})k(\nabla_{\partial_0} - i\nabla_{\partial_1} - j\nabla_{\partial_2} - k\nabla_{\partial_3})\} &= 0. \end{aligned}$$

Since

$$\nabla_{\partial_0} - i\nabla_{\partial_1} - j\nabla_{\partial_2} - k\nabla_{\partial_3} = \nabla_{\partial/\partial q}, \qquad \partial/\partial q = \partial_0 - i\partial_1 - j\partial_2 - k\partial_3,$$

we get the required equivalence. ■

Remark 2.6.4. We have

$$[(\operatorname{div}_{\mathbb{R}} S) = 0] \Leftrightarrow \begin{cases} \operatorname{Re}\{(S_{00}+iS_{01}+jS_{02}+kS_{03})\nabla_{\partial/\partial q}\} = 0,\\ \operatorname{Re}\{(S_{11}-iS_{01}-jS_{13}+kS_{12})i\nabla_{\partial/\partial q}\} = 0,\\ \operatorname{Re}\{(S_{22}+iS_{23}-jS_{02}-kS_{12})j\nabla_{\partial/\partial q}\} = 0,\\ \operatorname{Re}\{(S_{33}-iS_{23}+jS_{13}-kS_{03})k\nabla_{\partial/\partial q}\} = 0. \end{cases}$$

Proof. By (2.6.1) we get

$$\begin{aligned} s_{00}+s_{01}+s_{02}+s_{03} &= \tfrac14(S_{00}+iS_{01}+jS_{02}+kS_{03}),\\ s_{00}+s_{01}-s_{02}-s_{03} &= \tfrac14(S_{11}-iS_{01}-jS_{13}+kS_{12}),\\ s_{00}-s_{01}+s_{02}-s_{03} &= \tfrac14(S_{22}+iS_{23}-jS_{02}-kS_{12}),\\ s_{00}-s_{01}-s_{02}+s_{03} &= \tfrac14(S_{33}-iS_{23}+jS_{13}-kS_{03}). \end{aligned}$$

Now, the proof follows from Proposition 2.6.6. ■

Let us recall

Definition 2.6.6. A function $f : \Omega \to \mathbb{H}$ is said to be *antiregular in the sense of Fueter* in a domain $\Omega \subset \mathbb{H}$ if f is differentiable (in the usual sense) as a mapping of Ω, regarded as an open set in $\mathbb{R}^4$, with values in $\mathbb{R}^4$, and

$$D \cdot f := \tfrac{1}{4}(\partial_0 - i\partial_1 - j\partial_2 - k\partial_3)(f_0 + if_1 + jf_2 + kf_3) = 0$$

in Ω; here $\partial_k := \partial/\partial x^k,\ k = 0,1,2,3$.

Remark 2.6.5. Let $F = F_0 + iF_1 + jF_2 + kF_3 : \Omega \to \mathbb{H}$, where Ω is an open set in $\mathbb{H}$, be a differentiable mapping (i.e. each component is differentiable considered as a mapping $\mathbb{R}^4 \to \mathbb{R}$). Then $D \cdot F = 0$ if and only if

$$\begin{aligned}
&\partial_0 F_0 + \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3 = 0,\\
&\partial_1 F_0 - \partial_0 F_1 + \partial_2 F_3 - \partial_3 F_2 = 0,\\
&\partial_2 F_0 - \partial_0 F_2 - \partial_1 F_3 + \partial_3 F_1 = 0,\\
&\partial_3 F_0 - \partial_0 F_3 + \partial_1 F_2 - \partial_2 F_1 = 0.
\end{aligned} \tag{2.6.15}$$

Proof. Note that

$$\begin{aligned}
D \cdot F &:= \tfrac{1}{4}(\partial_0 - i\partial_1 - j\partial_2 - k\partial_3)(F_0 + iF_1 + jF_2 + kF_3)\\
&= \tfrac{1}{4}[(\partial_0 F_0 + \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) + i(\partial_0 F_1 - \partial_1 F_0 - \partial_2 F_3 + \partial_3 F_2)\\
&\quad + j(\partial_0 F_2 - \partial_2 F_0 - \partial_3 F_1 + \partial_1 F_3) + k(\partial_0 F_3 - \partial_3 F_0 - \partial_1 F_2 + \partial_2 F_1)].
\end{aligned}$$

Thus, if $D \cdot F = 0$, then the system (2.6.15) is satisfied and vice versa. ■

Theorem 2.6.2. *Let $\Phi : (M^4, g) \to (N^{4n}, h)$ be a smooth map between two smooth Riemannian manifolds. Assume (M^4, g) is locally conformally flat. If Φ is harmonic then the function*

$$\begin{aligned}
F_\Phi = F_0 + iF_1 + jF_2 + kF_3 &:= \tfrac{1}{32}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2)\\
&\quad + i\langle \Phi_0, \Phi_1\rangle + j\langle \Phi_0, \Phi_2\rangle + k\langle \Phi_0, \Phi_3\rangle
\end{aligned} \tag{2.6.16}$$

is antiregular in the sense of Fueter. Moreover, if Φ is a differentiable submersion almost everywhere and the function F_Φ is antiregular in the sense of Fueter, then Φ is harmonic.

Proof. Note that the system (2.6.12) can be written in the following very suggestive and condensed form:

$$\nabla_D F_\Phi = 0, \tag{2.6.17}$$

where $D = \frac{1}{4}(\partial_0 - i\partial_1 - j\partial_2 - k\partial_3)$ and F_Φ is the quaternion-valued function defined by (2.6.16). Since F_Φ is a scalar the equation (2.6.17) is equivalent to $D \cdot F_\Phi = 0$, which proves the theorem. ■

The above result is rather unexpected and it emphasizes the importance of the class of regular functions in the sense of Fueter.

Application to the 4-dimensional torus

Theorem 2.6.3. *On the torus $T^4 := \mathbb{R}^4/\Lambda$, where Λ is a lattice, consider a global linear coordinate system $q := x_0 + ix_1 + jx_2 + kx_3$. Then any harmonic map $\Phi : T^4 \to (N^{4n}, h)$*

satisfies

$$|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2 = a, \quad \langle\Phi_0, \Phi_1\rangle = b_1, \quad \langle\Phi_0, \Phi_2\rangle = b_2, \quad \langle\Phi_0, \Phi_3\rangle = b_3 \tag{2.6.18}$$

for suitable constants $a, b_1, b_2, b_3 \in \mathbb{R}$.

Proof. If Φ is harmonic then F_Φ is antiregular in the sense of Fueter. Let us recall that the properties of antiregular functions are analogous to those of regular functions in the sense of Fueter (see e.g. [16, 17, 31, 60]). Any antiregular function satisfies the maximum principle. Since the torus T^4 is compact, F_Φ has to be constant. Then by the definition of F_Φ we get (2.6.18), as required. ■

Consider a C^∞-mapping $\Phi : (T^4, g) \to (N^{4n}, h)$ and the Euclidean chart (x_0, x_1, x_2, x_3) on T^4. Then the derivatives of Φ with respect to the real variables $\Phi_0, \Phi_1, \Phi_2, \Phi_3$ can be considered as sections of the bundle $\Phi^{-1}TN^{4n}$.

COROLLARY 2.6.5. *If* $\Phi : (T^4, g) \to (N^{4n}, h)$ *is harmonic and non-constant then either* Φ_0 *or* (Φ_1, Φ_2, Φ_3) *has no zero on* T^4.

Proof. Indeed, otherwise at a point $p_1 \in T^4$, where $\Phi_0(p_1) = 0$, by Theorem 2.6.3, we would have

$$-|\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2 = a \le 0 \quad \text{and} \quad b_1 = b_2 = b_3 = 0$$

and at a point $p_2 \in T^4$ where $(\Phi_1(p_2), \Phi_2(p_2), \Phi_3(p_2)) = 0$ we would get

$$|\Phi_0|^2 = a \ge 0, \qquad b_1 = b_2 = b_3 = 0.$$

Then we would obtain

$$a = b_1 = b_2 = b_3 = 0.$$

This means that Φ would satisfy the following system of conditions:

$$\begin{cases} |\Phi_0|^2 = |\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2, \\ \langle\Phi_0, \Phi_1\rangle = \langle\Phi_0, \Phi_2\rangle = \langle\Phi_0, \Phi_3\rangle = 0 \end{cases} \tag{2.6.19}$$

at all points of T^4 with $\Phi_i := \partial\Phi/\partial x_i$, $i = 0, 1, 2, 3$.

But any smooth map $\Phi : (T^4, g) \to (N^{4n}, h)$ satisfying (2.6.19) has to be constant. Indeed, the system (2.6.19) must hold if we take instead of x_0 the variables x_1, x_2, x_3, respectively. Thus the system (2.6.19) has to be generalized to

$$\langle\Phi_i, \Phi_k\rangle = 0, \qquad i \ne k,\ i, k = 0, 1, 2, 3,$$

$$\begin{cases} |\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2 = 0, \\ |\Phi_1|^2 - |\Phi_0|^2 - |\Phi_2|^2 - |\Phi_3|^2 = 0, \\ |\Phi_2|^2 - |\Phi_0|^2 - |\Phi_1|^2 - |\Phi_3|^2 = 0, \\ |\Phi_3|^2 - |\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 = 0. \end{cases} \tag{2.6.20}$$

Note that the determinant of the system (2.6.20) satisfies

$$\det\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \neq 0.$$

Thus, Φ would be constant, which contradicts our assumption. ■

Let S^4 denote the 4-dimensional sphere.

COROLLARY 2.6.6. *Suppose that* $\Phi : (T^4, g) \to (S^4, h)$ *is a* C^∞*-mapping such that* $(\Phi_0, \Phi_1, \Phi_2, \Phi_3) \neq 0$ *on* T^4. *Then* $\deg(\Phi) = 0$.

Proof. The tangent bundle TS^4 is of rank 4. So, we can complexify this bundle. Denote by $TS^4_{\mathbb{C}}$ the complexification of TS^4. By the definition the bundle $\Phi^{-1}(TS^4_{\mathbb{C}})$ is also a complex vector bundle (now of rank 2).

Note that $(\Phi_0, \Phi_1, \Phi_2, \Phi_3)$ defines a non-zero section of the bundle $\Phi^{-1}TS^4$. So, if $(\Phi_0, \Phi_1, \Phi_2, \Phi_3) \neq 0$ on all T^4 then the bundle $\Phi^{-1}TS^4$ has a non-zero global section and this bundle is trivial.

Denote by $c_2(\Phi^{-1}TS^4_{\mathbb{C}})$ the second Chern class of the vector bundle $\Phi^{-1}(TS^4_{\mathbb{C}})$. Since the bundle $\Phi^{-1}(TS^4_{\mathbb{C}})$ is trivial we have

$$c_2(\Phi^{-1}TS^4_{\mathbb{C}}) = 0.$$

On the other hand (see [9, 48]),

$$c_2(TS^4_{\mathbb{C}}) = \frac{3}{4\pi^2} v_g^{S^4},$$

where $v_g^{S^4}$ is the volume form of S^4. Then we get

$$\begin{aligned} 0 &= \int\limits_{T^4} c_2(\Phi^{-1}TS^4_{\mathbb{C}}) = \int\limits_{T^4} \Phi^*[c_2(TS^4_{\mathbb{C}})] = \frac{3}{4\pi^2} \int\limits_{T^4} \Phi^*(v_g^{S^4}) \\ &= \frac{3}{4\pi^2} \deg(\Phi) \operatorname{Vol}(S^4) = 2\deg(\Phi) \end{aligned}$$

because by the definition of the topological degree we have

$$\int\limits_{T^4} \Phi^*(v_g^{S^4}) = \deg(\Phi) \operatorname{Vol}(S^4) \quad \text{and} \quad \operatorname{Vol}(S^4) = \frac{8}{3}\pi^2.$$

Thus any C^∞-mapping $\Phi : T^4 \to S^4$ with $(\Phi_0, \Phi_1, \Phi_2, \Phi_3) \neq 0$ has degree 0, as required. ■

III. Fueter–Hurwitz regular maps and Hurwitz pairs

III.1. Introduction. Let us recall the following well known list of results:

1. (Frobenius theorem, 1878) Any associative division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

2. (Hurwitz theorem, 1898) Any normed division algebra over $\mathbb{R}$ with unity is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.

3. (Skornyakov theorem, 1950; Bruck–Kleinfeld, 1951) Any alternative division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.

4. (Theorem of Kervaire, Bott–Milnor and Adams, 1958) Any division algebra over $\mathbb{R}$ has dimension 1, 2, 4 or 8.

These results gave rise to many interesting and important problems. One of them is the so-called "Hurwitz problem" (see e.g. [21, 22, 53, 54]). Originally it was stated as follows: for what values of s and n does there exist a formula

$$(x_1^2 + \ldots + x_s^2)(y_1^2 + \ldots + y_n^2) = z_1^2 + \ldots + z_n^2, \tag{3.1.1}$$

where $z_1, \ldots, z_n$ are homogeneous bilinear forms (with coefficients in a field $\mathbb{F}$) in the sets of variables $x_1, \ldots, x_s;\ y_1, \ldots, y_n$. The answer is well known when the field $\mathbb{F}$ (of characteristic $\neq 2$) is algebraically closed [21] or real closed [37]. For example, if $s = n$ then the only solution is $n = 1, 2, 4, 8$. The general Hurwitz problem, i.e. when we replace n on the right-hand side of (3.1.1) by an arbitrary r, is still open. The results of Hurwitz were the starting point for Ławrynowicz and Rembieliński to introduce the concept of the so-called Hurwitz pairs [45]. They developed the theory obtaining many interesting results [41–45]. Using the geometric concept of pseudo-Euclidean Hurwitz pairs, they gave their systematic classification in connection with real Clifford algebras. Moreover, they showed that the theory of Hurwitz pairs provided a convenient framework for some problems in mathematical physics (e.g. Dirac equation, Kałuża–Klein theories, spontaneous symmetry breaking and others).

In the theory of Hurwitz pairs an analogue to the classical Cauchy–Riemann equations can be defined. It is the generalized Fueter equation which we call the Fueter–Hurwitz equation. This equation first appeared in quaternionic analysis when searching for a proper notion of quaternionic holomorphy. The theory of mappings which are regular in the sense of Fueter is still being developed. In this chapter we find a special class of solutions of the Fueter–Hurwitz equation. Moreover, we define an index formula and we use it to derive a sufficiently general Cauchy integral formula for Fueter–Hurwitz regular mappings in the theory of Euclidean Hurwitz pairs. Such a formula is a generalization of the standard one known from complex analysis. The results obtained here are homogeneous dimensional and given as simple expressions in terms of the Hurwitz multiplication. They are generalizations of results established by Bartik, Ferreira, Markl and Souček [3] in the complex-quaternionic case and are related to analogous theorems of Brackx, Delanghe and Sommen [10] and Hestenes [20] for monogenic functions.

Further, we generalize the concept of supercomplex structure introduced by Ławrynowicz and Rembieliński [44] to pseudo-Euclidean Hurwitz pairs. We describe the basic algebraic and geometric properties of supercomplex structures and find a necessary and sufficient condition for their existence. We prove that if $O(n, k)$ denotes the orthogonal group preserving the norm $\|x\|^2 := x_1^2 + \ldots + x_n^2 - x_{n+1}^2 - \ldots - x_{n+k}^2$ then a complex structure J ($J \in O(n, k), J^2 = -I_{n+k}$, where I_{n+k} stands for the identity $(n+k)\times(n+k)$ matrix) exists if and only if n and k are even.

The concept of a supercomplex structure for Hurwitz pairs is strongly motivated by possible quantum mechanical applications of anisotropic Hilbert spaces (see e.g. [36, 43, 44]).

In the final section of this chapter we show the existence of a bilinear mapping $\star$:

$C(Q_S) \times C(Q_V) \to C(Q_V)$, where $(S, V, \circ)$ is a given Hurwitz pair, which makes the diagram

$$\begin{array}{ccc} S \times V & \xrightarrow{\circ \text{ (Hurwitz multiplication)}} & V \\ {\scriptstyle i_S \times i_V}\downarrow & & \downarrow{\scriptstyle i_V} \\ C(Q_S) \times C(Q_V) & \xrightarrow{\star = ?} & C(Q_V) \end{array}$$

commutative. Moreover, we prove that if such a mapping exists and satisfies the "algebraic Hurwitz condition"

$$N(x_S \star y_V) = N(x_S)N(y_V)$$

for any $x_S \in \Gamma_S$ and $y_V \in \Gamma_V$, where Γ denotes the Clifford group of the Clifford algebra $C(Q)$ and N is a spinor norm, then $\star$ is generated by the Hurwitz multiplication, i.e. $\star_{|S\times V} = \circ$. An example of a mapping $\star$ which does not satisfy the N-norm condition is given.

III.2. Hurwitz pairs—basic information. Let us recall the fundamental notions and basic results from the theory of Hurwitz pairs. More details can be found in [41–45].

Consider two real vector spaces S and V, equipped with non-degenerate pseudo-Euclidean real scalar products $(\ ,\ )_S$ and $(\ ,\ )_V$, respectively, with standard properties. For $f, g, h \in V$, $a, b, c \in S$ and $\alpha, \beta \in \mathbb{R}$ we suppose that

$$\begin{aligned} &(a,b)_S \in \mathbb{R}, && (f,g)_V \in \mathbb{R}, \\ &(b,a)_S = (a,b)_S, && (g,f)_V = (f,g)_V, \\ &(\alpha a,b)_S = \alpha(a,b)_S, && (\alpha f,g)_V = \alpha(f,g)_V, \\ &(a,b+c)_S = (a,b)_S + (a,c)_S, && (f,g+h)_V = (f,g)_V + (f,h)_V. \end{aligned} \tag{3.2.1}$$

In S and V we choose some bases (ε_α) and (e_j), respectively, with $\alpha = 1, \ldots, p = \dim S$, $j = 1, \ldots, n = \dim V$. We assume that $p \le n$. For

$$\eta \equiv [\eta_{\alpha\beta}] := [(\varepsilon_\alpha, \varepsilon_\beta)_S], \qquad \kappa \equiv [\kappa_{jk}] := [(e_j, e_k)_V] \tag{3.2.2}$$

by relations (3.2.1) we get immediately

$$\eta^{-1} \equiv [\eta^{\alpha\beta}], \quad \eta^T = \eta, \quad \kappa^{-1} \equiv [\kappa^{jk}], \quad \kappa^T = \kappa, \quad \det\ \eta \ne 0, \quad \det\ \kappa \ne 0.$$

Without any loss of generality we can choose the bases (ε_α) and (e_j) so that

$$\begin{aligned} \eta &= \operatorname{diag}(1, \overset{(r)}{\ldots}, 1, -1, \overset{(s)}{\ldots}, -1), \qquad r + s = p, \\ \kappa &= \operatorname{diag}(1, \overset{(l)}{\ldots}, 1, -1, \overset{(m)}{\ldots}, -1), \qquad l + m = n. \end{aligned} \tag{3.2.3}$$

Multiplication of elements of S by elements of V is defined as a mapping $\circ : S \times V \to V$ with the properties

(i) $(a + b) \circ f = a \circ f + b \circ f$ and $a \circ (f + g) = a \circ f + a \circ g$ for $f, g \in V$ and $a, b \in S$,
(ii) $(a, a)_S (f, g)_V = (a \circ f, a \circ g)_V$ (the generalized Hurwitz condition),
(iii) there exists a unit element ε_0 in S with respect to multiplication: $\varepsilon_0 \circ f = f$ for $f \in V$.

The $\mathbb{R}$-linearity of multiplication follows from (i): we have $\alpha(a \circ f) = a \circ (\alpha f)$ for $\alpha \in \mathbb{R}$. By (iii), multiplication of vectors of V by a real number α is identified with multiplication by $\alpha\varepsilon_0$.

The product $a \circ f$ is uniquely determined by the multiplication scheme for base vectors:

$$\varepsilon_\alpha \circ e_j = C^k_{\alpha j} e_k, \qquad \alpha = 1, \ldots, p,\ j, k = 1, \ldots, n. \tag{3.2.4}$$

Hereafter we shall require the *irreducibility* of the multiplication $S \times V \to V$, which means that it does not leave invariant proper subspaces of V. (This property is called *unsplittable* in [53] in the non-euclidean case.) In such a case we shall call (V, S) a *pseudo-Euclidean Hurwitz pair*.

In order to investigate the consequences of the most important condition (ii), let us recall that the generalized Hurwitz condition is equivalent to (see e.g. [45])

$$C_\alpha C^+_\beta + C_\beta C^+_\alpha = 2\eta_{\alpha\beta} I_n, \qquad \alpha, \beta = 1, \ldots, p = \dim S, \tag{3.2.5}$$

in matrix notation

$$C_\alpha := [C^k_{\alpha j}], \qquad C^+_\alpha := \kappa C^T_\alpha \kappa^{-1}, \tag{3.2.6}$$

where I_n stands for the identity $n \times n$ matrix.

Setting

$$C_\alpha := i\gamma_\alpha C_t, \qquad t \text{ fixed},\ \alpha = 1, \ldots, p,\ \alpha \neq t, \tag{3.2.7}$$

where i denotes the imaginary unit, we arrive at the following system equivalent to (3.2.5):

$$\begin{cases} C_t C^+_t = \eta_{tt} I_n, & t \text{ fixed}, \\ \gamma^+_\alpha = -\gamma_\alpha,\ \operatorname{Re}\gamma_\alpha = 0, & \gamma = 1, \ldots, p,\ \alpha \neq t, \\ \{\gamma_\alpha, \gamma_\beta\} = \gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\widetilde{\eta}_{\alpha\beta} I_n, & \alpha, \beta = 1, \ldots, p,\ \alpha, \beta \neq t, \end{cases} \tag{3.2.8}$$

where

$$\widetilde{\eta}_{\alpha\beta} = \eta_{\alpha\beta}/\eta_{tt}, \tag{3.2.9}$$

$\eta_{\alpha\beta}$ being chosen diagonal as in (3.2.3). Clearly, $\eta_{tt} = 1$ or -1.

From (3.2.3) it follows that γ_α are generators of the real Clifford algebra $C^{(r,s-1)}$ or $C^{(r-1,s)}$ with $(r, s-1)$ and $(r-1, s)$ determined by the signature of $\widetilde{\eta} := [\widetilde{\eta}_{\alpha\beta}]$ and by $r + s = p$.

Thus, following Ławrynowicz and Rembieliński [45], we have

LEMMA 3.2.1. *The problem of classifying the pseudo-Euclidean Hurwitz pairs* (V, S) *is equivalent to the classification problem for real Clifford algebras* $C^{(r,s)}$ *with purely imaginary and symmetric or antisymmetric generators* γ_α *according to* $\alpha \le r$ *or* $\alpha > r$, *given by the formulae*

$$\begin{cases} i\gamma_\alpha C_t = C_\alpha, & \alpha = 1, \ldots, r+s,\ \alpha \neq t, \\ C_t C^+_t = \eta_{tt} I_n, & t \text{ fixed}, \end{cases} \tag{3.2.10}$$

the matrices C_α *being determined by* (3.2.4) *and* (3.2.5). *The relationship is given by the formulae* (3.2.8) *and* (3.2.10).

COROLLARY 3.2.1 [45]. *Without any loss of generality, in Lemma* 3.2.1 *we may set* $C_t = I_n$ *and* $t = r$, *so that* $\eta_{tt} = 1$ *and* $\widetilde{\eta}_{\alpha\beta} = \eta_{\alpha\beta}$ *for* $\alpha, \beta \neq t$.

Lemma 3.2.2 [45]. *The pseudo-Euclidean Hurwitz pairs are of bidimension* (p, n), $p = \dim S = r' + (s' + 1)$, $n = \dim V$,

$$n = \begin{cases} 2^{[\frac{1}{2}p - \frac{1}{2}]} & \text{for } r' - s' \equiv 6, 7, 0 \pmod 8, \\ 2^{[\frac{1}{2}p + \frac{1}{2}]} & \text{for } r' - s' \equiv 1, 2, 3, 4, 5 \pmod 8, \end{cases}$$

where $[\,\cdot\,]$ *stands for "integer part of".*

Problem 3.2.1. Find all pairs of matrices (A, B), $A \in O(p)$, $B \in O(n)$, which preserve the Hurwitz multiplication, i.e. $As \circ Bv = s \circ v$ for any $s \in S$ and $v \in V$.

It is easy to prove

Lemma 3.2.3. *If* (A, B) *preserves the Hurwitz multiplication then it has to satisfy the following condition*:

$$(\star) \qquad A^{\alpha}_{\beta}(BC_{\alpha}) = C_{\beta}, \quad \alpha, \beta = 1, \ldots, p.$$

Denote by $G(S, V)$ the set of all pairs $(A, B) \in O(p) \times O(n)$ satisfying $(\star)$. Then we have

Lemma 3.2.4. *The set* $G(S, V)$ *with the multiplication*

$$(A_1, B_1)(A_2, B_2) = (A_1A_2, B_1B_2)$$

is a group with unit (I_p, I_n).

III.3. Fueter–Hurwitz equation. Given a pseudo-Euclidean Hurwitz pair $(V, S, \circ)$, let us consider a continuously differentiable V-valued mapping f with a domain Ω in S. It can be represented as

$$f(x) = f(x^{\alpha}\varepsilon_{\alpha}) = f^{k}(x)e_{k}.$$

Following Ławrynowicz and Rembieliński [45] we define the *generalized Fueter operator* D^+ as

$$D^+ := \varepsilon_{\alpha}\partial^{\alpha}, \quad \partial^{\alpha} = \frac{\partial}{\partial x_{\alpha}}, \quad x_{\alpha} = \eta_{\alpha\beta}x^{\beta}, \quad \alpha = 1, \ldots, p. \tag{3.3.1}$$

A mapping f as above is said to be *Fueter-type regular*, or *regular*, for short, in its domain, if it satisfies

$$D^+ \circ f = 0. \tag{3.3.2}$$

Theorem 3.3.1 [45]. *Any generalized Fueter operator* D^+ *has a conjugate* $D := \varepsilon^{+}_{\alpha}\partial^{\alpha}$, *where* ε^{+}_{α} *are defined by*

$$\varepsilon^{+}_{\alpha} \circ e_j = C^{+k}_{\alpha j},$$

with the property: *if* f *is regular*, *then* $D \circ f$ *is regular*, *so* $D \circ D^+ \circ f = D^+ \circ D \circ f$. *Moreover*, $D \circ D^+ = D^+ \circ D = \eta_{\alpha\beta}\partial^{\alpha}\partial^{\beta}$.

Ławrynowicz and Rembieliński showed that the generalized Fueter equation (3.3.2) can be transformed to a very suggestive form

$$\Big[\sum_{\alpha \neq r}(-i\gamma_{\alpha}\partial^{\alpha}) + I_r\partial^r\Big]\Psi = 0 \tag{3.3.3}$$

resembling the Dirac equation. Precisely they obtained

Theorem 3.3.2 [45]. *A continuously differentiable V-valued mapping $f = f^k e_k$ with domain in S is a solution of* (3.3.2) *if and only if the related mapping $\Psi = \kappa(f^1, \ldots, f^n)^T$ is a solution of* (3.3.3).

Corollary 3.3.1 [45]. *Any Dirac-like equation* (3.3.3) *implies the continuity equation*

$$\partial^r Q + \partial^\alpha j_\alpha = 0,$$

where

$$Q = (\overline{\Psi}, \Psi)_V, \quad j_\alpha = -i\Psi^+ \gamma_\alpha \Psi, \quad \Psi^+ = \Psi^T \kappa^{-1}.$$

Let us express the Fueter equation (3.3.2) in a more convenient form. By (3.2.4) we obtain

$$\begin{aligned}(D^+ \circ f = 0) &\Leftrightarrow \Big(\sum_{\alpha=1}^{p}\sum_{k=1}^{n}(\varepsilon_\alpha \partial^\alpha) \circ (f^k e_k) = 0\Big)\\ &\Leftrightarrow \Big(\sum_{\alpha,k}(\partial^\alpha f^k)(\varepsilon_\alpha \circ e_k) = 0\Big) \Leftrightarrow \Big(\sum_{\alpha,k,m}(\partial^\alpha f^k) C^m_{\alpha k} e_m = 0\Big)\\ &\Leftrightarrow \Big(\sum_{k=1}^{n}\sum_{m=1}^{n}\sum_{\alpha=1}^{p-1}[(\partial^\alpha f^k) C^m_{\alpha k} e_m + (\partial^p f^k) C^m_{pk} e_m] = 0\Big).\end{aligned}$$

By a suitable change of coordinates we can take $C_p = I_n$, where I_n is the identity $n \times n$ matrix (see e.g. [45]); then the last equation is equivalent to

$$\sum_{k=1}^{n}\sum_{\alpha=1}^{p-1}(\partial^\alpha f^k)\varepsilon_\alpha \circ e_k + \sum_{k=1}^{n}(\partial^p f^k)\varepsilon_p \circ e_k = 0.$$

Hence

$$(D^+ \circ f = 0) \Leftrightarrow \Big(\Big[\sum_{\alpha=1}^{p-1}\varepsilon_\alpha \partial^\alpha + \varepsilon_p \partial^p\Big] \circ = 0\Big).$$

Henceforth, the following equation will be called the *Fueter–Hurwitz equation* (F-H equation):

$$\Big(\sum_{\alpha=1}^{p-1}\varepsilon_\alpha \partial^\alpha + \varepsilon_p \partial^p\Big) \circ f = 0. \tag{3.3.4}$$

III.4. Special polynomial solutions of the Fueter–Hurwitz equation

Problem 3.4.1. Find non-constant solutions of the F-H equation.

Define a mapping $F_1 : S \to V$ given by

$$F_1(x,t) = F_1(x, \vec{t}_1, \ldots, \vec{t}_n) := \sum_{k=1}^{n}\Big(\sum_{\beta=1}^{p-1} t^\beta_k \varepsilon_\beta\Big) \circ x^p e_k - \sum_{k-1}^{n}\Big(\sum_{\gamma=1}^{p-1} t^\gamma_k x^\gamma\Big) e_k, \tag{3.4.1}$$

where $\vec{t}_k = (t^\beta_k)_{\beta=1,\ldots,p-1} \in \mathbb{R}^{p-1}$, $k = 1, \ldots, n$, are arbitrary given parameters. Since $\varepsilon_\beta \circ e_k = C^m_{\beta k} e_m$, the mapping F_1 is well defined for any system of parameters $(\vec{t}_1, \ldots, \vec{t}_n) \in \mathbb{R}^{p-1} \times \overset{(n)}{\cdots} \times \mathbb{R}^{p-1}$.

Let us notice that

$$\partial^p F_1 = \sum_{k=1}^{n} \Big(\sum_{\beta=1}^{p-1} t_k^{\beta} \varepsilon_\beta \Big) \circ e_k, \quad \partial^\alpha F_1 = - \sum_{k=1}^{n} t_k^{\alpha} e_k, \quad \alpha = 1, \ldots, p-1.$$

Hence

$$\begin{aligned} D^+ \circ F_1 &= \Big(\sum_{\alpha=1}^{p-1} \varepsilon_\alpha \partial^\alpha + \varepsilon_p \partial^p \Big) \circ F_1 = \sum_{\alpha=1}^{p-1} \varepsilon_\alpha \circ \Big(- \sum_{k=1}^{n} t_k^\alpha e_k \Big) + \sum_{k=1}^{n} \Big(\sum_{\beta=1}^{p-1} t_k^\beta \varepsilon_\beta \Big) \circ e_k \\ &= - \sum_{\alpha=1}^{p-1} \sum_{k=1}^{n} \varepsilon_\alpha t_k^\alpha \circ e_k + \sum_{\beta=1}^{p-1} \sum_{k=1}^{n} \varepsilon_\beta t_k^\beta \circ e_k = 0, \end{aligned}$$

so F_1 satisfies the F-H equation. We say that the mapping F_1 is a *special solution* to the Fueter–Hurwitz equation. The solution F_1 is a homogeneous polynomial of degree 1 in the variables $x^\gamma, \gamma = 1, \ldots, p$. We now look for polynomials F_m of degree $m > 1$ in the variables x^γ, which are solutions of the F-H equation.

First of all notice that if we define

$$\vec{t}_k := \sum_{\alpha=1}^{p-1} t_k^\alpha \varepsilon_\alpha, \quad k = 1, \ldots, n; \quad \vec{x} := \sum_{\beta=1}^{p-1} x^\beta \varepsilon_\beta,$$

then $F_1(x, \vec{t}_1, \ldots, \vec{t}_n)$ can be rewritten in the form

$$F_1(x, \vec{t}_1, \ldots, \vec{t}_n) = \sum_{k=1}^{n} (x^p \vec{t}_k - \langle \vec{t}_k, \vec{x} \rangle \varepsilon_p) \circ e_k, \tag{3.4.2}$$

where $\langle \cdot, \cdot \rangle$ stands for the usual scalar product in $\mathbb{R}^{p-1}$. Using the above notation define the following polynomial of degree 2:

$$F_2(x, \vec{t}_1, \ldots, \vec{t}_n) := \sum_{k=1}^{n} [(x^p)^2 \vec{t}_k \vec{t}_k - 2x^p \vec{t}_k \langle \vec{t}_k, \vec{x} \rangle + \langle \vec{t}_k, \vec{x} \rangle^2 \varepsilon_p] \circ e_k, \tag{3.4.3}$$

where $\vec{t}_k \vec{t}_k \circ e_k := \vec{t}_k \circ (\vec{t}_k \circ e_k), \ \vec{t}_k \in S$.

PROPOSITION 3.4.1. *F_2 is a well-defined mapping from S into V for any system $(\vec{t}_1, \ldots, \vec{t}_n) \in S \times \ldots \times S$ and satisfies the F-H equation.*

Proof. Indeed, we have

$$\partial^\alpha F_2 = \sum_{k=1}^{n} [2 \langle \vec{t}_k, \vec{x} \rangle t_k^\alpha \varepsilon_p - 2x^p \vec{t}_k t_k^\alpha] \circ e_k, \qquad \partial^p F_2 = \sum_{k=1}^{n} [2x^p \vec{t}_k \vec{t}_k - 2\vec{t}_k \langle \vec{t}_k, \vec{x} \rangle] \circ e_k.$$

Hence

$$\begin{aligned} D^+ \circ F_2 &= \sum_{\alpha=1}^{p-1} \varepsilon_\alpha \partial^\alpha F_2 + \varepsilon_p \partial^p F_2 \\ &= \sum_{\alpha=1}^{p-1} \sum_{k=1}^{n} [2 \langle \vec{t}_k, \vec{x} \rangle \varepsilon_\alpha t_k^\alpha - 2x^p \varepsilon_\alpha t_k^\alpha \vec{t}_k] \circ e_k + \sum_{k=1}^{n} [2x^p \vec{t}_k \vec{t}_k - 2\vec{t}_k \langle \vec{t}_k, \vec{x} \rangle] \circ e_k \\ &= \sum_{k=1}^{p} [2 \langle \vec{t}_k, \vec{x} \rangle \vec{t}_k - 2x^p \vec{t}_k \vec{t}_k] \circ e_k + \sum_{k=1}^{n} 2x^p \vec{t}_k \vec{t}_k - 2\vec{t}_k \langle \vec{t}_k, \vec{x} \rangle] \circ e_k = 0. \ \blacksquare \end{aligned}$$

For each $m = 1, 2, \dots,$ introduce

$$(3.4.4)\qquad \begin{cases} (\vec{t}_k)^{\circ m} := \vec{t}_k \overset{(m)}{\dots} \vec{t}_k, \\ (\vec{t}_k)^{\circ m} \circ e_k := \vec{t}_k \overset{(m-1)}{\dots} \vec{t}_k \circ (\vec{t}_k \circ e_k), \\ (\varepsilon_p)^{\circ m} = \varepsilon_p, \quad \varepsilon_p \circ e_k = e_k, \quad \vec{t}_k \varepsilon_p = \vec{t}_k, \quad k = 1, \dots, n. \end{cases}$$

Define

$$(3.4.5)\qquad F_m(x, \vec{t}_1, \dots, \vec{t}_n) = \sum_{k=1}^{n} [x^p \vec{t}_k - \langle \vec{t}_k, \vec{x} \rangle \varepsilon_p]^{\circ m} \circ e_k.$$

By (3.4.4), $F_m(x, \vec{t}_1, \dots, \vec{t}_n)$ are well-defined mappings from S into V for any system $(\vec{t}_1, \dots, \vec{t}_n) \in \mathbb{R}^{p-1} \times \overset{(n)}{\dots} \times \mathbb{R}^{p-1}$ for $m = 1, 2, \dots$ They are polynomials of degrees $m = 1, 2, \dots,$ respectively, in the variable $x \in S$.

LEMMA 3.4.1. *$F_m(x, \vec{t}_1, \dots, \vec{t}_n)$, defined by* (3.4.5), *are regular mappings*, *i.e. they satisfy the F-H equation for $m = 1, 2, \dots$*

Proof. By induction we get

$$\partial^\alpha F_m = -m \sum_{k=1}^{n} t_k^\alpha [x^p \vec{t}_k - \langle \vec{t}_k, \vec{x} \rangle \varepsilon_p]^{\circ m-1} \circ e_k,$$

$$\partial^p F_m = m \sum_{k=1}^{n} [x^p \vec{t}_k - \langle \vec{t}_k, \vec{x} \rangle \varepsilon_p]^{\circ m-1} \vec{t}_k \circ e_k.$$

Hence $D^+ \circ F_m(x, \vec{t}_1, \dots, \vec{t}_n) = 0$. ∎

By expanding the right-hand side of (3.4.5) in powers of t_k^α we obtain

$$(3.4.6)\qquad F_m(x, t) = \sum_{\Sigma_k m_k^\alpha = m} m! P_{m_1^\alpha \dots m_k^\alpha}(x) (t_1^\alpha)^{m_1^\alpha} \dots (t_k^\alpha)^{m_k^\alpha},$$

where

$$m_k^\alpha := m_k^1 + \dots + m_k^{p-1}, \qquad k = 1, \dots, n,$$

$$(t_l^\alpha)^{m_l^\alpha} := (t_l^1)^{m_l^1} \dots (t_l^{p-1})^{m_l^{p-1}}, \qquad l = 1, \dots, n,$$

and $P_{m_1^\alpha \dots m_k^\alpha}(x)$ are polynomials of degree m in $x^1, \dots, x^p$ with values in V.

On applying D^+ to the left-hand side of (3.4.6) and putting $\vec{t}_1, \dots, \vec{t}_n$ as independent variables we see that $P_{m_1^\alpha \dots m_k^\alpha}(x)$ are regular polynomials:

$$D^+ \circ P_{m_1^\alpha \dots m_k^\alpha}(x) = 0.$$

The exponential form $\exp[i(Ex_0 + \langle \vec{p}, \vec{x} \rangle]$ plays an important role in the solutions of the wave equation. Analogously, we introduce the following mapping:

$$(3.4.7)\qquad \exp\Big\{ i \Big[\sum_{k=1}^{n} (x^p \vec{t}_k - \langle \vec{t}_k, \vec{x} \rangle \varepsilon_p) \circ e_k \Big] \Big\}$$

$$:= \sum_{s=0}^{\infty} \frac{(i)^s}{s!} \Big[\sum_{k=1}^{n} (x^p \vec{t}_k - \langle \vec{t}_k, \vec{x} \rangle \varepsilon_p) \circ e_k \Big] = \sum_{s=0}^{\infty} \frac{(i)^s}{s!} F_s(x, t).$$

Of course, the function exp introduced above is a well-defined mapping from S into $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. It is also obvious that the exponential mapping is regular in the sense of

Fueter–Hurwitz:

$$D^{+}\circ\exp\Big\{i\Big[\sum_{k=1}^{n}(x^{p}\vec{t}_{k}-\langle\vec{t}_{k},\vec{x}\rangle\varepsilon_{p})\circ\Big]\Big\}=0,$$

because of the usual properties of the exponential map, since the individual terms $F_s(x,t)$ are regular.

III.5. Fourier representation of Fueter–Hurwitz regular mappings. The mapping (3.4.7) is a special solution of the F-H equation:

$$D^{+}\circ f=0,\qquad F:S\to V^{\mathbb{C}}.$$

A general solution of the F-H equation should be given by superposition of the special solutions (3.4.7) integrated over the parameters $\vec{t}_1,\dots,\vec{t}_n$, $\vec{t}_k=(t_k^1,\dots,t_k^{p-1})$, $k=1,\dots,n$:

$$\Phi(\mathbf{x})=\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}A(\vec{t}_1,\dots,\vec{t}_n)\exp\Big\{i\Big[\sum_{k-1}^{n}(x^{p}\vec{t}_{k}-\langle\vec{t}_{k},\vec{x}\rangle\varepsilon_{p})\circ e_k\Big]\Big\}\,d\mathbf{t},\tag{3.5.1}$$

where

$$d\mathbf{t}:=d\vec{t}_1\dots d\vec{t}_n:=dt_1^1\dots dt_1^{p-1}dt_2^1\dots dt_2^{p-1}\dots dt_n^1\dots dt_n^{p-1},$$

and A is a C^{∞}-mapping from $\mathbb{R}^{p-1}\times\overset{(n)}{\dots}\times\mathbb{R}^{p-1}$ into $\mathbb{R}$ satisfying the condition

$$\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}|A(\vec{t}_1,\dots,\vec{t}_n)|\,d\mathbf{t}<\infty.$$

It is clear that (3.5.1) is a solution of the F-H equation.

As is seen from (3.5.1), $\Phi(\mathbf{x})$ is expressed by a Fourier-type integral. By separating the factors containing x^p and $\vec{x}$, (3.5.1) can be written as

$$\Phi(\vec{x})=\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}A(\vec{t}_1,\dots,\vec{t}_n)e^{ix^{p}\Sigma_k\vec{t}_k\circ e_k}e^{-i\Sigma_k\langle\vec{t}_k,\vec{x}\rangle e_k}\,d\mathbf{t},\tag{3.5.2}$$

Putting $x^p=0$ into (3.5.2) the initial condition for $\Phi(\mathbf{x})$ is given by

$$\Phi_{|x^p=0}:=G(\vec{x})=\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}A(\vec{t}_1,\dots,\vec{t}_n)e^{-i\Sigma_k\langle\vec{t}_k,\vec{x}\rangle e_k}\,d\mathbf{t}.\tag{3.5.3}$$

It is clear that $G:S_{|x^p=0}\to V^{\mathbb{C}}$. The components $G^m(\vec{x})$ of $G(\vec{x})=G^m(\vec{x})e_m$ are as follows:

$$G^{m}(\vec{x})=\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}A(\vec{t}_1,\dots,\vec{t}_n)e^{-i\langle\vec{t}_m,\vec{x}\rangle}\,d\mathbf{t},\tag{3.5.4}$$

which are nothing but Fourier-type integrals. Indeed, by the definition (3.4.7) we have

$$\begin{aligned}e^{-i\Sigma_k\langle\vec{t}_k,\vec{x}\rangle e_k}&=\sum_{s=0}^{\infty}\frac{(-i)^s}{s!}\Big[\sum_{k=1}^{n}\langle\vec{t}_k,\vec{x}\rangle^{s}e_k\Big]\\&=\sum_{k=1}^{n}\Big[\sum_{s=0}^{\infty}\frac{(-i)^s}{s!}\langle\vec{t}_k,\vec{x}\rangle^{s}\Big]e_k=\sum_{k=1}^{n}e^{-i\langle\vec{t}_k,\vec{x}\rangle}e_k.\end{aligned}$$

Suppose that the initial condition $G(\vec{x})$ is given. We will try to determine $A(\vec{t}_1,\dots,\vec{t}_n)$. To do this assume that

$$A(\vec{t}_1,\dots,\vec{t}_n)=A_1(\vec{t}_1)\dots A_n(\vec{t}_n),\qquad \int_{-\infty}^{\infty} A_k(\vec{t}_k)d\vec{t}_k=1,\quad k=1,\dots,n. \tag{3.5.5}$$

Then

$$G^m(\vec{x})=\int_{-\infty}^{\infty} A_m(\vec{t}_m)e^{-i\langle\vec{t}_m,\vec{x}\rangle}d\vec{t}_m,\qquad m=1,\dots,n. \tag{3.5.6}$$

Applying the Fourier integral theorem to (3.5.6) we have

$$A_m(\vec{t}_m)=\frac{1}{(2\pi)^{p-1}}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} G^m(\vec{y}_m)e^{i\langle\vec{t}_m,\vec{y}_m\rangle}d\vec{y}_m, \tag{3.5.7}$$

where $d\vec{y}_m=dy_m^1\dots dy_m^{p-1}$. Hence, we obtain

$$A(\vec{t}_1,\dots,\vec{t}_n)=\frac{1}{(2\pi)^{n(p-1)}}\int_{-\infty}^{\infty} G^1(\vec{y}_1)\dots G^n(\vec{y}_n)e^{i\Sigma_{k=1}^n\langle\vec{t}_k,\vec{y}_k\rangle}\,d\vec{y}_1\dots d\vec{y}_n. \tag{3.5.8}$$

Inserting $A(\vec{t}_1,\dots,\vec{t}_n)$ given by (3.5.5), (3.5.7) and (3.5.8) into (3.5.2) we obtain

$$\begin{aligned}\varPhi(\mathbf{x})&\\ &=\frac{1}{(2\pi)^{n(p-1)}}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} G^1(\vec{y}_1)\dots G^n(\vec{y}_n)e^{i\Sigma_{k=1}^n\langle\vec{t}_k,\vec{y}_k\rangle}e^{i\Sigma_{l=1}^n(x^p\vec{t}_l-\langle\vec{t}_l,\vec{x}\rangle)\circ e_l}\,d\mathbf{y}\,d\mathbf{t}.\end{aligned} \tag{3.5.9}$$

THEOREM 3.5.1. *The $\varPhi(\mathbf{x})$ given by* (3.5.9) *satisfies the F-H equation and the initial condition*

$$\varPhi(\mathbf{x})_{|x^p=0}=\frac{1}{(2\pi)^{n(p-1)}}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} A(\vec{t}_1,\dots,\vec{t}_n)e^{-i\Sigma_{k=1}^n\langle\vec{t}_k,\vec{x}\rangle e_k}\,d\mathbf{t}=G(\vec{x}),$$

provided that $A(\vec{t}_1,\dots,\vec{t}_n)$ satisfies condition (3.5.5).

III.6. Integral representation of Fueter–Hurwitz regular mappings. Let us recall the notion of index and Cauchy integral formula from complex analysis:

If f is holomorphic in an open set $\varOmega\subset\mathbb{C}$, γ is a cycle in $\varOmega$ and $z_0\notin\langle\gamma\rangle$, then

$$\mathrm{Ind}_\gamma(z_0)\cdot f(z_0)=\frac{1}{2\pi i}\int_\gamma dz\,\frac{f(z)}{z-z_0}=\frac{1}{2\pi i}\int_\gamma dz\,\frac{\overline{z-z_0}}{\|z-z_0\|^2}f(z),$$

where

$$\mathrm{Ind}_\gamma(z_0):=\frac{1}{2\pi i}\int_\gamma dz\,\frac{1}{z-z_0}=\frac{1}{2\pi i}\int_\gamma dz\,\frac{\overline{z-z_0}}{\|z-z_0\|^2}.$$

To introduce an analogue of this formula in the theory of Hurwitz pairs we assume henceforth that

$$\eta_{\alpha\beta}=\delta_{\alpha\beta},\quad \alpha,\beta=1,\dots,p,$$

where $\delta_{\alpha\beta}$ denotes the Kronecker symbol. Then by Theorem 3.3.1 we have

$$D \circ D^+ = D^+ \circ D = \Delta,$$

where Δ denotes the usual Laplacian in $\mathbb{R}^p$.

Recall that the Hurwitz multiplication $\circ : S \times V \to V$ is not associative, but since the product is in V, we can multiply on the left by vectors in S as many times as we need.

LEMMA 3.6.1. *We have*

$$d := \tfrac{1}{2}(D \circ dx + dx^+ \circ D^+) = dx^\alpha \partial^\alpha,$$

where $dx = \varepsilon_\alpha dx^\alpha,\ dx^+ = \varepsilon^+_\beta dx^\beta$.

Proof. Let us remark that

$$D \circ dx \circ f = dx^\alpha (\partial^\beta f^k)[C_\alpha C^+_\beta]^r_k e_r, \quad dx^+ \circ D^+ \circ f = dx^\alpha (\partial^\beta f^k)[C_\beta C^+_\alpha]^r_k e_r$$

for any differentiable mapping $f : \Omega \to V,\ \Omega \subset S$ open. Then we obtain

$$\begin{aligned}(D \circ dx + dx^+ \circ D^+) \circ f &= dx^\alpha (\partial^\beta f^k)[C_\alpha C^+_\beta + C_\beta C^+_\alpha]^r_k e_r \\ &= 2dx^\alpha (\partial^\beta f^k)\delta_{\alpha\beta} e_k = 2(dx^\alpha \partial^\alpha) f. \ \blacksquare\end{aligned}$$

Remark 3.6.1. The operator d is the usual de Rham operator acting on the components of f.

In the complex case the cycle $\gamma := \exp(it),\ t \in [0, 2\pi]$, is a generator of the first homology group of $\mathbb{C}\backslash\{0\}$ and the equality $k = \mathrm{Ind}_\gamma(0)$ denotes that γ is homologous to $k \cdot \gamma$. We want to have a similar situation in the general case.

Now, let $l = l^k e_k \in V,\ l^k \in \mathbb{R},\ k = 1, \ldots, n$. Define for $p \geq 2$ the function $f_0 : S\backslash\{0\} \to V$ by

$$f_0(x) = l/\|x\|^{p-2}. \tag{3.6.1}$$

Then

$$g^+(x) := D \circ f_0 = -(p-2) \sum_{k,m=1}^{n} \sum_{\alpha=1}^{p} \frac{x^\alpha l^k}{\|x\|^p} C^{+m}_{\alpha k} e_m = -(p-2)\frac{x^+}{\|x\|^p} \circ l,$$

$$g(x) := D^+ \circ f_0 = -(p-2) \sum_{k,m=1}^{n} \sum_{\alpha=1}^{p} \frac{x^\alpha l^k}{\|x\|^p} C^{m}_{\alpha k} e_m = -(p-2)\frac{x}{\|x\|^p} \circ l,$$

where $x^+ := x^\alpha \varepsilon^+_\alpha$.

Since the function $1/\|x\|^{p-2}$ is harmonic in $\mathbb{R}^p \setminus \{0\}$, it is clear that

$$D \circ D^+ \circ f_0 = D^+ \circ D \circ f_0 = (\Delta f^k_0) e_k = \Delta f_0 = 0, \tag{3.6.2}$$

so $D \circ g = D^+ \circ g^+ = 0$.

Introduce the following $(p-1)$-form in S:

$$K_x := \sum_{i=1}^{p} (-1)^{i-1} \varepsilon_i dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^p,$$

where "$\widehat{\cdot}$" denotes that the corresponding term is omitted.

DEFINITION 6.1. Let Γ be a $(p-1)$-cycle in $S \setminus \{x_0\}$. Define the *index* of x_0 with respect to Γ and any fixed $l \in V$ by

$$\mathrm{Ind}_{\Gamma,l}(x_0) := \frac{1}{\sigma_{p-1}} \int_{\Gamma} \frac{(x-x_0)^+}{\|x-x_0\|^p} \circ K_x \circ l,$$

where σ_{p-1} denotes the $(p-1)$-volume of the $(p-1)$-unit sphere S_{p-1}.

LEMMA 3.6.2. $\mathrm{Ind}_{S^r_{p-1},l}(0) = l$, *where* S^r_{p-1} *is the* $(p-1)$*-sphere of radius* r (*with the induced orientation*) *taken as a cycle in* $S \setminus \{0\}$.

Proof. It can be checked by direct computation that

$$K_x = \frac{x}{\|x\|} d\sigma, \tag{3.6.3}$$

where $d\sigma$ is the standard surface element of the sphere. Indeed, the sphere S^r_{p-1} is described by the equation $F(x^1, x^2, \ldots, x^p) = (x^1)^2 + (x^2)^2 + \ldots + (x^p)^2 - r^2 = 0$.

From the classical formula we have

$$\int_{S^r_{p-1}} K_x \circ f(x) = \int_{S^r_{p-1}} [n_0 \circ f(x)]\, d\sigma,$$

where

$$n_0 := \frac{\sum_{i=1}^p (\partial F/\partial x^i)\varepsilon_i}{\sqrt{\sum_{j=1}^p (\partial F/\partial x^j)^2}} = \frac{x}{\|x\|}.$$

Using (3.6.3) we get

$$\begin{aligned}\mathrm{Ind}_{S_{p-1},e_k}(0) &= \frac{1}{\sigma_{p-1}} \int_{S^r_{p-1}} \frac{x^+}{\|x\|^p} \circ K_x \circ e_k = \frac{1}{\sigma_{p-1}} \int_{S^r_{p-1}} \frac{x^+}{\|x\|^p} \circ \frac{x}{\|x\|} \circ e_k d\sigma \\ &= \frac{1}{\sigma_{p-1} r^{p+1}} \int_{S^r_{p-1}} (x^\alpha \varepsilon^+_\alpha) \circ (x^\beta \varepsilon_\beta) \circ e_k\, d\sigma \\ &= \frac{1}{\sigma_{p-1} r^{p+1}} \int_{S^r_{p-1}} x^\alpha x^\beta C^r_{\beta m} C^{+m}_{\alpha k} e_r\, d\sigma.\end{aligned}$$

On the other hand, interchanging the indices α and β, we get

$$\mathrm{Ind}_{S^r_{p-1},e_k}(0) = \frac{1}{\sigma_{p-1} r^{p+1}} \int_{S^r_{p-1}} x^\alpha x^\beta [C^r_{\alpha m} C^{+m}_{\beta k}]^r_k e_r\, d\sigma.$$

Hence

$$\begin{aligned}2\,\mathrm{Ind}_{S^r_{p-1},e_k}(0) &= \frac{1}{\sigma_{p-1} r^{p+1}} \int_{S^r_{p-1}} x^\alpha x^\beta [C_\beta C^+_\alpha + C_\alpha C^+_\beta]^r_k e_r\, d\sigma \\ &= \frac{2}{\sigma_{p-1} r^{p+1}} \int_{S^r_{p-1}} x^\alpha x^\beta \delta_{\alpha\beta} e_k\, d\sigma \\ &= \frac{2}{\sigma_{p-1} r^{p+1}} \int_{S^r_{p-1}} \|x\|^2 e_k\, d\sigma = \frac{2}{\sigma_{p-1} r^{p+1}} \int_{S^r_{p-1}} e_k\, d\sigma = 2e_k.\end{aligned}$$

Finally, by linearity, $\mathrm{Ind}_{S^r_{p-1}}(0) = l$. ∎

If, in Definition 6.1, we restrict ourselves to only one basis vector, for example to e_1 and assume that $e_1 = 1$ and $n = p = 2$, then we obtain full analogy with the complex case.

LEMMA 3.6.3. *For any continuously differentiable mapping $f : \Omega \to V$, $\Omega \subset S$ open, we have*

$$d \circ \frac{x^+}{\|x\|^p} \circ K_x \circ f = \omega \frac{x^+}{\|x\|^p} \circ D^+ \circ f,$$

where $\omega = dx^1 \wedge \ldots \wedge dx^p$.

Proof. It is easy to check that

$$\frac{x^+}{\|x\|^p} \circ K_x \circ f = \sum_{\alpha,i}^{p} \sum_{k,s=1}^{n} (-1)^{i-1} dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^p \frac{x^\alpha f^k}{\|x\|^p} [C_i C_\alpha^+]_k^s e_s.$$

Then we have

$$\begin{aligned} D \circ dx \circ \frac{x^+}{\|x\|^p} \circ K_x \circ f \\ &= (\varepsilon_\gamma^+ \partial^\gamma) \circ (\varepsilon_\beta dx^\beta) \circ \left\{ (-1)^{i-1} dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^p \frac{x^\alpha f^k}{\|x\|^p} [C_i C_\alpha^+]_k^s e_s \right\} \\ &= \omega \sum_\beta \sum_{\alpha,\gamma,k,t} \partial^\gamma \left(\frac{x^\alpha f^k}{\|x\|^p} \right) [C_\beta C_\alpha^+ C_\beta C_\gamma^+]_k^t e_t. \end{aligned}$$

Analogously we get

$$dx^+ \circ D^+ \circ \frac{x^+}{\|x\|^p} \circ K_x \circ f = \omega \sum_\beta \sum_{\alpha,\gamma,k,t} \partial^\gamma \left(\frac{x^\alpha f^k}{\|x\|^p} \right) [C_\beta C_\alpha^+ C_\gamma C_\beta^+]_k^t e_t.$$

Hence

$$\begin{aligned} 2d \circ \frac{x^+}{\|x\|^p} \circ K_x \circ f &= \omega \sum_\beta \sum_{\alpha,\gamma,k,t} \partial^\gamma \left(\frac{x^\alpha f^k}{\|x\|^p} \right) [C_\beta C_\alpha^+ (C_\beta C_\gamma^+ + C_\gamma C_\beta^+)]_k^t e_t \\ &= 2\omega \sum_\beta \sum_{\alpha,\gamma,k,t} \partial^\gamma \left(\frac{x^+ f^k}{\|x\|^p} \right) [C_\beta C_\alpha^+]_k^t e_t \delta_{\beta\gamma} \\ &= 2\omega \sum_{\alpha,\beta,k,t} \partial^\beta \left(\frac{x^\alpha f^k}{\|x\|^p} \right) [C_\beta C_\alpha^+]_k^t e_t. \end{aligned}$$

Since

$$\begin{aligned} \partial^\beta \left(\frac{x^\alpha f^k}{\|x\|^p} \right) &= \partial^\beta \left(\frac{x^\alpha}{\|x\|^p} \right) f^k + \frac{x^\alpha}{\|x\|^p} (\partial^\beta f^k), \\ \partial^\beta \left(\frac{x^\alpha}{\|x\|^p} \right) &= \frac{\delta_{\beta\alpha} \|x\|^p - p x^\alpha x^\beta \|x\|^{p-2}}{\|x\|^{2p}}, \end{aligned}$$

we obtain

(3.6.4) $$2d\circ\frac{x^+}{\|x\|^p}\circ K_x\circ f = 2\omega\bigg(\frac{\delta_{\beta\alpha}\|x\|^p-px^\alpha x^\beta\|x\|^{p-2}}{\|x\|^{2p}}\bigg)f^k[C_\beta C_\alpha^+]_k^t e_t+2\omega\frac{x^\alpha}{\|x\|^p}(\partial^\beta f^k)[C_\beta C_\alpha^+]_k^t e_t.$$

Interchanging the indices α and β we get

(3.6.5) $$2d\circ\frac{x^+}{\|x\|^p}\circ K_x\circ f = 2\omega\bigg(\frac{\delta_{\beta\alpha}\|x\|^p-px^\alpha x^\beta\|x\|^{p-2}}{\|x\|^{2p}}\bigg)f^k[C_\alpha C_\beta^+]_k^t e_t+2\omega\frac{x^\beta}{\|x\|^p}(\partial^\alpha f^k)[C_\alpha C_\beta^+]_k^t e_t.$$

By (3.6.4) and (3.6.5) we have

$$\begin{aligned}
4d&\circ\frac{x^+}{\|x\|^p}\circ K_x\circ f\\
&=2\omega\bigg(\frac{\delta_{\beta\alpha}\|x\|^p-px^\alpha x^\beta\|x\|^{p-2}}{\|x\|^{2p}}\bigg)f^k[C_\alpha C_\beta^+ + C_\beta C_\alpha^+]_k^t e_t+4\omega\frac{x^\alpha}{\|x\|^p}(\partial^\beta f^k)[C_\beta C_\alpha^+]_k^t e_t\\
&=4\omega\bigg(\frac{\delta_{\beta\alpha}\|x\|^p-px^\alpha x^\beta\|x\|^{p-2}}{\|x\|^{2p}}\bigg)\delta_{\alpha\beta}f^k e_k+4\omega\frac{x^\alpha}{\|x\|^p}(\partial^\beta f^k)[C_\beta C_\alpha^+]_k^t e_t\\
&=4\omega\bigg(\frac{\delta_{\beta\alpha}\|x\|^p-px^\alpha x^\beta\|x\|^{p-2}}{\|x\|^{2p}}\bigg)f+4\omega\frac{x^\alpha}{\|x\|^p}(\partial^\beta f^k)[C_\beta C_\alpha^+]_k^t e_t\\
&=4\omega\frac{x^\alpha}{\|x\|^p}(\partial^\beta f^k)[C_\beta C_\alpha^+]_k^t e_t.
\end{aligned}$$

Finally, since we have

$$\frac{x^+}{\|x\|^p}\circ D^+\circ f=\frac{x^\alpha}{\|x\|^p}(\partial^\beta f^k)[C_\beta C_\alpha^+]_k^t e_t,$$

the result follows. ■

LEMMA 3.6.4. *If $f:\Omega\to V$, $\Omega\subset S$ open, is a regular mapping of Fueter type, then the form*

$$\frac{x^+}{\|x\|^p}\circ K_x\circ f$$

is closed with respect to the operator $d=\frac{1}{2}(D\circ dx+dx^+\circ D^+)$.

P r o o f. The lemma follows from Lemma 3.5.3. ■

R e m a r k 3.6.2. The form $\frac{x^+}{\|x\|^p}\circ K_x\circ l$ is closed, so the integral $\int_\Gamma\frac{(x-x_0)^+}{\|x-x_0\|^p}\circ K_x\circ l$ depends only on the homology class of Γ in $S\setminus\{x_0\}$.

THEOREM 3.6.2. *Let Γ be a $(p-1)$-cycle in S, $x_0\neq\langle\Gamma\rangle$. Then*

i) $\mathrm{Ind}_{\Gamma,l}(x_0)\in\mathbb{Z}l$,

ii) $\mathrm{Ind}_{\Gamma,l}(x_0)$, *as a function of x_0, is constant on every connected component of $S\setminus\langle\Gamma\rangle$,*

where $\mathbb{Z}$ denotes the set of integers.

Proof. (i) The $(p-1)$-dimensional unit sphere S_{p-1} is a deformation retract of $S\setminus\{0\}$ (see e.g. [59]), hence

$$H_{p-1}(S\setminus\{0\}) = H_{p-1}(S_{p-1}) = \mathbb{Z}.$$

This implies simultaneously that S_{p-1} is a generator of $H_{p-1}(S\setminus\{0\})$, so for every $(p-1)$-cycle Γ in $H_{p-1}(S\setminus\{0\})$ there exists $k\in\mathbb{Z}$ such that Γ is homologous to kS_{p-1} in $S\setminus\{0\}$. Hence

$$\mathrm{Ind}_{\Gamma,l} = \mathrm{Ind}_{kS_{p-1},l}(0) = k\,\mathrm{Ind}_{S_{p-1},l}(0) = kl.$$

The last equality follows from the linearity of the index in l and the result proved in Lemma 3.6.2.

(ii) The function $\mathrm{Ind}_{\Gamma,l}(x_0)$ has to be constant on every connected component of $S\setminus\langle\Gamma\rangle$, because it is continuous and discrete valued. ∎

Observation. If $\mathrm{Ind}_{\Gamma,l}(x_0) = kl$, since k does not depend on $l\neq 0$, we can define $\mathrm{Ind}_\Gamma(x_0) := k$, so $\mathrm{Ind}_{\Gamma,l}(x_0) = \mathrm{Ind}_\Gamma(x_0)l$.

Remark 3.6.3. If $f:\Omega\to V$, $\Omega\subset S$ open, is a regular mapping of Fueter type, then the form $K_x\circ f$ is closed with respect to the operator $d=\frac{1}{2}(D\circ dx + dx^+\circ D^+)$.

Hence, we have

THEOREM 3.6.2 (Cauchy). *If f is a regular mapping in a domain $\Omega\subset S, f:\Omega\to V$ and Γ is a $(p-1)$-dimensional cycle homologous to zero in Ω, then*

$$\int_\Gamma K_x\circ f(x) = 0.$$

THEOREM 3.6.3 (Cauchy Integral Formula). *Let $f:\Omega\to V$, $\Omega\subset S$ open, be a regular mapping in the sense of Fueter–Hurwitz and Γ a $(p-1)$-cycle homologically trivial in Ω. Then for every $x_0\in\Omega\setminus\langle\Gamma\rangle$ we have*

$$\mathrm{Ind}_\Gamma(x_0)f(x_0) = \frac{1}{\sigma_{p-1}}\int_\Gamma \frac{(x-x_0)^+}{\|x-x_0\|^p}\circ K_x\circ f(x).$$

Proof. Define $k_0 f(x_0) = \mathrm{Ind}_{\Gamma,f(x_0)}(x_0) = \mathrm{Ind}_\Gamma(x_0)f(x_0)$ and consider a sufficiently small $(p-1)$-sphere S^r_{p-1} with center x_0 and radius r. Then Γ is homologous in $\Omega\setminus\{x_0\}$ to $k_0S^r_{p-1}$ for some $k_0\in\mathbb{Z}$. Since $\frac{x^+}{\|x\|^p}\circ K_x\circ f(x)$ is a closed form by Lemma 3.6.4, integration over Γ can be replaced by integration over $k_0S^r_{p-1}$. So

$$\frac{1}{\sigma_{p-1}}\int_\Gamma \frac{(x-x_0)^+}{\|x-x_0\|^p}\circ K_x\circ f(x) = \frac{k_0}{\sigma_{p-1}}\int_{S^r_{p-1}} \frac{(x-x_0)^+}{\|x-x_0\|^p}\circ K_x\circ f(x).$$

The right-hand integral can be written as

$$\begin{aligned}\frac{k_0}{\sigma_{p-1}}\int_{S^r_{p-1}} \frac{(x-x_0)^+}{\|x-x_0\|^p}\circ K_x\circ f(x) &= \frac{k_0}{\sigma_{p-1}}\int_{S^r_{p-1}} \frac{(x-x_0)^+}{\|x-x_0\|^p}\circ K_x\circ f(x_0)\\ &\quad+\frac{k_0}{\sigma_{p-1}}\int_{S^r_{p-1}} \frac{(x-x_0)^+}{\|x-x_0\|^p}\circ K_x\circ [f(x)-f(x_0)]\\ &= I_1+I_2.\end{aligned}$$

By Lemma 3.6.2,

$$I_1 = \mathrm{Ind}_{\Gamma,f(x_0)}(x_0).$$

As in complex function theory, it is easy to see that $I_2 = 0$. The integrand in I_2 is closed in $\Omega \setminus \{x_0\}$, so I_2 does not depend on r for r small enough. Since $f(x)$ is continuous at x_0, for any $\varepsilon > 0$, there exists r so that $\|f(x) - f(x_0)\| < \varepsilon$ for $\|x - x_0\| < r$. Take $\varepsilon \to 0$ and the result follows as in Lemma 3.6.2, since the integral in the theorem is linear and continuous in f. ∎

COROLLARY 3.6.1. *The maximum principle and the mean value property hold for Fueter–Hurwitz regular mappings.*

Proof. Direct consequences of the previous theorem. ∎

The above facts are examples showing that the Fueter–Hurwitz analysis is a generalization of the complex and quaternionic Fueter analyses.

III.7. Anisotropic complex structure on the pseudo-Euclidean Hurwitz pairs. Let us set

DEFINITION 3.7.1. A *Hurwitz type vector space* $\mathbf{E}$ on (V,κ) is a p-dimensional subspace of the space $\mathrm{End}(V,\kappa)$ $(\dim \mathrm{End}\, V = \dim V)$ of endomorphisms of (V,κ), which consists of all endomorphisms E not leaving invariant proper subspaces of V, with the property

$$(Ef, Ef)_V = \|E\|^2 (f,f)_V \quad \text{for } f \in V,\ E \in \mathbf{E}, \tag{3.7.1}$$

where $\|E\| := (\mathrm{Tr}\, E^T E)^{1/2}$, $E^T E$ being considered in an arbitrary matrix representation of E in an orthonormal basis (e_j) of V. We assume that $\mathbf{E}$ contains the identity endomorphism E_0.

Consider next a system (γ_α) of $p-1$ imaginary $n \times n$ matrices determined by the formulae

$$\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\widehat{\eta}_{\alpha\beta} I_n, \quad \alpha,\beta = 1,\dots,p,\ \alpha,\beta \neq t,$$
$$\gamma_\alpha^+ = -\gamma_\alpha, \quad \mathrm{Re}\,\gamma_\alpha = 0, \quad \alpha = 1,\dots,p,\ \alpha \neq t,$$
$$\gamma_\alpha^+ := \kappa\gamma_\alpha^T\kappa^{-1},$$

where I_n is the identity $n\times n$ matrix and $\widehat{\eta}_{\alpha\beta}$ is determined by (3.2.8). Then the matrices γ_α generate a real Clifford algebra. Choose the basic endomorphisms $\{E_0, E_\alpha\}$, $\alpha = 1,\dots,p$, $\alpha \neq t$, in $\mathbf{E}$ so that

$$E_0 e_j = e_j, \quad E_\alpha e_j = i\gamma_{j\alpha}^k e_k, \quad \alpha = 1,\dots,p,\ \alpha \neq t,\ j,k = 1,\dots,n, \tag{3.7.2}$$

where i denotes the imaginary unit. The choice (3.7.2) is motivated by

LEMMA 3.7.1. *The endomorphisms E_0, E_α satisfy the relations*

$$E_0 = E_I, \quad E_\alpha e_j = C_{j\alpha}^k e_k, \quad E_I \textit{ is the identity endomorphism in } \mathbf{E}, \tag{3.7.3}$$

for $\alpha = 1,\dots,p$, $\alpha \neq t$, $j,k = 1,\dots,n$, where $C_{j\alpha}^k$ can be chosen as

$$C_\alpha = i\gamma_\alpha, \quad \alpha = 1,\dots,p,\ \alpha \neq t, \quad C_t = I_n.$$

Proof. The lemma follows directly from (3.2.7) and Corollary 3.1.1. ∎

Consider a fixed direction in $\mathbf{E}$ determined by the endomorphisms $E_\alpha,\ \alpha = 1,\dots,p$, $\alpha \neq t$. Define

$$\widetilde{n} := \sum_{\alpha=1,\,\alpha\neq t}^{p} E_\alpha n^\alpha, \qquad \sum_{\alpha,\beta=1,\,\alpha,\beta\neq t}^{p} \widehat{\eta}_{\alpha\beta} n^\alpha n^\beta = 1, \tag{3.7.4}$$

where (n^α) is a system of $p-1$ real numbers. Then we have

LEMMA 3.7.2. *The endomorphisms E_0 and $\tilde{n}$ replace* 1 *and i of C in the field of "numbers" $qE_0 + s\widetilde{n}$, where $q, s \in \mathbb{R}$:*

$$E^2 = E_0, \quad E_0\widetilde{n} = \widetilde{n}E_0 = \widetilde{n}, \quad \widetilde{n}^2 = -E_0. \tag{3.7.5}$$

Proof. We only prove the third equality. Notice that

$$\widetilde{n}^2(e_j) = \widetilde{n}(\widetilde{n}e_j) = E_\beta n^\beta (E_\alpha n^\alpha) e_j = -n^\alpha n^\beta \gamma^k_{j\alpha}\gamma^m_{k\beta} e_m = -n^\alpha n^\beta \{\gamma_\alpha, \gamma_\beta\}^m_j e_m.$$

On the other hand, we have $\widetilde{n}^2(e_j) = -n^\beta n^\alpha \{\gamma_\beta, \gamma_\alpha\}^m_j e_m$. Using the above equalities we obtain

$$\begin{aligned} 2\widetilde{n}^2(e_j) &= -n^\alpha n^\beta [\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha]^m_j e_m = -2n^\alpha n^\beta \widehat{\eta}_{\alpha\beta}[I_n]^m_j e_m \\ &= -2(n^\alpha n^\beta \widehat{\eta}_{\alpha\beta})\delta^m_j e_m = -2e_j = -2E_0(e_j). \end{aligned}$$

Hence $\widetilde{n}^2 = -E_0$, as required. ■

The endomorphism $\widetilde{n}$ is represented in the basis (e_j) by the matrix

$$J = in^\alpha \gamma_\alpha.$$

Now, we shall show some important properties of this matrix.

Remark 3.7.1. $J^2 = -I_n$.

Proof. On the one hand, by the definition we have

$$J^2 = (in^\alpha\gamma_\alpha)(in^\beta\gamma_\beta) = -n^\alpha n^\beta \gamma_\alpha\gamma_\beta.$$

On the other hand, changing the indices, we get $J^2 = -n^\beta n^\alpha \gamma_\beta\gamma_\alpha$. Thus

$$2J^2 = -n^\alpha n^\beta[\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha] = -2n^\alpha n^\beta \widehat{\eta}_{\alpha\beta} I_n = -2I_n. \ ■$$

Let us denote by $O(k,l)$ the group of orthogonal transformations of the space (V,κ) ($\kappa = \mathrm{diag}(1, \overset{(k)}{\dots}, 1, -1, \overset{(l)}{\dots}, -1)$). It is well known that a matrix B belongs to $O(k,l)$ if and only if

$$B^T\kappa B = \kappa \quad \text{or} \quad B\kappa B^T = \kappa. \tag{3.7.6}$$

By the definition of conjugation "+", given in (3.2.5), the above condition is equivalent to

$$B^+B = I_n \quad \text{or} \quad BB^+ = I_n.$$

Remark 3.7.2. $J \in O(k,l)$.

Proof. Directly by the definition of J we have

$$J\kappa J^T = -n^\alpha n^\beta \gamma_\alpha \kappa \gamma^T_\beta.$$

By (3.2.7) ($\gamma^+_\alpha = -\gamma_\alpha$) we get $\kappa\gamma^T_\beta\kappa^{-1} = -\gamma_\beta$. Thus

$$J\kappa J^T = n^\alpha n^\beta \gamma_\alpha\gamma_\beta\kappa.$$

On the other hand, changing the indices, we obtain

$$J\kappa J^T = n^\beta n^\alpha \gamma_\beta \gamma_\alpha \kappa.$$

Thus

$$2J\kappa J^T = n^\alpha n^\beta [\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma \alpha]\kappa = 2n^\alpha n^\beta \widehat{\eta}_{\alpha\beta} I_n \kappa = 2\kappa. \ \blacksquare$$

The standard complex structure in the Euclidean space E_n is the endomorphism represented by the matrix

$$J_0 = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}.$$

It is clear that $J_0 \in O(n)$.

Remark 3.7.3. For each pair (k, l) of positive integers such that $k + l = n$ we have $J_0 \notin O(k, l)$.

Proof. It suffices to show that $J_0\kappa \neq \kappa J_0$. Otherwise, we would have $J_0 \kappa J_0^T = \kappa J_0 J_0^T = \kappa$ and J_0 would belong to $O(k, l)$.

We divide our proof into three parts.

I. $k = l = n/2$. In this case we have

$$J_0\kappa = \begin{pmatrix} 0 & -I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}, \qquad \kappa J_0 = \begin{pmatrix} 0 & I_{n/2} \\ I_{n/2} & 0 \end{pmatrix},$$

so $J_0\kappa \neq \kappa J_0$.

II. $k < n/2$. Then

$$J_0\kappa = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \begin{pmatrix} I_k & & 0 \\ & -I & \\ 0 & & -I_{n/2} \end{pmatrix} = \begin{pmatrix} 0 & & -I_{n/2} \\ -I_k & & \\ & & 0 \\ & I & \end{pmatrix},$$

$$\kappa J_0 = \begin{pmatrix} & I_k & \\ 0 & & \\ & & -I \\ I_{n/2} & & 0 \end{pmatrix},$$

where I denotes $I_{n/2-k}$, so in this case $J_0\kappa \neq \kappa J_0$ as well.

III. $k > n/2$. Then

$$J_0\kappa = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \begin{pmatrix} I_{n/2} & & 0 \\ & I & \\ 0 & & \\ & & -I_l \end{pmatrix} = \begin{pmatrix} & & I & \\ & 0 & & \\ & & & -I_l \\ -I_{n/2} & & & 0 \end{pmatrix},$$

$$\kappa J_0 = \begin{pmatrix} 0 & & I_{n/2} & \\ -I & & & \\ & & & 0 \\ & I_l & & \end{pmatrix}.$$

where I denotes $I_{n/2-l}$. Again $J_0\kappa \neq \kappa J_0$. This completes the proof. $\blacksquare$

Problem 3.7.1. For which pairs (k, l) of positive integers does there exist a matrix $J \in O(k, l)$ satisfying $J^2 = -I_n$, $n = k + l$?

We are looking for a matrix $J \in \mathcal{M}(n)$ which satisfies

$$\text{a) } J^T \kappa J = \kappa, \quad \text{b) } J^2 = -I_n. \tag{3.7.7}$$

Notice that the above conditions are equivalent to

$$\text{a) } (\kappa J)^T = -\kappa J, \quad \text{b) } J^2 = -I_n. \tag{3.7.8}$$

LEMMA 3.7.3. *Let*

$$\kappa = \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix}, \qquad k, l \neq 0.$$

If $\mathbf{B} \in O(k,l)$, *then*

1. $\mathbf{B}$ *is of the form*

$$\mathbf{B} = \begin{pmatrix} A & C_1 \\ C_2 & B \end{pmatrix}, \tag{3.7.9}$$

where $A \in \mathcal{M}(k)$, $A \neq 0$; $B \in \mathcal{M}(l)$, $B \neq 0$; $C_1 \in \mathcal{M}(l \times k)$, $C_2 \in \mathcal{M}(k \times l)$ *and the following conditions are satisfied*:

$$\begin{aligned} &\text{a) } A^T A - C_2^T C_2 = I_k, && \text{b) } A^T C_1 - C_2^T B = 0, \\ &\text{c) } C_1^T A - B^T C_2 = 0, && \text{d) } B^T B - C_1^T C_1 = I_l. \end{aligned} \tag{3.7.10}$$

2. $\det \mathbf{B} = \pm 1$.

Proof. The condition 2) is a straightforward consequence of (3.7.6). To prove 1) assume that $\mathbf{B}$ is of the form (3.7.9). Then

$$\mathbf{B}^T = \begin{pmatrix} A^T & C_2^T \\ C_1^T & B^T \end{pmatrix}. \tag{3.7.11}$$

By (3.7.6), we have, say,

$$\begin{pmatrix} A^T & C_2^T \\ C_1^T & B^T \end{pmatrix} \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix} \begin{pmatrix} A & C_1 \\ C_2 & B \end{pmatrix} = \begin{pmatrix} A^T A - C_2^T C_2 & A^T C_1 - C_2^T B \\ C_1^T A - B^T C_2 & C_1^T C_1 - B^T B \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix}.$$

This is nothing but (3.7.10).

Assume that $A = 0$. Then by (3.7.10a) we would have $C_2^T C_2 = -I_k$. If $(a_1, \ldots, a_l)$ is the first column of C_2, then we would get $a_1^2 + \ldots + a_l^2 = -1$, which is impossible. Thus $A \neq 0$. Analogously, we show that $B \neq 0$. ■

THEOREM 3.7.1. *Let* κ *be as in Lemma* 3.7.3. *If* $J \in O(k,l)$ *and* J *satisfies* $J^2 = -I_n$, $n = k + l$, *then*

1) J *has the form*

$$J = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{3.7.12}$$

where $A \in \mathcal{M}(k)$, $A \neq 0$, $A^T = -A$; $B \in \mathcal{M}(l)$, $B \neq 0$, $B^T = -B$; $C \in \mathcal{M}(l \times k)$, *and the matrices* A, B, C *satisfy* (3.7.10) *with* $C_1 = C_2 = C$.

2) *The integers* k *and* l *are even.*

P r o o f. By the assumptions, J satisfies (3.7.7a) so we have

$$(\kappa J)^r_s = -(\kappa J)^s_r, \quad \sum_{m=1}^{n} \kappa^r_m J^m_s = -\sum_{w=1}^{n} \kappa^s_w J^w_r \quad \text{for } r,s=1,\dots,n.$$

Since κ is a diagonal matrix, the above equality is equivalent to

$$\kappa^r_r J^r_s = -\kappa^s_s J^s_r \quad \text{for } r,s=1,\dots,n. \tag{3.7.13}$$

By the assumption $\kappa = \mathrm{diag}(1,\overset{(k)}{\dots},1,-1,\overset{(l)}{\dots},-1)$, so by (3.7.13) we get:

I. If $r \le k$, $s \le k$ then $J^r_s = -J^s_r$.

II. If $r > k$, $s > k$ then $J^r_s = -J^s_r$.

III. If $r \le k$, $s > k$ then $J^r_s = J^s_r$.

IV. If $r > k$, $s \le k$ then $J^r_s = J^s_r$.

We conclude that J has the form (3.7.12). Thus

$$J^T = \begin{pmatrix} -A & C \\ C^T & -B \end{pmatrix}.$$

Using (3.7.8) we get

$$J^T \kappa J = \begin{pmatrix} -A^2 - CC^T & -AC - CB \\ C^T A + BC^T & C^T C + B^2 \end{pmatrix}$$

and

$$J^2 = \begin{pmatrix} A^2 + CC^T & AC + CB \\ C^T A + BC^T & C^T C + B^2 \end{pmatrix}.$$

Thus A, B, C satisfy (3.7.10) with $C_1 = C_2 = C$. Analogously to Lemma 3.7.3 we prove that $A, B \neq 0$.

In order to prove the second assertion of our theeorem we assume that k and l are odd ($k+l=n$, and by Lemma 3.2.2, n is always even). Since A and B are antisymmetric, we then have

$$\det A = \det B = 0. \tag{3.7.14}$$

We now show that (3.7.14) contradicts (3.7.10). Indeed, with the matrix A^2 we can associate a quadratic form F_{A^2} defined by $F_{A^2}(x,x) := \langle x, A^2 x\rangle$, where $\langle , \rangle$ denotes the usual scalar product. By (3.7.10a) we have

$$\begin{aligned} F_{A^2}(x,x) &= \langle x, (-I_k - CC^T)x\rangle = \langle x, -x - CC^T x\rangle = \langle x, -x\rangle - \langle x, CC^T x\rangle \\ &= -\|x\|^2 - \langle C^T x, C^T x\rangle = -\|x\|^2 - \|C^T x\|^2 < 0 \end{aligned}$$

for $x \neq 0$. The form F_{A^2} is thus negative definite, so $\det A^2 < 0$, which contradicts (3.7.14). ■

R e m a r k 3.7.4. If k and l are even integers ($k+l=n,\ k,l \neq 0$), then the matrix $J \in O(k,l)$ satisfying $J^2 = -I_n$ can be chosen as follows:

$$J = J^0 := \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 & 0 & \ldots & 0 \\ 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}. \tag{3.7.15}$$

Of course, $(J^0)^T = -J^0$.

Denote by $\mathcal{F}$ the family of all matrices $A \in \mathcal{M}(n)$ satisfying one of the equivalent conditions:

$$A^+ = -A, \quad \kappa A^T \kappa^{-1} = -A, \quad (A\kappa)^T = -(A\kappa),$$

where $\kappa^T = \kappa = \kappa^{-1}$.

Remark 3.7.5. Any $A \in \mathcal{F}$ satisfies

$$\operatorname{Tr} A = 0. \tag{3.7.16}$$

Proof. Indeed,

$$(A\kappa)^i_j = \sum_{m=1}^{n} A^i_m \kappa^m_j = A^i_j \kappa^j_j$$

because κ is diagonal. Now, since $A\kappa$ is antisymmetric, we get

$$0 = (A\kappa)^j_j = A^j_j \kappa^j_j \Rightarrow A^j_j = 0. \ \blacksquare$$

Corollary 3.7.1. *The matrices* $\gamma_\alpha,\ \alpha = 1, \ldots, p,\ \alpha \neq t$, *determined by* (3.2.6)–(3.2.8), *belong to* $\mathcal{F}$.

Corollary 3.7.2. *If* $\gamma_\alpha,\ \alpha = 1, \ldots, p, \alpha \neq t$, *are the matrices described by* (3.2.6)–(3.2.8) *and* (n^α) *is an arbitrary system of* $p-1$ *real numbers satisfying the condition* $\sum_{\alpha,\beta=1;\alpha,\beta\neq t}^{p} \widehat{\eta}_{\alpha\beta} n^\alpha n^\beta = 1$, *then*

$$\operatorname{Tr}(in^\alpha \gamma_\alpha) = 0. \tag{3.7.17}$$

Here the following problem arises:

Problem 3.7.2. Determine all matrices $C_\alpha,\ \alpha = 1, \ldots, p$, satisfying (3.2.4).

Lemma 3.7.4. *The general formula describing the admissible matrices* C'_α *satisfying* (3.2.4) *is*

$$C'_\alpha = \sum_\beta O^\beta_\alpha R C_\beta R^{-1}, \tag{3.7.18}$$

where $O \in O(\widehat{\eta}),\ R \in O(\kappa)$.

Proof. The matrices C_α only depend on the choice of bases in S and V. We shall show how the matrices C_α transform with the change of bases. Let

$$\varepsilon'_\alpha = O^\beta_\alpha \varepsilon_\beta, \quad e'_j = R^k_j e_k, \quad R \in O(\kappa), \quad O \in O(\widehat{\eta}),$$

and

$$F(\varepsilon'_\alpha, e_j) = C'^k_{\alpha j} e'_k.$$

Then

$$F(O_\alpha^\beta \varepsilon_\beta, R_j^k e_k) = C'^k_{\alpha j} R_k^m e_m,$$
$$O_\alpha^\beta R_j^k F(\varepsilon_\beta, e_k) = C'^k_{\alpha j} R_k^m e_m,$$
$$O_\alpha^\beta R_j^k C_{\beta k}^l e_l = C'^k_{\alpha j} R_k^m e_m.$$

Since $R \in O(\kappa)$, it follows that $\kappa R^T \kappa^{-1} = R^{-1},\ \kappa^{-1} = \kappa$, and

$$R_k^m (\kappa R^T \kappa)_m^w = \delta_k^w.$$

Thus

$$O_\alpha^\beta R_j^k C_{\beta k}^l e_l = C'^k_{\alpha j} R_k^m \delta_m^l e_l,$$
$$O_\alpha^\beta R_j^k C_{\beta k}^l = C'^k_{\alpha j} R_k^l.$$

Now, we multiply both sides by $(\kappa R^T \kappa)_l^s$:

$$O_\alpha^\beta R_j^k C_{\beta k}^l (\kappa R^T \kappa)_l^s = C'^k_{\alpha j} R_k^l (\kappa R^T \kappa)_l^s = C'^k_{\alpha j} \delta_k^s = C'^s_{\alpha j},$$
$$O_\alpha^\beta [R C_\beta \kappa R^T \kappa]_j^s = C'^s_{\alpha j},$$
$$O_\alpha^\beta R C_\beta R^{-1} = C'_\alpha,$$

as required. It is easy to see that if the matrices (C_α) satisfy (3.2.4) then so do the (C'_α). ■

COROLLARY 3.7.3. *The general formula describing the admissible matrices γ'_α satisfying* (3.2.7) *is*

$$\gamma'_\alpha = O_\alpha^\beta R \gamma_\beta R^{-1}, \tag{3.7.19}$$

where $R \in O(\kappa),\ O \in O(\widehat{\eta})$.

COROLLARY 3.7.4. *If (n^α) is an arbitrary system of numbers satisfying* (3.7.4) *and $\gamma_\alpha,\ \alpha = 1, \ldots, p,\ \alpha \neq t$, is an arbitrary system of matrices determined by* (3.2.6)–(3.2.8) *then, changing the base in the space (V, κ) by means of an orthogonal transformation $R \in O(\kappa)$, we have the following formula for the admissible matrices $J' \in O(k, l)$ satisfying $(J')^2 = -I_n$, $n = k + l$:*

$$J' = RJR^{-1},$$

where $J = in^\alpha \gamma_\alpha$.

Now, fix matrices $\gamma_\alpha,\ \alpha = 1, \ldots, p,\ \alpha \neq t$, and a system of $p-1$ real numbers (n^α) satisfying (3.7.4). Denote by $\mathrm{Or}(J^0) := \{M \in \mathcal{M}(n) :\ M = RJ^0R^{-1},\ R \in O(\kappa)\}$ the $O(\kappa)$-orbit of the matrix J^0. Further, let $\mathrm{Or}(J)$ denote the $O(\kappa)$-orbit of $J = in^\alpha \gamma_\alpha$. Let us compute the moments of J^0 and J. We have

$$\mathrm{Tr}\, J^{2k} = \mathrm{Tr}(J^2)^k = \mathrm{Tr}\,(-I_n)^k = (-1)^k\, \mathrm{Tr}\, I_n = n(-1)^k,$$
$$\mathrm{Tr}(J^0)^{2k} = \mathrm{Tr}(J^0)^2 = \mathrm{Tr}(-I_n)^k = n(-1)^k \quad \text{for } k = 1, \ldots, n/2.$$

Analogously, by Corollary 3.7.2, we have

$$\mathrm{Tr}\, J^{2k+1} = \mathrm{Tr}(J^{2k} \cdot J) = \mathrm{Tr}(-J) = 0$$

and, since J^0 is antisymmetric,

$$\mathrm{Tr}(J^0)^{2k+1} = \mathrm{Tr}(-J^0) = 0.$$

The matrices J and J^0 have the same moments, so they belong to the same orbit of $O(\kappa)$:

$$\mathrm{Or}(J^0) = \mathrm{Or}(J).$$

LEMMA 3.7.5. *Let n and p be positive integers determined by Lemma* 3.1.2, $n > 1$. *Then, with any system (n^α) of $p-1$ real numbers satisfying* (3.7.4) *we can associate a system of γ_α, $\alpha = 1, \ldots, p$, $\alpha \neq t$, of imaginary $n \times n$ matrices satisfying* (3.2.7) *so that*

$$in^\alpha \gamma_\alpha = J^0. \tag{3.7.20}$$

Proof. By the considerations preceding Lemma 3.7.5, for any system (n^α) of $p-1$ real numbers satisfying (3.7.4) and for any system (γ_α) of imaginary $n \times n$ matrices satisfying (3.2.7) the matrices $J = in^\alpha\gamma_\alpha$ and J^0 belong to the same $O(\kappa)$-orbit. Consequently, by the transitivity of the action of $O(\kappa)$ in this orbit, for each system of (n^α) in question there exists an orthogonal transformation of one matrix into the other and so the proof is complete. ∎

Let us pose the following problem:

PROBLEM 3.7.3. Describe the orbit $O(\kappa) \cdot J^0$.

Let Ω and Ω' belong to $O(\kappa) \cdot J^0$. Then $\Omega = AJ^0A^{-1}$, $\Omega' = BJ^0B^{-1}$, where $A, B \in O(\kappa)$. Notice that

$$(\Omega = \Omega') \Leftrightarrow [(A^{-1}B)J^0(A^{-1}B)^{-1} = J^0].$$

Introduce the following relation in $O(\kappa)$:

$$(A \sim B) \Leftrightarrow [(A^{-1}B)J^0(A^{-1}B)^{-1} = J^0].$$

It is clear that this is an equivalence relation. Then the set of different matrices Ω in the orbit $O(\kappa) \cdot J^0$ is isomorphic to the group $O(\kappa)/\sim \equiv O(\kappa)S(J^0)$, where $S(J^0) := \{A \in O(\kappa) : AJ^0A^{-1} = J^0\}$ is the stability group of J^0.

Let us recall that the endomorphism $\widetilde{n}$ is represented in the basis (e_j) by the matrix

$$J = in^\alpha \gamma_\alpha, \tag{3.7.21}$$

where

$$J = RJ^0R^{-1} \tag{3.7.22}$$

for some $R \in O(\kappa)$.

DEFINITION 3.7.2. The endomorphism $\widetilde{n}$ described by (3.2.3), (3.2.7), (3.7.3) and (3.7.4) will be called a *supercomplex structure* on (V, κ).

This definition is motivated by

LEMMA 3.7.6. *If a supercomplex structure $\widetilde{n}$ exists, then*

$$(Re)_{2j} = J(Re)_{2j-1} = \widetilde{n}(Re)_{2j-1}, \quad (Re)_{2j-1} = -J(Re)_{2j} = -\widetilde{n}(Re)_{2j}, \tag{3.7.23}$$

for $j = 1, \ldots, n/2$ and for some $R \in O(\kappa)$.

Proof. This is a straightforward consequence of Corollaries 3.7.3 and 3.7.4, Lemma 3.7.5, and (3.7.2), (3.7.4), (3.7.21). ∎

DEFINITION 3.7.3. $[(V,\kappa), J, \widetilde{n}, \bullet, \mathbf{E}]$ is a complex vector space $[(V,\kappa), J, \bullet]$ equipped with a supercomplex structure $(J, \widetilde{n})$ and a Hurwitz-type vector space $\mathbf{E}$ of endomorphisms $E: V \to V$ satisfying

$$(q + is) \bullet f = fq + (Jf)s \quad \text{for } f \in V \text{ and } q, s \in \mathbb{R}. \tag{3.7.24}$$

(By the definition it has to satisfy also the relations (3.7.2), (3.7.4) and (3.7.5).)

THEOREM 3.7.2. *Consider a pseudo-Euclidean Hurwitz pair* $[(V,\kappa),(S,\eta),\circ]$ *of bidimension* (n,p), $n > 1$, *and some orthonormal bases* (e_j) *in* V *and* (ε_α) *in* S. *Let* (n^α) *be an arbitrary system of real numbers* (3.7.4) *and* (γ_α) *a system of imaginary* $n \times n$ *matrices* (3.2.7)–(3.2.8) *with the property* (3.7.19), *which is possible under the assumption that* $\kappa = \operatorname{diag}(1, \overset{(k)}{\ldots}, 1, -1, \overset{(l)}{\ldots}, -1)$, $k, l \neq 0$. *Suppose that* f *is an arbitrary vector in* V *and let* $\sum_{j=1}^n e_j f_{\mathbb{R}}^j$ *be its decomposition* (*in* V). *Then this decomposition can be rearranged into the form*

$$f = \sum_{j=1}^{n/2} (Re)_{2j-1} f^{2j-1}, \quad \textit{where} \quad f^{2j-1} = E_0 f_{\mathbb{R}}^{2j-1} + \widetilde{n} f_{\mathbb{R}}^{2j}, \tag{3.7.25}$$

or

$$f = \sum_{j=1}^{n/2} (Re)_{2j} f^{2j}, \quad \textit{where} \quad f^{2j} = E_0 f_{\mathbb{R}}^{2j} - \widetilde{n} f_{\mathbb{R}}^{2j-1}, \tag{3.7.26}$$

for some $R \in O(\kappa)$, *where* $\widetilde{n} = \sum_{\alpha=1,\alpha\neq t}^p n^\alpha E_\alpha$.

P r o o f. The problem whose solution is formulated in Theorem 3.7.2 is well-posed by Lemma 3.2.2 , (3.7.2), (3.7.4), Theorem 3.7.1 and Lemma 3.7.5. By (3.7.2) and (3.7.4),

$$\widetilde{n} e_j = n^\alpha (i\gamma_{j\alpha}^k e_k) = (in^\alpha \gamma_\alpha)_j^k e_k = J_j^k e_k.$$

By Lemma 3.7.5, $\widetilde{n}(Re)_j = (J^0)_j^k e_k$. Using Lemma 3.7.6, we get

$$\widetilde{n}(Re)_{2j-1} = (J^0)_{2j-1}^k (Re)_k = (Re)_{2j}, \widetilde{n}(Re)_{2j} = (J^0)_{2j}^k (Re)_k = -(Re)_{2j-1}. \tag{3.7.27}$$

Thus, for every $f = \sum_{j=1}^n (Re)_j f_{\mathbb{R}}^J$ we get

$$\begin{aligned} f &= \sum_{j=1}^{n/2} [(Re)_{2j-1} f_{\mathbb{R}}^{2j-1} + (Re)_{2j} f_{\mathbb{R}}^{2j}] \\ &= \sum_{j=1}^{n/2} [(Re)_{2j-1} f_{\mathbb{R}}^{2j-1} + \widetilde{n}(Re)_{2j+1} f_{\mathbb{R}}^{2j}] = \sum_{j=1}^{n/2} (Re)_{2j-1} f^{2j-1}, \end{aligned}$$

where $f^{2j-1} := E_0 f_{\mathbb{R}}^{2j-1} + \widetilde{n} f_{\mathbb{R}}^{2j}$.

Analogously, we obtain (3.7.26). The uniqueness of these decompositions is a clear consequence of the uniqueness of $f = \sum_{j=1}^n e_j f_{\mathbb{R}}^j$. ■

From (3.7.25) and (3.7.26) we also deduce

LEMMA 3.7.7. *If* $\kappa = \mathrm{diag}(1, \overset{(k)}{\ldots}, 1, -1, \overset{(l)}{\ldots}, -1)$, $k, l \neq 0$, *then by Theorem* 3.7.2 *the decompositions* (3.7.25) *and* (3.7.26) *for* $f \in V$ *generate the decompositions*

$$V = \bigoplus_{j=1}^{n/2} C_j(E_0, \widetilde{n}, J) \tag{3.7.28}$$

or

$$V = \bigoplus_{j=1}^{n/2} \widetilde{C}_j(E_0, \widetilde{n}, J), \tag{3.7.29}$$

where $C_j(E_0, \widetilde{n}, J)$ *and* $\widetilde{C}_j(E_0, \widetilde{n}, J)$ *are the complex one-dimensional subspaces of* V *generated by* e_{2j-1} *and* e_{2j}, *respectively, for* $j = 1, \ldots, n/2$. *Their dependence on* $E_0, \widetilde{n}$ *and* J *is determined by* (3.7.2), (3.7.4) *and* (3.7.19).

On the other hand, with the help of the complex structure J we can introduce the complex scalar product $(\ , \ ) : V \times V \to \mathbb{C}$ as follows:

$$(f, g) = (f, g)_{\mathbb{R}} + i(Jf, g)_{\mathbb{R}} \quad \text{for } f, g \in V \tag{3.7.30}$$

(provided κ, the metric of V, satisfies the assumption of Lemma 3.7.7), where $(\ , \ )_{\mathbb{R}}$ denotes the usual (real) scalar product in V: $(f, g)_{\mathbb{R}} := \sum_{i=1}^{n} f^i g^i$ for $f = f^i e_i$, $g = g^i e_i$. Then we have

PROPOSITION 3.7.1. *The complex scalar product* $(\ , \ )$ *has the properties*

$$(f, g) = \overline{(g, f)}, \quad (f, g + h) = (f, g) + (f, h) \quad \textit{for } f, g \in V, \tag{3.7.31}$$

$$(f, zg) = z(f, g), \quad (f, f) = \|f\|^2_{\mathbb{R}} \quad \textit{for } f, g \in V \textit{ and } z \in \mathbb{C}, \tag{3.7.32}$$

$$(f, g) = \sum_{j=1}^{n/2} \overline{f^j_{\mathbb{C}}} g^j_{\mathbb{C}} \quad \textit{for } f, g \in V, \tag{3.7.33}$$

where the bar denotes complex conjugation and

$$f^j_{\mathbb{C}} = f^{2j-1}_{\mathbb{R}} + i f^{2j}_{\mathbb{R}}, \quad g^j_{\mathbb{C}} = g^{2j-1}_{\mathbb{R}} + i g^{2j-1}_{\mathbb{R}}, \quad j = 1, \ldots, n/2. \tag{3.7.34}$$

Proof. The properties (3.7.31) and (3.7.32) follow from (3.7.21) and (3.7.22) and from the definition of $(\ , \ )$ and $(\ , \ )_{\mathbb{R}}$. Indeed,

$$\begin{aligned}
(g, f) &= (g, f)_{\mathbb{R}} + i(Jg, f)_{\mathbb{R}} = (f, g)_{\mathbb{R}} + i(\widetilde{n}g, f)_{\mathbb{R}} = (f, g)_{\mathbb{R}} - n^\alpha(\gamma_\alpha g, f) \\
&= (f, g)_{\mathbb{R}} - n^\alpha \sum_{k=1}^{n} (\gamma_\alpha g)_k f_k = (f, g)_{\mathbb{R}} - n^\alpha \sum_{k=1}^{n} \Big(\sum_{m=1}^{n} \gamma^m_{\alpha k} g_m \Big) f_k \\
&= (f, g)_{\mathbb{R}} - n^\alpha \sum_{m=1}^{n} \sum_{k=1}^{n} g_m(-\gamma^k_{\alpha m} f_k) = (f, g)_{\mathbb{R}} + n^\alpha \sum_{m=1}^{n} g_m (\gamma_\alpha f)_m \\
&= (f, g)_{\mathbb{R}} + n^\alpha (g, \gamma_\alpha f)_{\mathbb{R}} = (f, g)_{\mathbb{R}} - n^\alpha (g, E_\alpha f)_{\mathbb{R}} = (f, g)_{\mathbb{R}} - i(g, \widetilde{n} f)_{\mathbb{R}} \\
&= (f, g)_{\mathbb{R}} - i(g, Jf)_{\mathbb{R}} = (f, g)_{\mathbb{R}} - i(Jf, g)_{\mathbb{R}} = \overline{(f, g)}.
\end{aligned}$$

In particular,

$$(f,f) = (f,f)_{\mathbb{R}} + i(Jf,f)_{\mathbb{R}} = \overline{(f,f)} = (f,f)_{\mathbb{R}} - i(Jf,f)_{\mathbb{R}}.$$

Hence $(Jf,f)_{\mathbb{R}} = 0$ and $(f,f) = (f,f)_{\mathbb{R}} = \|f\|_{\mathbb{R}}^2$. The remaining equalities in (3.7.31) and (3.7.32) are obvious.

To prove (3.7.33) we take (3.7.27):

$$\begin{aligned}
(f,g) &= (f,g)_{\mathbb{R}} + i(Jf,g)_{\mathbb{R}} = \sum_{k=1}^{n} f^k g^k + i(\widetilde{n}f,g)_{\mathbb{R}} \\
&= \sum_{k=1}^{n} f^k g^k + i(\widetilde{n}(f^k e_k),g)_{\mathbb{R}} \\
&= \sum_{k=1}^{n} f^k g^k + i\sum_{j=1}^{n/2} (f^{2j-1}\widetilde{n}(e_{2j-1}) + f^{2j}\widetilde{n}(e_{2j}),g)_{\mathbb{R}} \\
&= \sum_{k=1}^{n} f^k g^k + i\sum_{j=1}^{n/2} (f^{2j-1}e_{2j} - f^{2j}e_{2j-1},g)_{\mathbb{R}} \\
&= \sum_{k=1}^{n} f^k g^k + i\sum_{j=1}^{n/2} (f^{2j-1}g^{2j} - f^{2j}g^{2j-1}) \\
&= \sum_{j=1}^{n/2} [f^{2j-1}(g^{2j-1} + ig^{2j}) + f^{2j}(g^{2j} - ig^{2j-1})] \\
&= \sum_{j=1}^{n/2} (f^{2j-1} - if^{2j})(g^{2j-1} + ig^{2j}) = \sum_{j=1}^{n/2} \overline{f_{\mathbb{C}}^j} g_{\mathbb{C}}^j,
\end{aligned}$$

where $f_{\mathbb{C}}^j$ and $g_{\mathbb{C}}^j$ are defined by (3.7.34). ■

III.8. Pairs of Clifford algebras of Hurwitz type. Let $(S,V,\circ)$ be a Hurwitz pair. Suppose that the vector spaces S and V are equipped with non-degenerate quadratic quadratic forms Q_S and Q_V, respectively. In S and V we choose some bases (ε_α) and (e_j) with $\alpha = 1,\dots,p = \dim S$; $j = 1,\dots,n = \dim V$. Assume that $p \leq n$.

Let $C(Q_S)$ (resp. $C(Q_V)$) denote the Clifford algebra of (S,Q_S) (resp. (V,Q_V)). There are canonical injections $i_S : S \to C(Q_S)$ and $i_V : V \to C(Q_V)$. Then we get the following diagram:

$$\begin{array}{ccc}
S \times V & \xrightarrow{\ \circ\ \text{(Hurwitz multiplication)}\ } & V \\
\Big\downarrow{\scriptstyle i_S \times i_V} & & \Big\downarrow{\scriptstyle i_V} \\
C(Q_S) \times C(Q_V) & \xrightarrow{\ \star\, =\, ?\ } & C(Q_V)
\end{array} \tag{3.8.1}$$

It would be very interesting to complete the above diagram by a suitable mapping $C(Q_S) \times C(Q_V) \to C(Q_V)$. Define the mapping $\star : C(Q_S) \times C(Q_V) \to C(Q_V)$ by:

$$d1_S \star y_V := y_V,$$

$$(3.8.2)\qquad (\varepsilon_{i_1}\dots\varepsilon_{i_r})\star(e_{j_1}\dots e_{j_k}) := \begin{cases} e_{j_k}\dots e_{j_{r+1}}(\varepsilon_{i_r}\circ e_{j_r})\dots(\varepsilon_{i_1}\circ e_{j_1}) & \text{if } r<k,\\ (\varepsilon_{i_r}\circ e_{j_r})\dots(\varepsilon_{i_1}\circ e_{j_1}) & \text{if } r=k,\\ \varepsilon_{i_r}\circ[\varepsilon_{i_{r-1}}\circ[\dots\circ[\varepsilon_{i_{k+1}}\circ \\ [(\varepsilon_{i_k}\circ e_{j_k})\dots(\varepsilon_{i_1}\circ e_{j_1})]\dots] & \text{if } r>k,\end{cases}$$

$$(\varepsilon_{i_1}\dots\varepsilon_{i_r})\star 1_V := \|\varepsilon_{i_1}\|\dots\|\varepsilon_{i_r}\|1_V$$

for $1\le r\le p$, $1\le i_1<\ldots<i_r\le p$: $1\le k\le n$, $1\le j_1<\ldots<j_k\le n$. Then the required mapping $\star: C(Q_S)\times C(Q_V)\to C(Q_V)$ is defined by the bilinear extension of (3.8.2).

Remark 3.8.1. If (S,Q_S) is a Euclidean vector space then all $\|\varepsilon_i\|^2>0$. In this case the Clifford algebras $C(Q_S)$ and $C(Q_V)$ are considered to be real. But if (S,Q_S) is a pseudo-Euclidean vector space then there are some $\varepsilon_{i_1},\dots,\varepsilon_{i_r}$, $1\le r\le p$, such that $\|\varepsilon_{i_r}\|^2<0$, $1\le s\le r$. This time the Clifford algebras have to be treated as complex ones.

Proposition 3.8.1. *$\star$ is a well defined bilinear mapping. Moreover, $\star_{|S\times V}=\circ$, the Hurwitz multiplication, i.e. the diagram (3.8.1) is commutative.*

Lemma 3.8.1. *Let $x_S\in\Gamma_S$ and $y_V\in\Gamma_V$, where Γ_S (resp. Γ_V) denotes the Clifford group in $C(Q_S)$ (resp. $C(Q_V)$) and N is a spinor norm. Then*

$$(3.8.3)\qquad N_V(x_S\star y_V)=N_S(x_S)N_V(y_V).$$

Corollary 3.8.1.

$$\mathrm{Pin}_S\star\mathrm{Pin}_V\subset\mathrm{Pin}_V,\qquad \mathrm{Pin}_S\star\mathrm{Spin}_V\subset\mathrm{Spin}_V\,.$$

Theorem 3.8.1. *Assume that S and V are real vector spaces equipped with non-degenerate quadratic forms Q_S and Q_V, respectively. Denote by $C^{\mathbb{C}}(Q_S)$ (resp. $C^{\mathbb{C}}(Q_V)$) the complex Clifford algebras of (S,Q_S) (resp. (V,Q_V)). Suppose that there exists a bilinear mapping $\star: C^{\mathbb{C}}(Q_S)\times C^{\mathbb{C}}(Q_V)\to C^{\mathbb{C}}(Q_V)$ satisfying the condition (3.8.3). Then $\star$ is generated by the Hurwitz multiplication, i.e. $\star_{|S\times V}=\circ$, where $\circ: S\times V\to V$ is a bilinear mapping such that $\|s\circ v\|_V=\|s\|_S\|v\|_V$ for all $s\in S$ and all $v\in V$.*

Proof. Let $s\in S\subset\Gamma_S$ and $v\in V\subset\Gamma_V$. By definition of N we have

$$(3.8.4)\qquad N_V(s\star v)=N_S(s)N_V(v)=\|s\|_S^2\|v\|_V^2\in\mathbb{R}.$$

Let $(e_1,\dots,e_n)$ be an orthogonal base in V. Suppose

$$s\star v=a_0+\sum_{i=1}^n a^ie_i+\sum_{l=2}^n\sum_{i_1<\ldots<i_l}a_l^{i_1\dots i_l}e_{i_1}\dots e_{i_l}.$$

Then

$$N(s\star v)=a_0^2+\sum_{i=1}^n(a^i)^2Q_V(e_i)+\sum_{l=2}^n\sum_{i_1,\ldots,<i_l}(a_l^{i_1\dots i_l})^2Q_V(e_{i_1})\dots Q_V(e_{i_l}) + R(e_1,\dots,e_n),$$

where

$$R(e_1,\dots,e_n)=\sum b^ie_i+\sum_{i<j}b^{ij}e_ie_j+\ldots+\sum_{i_1<..<i_m}b^{i_1..i_m}e_{i_1}\dots e_{i_m}+be_1\dots e_n.$$

Since $N(s\star v)$ is a scalar, $R(e_1,\dots,e_n)$ must vanish. Then we get

$$N(s\star v)=a_0^2+\sum_{i=1}^{n}(a^i)^2Q_V(e_i)+\sum_{l=2}^{n}\sum_{i_1<\dots<i_l}(a_l^{i_1\dots i_l})^2Q_V(e_{i_1})\dots Q_V(e_{i_l}). \tag{3.8.5}$$

The multiplication $\star$ is bilinear so the coefficients a_0, a^i and $a_l^{i_1\dots i_l}$ are bilinear functions in s and v. By (3.8.4) the expression (3.8.5) should be separated into two parts, the first depending only on s and the second only on v. Then we can write

$$\begin{aligned}a_0^2+\sum_{i=1}^{n}(a^i)^2Q_V(e_i)+\sum_{l=2}^{n}\sum_{i_1<\dots<i_l}(a_l^{i_1\dots i_l})^2Q_V(e_{i_1})\dots Q_V(e_{i_l})=\|s\|_S^2\|v\|_V^2\\ =\|s\|_S^2\Big[c_0^2+\sum_{i=1}^{n}(c^i)^2Q_V(e_i)+\sum_{l=2}^{n}\sum_{i_1<\dots<i_l}(c_l^{i_1\dots i_l})^2Q_V(e_{i_1})\dots Q_V(e_{i_l})\Big].\end{aligned}$$

The coefficients $c_0, c^i, c_l^{i_1\dots i_l}$ are linear in v so, by continuity, we can write

$$c_0(v)=c_{0j}v^j,\quad c^i(v)=c^i_jv^j,\quad c_l^{i_1\dots i_l}(v)=c_{lj}^{i_1\dots i_l}v^j.$$

Thus, we get the following identity:

$$c_{0j}c_{0k}+\sum_{i=1}^{n}(c^i_jc^i_k-\delta^i_j\delta^i_k)Q_V(e_i)+\sum_{l=2}^{n}\sum_{i_1<\dots<i_l}c_{lj}^{i_1\dots i_l}c_{lk}^{i_1\dots i_l}Q_V(e_{i_1})\dots Q_V(e_{i_l})=0$$

for any $1\le j,\ k\le n$.

Take an orthogonal transformation $R\in O(Q_V)$. In a new base $e'=Re$ we have

$$\begin{aligned}c_{0j}c_{0k}+\sum_{i=1}^{n}(\widetilde{c}^i_j\widetilde{c}^i_k-\delta^i_j\delta^i_k)Q_V(Re_i)\\ +\sum_{l=2}^{n}\sum_{i_1<\dots<i_l}\widetilde{c}_{lj}^{i_1\dots i_l}\widetilde{c}_{lk}^{i_1\dots i_l}Q_V(Re_{i_1})\dots Q_V(Re_{i_l})=0.\end{aligned}$$

But $Q_V(Re_i)=Q_V(e_i)$. Thus the new coefficients $\widetilde{c}_j$ and $\widetilde{c}_{lj}^{i_1\dots i_l}$, obtained by the base change, satisfy the same identity as the previous ones. This is possible if and only if

$$\begin{cases}c_{0j}\equiv 0 \quad \text{for } j=1,\dots,n,\\ c^i_jc^i_k-\delta^i_j\delta^i_k\equiv 0\quad\text{for } 1\le i,j\le n,\\ c_{lj}^{i_1\dots i_l}\equiv 0\quad\text{for } l=2,\dots,n;\ 1\le i_1<\dots<i_l\le n;\ j=1,\dots,n.\end{cases}$$

Thus we get

$$s\star v=\|s\|_S\sum_{i=1,j=1}^{n}c^i_jv^je_i\in V\quad\textit{and}\quad\|s\|_S^2\|v\|_V^2=N_V(s\star v)=\|s\star v\|_V^2.$$

Then $\star_{|S\times V}$ satisfies the Hurwitz condition, as required. Moreover, define

$$w=(w^1,\dots,w^n),\quad w^i:=c^i_jv^j.$$

Then, finally, we obtain $s\star v=\|s\|_Sw$ or $s\star v=\|s\|_SOw$, where $O\in O(V,Q_V)$. ∎

EXAMPLE 3.8.1. We construct a bilinear map $\square : C^{\mathbb{C}}(Q_S)\times C^{\mathbb{C}}(Q_V)\to C^{\mathbb{C}}(Q_V)$ which does not satisfy the condition (3.8.3). Choose some bases (ε_α) and (e_j) in S and

V, respectively. Define

$$\text{(3.8.6)}\qquad \begin{cases} 1_S\Box 1_V := e_1\dots e_n, \\ 1_S\Box(e_{i_1}\dots e_{i_k}) := e_{i_1}\widehat{\dots}e_{i_k}, \\ 1_S\Box(e_1\dots e_n) := 1_V, \\ (\varepsilon_{j_1}\dots\varepsilon_{j_r})\Box(e_{i_1}\dots e_{1_k}) := \|\varepsilon_{j_1}\|\dots\|\varepsilon_{j_r}\|e_{i_1}\widehat{\dots}e_{i_k}, \\ (\varepsilon_{j_1}\dots\varepsilon_{j_r})\Box 1_V := \|\varepsilon_{j_1}\|\dots\|\varepsilon_{j_r}\|e_1\dots e_n, \\ (\varepsilon_{j_1}\dots\varepsilon_{j_r})\Box(e_1\dots e_n) := \|\varepsilon_{j_1}\|\dots\|\varepsilon_{j_r}\|1_V, \end{cases}$$

where "$\widehat{\cdot}$" is defined by

$$e_{i_1}\widehat{\dots}e_{i_r} := e_{j_1}\dots e_{j_s} \quad \text{with } j_1<\dots<j_s \text{ and } (i_1\dots i_r j_1\dots j_s) = (1\dots n).$$

The map $\Box : C^{\mathbb{C}}(Q_S)\times C^{\mathbb{C}}(Q_V) \to C^{\mathbb{C}}(Q_V)$ is defined by the bilinear extension of (3.8.6). It is easy to see that $\Box$ does not satisfy the condition (3.8.3). Indeed, take $s\in S$ and $v\in V$. We have

$$s\Box v = s^\alpha v^i(\varepsilon_\alpha\Box e_i) = s^\alpha v^i\|\varepsilon_\alpha\|_S e_1\dots\widehat{e_i}\dots e_n \notin V$$

and

$$N_V(s\Box v) = s^\alpha s^\beta\|\varepsilon_\alpha\|_S\|\varepsilon_\beta\|_S(v^i)^2 Q_V(e_1)\dots\widehat{Q_V(e_i)}\dots Q_V(e_n).$$

Suppose that $N_V(s\Box v) = N_S(s)N_V(v)$. Then we get

$$\sum_{\alpha,\beta} s^\alpha s^\beta\|\varepsilon_\alpha\|_S\|\varepsilon_\beta\|_S = \sum_\alpha (s^\alpha)^2\|\varepsilon_\alpha\|_S^2,$$

$$\sum_i (v^i)^2 Q_V(e_1)\dots\widehat{Q_V(e_i)}\dots Q_V(e_n) = \sum_i (v^i)^2 Q_V(e_i).$$

The above condition is equivalent to $\|\varepsilon_\alpha\|_S = 0$ and $\|e_1\|_V^2\dots\widehat{\|e_i\|_V^2}\dots\|e_n\|_V^2 = \|e_i\|_V^2$ but this is impossible.

Acknowledgements. This presentation grew out of a series of results obtained by the author on the Fueter regular mappings (quaternionic analysis), their connection with the harmonic maps, and on the so-called Fueter–Hurwitz regular mappings which are a generalization of holomorphic functions from complex analysis in the so-called "theory of Hurwitz pairs".

Several well known specialists in quaternionic and hypercomplex analysis, especially my friends Professor Stefano Marchiafava (Rome), Professors R. Michael Porter and Enrique Ramirez de Arellano (Mexico City), made many encouraging and useful suggestions during the preparation of this material. Their comments were essential in making several improvements and contributed to the removal of errors.

References

[1] J. Adem, *On the Hurwitz problem over an arbitrary field I, II*, Bol. Soc. Mat. Mexicana 25 (1980), 29–51; 26 (1981), 29–41.

[2] M. F. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology (Suppl. 1) 3 (1964), 3–38.

[3] V. Bartik, A. V. Ferreira, M. Markl and V. Souček, *Index and Cauchy integral formula in complex-quaternionic analysis*, Simon Stevin 59 (1985), 321–329.

[4] M. M. Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés Riemanniennes*, Bull. Soc. Math. France 83 (1955), 279–330.

[5] M. Berger et B. Gostiaux, *Géométrie différentielle*, Colin, Paris, 1972.

[6] A. L. Besse, *Manifolds All of Whose Geodesics are Closed*, Ergeb. Math. Grenzgeb. 13, Springer, 1978.

[7] —, *Einstein Manifolds*, Ergeb. Math. Grenzgeb. 10, Springer, Berlin, 1987.

[8] E. Bonan, *Sur les G-structures de type quaternionien*, Cahiers Topologie Géom. Différentielle 9 (1967), 389–461.

[9] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Springer, New York, 1982.

[10] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*, Res. Notes in Math. 75, Pitman, Boston, 1982.

[11] J. Chazarain and A. Piriou, *Introduction to the Theory of Partial Differential Equations*, North-Holland, Amsterdam, 1982.

[12] M. L. Curtis, *Matrix Groups*, Springer, 1984.

[13] J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps*, CBMS Regional Conf. Ser. in Math. 50, Amer. Math. Soc., 1983.

[14] —, —, *Another report on harmonic maps*, Bull. London Math. Soc. 20 (1988), 385–524.

[15] A. Fröhlich and A. McEvett, *Forms over rings with involution*, J. Algebra 12 (1969), 79–104.

[16] R. Fueter, *Die Funktionentheorie der Differentialgleichungen $\Delta = 0$ und $\Delta\Delta = 0$ mit vier reellen Variablen*, Comment. Math. Helv. 7 (1935), 307–330.

[17] —, *Über die analytische Darstellung der regulären Funktionen einer Quaternionnenvariablen*, ibid. 8 (1936), 371–378.

[18] A. Gray, *Vector cross product on manifolds*, Trans. Amer. Math. Soc. 14 (1968), 465–504.

[19] R. Harvey and H. B. Lawson (Jr.), *Calibrated geometries*, Acta Math. 148 (1982), 47–157.

[20] D. Hestenes, *Clifford Algebra to Geometric Calculus*, Reidel, Dordrecht, 1984.

[21] A. Hurwitz, *Über die Komposition der quadratischen Formen von beliebig vielen Variablen*, Nachr. Königl. Gesell. Wiss. Göttingen, Math.-Phys. Kl. 1898, 309–316; reprinted in: A. Hurwitz, *Mathematische Werke II*, Birkhäuser, Basel, 1933, 565–571.

[22] —, *Über die Komposition der quadratischen Formen*, Math. Ann. 88 (1923), 1–25; reprinted in: A. Hurwitz, *Mathematische Werke II*, Birkhäuser, Basel, 1933, 641–666.

[23] K. Imaeda, *Quaternionic formulation of classical electrodynamics and theory of functions of a biquaternion variable*, Report of Fundamental Physics Laboratory, Department of Electronic Science, Faculty of Science, Okayama University of Science, Feb. 1983.

[24] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vols. I–II, Interscience, 1963.

[25] V. Y. Kraines, *Topology of quaternionic manifolds*, Trans. Amer. Math. Soc. 122 (1966), 357–367.

[26] W. Królikowski, *Anisotropic complex structure on the pseudo-Euclidean Hurwitz pairs*, Ann. Polon. Math. 55 (1991), 225–240.

[27] —, *Quaternionic mappings and harmonicity*, preprint.

[28] —, *Pairs of Clifford algebras of the Hurwitz-type*, in: Proc. Internat. Sympos. on Generalizations of Complex Analysis and Their Applications in Physics, Warsaw, May 30 - July 1, 1994, J. Ławrynowicz (ed.), Banach Center Publ., to appear.

[29] —, *Fueter regular mappings and harmonicity*, Ann. Polon. Math., to appear.

[30] W. Królikowski and S. Marchiafava, *On Lichnerowicz smooth homotopy invariants for G-structures*, Note Mat. 13 (1993), 197–212.

[31] W. Królikowski and R. M. Porter, *Quaternionic regular and biregular functions in the sense of Fueter*, Ber. Univ. Jyväskylä Math. Inst. 55 (1993), 65–87.
[32] —, —, *Regular and biregular functions in the sense of Fueter—some problems*, Ann. Polon. Math. 59 (1994), 53–64.
[33] —, —, *Biregular quaternionic functions*, in: Clifford Algebras and Their Applications in Mathematical Physics, Proc. Deinze 1993, F. Brackx *et al*. (eds)., Kluwer Acad. Publ., Dordrecht, 1993, 129–135.
[34] W. Królikowski and E. Ramirez de Arellano, *Fueter–Hurwitz regular mappings and an integral representation*, in: Clifford Algebras and Their Applications in Mathematical Physics, Proc. Montpellier 1989, A. Micali *et al*. (eds.), Kluwer Acad. Publ., Dordrecht, 1992, 221–237.
[35] —, —, *On polynomial solutions of the Fueter–Hurwitz equation*, in: Contemp. Math. 137, Amer. Math. Soc., 1992, 297–305.
[36] W. Królikowski, D. Lambert, J. Ławrynowicz and J. Rembieliński, *The CR-problem and Hurwitz pairs applied to field theory and solid state physics*, Rep. Math. Phys. 29 (1991), 23–44.
[37] T. Y. Lam, *The Algebraic Theory of Quadratic Forms*, Benjamin, New York, 1973.
[38] A. Lichnerowicz, *Applications harmoniques et variétés Kähleriennes*, Rend. Sem. Mat. Fis. Milano 39 (1969), 1–12.
[39] —, *Applications harmoniques et variétés Kähleriennes*, Symp. Math. III (1970), Bologna, 341–402.
[40] —, *Théorie globale des connexions et des groupes d'holonomie*, Ed. Cremonese, 1955.
[41] J. Ławrynowicz and J. Rembieliński, *On the composition of non-degenerate quadratic forms with an arbitrary index*, Ann. Fac. Sci. Toulouse 10 (1989), 141–168 (reprinted in 11 (1990), 141–168, due to a printing error).
[42] —, —, *Hurwitz pairs equipped with complex structure*, in: Seminar on Deformations, Łódź–Warsaw 1982–84, Proceedings, Lecture Notes in Math. 1165, J. Ławrynowicz (ed.), Springer, Berlin, 1985, 184–197.
[43] —, —, *Pseudo-Euclidean Hurwitz pairs and the Kałuża–Klein theories*, J. Phys. A: Gen. 20 (1987), 5831–5848.
[44] —, —, *Supercomplex vector spaces and spontaneuos symmetry breaking*, in: Seminari di Geometria 1984, CNR (Università di Bologna), Bologna, S. Coen (ed.), 131–154.
[45] —, —, *Pseudo-Euclidean Hurwitz pairs and generalized Fueter equations*, in: Clifford Algebras and Their Applications in Mathematical Physics, Proceedings, Canterbury 1985, J. S. R. Chisholm and A. K. Common (eds.), Reidel, Dordrecht, 1986, 39–48.
[46] S. Marchiafava, *Su alcune sottvarietà che ha interesse considerare in una varietà Kähleriana quaternionale*, Rend. Mat. (7) 10 (1990), 493–529.
[47] E. Martinelli, *Variétés à structure quaternionienne généralisées*, Rev. Roumaine Math. Pures Appl. 10 (1965), 915–922.
[48] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton Univ. Press and Univ. of Tokyo Press, Princeton, N.J., 1974.
[49] R. Narasimhan, *Analysis on Real and Complex Manifolds*, North-Holland, Amsterdam, 1968.
[50] I. R. Portous, *Topological Geometry*, 2nd ed. Cambridge Univ. Press, Cambridge, 1981.
[51] L-S. Randriamihamison, *Algèbres de Clifford et paires de Hurwitz pseudo-euclidiennes*, thèse de docteur, L'Université Paul Sabatier de Toulouse (Sciences), 1988.
[52] S. M. Salomon, *Riemannian Geometry and Holonomy Groups*, Pitman Res. Notes Math. Ser. 201, Oxford, 1989.
[53] D. B. Shapiro, *Compositions of quadratic forms*, book preprint.
[54] —, *Spaces of similarities I. The Hurwitz problem*, J. Algebra 46 (1977), 148–170.
[55] —, *Products of sums of squares*, Exposition. Math. 2 (1984), 235–261.

[56] M. V. Shapiro and N. Vasilevski, *Quaternionic ψ-hyperholomorphic functions*, *singular integral operators and boundary value problems* (*I and II*), Complex Variables Theory Appl., to appear.

[57] A. J. Sommese, *Quaternionic manifolds*, Math. Ann. 212 (1975), 191–214.

[58] V. Souček, *Holomorphicity in quaternionic analysis*, Seminari di Geometria 1982–1983, Università di Bologna, Instituto de Geometria, Dipartimento de Matematica, 1984, 147–153.

[59] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[60] A. Sudbery, *Quaternionic analysis*, Math. Proc. Cambridge Philos. Soc. 85 (1979), 199–225.

[61] H. Tasaki, *Quaternionic submanifolds in quaternionic symmetric spaces*, Tôhoku Math. J. 38 (1986), 513–538.

[62] W. T. Wu, *Sur les classes caractéristiques des structures fibrés sphériques*, Publ. Instit. Math. Univ. Strasbourg 11, Hermann, Paris, 1952.