DIFFERENT ASPECTS OF DIFFERENTIABILITY
DISSERTATIONES MATHEMATICAE 340 (1995)

ON THE CONTINUATION OF THE IDEAS OF HEAVISIDE AND MIKUSIŃSKI IN OPERATIONAL CALCULUS

A. P. PRUDNIKOV

Computer Center, Russian Academy of Sciences
Vavilova St. 40, Moscow 117333, Russia

1. Introduction. In this survey we consider operational calculus as a branch of linear functional analysis. The purpose of this survey is to give some ideas what the modern theory of operational calculus is about. We shall concentrate mainly on the basic problems of the theory, occasionally touching upon analytic and algebraic aspects. In several cases they lead to work of Heaviside and Mikusiński.

1.1. *Historical remarks.* The last 30-years of 17th century is a period of time during which the first ideas of symbolic (operational) calculus were laid. In a historical article "Prodigious Mystery of Genuine Analysis", D. Przeworska-Rolewicz [PR4] mentions such scholars as I. Newton (1643–1727) and G. W. Leibniz (1646–1716).

Operational methods in analysis (operational calculi) date back to at least the 19th century, and are primarily concerned with methods of finding solutions of differential, integral and difference equations of physics and engineering.

Already in the nineteenth century, several mathematicians had published work on "symbolic calculus" as a system of operations with the symbol $D = \frac{d}{dt}$ (see, for example, M. Vashchenko-Zakharchenko [VZ], A. V. Letnikov [Let]). In some cases the meaning of functions of the operator D is clear. For example, $D^n = \frac{d^n}{dt^n}$ is the symbol for the operation of taking the nth derivative, and the meaning of $P_n(D)$ is plain, where $P_n(x)$ is an nth degree polynomial. The Taylor series

$$f(t+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(t)$$

1991 *Mathematics Subject Classification*: 01A60, 44A40.
The paper is in final form and no version of it will be published elsewhere.

can be written in the form

$$f(t+h) = e^{hD} f(t),$$

i.e., the exponential function e^{hD} of the operator D is the h-translation of the argument of $f(t)$. The inverse operation $\frac{1}{D}$ is clearly that of integration. Several mathematicians, including Liouville, Riemann, and others considered the operator D^{α} in which α is not necessarily an integer (fractional differentiation and integration, see [Sa]).

In his "Electromagnetic Theory", London (1899), the physicist O. Heaviside [He1,He2] inaugurated an operational calculus which he successfully applied to ordinary linear differential equations connected with electromagnetical problems. In his calculus, as is well known, certain operators occur whose meaning is not at all obvious. The interpretation of such operators as given by Heaviside himself is difficult to justify. The interpretation given by his successors is unclear with regard to its range of validity, since it is based upon the theory of Laplace transforms. The wide dissemination of Heaviside's operational calculus led to the appearance of numerous works aimed at giving it a basis. The original operator view-point of Heaviside was substantially displaced by the works of Carson [Car], Doetsch [Do], Van der Pol [VB] and others, who took either the Laplace transform or the Mellin integral as the basis of their investigations. This situation could not last long, however, because the successful development of functional analysis, and in particular, of the theory of linear operators contributed to the development of operator methods in mathematical analysis. Various theories have been put forward over the last half-century to explain the symbolic methods of Heaviside for solving ordinary and partial differential equations with constant and variable coefficients, and the methods themselves have been considerably developed.

The mathematical problems encountered are twofold: firstly, to find an analytical interpretation for the operations performed and to justify these operations in terms of the interpretation and, secondly, to provide an adequate theory of the Dirac's "impulse functions" which so frequently occur. A complete return to the original operator view-point was made by Mikusiński in 1944 (see J.G.M. [MJ1]). He has provided a strict operator basis for Heaviside's operational calculus without any reference to the theory of the Laplace transform. The theory of convolution quotients due to J. Mikusiński provides a clear and simple basis for an operational calculus. The theory of operational calculus on the basis of the convolution quotients is a legitimate child of the Polish Mathematical School. Among the mathematicians of this School active already in the last half-century who contributed to the renaissance of the operational calculus one should mention first of all J. Mikusiński, C. Ryll-Nardzewski, D. Przeworska-Rolewicz, P. Antosik, S. Bellert, R. Bittner, A. Kamiński, W. Kierat, R. Sikorski, K. Skórnik, W. Słowikowski, L. Włodarski. The pioneering works of mathematicians in the other countries have been published by L. Berg, A. Bleyer, T. K. Boehme, I. Dimovski, V. A. Ditkin, E. Gesztelyi, H.-J. Glaeske, L. Korevaar, G. L. Krabbe,

B. Stanković, R. Struble, Á. Száz, G. Temple, J. D. Weston, A. H. Zemanian and others.

Many articles concerning operational calculus and its applications have been published during the last decade. There is, for example, a series of papers on operational calculus published in connection with Mikusiński's works. Operational methods have also found wide application in a great variety of disciplines: mathematics, physics, mechanics, electrical engineering, informatics, etc. There is a large amount of literature concerned with both special problems in the theory of operational calculus and with its various applications. There are many monographs in which operational methods are considered.

History of the subject will be found in some well-known surveys (see [B5, Be1, Br1, Car, Coo, Cur, Di4, DV14, Fl, Fr, Ko3, Mc1, MJ11, MJ13, PR4, Ya]). The present survey work is mainly devoted to consideration of the most important results in operational calculus published recently, and in some more earlier works (see [Br1, DV14] for detailed references).

2. Linear operators and Heaviside's operational calculus. The need for a basis for operational calculus starting with the idea of a linear operator was considerable. A. I. Plesner [Pl] gave a rigorous description of the operational calculus, as based on contemporary operator theory. Plesner showed that operational calculus for the class $L^2(0,\infty)$ with weight $e^{-\rho x}$ can be based on his general theory of maximal operators. Until the appearance of this work, courses on operational calculus contained no introduction to the idea of the definition of an operator, without which it is impossible to present a rigorous description of this method.

Plesner considered the operator $D = i\frac{d}{dt}$, $t \geq 0$. The set of complex functions $\varphi(t)$, $t \geq 0$, satisfying the condition

$$\int_0^\infty |\varphi(t)|^2 e^{-2\rho t}\, dt < \infty$$

(the integral is in the Lebesgue sense) is the unitary space $H^{(\rho)}$, where

$$(\varphi_1, \varphi_2) = \int_0^\infty e^{-2\rho\xi} \varphi_1(\xi)\, \overline{\varphi_2(\xi)}\, d\xi$$

is the scalar product. The operator D and its domain of definition are defined by

$$D\varphi(t) = i\psi(t), \qquad \varphi(t) = \int_0^t \psi(\xi)\, d\xi,$$

where $\varphi(t) \in H^{(\rho)}$, $\psi(t) \in H^{(\rho)}$. Then $D_\rho = D - \rho i E$ is a Hermitian maximal operator in the space $H^{(\rho)}$. Each function $\varphi(t) \in H^{(\rho)}$ can be represented in the form

$$\varphi(t) = F(D_\rho) \cdot 1,$$

where $F(\sigma)$ is the boundary function of $F(\lambda)$ ($\lambda = \sigma + i\tau$, $\tau > 0$), an analytic function in the upper half-plane satisfying the condition

$$\int_{-\infty}^{\infty} \frac{|F(\lambda)|^2 d\sigma}{|\lambda + \rho i|^2} < \text{const},$$

and 1 is Heaviside's function.

The mapping $\varphi(t) \rightleftarrows F(\sigma)$ $(1 \rightleftarrows 1)$, $\varphi(t) \in H^{(\rho)}$, $F(\sigma) \in M^{(\rho)}$, is the basis of the operational calculus; the scalar product in $M^{(\rho)}$ is

$$(F_1, F_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F_1(\sigma)\overline{F_2(\sigma)}}{\sigma^2 + \rho^2}\, d\sigma.$$

R. A. Struble in his work [SR2] considered a collection of mappings which include many well-known linear operators, as dilatations, exponential shifts, algebraic derivatives, partial differential operators and pseudo-differential operators. The theory of representation of distributions by operators is presented and then used to explain the operational calculus for solving differential equations in the Mikusiński field.

In the article by V. A. Ditkin and A. P. Prudnikov [DV18] a method is suggested to develop an operational calculus based on a generalized notion of mapping in some function rings.

Let L be the set of all functions defined on the semi-axis $0 \le t < \infty$ and integrable in the Lebesgue sense in any finite interval $(0, A)$. The product of $f, g \in L$ is defined as

$$f * g = \int_0^t f(t - \xi) g(\xi)\, d\xi. \tag{2.1}$$

The set L forms a commutative ring with respect to addition and multiplication (2.1). This ring has no zero divisors and can be extended to the quotient field of Mikusiński $\mathfrak{M}$. Let ω be a linear operator defined on a linear set L_ω with values in L. Suppose that

1^0. $\omega f = 0 \Rightarrow f = 0$.

2^0. For all $f, g \in L_\omega$, there is $h \in L_\omega$ such that $\omega f * \omega g = \omega h$.

L_ω is a linear set and the operations of addition and multiplication by scalars are defined in L_ω a natural way.

We shall define the product of f, $g \in L_\omega$ as

$$fg = f \cdot g = \omega^{-1}(\omega f * \omega g) = h; \tag{2.2}$$

then L_ω forms a commutative ring with respect to addition and multiplication (2.2). Let $l_0 \in L_\omega$ be some fixed element. We define the linear operator U by

$$Uf = l_0 f, \qquad f \in L_\omega.$$

Obviously, for all $f,\, g \in L_\omega$ we have

$$U(fg) = Uf \cdot g = f \cdot U_g.$$

We shall write M_0 for the ideal in the ring L_ω generated by the element l_0. Obviously, if $F \in M_0$, then there exists an element $f \in L_\omega$ such that

$$F = Uf.$$

We introduce a new multiplication by

$$F \otimes G = U^{-1}(FG), \qquad (F, G \in M_0). \tag{2.3}$$

Here U^{-1} is the inverse operator for U. The set M_0 forms a commutative ring with respect to addition and multiplication (2.3). By adjoining elements of the form λl_0 (λ is any number) to M_0 we get the ring M. The element l_0 is a unit in the ring M. This ring M has no zero divisors. It can be extended to the quotient field $\mathfrak{M}_\omega$.

We put

$$Ul_0 = l_0 l_0 = \sigma \in M_0.$$

Then we have

$$\sigma \otimes F = U^{-1}(\sigma \cdot F) = U^{-1}(Ul_0 F) = l_0 F = UF, \qquad F \in M_0.$$

Let

$$\frac{l_0}{\sigma} = \Omega \in \mathfrak{M}_\omega \qquad (\mathfrak{M}_\omega \supset L_\omega).$$

For $F \in M$ we have $\Omega F = f = U^{-1}F$. The field $\mathfrak{M}_\omega$ is isomorphic to some subfield of $\mathfrak{M}$. In particular, $\omega = 1$ and $l_0 = 1$ and we have

$$Uf = \int_0^t f(u)\, du, \quad Ul_0 = t, \quad U^{-1} = \frac{d}{dt};$$

M_0 is the set of functions of the form

$$F(t) = \int_0^t f(u)\, du;$$

M is the set of functions of the form $\int_0^t f(u)\, du + C$, $f \in L$, C is any constant;

$$F \otimes G = \frac{d}{dt} \int_0^t F(t-u)G(u)\, du.$$

We get the classical operational calculus.

3. Mikusiński's operational calculus

3.1. *Mikusiński hypernumbers.* The work "Hypernumbers" by Jan Mikusiński [MJ1] was written and published in 1944 in Poland under wartime conditions. Only seven copies were made. In fact, it is the "first edition" of Mikusiński's "Operational Calculus".

A pair of elements (α, f), consisting of a complex number α and a vector f, belonging to the abstract space W, is called a *hypernumber*; it is written in the form

$$(\alpha, f) = \alpha + f.$$

The set of all possible pairs of complex numbers and vectors of the space W is called the *hypernumber space* and is denoted by $[W]$.

A particular realization of the spaces $[W]$ is the set C of complex-valued continuous functions $f(t)$ defined for $0 \le t < \infty$. Let us denote such functions by $\{f(t)\}$ or simply by f; whereas $f(t)$ will mean the value at t of the function $f(t)$. Mikusiński writes $\{f(t)\} \cdot \{g(t)\}$ or simply $f \cdot g$ for the convolution function

$$f \cdot g = \{f(t)\} \cdot \{g(t)\} = \{(f * g)(t)\} = \Big\{ \int_0^t f(t-s)g(s)\,ds \Big\}. \tag{3.1}$$

Hence C is a ring with respect to the addition $f + g$ and multiplication $f \cdot g$. The zero of this ring is represented by the function which is identically zero; this function is denoted by 0. This ring C is without zero divisors, that is, in C, $f \cdot g = 0$ implies either $f = 0$ or $g = 0$. This is a consequence of Titchmarsh's theorem. Hence, by introducing the convolution quotient $f/g = \frac{f}{g}$ of two function $f, g \in C$ with $g \ne 0$, we obtain a commutative field M. It is the Mikusiński field.

3.2. *The structure of the Mikusiński field.* We now apply the general concept of an equivalence relation to the Mikusiński field. We consider certain elementary ideas from the theory of fields.

It is possible to imbed C in a field M which contains no proper subfield containing the ring C. Moreover, the field M is uniquely determined by C; it is called the quotient field of the ring C. In order to construct M we begin by considering the set Q of all pairs $\frac{a}{b}$ where a and b are elements of C, $b \ne 0$. We define two pairs $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ of the set Q to be equivalent, written $\frac{a_1}{b_1} \sim \frac{a_2}{b_2}$, if $a_1 b_2 = a_2 b_1$. Clearly the relation thus defined is reflexive and symmetric; it is also transitive. Indeed, let

$$\frac{a_1}{b_1} \sim \frac{a_2}{b_2} \sim \frac{a_3}{b_3};$$

then

$$a_1 b_2 = b_1 a_2, \qquad a_2 b_3 = b_2 a_3,$$

and, multiplying the first equation by b_3 and the second by b_1 we obtain $a_1 b_2 b_3 = b_2 a_3 b_1$. Since C contains no divisors of zero we may cancel the factor b_2 in this last equation, obtaining $a_1 b_3 = a_3 b_1$, i.e., $\frac{a_1}{b_1} \sim \frac{a_3}{b_3}$. The equivalence relation thus defined partitions Q into classes of mutually equivalent elements. The set of all such equivalence classes is denoted by M, and we denote by $\{\frac{a}{b}\}$ the class containing the pair $\frac{a}{b}$.

We shall turn M into a field by defining, in a natural fashion, operations of addition and multiplication. The sum and product in M are defined by

$$\left\{\frac{a_1}{b_1}\right\} + \left\{\frac{a_2}{b_2}\right\} = \left\{\frac{a_1 b_2 + b_1 a_2}{b_1 b_2}\right\},$$
$$\left\{\frac{a_1}{b_1}\right\} \cdot \left\{\frac{a_2}{b_2}\right\} = \left\{\frac{a_1 a_2}{b_1 b_2}\right\}.$$

It is immediately seen that the operations thus defined do not depend upon the choice of the pairs $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ in the classes $\left\{\frac{a_1}{b_1}\right\}$ and $\left\{\frac{a_2}{b_2}\right\}$. The zero of M is the class $\left\{\frac{0}{b}\right\}$, the unit is the class $\left\{\frac{b}{b}\right\}$, and the inverse of the class $\left\{\frac{a}{b}\right\} \neq 0$ is the class $\left\{\frac{b}{a}\right\}$. Thus M is a field. Finally, assigning to each element of $a \in C$ the element $\left\{\frac{ac}{c}\right\}$ of M, we obtain the natural isomorphic imbedding of C in M. Let now $a, b \in C$ be arbitrary elements of C with $b \neq 0$. Any subfield of M containing C must also contain the element $\left\{\frac{ac}{c}\right\}\left\{\frac{bc}{c}\right\}^{-1} = \left\{\frac{ac}{c}\right\}\left\{\frac{c}{bc}\right\} = \left\{\frac{a}{b}\right\}$. Thus M is a minimal field containing C.

In the commutative field M we have: $\frac{a}{b} = \frac{c}{d}$ is equivalent to $ad = bc$, and, in particular, $\frac{a}{b} = c$ is equivalent to $a = bc$; moreover,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d},$$
$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d},$$
$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \quad (c \neq 0).$$

We note that in [DV1] the field $\overline{\mathfrak{M}}_0$ is considered. The elements of this field are functions of a complex variable p. Every function is expressible in the form $\frac{f^*(p)}{g^*(p)}$, where $f^*(p)$ and $g^*(p)$ are defined by absolutely convergent Laplace integrals. Hence, $\overline{\mathfrak{M}}_0$ is the extension to a field of the ring S^* of all functions $f^*(p)$ which can be expressed in the form of absolutely convergent Laplace integrals in which the sum and product have their usual meanings. Each function $F(p) \in \overline{\mathfrak{M}}_0$ corresponds to an operator $F(D)$. The function $F(p) = 1$ corresponds to the identity operator, and its operation on $f(t)$ leaves the function unchanged. If $F(p) = p$, the operation of $F(D)$ on $f(t)$ $[f(0) = 0]$ coincides with differentiation: $Df = f'(t)$. Finally, the sum and product of functions $F(p)$ and $G(p)$ of $\overline{\mathfrak{M}}_0$ correspond to the sum and product of the corresponding operators. The set of all operators $F(D)$ defined in this way is denoted by $\mathfrak{M}_0$.

It is clear that $\mathfrak{M}_0$ and $\overline{\mathfrak{M}}_0$ are isomorphic. The completeness of the operator field $\mathfrak{M}_0$ must be investigated, and the following theorem is proved: Let a linear operator B with domain in the set of functions $f(t)$ having Laplace transforms satisfy the following conditions:

1) $Bf(t)$ is also a function having a Laplace transform,

2) if $f(t)$ belongs to Ω_B (the domain of the operator B), then the product $\omega(D)f$ also belongs to Ω_B and $\omega(D)Bf = B\omega(D)f$, where $\omega(D)$ is any operator defined on all functions having Laplace transforms.

Then there is a function $F(p) \in \overline{\mathfrak{M}}_0$ such that, for all $f \in \omega_B$, we have $Bf = F(D)f$, i.e., the operator B belongs to the field $\overline{\mathfrak{M}}_0$.

Mikusiński noted that there is no need to obtain the field $\overline{\mathfrak{M}}_0$ by an extension of the ring of functions expressible as absolutely convergent Laplace integrals, and that, in fact, we can directly extend the ring L of functions of a real variable $f(t)$ in which multiplication is defined to be the convolution $f*g = \int_0^t f(t-u)g(u)\,d(u)$, and addition has its usual definition.

It follows from Titchmarsh's theorem that the ring L has no zero divisors and so it can be extended to the field $\mathfrak{M}_0$ of quotients. It is plain that $\mathfrak{M}_0$ is contained in $\mathfrak{M}$. Mikusiński's method is a radical return to the original operational point of view and it gives an elegant and rigorous basis for operational calculus. When this method is used no limitations are imposed on the behavior of $f(t)$ as $t \to \infty$, and so the range of application of Mikusiński's operational calculus is considerably wider than that of operational calculus based on the Laplace transform. The elements of the Mikusiński operator field can be considered as generalized functions.

The field of Mikusiński operators is interesting because of its topological, or rather convergence structure. That is, it can be endowed with two types of convergence:

I. $x_n \to x$ if $x_n = f_n/g$, $x = f/g$ $(f_n, f, g \in C)$ and $f_n \to f$ almost uniformly as $n \to \infty$.

II. $x_n \to x$ if $x_n = f_n/g_n$, $x = f/g$ $(f_n, f, g_n, g \in C)$ and $f_n \to f$, $g_n \to g$ almost uniformly as $n \to \infty$.

Both types are very useful in operational calculus but neither of them can be described by topology (see for example [Bu1, Bu2]).

Mikusiński's book [MJ10] presents a profound and detailed exposition of the new theory of operational calculus; it contains many interesting examples of applications of the theory, together with discussions of new problems related to the structure of the field of operators $\mathfrak{M}$. Operational functions are considered in detail, definitions are given of derivatives and integrals of such functions, limits of sequences of operators are defined, and some classes of series of operators are investigated. One interesting part of the book deals with operator differential equations and problems arising in connection with the solution of these equations.

3.3. *The interpretations of operational calculus by Mikusiński's successors.* After the publication of Mikusiński's book [MJ10], a whole series of articles devoted to the new theory appeared. The product in the original ring M is defined by the relation

$$F(t) * G(t) = \frac{d}{dt} \int_0^t F(t-u)G(u)du. \tag{3.3.1}$$

The derivative (3.3.1) does not permit any distinction between constants and function-constants. The extension of the ring coincides with the Mikusiński operator field.

A unified treatment is given by Bellert [B1, B5], Słowikowski [Sl3], and Bittner [Bi2] of operational methods as applied to various types of problems such as the solution of differential and difference equations with constant coefficients, Euler equations, difference-differential equations, Bernoulli's equations, and some classes of nonlinear equations. Let a linear operator T satisfying the conditions

$$T(X) \subset X, \tag{3.3.2}$$

$$T[\alpha x_1 + \beta x_2] = \alpha T x_1 + \beta T x_2 \tag{3.3.3}$$

be defined in a linear space X over the field Z of complex numbers. Here, α and β are complex numbers and x_1 and x_2 are arbitrary elements of X. Such an operation is called an *endomorphism.* A consequence of condition (3.3.2) is that the operation T can be iterated several times in the space X. Thus the operation $T^n[x]$ is defined by the formula

$$T^n[x] = T[T^{n-1}[x]], \qquad n = 1, 2, \ldots \tag{3.3.4}$$

We use the symbol T^0 to denote the identity transformation

$$T^0[x] = x, \tag{3.3.5}$$

or, using (3.3.5),

$$T^0 = 1. \tag{3.3.6}$$

Let the set $x_i = T^i[x]$, $i = 0, 1, \ldots, n$, form a system of linearly independent elements, i.e., the condition

$$\sum_{i=0}^{n} \alpha_i T^i x = 0 \tag{3.3.7}$$

implies that $\alpha_0 = \alpha_1 = \ldots = \alpha_n = 0$ or $x = 0$ (n is a positive integer). The set P of polynomial operators

$$W(T) = \alpha_0 + \alpha_1 T + \ldots + \alpha_n T^n \qquad (\alpha_i \in Z) \tag{3.3.8}$$

forms a commutative ring without zero divisors. The extension of the ring P to the field of quotients leads to rational operators of the form

$$\frac{P(T)}{Q(T)} = \frac{\alpha_0 + \alpha_1 T + \ldots + \alpha_n T^n}{\beta_0 + \beta_1 T + \ldots + \beta_m T^m}, \qquad Q \neq 0, \tag{3.3.9}$$

where α_i and β_i are complex numbers. The result of the operation $(P/Q)x$ on x is defined to be the element $y \in X$ satisfying the equation

$$Px = Qy. \tag{3.3.10}$$

This result is unique, but it is not always feasible in a linear set X. For example, if X is the set of functions defined and Lebesgue integrable on $(0, \infty)$, then the equation

$$Tx(t) = \int_0^t x(\tau)\,d\tau$$

cannot be solved in X. This equation is solvable if X is the set of generalized functions. The rational operator field is isomorphic to the field of rational functions. Because of this isomorphism the rational operator (3.3.9) has a unique expansion in vulgar fractions $1/(T-\lambda)^k$. The rational operator $1/T^i$ is of special importance. This operator is denoted by T^{-1}. Thus,

$$T^i T^{-i} = T^{-i} T^i = 1.$$

The operator $1/T$ is denoted by p:

$$p = \frac{1}{T}. \tag{3.3.11}$$

More general polynomial and rational operators can be introduced as above. Let

$$W(T) = A_0 + A_1 T + \ldots + A_n T^n, \tag{3.3.12}$$

where $A_0, A_1, \ldots, A_n$ are endomorphisms that commute with the endomorphism T, and where the $x_i = A_i T^i x$ form a linearly independent system of elements. The set of operators (3.3.12) is a commutative ring without zero divisors and can be extended to the field of quotients. The generalized rational operator P/Q can be expanded in vulgar fractions of the form

$$\frac{B}{(1-CT)^k},$$

where B and C are endomorphisms and k is a positive integer. One of the applications of generalized operators is in solving differential equations of the form

$$x^{(n)}(t) + A_1 x^{(n-1)}(t) + \ldots + A_n x(t) = \alpha(t), \tag{3.3.13}$$

where $A_1, \ldots, A_n$ are continuous endomorphisms and $x(t)$ and $\alpha(t)$ are continuous functions of a real variable t with values in the linear topological space X. Equations of this type are investigated in articles by Bittner [Bi2] and Słowikowski [Sł3]. Because of lack of space we consider only the algebra of operators and do not touch on their analysis. We only note that the basic ideas of analysis have been introduced: convergence, limit, continuity, series, etc.

I. *Heaviside operators*. Let X be the set of functions integrable on any finite interval $[0, t_0]$ and let

$$Tf(t) = \int_0^t f(\tau)\, d\tau. \tag{3.3.14}$$

If a function $f(t)$ has continuous derivatives $f^{(n)}(t)$, then

$$Tf'(t) = f(t) - f(0)$$

and

$$f^{(n)}(t) = p^n f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) p^{(n-k)}, \qquad n = 1, 2, \ldots$$

If $h(t)$ is the result of applying the operation P/Q to the unit function, the result of applying this operator to any integrable function $f(t)$ can be written as

$$\frac{P}{Q}[f(t)] = \frac{d}{dt} \int_0^t h(t-\tau) f(\tau) d\tau.$$

II. *Euler's equation*. Let

$$Tf(t) = \begin{cases} \int_1^t \frac{f(\tau)}{\tau}\, d\tau & \text{for } t \geq 1, \\ 0 & \text{for } t < 1. \end{cases} \tag{3.3.15}$$

The operation T thus defined satisfies conditions (3.3.2) and (3.3.3).

If $f(t)$ has continuous derivatives $f^{(n)}(t)$, it follows from the relation $T[tf'(t)] = f(t) - f(1)$ after the substitution $T^{-1} = p$ that

$$\begin{aligned} t^{n+1} f^{(n+1)}(t) = {} & p(p-1)\ldots(p-n)f(t) \\ & - p(p-1)\ldots(p-n)f(1) - \ldots \\ & - p(p-1)f^{(n-1)}(1) - pf^{(n)}(1), \end{aligned} \tag{3.3.16}$$

where n is a positive integer.

Let $f(t) = 1$ in (3.3.15). We have

$$T = p^{-1} = \int_1^t \frac{d\tau}{\tau} = \ln t.$$

Repeating the operation $T(1)$ ν times, we obtain

$$p^{-\nu} = \frac{[\ln t]^\nu}{\nu!}. \tag{3.3.17}$$

Writing $f(t) = t^\alpha$ in (3.3.15), we have

$$T(t^\alpha) = \int_1^t \tau^{\alpha-1} d\tau = \left.\frac{\tau^\alpha}{\alpha}\right|_1^t = \frac{1}{\alpha}(t^\alpha - 1).$$

Hence,

$$t^\alpha = \frac{p}{p-\alpha}. \tag{3.3.18}$$

We illustrate the method by solving Euler's equation

$$t^2 f''(t) - f(t) = \ln t.$$

Let $F(t) = 0$ and $f'(1) = 1$. From (3.3.16) we obtain the operational equation

$$p(p-1)f(t) - f(t) = p[1] + \ln t,$$

and so

$$f(t) = \frac{p}{p(p-1)-1} + \frac{p}{p(p-1)-1}(\ln t).$$

Returning to (3.3.17) and (3.3.18), we have

$$f(t) = \frac{\alpha_1^2 - 1}{\alpha_1^2(\alpha_1 - \alpha_2)}(t^{\alpha_1} - 1) + \frac{\alpha_2^2 - 1}{\alpha_2^2(\alpha_2 - \alpha_1)}(t^{\alpha_2} - 1) - \frac{\ln t}{\alpha_1\alpha_2},$$

where

$$\alpha_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}, \quad \alpha_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}.$$

III. Using the operation T defined by the relation

$$T \triangle_\lambda f(t) = f(t) - f(0), \tag{3.3.19}$$

where $\triangle_\lambda f(t) = f(t+\lambda) - f(t)$, and writing $T^{-1} = p$, we have $\triangle_\lambda f(t) = pf(t) - pf(0)$, or, more generally,

$$\triangle_\lambda^n f(t) = p^n f(t) - \sum_{\nu=0}^{n-1} p^{n-\nu} \triangle_\lambda^\nu f(0), \tag{3.3.20}$$

where $\triangle_\lambda^0 f(0) = f(0)$. The operation T defined by (3.3.19), in contrast to the previous operations, does not yield a unique result. However, if two functions $f_1(t)$ and $f_2(t)$ satisfy the same equation $T[\triangle_\lambda f(t)] = x(t)$, i.e., $T(\triangle_\lambda f_1) = x$ and $T(\triangle_\lambda f_2) = x$, then the difference $f_1(t) - f_2(t)$ is a periodic function with period λ. Hence, the results of applying the operation under consideration are "unique" up to a periodic function with period λ. Uniqueness can be obtained, for example, by limiting the domain of definition of the operation $Tf(t)$ to a set of step functions or by directly assuming that the argument t takes the integral values $0, 1, 2, \ldots$ We give below the simplest operational formulas.

1) Let $f(t) = (\alpha+1)^{t/\lambda}$ in (3.3.19). We have

$$\triangle_\lambda f(t) = (\alpha+1)^{\frac{t+\lambda}{\lambda}} - (\alpha+1)^{\frac{t}{\lambda}} = \alpha(\alpha+1)^{\frac{t}{\lambda}}$$

and

$$\alpha(\alpha+1)^{\frac{t}{\lambda}} = p(\alpha+1)^{\frac{t}{\lambda}} - p \cdot 1,$$

and, finally,

$$(\alpha+1)^{\frac{t}{\lambda}} = \frac{p}{p-\alpha}. \tag{3.3.21}$$

2) If $f(t)=t$ in (3.3.19), we have $\triangle_\lambda f(t) = t+\lambda-t=\lambda$ and $p^{-1}=t/\lambda$, and, more generally,

$$p^{-\nu} = \frac{1}{\nu!}\left(\frac{t}{\lambda}\right)^{(\nu)}, \tag{3.3.22}$$

where

$$\left(\frac{t}{\lambda}\right)^{(\nu)} = \frac{t}{\lambda}\left(\frac{t}{\lambda}-1\right)\left(\frac{t}{\lambda}-2\right)\cdots\left(\frac{t}{\lambda}-\nu+1\right).$$

Appropriate endomorphisms can similarly be used in the solution of certain difference equations with variable coefficients, difference-differential equations, and some nonlinear equations.

We note that Murav'ev [Mu] has developed α-operational calculus by using the ideas of α-differentiation.

The operational calculus of Mikusiński's type for matrix-functions is considered by S. Ichikawa and K. Akira [Ich]. The convolution $H=||h_{ij}(t)||$ of two functions $F=||f_{ij}(t)||$ and $G=||g_{ij}(t)||$ is defined by the formula $H=F*G$, where

$$h_{ij}(t) = \sum_k \int_0^t f_{ik}(t-\tau)g_{kj}(\tau)\,d\tau.$$

In the articles of Gesztelyi [Ge1] and Gesztelyi and Száz [Ge2] operational calculus based on the convolution

$$f*g = \int_\alpha^\beta f[\mu^{-1}(\mu(x)-\mu(t))]g(t)\,d\mu(t)$$

is considered, where $-\infty \le \alpha < \beta \le \infty$, $\mu(x)$ is a continuous monotone nondecreasing function, and $\lim_{x\to\alpha+0}\mu(x)=-\infty$, $\lim_{x\to\beta-0}\mu(x)=\infty$. L. Berg [Be4] introduced the integral operator

$$F(p) = (Af)(p) = \int_a^b A(p,t)f(t)\,dt,$$

where

$$A(p,t) = \rho(t)\sum_{h=0}^\infty V_k(t)\int_0^\infty V_k(\tau)e^{-p\tau}d\tau,$$

and used the product

$$f\cdot g = \int_c^d B(p,\,t)F(p)G(p)dp,$$

where $B(p,t)$ is the kernel of the integral operator, which is inverse to A.

3.4. *Titchmarsh's theorem on convolutions.* Mikusiński's work on operational calculus has led to important applications of a theorem proved by Titchmarsh concerning convolutions. This theorem was originally proved in connection with investigations of the distribution of zeros of certain types of entire functions.

TITCHMARSH'S THEOREM. *If $\varphi(t)$ and $\psi(t)$ are integrable and*

$$\int_0^x \varphi(t)\psi(x-t)\,dt = 0$$

almost everywhere in $0 < x < k$, then $\varphi(t) = 0$ almost everywhere in $(0,\lambda)$ and $\psi(t) = 0$ almost everywhere in $(0,\mu)$, where $\lambda + \mu \geq k$.

The proof given by Titchmarsh [Tit] is based on classical theorems concerning the growth and distribution of zeros of entire functions and is somewhat complex. A simpler proof given by Crum [Cr] is also based on analytical function theory. Not knowing of the results obtained by Titchmarsh and Crum, Schwartz stated the result of Titchmarsh's theorem as a hypothesis in 1947 and related it to problems in distribution theory. In the same year Dufresnoy, who had not seen the results of Crum either, confirmed Schwartz's hypothesis and published a sketch of proof [Du1]. This proof uses results from the theory of subharmonic functions. In the following year, Dufresnoy published his proof in detail (see [Du2]). A simplified proof was given by Mikusiński and Świerczkowski [MJ42]. In 1951, Mikusiński and Ryll-Nardzewski [MJ39] proposed a new proof of Titchmarsh's theorem using methods from the theory of functions of a real variable. A simpler proof was given by Mikusiński in [MJ27]. Mikusiński gave still another proof at the Second Hungarian Mathematical Congress in 1960 [MJ32]. A generalization of Titchmarsh's theorem to convolutions of functions of several variables was published by Lions [Li], and by Mikusiński and Ryll-Nardzewski [MJ30, MJ41]. It was proved by Kalisch [Ka] that Titchmarsh's theorem is equivalent to a theorem due to Brodskiĭ [Bro] concerning the operator $If = \int_0^x f(t)dt$. The proof given by Kalisch was simplified by Schreiber [Sch].

Consequences of Titchmarsh's theorem related to the convergence of series of squares of absolute values of Fourier coefficients of square-integrable functions were given by Wolibner [Wo]. Properties of convolutions were also investigated in other works (see, for example, [We5]). The Foiaş theorem on convolution (see [Fo, MJ34, MJ35]) was first proved by applying the representation theorem for linear continuous functionals and the Hahn-Banach extension theorem so that transfinite methods were involved. J. Mikusiński, in [MJ39], eliminated transfinite methods by using the representation theorem of functionals on L^1. A new proof of the Foiaş Theorem on the convolution of integrable functions was given by W. Kierat and K. Skórnik (see [Ki6, Sk11, Sk12]) without the use of transfinite methods and without the use of the theorem on representation of functionals on L^1.

3.5. *Operational calculus for functions of integral arguments.* Operational calculus has been successfully applied in discrete analysis in which functions of integral arguments $f(n)$, where n is an integer, are investigated. In the book of Ditkin and Prudnikov [DV10] the function $f(n)$ is continued to all real numbers by setting $\varphi(t) = f(n)$ for $n \leq t < n+1$. If we write $[t]$ for the integral part of t, we have $\varphi(t) = f([t]) \equiv f[t]$. The function $f[t]$ is called a step function. The general methods of operational calculus are applied to these functions. By this means, formulas are obtained for functions of integral arguments similar to the formulas for the transition from the operation of differentiation to the operator p; for discrete functions there are formulas for the transition from the difference operator to the operator $r = 1 - e^{-p}$. For example, the equation $f'(t) = pf(t) - pf(0)$ becomes

$$\nabla f[t] = rf[t] - rf[-1],$$

where

$$\nabla f[t] = f[t] - f[t-1].$$

The nth order difference operator can be expressed in terms of the operator r and its powers $r^2, r^3, \ldots, r^n$. Such relations can be used to solve finite difference equations with constant coefficients and in special cases with variable coefficients. Operational relations can be used in some cases to find sums of numerical series. It is easily proved, for example, that

$$\sum_{k=0}^{n} P_{n-k}(\cos\Theta)P_k(\cos\Theta) = \frac{\sin(n+1)\Theta}{\sin\Theta},$$

where $P_n(x)$ is a Legendre polynomial, or

$$\sum_{n=0}^{\infty}(-1)^n(2+n)(2\cdot 2+n)\ldots(2m+n)\lambda^n = \sum_{k=0}^{m}\frac{(m+k)!}{2^k k!}(1+\lambda)^{k-1-m}.$$

A special operational calculus can be developed for discrete functions (or, more precisely, for functions defined on the set of all nonnegative integers) without having recourse to the general theory of functions defined on the half-line $(0,\infty)$. Let N be the set of all functions $f(x)$ defined for $x = 0, 1, 2, \ldots$ The sum of functions $f(x)$ and $g(x)$ of N is defined as usual. The product is defined by the formula $f(x) * g(x) = h(x)$, where $h(0) = f(0)g(0)$, and

$$h(x) = \sum_{r=0}^{x} f(x-r)g(r) - \sum_{r=0}^{x-1} f(x-1-r)g(r)$$

for all integral $x > 0$. Then N is a commutative ring without zero divisors. Hence, N can be extended to the field of quotients $\mathfrak{N}$. The function $f(x) = 1$ for $x = 0, 1, 2, \ldots$ is the unit of the ring N, and hence also the unit of $\mathfrak{N}$. The elements of $\mathfrak{N}$ are called operators. It can be shown that the operator $1/x = s$ does not belong to the original ring N, i.e., this operator does not lead to a function of a discrete valued variable. In general, the product $s * f(x)$ is an operator. In order that the product $s * f(x)$ lead to a function, it is necessary and sufficient that $f(0) = 0$. In this case, $s*f(x) = f(x+1) - f(x) = \triangle f(x)$. The set of all functions of

N equal to zero for $x = 0$ forms an ideal of N. In this ideal the operator s coincides with the difference operator $\triangle$. In the general case, $s*[f(x)-f(0)] = \triangle f(x)$ and so (omitting the multiplication symbol $*$) we have $\triangle f(x) = sf(x) - sf(0)$. Hence

$$\triangle^n f(x) = s^n f(x) - s\triangle^{n-1} f(0) - s^2\triangle^{n-2} f(0) - \ldots - s^n f(0). \tag{3.5.1}$$

The operator $1/s = x$ is the summation operator

$$\frac{1}{s} f(x) = \sum_{r=0}^{x-1} f(r).$$

If we write $x^{(0)} = 1$ and $x^{(n)} = x(x-1)\ldots(x-n+1)$, we easily prove that

$$\frac{1}{s^n} = \frac{x^{(n)}}{n!}.$$

Hence,

$$\frac{1}{s^n} f(x) = \begin{cases} 0 & \text{for } x = 0, \\ \displaystyle\sum_{r=0}^{x-1} \frac{(x-1-r)^{(n-1)} f(r)}{(n-1)!} & \text{for } x \geq 1. \end{cases} \tag{3.5.2}$$

The relation

$$\frac{s}{(s-\lambda)^{n+1}} = \frac{x^{(n)}}{n!}(1+\lambda)^{(x-n)}, \qquad \lambda \neq -1, \tag{3.5.3}$$

is easily verified. Let

$$\eta_p(x) = \begin{cases} 0 & \text{if } x < p, \\ 1 & \text{if } x \geq p, \end{cases}$$

where x and p are positive integers.

Then $\eta_p(x) * \eta_q(x) = \eta_{p+q}(x)$ and $\eta_0(x) \equiv 1$. Hence, $\eta_p(x) = \eta_1^p(x)$. The functions $\eta_p(x)$ can be expressed in terms of the operator s:

$$\frac{1}{(s+1)^p} = \eta_p(x). \tag{3.5.4}$$

For any function $f(x) \in N$ we have

$$f(x) = s\sum_{r=0}^{n} \frac{f(r)}{(s+1)^{r+1}} = \bar{f}(s). \tag{3.5.5}$$

If we now let s be a complex number, this relation sets up a correspondence between the analytic function $f(s)$ and the function $f(x)$ if the series $\sum_{r=0}^{\infty} f(r) \times (s+1)^{r+1}$ converges. The product $f(x) * g(x)$ corresponds to the ordinary product $\bar{f}(s)\bar{g}(s)$, and, by the same token, constants are converted into constants, and, in particular, unity is converted into unity. It is plain that (3.5.5) plays the same role in the operational calculus of functions of a discrete variable that the Laplace-Carson integral plays in the operational calculus of functions defined on $(0, \infty)$. By analogy with the Heaviside calculus we can develop operational calculus for discrete functions further; we can thus derive methods for the solution of systems of difference equation with constant coefficients and for the solution of linear differential-difference equations and solve many other problems.

S. Bellert [B3] considered functions of integral valued arguments defined for all integers and equal to zero for all sufficiently large negative values of the argument. For every function $f(x)$ there is a number n_0 such that $f(x) = 0$ for all $x < n_0$. The sum of two functions is defined as usual, while the product is

$$f(x) * g(x) = \sum_{n=-\infty}^{\infty} f(x-k)g(k). \tag{3.5.6}$$

The set of all such functions forms a ring without zero divisors. The function $e(x)$ equal to one for $x = 0$ and zero elsewhere is the unit of the ring. The inverse of the function $q(x)$ equal to one for $x = 1$ and zero elsewhere is $\frac{e(x)}{q(x)} = p(x)$, where $p(-1) = 1$ and $p(x) = 0$ for $x \neq -1$. Hence, for all integers n we can write $q^n(x) = q(x) * q^{n-1}(x)$ and $q^0(x) = e(x)$, and for any function $f(x)$ we have

$$f(x) = \sum_{k=-\infty}^{\infty} f(k)q^k(x).$$

Bellert [B3] calls the set of all values of a function $f(x)$, i.e., the set of numbers

$$\{\ldots, 0, 0, \ldots, f(n_0), f(n_0-1), \ldots f(0), f(1), f(2), \ldots, f(k), \ldots\},$$

a "numerical operator". Starting from (3.5.6) he develops an algebra of numerical operators, establishes the relation between some operators and operators encountered in the theory of finite differences, and demonstrates the value of his theory in applications to the solution of linear difference equations. The rest of Bellert's work is devoted to an investigation of the applicability of his theory in the study of the stability of pulse control and to the investigation of transition functions in electrical systems disturbed by complex signals.

3.6. *Operational calculus on a finite interval.* Operational calculus on a finite interval, which is similar to operational calculus on the whole line, was developed by Mikusiński [MJ15]. The main difference between the two cases is that on a finite interval the original ring of functions has zero divisors. Let C_T be the ring of continuous complex valued functions defined in $0 \leq t \leq T$ with the usual definition of addition and with multiplication defined by the formula

$$ab = \int_0^t a(t-\tau)b(\tau)\,d\tau.$$

The divisors of zero in C_T are the functions which vanish in some neighborhood of zero. Let C_T^* be the set of functions which are not zero divisors. The elements $\frac{a}{b}$ $(a \in C_T,\ b \in C_T^*)$ form a ring of operators. This ring is isomorphic to the set of all functions locally integrable in the finite interval $0 \leq t \leq T$. A polynomial with operator coefficients can be expressed uniquely in the form $\prod_{i=1}^{n}(x - b_i)$ if and only if the operators $b_i - b_j$ $(i \neq j)$ are not zero divisors. Operational functions are defined and the analysis of operational functions is carried out by analogy with operational calculus for an infinite interval. The following operator classification is used:

1) right logarithm, if the operational function $e^{\lambda w}$ exists for $\lambda \geq 0$ but does not exist for $\lambda < 0$;

2) left logarithm, if the operational function $e^{\lambda w}$ exists for $\lambda \leq 0$ but does not exist for $\lambda > 0$;

3) two-sided logarithm, if the operational function $e^{\lambda w}$ exists for all real λ.

Problems in the theory of operational equations have been studied. In this theory an algebraic equation of degree n can have an infinite number of solutions. The usual uniqueness theorem for differential equations does not hold in this theory even for first-order equations.

Operational calculus on a finite interval can be used to solve classical partial differential equations with constant coefficients. In this case these equations can be solved in regions in which the variable t lies in a finite interval, and existence and uniqueness theorems can be proved for Cauchy's problem and for the mixed problem. For example, Tikhonov's theorem [Tik] concerning the uniqueness of the solution of the heat conduction equation

$$x_{\lambda\lambda}(\lambda, t) = x_{\lambda}(\lambda, t) \qquad (0 \leq t \leq t_0;\ -\infty < \lambda < \infty)$$

can be proved in the class of functions satisfying the condition

$$\max |x(\lambda, t)| e^{-C\lambda^2} \to 0 \qquad \text{as } |\lambda| \to \infty$$

for some C.

The following more general theorem was proved by Kierat and Mikusiński [Ki5].

THEOREM. *If w is a logarithm and α is a characteristic value, then the operator w is of the form $w = w_0 - \alpha s$, where w_0 is a two-sided logarithm and s is a differential operator.*

Antosik [An1] proved that the field of operators on the closed interval $0 \leq t \leq T$ is isomorphic to the field of operators on the open interval $0 \leq t < T$. Operational calculus is treated on the basis of the finite Laplace transform by G. Capriz (1953) and D. Mangeron (1946).

3.7. *Operational calculus for functions defined on the whole axis.* Operational calculus for functions defined on the whole axis is based on the two-sided Laplace transform

$$f^*(p) = \int_{-\infty}^{\infty} f(t) e^{-pt} dt, \qquad p = \sigma + i\tau.$$

This transform is investigated by Van der Pol and Bremmer [VB].

Operational calculus on the whole axis is developed by Ditkin [DV2] and the idea of the region of definition of an operator is introduced. Let S be the set of functions $f(x)$ defined almost everywhere on the real line $-\infty < x < \infty$, Lebesgue integrable on any finite interval, and such that there is at least one pair of values p_1 and p_2 for which the integrals

$$\int_a^\infty f(x)e^{-p_1 x}\,dx \quad \text{and} \quad \int_{-\infty}^b f(x)e^{-p_2 x}\,dx$$

converge.

Let S_+^* (S_-^*) denote the set of functions of a complex variable such that the following integrals are convergent:

$$f_1^*(p) = \int_{-\infty}^\infty f_1(x)e^{-px}\,dx, \quad f_1(x) = 0 \quad \text{for } x < a$$

$$\left(f_2^*(p) = \int_{-\infty}^\infty f_2(x)e^{-px}dx,\ f_2(x) = 0 \text{ for } x > b\right).$$

The sets S_+^* and S_-^* are linear (with the usual definitions of addition and multiplication by complex numbers). The direct sum $\mathfrak{M}$ of S_+^* and S_-^* contains a linear subset $\mathfrak{M}_0$ consisting of pairs of functions $(\Theta^*(p), \Theta^*(p))$, where

$$\Theta^*(p) = \int_a^b \Theta(x)e^{-px}\,dx,$$

and a and b are arbitrary real numbers [$\Theta(x)$ belongs to the intersection $S_+ \cap S_-$].

Theorem. *The linear sets S and S^* are isomorphic* (S^* *is the quotient set* $\mathfrak{M}/\mathfrak{M}_0$).

This theorem is used in the derivation of divisibility theory in $\mathfrak{M}$, on the basis of which a linear operator F (in general not single-valued) with domain of definition Ω_F is introduced which is the inverse image by the isomorphism $S \leftrightarrow S^*$ of the set of pairs $(f_1^*(p), f_2^*(p))$ of $\mathfrak{M}$, for which the pairs $(F_1(p)f_1^*(p), F_2(p)f_2^*(p))$ also belong to $\mathfrak{M}$, where $F_1(p)$ is a quotient of two functions of S_+^* and $F_2(p)$ is a quotient of two functions of S_-^*.

If $f \in \Omega_F$, then $Ff = g$, where g is the inverse image of a class g^*, the representation of which is the pair $(F_1(p)f_1^*(p), F_2(p)f_2^*(p))$. In some cases, F can be identified with $F(p)$.

Norris [No1, No2] uses Mikusiński's method as a basis for the development of operational calculus on the whole axis. The original set used for the construction of the operational calculus is the set C of all complex valued continuous functions $c(t)$ of real variable t defined on the whole axis $-\infty < t < \infty$, for each of which there exists a real number σ such that $c(t) = 0$ for all $t < \sigma$. Addition in C and multiplication by complex number have their ordinary definition. The product of two elements is defined to be the convolution

$$a(t) * b(t) = \int_{-\infty}^{\infty} a(x)b(t-x)\,dx.$$

These definitions of addition and multiplication convert C into a commutative ring without zero divisors which is used in the development of operational methods.

4. Operator differential equations and series. J. Mikusiński [MJ7, MJ10, MJ12, MJ17] has the greatest number of results for the linear differential equation of order n with constant coefficients of the form

$$a_n x^{(n)}(z) + a_{n-1}x^{(n-1)}(z) + \ldots + a_0 x(z) = f(z) \tag{4.1}$$

where a_i, $i = 0, 1, 2, \ldots, n$, are Mikusiński's operators and $f(z)$ is an operator function. In particular, he showed that the set of all solutions of this equation for $f = 0$ is a vector space over the operator's field.

Theorems on the existence of solution for a homogeneous equation of the first order also belong to Mikusiński. The solution of the homogeneous equation

$$a_n x^{(n)}(z) + a_{n-1}x^{(n-1)}(z) + \ldots + a_0 x(z) = 0 \tag{4.2}$$

is sought in the form of an exponential operator function $x(\lambda) = e^{\lambda w}$. The operator w is given by the characteristic equation

$$a_n w^n + a_{n-1}w^{n-1} + \ldots + a_0 = 0. \tag{4.3}$$

Let the roots of (4.3) be $w_1, w_2, \ldots, w_n$. The general solution of (4.2) will be the operator

$$x(\lambda) = \sum_{k=1}^{n} C_k e^{\lambda w_k}, \tag{4.4}$$

where C_k are arbitrary constant operators. An operator w is called a logarithm if the exponential operator function $e^{\lambda w}$ exists. In an article by W. Kierat and K. Skórnik [Ki7] a representation theory of the exponential operator function $e^{\lambda w}$ is given, together with a sufficient condition for it to be parametric. C. Ryll-Nardzewski [RN1] has proved that if $\limsup n^{\delta n}|a_n| < \infty$ for some $\delta > 1$, then the series $\sum a_n \lambda^n s^n$ is operational convergent for $\lambda \in \mathbb{C}$, where s is the differential operator. Moreover, if $\limsup n^n |a_n| > 0$ then the above series is not convergent for any $\lambda \in \mathbb{C}$. In an article on convergent power series of differential operators in the field of Mikusiński operators, Kierat [Ki1] shows that the above conditions are necessary neither for the convergence nor for the divergence of that series.

F. Stopp [Sto] used the field constructed by L. Berg which is isomorphic to the Mikusiński field to prove some theorems of Mikusiński about the existence of solution and the number of linearly independent solutions for differential equations of order n.

Most of the results we know about linear differential equations with variable coefficients are due to B. Stanković [St2, St3, St4]. In his works two directions can be seen. First he used the ideas of Ditkin of generalized Laplace transform. Secondly he considered a subset of operators with a richer topological structure in which it was possible to use the results of functional analysis. Stanković proved some theorems on the existence of solution of the nonhomogeneous equation $x'(z) + w(z)x(z) = h(z)$, $z_1 \le z \le z_2$, where $w(z)$ and $h(z)$ are operator functions. The unicity and existence of solution of the initial problem are proved only in some special cases of $w(z)$ and $h(z)$.

5. The generalized Laplace transform. Mikusiński's operational calculus eliminates the necessity of using Laplace transforms. The Laplace integral, however, is a natural means of expressing operators which have Laplace transforms. An operator $a \in \mathfrak{M}$ is called *Laplace transformable* if there exists a pair of functions $[F(t), G(t)]$ such that $a = F/G$ and the integrals $F^*(z) = \int_0^\infty F(t)e^{-zt}dt$ and $G^*(z) = \int_0^\infty G(t)e^{-zt}dt$ are absolutely convergent. The set of all Laplace-transformable operators forms a field $\mathfrak{M}_0 \subset \mathfrak{M}$ (see [DV10]). This field is isomorphic to the field $\overline{\mathfrak{M}}_0$ of all functions of a complex variable expressible in the form $\frac{F^*(z)}{G^*(z)}$, where $F^*(z)$ and $G^*(z)$ are expressible by Laplace integrals in some half-plane Re $z > \gamma$. In this isomorphism the operator $1/t = p$ corresponds to z, the function $f(t)$ corresponds to $\bar{f}(z) = z\int_0^\infty f(t)e^{-zt}dt$, and the operator $a = \frac{F}{G} \in \mathfrak{M}_0$ corresponds to $\bar{a}(z) = \frac{F^*(z)}{G^*(z)}$.

Laplace-transformable operators are a large and important class of operators encountered in applications. In many cases the isomorphism of the fields $\mathfrak{M}_0$ and $\overline{\mathfrak{M}}_0$ leads to simple derivations of various operational relations, rules and formulas. For example, we can identify the elements of $\mathfrak{M}_0$ and $\overline{\mathfrak{M}}$ which correspond in the isomorphism $p = z$, $f(t) = \bar{f}(p)$, and $a = \bar{a}(p)$.

The representation of the whole field $\overline{\mathfrak{M}}$ of Mikusiński operators by functions of a complex variable is considered in the book of Ditkin and Prudnikov [DV10]. It is proved that $\mathfrak{M}$ is isomorphic to a field $\overline{\mathfrak{M}}$ consisting of elements closely related to functions of the form $\frac{F^*_\omega(z)}{G^*_\omega(z)}$, where $F^*_\omega(z) = \int_0^\omega F(t)e^{-zt}\,dt$ and $G^*_\omega(z) = \int_0^\omega G(t)e^{-zt}\,dt$. This work is based on the idea of the generalized Laplace transform.

Let S^* be the set of all functions of a complex variable z expressible in the form of absolutely convergent Laplace integrals. The set S^* is a ring. We write J_ω for the set of all functions of S^* that can be expressed in the form $e^{-\omega z}F^*(z)$, where $F^*(z) \in S^*$ and $\omega \ge 0$. The set J_ω is an ideal ring in S^*. We form the residue ring with respect to the ideal J_ω: $S^*_\omega = S^*/J_\omega$, and denote the elements of S^*_ω by F_ω, G_ω, etc. We now form the direct sum of the sets S^*_ω in which ω runs over the ordered set of all positive numbers. The elements of the direct sum are denoted by $\{F_\omega\}$, $\{G_\omega\}, \ldots$ The direct sum is thus a ring. Among the elements $\{F_\omega\}$ of the direct sum of the sets S^*_ω we consider those for which the condition $F_\omega \supset F_{\omega_1}$ is satisfied whenever $\omega_1 > \omega$. Here F_ω and F_{ω_1} are considered to be sets from S^*. The set of all elements $\{F_\omega\}$ forms a ring L^*. Let $F(t)$ be any function locally integrable in $(0,\infty)$, and let $F^*_\omega = \int_0^\omega F(t)e^{-zt}dt$. Let F_ω be the residue class of the set S^*_ω containing $F^*_\omega(z)$. Since $F^*_{\omega_1}(z) - F^*_\omega(z) \in J_\omega$ for $\omega_1 > \omega$, it follows that $F_\omega \supset F_{\omega_1}$. Hence, $\{F_\omega\} \in L^*$. The element $\{F_\omega\}$ is called the *generalized Laplace transform* of $F(t)$. The properties of the usual Laplace transform carry over to the generalized Laplace transform, including the multiplication theorem (Borel's theorem). By means of the generalized Laplace transform every operator $a = \frac{F(t)}{G(t)} \in \mathfrak{M}$ can be made to correspond to an element $\{F_\omega\}/\{G_\omega\}$ of the field $\overline{\mathfrak{M}}$, where $\overline{\mathfrak{M}}$ is the extension of the ring L^* to the field of quotients. In this way the structure of Mikusiński's operator field and its relation to functions of a complex variable can be investigated.

The isomorphic representation of the field $\mathfrak{M}$ is realized by fundamental sequences $\{\varphi_n\}$ of functions $\varphi_n(z)$ analytic in the right half-plane (L. Berg [Be2]). A sequence $\varphi_1(z), \varphi_2(z), \ldots$ of functions analytic in some right half-plane is called *fundamental* if, as $|z| \to \infty$, the relation

$$\varphi_m(z) = \varphi_n(z) + O(e^{-nz}), \qquad \operatorname{Re} z \geq \lambda_{m,n}, \tag{5.1}$$

holds for any $m > n > 0$, $\varphi_0(z) = 0$. If addition and multiplication of fundamental sequences is defined termwise, we have thus obtained a ring. The null sequences $\varphi_n(z) = O\,(e^{-nz})$, $n = 1, 2, 3, \ldots$, form an ideal in this ring. We denote the ring of residues with respect to this ideal by R. In this ring two sequences $\{\varphi_n(z)\}$ and $\{\psi_n(z)\}$ are considered to be equal if $\varphi_n(z) - \psi_n(z) = O(e^{-nz})$, $n = 1, 2, 3, \ldots$ The ring M can be mapped homomorphically into the ring of residue classes R if a function $F(t) \in M$ is mapped into the residue class represented by the fundamental sequence with terms

$$\bar{F}_n(z) = z \int\limits_0^n F(t)e^{-zt}\,dt, \qquad n = 1, 2, 3, \ldots$$

If a residue class has a representation $\bar{F}(z)$ with terms that are independent of n, then $\bar{F}(z)$ is the unique representation. The set of all residue classes with such representations forms a subring of R which is isomorphic to the ring of all analytic functions bounded in some right half-plane. Hence, we can replace all residue classes of R with representations of the form $\{F(z)\}$ by the functions $F(z)$. For example, in the above homomorphism the function $t^n/n!$ is mapped into the function $1/z^n$. Conversely, the ring R can be mapped homomorphically into the field $\mathfrak{M}$ of quotients. The set R has no zero divisors. The corresponding field of quotients is isomorphic to the field $\mathfrak{M}$. Division of two classes can be performed term-by-term for their representations if we first omit the finite number of terms whose denominators can be equal to zero. Hence, the representation $\{\varphi_n\}/\{\psi_n\}$ of elements of the field of quotients has the form

$$\left\{\frac{\varphi_n}{\psi_n}\right\} = \left\{\frac{\varphi_\nu(z)}{\psi_\nu(z)}, \frac{\varphi_{\nu+1}(z)}{\psi_{\nu+1}(z)}, \dots\right\}$$

with ν sufficiently large. Two representations $\{\varphi_n/\psi_n\}$ and $\{\varphi_n^*/\psi_n^*\}$ determine the same element of the field of quotients if

$$\varphi_n(z)\,\psi_n^*(z) - \varphi_n^*(z)\,\psi_n(z) = O(e^{-zn})$$

for all $n > 0$. Hence, the Mikusiński operator field $\mathfrak{M}$ is isomorphic to the field of quotients with elements of the form $\{\varphi_n/\psi_n\}$ in which the analytic functions $\varphi_n(z)$ and $\psi_n(z)$ satisfy (5.1).

A definition is given of convergence of operators, and, in particular, the derivative $a'(\lambda)$ of an operation function $a(\lambda) = \frac{F(t;\lambda)}{G(t;\lambda)}$ is defined to be

$$a'(\lambda) = \frac{F'_\lambda * G - F * G'_\lambda}{G * G}.$$

It is proved, with certain limitations, that if $a'(\lambda) = 0$, $\alpha < \lambda < \beta$, then the operator $a(\lambda)$ is independent of the parameter λ [Be2]. The structure of the Mikusiński operator field has also been investigated by Weston [We1], Ryabtsev [Ry2], and Stanković and Skendžić [St4].

6. Operational calculus and generalized functions. Operational calculus can be based on Schwartz's theory of generalized functions [Sc2]. There are now several definitions of generalized functions (distributions in Schwartz's terminology). Sobolev [Sob] defines generalized functions to be linear functionals defined in certain function spaces. Schwartz also uses this definition. Mikusiński [MJ3] introduces generalized functions as forming the closure of the space of continuous functions in which an appropriate definition of weak convergence is defined. Generalized functions are investigated by C. Berge, A. Erdélyi, H. König, J. Korevaar, in the review by G. Temple, J. F. Colombeau and in other articles.

Schwartz introduces the idea of convolution for a certain class of generalized functions and considers the Laplace transform of generalized functions. These two fundamental ideas are the basis for the development of a theory of generalized functions. Moreover, we can use known methods to derive a rigorous theory of operator calculus, or, in the former terminology, operational calculus. Work has been carried out in this direction by Schwartz and Lavoine. Generalized functions are considered to be linear functionals in a basic space K of finite, infinitely differentiable test functions $\phi(x)$ of the real variable x.

P. Antosik was mainly interested in the sequential theory of distributions. In [An3] he proves that regular operations on distributions are continuous. He is also a coauthor of the book [An6]. The theory of tempered distributions in the book is based on Hermite polynomials [An5]. The equivalence of functional and sequential approaches to distributions is based on so-called diagonal theorems originated by J. Mikusiński and developed by Antosik [An4]. The diagonal theorem method replaces in many cases the Baire category arguments and leads to more general results [An7].

Many testing spaces which occur in the theory of generalized functions have the structure of scales of Hilbert spaces. W. Kierat [Ki2] considered the scales H_k of Hilbert spaces which are the completions of some spaces of smooth functions with respect to the norms generated by quadratic forms of the powers of Sturm-Liouville operators. In that paper Kierat investigated the expansion of the elements of H_k, H, H'_k and H' in Fourier series with respect to the eigenfunctions of Sturm-Liouville operators, where H is the projective limit of the spaces H_k. For the harmonic oscillator we obtain the scale of Hilbert spaces S_k, the Schwartz space S is the projective limit of S_k.

Kierat [Ki3] considered the spaces $A(\mathbb{R}^q)$ which are equipped with a locally convex barrelled topology having the following properties:

(a) $D(\mathbb{R}^q) \subset A(\mathbb{R}^q) \subset S(\mathbb{R}^q)$;

(b) $D(\mathbb{R}^q)$ is dense in $A(\mathbb{R}^q)$, the topology induced on $D(\mathbb{R}^q)$ from $A(\mathbb{R}^q)$ is weaker than the natural topology of $D(\mathbb{R}^q)$.

Let $A'(\mathbb{R}^q)$ denote the dual space of $A(\mathbb{R}^q)$ and let $\{\delta_\nu\}$ be a sequence of smooth functions which approximate the Dirac δ-distribution. Under some additional assumptions the following is proved.

THEREM. *A distribution T is in $A'(\mathbb{R}^q)$ if and only if $T * \delta_\nu$ converges in $A'(\mathbb{R}^q)$ with respect to the weak star topology.*

In [Ki3] examples of spaces of type $A(\mathbb{R}^q)$ are given. In [Ki4] it is shown that if

$$x^\mu \frac{\partial^{|\nu|}}{\partial x^\nu} f \in L^2(\mathbb{R}^q)$$

for $|\mu+\nu| \le k$, $k > q/2$, then $(\mathfrak{F}^{-1} \circ \mathfrak{F}) f(x) = f(x)$ for $x \in \mathbb{R}^q$, where $\mathfrak{F} f$ and $\mathfrak{F}^{-1} f$ denote the Fourier transform and the inverse Fourier transform of f.

K. Skórnik [Sk3] introduced hereditarily periodic distributions (of q real variables $\xi_1, \ldots, \xi_q$) which can also be characterized by their Fourier coefficients or by the integrals over their periods. Every periodic distribution is a sum of hereditarily periodic distributions of some variables $\xi_1, \ldots, \xi_q$. An estimation of Fourier coefficients is given (see also [Sk2]). A concept of a smooth integral is introduced (in [Sk3]) as a substitute for the integral from a fixed point to a variable point, which cannot be used in the case of distributions.

The tempered derivatives D^k of positive integer order k were introduced by J. Mikusiński in [MJ36]. He used this definition to determine the tempered distributions as tempered derivatives, of some order $k \in \mathbb{N}_0 = N \cup \{0\}$, of square integrable functions. In [An6] the tempered integrals S^k, $k \in \mathbb{N}_0$, are defined. The definitions of tempered integrals and tempered derivatives were extended to nonnegative orders by K. Skórnik [Sk4] (see also [Sk5]). The paper [Sk5] deals with a class of distributions having the property that the inverse operation to differentiation is unique. It is shown that if the tempered derivative is uniquely invertible in a class of distributions, then it can be extended to tempered derivatives of any nonnegative real orders so that the following equalities hold:

$$D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f, \quad S^{\alpha}S^{\beta}f = S^{\alpha+\beta}f \quad \text{for all } \alpha, \beta \geq 0.$$

In [Pi1] and [Pi2] the a-tempered integrals and a-tempered derivatives are defined. Using special sequences of a-tempered integrals and a-tempered derivatives the author characterized some subspaces of D'. Skórnik introduced local derivatives of functions of q real variables with values in Hilbert spaces (see [Sk1]) and in Banach spaces (see [Sk6, Sk7]). In [Sk9] it is shown that the local derivative and Sobolev's derivative are equivalent. In [Sk10] there is given a general form of functions f from $\mathbb{R}^q$ to a Banach space which are locally Bochner integrable and satisfy the differential equation $f^{(m)} = 0$, where m is a multi-index and the derivative is in the sense of Sobolev. The local derivatives of functions of one real variable can be found in [MJ37].

Mikusiński operational calculus or operational calculus using the Laplace transform are based on functions everywhere locally integrable in $[0,\infty]$, and so, functions important in applications which are everywhere locally integrable except for isolated points, for example $f(t) = 1/t^n$, $n = 1, 2, 3, \ldots$, are excluded. There are several articles devoted to the inclusion of such function in a generalized theory of generalized functions. There is reason to believe that such functions can be included in a general theory. Thus, the class of functions having power singularities at a finite number of points of $[0,\infty]$ is included among generalized functions. The corresponding generalized functions are obtained by the process of regularization. This regularization is defined up to addition of a functional concentrated at the singular points. Another method of including nonintegrable functions in operational calculus is based on ideas concerning the finite part of a divergent integral. These ideas were first introduced by Hadamard [Had] and D'Adhemar [DAd]. A series of articles appeared later in which consideration was given to the properties of finite parts of divergent integrals and to the application of Hadamard's theory in operational calculus and generalized functions. According to Hadamard, the finite part of a divergent integral

$$\int_a^b \frac{A(x)dx}{(x-a)^{p+\mu}},$$

where p is a positive integer, $0 < \mu < 1$, and $A(x)$ is p-times differentiable in the neighborhood of $x = a$, is the limit

$$\lim_{x\to a}\left(\int_x^b \frac{A(a)dx}{(x-a)^{p+\mu}} - \frac{B(x)}{(x-a)^{p+\mu-1}}\right),$$

where

$$B(x) = \sum_{\nu=0}^{p-1} \frac{(x-a)^{\nu}}{p+\mu-\nu-1} \frac{A^{(\nu)}(a)}{\nu!}.$$

The finite part of a divergent integral $\int_a^b f(x)dx$ is often written $\overline{\left|\int_a^b f(x)\,dx\right.}$ or P.F. $\int_a^b f(x)dx$ (la partie finie). Various modifications of the definition of the finite part of an integral were suggested later.

The simplest functions having power or logarithmic singularities at $x=0$ are of the form $f(x) = x^{\alpha} \ln^k x$, where α is arbitrary and k is a nonnegative integer; the finite parts of integrals of these functions are defined to be

$$\overline{\left|\int_a^b x^a \ln^k x\,dx\right.} = \frac{b^{\alpha+1}}{\alpha+1}\left[\ln^k b - \frac{k}{\alpha+1}\ln^{k-1} b \right.$$
$$\left. + \frac{k(k-1)}{(\alpha+1)^2}\ln^{k-2} b + \ldots + (-1)^k \frac{k!}{\alpha+1}\right] \quad \text{for } \alpha+1 \neq 0$$

and

$$\overline{\left|\int_a^b \frac{\ln^k x\,dx}{x}\right.} = \frac{\ln^{k+1} b}{k+1} \qquad \text{for } \alpha = -1.$$

For such functions we have

$$\overline{\left|\int_0^t f(x)\,dx\right.} = f(t). \tag{6.1}$$

The finite part of an integral has the following properties:

1. If $f(x)$ is integrable in (a,b), then

$$\overline{\left|\int_a^b f(x)\,dx\right.} = \int_a^b f(x)\,dx.$$

2. If α is any number, then

$$\overline{\left|\alpha\int_a^b f(x)\,dx\right.} = \alpha\,\overline{\left|\int_a^b f(x)\,dx\right.}..$$

3. If the finite parts $\overline{\left|\int_a^b f(x)\,dx\right.}$ and $\overline{\left|\int_a^b g(x)\,dx\right.}$ exist, then

$$\overline{\left|\int_a^b [f(x)+g(x)]\,dx\right.} = \overline{\left|\int_a^b f(x)\,dx\right.} + \overline{\left|\int_a^b g(x)\,dx\right.}.$$

4. We have

$$\overline{\left|\int_a^b f(x)\,dx\right.} = \overline{\left|\int_a^c f(x)\,dx\right.} + \overline{\left|\int_c^b f(x)\,dx\right.}, \qquad a < c < b.$$

There is a formula for $\overline{\left| \int_a^b f'(x)g(x)\,dx \right.}$ analogous to the formula for integration by parts. The idea of the principal part of a divergent integral was extended to cover multiple integrals by J. Gilly [Gi3]. Functions with discrete nonintegrable singularities can be included in general operational calculus theory by various methods. For example (see the book of Ditkin and Prudnikov, [DV15]), it can be proved for functions with a single singular point with a power or logarithmic singularity that, for sufficiently large n, the function

$$F_n(t) = \overline{\left| \frac{1}{(n-1)!} \int_0^t (t-u)^{n-1} f(u)\,du \right.}$$

belongs to the original ring M. Hence, $p^n F_n(t) \in \mathfrak{M}$. It can be proved that the operator $p^n F_n(t)$ is independent of n and so this operator depends only on the choice of $f(t)$. Hence, any function with a power or logarithmic singularity at $t = 0$ corresponds to the operator

$$a = p^n \overline{\left| \frac{1}{(n-1)!} \int_0^t (t-u)^{n-1} f(u)\,du \right.}. \tag{6.2}$$

This correspondence possesses the following properties. If $f(t)$ and $g(t)$ correspond to the operators a and b, and if λ and μ are numbers, then $\lambda f(t) + \mu g(t)$ corresponds to the operator $\lambda a + \mu b$; if $f(t)$ has no nonintegrable singularity, then a coincides with $f(t)$; finally, we can show that, if $f(t)$ is differentiable, then we have

$$f'(t) = p\left[f(t) - \sum_{r=0}^{\infty} \frac{(-1)^r f_r p^r}{r!} \right],$$

where

$$f_r = t^r f(t) - \overline{\left| \int_0^t [t^r f(t)]'\,dt \right.}.$$

These properties of the correspondence (6.2) mean that the operator $p^n F_n(t)$ can be identified with the function $f(t)$:

$$p^n \overline{\left| \int_0^t \frac{(t-u)^{n-1}}{(n-1)!} f(u)\,du \right.} = f(t).$$

Hence, there are functions with logarithmic or power singularities at $t = 0$ in the operator field $\mathfrak{M}$. Another method of including nonintegrable functions in the field $\mathfrak{M}$ is to define the finite part

$$\overline{\left| \int_0^t F(t-u)G(u)\,du \right.}$$

of the convolution of two functions. This type of convolution is investigated for several classes of functions by Gilly [Gi1, Gi2, Gi3]. If the product of $F(t)$ and

$G(t)$ is defined to be

$$F * G = \frac{d}{dt} \left| \overline{\int_0^t F(t-u)G(u)du} \right. ,$$

the original ring will contain nonintegrable functions. Finally, we can consider the finite part of the Laplace-Carson integral and write

$$\bar{f}(p) = p \left| \overline{\int_0^\infty f(t)e^{-pt}\,dt} \right. , \qquad \bar{g}(p) = p \left| \overline{\int_0^\infty g(t)e^{-pt}\,dt} \right. .$$

It has been proved that

$$\bar{f}(p)\bar{g}(p) = p \left| \overline{\int_0^\infty e^{-pt}\,dt \left| \overline{\int_0^t f(t-u)g(u)\,du} \right.} \right. .$$

This relation can also be used as a means of including several classes of nonintegrable functions in the general theory of operational calculus.

Schwartz [Sc2] uses the finite part of divergent integrals in the proof of the fact that functions with discrete nonintegrable singularities can be considered as generalized functions. He considers functionals

$$\left| \overline{\int f(t)\varphi(t)dt} \right. ,$$

where $\varphi \in K$ and at the same time defines the generalized pseudo-function $\left|\overline{f(t)}\right.$. It is clear that such considerations are limited to particular classes of functions $f(t)$. Lavoine [La] introduces the idea of Laplace transforms of pseudo-functions $\left|\overline{f(t)}\right.$, proves many properties of these transforms, and gives many examples of calculation of the Laplace transform for concrete pseudo-functions.

Unsolved problem. *Is it true that the field of quotients of M is isomorphic to the field of quotients of $\left|\overline{M}\right.$, where M is the Mikusiński ring and $\left|\overline{M}\right.$ is the field with the product defined by the relation*

$$f * g = \left| \overline{\int_0^t f(t-\tau)g(\tau)\,d\tau} \right. ?$$

The theory of operational calculus from the point of view of Schwartz distributions was considered by Sebastião e Silva [Se2, Se3]. He introduces the linear topological space F_ω of Schwartz distributions $\Phi(t)$ of the form $\frac{d^k}{dt^k} e^{kt} F(t)$, where $F(t)$ is bounded and equal to zero for $t \le 0$, and the linear topological space A_ω of functions $\phi(z)$ of the complex variable z defined and analytic for $\operatorname{Re} z > k$ with $\phi(z)/z^k$ bounded. The Laplace transform sets up an isomorphism between F_ω and A_ω. In view of this isomorphism, the theory of functions of the operator d/dt in the space F_ω is equivalent to the theory of functions of z as multipliers in A_ω.

In a paper by Fenyö [Fen], a relation is established between Mikusiński operators and Korevaar distributions [KJ1, KJ2, KJ3, KJ4]. An example is given of a Mikusiński operator which is not a distribution.

Some subsets of Laplace transforms of generalized functions in Hilbert space are considered by Nečas (1958).

Continuing the work of Lévy (1937), San Juan, and others, Doetsch considers the problem of finding when a linear functional $\mathcal{F}$ in E such that

$$\mathcal{F}\{F'\} = s\mathcal{F}\{F\} - F(+0) \tag{6.3}$$

(s is a complex number, $F(t)$ $(0 \leq t < \infty)$ is a function in some function space, and $F'(t)$ is its derivative) coincides with the Laplace transform

$$L\{F\} = \int_0^\infty e^{-st} F(t)\, dt.$$

Two cases are considered: $E = L^p(0,\infty)$, $p \geq 1$, and $E = U$ (here U is the space of functions $F(t)$ integrable in any interval $0 \leq t < T$, and such that $\lim_{T\to\infty} \int_0^T F(t)\, dt$ exists); the space U has the norm

$$||F|| = \sup\Big|\int_0^t F(t)\, dt\Big|.$$

The following theorem is proved:

Let $\mathcal{F}$ be a linear functional in E satisfying (6.3) *for some* s, $\operatorname{Re} s > 0$, *for the sequence of functions*

$$F(t) = \varphi_n(t) = e^{-\frac{t}{2}} t^n \in E \qquad (n = 0, 1, 2, \ldots).$$

Then $\mathcal{F} = L$ in the space E.

7. Convolutional calculus. The neologism “convolutional calculus” can only sporadically be met in mathematical writings. However, the term “convolution” is commonly used in analysis, though its meaning is far from univalent. Broadly speaking, a convolution is always conceived as a bilinear, commutative and associative operation in a linear space. In other words, it is a “multiplication” in a linear space, such that the space itself becomes a commutative and associative algebra. Among the mathematicians active in this topic recently one should mention first of all I. Dimovski. In his “Convolutional Calculus” [Di6] he considered a bilinear, commutative and associative operation $*$: $\mathfrak{X}\times\mathfrak{X}\to\mathfrak{X}$ in a linear space $\mathfrak{X}$ which is said to be a *convolution of a linear operator* $L : \mathfrak{X}\to\mathfrak{X}$ if and only if the relation $L(f * g) = (Lf) * g$ holds for all $f, g \in \mathfrak{X}$.

He successfully applied such convolutions to operational calculus and other problems of mathematical analysis.

7.1. *The Mikusiński ring.* The idea to introduce quotients with respect to the Duhamel convolution

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

is due to Volterra and Pérès [Vo1, Vo2].

The systematic development of the Heaviside operational calculus on such convolutional quotients has been done by Mikusiński [27].

THEOREM (Máté [Má]). *The multipliers of an annihilator-free algebra* $\mathfrak{A} = (\mathfrak{X}, *)$ *are linear operators in the linear space* $\mathfrak{X}$ *of the algebra. They form a commutative algebra with respect to the operator multiplication.*

THEOREM (Dimovski [Di6]). *If* $\mathfrak{A}$ *is an annihilator-free convolutional algebra, and* $\mathfrak{M}$ *is the algebra of its multipliers, then the mappings* $\mathfrak{A} \to \mathfrak{M}$ *and* $C \to \mathfrak{M}$, *defined as* $m \mapsto m*$ *and* $\lambda \mapsto [\lambda]$, *where* $(m*)f = m * f$ *and* $[\lambda]f = \lambda f$, *are isomorphic embeddings of algebras.*

Dimovski's book [Di6] presents a profound and detailed exposition of a new theory of convolutional calculus; it contains many interesting examples of applications of the theory, together with discussions of new problems related to the convolutional calculus.

Convolutional structures are considered in detail, and new definitions are given of methods based on the the notion of convolution of a linear operator. This unifies approaches from operational calculus, multiplier theory, algebraic analysis and spectral theory. The starting point is Mikusiński's approach to the Heaviside operational calculus and this is extended to local and nonlocal boundary value problems for linear differential operators.

Impressive corroboration of these ideas comes from the old work of Volterra and Pérès [Vo1, Vo2] (see also [Pe1, Pe2]), who described an algebraic approach in the treatment of the operational calculus ("calcul de composition"). D. Przeworska-Rolewicz [PPR2, PR3] developed an algebraic theory of right invertible operators without any convolution. A notion of an initial operator for a right invertible operator acting in a linear space is introduced. Using this notion a Taylor formula for right invertible operators is proved. Initial value problems and mixed boundary value problems for equations with right invertible operators are solved. There are given applications to ordinary and partial differential equations and to difference equations, all of them with variable coefficients.

7.2. *The convolutional Hilbert spaces.* The convolutional Hilbert spaces are considered by Ditkin [DV6].

Let $h(\xi)$ and $q(\xi)$ be positive functions defined for $0 < \xi < \infty$ and satisfying the conditions:

1^0. The integral $\int_0^\infty e^{-xh(\xi)}q(\xi)\,d\xi = \omega(x)$ is convergent, $\forall x > 0$.

2^0. If $\int_0^\infty e^{-xh\xi}f(x)\,dx = 0$, $\forall \xi > 0$, then $f(x) = 0$ almost everywhere.

Let S_0 be the set of all absolutely continuous functions $f(x)$, defined and finite for $0 < x < \infty$. If $f(x) \in S_0$ and $g(x) \in S_0$, then

$$f(x) * g(x) = \int_0^x f(x-y)g(y)\,dy \in S_0,$$

therefore the integral

$$\int_0^\infty (f(x) * g(x))\omega(x)\,dx = (f,g) \tag{7.1}$$

exists. Define the operator T_0 by

$$T_0 f = \int_0^\infty e^{-xh(\xi)} f(x)\,dx = \varphi(\xi), \qquad f \in S_0.$$

If we put $T_0 g = \psi(\xi)$ and $x + y = u$, $y = v$ in the integral

$$\varphi(\xi)\bar{\psi}(\xi) = \int_0^\infty \int_0^\infty e^{-(x+y)h(\xi)} f(x)\bar{g}(y)\,dx\,dy,$$

we get

$$\varphi(\xi)\bar{\psi}(\xi) = \int_0^\infty (f * \bar{g})\, e^{-xh(\xi)}\,dx = T_0(f * g).$$

Condition 1^0 and Fubini's theorem yield the convergence of the integral

$$\int_0^\infty \varphi(\xi)\bar{\psi}(\xi)q(\xi)\,d\xi$$

and the equality

$$\int_0^\infty \varphi(\xi)\bar{\psi}(\xi)q(\xi)d\xi = \int_0^\infty (f * \bar{g})\omega(x)\,dx = (f,g). \tag{7.2}$$

From (7.2) and 2^0 it follows that (7.1) has all the properties of a scalar product. The set S_0 becomes a pre-Hilbert space with this scalar product. When completed, it becomes the convolutional Hilbert space. The purpose of the convolutional Hilbert space, at least as the author perceives it, should consist in extracting various new linear problems of analysis. In particular, in [DV7] some integral transforms are considered.

8. On the method of perfect operators. In this part some principal questions of abstract perfect operator theory are touched that are elaborated in detail in two monographs by Prudnikov and Ryabtsev (see [Pr3, Pr4]).

Our purpose is to bring together in a single model two areas of generalized functions (distributions) and Mikusiński's operators, the first of which is essentially analytical in nature, the second essentially algebraic in nature. In this area two directions can be seen. First R. A. Struble [SR1] used the ideas of certain continuous linear transformations, called operator transformations, and a collection of related linear mappings. Schwartz distributions and Mikusiński operators are brought together in a collection of mappings which also include many well-known linear transformations. Those linear transformations are characterized as mappings which also commute with convolution. Secondly J. Weston [We1] considered the set of perfect operators with a richer topological structure in which it was possible to use results of topological algebra and functional analysis. Weston took the Laplace transform as basis of his investigations of perfect operators. In the above books ([Pr3, Pr4]), the authors have provided a strict perfect operator without any reference to the theory of the Laplace transform, and they used operator transformations of the type of Dimovski's convolution. W. Kierat and K. Skórnik [Ki8] showed in an elementary way that perfect operators are continuous in the space $C_0^{(\infty)}$.

8.1. *Perfect operators*: *fundamental concepts and propositions.* We begin by consideration of *abstract perfect operators*.

1^0. Let Ψ be a commutative algebra without zero divisors over a field K; we call it a *test* (or *basic*) algebra (usually we suppose that Ψ consists not only of zero). By a *perfect operator* on Ψ we call (J. Weston) any operator $a : \Psi \to \Psi$, commuting with any operator of multiplication by an element of Ψ:

$$(\forall \chi \in \Psi)(\forall \varphi \in \Psi) \qquad a(\chi * \varphi) = \chi * (a\varphi),$$

where $*$ denotes multiplication in Ψ.

2^0. Any operator $[\chi] = \chi *$ of *multiplication by* $\chi \in \Psi$ is, obviously, perfect; it is said to be *induced by the element* $\chi \in \Psi$: $[\chi]\varphi = \chi * \varphi$ $(\forall \varphi \in \Psi)$. Such an operator is uniquely defined by $\chi \in \Psi$ in virtue of the absence of zero divisors in Ψ.

Any operator $\breve{k} = k\cdot$ of *multiplication by* $k \in K$: $\breve{k}\varphi = k\varphi$ $(\forall \varphi \in \Psi)$ appears to be perfect too. It is uniquely defined by $k \in K$ (provided Ψ does not consist of zero only).

The equality $a = b$, the sum $a + b$ and the product ab of two perfect operators a and b, as well as the product ka of a perfect operator a by $k \in K$ are defined in the usual way: $a = b$ means $(\forall \varphi \in \Psi)$ $a\varphi = b\varphi$; $(\forall \varphi \in \Psi)(a + b)\varphi = a\varphi + b\varphi$, $(ka)\varphi = k(a\varphi)$, $(ab)\varphi = a(b\varphi)$.

3^0. Obviously, the sum $[\chi] + [\psi]$ of two operators induced by $\chi, \psi \in \Psi$ is the operator $[\chi + \psi]$ induced by $\chi + \psi \in \psi$; the product $\breve{k}[\chi]$ of the operator of multiplication by $k \in K$ with the operator induced by $[\chi] \in \Psi$ is the operator $[k\chi]$ induced by $k\chi \in \Psi$; the product $[\chi][\psi]$ of operators induced by $\chi, \psi \in \Psi$ is the operator $[\chi * \psi]$ induced by $\chi * \psi \in \Psi$.

Obviously also, the sum $\breve{k} + \breve{l}$ the product $\breve{k}\breve{l}$ of two operators of multiplication by $k, l \in K$ is the operator $(k + l)\breve{}$ or $(kl)\breve{}$ of multiplication by $k + l \in K$ and $kl \in K$, respectively.

LEMMA 1. *A perfect operator a commutes with any operator induced by an element $\chi \in \Psi$, i.e., $a[\chi] = [\chi]a$ $(\forall \chi \in \Psi)$, if and only if the product of a with any operator induced by an element $\chi \in \Psi$ is the operator induced by $a\chi \in \Psi$, that is, $a[\chi] = [a\chi]$ $(\forall \chi \in \Psi)$.*

Now we have the right to denote by the same symbol $*$ the product $\psi * \varphi$ of two elements $\psi, \varphi \in \Psi$, and the value $a * \varphi$ of a perfect operator a at an element $\varphi \in \Psi$, as well as the product $a * b$ of two perfect operators a, b.

Nonnegative integer powers of a perfect operator a will be denoted as follows: $a^{*1} = a$, $a^{*2} = a * a, \ldots, a^{*n} = a^{*n-1} * a, \ldots$. For $a \neq 0$ we assume $a^{*0} = \breve{I}$.

We can now omit the square brackets in the notation of an operator $[\chi]$ induced by an element $\chi \in \Psi$: $[\chi] = \chi$.

We can omit the symbol $\breve{}$ in the operator $\breve{k}$ of multiplication by $k \in K$, as well as the symbol $*$ in the notation of the value of $\breve{k}$ at $\varphi \in \Psi$ or the product of $\breve{k}$ by a perfect operator a: $\breve{k} * \varphi = \breve{k}\varphi = k\varphi$, $\breve{k} * a = ka$, but we should preserve the symbol $\breve{}$ in the notation of the sum of an operator $\breve{k}$ and a perfect operator a: $\breve{k} + a$.

For $k \in K$ we denote $\breve{k}^n = (k^n)\breve{}\,$ $(n = 1, 2, \ldots;$ for $k \neq 0$, $n = 0, \pm 1, \pm 2, \ldots)$.

LEMMA 2. *Any perfect operator a is linear*:

$$a * (\chi + \psi) = a * \chi + a * \psi \qquad (\forall \chi, \psi \in \Psi),$$
$$a * (k\varphi) = k(a * \varphi) \qquad (\forall k \in K,\ \varphi \in \Psi).$$

Let $P = P[\Psi]$ be set of all perfect operators on Ψ.

THEREM 1. *The set P of all perfect operators on Ψ is a commutative ring* (*an algebra over K*) *of linear operators with the identity and without zero divisors.*

By the properties

$$(a * b) * \varphi = a * (b * \varphi),\ (a + b) * \varphi = a * \varphi + b * \varphi\ (\forall a,\ b \in P,\ \varphi \in \Psi)$$

and

$$a * (\varphi + \psi) = a * \varphi + a * \psi\ (\forall a \in P,\ \varphi,\ \psi \in \Psi),$$

Ψ is a P-*module.*

The algebra $P = P[\Psi]$ contains the subalgebra of all operators induced by elements of the test algebra Ψ, which is *isomorphic* to the latter one, and is denoted also by Ψ: $P \supseteq \Psi$. Furthermore, $P = \Psi$ *if and only if* Ψ is an algebra *with identity.* Indeed, if $\breve{I}$ is the identity of the test algebra Ψ, then $[\breve{I}]$ is the identity operator and $(\forall a \in P = P[\Psi])$ $a = a * [\breve{I}] = [a * \breve{I}] \in \Psi$, that is, $P = \Psi$. And conversely, if $P = \Psi$, then Ψ and P are algebras with identity. Thus, in particular, $P[P] = P$.

The algebra $P = P[\Psi]$ consists only of zero if and only if the algebra Ψ consists only of zero.

The algebra $P = P[\Psi]$ contains the subring of all operators of multiplication by elements of K. This operator subring (if the test algebra Ψ does not consist only of zero) is a field, *isomorphic* to K and denoted also by K: $P \supseteq K$. (It may also be denoted by $K^{\smile}$: $P \supseteq K^{\smile}$ if necessary).

Furthermore, $P = K$ *if and only if* $\Psi = K$. Indeed, if $\Psi = K$, then Ψ is an algebra with identity, therefore, $P = \Psi = K$. And conversely, if $P = K$, then $(\forall\varphi \in \Psi)$ $(\exists k \in K)$ $[\varphi] = \breve{k} \in K$, hence $\Psi \subseteq K$. In particular, since in this case Ψ consists, as well as $P = K$, not only of zero, for any fixed $\chi \in \Psi$, $\chi \neq 0$, we have $(\exists I \in K | I \neq 0)$ $[\chi] = \breve{I}$, and therefore $(\forall k \in K)$ $\breve{k} = \frac{k}{l}\breve{l} = \left[\frac{k}{l}\chi\right] \in \Psi$, hence $K \subseteq \Psi$, so $\Psi = K$.

Thus, it is sufficient to consider the algebras Ψ and $P = P[\Psi]$ as *rings*, not distinguishing multiplication by elements of K as a separate operation.

LEMMA 3. *The mapping given by any operator* $a \in P$, $a \neq 0$, *is bijective* (*one-to-one*).

LEMMA 4. *The coincidence of the range of an operator* $a \in P = P[\Psi]$ (*where* Ψ *consists not only of zero*) *with the algebra* Ψ, *that is*, $a * \Psi = \Psi$, *is a necessary and sufficient condition for the existence and uniqueness of an inverse element for the element* a *of* P, *i.e., of the inverse operator* $a^{*-1} \in P$.

It is convenient to introduce the notation

$$a_* {}^* b = a * b^{*-1} \qquad (a, b \in P,\ b^{*-1} \in P).$$

LEMMA 5. *Any operator* $a \in P = P[\Psi]$ (*for nonzero* Ψ) *is uniquely defined by its value at an arbitrary nonzero element of* Ψ.

4^0. We note that by this lemma the ring $P = P[\Psi]$ of all perfect operators on an algebra Ψ (consisting not only of zero) is *isomorphic to the part* (possibly, proper) *subring* of the field $\bar{\Psi}$ of quotients of the ring Ψ, consisting of all fractions a which admit a representation of the form

$$a = \psi_* {}^* \varphi \qquad (\varphi, \psi \in \Psi,\ \varphi \neq 0),$$

for any fixed nonzero element φ, that is

$$(\forall\varphi \in \Psi | \varphi \neq 0)\ (\exists\psi \in \Psi) \qquad a = \psi * \varphi.$$

In fact, the field $\bar{\Psi}$ is isomorphic to the field of all operators of the form

$$a* = (\psi_* {}^* \varphi)* : \Psi \to \bar{\Psi} \qquad (\varphi, \psi \in \Psi,\ \varphi \neq 0);$$

at the same time, the ring P is isomorphic to the ring of all operators of the form

$$a* = (\psi_* {}^* \varphi)* : \Psi \to \Psi \qquad (\varphi, \psi \in \Psi,\ \varphi \neq 0).$$

Obviously also, by the same lemma *the field* $\bar{P}$ *of fractions* for the ring $P = P[\Psi]$ of all perfect operators on a nonzero algebra Ψ (generally speaking, $P \subseteq \bar{P}$) *is isomorphic to the field* $\bar{\Psi}$ of quotients for Ψ. This isomorphism maps an element $p_* {}^* q \in \bar{P}$ to $(p * \varphi)_* {}^* (q * \varphi) \in \bar{\Psi}$, where $\varphi \in \Psi$ is any fixed nonzero element, and its inverse maps any element $\psi_* {}^* \varphi \in \bar{\Psi}$ onto some element $[\psi]_* {}^* [\varphi] \in \bar{P}$, with an appropriate correspondence between actions on elements.

Therefore the ring $P = P[\Psi]$ is also isomorphic to a (possibly, proper) subring of the field P, consisting of all fractions a such that $(\forall q \in P | q \neq 0)$ $(\exists p \in P)$ $a = p_* {}^* q$.

The ring Ψ^* $(\Psi \subseteq \Psi^*)$ of fractions of the form $\psi_* {}^* q$ $(\psi \in \Psi,\ q \in P,\ q \neq 0)$ appears to be a $\bar{\Psi}$-module. This is a special case of a rather general proposition, given by L. Berg (1973).

If the ring $P = P[\Psi]$ is a field, then $(\forall \psi \in \Psi | \psi \neq 0)$ $(\exists \psi^{*-1} \in P)$ $\check{l} = \psi^{*-1} {*} \psi \in \Psi$, hence $P = \Psi$. Therefore, if we exclude the trivial case of coincidence $P = \Psi$, then the ring $P = P[\Psi]$ of all perfect operators on Ψ *cannot be a field* and therefore cannot be isomorphic to a field of quotients.

However, we emphasize that for the construction of the basis of a general operational calculus it is quite sufficient to extend the original linear system *not to the quotients field*, as in theory of Mikusiński, but only *to the ring* of perfect operators.

So, though we retain the possibility of using the quotients field as the final stage of generalization, all the general theory and applications of the operational calculus in terms of perfect operators are organized without reference to the concept of the quotients field.

In the above books a detailed exposition of the theory of perfect operators is given; and they contains discussion of problems related to algebraization of the original system, the main formula of operational calculus, the left finiteness, infinitesimal properties of the test algebra, infinitesimal properties of the ring of perfect operators, the structure of perfect operators, differentiation and integration of operator functions, operational calculus of several perfect operators. The method of perfect operators modifies, unifies and in an abstract form also generalizes the theory of generalized functions (Schwartz distributions) and the algebraic operational calculus.

9. The operational calculus of Bessel operators. In operational calculus using Bessel operators, the Bessel operator $B = \frac{d}{dt} t \frac{d}{dt}$ and some of its generalizations are considered. Operational-calculus theory for these operators was first considered by Ditkin [DV2]. In his work the operational calculus of the operator B is directly based on an extension of the ring C_2 of twice differentiable functions defined on the half-line $0 \leq t < \infty$ with piecewise continuous second derivatives. In C_2, product is defined by the formula

$$\varphi(t) * \psi(t) = \frac{d}{dt}\left\{ t \frac{d}{dt} \int_0^t d\xi \int_0^1 \varphi(x\xi)\psi[(1-x)(t-\xi)]\,dx \right\}, \tag{9.1}$$

and addition is defined in the usual way.

If $\varphi(t) \in C_2$ and $\psi(t) \in C_2$, the product (9.1) is also in C_2, it is commutative, associative, and distributive with respect to addition. Multiplication by scalars is defined in the usual way. The Mikusiński and Ryll-Nardzewski theorem can be used to prove that C_2 has no zero divisors. We denote the field extension of C_2 by

$N(C_2)$. Following Mikusiński, we call elements of this field operators. We write $1/t = B$ and have

$$\frac{1}{B}\psi(t) = \int_0^t \frac{d\xi}{\xi} \int_0^\xi \psi(u)\,du. \tag{9.2}$$

If $\varphi(t) \in C_2$ and $\varphi(0) = 0$, we have

$$B\varphi(t) = \frac{d}{dt}\left(t\frac{d\varphi}{dt}\right) = t\varphi''(t) + \varphi'(t). \tag{9.3}$$

Hence, if $\varphi(t) \in C_2$ and $\varphi(0) = 0$, the product $B\varphi$ is obtained by applying the operator $\frac{d}{dt}t\frac{d}{dt}$ to φ. From (9.2) we obtain $\frac{1}{B^n} = \frac{t^n}{(n!)^2}$, and thus

$$\frac{1}{B^{n+1}}\psi(t) = \frac{1}{(n!)^2}\int_0^t d\xi \int_0^1 \psi(x\xi)(1-x)^n(t-\xi)^n\,dx. \tag{9.4}$$

The equation $\frac{d}{dt}\left(t\frac{dy}{dt}\right) = ay$ is satisfied by the functions $I_0(2\sqrt{at})$ and $K_0(2\sqrt{at})$. It follows from (9.3) that

$$B[I_0(2\sqrt{at}) - 1] = aI_0(2\sqrt{at})$$

and

$$\frac{B}{B-a} = I_0(2\sqrt{at}). \tag{9.5}$$

This formula can be used to show, for example, that

$$\frac{B}{B+a} = J_0(2\sqrt{at}), \quad \frac{B}{B^2+a^2} = \operatorname{ber}(2\sqrt{at}), \quad \frac{aB}{B^2+a^2} = \operatorname{bei}(2\sqrt{at}),$$

and

$$\frac{B^2}{B^2-a^2} = \frac{1}{2}[I_0(2\sqrt{at}) + J_0(2\sqrt{at})],$$
$$\frac{aB}{B^2-a^2} = \frac{1}{2}[I_0(2\sqrt{at}) - J_0(2\sqrt{at})].$$

We now apply Mikusiński's theory to the operator ring $N(C_2)$ and in particular define the limit of a sequence of operators, the derivative of an operational function, series of operators, and integration of operational functions. Using this theory we can considerably extend the table of formulas for operational calculus using the operator $\frac{d}{dt}t\frac{d}{dt}$. We note that the operational calculus for the operator $\frac{d}{dt}t\frac{d}{dt}$ can also be based on the corresponding integral transformation. The analogue of the Laplace transform is here the Meijer transform

$$\bar{\varphi}(B) = 2\int_0^\infty \varphi(t)K_0(2\sqrt{Bt})\,dt. \tag{9.6}$$

If a measurable function $\varphi(t)$ satisfies the condition

$$|\varphi(t)| < Qe^{2\gamma_0\sqrt{t}}, \tag{9.7}$$

where Q and $\gamma_0 > 0$ are constants, the integral (9.6) converges and represents an analytic function in the region $\operatorname{Re}\sqrt{p} > \gamma_0$. We have the inverse transform

$$\varphi(t) = \frac{1}{2\pi i} \int\limits_L \bar{\varphi}(B) I_0(2\sqrt{Bt})\, dB, \tag{9.8}$$

where the path of integration L is any parabola $\operatorname{Re}\sqrt{B} = \gamma > \gamma_0$. Here $\varphi(t)$ must satisfy not only (9.7) but also the usual additional conditions; for example, it must be of bounded variation in a neighborhood of every point of the line $0 < t < \infty$. At points of discontinuity of $\varphi(t)$ the integral (9.8) converges to $\frac{1}{2}[\varphi(t+0) + \varphi(t-0)]$. The product theorem holds for functions satisfying (9.7). If $\varphi(t)$ and $\psi(t)$ satisfy (9.7), and

$$\omega(t) = \int\limits_0^t d\xi \int\limits_0^1 \varphi(x\xi)\psi[(1-x)(t-\xi)]\, dx,$$

then

$$\bar{\omega}(B) = 2 \int\limits_0^\infty \omega(t) K_0(2\sqrt{Bt})\, dt$$

exists and

$$\bar{\omega}(B) = \bar{\varphi}(B)\bar{\psi}(B).$$

The following relation exists between operator functions $F(B)$ $\left(B = \frac{d}{dt} t \frac{d}{dt}\right)$ and operator functions $F(D)$ $\left(D = \frac{d}{dt}\right)$. If $f(B) = \varphi(t)$ and $F(D) = f(t)$, we have

$$f(t) = \int\limits_0^\infty \varphi(t\xi) e^{-\xi}\, d\xi;$$

here $\varphi(t)$ satisfies (9.7).

Bessel operational calculus is considered in many articles. The operational calculus of the Bessel operator $B = \frac{d}{dt} t \frac{d}{dt}$ is applied in the solution of some problems of analysis (evaluation of integrals, summation of series, and the solution of ordinary differential equations with variable coefficients).

Meller [Me] investigated an operational calculus for several operators closely related to the Bessel operator $B = \frac{d}{dt} t \frac{d}{dt}$, and studied applications to the solution of differential equations, to the calculation of definite integrals, to the summation of series, and to the investigation of properties of special functions.

In [Pr1] an operational calculus for the operator $T = \frac{d}{dt} t \frac{d}{dt} t \frac{d}{dt}$ is constructed.

Osipov [Os2, Os3] establishes a relation between Bessel operators and Laguerre polynomials and Stirling numbers of the second kind. Further generalizations of the results in the operational calculus of Bessel operators are given in some of the well-known papers of E. Krätzel [Kra] and I. Dimovski [Di2, Di3]. In this connection let us note unsolved problems.

Let $V_0(x,k) = 1$, $V_1(x,k), \ldots, V_n(x,k)$ (k is a positive integer) be an orthonormal system of polynomials on the interval $0 \le x < \infty$ with respect to an ultra-exponential weight function $\xi(x,k)$, where

$$\xi(x,k)=\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma^k(s)}{x^s}\,ds,\qquad a,x,\operatorname{Re}s>0,$$

i.e.,

$$\int\limits_0^\infty V_n(x,k)V_m(x,k)\xi(x,k)\,dx=\delta_{m,n}.$$

If $k=1$ then $\xi(x,1)=e^{-x}$ and $V_n(x,1)=(-1)^nL_n(x)$ are the Laguerre polynomials. If $k=2$ then $\xi(x,2)=2K_0(2\sqrt{x})$, where $K_0(x)$ is the MacDonald function (modified Bessel function of the third kind), and $V_n(x,2)=V_n(x)$ is a new system of orthogonal polynomials, i.e.,

$$2\int\limits_0^\infty V_n(x)V_m(x)K_0(2\sqrt{x})dx=\delta_{m,n}.$$

PROBLEM 1. *Find a generating function, an analogue of Rodrigues' formula, a recurrence relation and a differential equation for the orthogonal polynomials* $V_n(x)$.

It should be noted that the moments are given by

$$2\int\limits_0^\infty K_0(2\sqrt{x})x^n dx=(n!)^2,$$

and that $V_n(x)$ can be found by means of the expansion of

$$\int\limits_0^\infty\frac{2K_0(2\sqrt{x})}{x-t}\,dt$$

in a continued fraction. The first three polynomials are

$$\begin{aligned}V_1(x)&=\frac{x-1}{\sqrt{3}},\\ V_2(x)&=\sqrt{\frac{3}{41}}\left(\frac{x^2}{4}-\frac{8}{3}x+\frac{5}{3}\right),\\ V_3(x)&=\sqrt{\frac{41}{2841}}\left(\frac{x^3}{36}-\frac{177}{164}x^2+\frac{267}{41}x-\frac{131}{41}\right).\end{aligned}$$

PROBLEM 2. *Find similar results for the orthogonal polynomials* $V_n(x,k)$ *with* $k>2$.

10. Operational calculus for several variables. The first work on operational calculus for problems involving two variables based on the two-dimensional Laplace transform

$$F(p,q) = \int_0^\infty\int_0^\infty e^{-px-qy} f(x,y)\,dx\,dy$$

was published by Humbert (1934). Voelker's dissertation and the works of Picone (1940) and Jaeger (1940), in which two-variable operational calculus methods were used in the solution of partial differential equations, appeared later. In 1941, Bernstein and Amerio simultaneously and independently published articles on the two-dimensional Laplace transform. Bernstein introduced the two-dimensional Laplace-Stieltjes integral, while Amerio considered the two-dimensional Laplace integral in the Lebesgue sense. Monographs containing rather detailed expositions of two-dimensional operational calculus and some new results were published by Doetsch and Voelker (1955), Poli and Delerue (1954), and Ditkin and Prudnikov (1958). In the first two articles the absolute convergence of the two-dimensional Laplace integral is used. In a dissertation by Burton (1961), the two-dimensional Laplace transform is applied in the solution of first-order hyperbolic systems of partial differential equations with constant coefficients.

The n-dimensional Laplace transform

$$F(p_1,\ldots,p_n) = \int_0^\infty \cdots \int_0^\infty e^{-\Sigma_{k=1}^n p_k x_k} f(x_1,x_2,\ldots,x_n)\,dx_1\,dx_2\ldots dx_n$$

is investigated by Delerue (1951). This work is developed further in several articles of B. Van der Pol and H. Bremmer. A generalization of the Laplace integral to the case of fractional integration in the Riemann-Liouville sense is given by Riesz. Mikusiński's method is used in the derivation of an operational calculus of several variables in Vasilach's works.

Leray considers Cauchy's problem and introduces a generalized Laplace transform $A(-\partial/\partial\xi,\xi)$ of the operator $a(x,\partial/\partial x)$ (the coefficients of a are polynomials in x) and establishes that the conjugate operator a^* and A have the same unitary solution $U^*(x,\xi)$ (up to the differentiation $-\partial/\partial\xi_0$).

The operational calculus for two variables based on the properties of a function ring in which products are defined by means of a Volterra-type transformation

$$f * g = \frac{\partial^2}{\partial x\partial y}\int_0^x\int_0^y f(x-\xi,y-\eta)g(\xi,\eta)\,d\xi\,d\eta$$

is considered by Ditkin and Prudnikov [DV15].

In this work operational calculus theory, based on the two-dimensional Laplace transform, is derived from the general theory by considering operators having a Laplace transform. Applications of operational calculus for several variables are described.

Recently the book "Multidimensional Integral Transformations" by Yu. A. Brychkov, H.-J. Glaeske, A. P. Prudnikov and Vu Kim Tuan has been published (see [Br2]).

11. Applications of operational calculus to optimization problems

11.1. *Nonlinear systems of automatic control.* It is well known that so-called Volterra series are an important tool in the characterization, analysis and design of analytic nonlinear systems. In a certain standardization they are of the form

$$y(t) = \sum_{n=1}^{\infty} y_n(t) = \sum_{n=1}^{\infty} \int_{R^n} h_n(\tau_1, \tau_2, \ldots, \tau_n) \prod_{j=1}^{n} x(t-\tau_j)d\tau_j,$$

where the series are assumed to be uniformly convergent. Here t denotes the time, x the input, y the output of the system and the kernels h_j, $j \in N$, characterize the system. By R^n we denote the n-dimensional Euclidean space, $n \in N = \{1, 2, \ldots\}$. The kernels h_n vanish if one of the variables is negative and the input function $x(t)$ vanishes if $t < 0$. For linear systems all kernels except h_1 vanish. An operational method for determining the kernels of the Volterra series was presented by J. K. Lubbok and V. S. Bansal (see [Br2] for detailed references).

11.2. *On a linear escape problem.* The foundations of the theory of linear differential games of escape were laid in the papers of L. S. Pontryagin and E. F. Mishchenko [Po1, Po2, Po3]. M. S. Nikol'skiĭ [Ni1] obtained certain new escape conditions differing from the escape conditions of [Po2, Po3]. The motion of a vector z described by the equation

$$\dot{z} = Cz - u - v + a$$

is considered in n-dimensional Euclidean space R^n, where $u \in P$ and $v \in Q$ are control vectors; a is a constant vector in R^n. The pursuer determines the vector $u(t)$; the evader determines the vector $v(t)$. P and Q are convex compact subset of R^n. A terminal linear subspace M of dimension $\leq n-2$ is given in R^n.

PROBLEM. *Find sufficient conditions in order that from any initial state $z_0 \notin M$ the evader, using his own information, can ensure the relation $z(t) \not\subset M$ for $t \geq 0$, i.e. escape on the entire half-axis* $[0, \infty)$.

The solution of this problem leads to a linear Volterra-type convolution equation of the first kind and it is investigated by Mikusiński's operational calculus [Ni2].

11.3. *On a linear problem of tracking motion.* M. S. Nikol'skiĭ [Ni3] considered the linear conflict-control object

$$\dot{x} = Ax + Bu + Cv, \tag{11.1}$$

where $x = R^n$ (n-dimensional Euclidean space), $u \in R^p$, and $v \in R^q$. The matrices A, B, and C are constants, and have dimensions $n \times n$, $n \times p$, and $n \times q$ respectively. On the control vector u and the perturbing vector v (the noise vector) we impose the geometric constraints $u \in P$ and $v \in Q$, where $P \subset R^p$ and $Q \subset R^q$ are nonempty compact sets with $0 \in Q$. The initial condition is $x(0) = x_0$. The motion of the point $x(t)$ from the initial point x_0 takes place in the time interval $I = [0, T]$, where $T > 0$, under the action of the Lebesgue-measurable controls

$u(t) \in P$ and $v(t) \in Q$, $t \in I$, and the corresponding solution $x(t)$ of equation (11.1) is sought in the class of absolutely continuous functions. Let π be a fixed nonzero matrix of dimension $m \times n$. For fixed measurable $u(t) \in P$ and $v(t) \in Q$, $t \in I$, and initial condition $x(0) = x_0$ the solution of equation (11.1) reduces to the form

$$\pi x(t) = \pi e^{tA} = \int_0^t \pi e^{(t-s)A}(Bu(s) + Cv(s))\,ds. \tag{11.2}$$

Consider for $t \in I$ the following convolution Volterra integral equation of the first kind:

$$\int_0^t \pi e^{(t-s)A} Bu(s)\,ds = -\int_0^t \pi e^{(t-s)A} Cv(s)\,ds, \tag{11.3}$$

for the unknown $u(s) \in P$ for arbitrary measurable $v(s) \in Q$, $s \in I$. M. S. Nikol'skiĭ [Ni3] takes up the question of obtaining effective sufficient conditions under which this equation $u(s) = \pi(s, v(r), 0 \le r \le s)$. For this he also uses the Mikusiński operational calculus.

12. Unsolved problems. Using the new convolutions in the construction of an operational calculus as a branch of nonlinear functional analysis consider the nonlinear differential equations:

1. The Riccati equation
$$y' + ay^2 = bx^m.$$
2. The Hopf equation
$$u_t + uu_x = 0.$$
3. The Burgers equation
$$u_t + uu_x - u_{xx} = 0.$$
4. The Korteweg–de Vries equation
$$u_t + uu_x + u_{xxx} = 0.$$

References

[Am] L. Amerio, *Sulla trasformata doppia di Laplace*, Atti Acad. Italia Mem. Cl. Sci. Fis. Mat. Natur. 12 (1941), 707–780.

[An1] P. Antosik, *The isomorphism of certain rings of operators*, Studia Math. 25 (1965), 227–230.

[An2] —, *On a topology of convergence*, Colloq. Math. 21 (1970), 205–209.

[An3] —, *The commutativity of the limit with regular operations*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), 325–327.

[An4] —, *On the Mikusiński diagonal theorem*, ibid. 15 (1971), 305–310.

[An5] P. Antosik and J. Mikusiński, *On the Hermite expansions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 787–791.

[An6] P. Antosik, J. Mikusiński and R. Sikorski, *Theory of Distributions. The Sequential Approach*, Elsevier and PWN, Amsterdam–Warszawa, 1973.

[An7] P. Antosik and C. Swartz, *Matrix Methods in Analysis*, Lecture Notes in Math. 1113, Springer, Berlin, 1985.

[B1] S. Bellert, *On foundations of operational calculus*, Bull. Acad. Polon. Sci. Cl. III 5 (1957), 855–858.

[B2] —, *Całka Fouriera i badanie czwórników elektrycznych metodą impulsową*, Zeszyty Nauk. Politechn. Warsz. 42 (1959), 79–109 (in Polish).

[B3] —, *Metoda operatorów liczbowych*, Rozpr. Elektrotechn. 5 (1959), 515–586 (in Polish).

[B4] —, *Rachunek operatorowy w przestrzeniach liniowych*, ibid. 6 (1960), 169–212 (in Polish).

[B5] —, *On the continuation of the idea of Heaviside in the operational calculus*, J. Franklin Inst. 276 (1963), 411–440.

[Be1] L. Berg, *Moderne Operatorenrechnung*, Z. Angew. Math. Mech. 39 (1959), 342–346.

[Be2] —, *Asymptotische Auffassung der Operatorenrechnung*, Studia Math. 21 (1962), 215–229.

[Be3] —, *Einführung in die Operatorenrechnung*, Deutsche Verlag Wiss., Berlin, 1962.

[Be4] —, *Konstruktion von Operatorenkörper*, Z. Angew. Math. Mech. 44 (1964), 3–4.

[Ber] D. L. Bernstein, *The double Laplace integral*, Duke Math. J. 8 (1941), 460–496.

[Bi1] R. Bittner, *On certain axiomatics for the operational calculus*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 7 (1959), 1–9.

[Bi2] —, *Operational calculus in linear spaces*, Studia Math. 20 (1961), 1–18.

[Bi3] —, *Algebraic and analytic properties of solutions of abstract differential equations*, Rozprawy Mat. 41 (1964).

[Bi4] —, *Rachunek operatorów w przestrzeniach liniowych*, PWN, Warszawa, 1974 (in Polish).

[Bi5] —, *Rachunek operatorów w grupach*, PWN, Warszawa, 1992 (in Polish).

[Boc] S. Bochner, *L. Schwartz, Théorie des distributions, t. I et II*, Bull. Amer. Math. Soc. 58 (1952), 78–85.

[Bo1] T. K. Boehme, *On Mikusiński operators*, Studia Math. 33 (1969), 127–140.

[Bo2] —, *The Mikusiński operators as a topological space*, Amer. J. Math. 98 (1976), 55–66.

[Bro] M. S. Brodskiĭ, *A problem suggested by I. M. Gel'fand*, Uspekhi Mat. Nauk 12 (2) (1957), 129–132.

[Br1] Yu. A. Brychkov, A. P. Prudnikov and V. S. Shishov, *Operational calculus*, in: Itogi Nauki, Mat. Analiz, 16, VINITI, Akad. Nauk SSSR, Moscow, 1979, 99–148 (in Russian).

[Br2] Yu. A. Brychkov, H.-J. Glaeske, A. P. Prudnikov and Vu Kim Tuan, *Multidimensional Integral Transformations*, Gordon and Breach, Reading, Mass., 1992.

[Bu1] J. Burzyk, *On convergence in the Mikusiński operational calculus*, Studia Math. 75 (1983), 313–333.

[Bu2] —, *On type II convergence in the Mikusiński operational calculus*, ibid. 77 (1983), 17–27.

[Bu3] —, *Nonharmonic solutions of the Laplace equation*, in: Generalized Functions, Convergence Structures and their Applications, Plenum Press, New York, 1988, 3–11.

[Bu4] —, *A Paley-Wiener type theorem for regular operators of bounded support*, Studia Math. 93 (1989), 187–200.

[Car] J. R. Carson, *Heaviside operational calculus*, Bell System Techn. J. 1 (1922), 43; 4 (1923), 685; 5 (1926), 50 and 5 (1926), 336.

[Col] J. F. Colombeau, *New Generalized Functions and Multiplication of Distributions*, North-Holland, Amsterdam, 1984.
[Coo] J. L. Cooper, *Heaviside and the operational calculus*, Math. Gazette 36 (315) (1952), 5–19.
[Cr] M. M. Crum, *On the resultant of two functions*, Quart. J. Math. Oxford 12 (1941), 108–111.
[Cur] H. B. Curry, *The Heaviside operational calculus*, Amer. Math. Monthly 50 (1943), 365–379.
[DAd] R. D'Adhemar, *Exercises et leçons d'analyse*, Paris, 1908.
[Dal] J. P. Dalton, *Symbolic operators*, Witwatersrand Univ. Press, Johannesburg, 1954.
[Del] J. Delsarte, *Le calcul linéaire*, Bull. Soc. Math. France, 1937, 42–53, and in: Oeuvres de Jean Delsarte, I, Paris, 1971, 411–422.
[Di1] I. H. Dimovski, *The uniqueness of the Mikusiński field*, Fiz.-Mat. Spis. 5 B"lgar. Akad. Nauk 5 (38) (1) (1962), 56–59.
[Di2] —, *Operational calculus for a class of differential operators*, C. R. Acad. Bulgare Sci. 19 (1966), 1111–1114.
[Di3] —, *On an operational calculus for a differential operator*, ibid. 21 (1968), 513–516.
[Di4] —, *The contemporary operational calculus*, Fiz.-Mat. Spis. B"lgar. Akad. Nauk 19 (1976), 54–71 (in Bulgarian).
[Di5] —, *Convolutional Calculus*, Publ. House of the Bulgar. Acad. Sci., Sofia, 1982.
[Di6] —, *Convolutional Calculus*, 2nd ed., Kluwer Academic Publishers, Dordrecht, 1990.
[DV1] V. A. Ditkin, *Operational calculus*, Uspekhi Mat. Nauk 2 (6) (1947), 72–158 (in Russian).
[DV2] —, *Operational calculus for functions defined on the whole real line*, Dokl. Akad. Nauk SSSR 112 (1957), 191–194 (in Russian).
[DV3] —, *Operational calculus theory*, ibid. 116 (1957), 15–17 (in Russian).
[DV4] —, *Operational calculus theory*, ibid. 123 (1958), 395–396 (in Russian).
[DV5] —, *The generalized Laplace transforms*, Inzh.-Fiz. Zh. 1 (11) (1958), 98–104 (in Russian).
[DV6] —, *The convolutional Hilbert spaces*, Dokl. Akad. Nauk SSSR 286 (1986), 528–530 (in Russian).
[DV7] —, *Integral representations in convolutional spaces*, ibid. 295 (1987), 791–795 (in Russian).
[DV8] V. A. Ditkin and P. I. Kuznetsov, *Operational Calculus Manual*, Gostekhizdat, Moscow, 1951 (in Russian).
[DV9] V. A. Ditkin and A. P. Prudnikov, *Operational Calculus for Two Variables and its Applications*, Fizmatgiz, Moscow, 1958 (in Russian).
[DV10] —, —, *Integral Transformation and Operational Calculus*, Fizmatgiz, Moscow, 1961 (in Russian).
[DV11] —, —, *The operational calculus of Bessel operators*, Zh. Vychisl. Mat. i Mat. Fiz. 2 (1962), 997–1018 (in Russian).
[DV12] —, —, *The operational calculus arising from the Bessel equations*, ibid. 3 (1963), 223–238 (in Russian).
[DV13] —, —, *Operational calculus for functions with integral arguments and its applications in discrete analysis*, Inzh.-Fiz. Zh. 7 (7) (1964), 101–115 (in Russian).
[DV14] —, —, *Operational calculus*, in: Itogi Nauki, Mat. Analiz, 1964, VINITI, Akad. Nauk SSSR, Moscow, 1966, 7–75 (in Russian).
[DV15] —, —, *Operational Calculus*, Izd. Vysshaya Shkola, Moscow, 1965 (in Russian).
[DV16] —, —, *Integral Transforms and Operational Calculus*, Pergamon Press, New York, 1965.

[DV17] V. A. Ditkin and A. P. Prudnikov, *Operational Calculus Manual*, Izd. Vysshaya Shkola, Moscow, 1965 (in Russian).

[DV18] —, —, *A method of justification for operational calculus*, Inzh.-Fiz. Zh. 24 (1973), 1114–1117 (in Russian).

[Do] G. Doetsch, *Handbuch der Laplace-transformation*, bd. I–IV, Birkhäuser, Basel, 1950–1966.

[Dr] M. Drobot and J. Mikusiński, *The translation operator and its applications to the statics of beams*, Uspekhi Mat. Nauk 13 (2) (1958), 73–92 (in Russian).

[Du1] J. Dufresnoy, *Sur le produit de composition de deux fonctions*, C. R. Acad. Sci. Paris 225 (1947), 857–859.

[Du2] —, *Autour du théorème de Phragmén-Lindelöf*, Bull. Sci. Math. 72 (1948), 17–22.

[Éf1] A. M. Éfros, *Some applications of operational calculus in analysis*, Mat. Sb. 42 (1935), 699–706 (in Russian).

[Éf2] A. M. Éfros and A. M. Danilevskiĭ, *Operational Calculus and Contour Integrals*, Khar'kov, 1937 (in Russian).

[Er1] A. Erdélyi, *On a generalisation of the Laplace transformation*, Proc. Edinburgh Math. Soc. 10 (1954), 53–55.

[Er2] —, *Operational Calculus*, Pasadena, 1955.

[Er3] —, *Lectures on Mikusiński's Theory of Operational Calculus and Generalized Functions*, Pasadena, 1959.

[Er4] —, *General asymptotic expansions of Laplace integrals*, Arch. Rational Mech. Anal. 7 (1961), 1–20.

[Er5] —, *Operational Calculus and Generalized Functions*, Holt, Rinehart and Winston, New York, 1962.

[Fen] S. Fenyö, *Über den Zusammenhang zwischen den Mikusińskischen Operatoren und den Distributionen*, Math. Nachr. 19 (1958), 161–164.

[Fis] B. Fisher, *The product of distributions*, Quart. J. Math. Oxford (2) 22 (1971), 291–298.

[Fl] H. G. Flegg, *A survey of the development of operational calculus*, Internat. J. Math. Ed. Sci. Tech. 2 (1971), 329–335.

[Fo] C. Foiaş, *Approximation des opérateurs de J. Mikusiński par des fonctions continues,* Studia Math. 21 (1961), 73–74.

[Fr] H. Freudenthal, *Operatoren van Heaviside tot Mikusiński*, Simon Stevin 33 (1959), 3–19.

[Ge1] E. Gesztelyi, *Generalized functions defined by fields which are isomorphic to the field of Mikusiński operators*, Publ. Math. Debrecen 16 (1969), 265–296.

[Ge2] E. Gesztelyi and Á. Száz, *On generalized convolution quotients*, ibid. 16 (1969), 297–305.

[Gi1] J. Gilly, *Comparaison entre la théorie de la composition et la transformation de Laplace-Carson*, C. R. Acad. Sci. Paris 218 (1944), 382–384.

[Gi2] —, *Sur une extension de la théorie de la composition de première espèce*, C. R. Acad. Sci. Paris 218 (1944), 100–102.

[Gi3] —, *Les parties finies d'intégrales et la transformation de Laplace-Carson*, Revue Scientifique 83 (1945), 259–270.

[Gl] H.-J. Glaeske, *Über die Konstruktion von Faltungen für Integraltransformationen mit orthogonalen Polynomen als Kern*, Wiss. Z. Friedrich-Schiller-Univ. Jena Math.-Naturwiss. R. 29 (1980), 213–220.

[Had] J. Hadamard, *Recherches sur les solutions fondamentales et l'intégration des équations linéaires aux dérivées partielles*, Ann. Ecole Norm. Sup. (3) 22 (1905), 101–141.

[Hau] F. Hausdorff, *Momentprobleme für ein endliches Interval*, Math. Z. 16 (1923), 220–248.

[He1] O. Heaviside, *Operators in mathematical physics*, Proc. Roy. Soc. A I, 52 (1893), 504–529 and II, 54 (1894), 105–143.

[He2] —, *Electromagnetic Theory*, London, 1899.

[Ich] S. Ichikawa and K. Akira, *Matrix operational calculus and its applications*, Mem. Eng. Kyoto Univ. 32 (1970), 210–222; 33 (1971), 80–93.

[Ka] G. K. Kalisch, *A functional analysis proof of Titchmarsh's theorem on convolution*, J. Math. Anal. Appl. 5 (1962), 176–183.

[Ka1] A. Kamiński, *Remarks on delta- and unit-sequences*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25 (1978), 25–30.

[Ka2] —, *On characterization of topological convergence*, in: Proc. Conf. on Convergence (Szczyrk 1979), Polska Akademia Nauk, Katowice, 1980, 50–70.

[Ka3] —, *Convolution, product and Fourier transform of distributions*, Studia Math. 74 (1982), 83–96.

[Kan] S. Kantorovitz, *On operational calculus and spectral operators*, University of Minnesota, 1962.

[Ki1] W. Kierat, *A remark on convergent power series of the differential operator in the field of Mikusiński operators*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21 (1973), 843–845.

[Ki2] —, *Applications of quadratic forms generated by Sturm-Lioville operators in the theory of generalized functions*, Prace Nauk. Uniw. Śląskiego 388 (1980).

[Ki3] —, *A sequential characterizations of distributions*, in: Proc. Second Conference on Convergence, Institute of Mathematics, Polish Academy of Sciences, Warszawa, 1981, 32–38.

[Ki4] —, *Some Fourier transform inversion theorem*, Studia Math. 77 (1984), 485–488.

[Ki5] W. Kierat and J. Mikusiński, *Sur un théorème de calcul opérationnel d'intervalle fini*, Studia Math. 25 (1964), 93–96.

[Ki6] W. Kierat and K. Skórnik, *A remark on the Foiaş theorem on convolution*, Bull. Polish Acad. Sci. Math. 34 (1986), 15–17.

[Ki7] —, —, *A remark on the operational exponential functions*, ibid. 35 (1987), 383–386.

[Ki8] —, —, *On continuity of perfect operators in the space* $C_0^{(\infty)}$, in: Complex Analysis and Applications (Sofia 87), 1989, 267–271.

[Kit] T. Kitagama, *The composition of permutable functions and the operational calculus*, Proc. Imp. Acad. Japan 13 (1937), 143–151 and 176–179.

[Koi] S. Koizumi, *On Heaviside's operational solution of a Volterra integral equation when its nucleus is a function of* $(x-\xi)$, Philos. Mag. 11 (1931), 432–441.

[Ko1] H. Komatsu, *An introduction to the theory of hyperfunctions*, in: Hyperfunctions and Pseudo-Differential Equations, Lecture Notes in Math. 287, Springer, Berlin, 1973, 3–40.

[Ko2] —, *Laplace transforms of hyperfunctions—A new foundation of the Heaviside calculus*, J. Fac. Sci. Univ. Tokyo I A 34 (1987), 805–820.

[Ko3] —, *Operational calculus in Japan*, in: Generalized Functions and Convergence, Memorial Volume for Professor Jan Mikusiński, World Scientific, Singapore, 1990, 59–70.

[Kön] H. König, *Neue Begründung der Theorie der Distributionen von L. Schwartz*, Math. Nachr. 9 (1953), 129–148.

[KJ1] J. Korevaar, *Distributions defined from the point of view of applied mathematics. I. Fundamental sequences*, Proc. Koninkl. Nederl. Akad. Wetensch. A 58 (1955), 368–378.

[KJ2] —, *Distributions defined by fundamental sequences. III. Convergence. Convolution. Definite integral. Inverse Laplace transformation*, ibid., 483–493.

[KJ3] —, *Distributions defined by fundamental sequences. IV. Multiplication and division. Substitution*, ibid., 494–503.

[KJ4] J. Korevaar, *Distributions defined by fundamental sequences*. ibid., 379–389, 663–674.

[Kr] G. L. Krabbe, *Operational Calculus*, Springer, Berlin, 1970.

[Krä] E. Krätzel, *Bemerkungen zur Meijer-Transformation und Anwendungen*, Math. Nachr. 30 (1965), 327–334.

[La] J. Lavoine, *Calcul symbolique. Distributions et pseudo-fonctions avec une table de nouvelles transformées de Laplace*, Centre Nat. Rech. Sci., Paris, 1959.

[Let] A. V. Letnikov, *The Theory of Differentiation with an Arbitrary Index*, Moscow, 1868 (in Russian).

[Lév] P. Lévy, *La calcul symbolique d'Heaviside*, Bull. Sci. Math. 50 (1926), 174–192.

[Li] T. L. Lions, *Supports de produit de composition*, C. R. Acad. Sci. Paris 232 (1951), 1622-1624.

[Má] L. Máté, *Multiplier operators and quotient algebra*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 13 (1965), 523–526.

[Mc1] N. W. McLachlan, *Historical note on Heaviside's operational method*, Math. Gaz. 22 (1938), 255–260.

[Mc2] —, *Modern Operational Calculus, with Applications in Technical Mathematics*, revised ed., Dover, New York, 1962.

[Me] N. A. Meller, *Operational calculus for the operator* $B_\alpha = t^{-\alpha}\frac{d}{dt}t^{\alpha+1}\frac{d}{dt}$, Numerical Mathematics, Collection 6 (1960), 161–168 (in Russian).

[MJ1] J. G. M. (J. Mikusiński), *Hiperliczby*, Cracow, 1944 (Edition of 7 copies); English transl.: *Hiperliczby-Hypernumbers*, University of California, Santa Barbara, 1983 and Studia Math. 77 (1983), 3–16.

[MJ2] J. Mikusiński, *L'anneau algébrique et ses applications dans l'analyse fonctionnelle (Première partie)*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 2 (1947), 1–48.

[MJ3] —, *Sur la méthode de généralisation de Laurent Schwartz et sur la convergence faible*, Fund. Math. 36 (1948), 235–239.

[MJ4] —, *L'anneau algébrique et ses applications dans l'analyse fonctionnelle* (*Deuxième partie*), Ann. Univ. Mariae Curie-Skłodowska Sect. A 3 (1949), 1–82.

[MJ5] —, *Sur les fondaments du calcul opératoire*, Studia Math. 11 (1950), 41–70.

[MJ6] —, *Remarks on the moment problem and a theorem of Picone*, Colloq. Math. 2 (1951), 138–141.

[MJ7] —, *Sur les équations différentielles du calcul opératoire et leurs applications aux équations classiques aux dérivées partielles*, Studia Math. 12 (1951), 227–270.

[MJ8] —, *Sur les fonctions exponentielles du calcul opératoire*, ibid. 12 (1951), 208–224.

[MJ9] —, *A new proof of Titchmarsh's theorem on convolution*, ibid. 13 1953), 56–58.

[MJ10] —, *Operational Calculus*, PWN, Warszawa, 1953 (in Polish, 1st ed.).

[MJ11] —, *O rachunku operatorów*, Zastos. Mat. 1 (1953), 28–40.

[MJ12] —, *Sur un type de conditions mixtes pour les équations aux dérivées partielles*, Studia Math. 13 (1953), 277–286.

[MJ13] —, *Rachunek operatorów na tle aktualnych kierunków matematyki*, Prace Mat. 1 (1955), 344–370 (in Polish).

[MJ14] —, *Une définition de distribution*, Bull. Acad. Polon. Sci. Cl. III 3 (1955), 589–591.

[MJ15] —, *Le calcul opérationnel d'intervalle fini*, Studia Math. 15 (1956), 225–251.

[MJ16] —, *Operational Calculus*, IL, Moscow, 1956 (Russian translation from Polish).

[MJ17] —, *Problèmes initiaux et mixtes pour les équations aux dérivées partielles considérées du point de vue du calcul opérationnel*, Cheskoslov. Mat. Zh. 6 (1956), 311–317.

[MJ18] —, *Sur la croissance de la fonction opérationelle* $\exp(-s^\alpha\lambda)$, Bull. Acad. Polon. Sci. Cl. III 4 (1956), 423–425.

[MJ19] —, *Sur la fonction opérationnelle* $\exp(-s^\alpha\lambda)$, ibid., 415–417.

[MJ20] —, *Sur quelques équations intégro-différentielles*, Studia Math. 15 (1956), 182–187.

[MJ21] J. Mikusiński, *The work of Polish mathematicians on generalized functions and operational calculus*, Uspekhi Mat. Nauk 11 (6) (1956), 169–172 (in Russian).

[MJ22] —, *Sur la fonction dont la transformée de Laplace est* e^{-s^α}, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 6 (1958), 691–693.

[MJ23] —, *On the function whose Laplace transform is* e^{-s^α}, Studia Math. 18 (1959), 191–198.

[MJ24] —, *Operational Calculus*, Pergamon Press and PWN, London–Warszawa, 1959 (transl. from the second Polish ed.).

[MJ25] —, *Operatorenrechnung und ihre Beziehungen zur modernen Mathematik*, Z. Angew. Math. Mech. 39 (1959), 381–384.

[MJ26] —, *Sur la convolution par* $\exp(t^2)$, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 7 (1959), 669–671.

[MJ27] —, *Une simple démonstration du théorème de Titchmarsh sur la convolution*, ibid., 715–717.

[MJ28] —, *On the convolution of functions of class* l_1, Bull. Inst. Politehn. Iasi 6 (3–4) (1960), 31–34.

[MJ29] —, *Remarks on the algebraic derivative in the operational calculus*, Studia Math. 19 (1960), 187–192.

[MJ30] —, *Convolution of functions of several variables*, ibid. 20 (1961), 301–312.

[MJ31] —, *Operations on distributions*, in: Proc. Internat. Sympos. Linear Spaces, Jerusalem, 1960, Academic Press and Pergamon Press, Jerusalem–Oxford, 1961, 335–339.

[MJ32] —, *Sur la convolution*, in: 2ème Congr. Math. Hongrois (Budapest, 1960), Budapest, 1961, 80–84.

[MJ33] —, *Operators and distributions*, Studia Math. Ser. Spec. 1 (1963), 77–78.

[MJ34] —, *An approximation theorem and its applications in operational calculus*, Studia Math. 27 (1966), 141–143.

[MJ35] —, *Convolution approximation and shift approximation*, ibid. 28 (1966), 1–8.

[MJ36] —, *A constructive theory of tempered distributions*, Bull. Acad. Pol. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 727–732.

[MJ37] —, *The Bochner Integral*, Birkhäuser, 1978.

[MJ38] —, *On the Foiaş theorem on the convolution*, Bull. Polish. Acad. Sci. Math. 33 (1985), 285–288.

[MJ39] J. Mikusiński and C. Ryll-Nardzewski, *Sur le produit de composition,* Studia Math. 12 (1951), 51–57.

[MJ40] —, —, *A theorem on bounded moments*, ibid. 13 (1953), 51–55.

[MJ41] —, —, *Un théorème sur le produit de composition des fonctions de plusieurs variables*, ibid., 62–68.

[MJ42] J. Mikusiński and S. Świerczkowski, *Twierdzenie Titchmarsha o splocie i teoria Dufresnoy*, Prace Mat. 4 (1960), 59–76 (in Polish).

[MP1] P. Mikusiński, *Convergence of Boehmians*, Japan J. Math. 9 (1983), 159–179.

[MP2] —, *Boehmians and generalized functions*, Acta Math. Hungar. 51 (1988), 271–281.

[Mu] P. A. Murav'ev, *The elements of* α*-operational calculus*, An. Ştiint. Univ. "Al. I. Cuza" Iaşi 10 (1964), 287–302.

[Ne] P. A. Nekrasov, *Generalized differentiation*, Mat. Sb. 14 (1888), 45–168 (in Russian).

[Ni1] M. S. Nikol'skiĭ, *On a method of escape*, Dokl. Akad. Nauk SSSR 15 (1974), 134–137 (in Russian).

[Ni2] —, *On a linear escape problem*, ibid., 1462–1466 (in Russian).

[Ni3] —, *On a linear problem of tracking motion*, ibid. 45 (1992), 193–197 (in Russian).

[No1] D. O. Norris, *A topology for Mikusiński operators*, PhD Diss., Ohio State Univ., 1963.

[No2] D. O. Norris, *A topology for Mikusiński operators*, Studia Math. 24 (1964), 245–255.

[OK1] J. O'Keefe, *Singularities of Hadamard's finite part of improper integrals in the distributions of Schwartz*, Rend. Circ. Mat. Palermo 6 (1957), 65–82.

[OK2] —, *Distribution theory of the operational calculus*, ibid., 157–170.

[Os1] S. Osipov, *Concerning the demonstration of Mikusiński's theorem on convolution with* $\exp(t^2)$, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 10 (1962), 209–210.

[Os2] —, *The expansion of polynomials in* B_α *operators*, Zh. Vychisl. Mat. i Mat. Fiz. 3 (1963), 191–194 (in Russian).

[Os3] —, *The expansion of polynomials in Bessel operators*, ibid. 4 (1964), 149–155 (in Russian).

[Pe1] J. Pérès, *Calcul symbolique d'Heaviside et calcul de composition de Vito Volterra*, C. R. Acad. Sci. Paris 217 (1943), 517–520.

[Pe2] —, *Quelques applications du calcul de composition de Volterra*, ibid., 585–588.

[Pi1] S. Pilipović, *A characterization of the space* D', Mat. Vesnik 6 (19) (34) (1982).

[Pi2] —, *The a-tempered derivative and some spaces of exponential distributions*, Publ. Inst. Math. (Beograd) 34 (48) (1983), 139–192.

[Pl] A.I. Plesner, *The inclusion of the Heaviside operational method in the spectral theory of maximal operators*, Dokl. Akad. Nauk SSSR 26 (1940), 10–12 (in Russian).

[Po1] L. S. Pontryagin, *A linear differential evasion game*, Dokl. Akad. Nauk SSSR 11 (1970), 359–361 (in Russian).

[Po2] L. S. Pontryagin and E. F. Mishchenko, *The problem of evasion in linear differential games*, Differentsial'nye Uravneniya 7 (1971), 335–342 (in Russian).

[Po3] —, *A linear differential evasion game*, Trudy Mat. Inst. Steklov. 112 (1973), 27–60 (in Russian).

[Pot] A. A. Potapenko, *Some applications of J. Mikusiński's operational calculus in the automatic control of systems subjected to random disturbances*, Vestnik Leningrad. Univ. 19 (1957), 129–134 (in Russian).

[Pr1] A. P. Prudnikov, *A contribution to operational calculus theory*, Dokl. Akad. Nauk SSSR 142 (1962), 794–797 (in Russian).

[Pr2] —, *The properties of functions satisfying the differential equation* $x^2y''' + 3xy'' + y' + x^2y = 0$, ibid. 144 (1962), 56–57 (in Russian).

[Pr3] A. P. Prudnikov and I. I. Ryabtsev, *Operational calculus of perfect operators*, Computing Center of the USSR Academy of Sciences, Moscow, 1991 (in Russian).

[Pr4] —, —, *Operational calculus of perfect operators (special problems)*, Computing Center of the USSR Academy of Sciences, Moscow, 1992 (in Russian).

[PR1] D. Przeworska-Rolewicz, *Equations with Transformed Argument. An Algebraic Approach*, PWN–Polish Scientific Publishers and Elsevier, Warszawa–Amsterdam, 1973.

[PR2] —, *Algebraic theory of right invertible operators*, Studia Math. 48 (1973), 129–144.

[PR3] —, *Algebraic Analysis*, PWN–Polish Scientific Publishers and D. Reidel, Warszawa–Dordrecht, 1988.

[PR4] —, *Prodigious mystery of genuine analysis*, in: Generalized Functions and Convergence, Memorial Volume for Professor Jan Mikusiński, World Scientific, Singapore, 1990, 51–58.

[Ry1] I. I. Ryabtsev, *On variable delay*, Izv. Vyssh. Uchebn. Zaved. Mat. 1 (8) (1959), 164–173 (in Russian).

[Ry2] —, *Local properties of Mikusiński operators*, ibid. 3 (1962), 143–150 (in Russian).

[RN1] C. Ryll-Nardzewski, *Sur la convergence des séries d'opérateurs*, Studia Math. 13 (1953), 37–40.

[RN2] —, *Sur les séries de puissances de l'opérateur différentiel*, ibid., 41–47.

[RN3] C. Ryll-Nardzewski, *Sur le corps des opérateurs de Mikusiński*, ibid. 14 (1954), 247–248.

[Sa] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach, Reading, Mass., 1993.

[Sch] M. Schreiber, *Remark on a paper of G. K. Kalisch "A functional analysis proof of Titchmarsh's theorem on convolution"*, J. Math. Anal. Appl. 7 (1963), 62–63.

[Sc1] L. Schwartz, *Généralisation de la notion de fonction, de dérivation, de transformation de Fourier, et applications mathématiques et physiques*, Ann. Univ. Grenoble 21 (1945), 57–74.

[Sc2] —, *Théorie des distributions I–II*, Hermann, Paris, 1950–51.

[Sc3] —, *Convolution. Les cours de Sorbonne*, Méthodes Math. de la Physique III, Paris, 1955.

[Sc4] —, *Transformation de Laplace*, Sorbonne, Paris, 1955.

[Sc5] —, *Méthodes mathématiques pour les sciences physiques*, Hermann, Paris, 1961.

[Se1] J. Sebastião e Silva, *Sui fondamenti delle teoria dei funzionali analitici*, Portug. Math. 12 (1953), 1–47.

[Se2] —, *Le calcul opérationnel au point de vue des distributions*, ibid. 14 (3–4) (1955), 105–132.

[Se3] —, *Sur une construction axiomatique de la théorie des distributions*, Révista Fac. Ci. Lisboa 2 (1955), 79–186.

[Sh] H.S. Shultz, *Operational calculus for functions of two variables*, SIAM J. Math. Anal. 9 (1978), 660–666.

[Si2] M. Da Silveira, *General operational calculus in N variables*, Portug. Math. 15 (1956), 49–69.

[Si1] —, *Correction to the article "General operational calculus in N variables"*, ibid. 16 (1–2) (1957), 41–42.

[Sk1] K. Skórnik, *The form of locally integrable function whose m-th local derivative vanishes almost everywhere*, Zeszyty Naukowe WSP w Katowicach, Sekcja Mat., Zeszyt 5 (1966), 127–152 (in Polish).

[Sk2] —, *An estimation of Fourier coefficients of periodic distributions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 581–585.

[Sk3] —, *Hereditarily periodic distributions*, Studia Math. 43 (1972), 245–272.

[Sk4] —, *On fractional integrals and derivatives of a class of generalized functions*, Dokl. Akad. Nauk SSSR 254 (1980), 541–543 (in Russian).

[Sk5] —, *On tempered integrals and derivatives of nonnegative orders*, Ann. Polon. Math. 40 (1981), 47–57.

[Sk6] —, *On local derivatives*, Dokl. Akad. Nauk SSSR 257 (2) (1981), 286–288 (in Russian).

[Sk7] —, *On local derivatives*, Ann. Polon. Math. 41 (1983), 283–292.

[Sk8] —, *On full derivatives*, ibid. 43 (1983), 323–325.

[Sk9] —, *Sobolev's and local derivatives*, Studia Math. 77 (1984), 509–517.

[Sk10] —, *On functions with vanishing local derivative*, Ann. Polon. Math. 47 (1986), 179–188.

[Sk11] —, *On the Foiaş theorem on convolution of continuous functions*, in: Complex Analysis and Applications (Sofia 85), 1986, 604–608.

[Sk12] —, *On the Foiaş theorem on convolution*, in: Complex Analysis and Applications (Sofia 87), 1989, 460–464.

[Sl1] W. Słowikowski, *A generalization of the theory of distributions*, Bull. Acad. Polon. Sci. Cl. III 1 (1955), 3–6.

[Sl2] —, *On the theory of operator systems*, ibid. 3 (1955), 137–142.

[Sl3] W. Słowikowski, *A generalization of Mikusiński's operational calculus*, ibid. 4 (1956), 633–638, 643–647.

[Sm] I. I. Smith and P. L. Alger, *A derivation of Heaviside's operational calculus based on the generalized functions of Schwartz*, Trans. AIEE 68 (1949), 939–944.

[Sob] S. Soboleff, *Méthode nouvelle à résoudre le problème de Cauchy pour les équations linéaires hyperboliques normales*, Mat. Sb. 1 (1936), 39–71.

[St1] B. Stanković, *Abbildung gewisser Operationen durch die zweidimensionale Laplace Transformation*, Publ. Inst. Math. Acad. Serbe Sci. 11 (1957), 1–8.

[St2] —, *Otvoreni problemi u teoriji operatora J. Mikusińskog*, Mat. Bibl. 25 (1963), 73–82.

[St3] —, *On the solvability of algebraic equations in the field of Mikusiński's operators*, Mat. Vesnik 8 (23) (1971), 391–394.

[St4] B. Stanković and M. Skendžić, *Opérateurs parfaits*, Publ. Inst. Math. (Beograd) 4 (1964), 57–62.

[Sto] F. Stopp, *Zur funktionentheoretischen Auffassung der Mikusińskischen Operatorenrechnung*, Math. Nachr. 32 (1966), 87–205.

[SR1] R. A. Struble, *An algebraic view of distributions and operators*, Studia Math. 37 (1971), 103–109.

[SR2] —, *Analytical and algebraic aspects of the operational calculus*, SIAM Rev. 19 (1977), 403–436.

[SR3] —, *A two-sided operational calculus*, Studia Math. 60 (1977), 239–254.

[Sz1] Á. Száz, *Krabbe's generalized functions as convolution quotients*, Publ. Math. 19 (1972), 287–290.

[Sz2] —, *The multiplier extensions of admissible vector modules and the Mikusiński-type convergence*, Serdica 3 (1977), 82–87.

[Te] G. Temple, *Theories and applications of generalized functions*, J. London Math. Soc. 28 (1953), 134–148.

[Tic] A. Tikhonov, *Theorem of the uniqueness of the solution of the heat conduction equation*, Dokl. Akad. Nauk SSSR 5 (1935), 294–300 (in Russian).

[Tit] E. C. Titchmarsh, *The zeros of certain integral functions*, Proc. London Math. Soc. 25 (1926), 283–302.

[Ur] K. Urbanik, *Sur la structure non-topologique du corps des opérateurs*, Studia Math. 14 (1954), 243–246.

[VB] B. Van der Pol and H. Bremmer, *Operational Calculus Based on the Two-sided Laplace Integral*, University Press, Cambridge, 1950.

[VZ] M. Vashchenko-Zakharchenko, *Symbolic Calculus and its Application to the Integration of Linear Differential Equations*, Kiev, 1862 (in Russian).

[Vl] V.S. Vladimirov, *Generalized Functions in Mathematical Physics*, Mir, 1979.

[Vo1] V. Volterra et J. Pérès, *Leçons sur la composition et les fonctions permutables*, Paris, 1924.

[Vo2] —, —, *Théorie générale des fonctionnelles*, Paris, 1936.

[We1] J. D. Weston, *An extension of the Laplace-transform calculus*, Rend. Circ. Mat. Palermo 6 (1957), 325–333.

[We2] —, *Operational calculus and generalized functions*, Proc. Roy. Soc. A 250 (1263) (1959), 460–471.

[We3] —, *Characterization of Laplace transforms and perfect operators,* Arch. Rational. Mech. Anal. 3 (1959), 348–354.

[We4] —, *On the representation of operators by convolution integrals*, Pacific J. Math. 10 (1960), 1453–1468.

[We5] —, *Time-invariant linear systems and a theorem of Titchmarsh*, in: Topics in Analysis, Colloquium on Mathematical Analysis, Jyväskylä 1970, Springer, Berlin, 1974, 376–383.

[WP] N. Wiener and R. Paley, *Fourier Transforms in the Complex Plane*, Nauka, Moscow, 1964 (Russian translation).

[Wl1] L. Włodarski, *Sur une formule de Efros*, Studia Math. 13 (1953), 183–187.

[Wl2] —, *Une remarque sur une classe de fonctions exponentielles du calcul opérationnel*, ibid., 188–189.

[Wlo] J. Wloka, *Distributionen und Operatoren*, Math. Ann. 140 (1960), 227–244.

[Wo] W. Wolibner, *Sur certains corollaires du théorème de Titchmarsh*, Studia Math. 14 (1953), 107–110.

[Ya] M. Yamazaki, *Operational calculus,* Soryuiron Kenkyu 7 (1955), 5–17.

[Yo] K. Yosida, *Operational Calculus—a Theory of Hyperfunctions*, Springer, Berlin, 1984.