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$L_{p,q}$ spaces

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CONTENTS

1. Introduction

The L_p spaces constitute a traditional and well-known class of Banach spaces which can be associated with an arbitrary measure. If this measure is disintegrable (Definition 2.1), then we may form a more general class, the $L_{p,q}$ spaces, by computing the L_p norms during the first stage of the integration procedure, and then by computing the L_q norm during the second stage. To illustrate this more precisely, let f be a measurable function on $[0,1]\times[0,1]$ which assumes values in a Banach space. If $1 \leqslant p < \infty$ and $1 \leqslant q < \infty$, define

$$\|f\|_{p,q} = \left(\int_0^1 \left(\int_0^1 \|f(x,y)\|^p dy\right)^{q/p} dx\right)^{1/q}$$

Then we have $\|f\|_{p,p} = \|f\|_p$ by the Fubini theorem, and we note that an analogous definition can be given when p or q is infinite (see Definition 3.1). Expressions of this sort have occurred in the literature from time to time ([6], p. 634 and p. 639; [7], p. 1174; [8], p. 655; [29], p. 628; [30], p. 2; and [31], pp. 177 ff.), but they do not seem to have received systematic study in their own right. This paper presents a body of basic theory surrounding these $L_{p,q}$ norms and the resulting $L_{p,q}$ spaces. It also gives an indication of the potential usefulness of these spaces by pointing out that certain well-known Banach spaces which have been considered in connection with induced representations of locally compact groups turn out to be subspaces of certain of the $L_{p,q}$ spaces. The simple recognition of this fact leads both to a clarification and to an extension of the current theory of induced representations.

The exposition is organized in the following way. In Section 2 we gather together the basic definitions, notation, and assumptions which will be used without further comment throughout the remainder of the paper. The only assumption which entails an essential (if mild) restriction of generality is that the principal measure μ be decomposable. In Section 3 the $L_{p,q}$ spaces are formally defined (Definitions 3.1 and 3.2), and a number of miscellaneous properties are obtained. These include the relationship between convergence in $L_{p,q}$ norm and convergence in measure (Proposition 3.11), the question of completeness (Theorem 3.12), the question of the density of simple functions (Propositions 3.14 and 3.15), the containment relations among the $L_{p,q}$ spaces (Proposition 3.17), the continuity properties of the map $(p, q) \mapsto \|f\|_{p,q}$ when the function f is fixed (Propositions 3.21 and 3.23, Corollary 3.24, and Proposition 3.25), and the question of the uniform convexity of the $L_{p,q}$ spaces (Theorem 3.28). In Section 4 the standard integral representations for bounded linear functionals on the L_p spaces are generalized to the $L_{p,q}$ setting. Attention is focussed upon the specification of general conditions under which the linear functional norm will equal the appropriate $L_{p,q}$ norm (Proposition 4.3 and Theorem 4.5), and upon a characterization of the circumstances under which a given linear functional will admit the standard integral representation (Theorem 4.9). As usual, the question of the reflexivity of the $L_{p,q}$ spaces can be decided in the light of these results (Corollary 4.13). Section 5 is devoted entirely to examples. Their primary purpose is to illustrate that the $L_{p,q}$ analogue of an L_p result may require a more complicated proof, or that it may be more complicated to state, or that it may simply be false. In the principal result of Section 6 (Theorem 6.11), a natural relationship is established between an arbitrary (σ-finite) measure space (X, S, μ) – with which is associated a σ-algebra of locally S-measurable subsets of X – and a σ-subring S' of the product σ-ring $S \times S$, on which the measure $\mu \times \mu$ may be regarded as disintegrable with respect to μ. (In fact, S' itself looks locally like a product σ-ring.) This relationship between μ on S and $\mu \times \mu$ on S' is sufficiently close that, whenever μ is disintegrable, a natural relationship will automatically exist between the given disintegration of μ and the standard disintegration of $\mu \times \mu$ with respect to μ (Theorem 6.12). This latter relationship is in turn sufficiently close as to ensure that all of the corresponding $L_{p,q}$ spaces are isometrically isomorphic (Proposition 6.13 and Theorem 6.14). We are thereby able to associate $L_{p,q}$ spaces with an arbitrary measure space which is equipped with a σ-algebra of locally measurable sets, and the section concludes with an informal discussion of the circumstances under which these generalized $L_{p,q}$ spaces can be represented as spaces of (equivalence classes of) functions defined on the original measure space. (We remark that one immediate consequence of this representation theory is the fact that each $L_{p,q}$ space associated with the measure μ may be isometrically embedded in the corresponding $L_{p,q}$ space associated with the measure $\mu \times \mu$ (see also Proposition 3.27). We require this fact in the proof of our uniform convexity result (Theorem 3.28).) Finally, Section 7 is devoted to an application of $L_{p,q}$ theory to locally compact groups. If a closed subgroup H of a locally compact group G is specified, then there exists a well-known natural disintegration of (left) Haar measure μ on G with respect to a suitably chosen measure on the space of (left) cosets of H. The well-known canonical representations of G on the L_p spaces possess natural analogues in the $L_{p,q}$ spaces (Theorem 7.4), and it is precisely these representations, when they are restricted to certain subspaces of the $L_{\infty,q}$ spaces, which turn out to have been previously studied in connection with induced representations of G. Our primary aim in Section 7 is to indicate how this observation enables some simplification of the current theory of induced representations to be made, and then to employ our $L_{p,q}$ dual space theory to sketch, without detailed proofs, a separate but parallel theory of induced representations in the dual space setting.

2. Notation and basic terminology

The purpose of this section is to review the principal definitions which will concern us, to establish notation, and to set out certain basic assumptions. These definitions, notation, and assumptions will be used frequently, and without further comment, throughout the remainder of the paper.

Let E and F be subsets of some fixed set X. Then by E^c we shall denote the complement of E (in X), by $E-F$ we shall denote the relative complement $E\cap F^c$, and by χ_E we shall denote the characteristic (or indicator) function of the set E. The symbol $\emptyset$ will denote the empty set.

Arrow symbols will be employed in the following ways: The symbol $\uparrow$ will signify "increase to a limit," and will occur in expressions such as $K_n\uparrow K$. When K and the K_n are sets, this expression will mean that $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots$, and that $K = \bigcup_{n=1}^{\infty} K_n$; when K and the K_n are extended real numbers ([25], §3, p. 34), it will mean that $K_1 \leqslant K_2 \leqslant K_3 \leqslant \ldots$, and that $K = \lim_{n\to\infty} K_n$; when K and the K_n are extended real valued functions, it will mean that $K_n(x)\uparrow K(x)$ for all (or for specified) points x in their common domain. In similar fashion the symbol $\downarrow$ will signify "decrease to a limit." Aside from its appearance in the standard functional notation $f\colon X\to Y$, the symbol $\to$ will be used (in expressions such as $K_n\to K$) to denote convergence of a sequence in a topological space; when K and the K_n are functions, the convergence will be pointwise (i.e., at all or at specified points in their common domain) unless another sort of convergence is indicated. The symbol $\mapsto$ will be used to specify the action of a function on a point in its domain. Thus, if f is a function, we shall treat the expressions f and $x\mapsto f(x)$ as synonymous; however, the latter expression will be used only when we do not wish to specify a symbol to denote the function. When f is a one-to-one correspondence, we shall write $\leftrightarrow$ in place of $\to$.

The symbol B will be used throughout this paper to denote an arbitrary real or complex Banach space, and we shall let $\|b\|$ denote the norm of an arbitrary element $b \in B$. We shall use $\boldsymbol{R}$ to denote the real numbers, and Φ to denote the field of scalars associated with B. In practice we shall treat the elements of Φ as complex numbers; the corresponding assertions for the case $\Phi = \boldsymbol{R}$ will always be obvious.

By a *measure space* we shall mean a triple (X, S, μ), where X is a set, where S is a σ-ring of subsets of X, and where μ is a measure on S, i.e., a countably additive set function, with domain S, which assumes values in the closed infinite interval $[0, \infty]$. A set $K \subset X$ will be called *measurable* if $K \in S$, and *locally measurable* if $K \cap E \in S$ for all $E \in S$. A measurable set K will be called *null* if $\mu(K) = 0$, and a locally measurable set K will be called *locally null* if $K \cap E$ is null for all $E \in S$. A property which holds for all points in X except (possibly) for a null set will be said to hold *almost everywhere* (*a.e.*), and a property which holds except (possibly) for a locally null set will be said to hold *locally almost everywhere* (*l.a.e.*). A function will be called *simple* if its range is a finite set, and a simple function $f\colon X \to B$ will be called *measurable* if the set $f^{-1}(\{b\})$ is measurable for each non-zero value b of f. An arbitrary function $f\colon X \to B$ will be called *measurable* (resp. *essentially measurable*) if it constitutes the a.e. (resp. l.a.e.) pointwise limit of a sequence of simple measurable functions, and f will be called *locally measurable* if $f\chi_E$ is measurable for all $E \in S$. In the special case $B = \boldsymbol{R}$, we shall allow a locally measurable function to assume the values $-\infty$ and $+\infty$. The function $f\colon X \to B$ will be called *null* if $f = 0$ a.e., and *locally null* if $f = 0$ l.a.e. Finally, if $0 < p < \infty$, the expression $\|f\|_p$ will as usual denote $\left(\int \|f(x)\|^p d\mu(x)\right)^{1/p}$ (cf. [25], p. 244), and the expression $\|f\|_\infty$ will denote $\inf\{a\colon \|f(x)\| \leqslant a \text{ l.a.e.}\}$ ([28], Definitions, p. 102).

When a measure is involved in the definition of a concept, we shall feel free to specify that measure in a natural way whenever we deal with the concept. Thus expressions such as μ-measurable, a.e. (μ), and $(\|f\|_p, \mu)$ will occur frequently. In fact we shall omit the reference to the measure only when it is represented by the symbol μ, or by no specified symbol.

We now present and discuss a key definition for this paper.

2.1. DEFINITION. Let (X, S, μ) and (Y, T, ν) be arbitrary measure spaces. Then we shall say that (X, S, μ) (or just μ) is *disintegrable* with respect to (Y, T, ν) (or just ν) if there exists a function $\psi\colon X \to Y$, and if there exists a family $\{\mu_y\}_{y \in Y}$ of measures on S such that

(2.1.1) for all $F \in T$, the set $\psi^{-1}(F)$ is a locally measurable subset of X;

(2.1.2) for all $E \in S$, the function $y \mapsto \mu_y(E)$ is ν-measurable; and

(2.1.3) for all $E \in S$, and for all F such that $F \in T$, or such that $F^c \in T$, we have $\mu(E \cap \psi^{-1}(F)) = \int_F \mu_y(E)\,d\nu(y)$.

The family $\{\mu_y\}_{y\in Y}$ will be called a *disintegration* of μ with respect to ν, and the octuple $(X, S, \mu, Y, T, \nu, \psi, \{\mu_y\}_{y\in Y})$ will be called a *disintegrable measure space.* ■

It is clear from (2.1.2) that (2.1.1) and (2.1.3) are valid for an arbitrary locally ν-measurable set F, and, in particular, we have $\mu(E) = \int \mu_y(E)\,d\nu(y)$ for all $E \in S$. Furthermore, an easy extension of this equality will give $\int f(x)\,d\mu(x) = \int\int f(x)\,d\mu_y(x)\,d\nu(y)$ for every (Bochner) integrable function $f\colon X \to B$, and also for every non-negative measurable function f on X.

We now present a typical example of a disintegrable measure space. The analysis of Section 6 will show that, in a certain sense, the examples of this sort will exhaust all disintegrable measure spaces.

2.2. Example. Let (X, S, μ) and (Y, T, ν) be two σ-finite measure spaces, let ψ_Y denote the canonical projection of $Y \times X$ onto Y (i.e., $\psi_Y(y, x) = y$ for all $(y, x) \in Y \times X$), and likewise let ψ_X denote the canonical projection of $Y \times X$ onto X. Given $y \in Y$, define a measure μ_y on the standard product σ-ring $T \times S$ ([10], p. 140), by letting $\mu_y(E) = \mu\big(\psi_X(E \cap (\{y\} \times X))\big)$ for all $E \in T \times S$. Then it is raedily deduced from the Fubini theorem that $(Y \times X, T \times S, \nu \times \mu, Y, T, \nu, \psi_Y, \{\mu_y\}_{y\in Y})$ constitutes a disintegrable measure space, and we shall call these specific μ_y the *standard disintegration* of $\nu \times \mu$ with respect to ν. ■

We have not seen Definition 2.1 elsewhere, although it clearly is similar in spirit to ([26], Definition 0, p. A424). In particular, it would have been more in keeping with previous lines of thought to have replaced assumption (2.1.3) by the assumption that, for all $y \in Y$, the measure μ_y is concentrated on the fibre $\psi^{-1}(\{y\})$ (i.e., that we have $\mu_y(E) = 0$ whenever the set $E \in S$ is disjoint from $\psi^{-1}(\{y\})$). (Assumption (2.1.3), which is slightly weaker, says in effect that, for $E \in S$, the function $y \mapsto \mu_y(E)$ resembles a conditional expectation (Definition 6.4).) We have two reasons for favoring Definition 2.1 as it stands. In the first place, condition (2.1.3) will be sufficient for us (in Theorem 6.12) to represent an arbitrary (σ-finite) disintegrable measure by a product measure with its standard disintegration, and this latter disintegration clearly does satisfy the stronger condition. In the second place, examples abound of disintegrations which satisfy precisely condition (2.1.3), and not (necessarily) the stronger condition. In the most common setting for the construction of disintegrations (see, for example, [5], Theorem 2, p. 408), the measures μ_y are obtained with the aid of a lifting for the measure ν ([5], Definition 1, p. 199). Whereas an arbitrary lifting is sufficient to obtain condition (2.1.3), it appears that the stronger condition can only be obtained by using a "strong" lifting ([5], Definition 3, p. 406). And whereas an arbitrary lifting will always exist in this setting ([5], Theorem 1, p. 206; Proposition 41, p. 337), the existence of a strong lifting is presently known only in special cases (see, for example, [13]).

For the remainder of this paper the disintegrable space $(X, S, \mu, Y, T, \nu, \psi, \{\mu_y\}_{y\in Y})$ will be fixed, and we now list some mild assumptions which we shall make about this space. These assumptions will not usually receive mention in the statements of results, although they will be freely used in proofs.

First, we shall assume that both (X, S, μ) and (Y, T, ν) are σ-finite and complete. Neither of these assumptions entails an essential restriction of generality, for we may always treat the sets which are not σ-finite as locally measurable, and we may always form the completion of any measure space ([10], Theorem B, p. 55). (Moreover, such adjustments will not essentially alter the conditions of Definition 2.1.)

Second, we shall assume that (X, S, μ) is decomposable (see [18], Definition 3.3, p. 200, and the discussion following, and in particular see [18], Proposition 3.5, p. 201; cf. [25], Problem 39, p. 243; and [27], Definition 3.1, p. 282). This assumption constitutes a genuine, if mild, restriction of generality, and so we shall adopt the policy of commenting informally, whenever we use it, upon circumstances under which it may be relaxed.

We now review the definition of decomposability with a formulation of that definition which will facilitate the comprehension of topics in this paper (and in particular of Theorem 6.11 and Proposition 6.13). Let us declare that a collection $R \subset S$ is *disjoint* (resp. *essentially disjoint*) if the intersection of any pair of distinct members of R is empty (resp. null). Then (X, S, μ) (or just μ) is *decomposable* if μ is σ-finite, and if we may *disjointize* every essentially disjoint family $R \subset S$, namely, if we may assign to each set $E \in R$ a measurable set $E' \subset E$ such that $E - E'$ is null, and such that the family $R' = \{E' \colon E \in R\}$ is disjoint. (We shall call R' a *disjointization* of R.) In particular, by Zorn's lemma, there exists at least one maximal essentially disjoint family $R \subset S$ which comprises sets of strictly positive finite measure. Any disjointization of such a family R will be called a *decomposition* for μ. Since μ is σ-finite, it is clear from [18], Proposition 3.5, p. 201, that this definition of decomposability does indeed constitute a reformulation of the more standard definition ([18], Definition 3.3, p. 200).

The expression "f is supported on E" will mean "$f(x) = 0$ whenever $x \notin E$". The expression "$1 \leqslant p, q \leqslant \infty$" will always mean "$1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$"; it will never mean "$1 \leqslant p$ and $q \leqslant \infty$". Similar expressions are, of course, to be likewise interpreted.

Finally, we shall habitually organize one or more ordinary paragraphs into a "superparagraph." Each superparagraph will begin with an expression such as "**2.1.** DEFINITION." or "**3.12.** THEOREM.", and it will end with the symbol ■. In particular, this symbol ■ will always signal the end of a proof.

3. Definition and basic properties of the $L_{p,q}$ spaces

The following definition is fundamental to this paper.

3.1. DEFINITION. Recalling the disintegrable measure space $(X, S, \mu, Y, T, \nu, \psi, \{\mu_y\}_{y\in Y})$, and recalling the Banach space B, we let $1 \leqslant p, q \leqslant \infty$, and we let $f: X \to B$ be a measurable function. Then we define the $L_{p,q}$ *norm*, $\|f\|_{p,q}$, *of* f by the formula

(3.1.1) $$\|f\|_{p,q} = \big(\big\|y \mapsto (\|f\|_p, \mu_y)\big\|_q, \nu\big).$$

If f is locally measurable, we define

(3.1.2) $$\|f\|_{p,q} = \sup_{E\in S} \|f\chi_E\|_{p,q},$$

and we let $\mathscr{L}_{p,q}(X, S, \mu, Y, T, \nu, \psi, \{\mu_y\}_{y\in Y}, B)$ denote the collection of those locally measurable functions $f: X \to B$ for which $\|f\|_{p,q}$ is finite. ■

We postpone to Lemma 3.6 the proof that the function $y \mapsto (\|f\|_\infty, \mu_y)$ is measurable whenever f is measurable, and hence that expression (3.1.1) makes sense for all values of p and q.

It is clear that (3.1.2) coincides with (3.1.1) when f is measurable, so that no ambiguity of notation will arise. We shall omit as many of the symbols after $\mathscr{L}_{p,q}$ (and after $L_{p,q}$ below) as can be omitted without loss of clarity; however, we shall omit the symbol B if and only if $B = \Phi$. The following facts are straightforward to verify.

(3.1.3) $\mathscr{L}_{p,q}(B)$ is a vector space.

(3.1.4) $\|\cdot\|_{p,q}$ determines a seminorm on $\mathscr{L}_{p,q}(B)$.

(3.1.5) $\|f\|_{p,q} = 0$ if and only if $f = 0$ l.a.e.

(3.1.6) If $f \in \mathscr{L}_{p,q}(B)$, then there is a set $E \in S$ which is such that $\|f\|_{p,q} = \|f\chi_E\|_{p,q}$. Moreover, if p and q are finite, then f vanishes l.a.e. outside of E, so that f is essentially measurable.

(3.1.7) If $\{f_n\}_{n=1}^\infty$ is a sequence of functions in $\mathscr{L}_{p,q}(B)$, then we have $\|f_n\|_{p,q} \to 0$ if and only if $\|f_n\chi_E\|_{p,q} \to 0$ for all $E \in S$.

3.2. DEFINITION. Adopting the notation of Definition 3.1, we let $L_{p,q}(B) = L_{p,q}(X, S, \mu, Y, T, \nu, \psi, \{\mu_y\}_{y\in Y}, B)$ denote the collection of equivalence classes of the functions in $\mathscr{L}_{p,q}(B)$ modulo locally null functions. ∎

From (3.1.3)–(3.1.5) it is evident that $L_{p,q}(B)$ is a normed linear space. We shall extend to this setting the abusive practice of treating the elements of $L_{p,q}(B)$ as functions. From (3.1.6) we may and shall always assume that, when p and q are finite, an arbitrary function $f \in L_{p,q}(B)$ is measurable.

3.3. EXAMPLE. Let ν be counting measure on the set of natural numbers $\{1, 2, \ldots\}$. Define $l_{p,q}(B) = L_{p,q}(\nu \times \nu, \nu, B)$, and define $l_{p,q} = l_{p,q}(\Phi)$.

When p and q are finite, we may characterize $l_{p,q}(B)$ as the collection of doubly indexed sequences $\{b_{mn}\}_{m,n=1}^{\infty}$ of elements of B for which the expression $\left(\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\|b_{mn}\|^p\right)^{q/p}\right)^{1/q}$ is finite; we may characterize $l_{p,q}(B)$ analogously when $p = \infty$ or $q = \infty$. The $l_{p,q}$ spaces constitute the obvious generalization of the familiar l_p spaces. ∎

The bulk of this section will be devoted to the presentation of miscellaneous properties of the $L_{p,q}$ spaces, including the most important: completeness (Theorem 3.12).

It is clear from the two stage computation of a μ integral (q.v.) that $L_{p,p}(B) = L_p(B)$ when $1 \leqslant p < \infty$. Our first set of lemmas will extend this observation to the case $p = \infty$, and they will simultaneously settle the question of measurability which arises in the definition of the $L_{\infty,q}$ norm.

3.4. LEMMA (L_∞ Monotone Convergence Theorem). *Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of non-negative locally measurable functions defined on an arbitrary measure space, and assume that $h_n \uparrow h$ l.a.e. Then we have $\|h_n\|_\infty \uparrow \|h\|_\infty$.*

Proof. Since $0 \leqslant h_1 \leqslant h_2 \leqslant \ldots \leqslant h$ l.a.e., it follows that $\|h_1\|_\infty \leqslant \|h_2\|_\infty \leqslant \ldots \leqslant \|h\|_\infty$. Therefore the limit $l = \lim_n \|h_n\|_\infty$ exists and is $\leqslant \|h\|_\infty$. But since we clearly have $h \leqslant l$ l.a.e., it follows that $\|h\|_\infty \leqslant l$. ∎

3.5. LEMMA (L_∞ Fatou's Lemma). *Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of non-negative locally measurable functions defined on an arbitrary measure space. Then we have $\|\liminf_n h_n\|_\infty \leqslant \liminf_n \|h_n\|_\infty$.*

Proof. For each fixed n, it is clear that we have $\inf_{k \geqslant n} h_k \leqslant h_j \leqslant \|h_j\|_\infty$ l.a.e. for all $j \geqslant n$, so that we obtain $\inf_{k \geqslant n} h_k \leqslant \inf_{k \geqslant n} \|h_k\|_\infty$ l.a.e. From this we obtain $\liminf_n h_n \leqslant \liminf_n \|h_n\|_\infty$ l.a.e., and the result follows. ∎

The corresponding assertion, that $\limsup_n \|h_n\|_\infty \leqslant \|\limsup_n h_n\|_\infty$, fails for the L_∞ norm, as does the dominated convergence theorem.

3.6. Lemma. *Let $f\colon X\to B$ be a measurable function. Then the map $y\mapsto(\|f\|_\infty, \mu_y)$ is ν-measurable.*

Proof. We shall assume initially that f is a non-negative simple measurable function. Omitting the trivial case $f\equiv 0$, we let $f=\sum_{i=1}^{n} c_i\chi_{E_i}$, where $E_i=f^{-1}(\{c_i\})$ for each i, and where $c_1>c_2>\ldots>c_n>0$. Then we have $(\|f\|_\infty, \mu_y)=\sum_{i=1}^{n} c_i\chi_{F_i}(y)$, where $F_1=\{y\in Y\colon \mu_y(E_1)>0\}$, and where $F_j=\{y\in Y\colon \mu_y(E_j)>0\}-(\bigcup_{i=1}^{j-1} F_i)$ for $j=2,3,\ldots,n$. Thus the map $y\mapsto(\|f\|_\infty, \mu_y)$ is ν-measurable, and this result extends to an arbitrary non-negative measurable function f by the straightforward use of Lemma 3.4. Finally, we obtain the result for the general function f by observing that $\|f\|_\infty$ equals the L_∞ norm of the non-negative function $\|f(\cdot)\|$. ∎

3.7. Lemma (L_∞ Fubini Theorem). *Let $f\colon X\to B$ be a measurable function. Then we have $(\|f\|_\infty, \mu)=\big(\big\|y\mapsto(\|f\|_\infty, \mu_y)\big\|_\infty, \nu\big)$.*

Proof. We shall begin, as in Lemma 3.6, by letting $f=\sum_{i=1}^{n} c_i\chi_{E_i}$, where $E_i=f^{-1}(\{c_i\})\in S$ for each i, where $\mu(E_1)>0$, and where $c_1>c_2>\ldots>c_n>0$. Then we clearly have $(\|f\|_\infty, \mu)=c_1$. Moreover, since $\mu(E_1)=\int\mu_y(E_1)\,d\nu(y)$, it follows that the set $F_1=\{y\in Y\colon \mu_y(E_1)>0\}$ has strictly positive ν measure. Since $(\|f\|_\infty, \mu_y)=c_1$ for all $y\in F_1$, the desired equality follows. In the general case it suffices once again to assume that f is non-negative, and in this case the result follows from three applications of Lemma 3.4. ∎

3.8. Proposition. *Let $f\colon X\to B$ be a locally measurable function. Then we have $\|f\|_\infty=\|f\|_{\infty,\infty}$, so that $L_{\infty,\infty}(B)=L_\infty(B)$.*

Proof. If $E\in S$, then we have $\|f\chi_E\|_\infty=\|f\chi_E\|_{\infty,\infty}$ by Lemma 3.7. Now it is clear that we may compute $\|g\|_\infty=\sup_{E\in S}\|g\chi_E\|_\infty$, and so the result follows. ∎

Our next definition and set of lemmas will lead up to the proof that the $L_{p,q}$ spaces are complete.

3.9. Definition. We shall say that a set $E\in S$ has *totally finite measure* if the function $y\mapsto\mu_y(E)$ is essentially bounded (with respect to ν), and if the set $\{y\in Y\colon \mu_y(E)>0\}$ has finite ν measure. ∎

The sets of totally finite measure form a convenient collection of "well behaved" sets on which the $L_{p,q}$ analogues of certain L_p properties are more likely to remain valid (for example, see Proposition 3.11 and Example 5.3; see also Examples 5.4 and 5.9). The following facts are straightforward to verify.

(3.9.1) If the set $E \in S$ has totally finite measure, then $\chi_E \in L_{p,q}$ $(= L_{p,q}(\Phi))$ for all values of p and q.

(3.9.2) The collection of sets in S which have totally finite measure constitutes a μ-ideal in S ([18], Definition 3.1, p. 199).

(3.9.3) Hence, given a set $E \in S$, there exists an increasing sequence of measurable sets of totally finite measure whose union is E.

We remark that (3.9.3) follows from (3.9.2) by [18], Proposition 3.2, p. 199, or by a simple direct argument. The monotone convergence theorem now implies that (3.1.2) will remain valid if the sets E in that statement are required to have totally finite measure; however, the same requirement would invalidate both (3.1.6) and (3.1.7).

For the next lemma, and for many of the remaining results of this section, the norm properties of $\|\cdot\|_{p,q}$ will not crucially come into play. Thus we may and shall consider the expression $\|f\|_{p,q}$ of Definition 3.1 to be defined for arbitrary $p, q > 0$. Although we shall refer to the *set* $L_{p,q}(B)$ (with the obvious definition) in Proposition 3.17 below, we shall treat $L_{p,q}(B)$ as a *space* only for $p, q \geqslant 1$.

3.10. LEMMA. *Let $0 < p, q, r, s < \infty$, and let $E \in S$ satisfy $\|\chi_E\|_{p,q} < \infty$. Then, for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that we have $\|\chi_F\|_{p,q} < \varepsilon$ for every measurable set $F \subset E$ which satisfies $\|\chi_F\|_{r,s} < \delta$.*

Proof. In order to make use of the triangle inequality, we shall assume initially that $r, s \geqslant 1$. If the conclusion were false, then there would exist a number $\varepsilon > 0$, and, for each $n = 1, 2, \ldots$, there would exist a set $F_n \subset E$ such that $\|\chi_{F_n}\|_{r,s} < 2^{-n}$, but such that $\|\chi_{F_n}\|_{p,q} \geqslant \varepsilon$. Let $G_n = \bigcup_{j=n}^{\infty} F_j$. Then we clearly have $\|\chi_{G_n}\|_{p,q} \geqslant \varepsilon$, while application of the triangle inequality and of the monotone convergence theorem gives $\|\chi_{G_n}\|_{r,s} \leqslant \sum_{j=n}^{\infty} \|\chi_{F_j}\|_{r,s} \leqslant 2^{-n+1}$. Now we see that $G_n \downarrow$ some set G, so that $\chi_{G_n} \downarrow \chi_G$. The finiteness of $\|\chi_E\|_{p,q}$ and of $\|\chi_{G_1}\|_{r,s}$ permits the application of the dominated convergence theorem, and we conclude that $\|\chi_G\|_{r,s} = \lim_n \|\chi_{G_n}\|_{r,s} = 0$, whereas (similarly) $\|\chi_G\|_{p,q} \geqslant \varepsilon$. In view of (3.1.5), this is impossible.

When $r < 1$ or $s < 1$, we find $r', s' \geqslant 1$ such that $s/r = s'/r'$. The desired conclusion now follows from the identity $\|\chi_F\|_{r',s'} = \|\chi_F\|_{r,s}^{s/s'}$, and from the special case considered above. ∎

Remark. The foregoing proof has been derived from a standard argument (e.g. [10], Theorem B, pp. 125–126).

3.11. PROPOSITION. *Let $0 < p, q \leqslant \infty$, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of B-valued measurable functions defined on X, and let $f\colon X \to B$ be a measurable function.*

(3.11.1) *If f and the f_n are supported on a set $E \in S$ of totally finite measure, then the condition $\|f_n - f\|_{p,q} \to 0$ implies that $f_n \to f$ in μ measure.*

(3.11.2) *If p and q are finite, and if there exists a non-negative measurable function h such that $\|h\|_{p,q} < \infty$, and such that $\|f_n(\cdot)\| \leqslant h$ a.e. for each n, then the condition $f_n \to f$ in μ measure implies that $\|f_n - f\|_{p,q} \to 0$.*

Remarks. We shall use the total finiteness assumption sparingly in the proof of (3.11.1). In particular, when p and q are finite, it will suffice to assume that the set E has finite measure. It is evident that this result will remain valid if the condition $f_n \to f$ (in $L_{p,q}$ norm or in μ measure) is everywhere replaced by the condition that f_n is Cauchy (or fundamental) in the same sense. In fact we shall need the Cauchy version for the $L_{p,}$ completeness proof (Theorem 3.12).

Proof. To prove (3.11.1), we let $a > 0$ be fixed, and we let $F_n = \{x \in X: \|f_n(x) - f(x)\| \geqslant a\}$ for all n. Our aim is to show that $\mu(F_n) \to 0$.

Let p and q be finite. Since $a\chi_{F_n} \leqslant \|f_n(\cdot) - f(\cdot)\|$, we obtain $\|\chi_{F_n}\|_{p,q} \leqslant a^{-1}\|f_n - f\|_{p,q}$, and so the insertion of the numbers $1, 1, p, q$ (in that order) into Lemma 3.10 gives the result.

Let $p = \infty$, and let q be finite. Define $M = \big(\|y \mapsto \mu_y(E)\|_\infty, \nu\big) < \infty$. Then, for ν-almost all $y \in Y$, we have

$$\begin{aligned}\mu_y(F_n) = (\|\chi_{F_n}\|_1, \mu_y) &\leqslant a^{-1}(\|f_n - f\|_1, \mu_y)\\ &\leqslant a^{-1}\mu_y(E)(\|f_n - f\|_\infty, \mu_y)\\ &\leqslant Ma^{-1}(\|f_n - f\|_\infty, \mu_y).\end{aligned}$$

If we now compute the L_q norms of these functions, we obtain $\|\chi_{F_n}\|_{1,q} \leqslant Ma^{-1}\|f_n - f\|_{\infty,q}$, and so the insertion of the numbers $1, 1, 1, q$ into Lemma 3.10 gives the result.

Let p be finite, and let $q = \infty$. Let $F = \{y \in Y: \mu_y(E) > 0\}$, and define $N = \nu(F) < \infty$. Then from (2.1.3) and from the inequality $\mu_y(F_n) \leqslant a^{-p}(\|f_n - f\|_p, \mu_y)^p$ we deduce that we have

$$\begin{aligned}\mu(F_n) &= \int_F \mu_y(F_n)\,d\nu(y)\\ &\leqslant a^{-p}\nu(F)\Big(\big\|y \mapsto (\|f_n - f\|_p, \mu_y)^p\big\|_\infty, \nu\Big)\\ &= a^{-p}\nu(F)\|f_n - f\|_{p,\infty}^p,\end{aligned}$$

and so the result follows.

Finally, in view of Proposition 3.8, the case $p = q = \infty$ is both trivial and well known.

To prove (3.11.2), we note that the standard arguments (e.g. [19], Theorem C, pp. 125–126) will carry over to this setting with little more than a change of notation. ■

Remark. Likewise, the well-known elaborations of (3.11.12) (see [7], Theorem 6, p. 122; and [19], Theorem C, p. 163) possess readily verifiable analogues in the $L_{p,q}$ setting. ∎

3.12. THEOREM. *Let $1 \leqslant p, q \leqslant \infty$. Then the space $L_{p,q}(B)$, equipped with the $L_{p,q}$ norm, is complete.*

Remark. We shall use the decomposability of (X, S, μ) in the proof below. However, this assumption is not necessary when all of the functions involved are measurable, namely, when p and q are finite; nor is it necessary when $p = q = \infty$.

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be an $L_{p,q}$ Cauchy sequence in $L_{p,q}(B)$. Our aim is to find a function $f \in L_{p,q}(B)$ such that $\|f_n - f\|_{p,q} \to 0$.

We shall assume initially that the f_n are supported on a set of totally finite measure. By the Cauchy version of (3.11.1), we conclude that the f_n are Cauchy in μ measure, and then from the Riesz–Weyl theorem we obtain a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges a.e. to some function f. Now let n be fixed. Then, for ν-almost all $y \in Y$, we have $\|f_n(\cdot) - f_{n_k}(\cdot)\| \to \|f_n(\cdot) - f(\cdot)\|$ a.e. (μ_y), and so we may apply Fatou's lemma (including Lemma 3.5 when $p = \infty$) to obtain $(\|f_n - f\|_p, \mu_y) \leqslant \liminf_k (\|f_n - f_{n_k}\|_p, \mu_y)$ a.e. (ν). The inequality is preserved if we compute the L_q norm of each of these functions, and a second application of Fatou's lemma (or of Lemma 3.5) now gives $\|f_n - f\|_{p,q} \leqslant \liminf_k \|f_n - f_{n_k}\|_{p,q}$. From this inequality it is clear that $\|f_n - f\|_{p,q} \to 0$, and hence that $f \in L_{p,q}(B)$.

Next we let $\{X_a\}_{a \in A}$ be a decomposition for μ, and we momentarily let $a \in A$ be fixed. Then it follows from (3.9.3) that X_a constitutes the union of a sequence of pairwise disjoint measurable sets such that each of these sets has totally finite measure. Hence we shall lose no generality by supposing that X_a itself has totally finite measure. Now the sequence $\{f_n \chi_{X_a}\}_{n=1}^{\infty}$ is $L_{p,q}$ Cauchy, and so, by the special case considered above, it will have an $L_{p,q}$ limit f_a. Let us now define $f(x) = f_a(x)$ if $x \in X_a$ for some a, and let us define $f(x) = 0$ otherwise. Then it is clear that $f\colon X \to B$ is locally measurable, and it remains to verify that $\|f_n - f\|_{p,q} \to 0$. It follows easily from the triangle inequality that $\|(f_n - f)\chi_E\|_{p,q} \to 0$ when E is the union of finitely many of the X_a, and it follows easily from the monotone convergence theorem (including Lemma 3.4) that the $L_{p,q}$ norm of an arbitrary function $g \in L_{p,q}(B)$ constitutes the supremum of $\|g\chi_E\|_{p,q}$ as E ranges through the totality of such finite unions. Again let n be fixed. Then, since we have $\|(f_n - f)\chi_E\|_{p,q} \leqslant \|f_n - f_m\|_{p,q} + \|(f_m - f)\chi_E\|_{p,q}$ for arbitrary m, we obtain $\|f_n - f\|_{p,q} \leqslant \sup_{m \geqslant n} \|f_n - f_m\|_{p,q}$ in the supremum. It is now clear that $\|f_n - f\|_{p,q} \to 0$, and hence also that $f \in L_{pq,}(B)$. ∎

If each of the functions f_n above is measurable, then so also is f, and we immediately obtain the following result.

3.13. COROLLARY. *Let $1 \leqslant p, q \leqslant \infty$. Then the collection of measurable functions in $L_{p,q}(B)$ constitutes a closed subspace of $L_{p,q}(B)$.* ∎

Remark. Since the measure space (X, S, μ) is decomposable, it is well known (and easily deduced from [7], Theorem 6, p. 335) that the collection of real valued locally measurable functions on X constitutes a complete lattice ([7], p. 43), with respect to the ordering $f \leqslant g$ l.a.e. This ordering descends to the real $L_{p,q}$ spaces, and all of its basic properties ([7], pp. 302–305), carry over from the L_p to the $L_{p,q}$ setting with identical proofs. In particular, the real space $L_{p,q}$ is a complete lattice with respect to this ordering ([7], Theorems 22–23, p. 302). (Cf. [27].) ∎

In the results below we examine the circumstances under which simple functions are dense.

3.14. PROPOSITION. *Let $1 \leqslant p, q < \infty$. Then the collection of B-valued simple measurable functions f such that f is supported on a set of totally finite measure constitutes a dense subspace of $L_{p,q}(B)$.*

Note. In particular, then, the collection of simple integrable functions in $L_{p,q}(B)$ is dense in $L_{p,q}(B)$.

Proof. Let $f \in L_{p,q}(B)$, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of B-valued simple measurable functions which converges pointwise to f. In view of (3.9.3), we may ensure that each of the f_n is supported on a set of totally finite measure (so that $f_n \in L_{p,q}(B)$), and we may also ensure that $\|f_n(\cdot)\| \leqslant 2\|f(\cdot)\|$ for all n. We then obtain $\|f_n - f\|_{p,q} \to 0$ from the dominated convergence theorem. ∎

3.15. PROPOSITION. *Let $1 \leqslant q \leqslant \infty$, and let N be a positive integer. Then the collection of simple functions in $L_{\infty,q}(\Phi^N)$ is dense in $L_{\infty,q}(\Phi^N)$.*

Proof. In view of Proposition 3.8, we may assume that q is finite. Moreover, it suffices to assume that $N = 1$, as we may treat each coordinate separately when $N > 1$. It also suffices to approximate an arbitrary non-negative function in $L_{\infty,q}$ by simple functions.

Therefore, let $g \in L_{\infty,q}$ be non-negative, and let $n = 1, 2, \ldots$ If $x \in X$ is such that $k2^{-n} \leqslant g(x) < (k+1)2^{-n}$ for some $k = 0, 1, \ldots, n2^n$, define $g_n(x) = k2^{-n}$; otherwise define $g_n(x) = 0$. Then g_n is simple, locally measurable, and $\leqslant g$, so that $g_n \in L_{\infty,q}$. To show that $\|g_n - g\|_{\infty,q} \to 0$, we let $E \in S$ be fixed. Then the finiteness of $\|g\|_{\infty,q}$ implies that of $(\|g\chi_E\|_\infty, \mu_y)$ for ν-almost all $y \in Y$, and we note that we have $(\|(g_n - g)\chi_E\|_\infty, \mu_y) \leqslant 2^{-n}$ whenever n is $\geqslant (\|g\chi_E\|_\infty, \mu_y)$. We also have $(\|(g_n - g)\chi_E\|_\infty, \mu_y) \leqslant (\|g\chi_E\|_\infty, \mu_y)$ for all $n = 1, 2, \ldots$, and for all $y \in Y$, and so we obtain $\|(g_n - g)\chi_E\|_{\infty,q} \to 0$

from the dominated convergence theorem. Finally, since E is an arbitrary member of S, we obtain $\|g_n - g\|_{\infty,q} \to 0$ from (3.1.7). ∎

If the function g above is measurable, then so also is each of the g_n, and we immediately obtain the following result.

3.16. COROLLARY. *Let $1 \leqslant q \leqslant \infty$, and let N be a positive integer. Then the collection of simple measurable functions in $L_{\infty,q}(\Phi^N)$ is dense in the space of measurable functions in $L_{\infty,q}(\Phi^N)$.* ∎

In the next series of results, we shall examine the containment relations among the $L_{p,q}$ spaces, and, along with these, we shall examine the map $(p, q) \mapsto \|f\|_{p,q}$, where $f\colon X \to B$ is a fixed locally measurable function. The norm properties of $\|\cdot\|_{p,q}$ will not be required, and so we recall the remarks which were made prior to Lemma 3.10.

3.17. PROPOSITION. *Let $0 < p, q \leqslant \infty$, and assume both that the set X is measurable, and that X has totally finite measure. Then we have $L_{r,s}(B) \subset L_{p,q}(B)$ whenever r and s simultaneously satisfy $p \leqslant r \leqslant \infty$ and $q \leqslant s \leqslant \infty$.*

Proof. It clearly suffices to verify that $L_{r,q}(B) \subset L_{p,q}(B)$, and that $L_{p,s}(B) \subset L_{p,q}(B)$. These containments follow easily from [5], Proposition 22, p. 239 (and its obvious generalization to the case $p < 1$ or $q < 1$). ∎

3.18. DEFINITION. If $f\colon X \to B$ is a locally measurable function, then we shall let D_f denote the set $\{(p, q) \in (0, \infty] \times (0, \infty]\colon \|f\|_{p,q} < \infty\}$. ∎

We shall endow the set D_f with the relativization of the standard product topology on $(0, \infty] \times (0, \infty]$.

In general, if the measurable function $f\colon X \to B$ is supported on a set of totally finite measure, then Proposition 3.17 asserts that D_f will have the "box-like" property that: If $(p, q) \in D_f$, then $(0, p] \times (0, q] \subset D_f$. In particular, we may conclude that D_f is arcwise connected. This weaker fact remains valid when f is not supported on a set of totally finite measure, and its proof will be clear from the following result.

3.19. PROPOSITION. *Let $f\colon X \to B$ be a locally measurable function, let $0 < p_1, q_1, p_2, q_2 \leqslant \infty$, and let $p_1 \leqslant p \leqslant p_2$. Under the convention $1/\infty = 0$, let $0 \leqslant \lambda \leqslant 1$ be such that $1/p = \lambda/p_1 + (1-\lambda)/p_2$. Then we have $\|f\|_{p,q} \leqslant \|f\|_{p_1,q_1}^{\lambda} \|f\|_{p_2,q_2}^{1-\lambda}$, where q is defined by the identity $1/q = \lambda/q_1 + (1-\lambda)/q_2$.*

Proof. In view of (3.1.2), we may assume that f is measurable. We then obtain the inequality $(\|f\|_p, \mu_y) \leqslant (\|f\|_{p_1}, \mu_y)^{\lambda} (\|f\|_{p_2}, \mu_y)^{1-\lambda}$ from the standard application of Hölder's inequality ([5], p. 237). With note of the fact that $\lambda q/q_1 + (1-\lambda) q/q_2 = 1$, we apply Hölder's inequality a second time to obtain

$$\int (\|f\|_p, \mu_y)^q d\nu(y) \leqslant \Big[\int (\|f\|_{p_1}, \mu_y)^{q_1} d\nu(y)\Big]^{\lambda q/q_1} \Big[\int (\|f\|_{p_2}, \mu_y)^{q_2} d\nu(y)\Big]^{(1-\lambda)q/q_2}$$

when q, q_1, and q_2 are finite, and an analogous inequality otherwise. The result follows. ∎

It is important to note that the set of pairs (p, q) which satisfy the conditions of Proposition 3.19 constitutes a Jordan (or simple) arc in the extended plane $[-\infty, \infty]\times[-\infty, \infty]$. Hence the following result is immediate.

3.20. COROLLARY. *Let $f: X\to B$ be a locally measurable function. Then the set D_f is arcwise connected.* ∎

In the three propositions below we investigate the continuity properties of the map $(p, q)\mapsto \|f\|_{p,q}$ for the fixed locally measurable function $f: X\to B$. The first of these will suggest the precise nature of the arc in D_f which will always connect two given points of D_f.

3.21. PROPOSITION. *Let $f: X\to B$ be a locally measurable function, and let A be any of the following sets:*

(3.21.1) *a vertical line $x = a$, or a horizontal line $y = b$ (where $0 < a, b \leqslant \infty$);*

(3.21.2) *a line $y = cx$ (where $0 < c < \infty$); or*

(3.21.3) *a hyperbola of the form $(x-a)(y-b) = ab$ (where $0 < |a|, |b| < \infty$).*

Then the map $(p, q)\mapsto \|f\|_{p,q}$ is continuous on the set $A\cap D_f$.

Note. We intend to include the "infinite end points" of A as part of A. Thus we consider the point (a, ∞) to be part of the hyperbola $(x-a)(y-b) = ab$, we consider the point (∞, ∞) to be part of the line $y = cx$, and so forth.

Proof. It is clear that the expression $\log\|f\|_{p,q}$, where $(p, q)\in A\cap D_f$, may be regarded either as a function of $1/p$ or of $1/q$. (We preserve the convention that $1/\infty = 0$.) The fact that this function is convex (and hence continuous ([25], pp. 108–110)) is obtained from Proposition 3.19 after a brief analysis of the set A. The result follows. ∎

The remaining continuity properties of the map $(p, q)\mapsto \|f\|_{p,q}$ essentially constitute corollaries of this result and of the following lemma.

3.22. LEMMA. *Let $f: X\to B$ be a locally measurable function, let $0 < p_n, q_n \leqslant \infty$ for $n = 1, 2, \ldots$, let $p_n\to p > 0$, and let $q_n\to q > 0$. Then $\|f\|_{p,q} \leqslant \liminf \|f\|_{p_n,q_n}$.*

Proof. We shall assume for notational purposes that p and q are finite, and also that $\|f\|_{p,q}$ is finite. Given $\varepsilon > 0$, we first approximate f (via the monotone convergence theorem) by a simple measurable function

h such that $0 \leqslant h(x) \leqslant \|f(x)\|$ for all $x \in X$, and such that $\|f\|_{p,q} \leqslant \|h\|_{p,q} + \varepsilon/4$. Let $h = \sum_{i=1}^{n} c_i \chi_{E_i}$, where the sets $E_i \in S$ are pairwise disjoint. We next approximate each function $y \mapsto \mu_y(E_i)$ by a simple ν-measurable function h_i such that $0 \leqslant h_i(y) \leqslant \mu_y(E_i)$ for all $y \in Y$, and such that $\|h\|_{p,q} \leqslant \left(\int \left(\sum_{i=1}^{n} c_i^p h_i(y)\right)^{q/p} d\nu(y)\right)^{1/q} + \varepsilon/4$. Let $h_i = \sum_{j=1}^{m} d_{ij}\chi_{F_j}$ for all $i = 1, 2, \ldots, n$, where the sets $F_j \in T$ are pairwise disjoint. We may assume that the c_i, the d_{ij}, and the $\nu(F_j)$ are all finite, so that, following routine algebraic manipulations, we obtain

$$\|f\|_{p,q} \leqslant \Big(\sum_{j=1}^{m}\Big[\Big(\sum_{i=1}^{n} c_i^p d_{ij}\Big)^{1/p}\Big]^q \nu(F_j)\Big)^{1/q} + \varepsilon/2$$

$$\text{(3.22.1)} \qquad \leqslant \Big(\sum_{j=1}^{m}\Big[\Big(\sum_{i=1}^{n} c_i^p d_{ij}\Big)^{1/p} - \delta\Big]^q \nu(F_j)\Big)^{1/q} + 3\varepsilon/4,$$

$$\text{(3.22.2)} \qquad \leqslant \Big(\sum_{j=1}^{m}\Big[\Big(\sum_{i=1}^{n} c_i^p d_{ij}\Big)^{1/p} - \delta\Big]^{q_k} \nu(F_j)\Big)^{1/q_k} + \varepsilon,$$

$$\text{(3.22.3)} \qquad \leqslant \Big(\sum_{j=1}^{m}\Big[\Big(\sum_{i=1}^{n} c_i^{p_k} d_{ij}\Big)^{1/p_k}\Big]^{q_k} \nu(F_j)\Big)^{1/q_k} + \varepsilon$$

$$\leqslant \|h\|_{p_k,q_k} + \varepsilon$$

$$\leqslant \|f\|_{p_k,q_k} + \varepsilon,$$

where $\delta > 0$ is chosen sufficiently small that (3.22.1) is valid, and where k is then chosen sufficiently large that first (3.22.2) and then (3.22.3) become valid. Since ε is arbitrary, the result follows.

The proofs of the remaining special cases of this lemma require little more than suitable changes of notation in the preceding proof as we have presented it. As usual, Lemma 3.4 must replace the monotone convergence theorem when the L_∞ norm is involved. ∎

3.23. PROPOSITION. *Let $f\colon X \to B$ be a locally measurable function. Then the map $\tau_f(p, q) = \|f\|_{p,q}$ is continuous on the set $A = \operatorname{int}(D_f) \cup D_f^c$, where $\operatorname{int}(D_f)$ denotes the (topological) interior of D_f, and where D_f^c denotes the complement of D_f in the set $(0, \infty] \times (0, \infty]$.*

Proof. Let $0 < p, q \leqslant \infty$. If $\|f\|_{p,q} = \infty$, then the continuity of τ_f on A at the point (p, q) follows at once from Lemma 3.22. It therefore remains to establish this fact when (p, q) lies interior to D_f.

We shall assume initially that p and q are finite. Given $\varepsilon > 0$, we use the continuity of τ_f along horizontal and vertical lines in D_f (Proposition 3.21) to enclose the point (p, q) in the interior of a rectangle

$R = [r_1, r_2] \times [s_1, s_2] \subset D_f$ in such a way that we have $\|f\|_{r_i,s_j} \leqslant (1+\varepsilon)\|f\|_{p,q}$ for each $i, j = 1, 2$. If $(r, s) \in R$, then two applications of Proposition 3.19 will establish the existence of numbers $0 \leqslant \lambda_{ij} \leqslant 1$ (where $i, j = 1, 2$) such that $\sum_{i,j=1}^{2} \lambda_{ij} = 1$, and such that $\|f\|_{r,s} \leqslant \prod_{i,j=1}^{2} \|f\|_{r_i,s_j}^{\lambda_{ij}}$. It therefore follows that $\|f\|_{r,s} \leqslant (1+\varepsilon)\|f\|_{p,q}$. As ε is arbitrary, this inequality, in conjunction with Lemma 3.22, completes the proof that τ_f is continuous at (p, q) ([25], Problem 14, p. 37).

When $p = \infty$ or $q = \infty$, the proof is conceptually identical, except for the fact that the point (p, q) will lie on an "edge" of the appropriate rectangle R. ■

3.24. COROLLARY. *Let f: $X \to B$ be a measurable function which is essentially bounded and which is supported on a set of totally finite measure. Then the map $(p, q) \mapsto \|f\|_{p,q}$ is continuous on $(0, \infty] \times (0, \infty]$.* ■

As an informal corollary to Propositions 3.12 and 3.23, we remark that the behavior of the map $(p, q) \mapsto \|f\|_{p,q}$ along that portion of each arc (3.21.1)–(3.21.3) which lies in $(0, \infty] \times (0, \infty]$ is analogous in every respect to the behavior of the standard map $p \mapsto \|f\|_p$ along the axis $(0, \infty]$.

The intuitively clear terminology of the next proposition will be explained precisely during the proof of that result.

3.25. PROPOSITION. *Let f: $X \to B$ be a measurable function which is supported on a set of totally finite measure. Then the map $\tau_f(p, q) = \|f\|_{p,q}$ is continuous from the lower left on $(0, \infty] \times (0, \infty]$, and it is also continuous from the upper right on D_f.*

Proof. Let $\{(p_n, q_n)\}_{n=1}^{\infty}$ be a sequence in $(0, \infty] \times (0, \infty]$, and let $(p, q) \in (0, \infty] \times (0, \infty]$.

To prove the first assertion, we must show that if $p_n \uparrow p$ and $q_n \uparrow q$, then $\|f\|_{p_n,q_n} \to \|f\|_{p,q}$. By Proposition 3.23, we may assume that $(p, q) \in D_f$, and we shall also assume initially that p and q are finite. In view of Propositions 3.17 and 3.21, the map τ_f will be continuous at (p, q) along the line segments which join (p, q) to the points $(0, q)$, $(0, 0)$, and $(p, 0)$. Therefore, given $\varepsilon > 0$, we may determine a closed rectangle $R \subset D_f$ (cf. Proposition 3.23) such that (p, q) constitutes the upper right-hand corner of R, and such that $\|f\|_{r,s} \leqslant (1+\varepsilon)\|f\|_{p,q}$ for each of the three remaining corners (r, s) of R. The proof now proceeds exactly as in Proposition 3.23.

When $p = \infty$ or $q = \infty$, the proof is conceptually identical, except that the "line segment which joins (p, q) to $(0, 0)$" must be replaced by a suitable arc which can be well defined in this setting. When $p = \infty$ and q is finite, any hyperbola of the form $(x-a)(y-q) = aq$, where

$a < 0$, will suffice; when p is finite and $q = \infty$, any hyperbola of the form $(x-p)(y-b) = pb$, where $b < 0$, will suffice; and when $p = q = \infty$, any straight line through the origin will suffice.

To prove the second assertion, we must show that if (p, q) and the (p_n, q_n) are all $\in D_f$, and if $p_n \downarrow p$ and $q_n \downarrow q$, then $\|f\|_{p_n,q_n} \to \|f\|_{p,q}$. However, this is an immediate consequence of Propositions 3.17, 3.21, and 3.23. ∎

It should be emphasized that the technique of Propositions 3.23 and 3.25 is as important as the results themselves, for it enables questions of joint continuity of the map τ_f to be reduced to questions of one-dimensional continuity along suitable arcs, and not necesarily just the arcs which were specified in Proposition 3.21. For example, it is a straightforward consequence of Proposition 3.21 that, when $f\colon X \to B$ is locally measurable, the topological boundary of D_f in $(0, \infty] \times (0, \infty]$ may be parametrized as a Jordan arc in some neighbourhood of an arbitrary boundary point (p, q). (We are excluding the degenerate case $D_f = \{(p, q)\}$.) As a result, the technique of Proposition 3.23 may be used to show that τ_f is continuous on D_f at a boundary point $(p, q) \in D_f$ if and only if it is continuous along the boundary of D_f at (p, q). (The details of the proofs of these assertions are somewhat tedious, owing to the proliferation of special cases.) The suggestion clearly emerges that if this technique cannot be employed to establish the continuity of τ_f on some subset of D_f, then τ_f will probably not be continuous on that subset, as can indeed be the case, even when the function f is supported on a set of totally finite measure (see Example 5.9).

3.26. PROBLEM. Characterize the circumstances under which the map τ_f would be continuous on D_f at some boundary point $(p, q) \in D_f$. ∎

The final result of this section (Theorem 3.28) treats the question of the uniform convexity of the $L_{p,q}$ spaces. We have been unable to find an elementary (or at least a more direct) proof of Theorem 3.28; instead, we shall use a major representation theorem from Section 6 in conjunction with the following result.

3.27. PROPOSITION [1]. *Let (X, S, μ) and (Y, T, ν) be arbitrary σ-finite measure spaces, and let $1 \leqslant p, q < \infty$. Then the space $L_{p,q}(\nu \times \mu, B)$ is isometrically isomorphic to the space $L_q(\nu, L_p(\mu, B))$.*

Remark. With straightforward changes to the proof below, this result may be extended to the case where p or q is infinite, provided that attention is entirely restricted to measurable functions. Example 5.2 essentially illustrates the pitfalls of further generalization.

Proof. Given a function $f \in L_{p,q}(\nu \times \mu, B)$ and a point $y \in Y$, define

[1] We are grateful to Mr. Ian Inglis for suggesting this result to us.

$U(f)(y)\colon x\mapsto f(y, x)$. Then it is clear that $U(f)(y)$ constitutes a well defined element of $L_p(\mu, B)$ for almost all $y \in Y$, and that the map U constitutes an isometric embedding of $L_{p,q}(\nu\times\mu, B)$ into $L_q(\nu, L_p(\mu, B))$. That U maps onto all of $L_q(\nu, L_p(\mu, B))$ is also clear if we note that a simple function $F\in L_q(\nu, L_p(\mu, B))$ constitutes the image of a (not necessarily simple) function $f\in L_{p,q}(\nu\times\mu, B)$. Then, given an arbitrary element F of $L_q(\nu, L_p(\mu, B))$, we let $\{F_n\}_{n=1}^{\infty}$ be a sequence of simple functions in $L_q(\nu, L_p(\mu, B))$ which converges to F in L_q norm. If $F_n = U(f_n)$ for each n, then the f_n comprise a Cauchy sequence in $L_{p,q}(\nu\times\mu, B)$, and it follows from the completeness of this space that they converge to a limit $f\in L_{p,q}(\nu\times\mu, B)$. Clearly we have $F = U(f)$, and so the proof is complete. ∎

3.28. THEOREM. *Let $1 < p, q < \infty$, and assume that the Banach space B is uniformly convex. Then so also is the space $L_{p,q}(B)$.*

Proof. It follows from Theorem 6.14 and Proposition 3.27 that $L_{p,q}(B)$ is isometrically isomorphic to a subspace of $L_q(\mu, L_p(\mu, B))$. Therefore it suffices to show that this latter space is uniformly convex. However, this follows at once from two applications of [4], Theorem 2, p. 504. ∎

It is clear from Proposition 3.27 (including its extension to infinite values of p and q) that the space $L_{p,q}(B)$ need not be uniformly convex in the absence of any one of the hypotheses of Theorem 3.28.

4. Integral representation of bounded linear functionals on $L_{p,q}(B)$

The structure of the dual spaces of the $L_{p,q}$ spaces is entirely analogous to that of the duals of the L_p spaces. Consequently our attention within this section will be devoted principally (1) to equating the appropriate linear functional and $L_{p,q}$ norms, and (2) to characterizing when a linear functional admits the standard integral representation. This latter question arises in the L_p setting only for the case of $L_\infty(B)$, and seems not to have been treated.

In the next paragraph we set out the notation which will be adopted within this section.

4.1. NOTATIONAL CONVENTIONS. If $1\leqslant p\leqslant\infty$, we shall let p' denote the unique number which is conjugate to p. (Thus, under the convention $1/\infty = 0$, we have $1/p+1/p' = 1$.) If V is a normed linear space, we shall let V^+ denote the space of linear functionals on V, and we shall let V^* denote the dual space of V. Given $v\in V$, and given $v^+\in V^+$, we shall write $\langle v, v^+\rangle$ in place of $v^+(v)$ whenever the former notation is more convenient. Without ambiguity we shall denote all linear functional norms by the same symbol $\|\cdot\|$ which is used for the norm in B. Finally, if $1 \leqslant p, q \leqslant \infty$, we shall let $I_{p,q} = \{E \in S\colon \chi_E \in L_{p,q}\}$. That $I_{p,q}$ constitutes a μ-ideal in S ([18], Definition 3.1, p. 199), is clear from (3.1.4), from (3.9.1), and from (3.9.2). ∎

We now recall precisely the sort of integral representation which we wish to obtain for certain linear functionals on the $L_{p,q}$ spaces. Let $1 \leqslant p, q \leqslant \infty$, let $f \in L_{p,q}(B)$, let $g\colon X \to B^*$ be a locally weak* measurable function (i.e., the function $x \mapsto \langle b, g(x)\rangle$ is locally measurable for all $b \in B$), and assume that $\|g(\cdot)\| \in L_{p',q'}$. Then it is readily verified that the function $x \mapsto \langle f(x), g(x)\rangle$ is locally measurable; moreover, from (3.1.2) and from two applications of Hölder's inequality we deduce the inequality $\|\langle f(\cdot), g(\cdot)\rangle\|_1 \leqslant \|g\|_{p',q'}\|f\|_{p,q}$, where $\|g\|_{p',q'} = \big\| \|g(\cdot)\| \big\|_{p',q'}$. It follows that the function $x \mapsto \langle f(x), g(x)\rangle$ is essentially measurable, so that we may define a (finite) μ-integral for this function without essential alteration to the standard procedure (e.g [25], §3, pp. 225 ff.; cf. [28], Definition, p. 91). Moreover, all of the basic properties of the standard μ-integral remain valid for this extended μ-integral, with the important exception that, in general, the disintegration of μ may not be used to compute it (see Example 5.2). Therefore, in spite of this limitation, it is clear that the expression $G(f) = \int \langle f(x), g(x)\rangle d\mu(x)$ determines an element $G \in L_{p,q}(B)^*$ in the whose linear functional norm $\|G\|$ is $\leqslant \|g\|_{p',q'}$.

We may now state more precisely the two questions which will concern us in this section: (1) When do we have $\|G\| = \|g\|_{p,q}$? and (2) Given $G \in L_{p,q}(B)^*$, when can we find a function g to represent G in the manner which was indicated above?

For the first of these questions we shall initially consider the case $B = \Phi$. For this case the function g is Φ-valued and locally measurable, and the expression $\langle f(x), g(x)\rangle$ reduces to the simple product $f(x)g(x)$.

4.2. Lemma. *Let $\varphi\colon Y \to \Phi$ be a (locally) ν-measurable function and let $f\colon X \to B$ be a μ-measurable function. Then, for ν-almost all $y \in Y$, we have either that $\int \varphi(\psi(x)) f(x) d\mu_y(x) = \varphi(y) \int f(x) d\mu_y(x)$, or that both integrals fail to exist. We also have $\big(\|(\varphi\circ\psi)f\|_\infty, \mu_y\big) = |\varphi(y)|\,(\|f\|_\infty, \mu_y)$ a.e. (ν).*

Proof. Let $\varphi = \chi_F$, where $F \in T$, and let $f = b\chi_E$, where $b \in B$, and where $E \in S$. Then it is readily deduced from assumption (2.1.3) that we have $\int_G \mu_y\big(E \cap \psi^{-1}(F)\big) d\nu(y) = \int_G \mu_y(E)\chi_F(y) d\nu(y)$ for all $G \in T$, so that $\mu_y\big(E \cap \psi^{-1}(F)\big) = \mu_y(E)\chi_F(y)$ a.e. (ν). Since we have $\varphi(\psi(x)) f(x) = b\chi_{E\cap\psi^{-1}(F)}(x)$ in this case, the desired equalities follow. By considering simple functions, and then by using the convergence theorems in the standard manner (in particular, as in Lemma 3.6), we may extend the result to arbitrary f for fixed $\varphi = \chi_F$, and thence to arbitrary φ. ∎

4.3. PROPOSITION. *Let* $1 \leqslant p, q \leqslant \infty$, *let* $g \in L_{p',q'}$, *and define the linear functional* $G \in L^*_{p,q}$ *by the identity* $G(f) = \int fgd\mu$ *for all* $f \in L_{p,q}$ (*where the nature of this* (*extended*) μ-*integral was discussed above*). *Then we have* $\|G\| = \|g\|_{p',q'}$.

Proof. It remains to show that $\|G\| \geqslant \|g\|_{p',q'}$. By (3.1.2) we may and shall assume throughout this proof that g is measurable, and not just locally measurable. We shall also maintain the convention $1/\infty = 0$ throughout.

Let $1 < p, q \leqslant \infty$. Eliminating the trivial case $\|g\|_{p',q'} = 0$, we define a function $f\colon X \to \Phi$ by the formula

$$f(x) = |g(x)|^{p'/p}\big(\overline{g(x)}/|g(x)|\big)\big(\textstyle\int |g(w)|^{p'} d\mu_{\psi(x)}(w)\big)^{\lambda}\, \|g\|_{p',q'}^{-q'/q},$$

where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$, where $\lambda = -(1/p') + (q'/qp')$, and where $x \in X$ is such that this formula makes sense, i.e., specifies a well defined scalar. For other $x \in X$, we define $f(x) = 0$. Then, in the light of Lemma 4.2, it is routinely verified that $\|f\|_{p,q} = 1$, and that $\int fgd\mu = \|g\|_{p',q'}$.

Let $1 < p \leqslant \infty$, and let $q = 1$. Given $\varepsilon > 0$, we deduce from the σ-finiteness of ν that there exists a set $F \in T$ such that $0 < \nu(F) < \infty$, and such that we have $(\|g\|_{p'}, \mu_y) \geqslant \|g\|_{p',\infty} - \varepsilon$ for all $y \in F$. We now define

$$f(x) = |g(x)|^{p'/p}\big(\overline{g(x)}/|g(x)|\big)\big(\textstyle\int |g(w)|^{p'} d\mu_{\psi(x)}(w)\big)^{-1/p}\nu(F)^{-1}\chi_F\big(\psi(x)\big)$$

when this expression makes sense, and we define $f(x) = 0$ otherwise. Again it is straightforward to check that $\|f\|_{p,1} = 1$, and that $\int fgd\mu \geqslant \|g\|_{p',\infty} - \varepsilon$. Since ε is arbitrary, the desired inequality follows.

Let $p = 1$, and let $1 < q \leqslant \infty$. By (3.9.3) we may assume that g is supported on a set of totally finite measure (Definition 3.9), and by Proposition 3.15 we may assume that g is simple. Hence, if we define the set $F = \{x \in X\colon 0 < |g(x)| = (\|g\|_\infty, \mu_{\psi(x)})\}$, we see that F has totally finite measure, and (what is most important) we see that $\mu_y(F) > 0$ whenever $(\|g\|_\infty, \mu_y) > 0$. Eliminating the trivial case $\|g\|_{\infty,q'} = 0$, we now define

$$f(x) = \mu_{\psi(x)}(F)^{-1}\chi_F(x)\big(\overline{g(x)}/|g(x)|\big)(\|g\|_\infty, \mu_{\psi(x)})^{q'/q}\|g\|_{\infty,q'}^{-q'/q}$$

when this expression makes sense, and we define $f(x) = 0$ otherwise. The identities $\|f\|_{1,q} = 1$ and $\int fgd\mu = \|g\|_{\infty,q'}$ are routine as usual.

Finally, in view of Proposition 3.8, the case $p = q = 1$ is well known, and it is also trivial ([28], Corollary 6.1.1, p. 156). ∎

4.4. DEFINITION. Let $1 \leqslant p, q \leqslant \infty$, and let $G \in L_{p,q}(B)^+$. Then the *set function determined by* G is the function $m_G\colon I_{p,q} \to B^+$ which is defined by the identity $\langle b, m_G(E)\rangle = G(b\chi_E)$ for all $b \in B$, and for all $E \in I_{p,q}$.

We shall say that G is *countably additive* if m_G is weak* countably additive (i.e., if the function $E \mapsto \langle b, m_G(E)\rangle$ is countably additive for all $b \in B$). ■

We shall write m in place of m_G whenever clarity will not be sacrificed. From inequalities which are similar to (4.5.5)–(4.5.7) below, we may readily deduce that $|m|(E) \leqslant \|G\|\,\|\chi_E\|_{p,q}$ for all $E \in I_{p,q}$, where $|m|$ denotes the (total) variation of m ([5], p. 32). Note also that if g represents G in the manner described earlier, then m will constitute the (weak*) indefinite integral of g with respect to μ on $I_{p,q}$ ([18], pp. 198–199).

4.5. Theorem. *Let $1 \leqslant p, q \leqslant \infty$, and let $g: X \to B^*$ be a locally weak* measurable function which is such that $\|g(\cdot)\|$ is locally measurable, and which is such that the function $x \mapsto \langle f(x), g(x)\rangle$ is integrable (possibly in the extended sense (q.v.)) for all $f \in L_{p,q}(B)$. Define $G(f) = \int \langle f(x), g(x)\rangle d\mu(x)$ for all $f \in L_{p,q}(B)$, and let m be the set function determined by G. If we may now assume that we have*

(4.5.1) $\quad |m|(E) = \int_E \|g(x)\| d\mu(x)$ *for every set E in a dense subset of $I_{p,q}$* ([18], Definition 3.1, p. 199),

then we may conclude that

(4.5.2) $$\|G\| = \|g\|_{p',q'} < \infty.$$

Proof. We shall assume initially that $\|g(\cdot)\| \in L_{p',q'}$. Under this assumption the dominated convergence theorem may be applied to yield the countable additivity of G. The countable additivity of $|m|$ now follows from [18], Lemma 4.6, p. 203, so that, from (4.5.1) and from [18], Proposition 3.2, p. 199, we obtain the equality $|m|(E) = \int_E \|g(x)\| d\mu(x)$ for all $E \in I_{p,q}$.

Since we have noted the inequality $\|G\| \leqslant \|g\|_{p',q'}$ in the previous discussion, it remains to establish that $\|G\| \geqslant \|g\|_{p',q'}$. This we shall accomplish by way of inequalities (4.5.3)–(4.5.8) below, and we shall now explain the notation which occurs in these inequalities. Given $\varepsilon > 0$, we first obtain from Proposition 4.3 a (non-negative) measurable function $h \in L_{p,q}$ such that $\|h\|_{p,q} \leqslant 1$, and such that (4.5.3) is satisfied. From the monotone convergence theorem we next obtain numbers $a_i > 0$ and pairwise disjoint sets $E_i \in S$, for $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^{n} a_i \chi_{E_i} \leqslant h$, and such that (4.5.4) is satisfied. Noting, then, the finiteness of $|m|(E_i)$ for each i, we next obtain pairwise disjoint sets $E_{ij} \in S$, for $j = 1, 2, \ldots, n_i$, such that $E_i = \bigcup_{j=1}^{n_i} E_{ij}$, and such that (4.5.5) is satisfied. For each i and j we now find an element $b_{ij} \in B$ such that $\|b_{ij}\| \leqslant 1$, and such that (4.5.6) is satisfied.

(The absence of absolute value signs in (4.5.6) is critical to this argument.) Define $f = \sum_{i=1}^{n} \sum_{j=1}^{n_i} a_i b_{ij} \chi_{E_{ij}}$. Then we have $\|f\|_{p,q} \leqslant 1$, so that (4.5.7) and (4.5.8) are immediate. The complete string of inequalities now follows.

$$\|g\|_{p',q'} \leqslant \int h(x)\|g(x)\|\,d\mu(x) + \varepsilon/4 \tag{4.5.3}$$

$$\leqslant \int \Big[\sum_{i=1}^{n} a_i \|g(x)\| \chi_{E_i}(x)\Big]\, d\mu(x) + \varepsilon/2 \tag{4.5.4}$$

$$= \sum_{i=1}^{n} a_i |m|(E_i) + \varepsilon/2$$

$$\leqslant \sum_{i=1}^{n} \sum_{j=1}^{n_i} a_i \|m(E_{ij})\| + 3\varepsilon/4 \tag{4.5.5}$$

$$\leqslant \sum_{i=1}^{n} \sum_{j=1}^{n_i} a_i \langle b_{ij}, m(E_{ij})\rangle + \varepsilon \tag{4.5.6}$$

$$= |G(f)| + \varepsilon \tag{4.5.7}$$

$$\leqslant \|G\| + \varepsilon. \tag{4.5.8}$$

Since ε is arbitrary, the desired inequality follows.

We shall complete the entire proof by showing that if $\|g\|_{p',q'} = \infty$, then there would exist a function $f_\infty \in L_{p,q}(B)$ such that $\int \langle f_\infty(x), g(x)\rangle\, d\mu(x) = \infty$, contrary to assumption. Let $M = \{E \in I_{p,q}\colon \|g\chi_E\|_{p',q'} < \infty\}$. Then M is a μ-ideal, and, from the special case above, we obtain $\|G_E\| = \|g\chi_E\|_{p',q'}$ for all $E \in M$, where $G_E(f) = G(f\chi_E)$ for all $f \in L_{p,q}(B)$. It follows at once from [18], Proposition 3.2, p. 199, that G cannot be bounded. In fact, if $E \in M$, then G cannot be bounded on the space of functions in $L_{p,q}(B)$ which vanish on E. Therefore, given a positive integer n, we may find a set $E_n \in M$ and a function $f_n \in L_{p,q}(B)$ such that E_n is disjoint from E, such that f_n is supported on E_n, such that $\|f_n\|_{p,q} \leqslant 2^{-n}$, such that $\langle f_n(x), g(x)\rangle \geqslant 0$ for all $x \in X$, and such that $G(f_n) \geqslant 1$. From this observation, in conjunction with the obvious induction argument (cf. [5], pp. 234–235), we may easily construct a function f_∞ with the properties specified above. ∎

When g is locally measurable, then (4.5.1) is automatically valid because of [7], Theorem 20 (a), p. 114 (cf. [18], p. 209). Hence the following corollary is immediate.

4.6. Corollary. *Let* $1 \leqslant p, q \leqslant \infty$. *Then the space* $L_{p',q'}(B^*)$ *is isometrically isomorphic to a (closed) norming subspace of* $L_{p,q}(B)^*$ (see [18], Definition 4.5, p. 203, and the discussion following). *Therefore its intersection with the unit ball of* $L_{p,q}(B)^*$ *is weak* dense in the unit ball.* ∎

If suitable definitions are made, then Theorem 4.5, and its proof, will extend at once to the more general form of integration which Dinculeanu considers in [5] (Chapter II, and in particular pp. 228 ff.). We obtain a generalization of his Theorem 7 (pp. 233–236) even in the L_p setting, since his arguments appear to rely crucially upon the stronger assumption that g is (locally) measurable. Moreover, the corresponding generalization of his Theorem 5 (pp. 228–232) will then follow as an immediate corollary.

In view of [18], Example 6.1, p. 213, it would be unreasonable to expect an automatic generalization of Proposition 4.3 from the scalar to the general case. Hypothesis (4.5.1) is about the weakest additional condition which can be imposed, since it is implied by (4.5.2) when p and q are finite. (However, we may obtain (4.5.1) when $p = \infty$ or $q = \infty$ by assuming that $\|G_E\| = \|g\chi_E\|_{p',q'}$ for all sets E in a dense subset of $I_{p,q}$, where $G_E(f) = G(f\chi_E)$ for all $f \in L_{p,q}(B)$.) The idea of the proof is to let $h \geqslant 0$ be the Radon–Nikodym derivative of $|m|$ with respect to μ ([18], Theorem 3.4, p. 200). The inequality $h \leqslant \|g(\cdot)\|$ l.a.e. is easily established, and we contradict (4.5.2) (or its strengthened version when $p = \infty$ or $q = \infty$) if we assume strict inequality on a set of positive measure. For a discussion of criteria which are allied to (4.5.1), see [18], p. 209.

We turn now to the question of when a linear functional $G \in L_{p,q}(B)^*$ may be represented by a B^*-valued locally weak* measurable function g. We shall not deviate from the standard practice of obtaining g as a Radon–Nikodym derivative of the set function m_G (Definition 4.4) with respect to μ. Therefore it will be feasible to study the action of G only upon the *measurable* functions in $L_{p,q}(B)$.

4.7. Definition. If $1 \leqslant p,\ q \leqslant \infty$, we shall let $L^m_{p,q}(B)$ denote the space of (equivalence classes of) measurable functions in $L_{p,q}(B)$. ■

That $L^m_{p,q}(B)$ is a Banach space follows from Corollary 3.13, and from (3.1.6) we obtain $L^m_{p,q}(B) = L_{p,q}(B)$ when p and q are finite. As usual, we shall write $L^m_{p,q}$ in place of $L^m_{p,q}(B)$ if and only if $B = \Phi$.

4.8. Definition. Let $1 \leqslant p,\ q \leqslant \infty$, and let $G \in L^m_{p,q}(B)^*$. Then we shall say that G has the *dominated convergence property* provided that, if we are given a sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $L^m_{p,q}(B)$, and if there exists a non-negative function $h \in L^m_{p,q}$ such that $\|f_n(\cdot)\| \leqslant h$ for all n, then the condition $f_n \to 0$ a.e. implies that $G(f_n) \to 0$. ■

In view of the dominated convergence theorem, the following facts are straightforward to verify.

(4.8.1) If $1 \leqslant p,\ q \leqslant \infty$, and if the linear functional $G \in L^m_{p,q}(B)^*$ has the dominated convergence property, then G is countably additive (Definition 4.4).

(4.8.2) If $1 \leqslant p, q < \infty$, then every linear functional $G \in L^m_{p,q}(B)^*$ has the dominated convergence property.

4.9. THEOREM. *Let $1 \leqslant p, q \leqslant \infty$, and let $G \in L^m_{p,q}(B)^*$. Then G has the dominated convergence property if and only if there exists a function $g\colon X \to B^*$ such that:*

(4.9.1) *g is locally weak* measurable;*

(4.9.2) *$\|g(\cdot)\|$ is locally measurable, and we have $\|G\| = \|g\|_{p',q'}$;*

and

(4.9.3) *for all $f \in L^m_{p,q}(B)$, we may compute $G(f) = \int \langle f(x), g(x)\rangle d\mu(x)$.*

Moreover, if the space B^* is separable or reflexive, then g may be chosen to be locally measurable, and not just locally weak measurable.*

Remark. The assumption that (X, S, μ) is decomposable will tacitly be made when we apply [18], Theorem 5.5, p. 212, below. However, when p' and q' are finite, the usual sort of argument (e.g. [7], p. 288) will render this assumption unnecessary.

Proof. If g satisfies (4.9.1)–(4.9.3), then an easy application of the dominated convergence theorem will establish that G has the dominated convergence property.

Now let G have the dominated convergence property, and let m be the set function determined by G. Then from (4.8.1) we obtain the countable additivity of G, and from the comment which followed Definition 4.4 we obtain the finiteness of $|m|$ on the μ-ideal $I_{p,q}$. Therefore we may apply [18], Theorem 5.5, p. 212, in conjunction with Theorem 4.5 to obtain a locally weak* measurable function $g\colon X \to B^*$ which satisfies $\langle b, m(E)\rangle = \int_E \langle b, g(x)\rangle d\mu(x)$ for all $b \in B$ and for all $E \in I_{p,q}$, and which is also such that, whenever $\|g\chi_E\|_{p',q'} < \infty$, we have $\|G_E\| = \|g\chi_E\|_{p',q'}$, where $G_E(f) = G(f\chi_E)$ for all $f \in L^m_{p,q}(B)$. It is now clear that we have $\|g\|_{p',q'} \leqslant \|G\| < \infty$, so that (4.9.2) and (4.9.3) follow.

Finally, when B^* is separable, the local measurability of g follows from a slight generalization of [23], Theorem 1.1, p. 278 (cf. [18], Lemma 4.7, p. 203). When B^* is reflexive, the local measurability of g may be obtained by applying Theorem 5.7 (p. 213) of [18] in place of Theorem 5.5. ∎

The dominated convergence property is neither an unexpected nor, we suspect, an avoidable criterion for the integral representation of bounded linear operators in any setting. For example, it can be seen to be equivalent to the standard assumption which is made in the setting of the Daniell integral ([25], pp. 286 ff.).

In the following corollary of Theorem 4.9, the first of several, we investigate conditions under which the converse of (4.8.1) is valid.

4.10. Corollary. *Let $1 \leqslant p$, $q \leqslant \infty$. Then the set of countably additive linear functionals in $L^m_{p,q}(B)^*$ coincides with the set of linear functionals in $L^m_{p,q}(B)^*$ which have the dominated convergence property if and only if the collection of simple functions in $L^m_{p,q}(B)$ is dense in $L^m_{p,q}(B)$.*

Proof. We note from the proof of Theorem 4.9 that only the countable additivity of G was required in order to obtain the Radon–Nikodym derivative g. The dominated convergence property was needed only to verify (4.9.3), which is automatically valid on the closure of the simple functions in $L^m_{p,q}(B)$. If this closure constitutes all of $L^m_{p,q}(B)$, then it follows that G will have the dominated convergence property.

Conversely, if the simple functions in $L^m_{p,q}(B)$ are not dense in $L^m_{p,q}(B)$, then the Hahn–Banach theorem will yield a non-zero linear functional $G \in L^m_{p,q}(B)^*$ which vanishes on the space of simple functions. That G is countably additive is trivial, whereas G could not have the dominated convergence property without being identically equal to zero. ∎

Thus from Example 5.4 we see that, even when $B = \Phi$, there will exist countably additive bounded linear functionals which do not have the dominated convergence property. From [7], Theorem 16, p. 296, we may readily deduce that there will also exist bounded linear functionals which are not even countably additive. In particular, such linear functionals may be defined on $L_\infty([0,1])$ (with Lebesgue measure), and we may obtain such linear functionals on the spaces $L_{p,q}([0,1]\times[0,1])$ (with product Lebesgue measure) when $p = \infty$ or $q = \infty$ by embedding $L_\infty([0,1])$ into $L_{p,q}([0,1]\times[0,1])$ in the obvious way, and then by appropriately using Proposition 3.17 and the Fubini theorem.

4.11. Definition. If $1 \leqslant p, q \leqslant \infty$, we shall let $L^w_{p',q'}(B^*)$ denote the collection of those bounded linear functionals on $L^m_{p,q}(B)$ which have the dominated convergence property. ∎

Of course this notation makes sense only in the light of Theorem 4.9. It is readily verified that $L^w_{p',q'}(B^*)$ constitutes a closed subspace of $L^m_{p,q}(B)^*$, and hence a Banach space in its own right. By exploiting the notion of a "type ϱ" function ([18], Definition 5.1, p. 209), we may realize $L^w_{p',q'}(B^*)$ as a space of equivalence classes of locally weak* measurable functions modulo functions which are l.a.e. equal to zero. It will thereby be seen to resemble the space $L_{p',q'}(B^*)$ more closely than might otherwise have been supposed. (That it is not identical to $L_{p',q'}(B^*)$, in general, is illustrated by [18], Example 6.3, p. 214.) We now summarize these remarks as part of the next corollary.

4.12. Corollary. *Let $1 \leqslant p$, $q \leqslant \infty$. Then we have*

$$L_{p',q'}(B^*) \subset L^w_{p',q'}(B^*); \tag{4.12.1}$$

and

(4.12.2) $$L^w_{p',q'}(B^*) \subset L^m_{p,q}(B)^*.$$

Moreover, these containments may be strict; however, we obtain equality in (4.12.1) *when* B^* *is separable or reflexive, and we obtain equality in* (4.12.2) *when both* p *and* q *are finite.*

Note. The notation in (4.12.1) is, of course, slightly abusive (cf. Corollary 4.6). ∎

4.13. COROLLARY. *Let* $1 < p, q < \infty$, *and let the Banach space* B *be reflexive. Then so also is the space* $L_{p,q}(B)$.

Proof. Cf. [7], Corollary 24, p. 67; Corollary 2, p. 288. ∎

Subject to two precuations, the principal results of this section will generalize in a natural and straightforward manner to linear operators on $L_{p,q}(B)$ which assume values in an arbitrary Banach space D. The first precaution is that, in order to apply [18], Theorem 5.5, p. 212, in the proof of the analogue of Theorem 4.9, it may be necessary to allow the function g to assume values in $\mathscr{L}(B, D^{**})$, and not just in $\mathscr{L}(B, D)$ (where $\mathscr{L}(B, D)$ denotes the space of bounded linear operators from B to D, and where $\mathscr{L}(B, D^{**})$ is defined likewise). The second precaution is that inequality (4.5.6) is meaningless in the more general setting. As a result it can happen that $\|G\| < \|g\|_{p',q'}$; however, the arguments of Theorem 4.5 may be used to establish the identity

$$\|g\|_{p',q'} = \sup\Big\{\sum_{i=1}^{n} \|G(b_i\chi_{E_i})\|:\ f = \sum_{i=1}^{n} b_i\chi_{E_i} \in L^m_{p,q}(B);\ \|f\|_{p,q} \leqslant 1\Big\}.$$

The remaining details present no real problems, and we omit them. (Cf. [5], pp. 255 ff.)

5. Examples in $L_{p,q}$ theory

In this section we collect together a few elementary examples. Their primary purpose is to illustrate certain aspects of L_p theory which do not generalize to the $L_{p,q}$ setting, and to elucidate certain of the previous arguments which may have appeared to be unnecessarily cumbersome.

In the next paragraph we set out some notation which will be common to many of the examples.

5.1. NOTATIONAL CONVENTIONS. We shall use the symbol I to denote the unit interval $[0, 1]$, and we shall use I^2 to denote $[0, 1]\times[0, 1]$. We shall let λ denote Lebesgue measure on the Lebesgue measurable subsets of $\boldsymbol{R}$, and we shall let λ^2 denote product Lebesgue measure $\lambda\times\lambda$. In deference to the customary notation for points along the x-axis, we shall denote the standard disintegration of λ^2 with respect to λ (see Example 2.2) by $\{\lambda_x\}_{x\in\boldsymbol{R}}$. A *standard elementary function* will be a Lebesgue measurable function $f\colon \boldsymbol{R}^2\to\boldsymbol{R}$ of the form $\sum_{n=1}^{\infty} c_n\chi_{E_n\times F_n}$, where the sets E_n ($\subset \boldsymbol{R}$) are pairwise disjoint. We note from the nonatomic nature of λ that we shall be free to choose the c_n, the $\lambda(E_n)$, and the $\lambda(F_n)$ arbitrarily to suit the purpose at hand. Moreover, when we specify or construct a standard elementary function, we shall use the symbols c_n, E_n, and F_n, as defined here, with no further comment. ■

We have made an apparently fussy distinction between measurable and essentially measurable functions, and, in marked contrast to the L_p norm, we have defined the $L_{p,q}$ norm in a two stage procedure. Our first example will illustrate the necessity for all of this.

5.2. Example. Given $z\in I$, we define $I_z^2 = I^2\times\{z\}$. Then we may clearly transfer λ^2 from I^2 to I_z^2, and we define (X, S, μ) (in this case) to be the direct sum of the resulting measure spaces ([27], Definition 3.1, p. 282). (Thus $X = \bigcup_{z\in I} I_z^2$, and the family $\{I_z^2\}_{z\in I}$ constitutes a decomposition for μ. It is well to visualize X not as the standard unit cube $[0,1]\times[0,1]\times\times[0,1]$, but rather as an extremely long (two-dimensional) ribbon.) We disintegrate μ with respect to λ on I by carrying over the standard disintegration of λ^2 on I^2 with respect to λ on I in the obvious manner. (In particular, then, we have $\psi(x, y, z) = x$ for all $(x, y, z)\in X$.) We denote the resulting disintegration by $\{\mu_x\}_{x\in I}$.

We now consider, for example, the set $N = \{(x, y, z)\in X\colon x = z\}$. This set N is locally null, and hence χ_N is an essentially measurable function which represents the zero linear functional on $L_{p,q}(\mu)$ for all values of p and q. However, we have $\mu_x(N) = 1$ for all $x\in I$. Thus the linear functional norm of χ_N would not equal $\|\chi_N\|_{p',q'}$ if (3.1.1) were used exclusively to compute the $L_{p,q}$ seminorms. Moreover, if we employed the formula $\int \mu_x(E)\,d\lambda(x)$ to extend μ to the smallest σ-ring which contains S and $\{N\}$, then we would obtain an essentially different measure space. (Even the L_p spaces would be different.)

It might nevertheless be suggested that Definition 3.1 was contrived artificially in order to facilitate the dual space representations. However, it is easy to construct a set K which is similar to N, but which is such that the function $x\mapsto\mu_x(K)$ is not measurable, or which is such that $\mu_x(K) = 1$ for all $x\in I$, whereas both of the functions $x\mapsto\mu_x(N\cap K)$ and $x\mapsto\mu_x(N\cup K)$ fail to be measurable. Thus the collection of locally measurable sets E such that the function $x\mapsto\mu_x(E)$ is well defined and measurable is far from being a σ-ring, and likewise the collection of locally measurable functions f such that the expression $(\|x \mapsto (\|f\|_p, \mu_x)\|_q, \lambda)$ is well defined and finite is far from being even a vector space. It would appear, then, that there is no useful way to apply (3.1.1) to locally measurable functions as well as to measurable functions.

On the other hand, when $p = \infty$ or $q = \infty$, locally measurable functions need to be considered, if only to serve as linear functionals. Therefore, in consideration of the dual space isometry results (Proposition 4.3 and Theorem 4.5), and in consideration of the canonical extension of any measure μ to the σ-algebra of locally measurable sets ([1], Exercise 1, p. 53; Exercise 21, p. 183; cf. [28], Scholium 3.10, p. 90), we suggest that formula (3.1.2) constitutes the unique natural choice for an extension of (3.1.1) to locally measurable functions. ∎

Our next example will illustrate that, whenever $p \neq q$, convergence in $L_{p,q}$ norm need not imply convergence in measure. Therefore, some sort of piecewise construction (in Theorem 3.12) of the limit of an $L_{p,q}$ Cauchy sequence of functions in $L_{p,q}(B)$ would seem to be unavoidable.

5.3. Example. Whenever $p \neq q$, there exists a standard elementary function $f \in L_{p,q}(\boldsymbol{R}^2, \lambda^2)$ such that the sequence of partial sums $f_m = \sum_{n=1}^{m} c_n \chi_{E_n \times F_n}$, where $m = 1, 2, \ldots$, converges to f in $L_{p,q}$ norm, but does not converge in λ^2 measure.

Assume that $0 < p < q < \infty$. Define $c_n = \lambda(E_n) = 1$, and define $\lambda(F_n) = n^{-1}$. Then we have $\|f\|_{p,q}^q = \sum_{n=1}^{\infty} n^{-q/p} < \infty$, we have $\|f_m - f\|_{p,q}^q = \sum_{n=m+1}^{\infty} n^{-q/p} \to 0$, and, whenever $0 < a \leqslant 1$, we have

$$\lambda^2\big(\{(x, y) \in \boldsymbol{R}^2 : |f_m(x, y) - f(x, y)| \geqslant a\}\big) = \sum_{n=m+1}^{\infty} n^{-1} = \infty.$$

It is also evident in this case that $\|f_m - f\|_{p,\infty} \to 0$.

Now assume that $0 < q < p < \infty$. Define $c_n = 1$, define $\lambda(E_n) = n^{-2p/(p-q)}$, and define $\lambda(F_n) = n/\lambda(E_n)$. Then we have $\|f\|_{p,q}^q = \sum_{n=1}^{\infty} n^{(q/p)-2} < \infty$, and the remaining conclusions are analogous to those above. Moreover, we may draw the same conclusion for the case $p = \infty$ by redefining $\lambda(E_n) = 2^{-n}$ in this case. ∎

Our next example will illustrate that the collection of simple functions need not be dense in the space $L_{p,q}(B)$, even when $B = \Phi$. A minor consequence is that we cannot reasonably compress inequalities (4.5.3) and (4.5.4) into a single inequality. In view of Propositions 3.14 and 3.15, we must have both $1 \leqslant p < \infty$ and $q = \infty$.

5.4. EXAMPLE. Whenever $1 \leqslant p < \infty$, there exists a standard elementary function $f \in L_{p,\infty}(I^2, \lambda^2)$ which cannot be approximated in the $L_{p,\infty}$ norm by any *bounded* function in $L_{p,\infty}(I^2, \lambda^2)$.

Define $c_n = 2^n$, define $\lambda(E_n) = 2^{-n}$, define $\lambda(F_n) = 2^{-np}$, and let $g: I^2 \to \Phi$ be a measurable function such that $(\|g\|_\infty, \lambda^2) =$ a finite constant M. Then, for any positive integer n such that $2^n \geqslant M$, and for λ-almost every point $x \in E_n$, we have $(\|f-g\|_p, \lambda_x) \geqslant ((2^n - M)^p 2^{-np})^{1/p} = 1 - M2^{-n}$. It follows that $\|f-g\|_{p,\infty} \geqslant 1$.

We remark that we have required the function f to be supported on I^2 rather than just on $\boldsymbol{R}^2$ in order to illustrate that even support on a set of totally finite measure will not necessarily render f approximable in the $L_{p,\infty}$ norm by a bounded function (see Definition 3.9 and the comment following it). ∎

Our next example will illustrate that Proposition 3.15 was special to the case $B = \Phi^N$, even when $X = I^2$. Thus, whenever $p = \infty$ or $q = \infty$, simple functions need not be dense in $L_{p,q}(B)$.

5.5. EXAMPLE. Let $1 \leqslant q \leqslant \infty$, let H be any infinite-dimensional Hilbert space, let $\{e_{ij}\}_{i,j=1}^{\infty}$ be a doubly indexed orthonormal subset of H, and let $\{E_i\}_{i=1}^{\infty}$ be a sequence of pairwise disjoint measurable subsets of I such that $\lambda(E_i) = 2^{-i}$ for each i. Define $f = \sum_{i,j=1}^{\infty} e_{ij}\chi_{E_i \times E_j}$. Then $\|f\|_{\infty,q} = 1$, so that $f \in L_{\infty,q}(I^2, \lambda^2, H)$, and we shall show that f cannot be approximated in the $L_{\infty,q}$ norm by any simple function $g \in L_{\infty,q}(I^2, \lambda^2, H)$.

To this end, let $x \in E_i$ for some i. If we had $(\|f-g\|_\infty, \lambda_x) < \sqrt{2}/2$, then, because g has only finitely many distinct values, it would follow from the triangle inequality that $\|e_{ij} - e_{ik}\| < \sqrt{2}$ for some $j \neq k$. However, this contradicts the equality $\|e_{ij} - e_{ik}\| = \sqrt{2}$ which is valid for any pair of orthonormal elements. Hence we must have $(\|f-g\|_\infty, \lambda_x) \geqslant \sqrt{2}/2$, and it follows that $\|f-g\|_{\infty,q} \geqslant \sqrt{2}/2$.

We remark that, unlike the situation in Example 5.4, the collection of bounded functions in the general space $L_{\infty,q}(B)$ is dense in $L_{\infty,q}(B)$, as is easily verified. ∎

We now consider some examples of the set D_f (Definition 3.18). The simplest such examples occur when the function f is of the form $(x, y) \mapsto f_1(x)f_2(y)$, where f_1 and f_2 are functions with domain $\boldsymbol{R}$, or when f is of the form χ_E, where $E \subset \boldsymbol{R}^2$. We see at once that D_f can be the cartesian product of any pair of intervals in $(0, \infty]$, including singletons. (In particular, D_f can be a singleton, or a horizontal or vertical line segment.) We also see that $D_f \cap \boldsymbol{R}^2$ can be any convex cone in $(0, \infty) \times (0, \infty)$

which emanates from the origin. Furthermore, we may automatically obtain additional examples by noting that

(5.5.1) the collection of subsets of $\boldsymbol{R}^2$ of the form $D_f \cap \boldsymbol{R}^2$ as f varies, is closed under the formation of finite intersections.

The following definition will clarify why this is true.

5.6. Definition. We shall say that two locally μ-measurable sets E_1 and E_2 are *totally disjoint* if there exist disjoint subsets F_1 and F_2 of Y such that $E_i \subset \psi^{-1}(F_i)$ for $i = 1, 2$. ∎

Thus, if the locally measurable functions f and g are supported on totally disjoint sets, and if $0 < p \leqslant \infty$, then we have $\|f+g\|_{p,q}^q = \|f\|_{p,q}^q + \|g\|_{p,q}^q$ when $0 < q < \infty$, and we also have $\|f+g\|_{p,\infty} = \max\{\|f\|_{p,\infty}, \|g\|_{p,\infty}\}$. It follows that $D_{f+g} = D_f \cap D_g$, whereupon (5.5.1) is readily verified.

In each of the examples above, the set $D_f \cap \boldsymbol{R}^2$ is convex. Our next example will illustrate that $D_f \cap \boldsymbol{R}^2$ need not be convex in general, even when the function f is supported on a set of totally finite measure. (However, it is clear from Proposition 3.19 that, with the usual convention $1/\infty = 0$, the set $\{(1/p, 1/q)\colon (p, q) \in D_f\}$ is convex.)

5.7. Example. Let a and b be (non-zero) real numbers, and let the standard elementary function f be defined by $c_n = n$, by $\lambda(E_n) = n^{-b-1}$, and by $\lambda(F_n) = n^{-a}$. Then we have $D_f \cap \boldsymbol{R}^2 = \{(p, q) \in (0, \infty) \times (0, \infty)\colon (p-a)(q-b) < ab\}$, and we note that this set is not convex. (In fact the "equinorm contour lines" for this function f will lie along the hyperbolas of the form $(x-a)(y-b') = ab'$, where $b' < b$, and including the degenerate case $b' = 0$ when $b > 0$.)

Continuing in the same vein, we see that D_f can be any of a variety of shapes bounded by hyperbolas of the form $(x-a)(y-b) = ab$ (where a and b vary), and that it can form part of such a hyperbola. (In particular, let $c_n = n^{-1}$, let $\lambda(E_n) = n^{b+1}$, and let $\lambda(F_n) = n^a$.) Therefore, while D_f is arcwise connected in all cases, it can happen that the *only* arc in D_f which will connect two given points of D_f is of the form specified in Proposition 3.21. ∎

5.8. Problem. Characterize the set D_f for an arbitrary locally measurable function $f\colon X \to B$. ∎

It is well known (and it is also a special case of Proposition 3.21) that the map $p \mapsto \|f\|_p$ is continuous on the set $\{p \in (0, \infty]\colon \|f\|_p < \infty\}$. Our final example will illustrate that this fact does not generalize to the $L_{p,q}$ setting, even when the function f is supported on a set of totally finite measure. It will also illustrate that the partial continuity results of Proposition 3.25 rely crucially upon the assumption of support on a set of totally finite measure.

5.9. Example. Let $0 < p, q < \infty$, let $p_n \to p$, let $q_n \to q$, and let $\varepsilon > 0$ be fixed. Suppose that we are able to construct a standard elementary function f which, for each $n = 1, 2, \ldots$, satisfies the following conditions:

(5.9.1) $\quad c_n^{q_n} \lambda(F_n)^{q_n/p_n} \lambda(E_n) = 1 + \varepsilon;$

(5.9.2) $\quad c_n^{q} \lambda(F_n)^{q/p} \lambda(E_n) = 2^{-n};$

and

(5.9.3) $\quad$ if $1 \leqslant i < n$, then $c_n^{q_i} \lambda(F_n)^{q_i/p_i} \lambda(E_n) \leqslant 2^{-n}$.

Then we may deduce from (5.9.3) that $f \in L_{p_n,q_n}(\boldsymbol{R}^2, \lambda^2)$ for all n, we may deduce from (5.9.2) that $\|f\|_{p,q}^q = \sum_{n=1}^{\infty} c_n^q \lambda(F_n)^{q/p} \lambda(E_n) = 1$, and we may likewise deduce from (5.9.1) that $\|f\|_{p_n,q_n}^{q_n} \geqslant 1 + \varepsilon$ for all n. Under these circumstances we clearly cannot have $\|f\|_{p_n,q_n} \to \|f\|_{p,q}$. (In fact we may readily verify that $\liminf_n \|f\|_{p_n,q_n} \geqslant (1+\varepsilon)^{1/q} > 1$, so that it is possible for the inequality of Lemma 3.22 to be strict.)

In order to obtain conditions (5.9.1)–(5.9.3) for a variety of choices of the p_n and the q_n, we shall adopt the following strategy. We solve for $\lambda(E_n)$ in (5.9.1) and in (5.9.2) to obtain

$$\lambda(E_n) = (1+\varepsilon) c_n^{-q_n} \lambda(F_n)^{-q_n/p_n} = 2^{-n} c_n^{-q} \lambda(F_n)^{-q/p}. \tag{5.9.4}$$

Assuming that $q_n \neq q$, we next solve for c_n in (5.9.4) to obtain

$$c_n = [(1+\varepsilon) 2^n]^{1/(q_n - q)} \lambda(F_n)^{-k_n/(q_n - q)}, \tag{5.9.5}$$

where, for each n, we define $k_n = q_n p_n^{-1} - q p^{-1}$. Finally, we insert expression (5.9.4) for $\lambda(E_n)$ and expression (5.9.5) for c_n into (5.9.3) to obtain

$$c_n^{q_i} \lambda(F_n)^{q_i/p_i} \lambda(E_n) = 2^{-n} \lambda(F_n)^{k_i} [(1+\varepsilon) 2^n \lambda(F_n)^{-k_n}]^{(q_i - q)/(q_n - q)}, \tag{5.9.6}$$

and we insert expression (5.9.5) for c_n into (5.9.4) to obtain

$$\lambda(E_n) = 2^{-n} \lambda(F_n)^{-q/p} [(1+\varepsilon) 2^n \lambda(F_n)^{-k_n}]^{-q/(q_n - q)}. \tag{5.9.7}$$

We now choose $\lambda(F_n)$ (either arbitrarily small or arbitrarily large) in such a way that expression (5.9.6) is $\leqslant 2^{-n}$ for each $i < n$, and also, whenever possible, in such a way that expression (5.9.7) is $\leqslant 2^{-n}$. (For if we can ensure that we have $\lambda(E_n) \leqslant 2^{-n}$ for all n, and if we can also ensure that the $\lambda(F_n)$ are bounded, then the function f will be supported on a set of totally finite measure.) Once $\lambda(F_n)$ is determined, we define c_n by (5.9.5), and we define $\lambda(E_n)$ by (5.9.7). Conditions (5.9.1)–(5.9.3) follow.

It remains, then, to obtain the $\lambda(F_n)$ as desired above, and it is clear that this is feasible only if we may ensure that the exponents of the $\lambda(F_n)$ in expression (5.9.6) will all have the same sign. To this end, let us impose the restriction that the p_n and the q_n must both converge in a strictly monotone fashion. With this restriction the desired consistency of sign is readily obtained, both in expression (5.9.6), and also (simultaneously) in expression (5.9.7). In particular, we may first specify either the p_n or the q_n in advance, and we may then select the members of the other sequence by an obvious induction argument. Moreover, if the p_n are specified in advance to obtain $p_n \downarrow p$ and $q_n \uparrow q$, or if the q_n are specified in advance to obtain $p_n \uparrow p$ and $q_n \downarrow q$, then the exponents of the $\lambda(F_n)$ in both (5.9.6) and (5.9.7) may all be chosen to be strictly positive. Hence the $\lambda(F_n)$ may be chosen to be arbitrarily small, so that the function f may be chosen to be supported on a set of totally finite measure.

We remark that, for the case $p_n \uparrow p$ and $q_n \uparrow q$, we may obtain support on a set of finite measure for the function f, so that even this relatively restrictive assumption about f is not sufficient to imply the conclusions of Proposition 3.25. On the other hand, it is interesting to note the different ways in which this example will break down if we select the p_n, etc. in accordance with the hypotheses of Proposition 3.21 or of Proposition 3.25. ∎

6. Structure of the $L_{p,q}$ spaces

A disintegrable measure space constitutes a natural generalization of a product measure space, and this fact lends intuitive appeal to the definition of the $L_{p,q}$ norm in terms of a "double integral." At the same time the generality of the $L_{p,q}$ concept appears to be limited. The principal results of this section are aimed, in part, at elucidating the minimum amount of machinery which is actually essential to the construction of the spaces $L_{p,q}(B)$, up to isometric isomorphism. Specifically, using only the collection $\{\psi^{-1}(F)\colon F \in T\}$ of locally measurable subsets of X, we shall associate with the space (X, S, μ) a subspace of $(X \times X,\ S \times S,\ \mu \times \mu)$ which is such that, given certain sets $E \in S$, a suitably defined "conditional expectation" of χ_E will be usefully related to the action of the standard product measure disintegration (see Example 2.2) upon the image of E in $S \times S$. One important consequence of this result is that the space $L_{p,q}(B)$, when it is defined, will be isometrically isomorphic to the associated $L_{p,q}$ space, and hence to a subspace of $L_{p,q}(\mu \times \mu, B)$. It is therefore reasonable to take this subspace as the definition of $L_{p,q}(B)$ when a disintegration of μ is not given, and we shall conclude the section with an informal discussion about the representation of this more general space $L_{p,q}(B)$ as a space of (equivalence classes of) B-valued, locally measurable functions defined on X itself, and equipped with an intrinsically defined "$L_{p,q}$" norm.

All of the material to follow is preliminary to Theorem 6.11, in which we shall construct the appropriate subspace of $(X \times X,\ S \times S,\ \mu \times \mu)$. None of these results (including Theorem 6.11 itself) will exploit the disintegrability of μ, and in fact we need only assume that (X, S, μ) is an arbitrary σ-finite measure space. For while the decomposability of μ will be used to promote conceptual clarity (especially within the somewhat formidable Theorem 6.11), it is not crucial to the principal conclusion of Theorem 6.11, as we shall remark.

Some of the following results are derived from and run parallel to the work of Maharam in [21]. Speaking loosely, we may say that Maharam treated the measure theoretic analogue of the question: When does a Banach space have a basis? We shall treat the analogue of the question: When does a subspace of a Banach space have a complement in that Banach space? Our own notion of "complementation" (to be compared, very roughly, with [7], p. 553) will immediately follow the list of notational conventions for this section.

6.1. Notational Conventions. If E and F are subsets of X, then we shall write $E \subset_\mu F$ when the set $E - F$ is locally null. We shall declare that E and F are *μ-equivalent*, and write $E =_\mu F$, when we have $E \subset_\mu F$ and $F \subset_\mu E$. If M is a collection of subsets of X, and if $K \subset X$, then we shall let $M|_K$ denote the collection $\{E \cap K\colon E \in M\}$, we shall let $\sigma(M)$ denote the σ-ring generated by M, and we shall let $[M]_\mu$ denote the collection of equivalence classes of the sets in M with respect to the equivalence relation $=_\mu$. We shall simplify notation by habitually referring to the elements of $[M]_\mu$ as sets rather than as equivalence classes of sets modulo null sets. Thus, if we refer to a *set* $E \in [M]_\mu$, we shall mean an (arbitrarily selected) member of some equivalence class in $[M]_\mu$.

In particular, let M_1 and M_2 be σ-subrings of S, and let $U\colon [M_1]_\mu \to [M_2]_\mu$ be a function. Then we shall call U a *ring homomorphism* if we have $U(\emptyset) =_\mu \emptyset$, and if, for all sets $E, F \in M_1$, we have $U(E \cup F) =_\mu U(E) \cup U(F)$ and $U(E \cap F) =_\mu U(E) \cap U(F)$; we shall call U a *σ-ring homomorphism* if U is a ring homomorphism, and if, for every sequence $\{E_n\}_{n=1}^\infty$ of sets in M_1, we have $U(\bigcup_{n=1}^\infty E_n) =_\mu \bigcup_{n=1}^\infty U(E_n)$. We note that if U is a one-to-one ring homomorphism, then U will be a σ-ring homomorphism if and only if the range of U constitutes a σ-ring (or, more precisely, if the collection of sets in M_2 whose μ-equivalence classes belong to the range of U constitutes a σ-ring). If, under these circumstances, the range of U is all of $[M_2]_\mu$, then we shall refer to U as a *σ-ring isomorphism*, we shall let $U^{-1}\colon [M_2]_\mu \to [M_1]_\mu$ be the standard inverse of U with respect to the composition of maps, and we shall define a measure $U(\mu)$ on M_2 by specifying that $U(\mu)(E) = \mu(U^{-1}(E))$ for all $E \in M_2$. Finally, we shall occasionally also apply the preceding definitions (with $=$ replacing $=_\mu$) to a function $U\colon M_1 \to M_2$.

(We remark that we have been using the specific measure μ for illustrative purposes; we may and shall use all of the above notation in connection with other measures.)

We recall now the following measure theoretic conventions about the symbol $\times$: When it appears between (the symbols for) two σ-rings, then we shall be indicating the standard product σ-ring ([10], p. 140); when it appears between two measures, we shall be indicating the standard product measure ([10], p. 145); and when it appears between two other sets, we shall be indicating the standard set theoretic cartesian product of those sets.

As we shall freely employ the axiom of choice, it will be sufficient for our purposes to define a *cardinal number* as an ordinal number $\varkappa$ each of whose members has strictly smaller cardinality than $\varkappa$. If A is any set, then we shall let $|A|$ denote the cardinality of A (that is, the uniquely determined cardinal number which can be put into a one-to-one correspondence with A). If γ is any ordinal number, then we shall let γ^+ denote the successor ordinal $\gamma \cup \{\gamma\}$.

Finally, we remark that we shall make frequent use of the phrase "standard σ-ring argument." By this phrase we shall intend a (straightforward) application of [22], Problem I.4.5, p. 19, the proof of which is entirely similar to that of the better known lemma on monotone classes ([10], Theorem B, p. 27). In most instances, arguments along different lines will also be possible. ∎

6.2. Definition. Let K be a measurable set of strictly positive finite measure. Then two measurable sets E and F, each $\subset K$, will be called *independent with respect to K* (or just *K-independent*) if we have $\mu(E \cap F) = \mu(K)^{-1}\mu(E)\mu(F)$. Two collections Q and R, each $\subset S|_K$, will be called *K-independent* if E and F are K-independent whenever $E \in Q$ and $F \in R$. A collection R of locally measurable sets will be called *complemented in K* if there exists a σ-ring $R^\perp \subset S|_K$ such that $K \in R^\perp$, such that $R|_K$ and $R^\perp$ are K-independent, and such that $[S|_K]_\mu = [\sigma(R|_K \cup R^\perp)]_\mu$. The σ-ring $R^\perp$ will be called *complementary to R in K*. ∎

The notation $R^\perp$ should not suggest that $R^\perp$ is in any way uniquely determined. The requirement that $K \in R^\perp$ is made for technical reasons, and we note that if $K \notin R^\perp$, then we could replace $R^\perp$ by $\sigma(R^\perp \cup \{K\})$ without damage to the other two properties. The definitions below are preliminary to Lemma 6.5, in which we indicate the significance of this notion of complementation.

6.3. DEFINITION. We shall say that a σ-ring $R \subset S$ is *σ-finite* if $\mu|_R$ (the restriction of μ to R) is a σ-finite measure. ∎

The σ-finiteness of R is precisely the assumption which is needed in order to adapt the notion of conditional expectation ([19], p. 341), from the setting of probability measures.

6.4. DEFINITION. Let R be a σ-finite σ-subring of S, and let h be a non-negative measurable function defined on X. Then an R-measurable function will be called the *conditional expectation of h with respect to R*, and will be denoted by $E^R h$, if we have

$$\int_G h d\mu = \int_G E^R h d\mu \quad \text{for all } G \in R. \tag{6.4.1}$$

If $E \in S$, then we shall let $k_E = E^R \chi_E$. ∎

Since R is σ-finite, the existence of $E^R h$ follows from the Radon-Nikodym theorem (see also [10], Exercise (7), p. 131). Moreover, equation (6.4.1) uniquely determines the equivalence class of $E^R h$ modulo null functions, so that reference to a function as "the" conditional expectation constitutes a tolerable abuse of terminology. We shall use precisely the following properties of the operator E^R.

(6.4.2) If h_1 and h_2 are non-negative measurable functions, then we have $E^R(h_1+h_2) = E^R h_1 + E^R h_2$ a.e.

(6.4.3) If the function h_1 in (6.4.2) is R-measurable, then we also have $E^R(h_1 h_2) = h_1 E^R h_2$ a.e.

(6.4.4) If $\{h_n\}_{n=1}^{\infty}$ is a sequence of non-negative measurable functions such that $h_n \uparrow h$ a.e., then we have $E^R h_n \uparrow E^R h$ a.e.

These properties are essentially well known (cf. [19], §25, pp. 347 ff.), and their proofs are entirely straightforward.

6.5. LEMMA. *Assume that μ is finite (so that S is a σ-algebra), let R be a σ-subalgebra of S which is complemented in X, and let $R^{\perp}$ be complementary to R in X. Then there exists a σ-ring isomorphism U: $[R \times R^{\perp}]_{\mu\times\mu} \to [S]_\mu$ for which we have*

$$U(F \times X) =_\mu F \quad \textit{for all } F \in R; \tag{6.5.1}$$

and

$$\mu_x(G) = \mu(X) k_{U(G)}(x) \quad \textit{a.e. for all } G \in R \times R^{\perp}, \tag{6.5.2}$$

where $\{\mu_x\}_{x\in X}$ denotes the standard disintegration of $\mu \times \mu$ with respect to μ.

Note. Thus the measures $U(\mu \times \mu)$ and μ differ only by the constant multiple $\mu(X)$.

Proof. Given $F \in R$, and given $E \in R^{\perp}$, we define $U(F \times E) = F \cap E$. There is now precisely one way to extend U to a ring homomorphism defined on $[Q]_{\mu \times \mu}$, where Q is the field which comprises the finite disjoint unions of rectangles of the form $F \times E$, and it is readily verfied both that this extension is well defined, and that it is indeed a ring homomorphism. It is also clear that we have $(\mu \times \mu)(G) = \mu(X)\mu(U(G))$ for all $G \in Q$.

We now employ a standard device (cf. [25], p. 322). Under the correspondence $G \leftrightarrow \chi_G$, we embed $[R \times R^{\perp}]_{\mu \times \mu}$ into $L_1(\mu \times \mu)$, and similarly we embed $[S]_\mu$ into $L_1(\mu)$. By the Carathéodory extension theorem, the set $[Q]_{\mu \times \mu}$ is dense in $[R \times R^{\perp}]_{\mu \times \mu}$ with respect to the metric topology which is induced by this embedding. Since $[S]_\mu$ is complete, and since U is essentially an isometry on $[Q]_{\mu \times \mu}$, it follows that U extends uniquely to $[R \times R^{\perp}]_{\mu \times \mu}$ in such a way that we have $(\mu \times \mu)(G) = \mu(X)\mu(U(G))$ for all $G \in R \times R^{\perp}$. Since the Boolean operations are continuous ([10], Theorem A, p. 168) it follows that the extended U is a ring homomorphism. But it is clear that U is one-to-one, and, since $X \in R^{\perp}$, that U maps onto $[S]_\mu$. Hence U is a σ-ring isomorphism.

Since property (6.5.1) is evident, it remains only to verify (6.5.2). To this end, let $F \in R$, and let $E \in R^{\perp}$. Then an easy computation will show that $k_E \equiv \mu(E)\mu(X)^{-1}$ a.e., so that an application of (6.4.3) will yield equality (6.5.2) for $G = F \times E$. That this equality is valid for an arbitrary set $G \in R \times R^{\perp}$ now follows from a standard σ-ring argument. ∎

In the light of this result we may now outline our overall strategy. We shall produce a decomposition $\{X_\alpha\}_{\alpha \in A}$ for (X, S, μ) which is such that the collection of inverse images of measurable subsets of Y is complemented in each of the X_α. We shall then use Lemma 6.5 to represent each X_α by a (genuine) measurable rectangle in $X \times X$, and we shall do this, moreover, in such a way that the "overhang" between any two of these (pairwise disjoint) rectangles resembles exactly the overhang between the corresponding pair of X_α. (We informally describe the "overhang" between two (disjoint) sets to be the extent of their deviation from total disjointness (see Definition 5.6). In the absence of a disintegration for μ, it can be determined by an examination of conditional expectations.) It will therefore be clear from Lemma 6.5 that, locally, an arbitrary σ-finite (disintegrable) measure μ looks exactly like a product measure; at the same time, the preservation of the overhang between different local elements will ensure the global isometric embedding of $L_{p,q}(B)$ into $L_{p,q}(\mu \times \mu, B)$ for all values of p and q (see Theorem 6.14).

Our immediate goal, then, is to establish that complemented sets occur in sufficient abundance to exhaust essentially all of X.

6.6. Definition. Let R be an arbitrary collection of locally measurable subsets of X. Then a set $K \in S$ will be called an *R-atom* (*for* μ) if $\mu(K) > 0$, and if $[S|_K]_\mu = [R|_K]_\mu$. ∎

We note that if K is an R-atom of finite measure, then the σ-ring $R^\perp = \{\emptyset, K\}$ is complementary to R in K. In the special case $R = \{\emptyset, X\}$, an R-atom constitutes an atom in the ordinary sense ([9], p. 138), so that the following result generalizes the well-known lemma of Halmos ([19], Lemma 2, p. 139) on non-atomic measures.

6.7. LEMMA. *Let R be a σ-finite σ-subring of S, let $K \in S$ contain no R-atoms, and let h be an R-measurable function on X which is such that $0 \leqslant h \leqslant k_K$ a.e. Then there exists a measurable set $E \subset K$ such that $h = k_E$ a.e.*

Note. We have $k_K \leqslant 1$ a.e. (see Definition 6.4).

Proof. We shall assume initially that $\mu(K) < \infty$. Using induction, we produce a sequence $\{E_n\}_{n=0}^{\infty}$ of pairwise disjoint measurable subsets of K such that $E_0 = \emptyset$, and such that, for each $n = 1, 2, \ldots$, the set E_n is selected from the collection of measurable sets $G \subset K - \bigcup_{i=0}^{n-1} E_i$ which satisfy $k_G \leqslant h - \sum_{i=0}^{n-1} k_{E_i}$ a.e. Moreover, if s_n denotes the supremum of the $\mu(G)$ for all such sets G, then E_n is selected in such a way that $\mu(E_n) \geqslant s_n/2$. We now define $E = \bigcup_{n=1}^{\infty} E_n$, and conclude, by combining (6.4.2) and (6.4.4), that $k_E = \sum_{n=1}^{\infty} k_{E_n} \leqslant h$ a.e. Let $K' = K - E$, and let $h' = h - k_E$. Then from (6.4.2) we obtain $0 \leqslant h' \leqslant k_{K'}$ a.e. Suppose that we did not have $h' = 0$ a.e. Then there would exist a measurable set $E' \subset K'$ such that $\mu(E') > 0$, and such that $k_{E'} \leqslant h'$ a.e. But from the finiteness of $\mu(K)$ we must conclude that $s_n \to 0$, so that E' could not have strictly positive measure. Hence we must have $h' = 0$ a.e., as desired.

(We remark that the somewhat technical construction of the E_n may be circumvented if resort is made to the axiom of choice in the form of Zorn's lemma.)

It remains, then, to produce the set E' as desired above, and this we accomplish by an easy adaptation to this setting of the argument in the proof of Lemma 2 on p. 110 of [21]. The one point which needs to be checked is the existence of a measurable set $G \subset K'$ for which the set $\{x \in X\colon 0 < k_G(x) < k_{K'}(x)\}$ has strictly positive measure. But if no such set G existed, then, for all $G \in S|_{K'}$, there would exist a set $F \in R$ such that $k_G = k_{K'}\chi_F$ a.e. A routine computation would then show that $G =_\mu K' \cap F$, so that K' would be an R-atom, contrary to assumption. (We remark that the "principle of exhaustion" which is mentioned on p. 110 of [21] may be understood to be an application of [18], Proposition 3.2, p. 199.)

If $\mu(K) = \infty$, then we let $K_n \uparrow K$, where $\mu(K_n) < \infty$ for each $n = 1, 2, \ldots$ An easy induction argument, based upon the special case above, will establish the existence of a sequence $\{E_n\}_{n=1}^{\infty}$ in S such that $E_n \subset K_n$ for all n, such that $k_{E_n}(x) = \min\{h(x), k_{K_n}(x)\}$ a.e. for all n, and such that $E_n \uparrow$ some set E. The equality $h = k_E$ a.e. now follows immediately from (6.4.4). ■

6.8. LEMMA. *Let $X \in S$ (so that μ is totally σ-finite), let R be a σ-finite σ-subalgebra of S, and assume that X contains no R-atoms for μ. Let L denote the Lebesgue measurable subsets of $[0, 1]$, let λ denote Lebesgue measure on L, and let $\{E_n\}_{n=1}^{\infty}$ be a sequence of (not necessarily distinct) sets in S. Then there exists a σ-subalgebra S' of S such that $R \subset S'$, such that $E_n \in S'$ for all n, and such that there exists a σ-ring isomorphism U: $[R \times L]_{\mu\times\lambda} \to [S']_\mu$ for which we have*

(6.8.1) $$U(F \times [0, 1]) =_\mu F \quad \textit{for all } F \in R;$$

and

(6.8.2) $$\lambda_x(G) = k_{U(G)}(x) \quad \textit{a.e. for all } G \in R \times L,$$

where $\{\lambda_x\}_{x\in X}$ denotes the standard disintegration of $\mu \times \lambda$ with respect to μ.

Note. Thus the measure $U(\mu \times \lambda)$ coincides with μ on S'.

Proof. Let R^ι (ι to suggest "inverse image") denote the σ-algebra $\{F \times [0, 1]: F \in R\}$. Then, in view of the Carathéodory extension theorem, and in view of the fact that λ is a non-atomic measure, it is not difficult to verify that $X \times [0, 1]$ contains no R^ι-atoms for $\mu \times \lambda$. It is also straightforward to verify the identity $(E^{R^\iota}\chi_G)(x, t) = \lambda_x(G)$ a.e. $(\mu \times \lambda)$ for all $G \in R \times L$. Therefore, given a set $K \in R \times L$, and given an R-measurable function h on X which is such that $0 \leqslant h(x) \leqslant \lambda_x(K)$ a.e., we may use Lemma 6.7 to obtain a set $G \subset K$ in $R \times L$ such that $h(x) = \lambda_x(G)$ a.e.

We shall initially define U upon sets rather than upon equivalence classes of sets, and we shall do this is an induction procedure which will require precisely two simple arguments. For the first of these, let $K \in R \times L$, and let $K' \in S$ be such that $k_{K'}(x) = \lambda_x(K)$ a.e. Now K' contains no R-atoms for μ. Therefore, given any set $G \subset K$ in $R \times L$, we may obtain from Lemma 6.7 a measurable set $G' \subset K'$ such that $k_{G'}(x) = \lambda_x(G)$ a.e. Furthermore, it follows from (6.4.2) that $k_{K'-G'}(x) = \lambda_x(K - G)$ a.e. For the second of these arguments, let $P = \{G_1, \ldots, G_n\} \subset R \times L$ be a partition of $X \times [0, 1]$ (i.e., the G_i are pairwise disjoint, and we have $X \times [0, 1] = \bigcup_{i=1}^{n} G_i$). Similarly, let $P' = \{G_1', \ldots, G_n'\} \subset S$ be a partition of X, and let P' be such that we have $k_{G_i'}(x) = \lambda_x(G_i)$ a.e. for $i = 1, 2, \ldots, n$. Now a typical set $G \in Q = \sigma(R^\iota \cup P)$ is of the form $\bigcup_{i=1}^{n} G_i \cap H_i$, where $H_i \in R^\iota$ for each i.

Let us define $U(G) = \bigcup_{i=1}^{n} G'_i \cap F_i$, where, for each i, the set $F_i \in R$ is determined by the identity $F_i \times [0, 1] = H_i$. Then, by using (6.4.2) and (6.4.3), we may readily verify that U constitutes a σ-ring isomorphism from Q onto $Q' = \sigma(R \cup P')$, and that we have $\lambda_x(G) = k_{U(G)}(x)$ a.e. for every set $G \in Q$.

We note as well that, because of the applicability of Lemma 6.7 to subsets of $X \times [0, 1]$, both of these arguments may be reversed (i.e., so that we may obtain sets in $R \times L$ which correspond, as above, to given sets in S).

We turn now to the actual construction of the desired σ-ring isomorphism U. Let $M = \{G_n\}_{n=1}^{\infty}$ be a countable subset of $R \times L$ which is such that $[R \times L]_{\mu \times \lambda} = [\sigma(R' \cup M)]_{\mu \times \lambda}$. Now let P_1 be the partition $\{G_1, G_1^c\}$. Using the first argument above, we obtain a partition $U(P_1) = \{U(G_1), U(G_1^c)\}$ of X such that $k_{U(G_1)}(x) = \lambda_x(G_1)$ a.e., and then, using the second argument above, we define the σ-ring isomorphism U from $Q_1 = \sigma(R' \cup P_1)$ onto $\sigma(R \cup U(P_1))$. Now let P'_1 be the least common refinement of the partitions $\{E_1, E_1^c\}$ and $U(P_1)$. (Thus we have $P'_1 = \{F_1, F_2, F_3, F_4\}$, where, let us say, $F_1 = E_1 \cap U(G_1)$, where $F_2 = E_1^c \cap U(G_1)$, and so on.) As before, we obtain a partition $U^{-1}(P'_1) = \{U^{-1}(F_1), \ldots, U^{-1}(F_4)\}$ of $X \times [0, 1]$ such that $G_1 = U^{-1}(F_1) \cup U^{-1}(F_2)$, such that $\lambda_x(U^{-1}(F_1)) = k_{F_1}(x)$ a.e., and also such that analogous identities hold for F_3 and F_4. As before, we obtain the σ-ring isomorphism U^{-1} from $Q'_1 = \sigma(R \cup P'_1)$ onto $\sigma(R' \cup U^{-1}(P'_1))$, and it is readily verified that this mapping U^{-1} does indeed extend the original mapping U^{-1} on $\sigma(R \cup U(P_1))$. We are now ready to begin a second cycle. Let P_2 be the least common refinement of the partitions $\{G_2, G_2^c\}$ and $U^{-1}(P'_1)$, and extend U from $\sigma(R' \cup U^{-1}(P'_1))$ to $Q_2 = \sigma(R' \cup P_2)$ just as we did for U^{-1} above. Continuing back and forth by the obvious induction procedure, we obtain an increasing sequence $\{Q_n\}_{n=1}^{\infty}$ of σ-subalgebras of $R \times L$ such that $G_n \in Q_n$ for all n, and we obtain a ring isomorphism U defined on the field $Q = \bigcup_{n=1}^{\infty} Q_n$ which is such that $E_n \in U(Q_{n+1})$ for all n, and which is such that we have $\lambda_x(G) = k_{U(G)}(x)$ a.e. for all $G \in Q$. These last equalities imply, in particular, that $\mu(U(G)) = 0$ if and only if $(\mu \times \lambda)(G) = 0$, so that U automatically determines a ring isomorphism defined on $[Q]_{\mu \times \lambda}$, and this we shall also denote by U.

Let $S' = \sigma(U(Q))$. Then, if μ is finite, we may follow the procedure of Lemma 6.5 to extend U to an isometry, and hence a σ-ring isomorphism, from $[\sigma(Q)]_{\mu \times \lambda} = [R \times L]_{\mu \times \lambda}$ onto $[S']_{\mu}$. Property (6.8.1) is evident, and we obtain property (6.8.2) from a standard σ-ring argument. If μ is not finite, we let $X \times [0, 1] = \bigcup_{n=1}^{\infty} H_n$, where the $H_n \in R'$ are pairwise disjoint, and where $(\mu \times \lambda)(H_n) < \infty$ for each n. We then extend U, in the manner described above, on each ring $[Q|_{H_n}]_{\mu\times\lambda}$ separately, and we piece together the resulting mappings in the one obvious way. ■

6.9. LEMMA. *Let T^ι (ι again to suggest "inverse image" (see Theorem 6.12)) be a σ-algebra of locally measurable subsets of X. Then the collection of sets $E \in S$ such that T^ι is complemented in E constitutes a dense subset of S* ([18], Definition 3.1, p. 199), *so that there exists a decomposition $\{X_a\}_{a\in A}$ of (X, S, μ) such that T^ι is complemented in each of the X_a.*

Remarks. Of course when μ is totally σ-finite, then X admits the usual sort of decomposition into at most countably many T^ι-atoms together with at most one non-null set E which contains no T^ι-atoms. If, in addition, $\mu|_E$ is finite and separable, then the proof below shows that T^ι is automatically complemented in E.

It is also worth mentioning that in all cases we shall construct the complementary σ-ring $R^\perp$ in such a way that μ is non-atomic on $R^\perp$. Further discussion follows Theorem 6.14.

Proof. In view of [18], Proposition 3.2, p. 199, it suffices to show that every non-null set $K \in S$ contains a non-null measurable subset in which T^ι is complemented. Since T^ι is trivially complemented in a T^ι-atom, it suffices to assume that K contains no T^ι-atoms.

Given any set $E \in S$, and given any σ-ring Q such that $T^\iota|_E \subset Q \subset S|_E$, let us define $\varkappa(E;Q)$ to be the smallest cardinal number θ for which there exists a set $M \subset Q$ such that $|M| = \theta$, and such that $[Q]_\mu = [\sigma(T^\iota|_E \cup M)]_\mu$. (We may informally describe $\varkappa(E;Q)$ as the minimum number of sets in Q which must be added to $T^\iota|_E$ in order to generate all of Q, up to μ-equivalence.) Now to each non-null measurable subset E of K we associate the cardinal number $\varkappa(E; S|_E)$, we let $\varkappa$ be the smallest member of the resulting set of cardinal numbers, and then we single out a specific non-null measurable subset E of K for which $\varkappa(E; S|_E) = \varkappa$. The minimality of $\varkappa$ will now enable us to show that T^ι is complemented in E, and to show it by simply iterating the argument of Lemma 6.8, possibly transfinitely often. (Cf. [21], Theorem 2, p. 111.)

Since K contains no T^ι-atoms, it follows that $\varkappa \geqslant \aleph_0$. If $\varkappa = \aleph_0$, then we let $T^\iota|_E$ play the role of R in Lemma 6.8, and we obtain from that lemma a σ-ring isomorphism U: $[R\times L]_{\mu\times\lambda} \to [S|_E]_\mu$ for which we have $\lambda_x(G) = k_{U(G)}(x)$ a.e. whenever $G \in R\times L$. (The symbols L, λ, and λ_x were defined in the statement of Lemma 6.8.) If we now define $R^\perp$ to be the σ-ring of subsets of E of the form $U(E\times H)$, where $H \in L$, then it is easily seen that $R^\perp$ is complementary to T^ι in E.

If $\varkappa > \aleph_0$, then we choose any subset $M = \{E_\gamma\}_{\gamma\in\varkappa}$ of $S|_E$ for which we have $[S|_E]_\mu = [\sigma(T^\iota|_E \cup M)]_\mu$. Given $\gamma \in \varkappa$, we shall construct σ-rings

$R^{\perp}(\gamma)$ and $S(\gamma)$ with the following properties:

(6.9.1) $R^{\perp}(0) = \{\emptyset, E\}$;

(6.9.2) $R^{\perp}(\gamma)$ and $T'|_E$ are independent with respect to E ;

(6.9.3) if $\beta < \gamma$, then $R^{\perp}(\beta) \subset R^{\perp}(\gamma)$;

(6.9.4) $S(\gamma) = \sigma(T'|_E \cup R^{\perp}(\gamma))$;

(6.9.5) $E_\gamma \in S(\gamma^+)$ for all $\gamma \in \varkappa$;

and

(6.9.6) $\varkappa(E; S(\gamma)) \leqslant \max\{|\gamma|, \aleph_0\}$ for all $\gamma \in \varkappa$.

By transfinite induction it suffices to fix $\gamma \in \varkappa$, and to assume that $R^{\perp}(\beta)$ and $S(\beta)$ have already been constructed for all $\beta < \gamma$. If γ is a limit ordinal, we define $R^{\perp}(\gamma) = \sigma\big(\bigcup_{\beta<\gamma} R^{\perp}(\beta)\big)$, so that $S(\gamma) = \sigma\big(\bigcup_{\beta<\gamma} S(\beta)\big)$. Then (6.9.3) and (6.9.5) are immediate, while (6.9.2) follows from a standard σ-ring argument (and is essentially an instance of a well-known result ([22], Proposition IV.4.1, p. 126). Moreover, it is obvious that $\varkappa(E; S(\gamma)) \leqslant |\gamma \times \gamma| = |\gamma|$, so that (6.9.6) follows. If $\gamma = \beta^+$ is a successor ordinal, and if $E_\beta \in S(\beta)$, then we may let $R^{\perp}(\gamma) = R^{\perp}(\beta)$. Otherwise we note that E contains no $S(\beta)$-atoms, for if $F \subset E$ were an $S(\beta)$-atom, then, by [10], Theorem E, p. 25, we would have $\varkappa(F; S|_F) = \varkappa(F; S(\beta)|_F) \leqslant \varkappa(E; S(\beta)) < \varkappa$, contrary to the minimality of $\varkappa$. Therefore we may let $S(\beta)$ play the role of R in Lemma 6.8, so that, exactly as for the case $\varkappa = \aleph_0$, we obtain a σ-ring $S(\beta)^{\perp} \subset S|_E$ such that $S(\beta)$ and $S(\beta)^{\perp}$ are E-independent, and such that $E_\beta \in \sigma(S(\beta) \cup S(\beta)^{\perp})$. We now define $R^{\perp}(\gamma) = \sigma(R^{\perp}(\beta) \cup S(\beta)^{\perp})$, so that $S(\gamma) = \sigma(S(\beta) \cup S(\beta)^{\perp})$. Then (6.9.3), (6.9.5), and (6.9.6) are immediate, while (6.9.2) follows from a standard σ-ring argument (see [22], Corollary, p. 126). The construction of the $R^{\perp}(\gamma)$ is now complete, and, in view of the arguments above, it is clear that the σ-ring $R^{\perp} = \sigma\big(\bigcup_{\gamma\in\varkappa} R^{\perp}(\gamma)\big)$ is complementary to T' in E. ∎

The transfinite induction argument of this lemma was inspired by a suggestion in [21], Theorem 1, p. 109. The argument of Lemma 6.8 generalizes the somewhat more intricate argument which Maharam used to prove Lemma 1 on p. 110 of [21]. Maharam's argument may be adapted to this setting, but it would require that the sets E_n of Lemma 6.8 constitute a single set. Thus it could have been used in the proof of Lemma 6.9 for the case $\varkappa > \aleph_0$, but (apparently) not for the case $\varkappa = \aleph_0$.

One final lemma of a purely technical nature will complete the preparation for Theorem 6.11.

6.10. LEMMA. *Let (X, S, μ) and (X', S', μ') be arbitrary σ-finite measure spaces, and let U be a σ-ring isomorphism from $[S]_\mu$ onto $[S']_{\mu'}$. Then,* *for every Banach space B, there exists a one-to-one correspondence U_B between the associated spaces of equivalence classes of B-valued measurable functions modulo null functions which is uniquely determined by the following three properties:*

(6.10.1) $U_B(b\chi_E) = b\chi_{U(E)}$ *a.e.* (μ') *for all* $b \in B$, *and for all* $E \in S$;

(6.10.2) U_B *is linear;*

and

(6.10.3) *if* $f_n \to f$ *a.e.* (μ), *then* $U_B(f_n) \to U_B(f)$ *a.e.* (μ').

Moreover, the mapping U_B possesses three additional properties. Let f: $X \to B$ be measurable.

(6.10.4) *If* h: $X \to \Phi$ *is measurable, then* $U_B(hf) = U_\Phi(h)\,U_B(f)$ *a.e.* (μ').

(6.10.5) *If D is a Banach space, and if φ: $B \to D$ is a continuous function, then* $U_D(\varphi \circ f) = \varphi \circ U_B(f)$ *a.e.* (μ'). *(In particular, we have* $U_{\mathbf{R}}(\|f(\cdot)\|) = \|U_B(f)(\cdot)\|$ *a.e.* (μ').*)*

(6.10.6) $(\|U(f)\|_\infty, \mu') = (\|f\|_\infty, \mu)$.

Remarks. This sort of result is well known for the case $B = \Phi$ (cf. [19], §31.1, pp. 430–432; and [25], Problem 7, p. 319). Since the standard proofs employ properties which are special to the real numbers, we shall present an entirely different approach. While we have stated the result in precisely the generality which we shall require, two additional facts are worth noting: The definition of U_B below, as well as properties (6.10.1)–(6.10.5), requires only that U be a σ-ring homomorphism; moreover, in the presence of decomposability assumptions, it is clear that U_B can be extended to (equivalence classes of) locally measurable functions (see [18], Proposition 3.5, p. 201).

Proof. Without ambiguity we shall hereafter denote all of the functions U_B (as B varies) by the one symbol U.

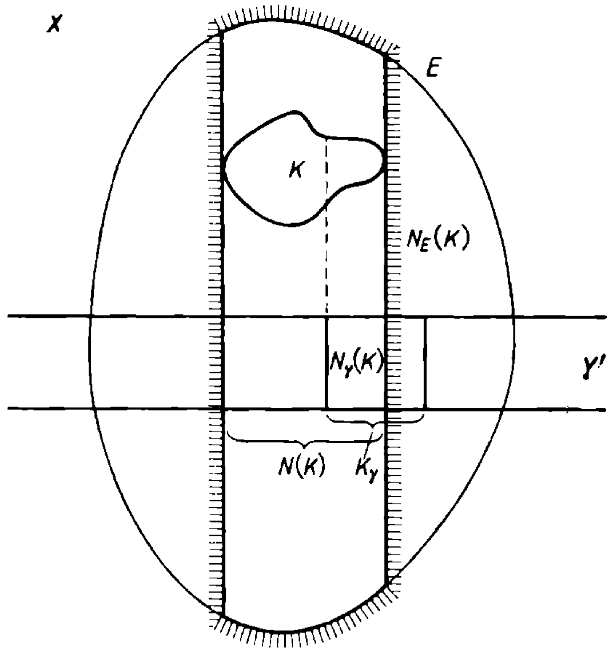
If $f = \sum_{i=1}^{n} b_i \chi_{E_i}$ is a B-valued measurable function, we define $U(f) = \sum_{i=1}^{n} b_i \chi_{U(E_i)}$. Then, because U is a ring homomorphism, it is clear that the equivalence class of $U(f)$ is uniquely determined independently of the representation of f.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of B-valued simple measurable functions such that $f_n \to 0$ a.e. (μ), and let $E \in S$ be such that f_n is supported on E for all n. Then, using Egoroff's theorem, we produce a sequence $\{E_j\}_{j=1}^{\infty}$ of pairwise disjoint measurable subsets of E such that $E =_\mu \bigcup_{j=1}^{\infty} E_j$, and such that $f_n \to 0$ uniformly on E_j for each j. Now if, for a fixed number $\varepsilon > 0$, we have $\|f_n(x)\| < \varepsilon$ for all $x \in E_j$, then it is clear that we also have $\|U(f_n)(x')\| < \varepsilon$ for (μ'-almost) all $x' \in U(E_j)$. Since the $U(E_j)$ exhaust $U(E)$, we obtain $U(f_n) \to 0$ a.e. (μ').

We are now prepared to define $U(f)(x') = \lim_{n\to\infty} U(f_n)(x')$ a.e. (μ'), where $f\colon X \to B$ is an arbitrary measurable function, and where $\{f_n\}_{n=1}^{\infty}$ is any sequence of simple measurable functions which converges pointwise to f a.e. (μ). In view of the above observation, and in view of the completeness of B, it is straightforward to verify that $U(f)$ is well defined a.e. (μ'), that the equivalence class of $U(f)$ is independent of the choice of the f_n (and hence of the equivalence class of f), and that the function U possesses all of the stated properties. (In particular, the argument of the last paragraph will essentially establish (6.10.3), and Lemma 3.4 will establish (6.10.6).) ∎

We are now in a position to convert Lemma 6.5 into global representation for (X, S, μ). However, we cannot simply represent each set X_a (in a suitable decomposition for μ) by the rectangle $X_a \times X_a$, because this would destroy the information about the "overhang" between different X_a. In order to preserve this information, we shall construct a "baseline" $Y' \subset X$ (which is to serve as a substitute for Y), we shall project each (sufficiently "thin") set $K \in S$ onto a set $N(K) \subset Y'$ (which is to serve as a substitute for the set $\{y \in Y\colon \mu_Y(K) > 0\}$), and we shall then represent X_a by the rectangle $N(X_a) \times X_a$ in place of $X_a \times X_a$. (And while it will be true that $N(N(K)) =_\mu N(K)$, i.e., that N is a genuine projection, we shall not need to make use of this fact.)

To illustrate these ideas more concretely, let us consider the case where $X = \boldsymbol{R}^2$ equipped with product Lebesgue measure $\lambda \times \lambda$, where ψ is the natural projection onto $Y = \boldsymbol{R} =$ the x-axis equipped with Lebesgue measure λ, and where T' constitutes the collection of ψ inverse images of the Lebesgue measurable subsets of Y. Then the first part of Theorem 6.11 says, in effect, that we may find a set $Y' \subset X$ to replace Y (for example, $Y' = \boldsymbol{R} \times [0, 1]$ equipped with product Lebesgue measure), and, given K in S (or, more properly, in the S^τ of Theorem 6.11), we may find a set $N(K) \subset Y'$ to replace the set $P(K) = \{x \in \boldsymbol{R}\colon \lambda_x(K) > 0\}$. (In this case we may take $N(K) = P(K) \times [0, 1]$.) Moreover, we may obtain the sets $N(K)$ (as K varies) in such a way that whenever two sets $K_1, K_2 \in S^\tau$ "overhang" one another (i.e., whenever $P(K_1) \cap P(K_2)$ is non-null), then the corresponding intersection $N(K_1) \cap N(K_2)$ is also non-null, even though K_1 and K_2 may be disjoint. The following diagram illustrates some of the entities of Theorem 6.11 as they apply to this example.



6.11. THEOREM. *Let T' be a σ-algebra of locally measurable subsets of X, let S^τ (τ to suggest "thin") denote the collection of those sets $E \in S$ for which $T'|_E$ is σ-finite* (see Definition 6.3), *and then, given any set $E \in S^\tau$ which contains some measurable set K, let $N_E(K)$ denote the set $\{x \in E: E^Q\chi_K(x) > 0\}$, where $Q = T'|_E$. Then there exists a locally measurable set $Y' \subset X$, and, for every set $K \in S^\tau$, there exists a set $N(K) \in S^\tau$ such that*

(6.11.1) $N(K) \subset_\mu Y'$;

(6.11.2) $\mu(K) > 0$ *if and only if* $\mu(N(K)) > 0$;

and

(6.11.3) *for any set $E \in S^\tau$ which contains $K \cup N(K)$, we have $N(K) =_\mu N_E(K) \cap Y'$.*

It follows that, for $G \in T'$, the correspondence $K \cap G \leftrightarrow N(K) \cap G$ determines a σ-ring isomorphism U_K: $[T'|_K]_\mu \to [T'|_{N(K)}]_\mu$.

Now let $\{X_a\}_{a \in A}$ be a decomposition for (X, S, μ) which is such that T' is complemented in each of the X_a (see Lemma 6.9), *let $U_a = U_{X_a}$ for all $a \in A$, and let $R = \sigma(\bigcup_a T'|_{X_a})$. Then there exists a σ-ring $S' \subset S \times S$, and there exists a σ-ring isomorphism U: $[S]_\mu \to [S']_\mu$ which is such that, for all $a \in A$, we have*

(6.11.4) $U(X_a \cap G) =_\mu (N(X_a) \cap G) \times X_a$ *for all $G \in T'$*;

and

(6.11.5) $\quad \mu_x(U(E)) = \mu(X_\alpha)\, U_\alpha(k_E)(x) \quad$ *a.e. for all* $E \in S|_{X_\alpha}$,

where $\{\mu_x\}_{x\in X}$ *denotes the standard disintegration of* $\mu \times \mu$ *with respect to* μ.

Remarks. We recall from Definition 6.4 that $k_E = E^R \chi_E$, and we recall the construction of the function $U_\alpha(k_E)$ in Lemma 6.10. The assumption of decomposability (q.v.) is not strictly necessary to the construction of the σ-ring isomorphism U. We preserve it in this proof because it clarifies the construction of U, and because the additional arguments which would be needed in its absence are entirely routine.

Proof. It is worth pointing out that if $E^{T^l}\chi_K$ were well defined for all $K \in S$ (e.g. when μ is finite), then we could let $Y' = X$, and we could let $N(K) = \{x \in X\colon E^{T^l}\chi_K(x) > 0\}$. Our construction of $N(K)$ in the general setting is based upon this distinguished special case.

For reference purposes we first make note of two nearly trivial properties of the set $N_E(K)$ which was defined above.

(6.11.6) $\quad$ For all $G \in T^l$, we have $N_E(K \cap G) =_\mu N_E(K) \cap G$.

(6.11.7) $\quad$ If $F \in S^r$, and if $E \subset F$, then we have $N_E(K) =_\mu N_F(K) \cap E$.

We now obtain the set Y' by means of Zorn's lemma. Let us declare that the sets $K_1, K_2 \in S$ are *essentially totally disjoint* (cf. Definition 5.6) if there exist disjoint sets $G_1, G_2 \in T^l$ such that $K_i \subset_\mu G_i$ for $i = 1, 2$. Let $\{K_\gamma\}_{\gamma\in C}$ be a maximal family of non-null sets $K_\gamma \in S^r$ which are pairwise essentially totally disjoint, and define $Y' = \sup_{\gamma\in C} K_\gamma$, where the supremum is taken with respect to the ordering $\subset_\mu$ (see [27], Definition 2.6, p. 279; Lemma 3.2.2, p. 284).

We next obtain the set $N(K)$ for an arbitrary set $K \in S^r$. For every $\gamma \in C$, let us consider the set $N_\gamma(K) = K_\gamma \cap N_{K\cup K_\gamma}(K)$. Clearly $N_\gamma(K)$ is non-null if and only if the "companion" set $K \cap N_{K\cup K_\gamma}(K_\gamma)$ is non-null. Moreover, because the K_γ are pairwise essentially totally disjoint, it follows that, as γ varies, the pairwise intersections of these companion sets will all be null. Therefore, because μ is σ-finite, we see that at most countably many of the companion sets can be non-null, i.e., that at most countably many of the $N_\gamma(K)$ can be non-null. Define $N(K)$ to be the (countable) union of the non-null $N_\gamma(K)$. Then property (6.11.1) is evident, and it is also clear that $N(K)$ is null whenever K is. Moreover, as the sets K and K_γ are essentially totally disjoint whenever $N_\gamma(K)$ is null, we cannot simultaneously have $\mu(K) > 0$ and $\mu(N(K)) = 0$ without violating the maximality of the K_γ. Thus (6.11.2) is established. Finally, let $E \in S^r$ be $\supset K \cup N(K)$. Then, in view of (6.11.7), it is not difficult to verify that $N_\gamma(K) =_\mu N_E(K) \cap K_\gamma$ for all $\gamma \in C$, and so (6.11.3) follows.

Now it is immediate from (6.11.3) and from (6.11.6) that we have $N(K \cap G) =_\mu N(K) \cap G$ for all $G \in T'$. It follows, in particular, that $\mu(K \cap G) > 0$ if and only if $\mu(N(K) \cap G) > 0$, so that the correspondence $K \cap G \leftrightarrow N(K) \cap G$ does indeed determine a σ-ring isomorphism U_K: $[T'|_K]_\mu \to [T'|_{N(K)}]_\mu$.

We now define U on the measurable subsets of X_a for each a separately. Let $R^\perp(a)$ be complementary to R (and hence to T') in X_a (see Definition 6.3). Then from Lemma 6.5 we obtain a σ-ring isomorphism U_1: $[S|_{X_a}]_\mu \to [T'|_{X_a} \times R^\perp(a)]_{\mu\times\mu}$ such that $U_1(F) = F \times X_a$ for all $F \in T'|_{X_a}$, and such that $\mu_x(U_1(E)) = \mu(X_a) k_E(x)$ a.e. for all $E \in S|_{X_a}$. Now it follows easily from the Fubini and Radon–Nikodym theorems that the measures $\mu \times \mu$ and $U_a(\mu) \times \mu$ share the same null sets in $T'|_{N(X_a)} \times R^\perp(a)$. Therefore, in the spirit of Lemma 6.5, we may easily construct a σ-ring isomorphism U_2: $[T'|_{X_a} \times R^\perp(a)]_{\mu\times\mu} \to [T'|_{N(X_a)} \times R^\perp(a)]_{\mu\times\mu}$ which is such that $U_2(F \times E) = U_a(F) \times E$ for all $F \in T'|_{X_a}$, and for all $E \in R^\perp(a)$, and which therefore satisfies $\mu_x(U_2(G)) = U_a(w \mapsto \mu_w(G))(x)$ a.e. for every set $G \in T'|_{X_a} \times R^\perp(a)$. If we now define $U(E) = U_2(U_1(E))$ for all $E \in S|_{X_a}$, then property (6.11.4) is evident, while property (6.11.5) follows at once from (6.10.2).

Finally, we define $S' = \sigma\left(\bigcup_a \left(T'|_{N(X_a)} \times R^\perp(a)\right)\right)$, we extend U in the one obvious way from each of the $S|_{X_a}$ to all of S, and we thereby obtain the desired global σ-ring isomorphism U: $[S]_\mu \to [S']_{\mu\times\mu}$. ∎

It is clear that we may represent (X, S, μ) and T' by an explicitly defined disintegrable measure space $(X', S', \mu', Y', T', \nu', \psi', \{\mu_{y'}\}_{y' \in Y'})$, and, with the possible exception of T', there are obvious natural choices for all of the entities involved. The obvious choice for T' is $T'|_{Y'}$; the *correct* choice is $T = \sigma(\bigcup_\gamma T'|_{K_\gamma})$. The reason for the addition of the K_γ is to reduce the sets in $T'|_{Y'}$ to manageable size (or, more precisely, to make $T' \subset S$); however, we note that locally the two σ-rings are essentially the same (or, more precisely, that $[T'|_E]_\mu = [(T'|_{Y'})|_E]_\mu$ for all $E \in S$, as is readily verified).

In the following result we examine what this theorem says in the special case that μ is disintegrable. This special case will help to clarify the motivation behind the general result, which is somewhat lacking in intuitive appeal. We shall preserve without comment all of the notation of Theorem 6.11.

6.12. THEOREM. *Let $T' = \{\psi^{-1}(F)\colon F \in T$ or $F^c \in T\}$, let $a \in A$ be fixed, let $Y_a = \{y \in Y\colon \mu_y(X_a) > 0\}$, and let $Y'_a = N(X_a)$. Then, for $F \in T$, the correspondence $Y_a \cap F \leftrightarrow Y'_a \cap \psi^{-1}(F)$ determines a σ-ring isomorphism V_a: $[T|_{Y_a}]_\nu \to [T'|_{Y'_a}]_\mu$. Furthermore, for every set $E \in S|_{X_a}$, we have*

$$\mu_x(U(E)) = \mu(X_a)\mu_{\psi(x)}(X_a)^{-1} V_a(y \mapsto \mu_y(E))(x)$$

a.e., and, for every index $\beta \in A$, we have $V_\alpha = V_\beta$ on their common domain $[T|_{Y_\alpha \cap Y_\beta}]_\nu$.

Alternatively, there exists a σ-ring $S'' \subset T \times S$, and there exists a σ-ring isomorphism $V: [S]_\mu \to [S'']_{\nu\times\mu}$ which is such that, for all $\alpha \in A$, we have

$$(6.12.1) \qquad V(X_\alpha \cap \psi^{-1}(F)) =_\mu (Y_\alpha \cap F) \times X_\alpha \quad \text{for all } F \in T;$$

and

$$(6.12.2) \qquad \mu_y(V(E)) = \mu(X_\alpha)\mu_y(X_\alpha)^{-1}\mu_y(E) \quad \text{a.e. } (\nu) \text{ for all } E \in S|_{X_\alpha},$$

where $\{\mu_y\}_{y\in Y}$ (also) denotes the standard disintegration of $\nu \times \mu$ with respect to ν.

Note. The assumption of decomposability is not crucial to the construction of V, just as it was not crucial to that of U in Theorem 6.11.

Proof. We begin by establishing the appropriate link between the "concrete" notion of disintegration and the "abstract" notion of conditional expectation. To this end, let $K \in S^r$ be fixed, and define $Y_K = \{y \in Y: \mu_y(K) > 0\}$. Then, in view of the key assumption (2.1.3), it is easy to see that the correspondence $Y_K \cap F \leftrightarrow K \cap \psi^{-1}(F)$ determines a σ-ring isomorphism $V_K: [T|_{Y_K}]_\nu \to [T^r|_K]_\mu$. Now let $Q = T^r|_K$, and let E be any measurable subset of K. Then, for all $F \in T$, we have

$$\mu(E \cap \psi^{-1}(F)) = \int_{\psi^{-1}(F)} E^Q \chi_E(x)\, d\mu(x) = \int_F \mu_y(E)\, d\nu(y)$$
$$= \int_{\psi^{-1}(F)} V_K(y \mapsto \mu_y(E))(x)\, dV_K(\nu)(x)$$

(see Definition 6.4, assumption (2.1.3), and [25], Problem 7.b, p. 319). We thus obtain the desired link:

$$(6.12.3) \qquad E^Q \chi_E(x) = g_K(x) V_K(y \mapsto \mu_y(E)) \quad \text{a.e. for all } E \in S|_K,$$

where $g_K: K \to (0, \infty)$ is the Radon–Nikodym derivative of $V_K(\nu)$ with respect to μ.

In particular, the relation $V_\alpha(F) = U_\alpha(V_{X_\alpha}(F))$ for all $F \in T|_{Y_\alpha}$ (and for all $\alpha \in A$) shows that V_α is indeed a σ-ring isomorphism, and we shall now establish the two stated assertions about V_α. For the first of these, we allow X_α to play the role of the set K in equality (6.12.3), we apply U_α to both sides of that equality, and then from (6.10.4) we obtain $U_\alpha(k_E)(x) = U_\alpha(g_{X_\alpha})(x) V_\alpha(y \mapsto \mu_y(E))(x)$ a.e. In particular, by setting $E = X_\alpha$, we may explicitly compute $U_\alpha(g_{X_\alpha})(x) = \mu_{\psi(x)}(X_\alpha)^{-1}$ a.e., so that, for an arbitrary set $E \in S|_{X_\alpha}$ we obtain $\mu_x(U(E)) = \mu(X_\alpha)\mu_{\psi(x)}(X_\alpha)^{-1} V_\alpha(y \mapsto \mu_y(E))(x)$ a.e. as a direct consequence of Theorem 6.11. For the second of these assertions, let $\beta \in A$, and let $E \in S^r$ be $\supset X_\alpha \cup X_\beta \cup Y'_\alpha \cup Y'_\beta$. Then, by allowing E to play the role of the set K in (6.12.3), we may easily conclude that $N_E(X_a) =_\mu E \cap \psi^{-1}(Y_a)$, and likewise for β. Therefore, if $F \in T$ is $\subset Y_a \cap Y_\beta$, we obtain $E \cap \psi^{-1}(F) \subset_\mu N_E(X_a) \cap N_E(X_\beta)$, and so it follows from (6.11.3) that we have $E \cap \psi^{-1}(F) \cap Y' \subset_\mu Y'_a \cap Y'_\beta$. It is now clear that $V_a(F)$ and $V_\beta(F)$ are each $=_\mu \psi^{-1}(F) \cap Y'_a \cap Y'_\beta$, i.e., that $V_a = V_\beta$ on their common domain.

The construction of the alternate σ-ring isomorphism V is virtually identical to that of U. It is simply a matter of using Y as a baseline in place of Y'. Thus we replace each set $Y_{a'}$ by Y_a, we replace each "projection" U_a by the corresponding mapping onto $[T|_{Y_a}]_\nu$, which is precisely $V_{X_a}^{-1}$, and we consequently obtain $\mu_y(V(E)) = \mu(X_a) V_{X_a}^{-1}(k_E)(y)$ a.e. (ν) for every set $E \in S|_{X_a}$, and for every index $a \in A$. Property (6.12.1) is (again) evident, while property (6.12.2) may be obtained from the above equality by a brief computation in the spirit of the last paragraph. (Alternatively, we could obtain V by composing U with a "change of baseline" σ-ring isomorphism. It would resemble the map U_2 which was defined in the proof of Theorem 6.11.) ∎

It remains now only to establish that the conclusions of this theorem are sufficient to ensure that the corresponding $L_{p,q}$ spaces are isometrically isomorphic for all values of p and q. This we accomplish in the following general result.

6.13. PROPOSITION. *Let $(X, S, \mu, Y, T, \nu, \psi, \{\mu_y\}_{y \in Y})$ and $(X', S', \mu', Y', T', \nu', \psi', \{\mu_{y'}\}_{y' \in Y'})$ be disintegrable measure spaces, and assume that both μ and μ' are decomposable. Suppose that there exists a σ-ring isomorphism $U\colon [S]_\mu \to [S']_{\mu'}$, and suppose that we may associate with U a decomposition $\{X_a\}_{a \in A}$ for μ which has the following property: namely, that, for all $a \in A$, we may specify a σ-ring isomorphism $V_a\colon [T|_{Y_a}]_\nu \to [T'|_{Y'_a}]_{\nu'}$, where $Y'_a = \{y \in Y\colon \mu_y(X_a) > 0\}$, and where $Y'_a = \{y' \in Y'\colon \mu_{y'}(U(X_a)) > 0\}$, and we may specify a non-negative measurable function h_a on Y'_a such that $\mu_{y'}(U(E)) = h_a(y') V_a(y \mapsto \mu_y(E))(y')$ a.e. (ν') for every set $E \in S|_{X_a}$. Assume, moreover, that, for all $a, \beta \in A$, we have $V_a = V_\beta$ on their common domain $[T|_{Y_a \cap Y_\beta}]_\nu$. Then, for $1 \leqslant p, q \leqslant \infty$, the spaces $L_{p,q}(\mu, B)$ and $L_{p,q}(\mu', B)$ are isometrically isomorphic.*

Remarks. All of the arguments in the proof below are valid for arbitrary $p, q > 0$. As usual, the assumption of decomposability (q.v.) is required only for the purpose of piecing together a family of locally defined measurable functions into a globally defined (locally) measurable function; thus it may be abandoned when both p and q are finite.

Proof. If $f\colon X \to B$ is an arbitrary locally measurable function, then we may and shall determine a locally measurable function $I(f)\colon X' \to B$ by specifying its values on each set $U(X_a)$ separately. Thus, let $a \in A$ be fixed, let $g_a\colon Y'_a \to (0, \infty)$ be the Radon–Nikodym derivative of ν'

(restricted to $T'|_{Y_a}$) with respect to $V_a(\nu)$, and, for (μ'-almost) all $x' \in U(X_a)$, define

$$I(f)(x') = g_a(\psi'(x'))^{-1/q} h_a(\psi'(x'))^{-1/p} U(f\chi_{X_a})(x'),$$

under the usual convention $1/\infty = 0$. Then it is clear (in particular, from (6.10.2)) that this mapping I will determine an isometric isomorphism between the spaces $L_{p,q}(\mu, B)$ and $L_{p,q}(\mu', B)$, provided that we can establish the identity $\|f\|_{p,q} = \|I(f)\|_{p,q}$ for all $f \in L_{p,q}(B)$.

We shall obtain this identity first for a non-negative simple measurable function $f = \sum_{i=1}^{n} c_i \chi_{E_i}$. Let $K = \bigcup_{i=1}^{n} E_i$, let $\{a_j\}_{j=1}^{\infty}$ be any sequence in A such that $K \subset_\mu \bigcup_{j=1}^{\infty} X_{a_j}$, let $H = \bigcup_{j=1}^{\infty} Y_{a_j}$, and let $H' = \bigcup_{j=1}^{\infty} Y'_{a_j}$. Then it is clear from the compatibility of the V_{a_j} that there is a uniquely determined σ-ring isomorphism V: $[T|_H]_\nu \to [T'|_{H'}]_{\nu'}$ which extends each of the V_{a_j}; moreover, if g: $H' \to (0, \infty)$ is the Radon–Nikodym derivative of ν' (restricted to $T'|_{H'}$) with respect to $V(\nu)$, then it is clear that, for each j, we have $g = g_{a_j}$ a.e. (ν') on Y'_{a_j}. Accordingly, when $p < \infty$, we may readily deduce the identity

(6.13.1) $\quad (\|I(f)\|_p, \mu_{y'}) = g(y')^{-1/q} V(y \mapsto (\|f\|_p, \mu_y))(y') \quad$ a.e. (ν')

by using Lemma 4.2 together with (6.10.1)–(6.10.3) and (6.10.5); furthermore, it is easy to see that we have $\{y' \in Y'\colon \mu_{y'}(U(E)) > 0\} =_\mu V(\{y \in Y\colon \mu_y(E) > 0\})$ for every measurable set $E \subset K$, so that, when $p = \infty$, we may deduce (6.13.1) by representing f as, for example, in Lemma 3.6. In view of (6.10.6), the indentity $\|f\|_{p,q} = \|I(f)\|_{p,q}$ now follows easily in both cases.

With suitable definitions for V and g (and it is clear how to make them), we may extend the identity (6.13.1) to any fixed measurable function in $L_{p,q}(\mu, B)$ by a routine use of the monotone convergence theorem (including Lemma 3.4) together with (6.10.3) and (6.10.5). The isometry itself then follows just as it did for the simple function f. Consequently, if $f \in L_{p,q}(\mu, B)$ is completely arbitrary, we obtain $\|f\chi_E\|_{p,q} = \|I(f)\chi_{U(E)}\|_{p,q}$ for all $E \in S$ from (6.10.4), and so the identity $\|f\|_{p,q} = \|I(f)\|_{p,q}$ follows immediately from (3.1.2). ∎

In view of Theorem 6.12, the following theorem constitutes an immediate corollary of Proposition 6.13, and, just as in Proposition 6.13, the assumption that (X, S, μ) be decomposable is not crucial when both p and q are finite.

6.14. THEOREM. *There exists a σ-ring $S' \subset S \times S$, and there exists a σ-ring $T' \subset S$ such that, for $1 \leqslant p$, $q \leqslant \infty$, the space $L_{p,q}(\mu, B)$ is isometrically isomorphic to the space $L_{p,q}(X \times X, S', \mu \times \mu, X, T', \mu, B) \subset L_{p,q}(\mu \times \mu, B)$.*

Alternatively, there exists a σ-ring $S'' \subset T \times S$ such that, for $1 \leqslant p, q \leqslant \infty$, the space $L_{p,q}(\mu, B)$ is isometrically isomorphic to the space $L_{p,q}(Y \times X, S'', \nu \times \mu, Y, T, \nu, B) \subset L_{p,q}(\nu \times \mu, B)$. ∎

This theorem clearly provides information which is relevant to the problem of classifying the $L_{p,q}$ spaces, as Banach spaces, up to isomorphism and up to isometric isomorphism. It is implicit in Lemma 6.8, and more explicit in [21], that the original decomposition $\{X_a\}_{a \in A}$ in Theorem 6.11 may be chosen in such a way that, for each $a \in A$, the restriction of μ to $T'|_{X_a}$ is equivalent to product Lebesgue measure on a huge cube ([10], §38, pp. 154 ff.). (By "equivalent" we mean that there exists a σ-ring isomorphism between the corresponding σ-rings modulo null sets, and by a "huge cube" we mean a cartesian product each of whose factors is $[0, 1]$.) The number of factors in this cube might be uncountable ([10], Exercise (2), p. 158), or it might be zero (namely, when $[T'|_{X_a}]_\mu = [\{\emptyset, X_a\}]_\mu$). We may likewise ensure that the restriction of μ to the complementary σ-ring $R^\perp(a)$ is equivalent to product Lebesgue measure on a (possibly different) huge cube. (In fact both of these equivalences may be obtained by the transfinite induction procedure of Lemma 6.9.) From this point of view, the classification problem is naturally divided into two parts: locally, a study of the $L_{p,q}$ spaces which are associated with product Lebesgue measure on huge cubes; and globally, a study of the various configurations of "overhang" which emerge when a collection of huge cubes is piled into a heap. (From an alternate point of view, the classification problem may be subsumed under the problem of classifying the subspaces of the L_p spaces (see Theorem 6.14 and Proposition 3.27).)

We shall conclude this section with a discussion of the more general definition of an $L_{p,q}$ space which the foregoing results make possible. Given a (decomposable) measure space (X, S, μ), given a σ-algebra T' of locally measurable subsets of X, and given $1 \leqslant p, q \leqslant \infty$, let us formally define $L_{p,q}(X, S, \mu, T', B)$ to be the space $L_{p,q}(X \times X, S', \mu \times \mu, X, T', \mu, B)$ which was mentioned in Theorem 6.14 and explicitly constructed in Theorem 6.11. Then Theorem 6.14 states precisely that, up to isometric isomorphism, the more general definition coincides precisely with our original definition whenever the latter can be given.

The question which arises naturally is: Can we represent these new $L_{p,q}$ spaces just as we did the old ones, namely, by spaces of equivalence classes of (locally) measurable functions which are defined on X itself? We shall now indicate an answer to this question for the case $p < \infty$.

Let f: $X \to B$ be a measurable function. Then, preserving all of the notation of Theorem 6.11, we define

$$\|f\|^R_{p,q} = \left(\left\| \left[\sum U_a\left(E^R\left(\|f(\cdot)\|^p \chi_{X_a}(\cdot)\right)\right)\right]^{1/p} \right\|_q, \mu\right), \tag{6.14.1}$$

where the sum is taken over those (countably many) $\alpha \in A$ for which $f\chi_{X_\alpha}$ is not a null function. If f is locally measurable, we define

$$\|f\|_{p,q}^R = \sup_{E \in S} \|f\chi_E\|_{p,q}^{R}. \tag{6.14.2}$$

Now it is clear that $\|f\|_{p,q}^R = 0$ if and only if $f = 0$ l.a.e. Therefore, the mapping $\|\cdot\|_{p,q}^R$ may be assumed to act upon equivalence classes of locally measurable functions modulo locally null functions, and we may form the space $L_{p,q}^R(\mu, B)$ of those equivalence classes for which expression (6.14.2) is finite. Proving virtually nothing about this space, we may establish that it is isometrically isomorphic to the $L_{p,q}$ space which we have already defined. To this end, let f: $X \to B$ be locally measurable. Then, in the spirit of Proposition 6.13, we may specify a locally measurable function $I(f)$ on $X \times X$ by defining $I(f)(x') = \mu(X_\alpha)^{-1/p}\, U(f\chi_{X_\alpha})(x')$ for (μ'-almost) all $x' \in U(X_\alpha)$, and for all $\alpha \in A$. With the techniques of this section it is straightforward to verify that I determines the desired isometric isomorphism.

Now it may be inconvenient to use R (and hence the sets X_α of Theorem 6.11) to obtain the norm $\|\cdot\|_{p,q}^R$. Therefore, let $\{X_\gamma\}_{\gamma\in C}$ be any other decomposition for (X, S, μ), let $R' = \sigma\left(\bigcup_\gamma T'|_{X_\gamma}\right)$, and replace R by this σ-ring R' in all of the definitions of the last paragraph. We obtain a space $L_{p,q}^{R'}(\mu, B)$ which turns out to be isometrically isomorphic to $L_{p,q}^R(\mu, B)$, and we shall now indicate how to construct a natural isomorphism between the two spaces. Given $\gamma \in C$, we may assume without loss of generality that $X_\gamma \subset X_\alpha$ for some $\alpha \in A$. Let $H_\gamma = N_{X_\alpha}(X_\gamma)$. Then, for $G \in T'$, the correspondence $H_\gamma \cap G \leftrightarrow X_\gamma \cap G$ determines a σ-ring isomorphism W_γ: $[T'|_{H_\gamma}]_\mu \to [T'|_{X_\gamma}]_\mu$, and we let h_γ: $X_\gamma \to (0, \infty)$ be the Radon–Nikodym derivative of $W_\gamma(\mu)$ with respect to μ (restricted to $T'|_{X_\gamma}$). If we are now given a locally measurable function f: $X \to B$, then we may specify a locally measurable function $I(f)$ on X by defining $I(f)(x) = h_\gamma(x)^{-1/p} f(x)$ for all $x \in X_\gamma$, and for all $\gamma \in C$. With the techniques of this section it is (again) straightforward to verify that $\|I(f)\|_{p,q}^{R'} = \|f\|_{p,q}^R$, and hence that I determines the desired isometric isomorphism.

Continuing in the same vein, we may use the W_γ and the h_γ to show, via Proposition 6.13, that if a different baseline (Y') and a different decomposition ($\{X_\alpha\}_{\alpha\in A}$) are chosen in Theorem 6.11, then the resulting $L_{p,q}$ spaces will still be the same, up to isometric isomorphism, for all values of p and q. Thus it may be said that the procedure of Theorem 6.11 uniquely determines each of the spaces $L_{p,q}(X, S, \mu, T', B)$, independently of how the various ancillary entities within that theorem are chosen.

At the same time there does not appear to be a natural choice for a specific space equipped with a specific norm which we may call "the" space $L_{p,q}(X, S, \mu, T', B)$. One obvious *candidate* for a natural expression (which would be associated with a function f on X) is

$$\|f\|_{p,q} = \left(\left\| \left[E^{T'}(\|f(\cdot)\|^p)\right]^{1/p} \right\|_q, \mu\right); \tag{6.14.3}$$

however, because the sets in T' may be too "large", the conditional expectation operator $E^{T'}$ is not well defined in general. In this light the definition of R (or of R') may be seen to be an artificial device for cutting the sets in T' "down to size" so that a conditional expectation operator (of some usefulness) might actually be obtained. On the other hand, since E^R is not really the same as $E^{T'}$, its use in the definition of an "$L_{p,q}$ norm" cannot be expected to yield a natural expression. We may, of course, let R' (as above) be $= T'$ precisely when T' is itself a σ-finite σ-subring of S (and in particular when μ is finite). In this case we are also free to let $Y' = X$, and under these circumstances it is clear that the less natural expression (6.14.1) will reduce to the more natural expression (6.14.3).

We remark, finally, that we are unable to extend these observations to the case $p = \infty$ without introducing ancillary measures directly into the space X itself (cf. [26], Definition 0, p. 424). However, it is not always possible to do this in a satisfactory manner ([10], Complement and Detail No. 1, p. 370), and so we omit further discussion.

7. An application in group representation theory

The aim of this section is to illustrate, for the most part without detailed proofs, that $L_{p,q}$ spaces arise naturally in connection with locally compact groups, and that they play an especially useful role in the study of induced representations of these groups.

The notation within this section coincides as far as possible with that of Rieffel in [24], pp. 481 ff., and detailed existence proofs of the various entities posited may be found in [2].

7.1. Notational Conventions. Throughout this section the role of X will be played by an arbitrary locally compact Hausdorff topological group G. We shall let S denote the Borel subsets of G (that is, the σ-ring generated by the compact subsets of G), and we shall let μ be a fixed left Haar measure defined on S. To obtain the space (Y, T, ν) in this setting, we specify a closed subgroup H of G, and we equip this subgroup with a left Haar measure β. Then the role of Y will be played by the set of left cosets of H in G, which we shall denote simply by G/H. We shall let ψ be the canonical projection of G onto G/H, and we shall equip G/H with the quotient topology determined by ψ. Note, then, that G/H constitutes a locally compact Hausdorff space in its own right. We shall let T denote the Borel subsets of G/H, and we shall let $\nu = \lambda$ be a fixed quasi-invariant measure defined on T. All of these measures are assumed to be regular Borel measures, so that, since such measures are always decomposable ([5], Proposition 41, p. 337), we are free to apply all of our previous theory. If $x \in G$, we shall abbreviate $\psi(x)$ to $\dot{x}$, and we shall also denote an arbitrary element of G/H by $\dot{x}$ with the understanding that, unless it is otherwise specified, x is to denote an arbitrary element of the set $\psi^{-1}(\{\dot{x}\})$.

In order to obtain an explicit disintegration of μ with respect to λ, we let Δ and δ denote the modular functions of G and H, respectively, and we let $\varrho: G \to (0, \infty)$ be a continuous function with the property that $\varrho(xs) = \dfrac{\delta(s)}{\Delta(s)}\,\varrho(x)$ for all $x \in G$ and $s \in H$. Then it is always possible to determine the measure λ in such a way that a disintegration $\{\mu_{\dot{x}}\}_{\dot{x}\in G/H}$ of μ with respect to λ can be defined by the formula

$$\mu_{\dot{x}}(E) = \varrho(x)^{-1} \int\limits_{H} \frac{\Delta(s)}{\delta(s)}\, \chi_E(xs)\, d\beta(s)$$

for all $E \in S$, and for all $\dot{x} \in G/H$ (cf. [24], Proposition 10.1, p. 482). Notice that this expression is independent of the choice of $x \in \psi^{-1}(\{\dot{x}\})$ precisely because the modular functions are homomorphisms. The $L_{p,q}$ spaces which we shall discuss in this section will without exception be defined in terms of this disintegration.

We shall preserve the notational conventions of Section 4, particularly in connection with the Banach space B and its dual space B^*. We shall let $C_c(B)$ denote the space of continuous functions of compact support from G to B.

In practice, H and B arise because of the existence of a strongly continuous isometric representation of H on B. By this we mean a group homomorphism $s \mapsto U_s$ from H into the group of isometric isomorphisms of B onto B which has the property that, for each $b \in B$, the map $s \mapsto U_s(b)$ from H into B is continuous. We shall assume that such a representation has been given, and we shall hereafter abbreviate $U_s(b)$ to sb. We now recall that to every representation of H on B there corresponds a canonical representation of H on B^* which, for $s \in H$ and $b^* \in B^*$, is defined by $\langle b, sb^*\rangle = \langle s^{-1} b, b^*\rangle$ for all $b \in B$. We shall refer to this representation of H on B^* as the *dual* of the original representation of H on B. Clearly, if the original representation is isometric, then so also is the dual representation; and if the original representation is strongly continuous, then the dual representation is weak* continuous in the sense that, for each $b^* \in B^*$, the map $s \mapsto sb^*$ from H into B^* is continuous in the weak* topology on B^*. We remark that we shall not consider any continuity properties which a representation may possess other than the two which have been described in this paragraph. ∎

Our primary consideration is the problem of inducing a representation of G in a natural way from the given representation of H. This problem has received extensive treatment to date, and among the best known papers which deal with it we include [3], [15], [16], [20], and [24]. The general procedure has consisted of the painful de novo construction of a class of $L_p(B)$-like Banach spaces; on each of these spaces a strongly continuous isometric representation has been constructed which bears a natural relationship to the original (strongly continuous isometric) representation of H on B. These spaces in fact turn out to be subspaces of the $L_{\infty,q}$ spaces; moreover, the induced representations of G constitute precisely the restrictions to these subspaces of the analogues in the $L_{p,q}$ spaces of the well-known canonical representations of G on the L_p spaces. Two principal consequences flow from this observation: first, an extension to the general setting of Rieffel's relatively simple proof ([24], Lemma 10.10, p. 490) that the induced representations are strongly continuous; and second, an extension of the entire theory to a parallel theory associated with the dual representation of H on B^*. Thus our immediate concern is to obtain these canonical representations of G on the $L_{p,q}$ spaces, and this we accomplish in the three results below. Because of the basic nature of these results, complete proofs are included.

7.2. LEMMA. *Every compact subset K of G has totally finite measure* (Definition 3.9).

Proof. The function $\varrho(x)^{-1}$ is continuous on K, and so is bounded there by a constant r. Let $C = (K^{-1}K)\cap H$. Since H is closed, it follows that C will be compact in H, so that the ratio $\Delta(s)/\delta(s)$, which is continuous on H, will be bounded on C by a constant d. Now the expression $\mu_{\dot{x}}(K)$ vanishes unless $\dot{x} \in \psi(K)$, a set which is compact and which therefore has finite λ measure. If, then, we have $\mu_{\dot{x}}(K) > 0$, it follows that we may choose x to belong to K, so that $(x^{-1}K)\cap H \subset C$. It is now easy to see that we have $\mu_{\dot{x}}(K) \leqslant rd\beta(C) < \infty$, so that the map $\dot{x} \mapsto \mu_{\dot{x}}(K)$ is uniformly bounded. ∎

7.3. Proposition. *If $1 \leqslant p,\ q \leqslant \infty$, then $C_c(B) \subset L_{p,q}(B)$; if, moreover, both p and q are finite, then $C_c(B)$ is dense in $L_{p,q}(B)$.*

Proof. Since the functions in $C_c(B)$ are bounded, the first assertion is an immediate consequence of Lemma 7.2 and of (3.9.1). For the second assertion we consider first a characteristic function $\chi_E \in L_{p,q}(\Phi)$. If E has compact closure, then, in view of Lemma 7.2 and of the regularity of μ, a straightforward application of Lemma 3.10 will yield the existence, for every $\varepsilon > 0$, of a compact set $K \subset E$ and an open set $U \supset E$ such that $\|\chi_{U-K}\|_{p,q} < \varepsilon$. If $f \in C_c(\Phi)$ is chosen to be such that $\chi_K \leqslant f \leqslant \chi_U$, then we clearly obtain $\|\chi_E - f\|_{p,q} < \varepsilon$ as well. Moreover, since every set $E \in S$ is contained in a countable union of compact sets, the extension of this approximation to an arbitrary characteristic function in $L_{p,q}(\Phi)$ follows from the dominated convergence theorem. In view of Proposition 3.14, the further extension to simple and thence to arbitrary functions in $L_{p,q}(B)$ constitutes a standard triangle inequality argument. ∎

7.4. Theorem. *Let $1 \leqslant p,\ q \leqslant \infty$. Then, under the convention that $1/\infty = 0$, the representation of G on $L_{p,q}(B)$ which is given by*

$$(zf)(x) = \left(\frac{\varrho(x)}{\varrho(z^{-1}x)}\right)^{1/p-1/q} f(z^{-1}x),$$

where $z, x \in G$, and where $f \in L_{p,q}(B)$, is isometric, and it is strongly continuous if both p and q are finite.

Proof. By verifying the isometry, we also verify that $zf \in L_{p,q}(B)$, so that we clearly do obtain an isometric representation of G on $L_{p,q}(B)$. When p and q are finite, we have

$$\begin{aligned}
\|zf\|_{p,q}^q &= \int \left[\varrho(x)^{-1} \int_H \frac{\Delta(s)}{\delta(s)} \left(\frac{\varrho(xs)}{\varrho(z^{-1}xs)}\right)^{1-p/q} \|f(z^{-1}xs)\|^p \, d\beta(s)\right]^{q/p} d\lambda(\dot{x}) \\
&= \int \varrho(x)^{-1} \left[\varrho(x)^{p/q-1} \int_H \frac{\Delta(s)}{\delta(s)} \left(\frac{\varrho(x)}{\varrho(z^{-1}x)}\right)^{1-p/q} \|f(z^{-1}xs)\|^p \, d\beta(s)\right]^{q/p} d\lambda(\dot{x}) \\
&= \int \varrho(x)^{-1} \left[\int_H \frac{\Delta(s)}{\delta(s)} \left(\frac{1}{\varrho(x)}\right)^{1-p/q} \|f(xs)\|^p \, d\beta(s)\right]^{q/p} d\lambda(\dot{x}) \\
&= \int \left[\varrho(x)^{-1} \int_H \frac{\Delta(s)}{\delta(s)} \|f(xs)\|^p \, d\beta(s)\right]^{q/p} d\lambda(\dot{x}) = \|f\|_{p,q}^q.
\end{aligned}$$

The quasi-invariance of the measure λ came into play in the third of the equalities above, and this equality follows specifically from [2], Theorem 2c, p. 56. When p or q is infinite, we fix a set $E \in S$ and observe, by equalities which are analogous to those above, that we have $\|(zf)\chi_E\|_{p,q}$

$= \|f(\chi_{z^{-1}E})\|_{p,q}$. Since $z^{-1}E$ varies throughout S as E does, we conclude in this case as well that $\|zf\|_{p,q} = \|f\|_{p,q}$.

We shall now establish that this representation is strongly continuous when p and q are finite. Because of the density of $C_c(B)$ in $L_{p,q}(B)$, it will suffice to verify the strong continuity for an arbitrary function $f \in C_c(B)$; moreover, it will suffice to verify this continuity at the identity element e of G. Let f be supported on the compact set K, let $M > 0$ be an upper bound for the continuous function $\varrho(x)^{1/p-1/q}$ on K, let U be an arbitrary compact neighbourhood of e, and define $N = \|\chi_{U^{-1}K}\|_{p,q}$. Note that N is finite because the set $U^{-1}K$ is compact, and hence of totally finite measure. Let $\varepsilon > 0$, define $h(x) = f(x)\varrho(x)^{-(1/p-1/q)}$, and let V be a neighbourhood of e such that $V \subset U$, and such that we have $\|h(x)-h(y)\| \leqslant \varepsilon/MN$ whenever $xy^{-1} \in V$ ([11], Theorem 15.4, p. 185). (We are eliminating the trivial case $N = 0$.) Since, for each $z \in V$, the function $zf-f$ is supported on the set $U^{-1}K$, a simple computation will show that we have $\|zf-f\|_{p,q} \leqslant (\varepsilon/N)\|\chi_{U^{-1}K}\|_{p,q} = \varepsilon$. The strong continuity of the representation at e is thus established. (Cf. [11], Theorem 20.4, p. 285.) ∎

For the following definition we recall that we have been given a strongly continuous isometric representation of the closed subgroup H on the Banach space B.

7.5. Definition. For $1 \leqslant q \leqslant \infty$, we define $K_q(B)$ to be the set of those equivalence classes in $L_{\infty,q}(B)$ at least one of whose members h satisfies the equality $h(xs) = s^{-1}h(x)$ for all $x \in G$, and for all $s \in H$. ∎

Note, then, that we have $\|h(xs)\| = \|h(x)\|$, so that h is constant in norm on the left cosets of H. Thus we have $(\|h\|_\infty, \mu_{\dot{x}}) = \|h(x)\|$ for any choice of $x \in \psi^{-1}(\{\dot{x}\})$, and from this it follows that we may compute $\|h\|_{\infty,q} = \left(\int \|h(x)\|^q d\lambda(\dot{x})\right)^{1/q}$ (and analogously when $q = \infty$), even though h is in general only locally measurable.

It is now clear that these spaces $K_q(B)$ (essentially) constitute precisely the "$L_p(B)$-type" spaces which have been considered in the previous literature, that the corresponding representations of G which were described in Theorem 7.4 may be restricted to these spaces (i.e., that $K_q(B)$ constitutes a *submodule* of $L_{\infty,q}(B)$ for all q), and that these restrictions (essentially) coincide precisely with the induced representations of G which have been considered in the previous literature.

In this light, the fact that these induced representations are isometric follows at once from Theorem 7.4, while the proof that each space $K_q(B)$ is complete reduces to a demonstration that it is closed in $L_{\infty,q}(B)$. However, we confess that the proofs are not thereby greatly simplified (cf. [24], Lemma 10.3, p. 483; and [16], Theorem 2, p. 186).

It is with the proof that the induced representations are continuous when q is finite that an appreciable clarification can be achieved. The general proofs to date have all been of a highly technical nature (for a detailed example, see [16], Theorem 3, p. 189). In the special case $q = 1$, Rieffel ([24], Theorem 10.4, p. 484) defined a bounded, surjective linear map which we shall call Π_1: $L_1(B) \to K_1(B)$, and which possesses two key properties: namely, that it is *intertwining* (i.e., that we have $\Pi_1(zf) = z\Pi_1(f)$ for all $z \in G$, and for all $f \in L_1(B)$), and that the quotient norm which it determines coincides precisely with the $L_{\infty,1}$ norm on $K_1(B)$. In the light of these properties, it is entirely trivial to establish that the strong continuity of the representation of G on $L_1(B)$ implies the same for the induced representation of G on $K_1(B)$ ([24], Lemma 10.10, p. 490). Now this proof generalizes immediately to all (finite) q once we recognize that the exact same integral equation which was used to define Π_1 will also define a map Π_q: $L_{1,q}(B) \to K_q(B)$ with analogous properties. The technical details of the proof need to be altered only trivially by the occasional insertion of expressions which involve the function ϱ (in the spirit of Theorem 7.4). Moreover, because the quotient of a complete space is also complete, the completeness of the K_q spaces (for finite q) follows as an immediate corollary.

(We remark that inessential differences in detail occur between Rieffel's definitions and our own. The definition of Π_q which is appropriate for our definitions is:

$$\Pi_q(f)(x) = \varrho(x)^{-1} \int\limits_H \frac{\Delta(s)}{\delta(s)} sf(xs)\, d\beta(s) \qquad \text{for all } x \in G,$$

where $f \in L_{1,q}(B)$. When $q = \infty$, this integral is well defined for all $f \in L^m_{1,\infty}(B)$ (Definition 4.7), and determines a map Π_∞ whose range we might denote by $K^m_\infty(B)$ — it is, namely, the space of those functions in $K_\infty(B)$ which are supported on the ψ inverse image of a measurable subset of G/H. The properties of Π_∞ are entirely analogous to those of the other Π_q.)

We shall now outline a theory for the dual representation of H on B^* which is parallel (we are tempted to say "dual") to the present theory for that of H on B. In order to give a flavor of the methods which are involved when strong continuity is replaced by weak* continuity, we shall present in the following result a weak* analogue of Theorem 7.4. For this result it will be notationally convenient to treat the elements of the space $L^w_{p',q'}(B^*)$ (Definition 4.11) as functions which satisfy (4.9.1)–(4.9.3).

7.6. THEOREM. *Let $1 \leqslant p, q \leqslant \infty$, and let p' and q' denote the respective conjugate exponents* (Notational Conventions 4.1). *Then the representation of G on $L_{p,q}(B)$ may be restricted to $L^m_{p,q}(B)$; likewise, the* *corresponding dual representation on $L^m_{p,q}(B)^*$ may be restricted to $L^w_{p',q'}(B^*)$, and on this space it is given by*

$$(zg)(x) = \left(\frac{\varrho(x)}{\varrho(z^{-1}x)}\right)^{1/p'-1/q'} g(z^{-1}x),$$

where $z, x \in G$, and where $g \in L^w_{p',q'}(B^*)$. (As before, we adopt the convention that $1/\infty = 0$.) Moreover, this representation of G on $L^w_{p',q'}(B^*)$ is weak continuous when p and q are both finite, and it is also weak* continuous when p' and q' are both finite.*

Proof. The first assertion is obvious. For the second assertion, an easy computation will establish that we have $\langle f, zg\rangle = \langle z^{-1}f, g\rangle$ for all $z \in G$, and for all $f \in L^m_{p,q}(B)$, so that, in particular, the function zg determines a linear functional on $L^m_{p,q}(B)$. Thus we need only to verify that $zg \in L^w_{p',q'}(B^*)$, and the only non-trivial part of this verification is the equality $\|zg\|_{p',q'} = \|zg\|$ $(< \infty)$, where $\|zg\|$ denotes the linear functional norm of zg. However, we have $\|zg\| = \|g\| < \infty$ because the dual of an isometric representation is isometric, we have $\|g\| = \|g\|_{p',q'}$ because g satisfies (4.9.2), and we have $\|g\|_{p',q'} = \|zg\|_{p',q'}$ by precisely the arguments of Theorem 7.4.

When p and q are finite, the weak* continuity of this representation follows from the strong continuity of the representation of G on $L^m_{p,q}(B) = L_{p,q}(B)$ (Theorem 7.4). We now present a weak* analogue of the argument of Theorem 7.4 to establish this fact when p' and q' are finite. Again it suffices to verify the weak* continuity at the identity e of G, and to this end we let $f \in L^m_{p,q}(B)$ and $g \in L^w_{p',q'}(B^*)$ be fixed. Then it follows from (3.1.6) that g may be taken to be supported on a set $E \in S$, and a straightforward application of the dominated convergence theorem will show that it suffices to assume that E is compact, and that $\|g(\cdot)\|$ is bounded by a fixed constant $M > 0$. Let $\varepsilon > 0$, and let U be an arbitrary compact neighbourhood of e. Then it follows from Lemma 2.2, from (3.9.1), and then from Lemma 3.10 that there exists a number $\delta > 0$ such that we have $\|\chi_F\|_{p,q} \leqslant \varepsilon/6M\|f\|_{p,q}$ for every measurable set $F \subset E$ which satisfies $\mu(F) < \delta$. (We are eliminating the trivial case $\|f\|_{p,q} = 0$.) Having fixed such a δ, we now employ Lusin's criterion of measurability to specify a compact set $K \subset UE$ such that $\mu(UE-K) < \delta$, and such that $f|_K$ is continuous. Let $N > 0$ be an upper bound for the continuous function $\varrho(x)^{1/p-1/q}$ on K, and define $h(x) = f(x)\varrho(x)^{-(1/p-1/q)}$. Then h is continuous on K, and so we may find a finite open cover $U_1, \ldots, U_n$ of K such that, for each i, the diameter of the set $h(U_i \cap K)$ is $\leqslant \varepsilon/3MN\mu(E)$. (We are eliminating the trivial case $\mu(E) = 0$.) Lebesgue's covering lemma will generalize to groups (see [14], pp. 154 ff.; and [11], Theorem 4.10, p. 20), and so we may find a neighbourhood V of e such that $V \subset U$, and such that, for all $x \in K$, the set Vx is contained in one of the U_i.

We shall now establish that we have $|\langle f, zg\rangle - \langle f, g\rangle| \leqslant \varepsilon$ for all $z \in V$. We notice first of all that

$$|\langle f, zg\rangle - \langle f, g\rangle| = |\langle z^{-1}f - f, g\rangle| = \left| \int_E \left\langle \left(\frac{\varrho(x)}{\varrho(zx)}\right)^{1/p-1/q} f(zx) - f(x), g(x) \right\rangle d\mu(x) \right|.$$

We shall divide E into three disjoint sets and consider the above integral on each of these sets separately. First let $F = K \cap E$. Then we have $E - F = (UE \cap E) - (K \cap E) = (UE - K) \cap E$, so that $\mu(E - F) \leqslant \mu(UE - K) < \delta$. It now follows from Hölder's inequality that the integral on $E - F$ is bounded by $\|g\chi_{E-F}\|_{p',q'}(\|z^{-1}f\|_{p,q} + \|f\|_{p,q}) \leqslant M\|\chi_{E-F}\|_{p',q'}(2\|f\|_{p,q}) \leqslant \varepsilon/3$. Next let $F_1 = F - (z^{-1}K) = \{x \in F: zx \notin K\}$, and let $F_2 = F - F_1$. Since $z \in V \subset U$, it follows easily that we have $F_1 \subset z^{-1}(UE - K)$, so that $\mu(F_1) \leqslant \mu(z^{-1}(UE - K)) = \mu(UE - K) < \delta$. We may threfore replace $E - F$ by F_1 in the above argument, and we may thereby conclude that the integral on F_1 is $\leqslant \varepsilon/3$. Finally, we consider the integral on F_2. For $x \in F_2$, we have $zx \in K$, so that $\|h(zx) - h(x)\| \leqslant \varepsilon/3MN\mu(E)$. Hence the integral on F_2 is bounded by

$$\int_{F_2} \|g(x)\| \left\| \left(\frac{\varrho(x)}{\varrho(zx)}\right)^{1/p-1/q} f(zx) - f(x) \right\| d\mu(x) \leqslant M\big(\varepsilon/3M\mu(E)\big)\mu(F_2) \leqslant \varepsilon/3.$$

Putting all of this together, we obtain the desired inequality $|\langle f, zg\rangle - \langle f, g\rangle| \leqslant \varepsilon$, and, since $z \in V$ was arbitrary, the weak* continuity is established. ∎

We remark that we have been unable to obtain a proof for this result by exploiting the existence of a weak* dense subspace (for example, $L_{p',q'}(B^*)$ (Corollary 4.6)) on which the representation is known to be strongly continuous. The key to the above argument, and to that of Theorem 7.4, lies in the appropriate application of the dominated convergence theorem, and our techniques appear to fail in precisely those two cases (namely, $p = 1$, $q = \infty$ and $p = \infty$, $q = 1$) when the dominated convergence theorem can be applied neither to the $L_{p,q}$ norm nor to the $L_{p',q'}$ norm.

In order to clarify the parallel between the norm and the weak* theories, we shall hereafter regard the space $L^w_{p',q'}(B^*)$ not as a space of linear functionals (Definition 4.11), but rather as a space of equivalence classes of functions g such that g satisfies (4.9.1)–(4.9.3), and such that the equivalence is determined by the relation "having the same action on $L_{p,q}(B)$ (as specified by (4.9.3))" instead of the relation "equal l.a.e.". That these two relations do not coincide in general is illustrated by [18], Example 6.3, p. 214. In this light, $L^w_{p',q'}(B^*)$ may be regarded as isometrically embedded in (i.e., isometrically isomorphic to a subspace of) $L_{p,q}(B)^*$

as well as $L^m_{p,q}(B)^*$ (although, of course, Theorem 4.9 does not apply to the former embedding). We shall maintain the notational convenience of regarding the elements of $L^w_{p',q'}(B^*)$ as functions rather than as equivalence classes of functions.

We are now prepared for the definition of the weak* analogue of the space $K_q(B)$.

7.7. Definition. For $1 \leqslant q \leqslant \infty$, we define $K^w_q(B^*)$ to be the set of those equivalence classes in $L^w_{\infty,q}(B^*)$ at least one of whose members h satisfies the equality $h(xs) = s^{-1}h(x)$ for all $x \in G$, and for all $s \in H$. ∎

As before, the corresponding representations of G which were described in Theorem 7.6 may be restricted to these spaces $K^w_q(B^*)$, and we thereby obtain isometric induced representations of G corresponding to the dual representation on H on B^*. As before, when q is finite, the analysis of these representations is facilitated by the definition of an intertwining quotient map Π^w_q: $L^w_{1,q}(B^*) \to K^w_q(B^*)$ which in this case will also be weak* continuous (i.e., continuous with respect to the weak* topologies on $L^w_{1,q}(B^*)$ and $K^w_q(B^*)$). The integral formulas by which Π^w_q and Π_q are defined are identical, except for the fact that the formula which is associated with Π^w_q constitutes a weak* integral instead of a strong (i.e., a Bochner) integral. The completeness of the space $K^w_q(B^*)$ now follows from the fact that Π^w_q constitutes a quotient map, while the weak* continuity of the induced representation of G on $K^w_q(B^*)$ follows from Theorem 7.6 together with the weak* continuity of Π^w_q. (When $q = \infty$, a map Π^w_∞ can be defined which is likewise analogous to the map Π_∞.)

Although we have discussed the norm and weak* theories separately, a number of intimate links exist between them. The following theorem provides the principal link.

7.8. Theorem. *Let $1 \leqslant q < \infty$, let q' be conjugate to q, and let $\Gamma \in K_q(B)^*$. Then there exists a function g: $G \to B^*$ such that:*

(7.8.1) *g is locally weak* measurable, and we have $g(xs) = s^{-1}g(x)$ for all $x \in G$ and $s \in H$;*

(7.8.2) *$\|g(\cdot)\|$ is locally measurable, and we have $\|\Gamma\| = \left(\int \|g(x)\|^{q'} d\lambda(\dot{x})\right)^{1/q'}$ (and analogously when $q' = \infty$);*

and

(7.8.3) *for all $f \in K_q(B)$, we may compute $\Gamma(f) = \int \langle f(x), g(x)\rangle d\lambda(\dot{x})$.*

Moreover, if the space B^* is separable or reflexive, then g may be chosen to be locally measurable, and not just locally weak measurable.* ∎

In a word, we have $K_q(B)^* = K^w_{q'}(B^*)$ when $1 \leqslant q < \infty$. When $q = \infty$, the most we can expect, and can obtain, is an isometric embedding of the space $K^w_1(B^*)$ in the space $K_\infty(B)^*$.

We have omitted our somewhat gruelling proof of Theorem 7.8. In outline, the function g is derived by first obtaining from Theorem 4.9 a function which represents $\Pi_q^*(\Gamma) \in L_{\infty,q'}^w(B^*)$, where Π_q^* denotes the adjoint of Π_q, and by then employing the principal result of [12] to convert this function into the desired function g. It is worth remarking that if the existence of a measurable cross section from G/H into G can be assumed, then it follows readily that $K_q(B)$ is isometrically isomorphic to the space $L_q(G/H, \lambda, B)$ for all values of q (cf. [15], Lemma 1, p. 168). Under these circumstances, Theorem 7.8 follows equally readily from Theorem 4.9 applied to $L_q(G/H, \lambda, B)$.

This identification of $K_{q'}^w(B^*)$ with (at least) a subspace of $K_q(B)^*$ gives rise to two natural questions. First, is there any relationship between the weak* topology which $K_{q'}^w(B^*)$ inherits from $K_q(B)^*$ and that which it inherits from $L_{\infty,q'}^w(B^*) \subset L_{1,q}(B)^*$? They are identical, since the (injective) adjoint map Π_q^* is weak* continuous in both directions. Second, is there any relationship between the induced representation of G on $K_{q'}^w(B^*)$ and the dual of the induced representation of G on $K_q(B)$? They are identical, as an easy computation will show. Thus, in particular, we obtain an easy proof of the weak* continuity of the induced representation of G on $K_{q'}^w(B^*)$ when $1 < q' \leqslant \infty$.

The final link between the norm and the weak* theories is that which exists between the maps Π_q and $\Pi_{q'}^w$ (where q and q' remain conjugate). Let us now regard Π_q as having codomain $L_{\infty,q}(B)$ (instead of its range, as we did above). Then the domain of the adjoint Π_q^* becomes $L_{\infty,q}(B)^*$, and this space contains the domain $L_{1,q'}^w(B^*)$ of $\Pi_{q'}^w$. A computation which is only slightly tedious will now establish that, upon restriction to the subspace $L_{1,q'}^w(B^*)$, the map Π_q^* coincides precisely with $\Pi_{q'}^w$. (In fact, this identification provides possibly the easiest method for establishing the weak* continuity of $\Pi_{q'}^w$.)

We remark that when the space B^* is separable or reflexive, then it follows from Corollary 4.12 that we have $L_{p,q}^w(B^*) = L_{p,q}(B^*)$ for all values of p and q. Hence we also have $K_q^w(B^*) = K_q(B^*)$ for all values of q, so that the induced representation of G will be strongly continuous when q is finite, even though the dual representation of H on B^* need not be strongly continuous. (However, when B (and hence B^*) are reflexive, then the strong continuity of the original representation of H on B will imply the same for the dual representation ([24], Corollary 8.8, p. 473).)

The details of the proofs which were omitted from this section may be found in the author's dissertation [17].

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