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Spectrum of L

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CONTENTS

0. Motivation, results to be used in the sequel

As concerns motivations which led us to studying the object which we pursue in this paper, the spectrum of L, i.e. the class of lengths of maximal pairwise elementarily equivalent $\subseteq$-increasing chais of constructible levels, they come from different sources.

First of all, such sequences coincide with sequences of (standard) models into which first element of the sequence can be elementarily embedded.

Secondly, the investigations of Mostowski [6] and Wilmers [7] have shown a number of elements of the spectrum but didnt say too much about the whole entity.

Finally, from the Hanf-number properties of its supremum (one studied here and another from [2]).

We obtained most results of this paper independently. Rasmussen in his thesis submitted to the University of Leeds in 1973, Marek, later, in manuscripts circulated starting from 1975. Rasmussen circulated in 1974 a paper containing a portion of the thesis. Due to M. Srebrny and J. Derrick, we put the work together. Some of the results were obtained after the work of each was known to the other.

We are grateful to our collegues: F. Drake, J. Derrick (thesis advisors of the second author), A. Krawczyk and M. Srebrny for valuable discussions.

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We assume throughout the paper, as our metatheory, the Kelley–Morse theory of classes. This has several – perhaps not so grave – consequences: Existence of arbitrarily large α such that L_α is a model for the Zermelo–Fraenkel set theory, existence of satisfaction class for V, finally freedom in dealing (as we do) with classes not being concerned with their definition. Most of the results of the paper can be proved in weaker theory, namely $\mathrm{ZF}+(\alpha)(E_\beta)(\beta > \alpha \;\&\; L_\beta \models \mathrm{ZF})$.The spectrum itself being independent from the (true) theory of V could be successfuly investigated in ZF only.

One thing which may seem slightly odd is that *we assume $V = L$ throughout the paper*. For those who do not like it (quite a number of people in the age when elephantine cardinals run freely) we suggest that they add a number of superscripts L in appropriate places.

We study here mostly model theory of constructible levels L_α's. These are defined as follows:

$L_0 = 0$;

$L_{\alpha+1} = \mathrm{Def}\, L_\alpha$ (the family of parametrically definable subsets of L_α);

$L_\lambda = \bigcup_{\nu<\lambda} L_\nu$ (λ limit);

$L = \bigcup_{\nu\in On} L_\nu$.

Not only L possesses Σ_1-definable well-ordering but also for each α, L_α possesses a definable well-ordering. This results in that, for each α, there is a $\beta \leqslant \alpha$ such that $L_\beta \equiv L_\alpha$ and L_β is pointwise definable. From this we immediately derive the following version of Tarski's theorem.

PROPOSITION 0.0. *For every* α, $\mathrm{Th}(L_\alpha)$ *is not definable over* L_α.

Our assumption, $V = L$, has the following consequence: Define $H_\varkappa = \{x\colon |\mathrm{TC}(x)| < \varkappa\}$; thus $H_\varkappa$ is the set of those sets which are hereditarily of cardinality less than $\varkappa$ (in the case $\varkappa = \omega_1$ we use symbol HC to denote $H_\varkappa$). Then, by Gödel's fundamental result, in L, $x \subseteq L_\alpha \Rightarrow x \in L_{\alpha^+}$.

Therefore we have the following

PROPOSITION 0.1. $(\varkappa)(\mathrm{Card}(\varkappa) \Rightarrow H_\varkappa = L_\varkappa)$.

Thus, in particular, HC is simply L_{ω_1}. Proposition 0.1 is provable in $\mathrm{ZF} + V = L$ and so holds in its models.

In this paper we are going to consider various set theories: These are: ZF, Zermelo–Fraenkel set theory, $\mathrm{ZF} + V = L$, Zermelo–Fraenkel set theory with the axiom of constructibility, ZF^-, Zermelo–Fraenkel set theory without powerset axiom, $\mathrm{ZF}^- + V = \mathrm{HC}$, where $V = \mathrm{HC}$ means "everything is denumerable". (Notice that HC satisfies $\mathrm{ZF}^- + V = \mathrm{HC}$.) Another important theory is KP, the Kripke–Platek set theory. In all these theories the class L is an inner model of it and the standard models for $T + V = L$ (T one of the above) are always of the form L_α.

Throughout the paper, with one important exception when we explicitly state it (Section 7), the word "model" means "standard model" (i.e. transitive model) or at least "well-founded model".

We are going to use in the paper the following (only a small part of it but anyway):

PROPOSITION 0.3. *If* $\xi \geqslant \omega_n (n \in \omega)$, $X \prec L_\xi$, $|X| \geqslant \omega_n$, *then* $L_{\omega_n} \subseteq X$.

Proof. By induction. For $n = 0$ the result is obvious. Assume it is valid for n. To show it for $n+1$, notice that $L_{\omega_n} \subseteq X$ by inductive assumption, and so $L_{\omega_n} \in X$ too (as ω_n is definable in L_ξ for every $\xi > \omega_n$). Prove first that $X \cap L_{\omega_{n+1}}$ is transitive which is fairly obvious as $x \in L_{\omega_{n+1}} \Rightarrow |x| \leqslant \omega_n$ and $\omega_n \in X$. Now if $X \cap L_{\omega_{n+1}} \neq L_{\omega_{n+1}}$, then there is a transitive $y \notin L_{\omega_{n+1}}$ such that the contraction function, π^{-1}, brings y down to $L_{\omega_{n+1}}$. Thus:

$$\pi X \models \text{"}|\pi y| = \omega_n\text{"} \quad \text{so} \quad X \models |y| = \pi^{-1}\omega_n$$

but $\pi^{-1}\omega_n = \omega_n$ so $X \models |y| = \omega_n$ and so $L_\xi \models |y| = \omega_n$ which means (as y is transitive) $y \in L_{\omega_{n+1}}$ which is a contradiction. ■

One of the most important tools of this paper are results concerning the so-called *stable sets.*

They are transitive sets X such that $X \prec_1 L$, i.e. whenever $\psi \in \Delta_0$, $(Ex)\psi(x, \vec{y})$ (where $\vec{y} \in X$) then there is $z \in X$ such that $\psi(z, \vec{y})$.

Now – with the definition we accepted – stable sets are always of the form L_α for some α. Extremely important property of stable sets is their definability property. Since they are all constructible levels, they are well-ordered by inclusion. Let σ_α be their enumeration function (which is continuous).

The result on $\sigma_{(\cdot)}$ (proved in [1], [3]) is: Elements of $L_{\sigma_{\alpha+1}}$ are exactly those which are Σ_1-definable in L using parameters from $L_{\sigma_\alpha} \cup \{L_{\sigma_\alpha}\}$ (or if you wish just ordinals from $\sigma_\alpha + 1$). In particular, L_{σ_0} consists exactly of elements Σ_1-definable in L(and in fact in V). σ_0 is equal to δ_2, first non-Δ^1_2-definable ordinal (due to Kripke and Platek; for the proof of that result see [1] or [3]). Important fact is that the equality $\sigma_0 = \delta_2$ is obtained in ZF and thus holds in its models. Another interesting fact is that L_{σ_0} is the largest Σ_1-pointwise definable level (actually the largest Σ_1 pointwise definable p.r. closed set). This follows from the persistence of Σ_1 formulas upwards. There are Σ_1-pointwise definable sets smaller than L_{σ_0}, for instance $L_{\delta_2^{L_\alpha}}$, where L_α is the least model of ZF. We will meet them in large numbers in Section 6. The following fact is useful:

THEOREM 0.4. *If* $\alpha > \delta_2$, $L_\alpha \models$ ZF, *then* $\delta_2^{L_\alpha} = \delta_2$.

Proof. By the basis property of Δ^1_2-sets of natural numbers we have $\delta_2 \leqslant \delta_2^{L_\alpha}$. But $L_{\delta_2^{L_\alpha}}$ is Σ_1-pointwise definable. Thus $\delta_2^{L_\alpha} \leqslant \delta_2$. ■

For δ_n, the first non-Δ^1_n-ordinal, $n \geqslant 3$, the situation is more complicated; If $\alpha < \delta_n$, then $\delta_n^{L_\alpha} < \delta_n$ anyway. If $\alpha > \omega_1$, then simply because $\mathscr{P}(\omega)^{L_\alpha} = \mathscr{P}(\omega)$ we have:

PROPOSITION 0.5. *If* $\alpha > \omega_1$, $L_\alpha \models$ ZF, *then*

$$\delta_n^{L_\alpha} = \delta_n.$$

But what about $\delta_3^{L_\alpha}$ for $\delta_3 < \alpha < \omega_1$. Is it δ_3? It is not the case. Here we have rather unexpected phenomenon:

PROPOSITION 0.6. (1) *Given* $\alpha < \omega_1$, *there is* $\alpha < \beta < \omega_1$ *such that* $L_\beta \models$ ZF, *and* $L_\beta \models$ "*α is aΔ^1_3 ordinal*".

(2) *Thus, given* $X \subseteq \omega$, *there is* $\beta < \omega_1$ *such that* $L_\beta \models$ ZF *and* $L_\beta \models$ "*x is aΔ^1_3 set of natural numbers*".

Proof. (1) Let $(\cdot)^+$ mean here "next admissible ordinal". We firstly pick $\gamma > \alpha$ such that γ is denumerable in L_{γ^+}. It is well known that there is a cofinal-in-ω_1 set of such γ's. (It follows from the following: If γ is a non-gap, i.e. $(L_{\gamma+1} - L_\gamma) \cap \mathscr{P}(\omega) \neq 0$, then γ is denumerable in L_{γ^+}; see for instance [4].)

Now consider $\beta = \theta_{\gamma+1}$, the height of the $(\gamma+1)$st model of ZF. Clearly, $\theta_\gamma \geqslant \gamma$ and $\theta_{\gamma+1} > \gamma^+$ since θ_α is never a next admissible ordinal. Thus γ is denumerable in L_β and, by construction, $L_\beta \models$ "There is only denumerably many transitive models of ZF + $V = L$". Now the type of those models in L_β is $\gamma+1$. If we show that L_β satisfies "$\gamma+1$ is a Δ^1_3 ordinal", then we are done as $\alpha \leqslant \gamma$ and Δ^1_3 ordinals form a segment. Now the sequence $\langle L_{\theta_\nu} : \nu \leqslant \gamma \rangle$ is an element of L_β and is denumerable in L_β (as γ is denumerable there). Also all θ_ν $(\nu \leqslant \gamma)$ must be denumerable in L_β – otherwise $L_\beta \models$ "There is non-denumerably many transitive models of ZF + $V = L$" (by Skolem–Löwenheim's argument within L_β).

Therefore, if τ is $\omega_1^{L_\beta}$, then:

(1) $\gamma < \tau$.

(2) For all $\nu \leqslant \gamma$, $\theta_\nu < \tau$.

It is not difficult to see that the predicate "ϱ is a type of all transitive models of ZF + $V = L$" – if non-empty – is Σ_2. But in our case – as all transitive models of ZF + $V = L$ (in L_β) are in L_τ – it is Σ_2 over L_τ. This is translated to Σ^1_3 predicate and by the basis property there is a Δ^1_3 ordinal satisfying this predicate. Thus $\gamma+1$ is a Δ^1_3 ordinal in L_β and so α is Δ^1_3 ordinal in L_β.

(2) follows immediately from (1) and the following fact due to Shoenfield: If $V = L$, then the Δ^1_n sets of natural numbers are exactly elements of L_{δ_n} $(n \geqslant 2)$. ∎

We gave only the sketch of the proof of the strange result [6], still it should be clear that using the same method and somewhat more sophisticated theory $T_{\mathrm{Spec}_n} = \mathrm{ZF} + (\mathrm{Th}\, L_{\delta_n})^{\underline{L_{\delta_n}}}$ ($\underline{L_{\delta_n}}$ being a defined term) and the following theory $T_n = T_{\mathrm{Spec}_n} +$ "T_{Spec_n} has only denumerably many models" we can keep δ_n fixed while making a given denumerable ordinal analytical. (Not Δ^1_{n+1} though.)

Interesting feature of δ_n is that $L_{\delta_n} \prec_{n-1} L_{\omega_1}$. (In the case $n = 2$ it is stability of L_{δ_2} plus Levy's absoluteness lemma.)

In particular, $\delta_\infty = \bigcup_n \delta_n$, first non-analytical ordinal has the following property:

$$L_{\delta_\infty} \prec L_{\omega_1}.$$

It is also the least such.

Stability has as its consequence the following important fact (notice that via Gödelization theories are always sets of natural numbers).

PROPOSITION 0.7. *Assume that L_α is stable, T a theory in L_{ST}, $T \in L_\alpha$. If there is $\beta \geqslant \alpha$ such that $L_\beta \models T$, then $\{\nu \in \alpha : L_\nu \models T\}$ is cofinal in α and Δ_1 definable in L_α and therefore it has order type α.*

Proof. Given $\varrho \in \alpha$, the statement

$$(E\xi)(\varrho \in \xi \,\&\, L_\xi \models T)$$

is true and Σ_1. Thus there is ξ in α with this property (notice that ϱ and T are parameters from L_α).

This gives cofinality. Since stable sets are admissible, we get the rest. ∎

Finally we come to the questions of definability. There are numerous possible notions of definability (in an unpublished memoir Mostowski compared several of those). We shall use some of them. The first (of which all others will be derived) is the notion of the so-called "strong definability in the theory T".

Assume that T has arbitrarily large transitive models. An ordinal ξ is *strongly definable in* T iff there is parameter-free formula $\psi(\cdot)$ such that:

Whenever $\nu > \xi$, $L_\nu \models T$, then $L_\nu \models (E!\ x)\psi$ and $L_\nu \models \psi[\xi]$.

Whenever $\nu \leqslant \xi$, $L_\nu \models T$, then $L_\nu \models \neg (Ex)\psi$.

The following will be used:

THEOREM 0.8. *If ξ is strongly definable in T, L_α is stable, $T \in L_\alpha$, then $\xi < \alpha$.*

Proof. Since $T \in L_\alpha$, $T \cup \{\ulcorner Ex\psi \urcorner\} \in L_\alpha$, because L_α is admissible. By stability, if there is a model L_η of $T \cup \{\ulcorner Ex\psi \urcorner\}$, then there is one in L_α. But then $\xi < \alpha$. ∎

In particular, an ordinal strongly definable in any theory is denumerable and ordinals strongly definable in Δ^1_2 theories are all in L_{δ_2} (the same holds for $n > 2$ as well and for analytical theories and L_{δ_α}).

It is an old result of Suzuki that ordinals strongly definable in, say, ZF are cofinal in δ_2 but do not fill it up. We will come back to that question in Section 6.

If one was wondering how much depends is this paper on the choice of particular hierarchy L_α and not, say, on F_α of Gödel or J_α of Jensen, then we must disappoint him; in our case (of models of KP) $L_\alpha = J_\alpha = F_\alpha$.

One more convention: If T is a theory, then θ^T_α is the height of the αth model of T of the form L_ξ.

1. Slicing L_α's

If α is an ordinal, then L_α, αth level of constructible hierarchy, may be but not necessarily is pointwise definable. Clearly, each pointwise definable level must be denumerable. Since for every $\alpha \in On$, L_α posesses a definable well-ordering, thus definable Skolem functions, therefore $\mathrm{Def}\, L_\alpha \prec L_\alpha$. Now $\mathrm{Def}\, L_\alpha \simeq L_\xi$ for some ξ. Unique ξ with this property is nothing else but $f(\alpha)$ or $g(\alpha)$ of Mostowski [6] and Wilmers [7] (they were discussing the models of ZF only but there is no reason to restrict ourselves to that case). Let $\eta(\alpha)$ be unique ξ as above. The contraction function π^{-1} contracts $\mathrm{Def}\, L_\alpha$ onto $L_{\eta(\alpha)}$ and so the inverse function π imbeds elementarily $L_{\eta(\alpha)}$ into L_α. Depending on the theory of L_α, π may be the identity map or not. Suppose that α is limit or even stronger that L_α models KP theory (cf. [1]).

Proposition 1.0 (Devlin). *If $L_\alpha \models \mathrm{KP} + V = \mathrm{HC}$, then π is the identity.*

Proof of it is quite simple, see [5].

Proposition 1.1. *If $L_\alpha \models \mathrm{KP} +$ "ω_1 exists" and L_α is not pointwise definable, then π is not an identity.*

Proof. Assume that $\pi\colon L_{\eta(\alpha)} \to L_\alpha$ is an identity map (and elementary imbedding). Thus $\eta(\alpha) \neq \alpha$ since $L_{\eta(\alpha)}$ is pointwise definable but L_α is not. Thus $L_{\eta(\alpha)} \in L_\alpha$.

Now we have the following fact which we leave to the reader:

If ξ is limit, $\alpha \in \xi$, then:

L_α is pointwise definable iff $L_\xi \models$ "L_α is pointwise definable".

Now, if π is an identity, then $\omega_1^{L_\alpha} = \omega_1^{L_{\eta(\alpha)}}$ and since $L_\alpha \models$ "$\omega_1^{L_{\eta(\alpha)}}$ is denumerable", therefore $L_\alpha \models$ "ω_1 is denumerable" which is an absurd. ∎

Note that $\mathrm{KP} + V = L \vdash V = \mathrm{HC} \vee$ "ω_1 exists" so Propositions 0 and 1 completely characterize the situation.

We note that there is a unique π elementarily imbedding $L_{\eta(\alpha)}$ into L_α since the element with the definition $\psi(\cdot)$ in $L_{\eta(\alpha)}$ (note $\mathrm{Def}\, L_{\eta(\alpha)} = L_{\eta(\alpha)}$) must be mapped on the element with the definition $\psi(\cdot)$ in L_α. In general, by simple cardinality argument, elementary imbedding of L_α into L_β is not unique. One easily produces pairs $\langle L_\alpha, L_\beta \rangle$ with as many imbeddings as one wants.

Proposition 1.2. *If $L_\alpha \models \mathrm{ZF}^-$, $\pi\colon L_{\eta(\alpha)} \to L_\alpha$ a canonical elementary embedding, $\zeta = \bigcup_{\varrho < \eta(\alpha)} \pi(\varrho)$, then $L_\zeta \prec L_\alpha$.*

Proof. Assume $L_\alpha \models (Ex)\varphi(x, \vec{y})$, $\vec{y} \in L_\zeta$. Pick ϱ definable such that $\vec{y} \in L_\varrho$. Let μ be a least such that L_μ reflects both $Ex\varphi(x, \cdot)$ and $\varphi(\cdot, \cdot)$ for all parameters from L_ϱ. This can be converted to a definition of μ in L_α. Thus the existential quantifier can be bounded by L_μ and since $\mu < \zeta$, $L_\alpha \models (Ex)_{L_\zeta} \varphi(x, y)$ which by Tarski's test completes the proof. ∎

To see that the assumption $L_\alpha \models ZF^-$ was necessary note that $L_\alpha \prec L_\beta$ ($\alpha < \beta$) implies that both L_α and L_β satisfy ZF^- (just for the completeness note that if in addition $\alpha^+ < \beta$, then L_α and L_β are models of ZF).

PROPOSITION 1.3. *If ζ from Proposition 1.2 is less than α, T is a recursive theory, $L_\alpha \models T$, then $\alpha = \theta^T_\alpha$.*

Proof. All recursive sets are in L_α, because $L_\zeta \prec L_\alpha$, $\zeta < \alpha$, and thus $\omega+1 < \alpha$ and all recursive sets are in $L_{\omega+1}$. Now, given μ definable in L_α, $L_\alpha \models$ "There is transitive model of $T+V=L$ containing μ". (Because L_ζ is that model.) The same is true in L_ζ and so $L_\zeta \models$ "There are arbitrarily large transitive models of $T+V=L$". Since L_ζ satisfies KP, we have $\zeta = \theta^T_\zeta$ and thus $\alpha = \theta^T_\alpha$. ∎

Though we know that the imbedding of $L_{\eta(\alpha)}$ into L_α is unique we want to know something more about it; we assume now that $L_\alpha \models ZF^- +$"ω_1 exists". The imbedding π maps the ordinals of $L_{\eta(\alpha)}$, i.e. $\eta(\alpha)$, into ordinals of L_α, i.e. α. Since π is elementary and so $\pi(\alpha+1) = \pi(\alpha)+1$, $\eta(\alpha)$ is "sliced" by π into segments (pieces), namely counterimages of maximal segments included in $\pi * L_\alpha$. Since π preserves rank, actually the whole $L_{\eta(\alpha)}$ is "sliced" by π.

LEMMA 1.4. *Imbedding π: $L_{\eta(\alpha)} \to L_\alpha$ slices $L_{\eta(\alpha)}$ into equal pieces of length $\omega_1^{L_{\eta(\alpha)}}$.*

Proof. Since the ordinal sum and difference of definable ordinals is again definable, the length of every piece is the same and actually equal to the least ordinal moved by π.

Thus we need to show that $\omega_1^{L_{\eta(\alpha)}}$ is the first ordinal by π, i.e. for each $\nu < \omega_1^{L_{\eta(\alpha)}}$ $\pi(\nu) = \nu$ but $\pi(\omega_1^{L_{\eta(\alpha)}}) \neq \omega_1^{L_{\eta(\alpha)}}$. The second fact was shown in Proposition 1.1. To show the first one we employ the reasoning of Proposition 1.0. Since it was not given, we show the following:

SUBLEMMA 1.5. *If $X \prec L_\xi$, $L_\xi \models ZF^- +$"ω_1 exists", then $X \cap (HC)^{L_\xi}$ is transitive.*

Proof. If $a \in X \cap (HC)^{L_\xi}$, then $TC(a) \in X \cap (HC)^{L_\xi}$ and $L_\xi \models$ "There is a mapping of ω onto $TC(a)$". One such map f is in X. Since $\omega \subseteq X$, therefore $Rf \subseteq X$. Thus $TC(a) \subseteq X$. ∎

Coming back to our Lemma 1.4, $(\mathrm{Def}\, L_\alpha) \cap (HC)^{L_\alpha} = (\mathrm{Def}\, L_\alpha) \cap L^{L_\alpha}_{\omega_1}$ is transitive. Thus π^{-1} is identity on it and so π is identity on each element of $\omega_1^{L_{\eta(\alpha)}}$. ∎

Once we know that the first element which is moved is $\omega_1^{L_{\eta(\alpha)}}$ we can depict the situation as follows:

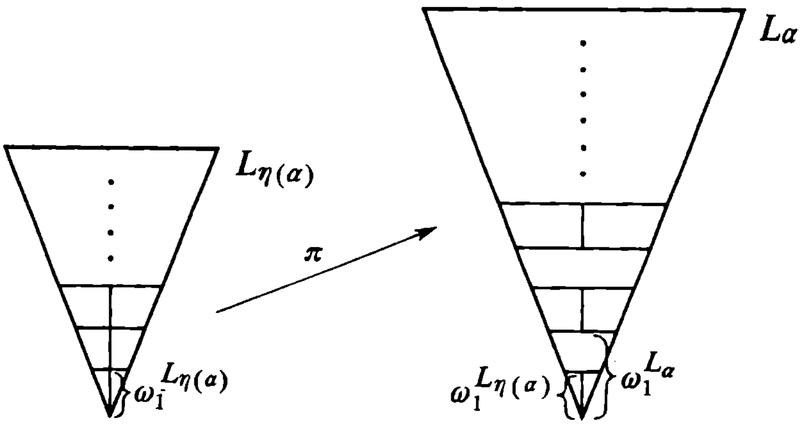

Fig. 1

One easily shows, that the segment of $<_L$ on which $L_{\eta(\alpha)}$ is sliced also have the length $\omega_1^{L_{\eta(\alpha)}}$.

Lemma 1.6. $L_{\omega_1}^{L_{\eta(\alpha)}} \prec L_{\omega_1}^{L_\alpha}$.

Proof. π is monotone and $\pi(\omega_1^{L_{\eta(\alpha)}}) = \omega_1^{L_\alpha}$.

If $L_{\omega_1}^{L_\alpha} \models (Ex)\psi(x,\bar{y})$, $\bar{y} \in L_{\omega_1}^{L_{\eta(\alpha)}}$, then $L_\alpha \models (Ex)_{L_{\omega_1}} \varphi^{\underline{L_{\omega_1}}}(x,\bar{y})$, thus $L_{\eta(\alpha)} \models (Ex)_{L_{\omega_1}}\left(x, \varphi^{\underline{L_{\omega_1}}}(x,\pi^{-1}(\bar{y}))\right)$.

Picking the appropriate x in $L_{\omega_1}^{L_{\eta(\alpha)}}$ and using π (it is identity on $L_{\omega_1}^{L_{\eta(\alpha)}}$), we get the required x. ∎

For completeness sake note that in that case $L_{\omega_1}^{L_{\eta(\alpha)}}$ is *not* pointwise definable (though $L_{\eta(\alpha)}$ is), in particular, $\omega_1^{L_{\eta(\alpha)}}$ is a critical point in the enumeration of all ζ's such that $L_\zeta \prec L_{\omega_1}^{L_\alpha}$.

Coming back to the study of π. We know that π slices $\eta(\alpha)$ into equal pieces of length $\omega_1^{L_{\eta(\alpha)}}$ and that at the beginning they appear (at L_α) at equal distances $\omega_1^{L_\alpha}$. This happens until $\omega_1^{L_{\eta(\alpha)}}$ piece when there is a jump of the length $(\omega_1^{L_\alpha})^2$. We have in fact

Theorem 1.7. *If $\zeta = (\omega_1^{L_{\eta(\alpha)}})^{\varrho_1} \cdot \zeta_1 + \ldots + (\omega_1^{L_{\eta(\alpha)}})^{\varrho_n} \cdot \zeta_n$ is the Cantor's decomposition of $\zeta < \eta(\alpha)$ into the powers of $\omega_1^{L_{\eta(\alpha)}}$, then $\pi(\zeta) = (\omega_1^{L_\alpha})^{\pi(\varrho_1)} \cdot \zeta_1 + \ldots + (\omega_1^{L_\alpha})^{\pi(\varrho_n)} \cdot \zeta_n$.*

Proof. First note that Cantor's decomposition of ordinals is absolute with respect to transitive models of ZF (since ordinal division is). Now, having in mind that $\pi(\gamma^\delta) = \pi(\gamma)^{\pi(\delta)}$ and that $\pi(\omega_1^{L_{\eta(\alpha)}}) = \omega_1^{L_\alpha}$ and that for $\zeta < \omega_1^{L_{\eta(\alpha)}}$, $\pi(\zeta) = \zeta$, we are done. ∎

Corollary 1.8. *If $\mu < \alpha$ has in its Cantor's decomposition in powers of $\omega_1^{L_\alpha}$ at least one coefficient greater than $\omega_1^{L_{\eta(\alpha)}}$, then $\mu \notin R\pi$.*

Actually we could define a notion of hereditary coefficient of μ as coefficient of μ or coefficient of any exponent of the decomposition of μ, etc. Clearly, if $\mu \in R\pi$, all the hereditary coefficients must be less than $\omega_1^{L_{\eta(\alpha)}}$. Note that the result of 1.7 though quite trivial allows to calculate $\pi(\zeta)$ from the values of π on its exponents. Thus for non-epsilon w.r.t. $\omega_1^{L_{\eta(\alpha)}}$ numbers σ, value $\pi(\sigma)$ is determined by $\pi \restriction \sigma$. Every cardinal greater than ω_1 is clearly an epsilon number w.r.t. ω_1. Thus in this place the jump cannot be estimated from below. One can develop a whole theory of how π works but this does not shed new light on its properties.

We consider now the continuity properties of π.

Proposition 1.9. (a) *If* $L_{\eta(\alpha)} \models$ "$cf(\varrho) = \omega$", *then* π *is continuous in* ϱ.

(b) *If* $L_{\eta(\alpha)} \models$ "$cf(\varrho) > \omega$", *then* π *is discontinuous at* ϱ.

Proof. (a) Let $\varrho = \bigcup_{n<\omega} f_n$, $f \in L_{\eta(\alpha)}$. Then $\pi(\varrho) = \bigcup_{\pi(n)<\pi(\omega)} (\pi f)_{\pi(n)} = \bigcup_{n<\omega} (\pi(f))_n$ which gives desired continuity.

(b) If $L_{\eta(\omega)} \models cf(\varrho) = \varkappa$, $\varkappa \geqslant \omega_1$, then $L_{\eta(\alpha)} \models \varrho = \bigcup_{\zeta<\varkappa} f_\zeta$ for some $f \in L_{\eta(\alpha)}$.

Thus $L_\alpha \models \pi\varrho = \bigcup_{\pi(\zeta)<\pi(\varkappa)} (\pi f)_{\pi(\zeta)}$. But $\pi(\varkappa) > \varkappa$ and f is monotone thus $\bigcup_{\zeta<\varkappa} (\pi f)_{\pi(\zeta)} < \bigcup_{\zeta<\pi(\varkappa)} (\pi f)_\zeta$ which shows discontinuity. ∎

We say that α is *sliceable* iff there is $\zeta > \alpha$ such that there is elementary imbedding of L_α into L_ζ. We close this section with the following observation:

Proposition 1.10. *If α is strongly definable* (*in* ZF *say*) $L_\alpha \models$ ZF, *then* L_α *is not sliceable.*

Proof. Assume $\pi\colon L_\alpha \to L_\zeta$. Thus $\alpha < \zeta$ and for the formula $\psi(\cdot)$ strongly defining α in $\mathrm{ZF}+V=L$, $L_\zeta \models (Ex)\psi(x)$ but $L_\alpha \models \neg(Ex)\psi(x)$. ∎

The question: Into how many models can a given pointwise definable model be "sliced-in" or more precisely: what is their order type under inclusion, is the principal question we deal with in this paper.

2. Hereditarily countable, definable elements

Lemma 2.1. $L_{\theta_\alpha} \equiv L_{\theta_\beta}$ *iff the ordinals definable and denumerable in* L_{θ_α} *and in* L_{θ_β} *coincide.*

Proof. If $L_{\theta_\alpha} \equiv L_{\theta_\beta}$, then, since by the results of Section 1, the ordinals denumerable and definable in L_{θ_α} are just the elements of $\omega_1^{L_{\eta(\theta_\alpha)}}$ and since $\eta(\theta_\alpha) = \eta(\theta_\beta)$, we have $\Rightarrow$.

Now assume that $L_{\theta_\alpha} \not\equiv L_{\theta_\beta}$. Then $\eta(\theta_\alpha)$ is less than $\eta(\theta_\beta)$ or conversely. By symmetry consider the case $\eta(\theta_\alpha) < \eta(\theta_\beta)$. Then $\mathrm{Th}(L_{\theta_\alpha}) \in L_{\eta(\theta_\alpha)+2} \subseteq L_{\eta(\theta_\beta)}$. Also, being a set of integers $\mathrm{Th}(L_{\theta_\alpha})$ has denumerable constructible order in $L_{\eta(\theta_\beta)}$. Call it ζ. But then if $\zeta < \omega_1^{L_{\eta(\theta_\alpha)}}$, then we find that $\mathrm{Th}(L_{\theta_\alpha}) = \mathrm{Th}(L_{\eta(\theta_\alpha)}) \in L_{\eta(\theta_\alpha)}$, clearly contradicting Tarski's truth undefinability theorem. ∎

The fact that we were showing the lemma for the case of models of ZF is not important; ZF^- would do as well. Also let us note the following proposition with the same (as above) proof:

PROPOSITION 2.2. *If $L_{\theta_\alpha} \equiv L_{\theta_\beta}$, a is hereditarily countable in both models and is definable in both of them, then there is a formula Φ which defines a in both models L_{θ_α} and L_{θ_β}.*

(Actually 2.2 is again an equivalence.) ∎

We now look at the hereditarily countable elements in $L_\varkappa$, $\varkappa$ being a cardinal.

Since ω_1 is uniformly definable in L_α, for every $\alpha > \omega_1$ we have the following result about $(\mathrm{Def}\, L_\varkappa) \cap L_{\omega_1}$:

LEMMA 2.3. $\mathrm{Def}\, L_{\omega_1} \subsetneqq (\mathrm{Def}\, L_\alpha) \cap L_{\omega_1}$ *for all* $\alpha > \omega_1$.

Proof. Inclusion is obvious in view of our previous remark. By definability of L_{ω_1} in L_α (as the set HC of hereditarily countable elements) we find that the inclusion is proper. ∎

Similarly we find that $(\mathrm{Def}\, L_{\omega_2}) \cap L_{\omega_1} \subsetneqq (\mathrm{Def}\, L_{\omega_3}) \cap L_{\omega_1}$, etc. and finally we have:

PROPOSITION 2.4. *If $\varkappa$ is definable cardinal, then*

$$(\mathrm{Def}\, L_\varkappa) \cap L_{\omega_1} \subsetneqq \mathrm{Def}\, L \cap L_{\omega_1}. \blacksquare$$

Now, is it true that if $\varkappa, \lambda$ are definable $\varkappa < \lambda$, then $(\mathrm{Def}\, L_\varkappa) \cap L_{\omega_1} \subsetneqq (\mathrm{Def}\, L_\lambda) \cap L_{\omega_1}$? Momentary reflection shows that it is not the case; indeed we have pairs $\langle \varkappa_i, \lambda_i \rangle$, $i = 0, 1, 2$, for which all three possibilities: $\subsetneqq$, $=$, $\supsetneqq$ hold.

PROPOSITION 2.5. *For each ordinal ξ there is a cardinal η such that $\xi \in \mathrm{Def}\, L_\eta$.*

Proof. Pick least $\zeta > \xi$ such that $\omega_\zeta = \zeta$. Then ξ is definable in $L_{\omega_{\zeta+\zeta+1}}$ as the difference between the index of the largest cardinal and the index of the largest fixed point of the aleph function. ∎

PROPOSITION 2.6. *If $L_\varkappa \prec L$, then* $\mathrm{Def}\, L_\varkappa = \mathrm{Def}\, L$.

Note now the following:

PROPOSITION 2.7. *For every $\zeta \in \omega_1$ there is a cardinal η such that* $\zeta + 1 \subseteq \mathrm{Def}\, L_\eta$.

Proof. If $\zeta \in (\mathrm{Def}\, L_\eta) \cap L_{\omega_1}$, then there is a function enumerating ζ which is also definable in L_η. But since $\omega \subseteq \mathrm{Def}\, L_\eta$, $\zeta + 1 \subseteq (\mathrm{Def}\, L_\eta) \cap L_{\omega_1}$. ∎

THEOREM 2.8. *There are pairs of definable cardinals* $\langle \varkappa_i, \lambda_i \rangle$, $i = 0, 1, 2$, $\varkappa_i < \lambda_i$ *such that*,

(1) $$(\mathrm{Def}\, L_{\varkappa_0}) \cap L_{\omega_1} \subsetneq (\mathrm{Def}\, L_{\lambda_0}) \cap L_{\omega_1},$$

(2) $$(\mathrm{Def}\, L_{\varkappa_1}) \cap L_{\omega_1} = (\mathrm{Def}\, L_{\lambda_1}) \cap L_{\omega_1},$$

(3) $$(\mathrm{Def}\, L_{\lambda_2}) \cap L_{\omega_1} \subsetneq (\mathrm{Def}\, L_{\varkappa_2}) \cap L_{\omega_1}.$$

Proof. (1) Take $\varkappa_0 = \omega_1$, $\lambda_0 = \omega_2$.

(2) To see this it is enough to construct one pair of cardinals $\varkappa < \lambda$ such that $(\mathrm{Def}\, L_\varkappa) \cap L_{\omega_1} = (\mathrm{Def}\, L_\lambda) \cap L_{\omega_1}$, because the lexicographically least is definable.

Thus pick $\varkappa$ as the least ζ such that $L_\zeta \prec L$ and λ as the least $\zeta > \varkappa$ such that $L_\zeta \prec L$.

(3) Let $\zeta = \mathrm{Def}\, L \cap \omega_1$. By Lemma 2.7, ζ (being denumerable) is definable, together with all its elements, in some L_η. Thus picking $L_\varkappa$, $\varkappa > \eta$ such that $L_\varkappa \prec L$ we have $(\mathrm{Def}\, L_\eta) \cap L_{\omega_1} \supsetneq (\mathrm{Def}\, L) \cap L_{\omega_1}$ and $(\mathrm{Def}\, L) \cap L_{\omega_1} = (\mathrm{Def}\, L_\varkappa) \cap L_{\omega_1}$. Thus a pair $\langle \varkappa, \lambda \rangle$ such that $\varkappa < \lambda$ but $(\mathrm{Def}\, L_\varkappa) \cap \cap L_{\omega_1} \supsetneq (\mathrm{Def}\, L_\lambda) \cap L_{\omega_1}$ exists; again we pick a lexicographically first such (which will be not the pair we constructed!). ∎

The following is the obvious interconnection between Lemma 2.1 and others:

PROPOSITION 2.9. *For every* $\alpha \geq \omega_1$, $(\mathrm{Def}\, L_\alpha) \cap L_{\omega_1} \prec L_{\omega_1}$. ∎

Since the latter model satisfies $V = \mathrm{HC}$, so does the former and so it is transitive and of the form L_ν and thus we have

PROPOSITION 2.10. (a) *Elementary submodels of* L_{ω_1} *are well-ordered by inclusion.*

(b) *Structures of form* $(\mathrm{Def}\, L_\alpha) \cap L_{\omega_1}$, $\alpha > \omega$, *are well-ordered by inclusion and are all among those considered in* (a). ∎

The classes considered in (a) and (b) do not coincide; in particular, the second elementary substructure $L_\nu \prec L_{\omega_1}$ is not of the form $(\mathrm{Def}\, L_\alpha) \cap L_{\omega_1}$.

3. Spectrum of L

We start with the following motivation; given L_α and L_β, we have $L_\alpha \equiv L_\beta$ iff $\eta(\alpha) = \eta(\beta)$, and so there is an elementary imbedding of $L_{\eta(\alpha)}$ into L_β. Now we ask for a pointwise definable constructible level L_ν, how many elementary imbeddings there are of L_ν into constructible levels. Given such L_ν, we can order in natural way (according to inclusion) those levels in which L_ν can be imbedded. This coincides with taking a complete theory T (in our case the $\mathrm{Th}(L_\nu)$) and considering all constructible levels modelling it. In this way, for complete T, $I(T) = tp\{\alpha\colon L_\alpha \models T\}$, $I_\alpha = I(\mathrm{Th}(L_\alpha))$ and $\mathrm{Sp}\, L$

$= \{\alpha\colon (ET)(\mathrm{ZF} \subseteq T \,\&\, I(T) = \alpha)\}$. Thus the *spectrum of* L, $\mathrm{Sp}\,L$, is the class of types of transitive models for complete extensions of $\mathrm{ZF}+V=L$.

An useful tool to investigate types is the notion of strongly definable ordinal (see Section 0).

We say that a theory T satisfies Wilmers conditions (W) iff the following W1, W2, W3 are true.

W1: $\mathrm{ZF} \subseteq T$.

W2: There are arbitrarily large models of T of the form L_α.

W3: There is a formula $\psi(\cdot)$ of L_{ST} such that $T \vdash Ex\psi(x)$ and whenever $L_\xi \models T$ and $L_\xi \models \psi[S]$, then $S = T$.

Note that T having property W3 is not complete. Otherwise the least transitive model of T is of form L_ξ and $\mathrm{Th}(L_\xi) \in L_\xi$ which is absurd since L_ξ is pointwise definable.

THEOREM 3.1 (Wilmers [7]). *If T satisfies* W *and α is strongly definable in T, then $\alpha \in \mathrm{Sp}\,L$.*

Proof. [As this is a slight strengthening of the original result of Wilmers (he proved only case (i)) we give the proof.]

Let γ_α be the least ordinal with the following property: There is pairwise elementary equivalent sequence $\langle L_{\varrho_\beta}\colon \beta < \alpha\rangle$ such that: $\gamma_\alpha = \bigcup_{\beta<\alpha} \varrho_\beta$ and, moreover, $(\beta)_\alpha(L_{\varrho_\beta} \models T)$. Since α is strongly definable in T, so is γ_α. (This is an unpleasant callygraphical exercise.)

Now consider two cases:

(i) $\alpha < \theta^T_\alpha$.

Let S be any sequence $\langle L_{\varrho_\beta}\colon \beta < \alpha\rangle$ such that

1° $\beta_1 < \beta_2 < \alpha \Rightarrow L_{\varrho_{\beta_1}} \equiv L_{\varrho_{\beta_2}}$,

2° $L_{\varrho 0} \models T$,

3° $\bigcup_{\beta<\alpha} \varrho_\beta = \gamma_\alpha$.

We show that this sequence cannot be extended.

Subcase (1). $\alpha = \eta+1$.

Then $\gamma_\alpha = \varrho_\eta$ and so ϱ_η is strongly definable in T. The strong definition of ϱ_η is void in L_{ϱ_0} but non-void in any $L_\xi \models T$ for $\xi > \varrho_\eta$. Thus S cannot be extended.

Subcase (2). α is limit.

If $L_{\gamma_\alpha} \not\models T$, then the reasoning of case (1) is good; The strong definition of γ_α is void in L_{ϱ_0} but non-void in any $L_\xi \models T$, $\xi > \gamma_\alpha$.

So assume $L_{\gamma_\alpha} \models T$. We derive a contradiction assuming S can be extended. As before we cannot extend S beyond L_{γ_α}. So we show that $L_{\varrho_0} \not\equiv L_{\gamma_\alpha}$. We derive an even "better" contradiction showing that L_{γ_α} is not a model of replacement. (Contradicting W1.) Indeed, $\alpha < \theta_\alpha \leqslant \gamma_\alpha$. Thus $\alpha < \gamma_\alpha$. Then the class of all models of the theory $\mathrm{Th}(L_{\varrho 0})$ below γ_α which has the type α is not bounded below γ_α contradicting replacement in γ_α. (Note that we may assume that S consists of all models of $\mathrm{Th}(L_{\varrho 0})$ below γ_α – otherwise such complete sequence would be of length bigger than α and it would have an initial segment of length α contradicting the definition of γ_α.)

(ii) $\alpha = \theta_\alpha^T$.

Consider γ_α. Then $\theta_\alpha^T \leqslant \gamma_\alpha$ (just by the definition of γ_α).

Subcase (1). $\theta_\alpha^T < \gamma_\alpha$, i.e. $\alpha < \gamma_\alpha$.

Then the same trick as used in (i) subcase (2) works. Any sequence of length α with limit γ_α cannot be extended neither beyond γ_α nor to γ_α.

Subcase (2). $\alpha = \theta_\alpha^T = \gamma_\alpha$.

Consider the following theory T_1; T_1 is the L-first complete theory such that $L_\alpha \models$ "There are arbitrarily large ξ's such that $L_\xi \models T_1$" and $T \subseteq T_1$. Then T_1 is definable in L_α so $T_1 \neq \mathrm{Th}(L_\alpha)$. Since replacement holds in L_α, so $tp\{\xi\colon \xi < \alpha \,\&\, L_\xi \models T_1\} = \alpha$.

Take as S_1 the sequence of models of T_1 below α. Then as before S_1 cannot be extended beyond α and α is also excluded. Thus again $\alpha \in \mathrm{Sp}\,L$. ∎

Definitely $\mathrm{Sp}\,L$ is of the cardinality $\leqslant \omega_1$ (remember that $V = L$ is assumed thus continuum hypothesis). In order to proceed with showing that $\mathrm{Sp}\,L$ is quite large we have the following:

Lemma 3.2. *If* $\alpha \in \omega_1$, *then*

$$\alpha \in \mathrm{Sp}\,L \Leftrightarrow L_{\omega_1} \models \text{“}\alpha \in \mathrm{Sp}\,L\text{”}.$$

Proof. $\Rightarrow$. If $\beta \geqslant \omega_1$, then by the stability of ω_1 the type of the set $\{\gamma \in \omega_1\colon L_\gamma \equiv L_\beta\}$ is ω_1. Thus if $\alpha \in (\mathrm{Sp}\,L) \cap \omega_1$, then any sequence witnessing this must consist only of denumerable models and is itself denumerable, i.e. it is hereditarily denumerable and as $L_{\omega_1} = \mathrm{HC}$ we find that $L_{\omega_1} \models$ “$\alpha \in \mathrm{Sp}\,L$”.

$\Leftarrow$. Conversely, let $L_{\omega_1} \models$ “$\alpha \in \mathrm{Sp}\,L$”. Let S be a sequence of type α making α to be in $\mathrm{Sp}\,L$, $S \in L_{\omega_1}$. If we were able to extend S beyond ω_1, then by previous remark there are arbitrarily large $\xi < \omega_1$ such that $L_\xi \equiv L_{\varrho_0}$ ($S = \langle L_{\varrho_\eta}\colon \eta \in \alpha \rangle$). But then S may be extended within ω_1 since $\equiv$ is absolute w.r.t. L_{ω_1}, a contradiction. ∎

Corollary 3.3. $\omega_1 \subseteq \mathrm{Sp}\,L \Leftrightarrow L_{\omega_1} \models (\alpha)\,(\alpha \in \mathrm{Sp}\,L)$.

Now look: $L_{\delta_\chi} \prec L_{\omega_1}$ (see Section 0) and so we have:

Corollary 3.4. $\omega_1 \subseteq \mathrm{Sp}\,L \Leftrightarrow \delta_\chi \subseteq \mathrm{Sp}\,L$.

We remind now the following:

Theorem 3.5 (Mostowski). *If T satisfies* W3 *and* $T \supseteq \mathrm{KP}$, *then all elements of the least transitive model of $T + V = L$ are strongly definable in T.*

Proof. The least transitive model of $T + V = L$ is pointwise definable and of form L_ξ. Let z be an element of that model, and let ψ_z be its definition on L_ξ. We write the following strong definition of z in T: (There is no transitive model of $T+V=L\,\&\,\psi_z(\cdot)$) $\vee$ (There is least transitive model x of $T+V=L\,\&\,\psi_z^{(x)}(\cdot)$). ■

We produce now the theory T satisfying the Wilmers condition (W) and such that its least transitive model contains whole L_{δ_x}.

Treat $\underline{L_{\omega_1}}$ as defined term in set theory and consider the following theory T_{Spec} (being a fragment of true set theory):

$$T_{\mathrm{Spec}} = \mathrm{ZF} \cup \{V=L\} \cup \{\Phi^{\underline{L_{\omega_1}}}\colon \Phi \in \mathrm{Th}(L_{\omega_1})\}.$$

LEMMA 3.6. *If* $L_\xi \models T_{\mathrm{Spec}}$, *then* $L_{\omega_1^{L_\xi}} \equiv L_{\omega_1}$.

Proof. It is enough to note that $\mathrm{Th}(L_{\omega_1})$ is complete and $\Phi^{\underline{L_{\omega_1}}} \in \mathrm{Th}(L_\xi) \Rightarrow L_{\omega_1^{L_\xi}} \models \Phi$. ■

LEMMA 3.7. T_{Spec} *satisfies* (W).

Proof. W1 is obvious. W2 holds, because T_{Spec} is true in L. To see W3 notice that the definition of T_{Spec} over L_ξ gives $\mathrm{ZF}\cup\{V=L\}\cup \cup\{\Phi^{\underline{L_{\omega_1}}}\colon \Phi\in L_{\omega_1^{L_\xi}}\}$, but $L_{\omega_1^{L_\xi}} \equiv L_{\omega_1}$ so it is good old T_{Spec}. ■

THEOREM 3.8. $(\alpha)_{\delta_\gamma}$ $(\alpha\in \mathrm{Sp}\,L)$.

Proof. By Theorem 1 it is enough to find a theory T such that all analytic ordinals are strongly definable in T and such that T satisfies condition (W). T_{Spec} is a good candidate. We already know that T_{Spec} satisfies condition (W), so it is enough to show that $\delta_x < \theta_0^{T_{\mathrm{Spec}}}$ which by Mostowski result implies the rest. To show this let us note that $L_\xi \models T_{\mathrm{Spec}} \Rightarrow L_{\omega_1^{L_\xi}} \equiv L_{\omega_1}$. But $L_{\delta_\prime}$ is the least model elementarily equivalent to L_{ω_1} and so the least model of T_{Spec} contains L_{δ_x}. ■

Just for completeness of argument note that if $\beta=\theta_0^{T_{\mathrm{Spec}}}$, then $\omega_1^{L_\beta} > \delta_x$, moreover, in the enumeration of models of $\mathrm{Th}(L_{\omega_1})$, β is a critical point.

Thus we get:

COROLLARY 3.9. $(\alpha)_{\omega_1}$ $(\alpha\in \mathrm{Sp}\,L)$.

Proof. By Theorem 3.8 and Corollary 3.4. ■

Let us note that the use of $\mathrm{Sp}\,L$ is slightly informal, because the way we formulated it, $On \in \mathrm{Sp}\,L$. But it is just one object and since we show that $\mathrm{Sp}\,L$ is uncountable, we can digest this informality.

COROLLARY 3.10. $|\mathrm{Sp}\,L| = \omega_1$. ■

One asks immediately: is $\mathrm{Sp}\,L$ a segment? We show it is not by a trivial argument namely seeing that the theory of L has type On. But leaving alone On we proceed as follows: Let T be $<_L$ first complete theory having at least ω_1 models of the form L_α. Then T is definable in every L_{θ_ξ} with $\theta_\xi \geqslant \omega_1$.

Thus T has exactly ω_1 models (actually this type of reasoning is used "ad nauseam" in this paper). The reader may easily prove that $\omega_2, \omega_3, \ldots$ $\ldots, \omega_\omega \in \mathrm{Sp}\, L$. Some closure conditions for $\mathrm{Sp}\, L$ will be given below.

To extend our initial segment of $\mathrm{Sp}\, L$ beyond ω_1 we introduce the following convention (for a moment). A class $X \subseteq \mathrm{Mod}_{\mathrm{ZF}}$ is good iff (1) X is definable in ZF, say by $\Psi(\cdot)$. (2) For each $L_\xi \in X$, Ψ defines in L_ξ the segment $X \cap L_\xi$.

An example of the good class is $X^{\omega_1} = \{L_{\theta_\alpha}: \alpha > \omega_1\}$; also $X^{\omega_2} = \{L_{\theta_\alpha}: \alpha > \omega_2\}$ is a good class etc.

For a good class X define $I^X(\theta_\xi)$ as follows:

$$I^X(\theta_\xi) = tp\{\theta_\beta\colon L_{\theta_\beta} \models T \,\&\, L_{\theta_\beta} \in X\}, \qquad I^X(T) = tp\{\theta_\beta\colon L_{\theta_\beta} \in X \,\&\, L_{\theta_\beta} \models T\}.$$

Finally, set $X^\alpha = \{L_{\theta_\beta}\colon \theta_\beta > \alpha\}$.

The notion of strongly definable in X is trivially generalized. The following are "relativized" versions of the Wilmers and Mostowski theorems.

THEOREM 3.11. (a) *If X is good, α strongly definable in X, then there is an ordinal $\theta_\xi \in X$ such that $I^X(\theta_\xi) = \alpha$.*

(b) *The least element of X is strongly definable in X.*

We leave the proof to the reader. ∎

Yet, however, the class X^{ω_1} is good, and, what is more important, anything strongly definable is strongly definable in X. Thus we get the following

PROPOSITION 3.12. *If α is strongly definable, then $\omega_1 + \alpha \in \mathrm{Sp}\, L$.*

Proof. Given a theory T with exactly α models above ω_1, we note that it has exactly ω_1 models below ω_1. ∎

So even without more information on closure properties of $\mathrm{Sp}\, L$ we are now able to conclude that $\omega_1 + \omega_1$, $\omega_1 \cdot \omega_1$, least solution of the equation $\xi = \omega_\xi$, etc. are in $\mathrm{Sp}\, L$.

All this is easily obtained using appropriate good classes.

4. The width of elements of spectrum

As defined in Section 3, the spectrum of L, denoted by $\mathrm{Sp}\, L$, is the class of ordinals which are values of the function $I(\theta_\xi^{\mathrm{ZF}})$ and thus $\mathrm{Sp}\, L = \{I(T)\colon T \text{ complete, extending } \mathrm{ZF} \,\&\, (Ev)(T = \mathrm{Th}(L_v))\}$. (Remember this is slightly informal as $On \in \mathrm{Sp}\, L$.)

Given $\alpha \in \mathrm{Sp}\, L$, we ask how many theories T are there such that $I(T) = \alpha$. Fortunately or not the answer is always the same; the width of any ordinal v in the spectrum of L, namely the power of the set of those T which generate the ordinal v is always ω_1. For the case $v < \omega_1$ and $v = On$ we give very simple proof. General case is more complicated. Yet the proof of the main result of this section allows to understand better what $\operatorname{Sp} L$ is.

LEMMA 4.1. *If $L_{\theta_{\alpha_1}}$, $L_{\theta_{\alpha_2}}$ are models of the theory* "ZF + *there exists denumerably many transitive models of* $\mathrm{ZF}+V=L$", *then $\theta_{\alpha_1}=\theta_{\alpha_2}$ or $L_{\theta_{\alpha_1}} \not\equiv L_{\theta_{\alpha_2}}$.*

Proof. Assume $\theta_{\alpha_1} \neq \theta_{\alpha_2}$. We can choose a formula $\Psi(\cdot)$ which defines α_1 in $L_{\theta_{\alpha_1}}$ and α_2 in $L_{\theta_{\alpha_2}}$. (For instance, "The type of transitive models of $\mathrm{ZF}+V=L$".) By the assumption both models satisfy: "The unique ξ such that Ψ is denumerable". Consider the following $\Phi(\cdot)$: "The L-first well-ordering of ω, X, such that X is isomorphic to the unique α such that $\Psi(\alpha)$". As $\alpha_1 \neq \alpha_2$ then objects defined by Φ in $L_{\theta_{\alpha_1}}$ and in $L_{\theta_{\alpha_2}}$ must be different. Thus for some $n\in\omega$, $n\in$"Unique X such that $L_{\theta_{\alpha_1}} \models \Phi[X]$" $\Leftrightarrow n\notin$"Unique X such that $L_{\theta_{\alpha_2}} \models \Phi[X]$". Thus $L_{\theta_{\alpha_1}} \not\equiv L_{\theta_{\alpha_2}}$. ∎

Since $\mathrm{ZF}+V=L+$ "There is denumerably many transitive models of $\mathrm{ZF}+V=L$" has ω_1 models (consider $L_{\theta_{\alpha+1}}$ for those α's which are denumerable in L_{α^+} – next admissible set), their theories witness to the fact that the width of 1 is ω_1. The above proof can be "squeezed" a bit to give way to all denumerable ordinals. Indeed, assume that there is $\nu<\omega_1$ such that $|\{T\colon I(T)=\nu\}| \leqslant \omega$ (but not 0, this case must be treated separately). Then the least such ν is definable in L_{ω_1}, hence analytical and so, for some n, ν and the upper bound of all complete non-extendable sequences of length ν are smaller than δ_n. Take the following theory T_{Spec_n} in ZF language: $\mathrm{ZF}+V=L+(\mathrm{Th}_{L_{\delta_n}})^{\underline{L_{\delta_n}}}$ (where $\underline{L_{\delta_n}}$ is a defined term, denoting L_{δ_n}). Then again the theory T_{Spec_n} satisfies condition (W) – a variant of the proof for T_{Spec} works. So all the ordinals strongly definable in T_{Spec_n} are in the spectrum of L and, moreover, given α strongly definable in T_{Spec_n}, we can choose a complete extension $T' \supseteq T_{\mathrm{Spec}_n}$ such that $I(T')=\alpha$. Now, our ν is strongly definable in T_{Spec_n} (being Δ^1_n-ordinal, all of them are strongly definable in T_{Spec_n} being elements of the least-transitive model of T_{Spec_n}). So there is $T' \supseteq T_{\mathrm{Spec}_n}$ such that $I(T')=\nu$. But the sequence of models of T' starts after all the sequences for ν are finished! A contradiction! Thus we get

THEOREM 4.2. *For every $1 \leqslant \alpha \leqslant \omega_1$ the width of α is ω_1.* ∎

The case of $\alpha = On$ we treat as follows:

Let $Q\alpha\Psi$ denote: "There are arbitrarily big α such that Ψ". We have the following:

LEMMA 4.3. *For every α, $Q\beta$ $(\alpha\in \operatorname{Def} L_{\theta_\beta})$.*

Proof. Given γ greater than α, pick first $\nu \geqslant \gamma$ such that $\theta_\nu=\nu$. Then α is definable in $L_{\theta_{\nu+\alpha+1}}$ as the difference between the type of well-founded models of $\mathrm{ZF}+V=L$ and larger fixed point for the enumeration of such. ∎

LEMMA 4.4. *For every $\alpha\in\omega_1$, $Q\beta$ $(\alpha+1 \subseteq \operatorname{Def} L_{\theta_\beta})$.*

Proof. Once α is denumerable in L_{θ_β} and definable, then there is an enumerating function for α which is definable in L_{θ_β}. But then all elements of α are also in $\operatorname{Def} L_{\theta_\beta}$. ∎

Now we are able to prove:

THEOREM 4.5. *The width of On is ω_1.*

Proof. Assume the width of $On \leqslant \omega$. Then there must be a bound ν below ω_1 on $\eta(\theta_\alpha)$ such that $I(\theta_\alpha) = On$. In particular, each $\omega_1^{L_{\eta(\theta_\alpha)}} < \nu$ for those θ_α's.

Now take those L_{θ_ξ} in which ν is denumerable and definable. Once ν is denumerable and definable in one transitive model of a complete theory, it must be such in all of them. Since there are ω_1 theories and On models, there must be On of them satisfying same theory and in all of them ν is denumerable and definable so less than ω_1 of some pointwise definable model L_{θ_α} such that $I(\theta_\alpha) = On$. Thus $\nu < \nu$, a contradiction. ∎

By a refinement of this reasoning we will prove that every $\nu \in \operatorname{Sp} L$ has width ω_1. The case of $\nu = 0$ we treat separately.

LEMMA 4.6. *If $T \supseteq \mathrm{ZF} + V = L$ has in L_{ω_1} only non-standard models, then T has no standard model.*

Proof. By the stability of L_{ω_1}. ∎

Actually we could take instead of ω_1 any stable ξ such that $T \in L_\xi$.

LEMMA 4.7. *Assume that there is $\alpha \in \omega_1$ such that (T) ("T is complete extension of $\mathrm{ZF} + V = L$ with only non-standard models" $\Rightarrow T \in L_\alpha$).*

Then there is an analytic α (i.e. $\alpha < \delta_\omega$) with this property.

Proof. Because $L_{\delta_\alpha} \prec_\exists L_{\omega_1}$. ∎

THEOREM 4.8. *The width of 0 is ω_1.*

Proof. Otherwise the width of 0 is $\leqslant \omega$ and so by Lemmas 4.6 and 4.7 there must be $n \in \omega$ such that all the complete extensions with only non-standard models are in L_{δ_n} (and so they are Δ_n^1). Now consider our theory T_{Spec_n}. The theory T_{Spec_n} + "There exists a derivation of $0 = 1$ from T_n" has no standard model and cannot be in L_{δ_n} being a complete Σ_n^1 set. By standard reasoning this theory is consistent. ∎

Now we start the general attack on arbitrary β in $\operatorname{Sp} L$. What we need to construct is a – cofinal in ω_1 – sequence of theories each realizing β as its type. The sequence of such theories will be constructed by induction. The basis of induction being that at least one such theory exists is given by the fact that $\beta \in \operatorname{Sp} L$.

The theories we are going to produce will be ordered by $<_L$ in type ω_1. Thus speaking of the order type of theories we mean their ordering under $<_L$. Our inductive assumption looks like this:

There is a sequence $\langle Q_\eta \colon \eta < \alpha \rangle$ of theories such that $I(Q_\eta) = \beta$ for all $\eta < \alpha$.

Now, α is denumerable and the sequence $\langle Q_\eta\colon \eta < \alpha\rangle$ is hereditarily countable. Consider the following class X of models of $\mathrm{ZF}+V=L$: $L_{\theta_\xi} \in X \Leftrightarrow \alpha$ is denumerable and definable in $L_{\theta_\xi} \& (\eta)_\alpha (Q_\eta$ is definable in $L_{\theta_\xi}) \& \langle Q_\eta\colon \eta < \alpha\rangle$ is definable in L_{θ_ξ}. (Actually the clauses 1 and 3 imply 2.) Remember that $Q_\eta \subseteq \omega$ and so we have:

LEMMA 4.9. *The class X is closed with respect to elementary equivalence of constructible levels.*

Proof. By 2.1 since X consists of those models in which all elements under consideration are both definable and (hereditarily) countable. ■

For each Q_η, S_η is the full sequence of transitive models of Q_η. Let ζ_η be the supremum of heights of terms of S_η and finally $\lambda = \bigcup\{\zeta_\eta\colon \eta < \alpha\}$.

We are going to produce Q_α as the $<_L$-first complete theory with the following properties (A), (B) and (C).

(A) $\mathrm{Mod}(Q_\alpha) \subseteq X$ (meaning transitive models of Q),

(B) $I(Q_\alpha) \geqslant \beta$,

(C) Q_α has no more than β models in $L_{\lambda+1}$.

For a moment our main task is to show that a theory satisfying (A), (B) and (C) exists at all. Afterwards we show that its type is β.

The first step is to show that for each $L_{\theta_\alpha} \in X$, $X \cap L_{\theta_\alpha}$ is definable in L_{θ_α}. Indeed, once the elements under consideration are definable in L_{θ_α} we can easily write that they are definable and denumerable somewhere. The details are easy to fill up and so we have:

LEMMA 4.10. *For each $x \in X$, $x \cap X$ is definable in x. This definition may be chosen uniformly for each "elementary" class included in X.* ■

Note that by 4.9 an elementary class is disjoint from X or all its transitive elements are in X.

It is clear that we cannot hope to have one uniform definition for X itself.

Now define $X^1 = X \cap \{L_{\theta_\xi}\colon I(Q_\xi) \geqslant \beta\}$. X^1 is definitely non-empty as for given $z \in \mathrm{HC}$ (in our case $\{\alpha, \langle Q_\eta\colon \eta < \alpha\rangle\}$) there is a theory T such that $I(T) = On$ and z is definable and denumerable in all models of T. This follows by reasoning used to prove that width of On is ω_1.

Once we know that, consider θ_ϱ the least ordinal number such that: $\theta_\varrho > \lambda$ and $L_{\theta_\varrho} \in X^1$.

Let $T = \mathrm{Th}(L_{\theta_\varrho})$; we show that T satisfies conditions (A), (B) and (C). (A) and (B) are obvious by construction. Consider L_{θ_ϱ}. The following – for a moment informal – statement Φ is true in L_{θ_ϱ}: $(\eta)_\alpha (I(Q_\alpha) = I(Q_0) \&$ "Above λ there is no element of X^1").

Now, β being definable in L_{θ_ϱ} (as $I(Q_0)$ – remember Q_0 is definable in L_{θ_ϱ} by construction) we see that X^1 (or more precisely $X^1 \cap L_{\theta_\varrho}$) is definable in L_{θ_ϱ}. Using the definition of $\langle Q_\eta\colon \eta < \alpha\rangle$ in L_{θ_ϱ}, we are able to produce a definition of λ in L_{θ_ϱ}. (Let $\Psi(\cdot)$ be this definition.) Thus we see that the statement Φ can be formalized. Thus we have $L_{\theta_\varrho} \models \Phi$ and so any model of $\mathrm{Th}(L_{\theta_\varrho})$ also satisfies Φ.

We know that the theory T has at least β models and let us remind that by our standard convention θ_ξ^T is the height of ξth model of T.

Let $\lambda_\xi^{Q_\eta}$ be supremum of heights of first ξ models of Q_η.

A picture could (?) help:

$$\begin{array}{lllll}
 & Q_0 & Q_1 & & Q_\eta \ \cdots \quad (\eta < \alpha) \\
0 & L_{\theta_0^{Q_0}} & L_{\theta_0^{Q_1}} & & L_{\theta_0^{Q_\eta}} \\
1 & & & & \\
2 & & & & \\
\vdots & & & & \\
\xi & L_{\theta_\xi^{Q_0}} & L_{\theta_\xi^{Q_1}} & & L_{\theta_\xi^{Q_\eta}} \\
\vdots & & & & \\
(\xi < \beta) & & & &
\end{array}$$

Thus we have matrix $\beta \times \alpha$. In the ηth column stand the consecutive models for Q_η, i.e. S_η.

The ξth row consists of ξth models of $Q_0, Q_1, \ldots, Q_\eta$ $(\eta < \alpha)$.

Set $\lambda_\xi = \bigcup_{\eta < \alpha} \lambda_\xi^{Q_\eta}$. λ_ξ is the supremum of heights of all models which appear in our matrix before the ξth row. In this way, λ is nothing else but λ_β.

In order to show that T satisfies (C) it is enought to prove that $\lambda+1 < \theta_\beta^T$ (since between $\lambda+1$ and θ_β^T there is no model of a theory having at least β models).

At this point we show by induction on ξ $(\xi \leqslant \beta)$ the following

LEMMA 4.11. $\lambda_\xi < \theta_\xi^T$.

Proof. $\xi = 0$. Since every Q_η is definable in T, Q_η is in every model of T. But then the least transitive model of Q_η (which is a contraction of a relation which is arithmetic in Q_η) belongs to the least transitive model of T. Thus $\lambda_0 \leqslant \theta_0^T$ but, as $\langle Q_\eta \colon \eta < \alpha \rangle$ is definable in T, λ_0 must be strictly less than θ_0^T (we use replacement here).

ξ is limit and our inductive assumption is: for all $\gamma < \xi$, $\lambda_\gamma < \theta_\gamma^T$. Then clearly $\lambda_\xi \leqslant \theta_\xi^T$. But $L_{\theta_\varrho} \models T$ and $L_{\theta_\varrho} \models Ex\Psi$. (Remember that Ψ was a definition of λ in L_{θ_ϱ}.) Thus $L_{\theta_\gamma^T}\ T \models (Ex)\Psi$.

But the object defined by Ψ in $L_{\theta_\xi^T}$ is nothing else but λ_ξ. Thus $\lambda_\xi < \theta_\xi^T$.

Finally we consider the case where $\xi = \gamma+1$ for some γ. It is enough just as in the case of ξ limit to show that

$$\lambda_{\gamma+1} \leqslant \theta_{\gamma+1}^T.$$

So by way of contradiction assume that $\theta^T_{\gamma+1} < \lambda_{\gamma+1}$. Then we have $\lambda_\gamma < \theta^T_\gamma < \theta^T_{\gamma+1} < \lambda_{\gamma+1}$. But $L_{\theta^T_{\gamma+1}} \models I(Q_\eta) = I(Q_0)$ for all $\eta < \alpha$ and since $\lambda_{\gamma+1}$ is the supremum of the heights of models of Q_η of length $\gamma+1$ (all $\eta < \alpha$), therefore there must be $\eta < \alpha$ such that the sequence of models of Q_η below $\theta^T_{\gamma+1}$ has length at most γ. Thus *all* the sequences of models for consecutive Q_η have length γ. Thus in $L_{\theta^T_{\gamma+1}}$ the formula Ψ again defines λ_γ. Now the contradiction is immediate since in $L_{\theta^T_{\gamma+1}}$ there is a theory with $\gamma+1$ models which has a model above λ_γ (it is T itself). Thus $L_{\theta^T_{\gamma+1}} \models \neg\Phi$. A contradiction. ∎

Thus we get a candidate for Q_α, the $<_L$-first complete extension of ZF making (A), (B), (C) true. Call it Q_α which is reasonable in view of the following:

LEMMA 4.12. $I(Q_\alpha) = \beta$.

Proof. By choice of β, $I(Q_\alpha) \geqslant \beta$. Remember that α is definable and countable in every model of Q_α and same for $\langle Q_\eta \colon \eta < \alpha\rangle$. So let us consider the sequence $\langle L_{\theta^{Q_\alpha}_\xi} \colon \xi < \beta\rangle$ and show it cannot be extended. Otherwise $\theta^{Q_\alpha}_\beta$ exists. First consider $\bigcup_{\xi<\beta} \theta^{Q_\alpha}_\xi$. There are two cases: β non-limit and β limit.

If β *is limit*, set $\nu = \bigcup_{\xi<\beta} \theta^{Q_\alpha}_\xi$. We show that $L_\nu \not\models Q_\alpha$. Indeed, if so then β is definable in L_ν (as $I(Q_0)$) and the replacement would be violated in L_ν. Thus $\nu < \theta^{Q_\alpha}_\beta$. In particular, $\lambda < \theta^{Q_\alpha}_\beta$ and so $\theta^{Q_\alpha}_\beta > \lambda+1$. Now we simply show that Q_α is definable in $L_{\theta^{Q_\alpha}_\beta}$. Indeed, the following defines Q_α in $L_{\theta^{Q_\alpha}_\beta}$: "$<_L$-first theory satisfying (A), (B) and (C)". This is a clear contradiction.

If β *is not limit* we proceed similarly; we leave this case to the reader. ∎

All our work until now is summed up in the following:

THEOREM 4.13. *Given $\beta \in \operatorname{Sp} L$, the width of β is ω_1.*

Proof. We have shown that in the process of construction any denumerable sequence of theories with the type β can be prolonged. Thus the sequence of theories having β models is cofinal in ω_1 and so has the type ω_1. ∎

The rest of this section will be devoted to the analysis of the above proof. First let us note the following

THEOREM 4.14. *If $\beta+1 \in \operatorname{Sp} L$, then $\beta \in \operatorname{Sp} L$.*

Proof. Let T be such that $I(T) = \beta+1$.

Let X be the class of those L_{θ_ν} in which T is definable. Clearly, X is closed under elementary equivalence and "$x \in X$" is definable in every element of X. Now as in the proof of Theorem 4.13 pick $<_L$-first T^1 such that:

$$I(T^1) = \beta+1, \quad \operatorname{Mod}(T^1) \subseteq X$$

(meaning transitive models of T^1). Consider $\theta_\beta^{T^1}$, i.e. $(\beta+1)$st model of T^1. T^1 is definable in every model L_{θ_ξ} whenever $L_{\theta_\xi} \in X$, $\theta_\xi > \theta_\beta^{T^1}$. Thus β, being definable from $\beta+1 = I(T^1)$, is also definable there.

Thus the following can be written in L_{ST} and is satisfied in $L_{\theta_\beta^{T^1}}$:

$I(T^1) = \beta$ & "all models of T^1 are in X". In particular, $L_{\theta_\beta^{T^1}} \models (\text{Ŀ}T^2)\ (I(T^2) = \beta$ & "all the models of T^2 are in X").

But look: T^1 is not definable in $L_{\theta_\beta^{T^1}}$ and so the $<_L$-first object T_0 making the above existential formula true is not T^1. T_0 has β models within $L_{\theta_\beta^{T^1}}$. We show that it cannot be prolonged. Otherwise there is $\theta_\xi > \theta_\beta^{T^1}$ such that $L_{\theta_\xi} \models T_0$. (Note that $\theta_\xi = \theta_\beta^{T^1}$ is automatically excluded as $T_0 \neq T^1$.)

Now since some models of T_0 are in X, all of them are in X and so, in L_{θ_ξ}: T is definable, $\beta+1$ is definable, β is definable, T^1 is definable and finally $\theta_{\beta+1}^{T^1}$ is definable so T_0 is definable.

So T_0 is definable in L_{θ_ξ}, so $L_{\theta_\xi} \not\models T_0$. A contradiction. ∎

In view of the fact that $\operatorname{Sp} L$ is not a segment Theorem 4.14 says that every gap must finish in a limit place. Actually by the same method stronger results could be obtained, as we will see later. Also a gap is $\operatorname{Sp} L$ cannot be finite.

The reasoning of our Theorem 4.13 gives another result:

Namely, define $\operatorname{iSp} L$ (incomplete spectrum of L) as follows: We extend $I(T)$ to, possibly incomplete, $T \supseteq \mathrm{ZF}$ and put $\operatorname{iSp} L = \{I(T);\ T \in L\}$. One would suspect that $\operatorname{iSp} L$ is larger than $\operatorname{Sp} L$ but it is not the case;

THEOREM 4.15. $\operatorname{iSp} L = \operatorname{Sp} L$.

Proof. Clearly, $\operatorname{Sp} L \subseteq \operatorname{iSp} L$. Let $\alpha \in \operatorname{iSp} L$. We want complete T such that $I(T) = \alpha$. So let T_0 be $<_L$-first theory with α models of form L_ξ. Let ϱ be supremum of the heights of models of T_0. The reader is right in his suspicion that we take X as $\{L_{\theta_\xi}\colon T_0$ is definable in $L_{\theta_\xi}\}$. As before X is closed under $\equiv$ (for transitive models).

Moreover, "$x \in X$" is definable in all elements of X. Now take as T the $<_L$-first complete extension of $\mathrm{ZF} + V = L$ such that

(a) Transitive models of T belong to X,

(b) $I(T) \geqslant \beta$,

(c) T has no more than β models in $L_{\varrho+1}$.

Actually by choice of ϱ, if (a), (b) are satisfied, then (c) is satisfied. Routine proof shows that T with properties (a), (b), (c) exists and that it has exactly β models. This completes the argument. ∎

Slight variation of the proof shows that we really do not use the fact that we deal with models of ZF. And in fact if T is a recursive theory, $\mathrm{KP} \subseteq T$, then we may deal as easily with $\operatorname{Sp}_T L$ (thus $\operatorname{Sp} L$ is $\operatorname{Sp}_{\mathrm{ZF}} L$).

If one works a bit the proof of the following may be obtained.

THEOREM 4.16. *If T is a recursive extension of* KP *which is true (what we use really is that they are arbitrarily large levels modelling T), then* $\mathrm{Sp}_T L = \mathrm{Sp}\, L$. ∎

But then once we realize this, a different thing comes to consideration: In the proof of the main result of this section we never used the fact that the function $\theta_\xi^{Q_n}$ is generated by ZF set theory. We just used it to produce the sequence of ordinals $\langle \lambda_\xi : \xi < \beta \rangle$. Once those ordinals were produced we produced a theory T containing ZF such that:

$$\lambda_\xi < \theta_\xi^T$$

this was in turn used to show that $I(T) = \beta$.

So let $f_a(\alpha) = \delta$ be a Δ_1 function defined on an initial segment of ordinals with hereditarily countable parameter a. (Think about f as, say, θ_α^T, where T is our parameter.)

Then by following the argument in the proof of Theorem 4.13 we can show that if β is domain of such function, then $\beta \in \mathrm{Sp}\, L$. Actually we can prove that there is a theory T such that its enumeration function majorizes $f_a(\xi)$ for $\xi \in \beta$. Thus we sketched the proof of the half of the following:

THEOREM 4.17. $\mathrm{Sp}\, L$ *is the class of ordinals being domains of Δ_1-functions from initial segments of On into On with real parameters.*

The proof of the other half is trivial as $\theta_{(\cdot)}^T$ is a Δ_1-function defined on initial segment of *On* into *On* in the parameter T. This result generalizes both Theorems 4.15 and 4.16. ∎

5. Non-uniform strong definability

In this section we discuss properties of the uncountable members of $\mathrm{Sp}\, L$. The methods used are an extension of those of Wilmers (see Section 3) and some reasonings of Section 4.

First we introduce a weaker form of the notion of a "good" class (introduced in Section 3). Let X be a class included in $\mathrm{Mod}_{\mathrm{ZF}}$, $P(\cdot)$ a property. We will say that P is *non-uniformly definable* for X iff for each $\alpha_\theta \in X$ there is a formula $\psi(\cdot)$ such that for any model $L_\nu \in X$, if $L_\nu \equiv L_\theta$, then ψ defines $P \cap L_\nu$ in L_ν. We shall say that a class X is *usable* iff the property "$x \in X$" is non-uniformly definable for X. Note that we encountered usable classes in Section 4.

Also we feel obliged to warn the reader that in the definition of P in elements of X the same formula, φ, defines P in elementarily equivalent models in X but different formulae may work in pairs of models which are not elementarily equivalent.

These concepts will be very useful, because we know (Section 2) that in elementarily equivalent models the same sets are definable and hereditarily countable (in the sense of the models); moreover, the same sets (heriditarily countable in the sense of the models) are defined by the same formulae.

Now suppose $a \in \mathrm{HC}$ (i.e. $a \in L_{\omega_1}$). Let X_a be the class of models of ZF in which a is both definable and hereditarily countable. Then using the above remarks the following is obvious:

LEMMA 5.1. (1) *If* $a \in \mathrm{HC}$, *then the class* X_a *is a usable class.*

(2) X_a *is closed under elementary equivalence.* ∎

The notation X_a will be used extensively in what follows, usually with a as a theory (i.e. set of natural numbers) or a sequence of theories.

The following will be useful:

If X is a class of models of ZF and $\alpha \in On$, $X(\alpha)$ denotes $\{L_\nu \in X: \nu \geqslant \alpha\}$.

In particular, $X_a(\alpha) = \{L_\nu: \nu \geqslant \alpha$ and a is definable and hereditarily countable in $L_\nu\}$.

Similarly as the strong definability was a variant of definability we introduce now the concept of a non-uniform strongly definable element for a class. (Before we dealt with strongly definable ordinals but as $V = L$ both notions boil down to the same thing.) We say that a set a is *non-uniformly strongly definable* for a class X (n.u.s.d. for X, for short) if for each $L_\theta \in X$ there is a formula $\varphi(\cdot)$ such that for any model $L_\nu \in X$ whenever $L_\nu \equiv L_\theta$, then if $a \in L_\nu$, then φ defines a in L_ν, otherwise $L_\nu \models \neg Ex\varphi(x)$.

This is the non-uniform analogue of the concept of strongly definable element. It is equivalent to say that a set a is n.u.s.d. for a class X iff a is strongly definable in every $Y \cap X$, where Y is an elementary class defined by a complete theory.

The following is again straight-forward:

LEMMA 5.2. (1) *If a set* a *is n.u.s.d. for* $X(\alpha)$ *and* $\beta > \alpha$, *then* a *is n.u.s.d. for* $X(\beta)$.

(2) *If* α *is n.u.s.d. for* X *and* X *is usable, then* $X(\alpha)$ *is usable.*

(3) *If* $a \in \mathrm{HC}$, *then* a *is n.u.s.d. for* X_a. ∎

At this moment we wish to obtain results for these new concepts similar to those of Wilmers (see Section 3). Suppose that X is a usable class and that we want to show that for an ordinal β there is a complete theory T (extending ZF) such that

$$I^X(T) = \beta.$$

Such a theory, if it exists, must clearly have at least β models in X. The following lemma shows that classes of the form X_a, where $a \in \mathrm{HC}$, always contain models of such a theory.

LEMMA 5.3. *Given X_a ($a \in \mathrm{HC}$), then there is a complete extension T of* ZF *such that:*

(A) *All models of T are in X_a;*

(B) *$I^{X_a}(T) \geqslant \beta$ (and hence $I(T) \geqslant \beta$ by Lemma* 5.1(2)).

Proof. It was shown (in Section 4) that there are ω_1 distinct theories for which $I(T) = On$. The heights of the minimal models of these theories are cofinal in ω_1. It is therefore possible to find a theory in whose minimal model a is both definable and hereditarily countable. (Simply by choosing a theory whose minimal model contains both a and an injection of a into ω.) Such a theory satisfies both (A) and (B). ∎

Now we can prove a result analogous to the theorem of Wilmers (see Section 3).

THEOREM 5.4. *Suppose the ordinals α and β are n.u.s.d. for a class $X_a(\alpha)$ and suppose there is a complete extension T of* ZF *whose models are in X_a and which satisfies:*

(A) *All models of T are in X_a.*

(B) *$I(T) \geqslant \beta$.*

(C) *There are no more than β models of T in L_α.*

Then $\beta \in \mathrm{Sp}\, L$.

Proof. For each complete extension T of ZF satisfying (A), (B) and (C), let ϱ_β^T be the supremum of the ordinals of the first β models of T.

Let ϱ_β denote the least of such ordinals, i.e.

$$(*) \qquad \varrho_\beta = \bigcap \{\varrho_\nu^T\colon \text{models of } T \text{ are in } X_a \text{ and } I(T) \geqslant \beta \text{ and } T \text{ has no more than } \beta \text{ models in } L_\alpha\}.$$

Now let $\lambda = \max(\alpha, \varrho_\beta)$ and consider the class $X_a(\lambda)$. Since $\lambda \geqslant \alpha$ and α is n.u.s.d. for $X_a(\alpha)$, α is n.u.s.d. for $X_a(\lambda)$ (by Lemma 5.2). Similarly, since $\beta \leqslant \varrho_\beta$ (obviously) and $\varrho_\beta \leqslant \lambda$, β is n.u.s.d. for $X_a(\lambda)$. Further, a is n.u.s.d. for $X_a(\lambda)$ (again by Lemma 5.2) and so $X_a(\lambda)$ is usable class.

Thus, given definitions of α, β and a, it is clear that $(*)$ can be formalized and since the same definition for α, β, a work in pairs of elementarily equivalent models from $X_a(\alpha)$, therefore the same formula defines ϱ_β is every such pair of models. That means that ϱ_β is n.u.s.d. for $X_a(\lambda)$ and therefore λ is also n.u.s.d. for $X_a(\lambda)$.

Now let Q denote the $<_L$-first complete theory satisfying (A), (B), (C) and the following:

(D) *The supremum of heights of the first β models of Q is ϱ_β.*

The conditions of the theorem and the definition of ϱ_β ensure that Q exists. Just as in the case of our proof of 3.1 we claim that

$$I(Q) = \beta.$$

We already know that $I(Q) \geqslant \beta$ and that Q has no more than β models in L_a, and hence it must have exactly β models in L_λ. (By the definition of L_λ.) Now consider $X_a(\lambda)$.

Q has no model in this class for as we have seen a, β, ϱ_β, and finally λ are n.u.s.d. for $X(\lambda)$. In particular, therefore, if $L_\xi \in X_a(\lambda)$, then a, α, β, ϱ_β, λ are definable in L_ξ and thus so is the theory Q. Thus $Q \neq \mathrm{Th}(L_\xi)$ (see Section 0).

So Q has no models in $X_a(\lambda)$ and exactly β models in L_λ. As $\mathrm{Mod}(Q) \subseteq X_a$, this means that $I(Q) = \beta$. ∎

The above result will be used constantly below. We have already shown that $\mathrm{Sp}\,L$ does not consists of an initial segment of the ordinals but that it contains all the countable ordinals. Let δ denote the first ordinal not in $\mathrm{Sp}\,L$. Since $|\mathrm{Sp}\,L| = \omega_1$, δ also has cardinality ω_1. The following gives further information about this ordinal δ.

THEOREM 5.5. *δ is stable ordinal.*

Proof. Assume δ is not stable. Then, since the enumeration function of stable ordinals is continuous, therefore there is the largest stable ordinal, δ^- say, below δ. As δ is not stable it must be Σ_1 definable (in L) from δ^- and parameters in δ^- (see [3], Theorem 4.4). That is, for some Δ_0 formula φ:

$$(*) \qquad x = \delta \Leftrightarrow (Ey)\varphi(x, y, \alpha_1, \ldots, \alpha_n, \delta^-),$$

where $\alpha_1, \ldots, \alpha_n$ are in δ^-. By the definition of δ, $\delta^-, \alpha_1, \ldots, \alpha_n \in \mathrm{Sp}\,L$ and so there are theories $T_0, \ldots, T_n$ such that:

$$I(T_i) = \alpha_i \quad (i = 1, \ldots, n), \qquad I(T_0) = \delta^- .$$

Let $S = \langle T_0, \ldots, T_n\rangle$ and consider the class $X_S(\delta^-)$.

The sequence, S, and hence the theories $T_0, \ldots, T_n$ are all definable in each model in this class. Moreover, by Proposition 0.7 all the models of $T_0, \ldots, T_n$ are in L_{δ^-}. If follows that $\delta^-, \alpha_1, \ldots, \alpha_n$ are all n.u.s.d. for $X_S(\delta^-)$ (the definition of α_i is $I(T_i)$, the definition of δ^- is slightly more complicated).

Now pick the $<_L$-first y making $(*)$ true, call it y_0. Thus $\varphi(\delta, y_0, \alpha_1, \ldots, \alpha_n, \delta^-)$.

Consider the class $X^1 = \{L_\nu \in X_S(\delta^-) \colon \delta, y_0 \in L_\nu\}$. As the formula φ is Δ_0 it is easy to see that X^1 is a usable class. Now let T be the $<_L$-first complete theory satisfying:

(A) *Some models of T are in X^1.*

(B) $I(T) \geqslant \delta$.

(C) *T has no more than δ models in L_α, where α is* $\mathrm{Mod}_L(y_0)$.

We need to show that T exists. By Theorem 5.4 there is a theory satisfying (A) and (B). Any theory which satisfies (A) and (B) must also satisfy (C). For suppose a theory T satisfies (A) and (B); then T has a model L_ξ containing

y_0. Also, by (A), $L_\xi \in X_S(\delta^-)$ and $\alpha_1, \ldots, \alpha_n$, δ^- are definable in L_ξ, by formulas $\varphi_1, \ldots, \varphi_n$, φ_0 say. Then $L_\xi \models (Ex_0, \ldots, x_n)\,(\bigwedge\!\!\bigwedge \varphi_i(x_i) \,\&\, (Ey, y_2)\, \varphi(y_1 y_2, \vec{x}))$, where φ is the formula from (*).

This sentence must be true in all models of T; suppose that L_η is a model of T with $\eta > \delta^-$. Then $\varphi_0 \ldots \varphi_n$ define δ^-, $\alpha_1, \ldots, \alpha_n$ in L_η so

$$L_\eta \models (Ey_1 y_2)\, \varphi(y_1 y_2, \vec{\alpha}, \delta^-).$$

Thus $y_0 \in L_\eta$ and so $\eta > \alpha$.

Using Proposition 0.7, we see that T has no more than δ^- models in L_α and (C) is satisfied (as $\delta^- < \delta$). As in the preceding theorem we can easily show that $I(T) = \delta$ contradicting the definition of δ. Thus δ must be stable. ∎

Similar reasoning gives stronger result:

THEOREM 5.6. *δ is a critical point in the enumeration of stable ordinals.*

Proof. Assume not. Then δ is the αth stable ordinal for some $\alpha < \delta$. But then $\alpha \in \operatorname{Sp} L$ and so let T be a theory such that $I(T) = \alpha$. Then, since all models of T are in L_δ (by Proposition 0.7), α is n.u.s.d. for the class $X_T(\delta)$ (the definition is "$I(T)$"). Moreover, the set $\{\xi\colon \xi \text{ is stable} \,\&\, \xi \leqslant \delta\}$ is absolute for this class (see Lemma 4.11 of [3] and the fact that if $L_\alpha \prec_1 L$, then for all $\beta > \alpha$, $L_\alpha \prec_1 L_\beta$).

But then δ is n.u.s.d. for $X_T(\delta)$ (the definition is "αth stable ordinal") So, by Theorem 4, $\delta \in \operatorname{Sp} L$ contradicting the definition of δ. (Note that conditions (A), (B), (C) are satisfied by virtue of Proposition 0.7 and Lemma 5.2.) Thus δ is δth stable ordinal. ∎

The length of the first gap in $\operatorname{Sp} L$ may be now estimated: Let δ^* the first stable ordinal above δ.

THEOREM 5.7. $\delta^* \in \operatorname{Sp} L$.

Proof. We note that the class of stable ordinals below given stable ordinal τ is absolute for models of ZF containing τ. Thus the formula "x is the least ordinal not in $\operatorname{Sp} L$" strongly defines δ for the class $\operatorname{Mod}_{ZF}(\delta^*)$. Thus δ^* is also strongly definable (and hence n.u.s.d.) for this class – it is defined as the first stable ordinal after δ. Hence, by Theorem 5.4, $\delta^* \in \operatorname{Sp} L$. ∎

Note that $L_{\delta^*} \not\models$ ZF.

The methods above can also be used to investigate some closure properties of $\operatorname{Sp} L$. Although Theorem 4.17 implies most of them we will still prove them directly using Theorem 4.13.

Suppose that $P(\cdot)$ is a Π_1-property of ordinals such that:

(i) $\{\varkappa\colon P(\varkappa)\}$ is unbounded,

(ii) $(\varkappa)(P(\varkappa) \Rightarrow \varkappa$ is stable).

Examples of such P can be given now; for instance, the weakest such property is "$\varkappa$ is stable" which is Π_1 (see footnote (3), p. 187 of [3]). But, for instance: "$\varkappa$ is cardinal", "$\varkappa$ is solution of the equation $\omega_\alpha = \alpha$", "$\varkappa$ is inaccessible cardinal" (under appropriate assumptions) may serve as examples of such properties.

For each ordinal α, let $P^+(\alpha)$ be the least ordinal above α with the property P. We have the following:

THEOREM 5.8. *If P is Π_1 and has properties* (i) *and* (ii) *above, then whenever $\beta \in \mathrm{Sp}\, L$, $P^+(\beta) \in \mathrm{Sp}\, L$.*

Proof. If – by chance – $P^+(\beta)$ is denumerable, then there is nothing to prove (by Corollary 3.9). So assume $P^+(\beta) \geqslant \omega_1$. Let T be such that

$$I(T) = \beta.$$

Then β is n.u.s.d. for the class $X_T(P^+(\beta))$ (as usual is defined by "$I(T)$"). Further, since $P(\beta)$ is Π_1 and $P^+(\beta)$ is stable, the class:

$$\{\varkappa\colon P(\varkappa) \wedge \varkappa \leqslant P^+(\beta)\}$$

is absolute for $X_T(P^+(\beta))$. Hence $P^+(\beta)$ is n.u.s.d. for $X_T(P^+(\beta))$ (again as in the proof of Theorem 7.7 we have the following definition: "The first ordinal with the property $P(\cdot)$ above β"). So once again we apply Theorem 5.3 to show $P^+(\beta) \in \mathrm{Sp}\, L$ (again we use Proposition 0.7 and Lemma 5.2 to check (A), (B) and (C)). ■

Looking now at the Π_1 properties exhibited above, we see immediately the following:

COROLLARY 5.9. $\mathrm{Sp}\, L$ *is closed under*:

(a) *Successor stable ordinal*;

(b) *Successor cardinal*;

(c) *Successor solution of the equation $\omega_\xi = \xi$*;

(d) *Successor inaccessible cardinal (provided it exists).*

In fact, if P is Π_1 property satisfying (i) *and* (ii) *and P^α is the enumeration of the ordinals with the property $P(\cdot)$, then whenever $\alpha \in \mathrm{Sp}\, L$, $P^\alpha \in \mathrm{Sp}\, L$. Thus if $\alpha \in \mathrm{Sp}\, L$, α-th stable ordinal, α-th cardinal, etc. are in* $\mathrm{Sp}\, L$.

Now we could use Theorem 4.17 to prove "ordinary" closure properties of $\mathrm{Sp}\, L$. The proof of the theorem below shows this directly.

We say that n-ary function $f\colon On^n \to On$ *is non-decreasing* iff for all $\alpha_1, \ldots, \alpha_n$,

$$\max(\alpha_1, \ldots, \alpha_n) \leqslant f(\alpha_1, \ldots, \alpha_n).$$

Also f is non-uniformly definable for a class X iff the predicate $f(\alpha_1, \ldots, \alpha_n) = \beta$ is non-uniformly definable for X.

THEOREM 5.10. *Suppose $\alpha_1, \ldots, \alpha_n \in \mathrm{Sp}\, L$ and $T_1, \ldots, T_n$ are such that $\alpha_i = I(T_i)$ $(i = 1, \ldots, n)$. If f is non-decreasing, non-uniformly definable for $X_{\langle T_1, \ldots, T_n \rangle}$ and $f(\alpha_1, \ldots, \alpha_n) = \beta$, then $\beta \in \mathrm{Sp}\, L$.*

Proof. Again we will reduce it to Theorem 5.3.

Let $\varrho_i = \bigcup\{\xi\colon L_\xi \models T_i\}$, $\varrho = \bigcup_{i=1}^{n} \varrho_i$.

Consider $X_{\langle T_1,\ldots,T_n\rangle}(\varrho)$. $\alpha_1, \ldots, \alpha_n$ are n.u.s.d. for $X_{\langle T_1,\ldots,T_n\rangle}(\varrho)$ and thus so are $\varrho_1, \ldots, \varrho_n$ and finally so is ϱ. Take $\gamma = \max(\beta, \varrho)$. Then both β and ϱ are n.u.s.d. for $X_{\langle T_1,\ldots,T_n\rangle}(\gamma)$. (Since f is non-uniformly definable for this class.) So γ is n.u.s.d. for $X_{\langle T_1,\ldots,T_n\rangle}(\gamma)$. We now wish to appeal to Theorem 5.3 to show $\beta \in \mathrm{Sp}\,L$ but we need to show that there is a theory satisfying conditions (A), (B) and (C). This can be done: Lemma 5.2 takes care of (A) and (B) and the proof that there is a theory satisfying (C) also follows the lines of a similar proof given in Theorem 4.8. We leave the details to the reader. Thus $\beta \in \mathrm{Sp}\,L$. ∎

COROLLARY 5.11. *$\mathrm{Sp}\,L$ is closed under ordinal addition, multiplication, exponentiation, etc.* ∎

6. Solution to a problem of Wilmers

In [7], G. Wilmers, while considering the ordinal μ, the first non-strongly-definable ordinal (in ZF) asks whether $L_\mu \models \mathrm{ZF}$. He shows simultaneously that

$$\mu = \bigcup_{\xi<\mu} \theta_\mu .$$

We settle the problem in negative showing that

$$\mu = \delta_2^{L_\xi}, \quad \text{where } \xi = \theta_\mu .$$

We have first the following:

LEMMA 6.1. *If η is strongly definable in* ZF, *then η is denumerable in $L_{\theta_\eta+2}$.*

Proof. Let $\psi_\eta(\cdot)$ strongly define η in ZF. Since $\eta < \theta_{\eta+1}$, therefore $L_{\theta_{\eta+2}}$ satisfies "There is a denumerable model of $\mathrm{ZF}+V=L+(Ex)\psi_\eta$" (as $L_{\theta_{\eta+1}} \in L_{\theta_{\eta+2}}$ and Skolem–Löwenheim holds in $L_{\theta_{\eta+2}}$). By strong definability of η, η is denumerable in $L_{\theta_{\eta+2}}$. ∎

The ordinal μ is limit since $\mu = \bigcup_{\xi<\mu} \theta_\xi$ and all ordinals on the R.H.S. are limit. Also if $\eta < \mu$, then $\theta_\eta < \mu$ and $\theta_{\eta+2} < \mu$. Thus we have:

LEMMA 6.2. $L_\mu \models V = \mathrm{HC}$.

Proof. Given $\eta < \mu$, then $\theta_{\eta+2} < \mu$ and since η is denumerable in $L_{\theta_{\eta+2}}$, it is denumerable in L_μ as well. ∎

This settles the original question of Wilmers since no model of ZF can satisfy $V = \mathrm{HC}$.

In order to prove the promised equality we discuss the properties of strongly definable ordinals.

LEMMA 6.3. *If* $\xi < \mu$, *then* $L_{\theta_\mu} \models$ "ξ *is strongly definable in* ZF + $V = L$" *and so* $L_{\theta_\mu} \models$ "$\xi < \delta_2$".

Proof. If $\psi_\xi(\cdot)$ strongly defines ξ in ZF, then the same formula serves as strong definition for those transitive models of ZF + $V = L$ which are inside of L_{θ_μ}. Thus in L_{θ_μ}, ξ is strongly definable in recursive theory so by Theorem 0.8 (which was proved in ZF and so holds in L_{θ_μ}), $L_{\theta_\mu} \models$ "$\xi < \delta_2$". ∎

The following is an important property of δ_2:

LEMMA 6.4. *If* $\alpha < \delta_2$, *then there is* $\beta < \delta_2$ *such that for all* $\gamma > \beta$, $L_\gamma \not\equiv L_\alpha$.

Proof. Since $\alpha < \delta_2$, there is Δ_0 formula ψ such that $(Ex)\psi(x, \cdot)$ defines α. Since $(Ex)\psi(x, \alpha)$, there must be x in L_{δ_2} such that $\psi(x, \alpha)$. Let β' be least such that there is $x \in L_{\beta'+1} - L_{\beta'}$ for which $\psi(x, \alpha)$ and finally let $\beta = \max(\beta', \alpha)$. For any $\gamma > \beta$, $L_\gamma \models (Ey)(Ex)\psi(x, y)$. So it is enough to show that $L_\alpha \models \neg(Ey)(Ex)\psi(x, y)$. Indeed, if L_α satisfies $(Ey)(Ex)\psi(x, y)$, then for some y in L_α $(Ex)\psi(x, y)$ contradicting the fact that $(Ex)\psi(x, \cdot)$ defines α. ∎

COROLLARY 6.5. *If* T *is complete*, $T \in L_{\delta_2}$, *then* $I(T) < \delta_2$ *and the complete sequence for* T, S_T *also belongs to* L_{δ_2}.

Proof. If $T \in L_{\delta_2}$, then if T has a transitive model, then it has one in L_{δ_2}. By Lemma 6.4 *all* the models of T must be in L_{δ_2} and they are bounded below δ_2. By Δ_1-separation $S_T \in L_{\delta_2}$. ∎

The fact that $T \in L_{\delta_2}$ is equivalent to: $T \in \Delta^1_2$. Thus we proved in Corollary 6.5 that complete Δ^1_2 theories have their complete sequences in L_{δ_2}. One could suspect that a similar phenomenon occurs for $n > 2$ as well. This is not true.

THEOREM 6.6. *There is a complete* Δ^1_4 *theory* T *extending* ZF + $V = L$ *such that* $\{\alpha\colon L_\alpha \models T\}$ *is unbounded in* L_{ω_1}.

Proof. Indeed, there is a complete theory T such that models of T are unbounded in L_{ω_1}. Thus for some complete T,

$$L_{\omega_1} \models (\alpha)(E\beta)(\beta > \alpha \,\&\, L_\beta \models T).$$

The formula after the satisfaction sign is Π_2 and can be easily written down as Π^1_3 formula. Thus there is Δ^1_4 T making it true. ∎

The theory constructed in Theorem 6.6 does not satisfy us completely, because we do not know if a Δ^1_3 theory T (which is unbounded in L_{ω_1}) can be found. Again let us notice that the complete sequence for T, S_T, has the length ω_1 (just another proof that $\omega_1 \in \mathrm{Sp}\, L$), because we have:

THEOREM 6.7. *If* $\alpha \geq \omega_1$, *then* $\mathrm{Th}(L_\alpha)$ *is not analytical* (*i.e. it does not belong to* L_{δ_r}).

Proof. $\mathrm{Th}(L_{\omega_1}) \notin L_{\delta_\chi}$ since $L_{\delta_\chi} \dashv\exists L_{\omega_1}$ and L_δ is pointwise definable. Assume now $\alpha > \omega_1$. Then ω_1 is definable in L_α (in fact uniformly; namely as the least non-denumerable ordinal). Thus $\mathrm{Th}(L_{\omega_1})$ is recursive in $\mathrm{Th}(L_\alpha)$. As L_{δ_χ} is closed under relative recursivity we are done. ∎

Interesting (?) fact is that by the reasoning as above, for the Δ^1_4 theory T constructed in the proof of Theorem 6, δ_χ is not definable in the models of T.

We come back to the problem of the least non-strongly-definable ordinal (in ZF).

We know now that if $\xi < \mu$, then $\xi < \delta_2^{L_{\theta_\mu}}$.

Thus $\mu \leqslant \delta_2^{L_{\theta_\mu}}$. (Note that μ is anyway Σ_1 definable in L – since it is in L_{δ_2} – but the Σ_1 definition need not work in a given model of ZF.) If $\mu < \delta_2^{L_{\theta_\mu}}$, then, in L_{θ_μ}, μ itself is Σ_1-definable. Let $(Ex)\psi(x, \cdot)$ be a Σ_1 definition of μ in L_{θ_μ}. Then the least transitive model of $(Ey)(Ex)\psi(x, y) + V = L$ is L_ξ for some $\xi > \mu$ but $\xi < \theta_\mu$. But μ is the supremum of the heights of transitive models of ZF in L_{θ_μ} (by the definition of θ_μ and the fact that $\mu = \bigcup_{\nu<\mu} \theta_\nu$). Therefore μ is also the supremum of heights of models of ZF in L_ξ. Now we simply produce the following strong definition Ξ of μ (we give it in English but it can be formalized).

"z is the supremum of heights of transitive models of $\mathrm{ZF} + V = L$ in the least transitive model of $\mathrm{KP} + V = L + (Ey)(Ex)\psi(x, y)$."

Since below μ there is no model of $(Ey)(Ex)\psi(x, y)$ and above μ in any model of $\mathrm{ZF}(E!\ z)\Xi(z)$ and $\Xi(\mu)$, we produced a strong definition of μ in ZF, contradiction. Thus we proved the following:

THEOREM 6.8. $\mu = \delta_2^{L_{\theta_\mu}}$. ∎

Even though μ is not strongly definable, θ_μ is in fact strongly definable. We show this using the following:

LEMMA 6.9. *If α is strongly definable in* ZF, *$\alpha < \mu$, then θ_α is Δ^1_2-ordinal in $L_{\theta_{\alpha+1}}$ and so δ_2 of L_{θ_α} is less than δ_2 of $L_{\theta_{\alpha+1}}$.*

Proof. We consider two cases.

(a) $\alpha < \theta_\alpha$. Let $\psi_\alpha(\cdot)$ strongly define α in ZF. Thus $L_{\theta_\alpha} \models Ex\psi_\alpha(x)$. Thus in $L_{\theta_{\alpha+1}}$ there must be a model of ZF+"$Ex\psi_\alpha(x)$" below its δ_2. But δ_2 of $L_{\theta_{\alpha+1}}$ is stable in $L_{\theta_{\alpha+1}}$ (i.e. it is its 1-elementary submodel) so α is less than δ_2 of $L_{\theta_{\alpha+1}}$ and by stability, θ_α is less than that ordinal. Thus $\theta_\alpha < \delta_2$ of $L_{\theta_{\alpha+1}}$ and thus δ_2 of $L_{\theta_\alpha} < \delta_2$ of $L_{\theta_{\alpha+1}}$.

(b) $\alpha = \theta_\alpha$. Let $\xi < \alpha$ be given. As it is strongly definable, we may proceed as in case (a) and find a model of $\mathrm{ZF} + V = L + Ex\psi_\xi$ below δ_2 of $L_{\theta_{\alpha+1}}$. This shows that $\xi < \delta_2$ of $L_{\theta_{\alpha+1}}$ and so $\alpha \leqslant \delta_2$ of $L_{\theta_{\alpha+1}}$. But if $\alpha = \delta_2$ of $L_{\theta_{\alpha+1}}$, then we get the contradiction as $L_\alpha \models \mathrm{ZF}$ ($\alpha = \theta_\alpha$) and, on the other hand, ZF $\vdash$ "L_{δ_2} satisfies $V = \mathrm{HC}$". Thus $\theta_\alpha < \delta_2$ of $L_{\theta_{\alpha+1}}$ and as above δ_2 of $L_{\theta_\alpha} < \delta_2$ of $L_{\theta_{\alpha+1}}$. ∎

THEOREM 6.10. *θ_μ is strongly definable* (*in* ZF).

Proof. There are two cases possible:

(a) δ_2 of $L_{\theta_\mu} = \delta_2$ of $L_{\theta_{\mu+1}}$. Then, by Lemma 6.9, μ is least such! Writing a strong definition of θ_μ is tedious calygraphical exercise:

"There is ξ such that δ_2 of L_{θ_ξ} is equal to δ_2 of $L_{\theta_{\xi+1}}$ and our x is θ of least such or there is no such ξ and there is $L_\alpha \models$ ZF such that δ_2 of L_α is δ_2 and x is this α".

(b) δ_2 of $L_{\theta_\mu} < \delta_2$ of $L_{\theta_{\mu+1}}$. Then θ_μ is Σ_1 definable in $L_{\theta_{\mu+1}}$ but, obviously, not in L_{θ_μ}. Pick a Σ_1 definition ψ of θ_μ in $L_{\theta_{\mu+1}}$ and write:

"x is the height of largest transitive model of $\mathrm{ZF} + V = L + \neg(Ey)\psi(y)$".

Since ψ is Σ_1, that will do. ∎

Note that the obvious trial: "δ_2 of what we defined above" will not produce strong definition of μ. It will not work in L_{θ_μ} (where μ belongs).

Since μ is not strongly definable but θ_μ is, we have justified our claim that not all Δ^1_2-ordinals are strongly definable. The fact is that strongly definable ordinals (in ZF, but any other recursive, true theory works as well) are cofinal in δ_2. To see this one has to look at Σ_1 definitions of ordinals and see that if $(Ex)\psi(x, \cdot)$ is a definition of α, then $\Phi(\nu)$:

$$\text{"}(Ex)_{L_\nu}(Ey)_{L_\nu}\psi(x, y) \text{ and } \nu \text{ is least such"}$$

strongly defines an ordinal greater than α in ZF.

We finally look at the first model L_ν such that $I(\mathrm{Th}(L_\nu)) > 1$. Then $I(\mathrm{Th}(L_\nu)) = 2$ and it just means that for some $\alpha > \nu$, there is an elementary imbedding $\pi^{-1}\colon L_\nu \to L_\alpha$. The image of π^{-1} cannot be bounded in L_α. (Since it would imply $I(\mathrm{Th}(L_\alpha)) > 2$.)

Now by results of Section 1, π^{-1} is constant on elements of $L_{\omega_1^{L_\nu}}$. Thus L_α satisfies "There is a model of ZF containing ξ" (for each $\xi < \omega_1^{L_\nu}$) so the same holds in L_ν. However, if there would be, in L_ν, more than ω_1 models of $\mathrm{ZF} + V = L$, then at least two of them would be elementarily equivalent (use cardinality argument inside of L_ν) and so this would contradict the definition of L_ν. Thus in L_ν it is true that there is exactly ω_1 transitive models of $\mathrm{ZF} + V = L$. This shows that, contrary to a claim of Mostowski, $\nu \neq \theta_\nu$.

However, we have proved:

THEOREM 6.11. $\nu = \theta_{\omega_1^{L_\nu}}$. ∎

Using Theorem 6.11, one can estimate the size of the least η such that θ_η is not strongly definable; it is smaller that ν.

7. Supremum of spectrum of L

Informally we say that On belongs to the spectrum but obviously, as long as we are speaking about complete sequences which are sets, then the elements of spectrum are ordinals and their supremum is an ordinal. We have shown that $\alpha \in \mathrm{Sp}\, L \Rightarrow \omega_\alpha \in \mathrm{Sp}\, L$ and so $\varrho = \bigcup (\mathrm{Sp}\, L \cap V)$ is a cardinal. (We hope the reader forgives us the horrible formula $\mathrm{Sp}\, L \cap V$.) It was shown in [2] that its cofinality character is ω_1. For completeness sake we give here a (different) proof:

THEOREM 7.1. $cf(\varrho) = \omega_1$.

Proof. We just show that any map $f\colon \omega \to \varrho$ is bounded in ϱ. Since $\alpha \in \mathrm{Sp}\, L \Rightarrow \varrho_\alpha \in \mathrm{Sp}\, L$, we can assume that:

(a) for each $n \in \omega$, $f(n) \in \mathrm{Sp}\, L$;

(b) for each $n \in \omega$, $f(n)$ is stable.

Pick for each n, T_n a theory such that $I(T_n) = f(n)$. The sequence $U = \langle T_n\colon n \in \omega \rangle$ is in HC and so we consider the class $X = X_U$. Set $\delta = \bigcup_n f(n)$ and consider $X(\delta)$. δ is stable (as limit of stables) and n.u.s.d. for $X(\delta)$. Reasoning as many times before, we find $\delta \in \mathrm{Sp}\, L$. ■

Actually if we are slightly more efficient in writing suitable definitions we show that:

PROPOSITION 7.2. *$\mathrm{Sp}\, L$ is closed under denumerable limits.*

It was shown in [2] that ϱ is the supremum of Π_1- (and Σ_2-) definable cardinals. Here we give another model-theoretic characterization of ϱ.

In what follows the word "model" exceptionally means "relational structure".

A model $\langle \mathfrak{A}, < \rangle$ is partly well-ordered if $<$ is a well-ordering of its field. For instance, $\langle L_\alpha, \in, \in \restriction \alpha \rangle$ is partly well-ordered. Languages under consideration are denumerable thus every theory is a real.

We say that $\varkappa$ is *Hanf number for well-orderings* if for every theory T if T has partly well-ordered model of cardinality $\varkappa$, then it has partly well-ordered model in every cardinality.

Note that an elementary submodel of partly well-ordered model is again partly well-ordered.

THEOREM 7.3. *If T does not have partly well-ordered models in every cardinality, then there must be λ such that T has no partly well-ordered models of cardinality above λ.*

Proof. By the above remark and downward Skolem–Löwenheim. ■

THEOREM 7.4. *Suppose that T a theory in a denumerable language does not have partly well-ordered models in every cardinality and that $\varkappa$ is the supremum of cardinalities of partly well-ordered models of T. Then $\varkappa < \varrho$.*

Proof. Consider X_T and note that the predicale Mod_T is definable for X_T. Since in ZF the notion of well-ordering is absolute w. r. t. transitive models, the predicate $\mathrm{mod}_T(\cdot)$ meaning "$(\cdot)$ is partly well-ordered model of T" is definable for X_T.

But every partly well-ordered model of T has cardinality $\leqslant \varkappa$ thus is isomorphic to a structure belonging to $I_{\varkappa^+}$ (i.e. $H_{\varkappa^+}$).

Thus $\varkappa^+$ is non-uniformly strongly definable in $X_T(\varkappa^+)$ as follows:

(Ey) (y is supremum of powers of partly well-ordered models of $T \& x = y^+$)

and so the class $X_T(\varkappa^+)$ is usable. Using the argument of Section 5 we show that $\varkappa^+$ belongs to $\mathrm{Sp}\, L$. Thus $\varkappa < \varrho$. ∎

We show now the following:

THEOREM 7.5. *ϱ is a Hanf number for well-orderings.*

Proof. Clearly, in view of Theorem 7.4, the Hanf number for well-orderings is $\leqslant \varrho$. Conversely, let $\lambda < \varrho$. We can assume $\lambda \in \mathrm{Sp}\, L$, λ cardinal. Take $L = L_{ST}\{<\}$ and a theory T_0 such that $I(T_0) = \lambda + 1$ (this is possible by closure properties of $\mathrm{Sp}\, L$). Set $T = T_0$ "$<$ is $\in \upharpoonright On$"; then well founded models uniquely expand to partly well-ordered models of T. But T has all well founded models (up to isomorphism) in L_{λ^+}, one of them in $L_{\lambda^+} - L_\lambda$. Thus we pointed a theory without partly well-ordered models in every cardinality but with partly well-ordered models above λ. ∎

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