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# Boolean algebras of projections and ranges of spectral measures

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# Introduction

Since its conception the notion of a selfadjoint operator $T$ in a Hilbert space $H$ has been intimately connected with its resolution of the identity. This is a suitable family of selfadjoint projections $P_\lambda$ indexed by the real numbers $\lambda \in \mathbb{R}$ so that $T = \int_{\mathbb{R}} \lambda \, dP_\lambda$, where the "integral" is defined as a certain limit (in the strong operator topology) of Riemann–Stieltjes sums. Utilizing the connection between functions of bounded variation on $\mathbb{R}$ and regular, $\mathbb{C}$-valued measures defined on the Borel subsets $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$, many authors prefer to re-interpret the integral $T = \int_{\mathbb{R}} \lambda \, dP_\lambda$ as a genuine operator-valued integral $\int_{\mathbb{R}} \lambda \, dP(\lambda)$, where now $P : \mathcal{B}(\mathbb{R}) \to L(H)$ denotes a projection-valued measure $P$ taking its values in the space $L(H)$ of all bounded linear operators on $H$, which is $\sigma$-*additive* with respect to the strong operator topology and has the *multiplicative* property (i.e. $P(E)P(F) = P(E \cap F)$ for all $E, F \in \mathcal{B}(\mathbb{R})$ and $P(\mathbb{R}) = I$, the identity operator on $H$). Moreover, the support of $P$ is precisely the spectrum $\sigma(T)$ of $T$. Such measures are called (*selfadjoint*) *spectral measures*. If $P$ is defined on $\mathcal{B}(\mathbb{C})$ rather than $\mathcal{B}(\mathbb{R})$, then the analogous operators $\int_{\mathbb{C}} \lambda \, dP(\lambda)$ correspond to normal operators.

The first and most obvious extension is to relax the requirement that $P$ should take its values in the selfadjoint projections, since the concept of selfadjointness has no analogue in Banach spaces. However, this is not a genuine extension: the Mackey–Wermer theorem asserts that every spectral measure $P : \mathcal{B}(\mathbb{C}) \to L(H)$ is similar to a selfadjoint spectral measure in $H$. A more useful extension is to allow $\mathcal{B}(\mathbb{C})$ to be replaced by an arbitrary $\sigma$-algebra $\Sigma$ of subsets of some non-empty set $\Omega$. This has more interesting consequences. It gives the possibility of realizing the range $P(\Sigma) = \{P(E) : E \in \Sigma\} \subseteq L(H)$ of a spectral measure $P : \Sigma \to L(H)$ as a Boolean algebra of commuting idempotents, with respect to the partial order of range inclusion, which has certain specific properties. The essential features are that $P(\Sigma)$ is uniformly bounded, that it is $\sigma$-complete as an abstract Boolean algebra (i.e. countable families $\{P_n\}_{n=1}^\infty$ in $P(\Sigma)$ always have a supremum $\bigvee_n P_n$ and an infimum $\bigwedge_n P_n$ with respect to the partial order in $P(\Sigma)$) and that there is a close connection between the order properties of countable families of projections and the topology of $H$, namely

$$\Big(\bigwedge_n P_n\Big)H = \bigcap_{n=1}^{\infty} P_n H \quad \text{and} \quad \Big(\bigvee_n P_n\Big)H = \overline{\mathrm{sp}}\Big\{\bigcup_{n=1}^{\infty} P_n H\Big\}.$$

Moreover, and most importantly, every such Boolean algebra is the range of some spectral measure defined on a $\sigma$-algebra (which *cannot* always be chosen as $\mathcal{B}(\mathbb{C})$!).

In the 1950's N. Dunford initiated the study of spectral and scalar-type spectral operators in a general Banach space $X$. These are the analogues of selfadjoint and normal operators in Hilbert space. As in the Hilbert space setting, the fundamental concept is again that of a spectral measure $P : \Sigma \to L(X)$. The intimate connection between spectral measures and Boolean algebras of projections in $L(X)$ with the special properties listed above was exposed in two penetrating papers by W. Bade [1, 2], which later formed a significant part of the classic monograph [7]. As successful as the theory turned out to be, it was also quickly realized that there were certain inherent limitations. Particular operators which "ought to be" scalar-type spectral operators failed to be so. For instance, the operator $T : \ell^\infty \to \ell^\infty$ given by

$$T(\{x_n\}_{n=1}^\infty) = \{n^{-1}x_n\}_{n=1}^\infty, \quad \{x_n\}_{n=1}^\infty \in \ell^\infty,$$

is an infinite version of a diagonal matrix and so ought to be within the class of scalar-type spectral operators. The problem is that the set function $P : \mathcal{B}(\mathbb{R}) \to L(\ell^\infty)$ defined by

$$P(E)(\{x_n\}_{n=1}^\infty) = \{\chi_E(n^{-1})x_n\}_{n=1}^\infty, \quad \{x_n\}_{n=1}^\infty \in \ell^\infty,$$

for each $E \in \mathcal{B}(\mathbb{R})$, which is the candidate for the resolution for the identity of $T$, fails to be $\sigma$-additive for the strong operator topology in $L(\ell^\infty)$. However, if $\ell^\infty$ is equipped with its weak-$*$ topology $\sigma(\ell^\infty, \ell^1)$ rather than its norm topology, then this difficulty disappears and $T = \int_{\mathbb{R}} \lambda \, dP(\lambda)$ is indeed a scalar-type spectral operator. Of course, $(\ell^\infty, \sigma(\ell^\infty, \ell^1))$ is a locally convex Hausdorff space (briefly, lcHs) and no longer a Banach space.

Examples of the above kind, and many others, led to the theory of spectral operators and spectral measures being developed in the setting of lc-spaces. A priori this presents no difficulties as the definitions in the Banach space setting are purely of an order theoretic and topological nature, and hence have immediate extensions to lc-spaces. Nevertheless, many important results from the Banach space setting were established in the lc-setting as late as the 1980's. The difficulty was that although the statements of the expected (more general) results were relatively straightforward to formulate, their proofs were resistant to the methods and techniques employed for Banach spaces, e.g. Banach algebra techniques, existence of Bade functionals, the fact that only bounded functions generate continuous operators via integration, etc. Unfortunately, the corresponding locally convex algebra techniques are not always effective, Bade functionals simply do not exist in the non-normable setting (not even in Fréchet spaces) and plenty of unbounded functions exist which generate continuous (everywhere defined) operators via integration. The new techniques needed came from the theory of lc-Riesz spaces and the general theory of vector measures. These methods are based on principles which can be naturally adapted to the types of problems which arise for spectral measures in the non-normable setting. Roughly speaking, the class of Banach spaces is too special and too "nice" (unfortunately) to be a reliable guide within the class of lc-spaces. It is often unclear what the "most appropriate" notion or extension of a classical Banach space concept should be in the non-normable setting. Moreover, certain fundamental features arise for non-normable spaces which are simply not present in Banach spaces. Something as basic as the uniform boundedness of the range of a spectral measure (interpreted as equicontinuity in a lcHs) fails to hold in general. For instance, the spectral measure $P$ in $(\ell^\infty, \sigma(\ell^\infty, \ell^1))$ given above does not have equicontinuous range. Or, the relative weak compactness of $P(\Sigma)$, which is a basic fact in the Banach space setting, may fail for spectral measures (even equicontinuous ones!)

in lc-spaces. Or, in separable lc-spaces it is not always the case that, as in a separable Banach space $X$, the range of every spectral measure is a closed subset of $L(X)$ for the strong operator topology. And so on.

Granted that obstacles and difficulties of the type listed above (and many others) exist and are important to understand and clarify, it seems worthwhile to have available a detailed and comprehensive investigation of the essential properties and phenomena associated with the class of spectral measures in lcH-spaces. Wherever possible, suitable examples are provided to highlight important phenomena. Throughout, we have attempted to show, in a systematic way, precisely when certain basic underlying assumptions are crucial rather than just a convenience as is often the case in the literature. For instance, when is equicontinuity of the spectral measure essential and when not? When is it crucial that the lcHs $X$ be complete (or quasicomplete, or sequentially complete) and when not? When is Bade $\sigma$-completeness sufficient and when is the stronger property of Bade completeness actually needed? And so on. At various (relevant) places throughout the paper we pose questions which, as far as we are aware, are unsolved.

## 1. Preliminaries

Given a lcHs $Y$ let $\mathcal{P}(Y)$ denote the family of all continuous seminorms on $Y$ and $Y'$ denote the space of all continuous linear functionals on $Y$. We write $Y_\sigma$ to denote $Y$ equipped with its weak topology $\sigma(Y,Y')$. By a *vector measure* in $Y$ we mean a $\sigma$-additive map $m:\Sigma\to Y$ whose domain $\Sigma$ is a $\sigma$-algebra of subsets of some non-empty set $\Omega$. The Orlicz–Pettis lemma [14; I Theorem 1.3] implies that $m$ is $\sigma$-additive in $Y$ iff it is $\sigma$-additive in $Y_\sigma$, that is, iff the set function $\langle m,y'\rangle:\Sigma\to\mathbb{C}$ defined by $E\mapsto\langle m(E),y'\rangle$, for $E\in\Sigma$, is $\sigma$-additive for each $y'\in Y'$. In particular, it follows that the range $m(\Sigma)=\{m(E):E\in\Sigma\}$ is a bounded subset of $Y$. In some cases more is true.

LEMMA 1.1 ([33]). *Let $Y$ be a lcHs which is quasicomplete and $m:\Sigma\to Y$ be a vector measure. Then $m(\Sigma)$ is a relatively weakly compact* (*briefly*, *$w$-compact*) *subset of $Y$* (*i.e. the closure of $m(\Sigma)$ in $Y_\sigma$ is compact in $Y_\sigma$*).

Remark 1.2. The quasicompleteness of $Y$ is not necessary for the conclusion of Lemma 1.1 to hold. Indeed, let $X$ be any Banach space and $Y=X_\sigma$. Given a vector measure $m:\Sigma\to Y$ let $m_X$ denote $m$ considered as taking its values in $X$. By Lemma 1.1 applied to $m_X$ it follows easily that $m(\Sigma)$ is relatively $w$-compact in $Y$. By choosing $X$ suitably (e.g. $c_0$ or $L^1([0,1])$) it is possible to arrange it so that $Y$ is not quasicomplete (indeed, not even sequentially complete with $X=c_0$). ■

Let $\Lambda$ be a topological Hausdorff space and $Z\subseteq\Lambda$. Then $[Z]$ denotes the set of all elements in $\Lambda$ which are the limits of some sequences of points from $Z$. A set $Z\subseteq\Lambda$ is called *sequentially closed* if $Z=[Z]$. The *sequential closure* of a set $Z\subseteq\Lambda$ is the smallest sequentially closed subset of $\Lambda$ which contains $Z$. It is always equipped with the relative topology from $\Lambda$.

Given a lcHs $Y$ and a vector measure $m:\Sigma\to Y$ let $[Y]_m$ denote the sequential closure (in $Y$) of the linear hull of $m(\Sigma)$. Then $[Y]_m$ is a vector subspace of $Y$; in a certain sense it is the "smallest" subspace of $Y$ which contains $m(\Sigma)$ and in which it is possible to do some reasonable analysis; see [20–23]. A subset of $[Y]_m$ can be relatively $w$-compact in $Y$ without being relatively $w$-compact in $[Y]_m$. Indeed, if $Y$ is the lcHs of all $\mathbb{C}$-valued functions on $\Omega = [0,1]$ equipped with the topology of pointwise convergence on $\Omega$ and $m : \Sigma \to Y$ (where $\Sigma$ is the $\sigma$-algebra of Borel subsets of $\Omega$) is the vector measure given by $m(E) = \chi_E$, for $E \in \Sigma$, then $[Y]_m$ is the sequentially complete subspace of $Y$ consisting of all Borel measurable functions. The completeness of $Y$ implies that $m(\Sigma)$ is relatively $w$-compact in $Y$; see Lemma 1.1. However, it is shown in [12; Example 4] that $m(\Sigma)$ is *not* relatively $w$-compact as a subset of $[Y]_m$. Fortunately, the converse situation is more satisfactory. Indeed, since $[Y]_m$ has the relative topology from $Y$ it is routine to check that a subset of $[Y]_m$ which is relatively $w$-compact in $[Y]_m$ is also relatively $w$-compact in $Y$. Combining this comment with Lemma 1.1 gives the following criterion, which is quite effective for a *particular* vector measure $m$ in situations where Lemma 1.1 is not itself applicable. The point is that $Y$ may be quite large (and not be quasicomplete) whereas $m$ only takes its values in a "very small" part $[Y]_m$ of $Y$ (which may be quasicomplete).

LEMMA 1.3. *Let $Y$ be a lcHs and $m : \Sigma \to Y$ be a vector measure. If either of the lcH-spaces $[Y]_m$ or $Y$ is quasicomplete, then $m(\Sigma)$ is a relatively $w$-compact subset of $Y$.*

Let $Y$ be a lcHs and $m : \Sigma \to Y$ be a vector measure. A $\Sigma$-measurable function $f : \Omega \to \mathbb{C}$ is called *$m$-integrable* if it is integrable with respect to $\langle m, y'\rangle$, for each $y' \in Y'$, and if, for each $E \in \Sigma$, there is an element of $Y$, denoted by $\int_E f\,dm$, which satisfies $\langle \int_E f\,dm,\, y'\rangle = \int_E f\,d\langle m,\, y'\rangle$ for every $y' \in Y'$. The linear space of all $m$-integrable functions is denoted by $\mathcal{L}(m)$; it always contains the space $\mathrm{sim}(\Sigma)$ of all $\Sigma$-simple functions. Each $q \in \mathcal{P}(Y)$ induces a seminorm $q(m)$ in $\mathcal{L}(m)$ via the formula

$$q(m) : f \mapsto \sup\Big\{ \int_\Omega |f|\,d|\langle m,\, y'\rangle| : y' \in U_q^0 \Big\}, \qquad f \in \mathcal{L}(m), \tag{1}$$

where $|\nu|$ is the total variation measure of a measure $\nu : \Sigma \to \mathbb{C}$ and $U_q^0 \subseteq Y'$ is the polar of the closed $q$-unit ball $U_q = q^{-1}([0,1])$. The seminorms (1), as $q$ varies through $\mathcal{P}(Y)$, define a lc-topology $\tau(m)$ in $\mathcal{L}(m)$. Since $\tau(m)$ may not be Hausdorff we form the usual quotient space of $\mathcal{L}(m)$ with respect to the closed subspace $\bigcap_{q\in\mathcal{P}(Y)} q(m)^{-1}(\{0\})$. The resulting lcHs (with topology again denoted by $\tau(m)$) is denoted by $\mathcal{L}^1(m)$; it can be identified with equivalence classes of functions from $\mathcal{L}(m)$ modulo $m$-null functions, where $f \in \mathcal{L}(m)$ is *$m$-null* whenever $\int_E f\,dm = 0$ for all $E \in \Sigma$. In particular, $E \in \Sigma$ is $m$-null iff $m(F) = 0$ for every $F \in \Sigma$ with $F \subseteq E$. All of these definitions and further properties of $\mathcal{L}^1(m)$ can be found in [14].

Let $\Sigma(m)$ denote the subset of $\mathcal{L}^1(m)$ corresponding to $\{\chi_E : E \in \Sigma\} \subseteq \mathcal{L}(m)$. Elements of $\Sigma(m)$ will be identified with equivalence classes of elements from $\Sigma$. The topology $\tau(m)$ of $\mathcal{L}^1(m)$ induces a topology on $\Sigma(m)$ by restriction (again denoted by $\tau(m)$). Given a $\mathbb{R}$-valued $f \in \mathcal{L}(m)$, let $A(f) = |\mathbf{1} - f|^{-1}((-\infty, 1//2])$ where $\mathbf{1}$ is the function constantly equal to 1 on $\Omega$. The inequality $|\chi_E - \chi_{A(\mathrm{Re}(f))}| \le 2|\chi_E - f|$, valid for any $E \in \Sigma$ and $f \in \mathcal{L}(m)$, implies that $\Sigma(m)$ is always a $\tau(m)$-closed subset of $\mathcal{L}^1(m)$. The vector measure $m$ is called a *closed measure* [14] if $(\Sigma(m), \tau(m))$ is a complete topological space.

Lemma 1.4 ([6; Proposition 1.1]). *Let $Y$ be a lcHs and $m : \Sigma \to Y$ be a vector measure. Then $m$ is a closed measure iff $\Sigma(m)$ is complete as an abstract Boolean algebra and whenever $\{E_\alpha\} \subseteq \Sigma(m)$ is downwards filtering to* 0 *it follows that $m(E_\alpha) \xrightarrow{\alpha} 0$ in $X$.*

The quasicompleteness of $Y$ (a standing assumption in [6]) is not needed for the proof of Proposition 1.1 given there.

A connection between the completeness of $(\Sigma(m), \tau(m))$ and that of the lcHs $\mathcal{L}^1(m)$ is given by the following

Lemma 1.5 ([28; Theorem 2]). *Let $Y$ be a lcHs and $m : \Sigma \to Y$ be a vector measure such that $[Y]_m$ is sequentially complete. Then $m$ is a closed measure iff $\mathcal{L}^1(m)$ is complete as a lcHs.*

Remark 1.6. Since $\Sigma(m)$ is a closed subset of $\mathcal{L}^1(m)$ it is clear that the completeness of $\mathcal{L}^1(m)$ always implies that $m$ is a closed measure, independently of whether or not $[Y]_m$ is sequentially complete. Examples of measures $m$ where $[Y]_m$ is not sequentially complete but $\mathcal{L}^1(m)$ is nevertheless complete can be found in [28]. To produce examples where $m$ is a closed measure but $\mathcal{L}^1(m)$ is not complete is relatively straightforward. For instance, let $Y$ denote the vector subspace of the Banach space $L^1([0,1])$ consisting of the simple functions based on the Lebesgue measurable sets $\Sigma$. Define $m : \Sigma \to Y$ by $m(E) = \chi_E$, for $E \in \Sigma$. Then $(\Sigma(m), \tau(m))$ can be identified with the set $\{\chi_E : E \in \Sigma\} \subseteq L^1([0,1])$ equipped with the norm topology, and hence is a complete space. That is, $m$ is a closed measure. However, $\mathcal{L}^1(m) = \mathrm{sim}(\Sigma)$ and the topology $\tau(m)$ is precisely the norm topology from $L^1([0,1])$, which shows that $\mathcal{L}^1(m)$ is not complete. In this case $[Y]_m = Y$ is not sequentially complete. ■

For *closed* measures it is possible to relax the completeness assumptions of Lemma 1.3.

Proposition 1.7. *Let $Y$ be a lcHs and $m : \Sigma \to Y$ be a vector measure. If $\mathcal{L}^\infty(m) \subseteq \mathcal{L}^1(m)$ and $m$ is a closed measure, then $m(\Sigma)$ is relatively $w$-compact.*

Proof. Let $\mathrm{ca}(\Sigma)$ denote the space of all $\mathbb{C}$-valued, $\sigma$-additive measures on $\Sigma$. By [14; p. 21] there is a family of non-negative measures $\Lambda \subseteq \mathrm{ca}(\Sigma)$ which is equivalent to $m$ (in the sense of [14; p. 21]). Let $\Gamma$ be the set of all $\mu \in \mathrm{ca}(\Sigma)$ for which there exists a $\lambda \in \Lambda$ with $\mu \ll \lambda$. A $\Sigma$-measurable function $f : \Omega \to \mathbb{C}$ is called $\Lambda$-*integrable* if it is $\lambda$-integrable for all $\lambda \in \Lambda$. Two such functions $f$ and $g$ are called $\Lambda$-*equivalent* if $\int_\Omega |f - g|\, d\lambda = 0$ for all $\lambda \in \Lambda$. Then $\mathcal{L}^1(\Lambda)$ denotes the space of all ($\Lambda$-equivalence classes of) $\Lambda$-integrable functions; see [14; p. 38]. Moreover, $\mathcal{L}^1(m) \subseteq \mathcal{L}^1(\Lambda)$ as vector spaces and this inclusion may be *strict*; the claim in [14; p. 67] that $\mathcal{L}^1(m) = \mathcal{L}^1(\Lambda)$ is *incorrect*. Indeed, if $m$ is the vector measure in Example 1 of [14; p. 31], then $\lambda = \sum_{t=1}^{\infty} t^{-3}\delta_t$ (where $\delta_t$ is the Dirac point mass at $t \in T = \mathbb{N}$) is a finite, positive measure equivalent with $m$ and so we can choose $\Lambda = \{\lambda\}$. Then $f(t) = t$, for $t \in T$, belongs to $\mathcal{L}^1(\Lambda)$ but not to $\mathcal{L}^1(m)$, as noted in [12]. Back to the general setup, since $Y' \circ m = \{\langle m, y'\rangle : y' \in Y'\} \subseteq \Gamma$, the $\sigma(Y' \circ m)$ topology is weaker than the $\sigma(\Gamma)$ topology on $\mathcal{L}^1(\Lambda)$, which is defined to be the weakest lcH-topology on $\mathcal{L}^1(\Lambda)$ which makes each functional $f \mapsto \int_\Omega f\, d\nu$, for $f \in \mathcal{L}^1(\Lambda)$, continuous for every $\nu \in \Gamma$. These remarks establish the following fact which is the correct version of [14; IV, Theorem 1.1].

Fact. *The integration map* $I_m : \mathcal{L}^1(m) \to Y$, *defined by* $I_m f = \int_\Omega f\,dm$ *for* $f \in \mathcal{L}^1(m)$, *is continuous between the* $\sigma(\Gamma)$ *topology on* $\mathcal{L}^1(m) \subseteq \mathcal{L}^1(\Lambda)$ *and the weak topology on* $Y$.

For each $p \in \mathcal{P}(Y)$ let $\lambda_p \in \mathrm{ca}(\Sigma)$ be a finite, positive measure equivalent to $p(m)$ (see [14; II Theorem 1.1]), and let $\Lambda$ now denote the particular family $\{\lambda_p : p \in \mathcal{P}(Y)\}$. Since $m$ is a closed measure, $\Sigma(m) = \Sigma(\Lambda)$ is $\tau(\Lambda)$-complete; see [14; pp. 71–72]. Accordingly, if $\mathcal{L}_{[0,1]}(\Lambda)$ denotes the set of those elements $f \in \mathcal{L}^1(\Lambda)$ such that $\{w \in \Omega : f(w) \notin [0,1]\}$ is $\Lambda$-null, then $\mathcal{L}_{[0,1]}(\Lambda)$ is a $\sigma(\Gamma)$ compact subset of $\mathcal{L}^1(\Lambda)$ (see [14; Corollary 1, p. 55]). Since the $m$-null sets and $\Lambda$-null sets coincide the assumption $\mathcal{L}^\infty(m) \subseteq \mathcal{L}^1(m)$ implies that $\mathcal{L}_{[0,1]}(\Lambda) \subseteq \mathcal{L}^1(m)$ and so $\mathcal{L}_{[0,1]}(\Lambda)$ is a compact subset of $(\mathcal{L}^1(m), \sigma(\Gamma))$. Then the above Fact implies that $I_m(\mathcal{L}_{[0,1]}(\Lambda))$ is $w$-compact in $Y$. Since $\Sigma(m) \subseteq \mathcal{L}_{[0,1]}(\Lambda)$ the proof of Proposition 1.7 is complete. ■

Remark 1.8. (i) We note that $\mathcal{L}^\infty(m) \subseteq \mathcal{L}^1(m)$ whenever $[Y]_m$ is sequentially complete [20; Proposition 2.2]. However, the sequential completeness of $[Y]_m$ is not necessary for the inclusion $\mathcal{L}^\infty(m) \subseteq \mathcal{L}^1(m)$ to hold [20; Example 3.4(i)].

(ii) Let $Y$ denote the space of all $\mathbb{C}$-valued Borel functions on $\Omega = [0,1]$, equipped with the topology of pointwise convergence on $\Omega$. Let $\Sigma$ be the family of Borel subsets of $\Omega$ and define a vector measure $m : \Sigma \to Y$ by $m(E) = \chi_E$, for $E \in \Sigma$. Then $\mathcal{L}^\infty(m) \subseteq \mathcal{L}^1(m)$ since $Y$ is sequentially complete. However, it is shown in [12; Example 4] that $m(\Sigma)$ is *not* relatively $w$-compact. This shows that it is not possible to omit the closedness of $m$ (in general) in Proposition 1.7.

(iii) Let $Z$ denote $L^\infty([0,1])$ equipped with its weak-$*$ topology $\sigma(L^\infty, L^1)$ and $Y$ denote the subspace of all $\Sigma$-simple functions, where $\Sigma$ is the family of Borel subsets of $\Omega = [0,1]$. Define a vector measure $m : \Sigma \to Y$ by $m(E) = \chi_E$ for $E \in \Sigma$. Then it is routine to check that $\mathcal{L}^1(m) = \mathrm{sim}(\Sigma)$, as a vector space, and $\int_E f\,dm = \chi_E f$ for each $E \in \Sigma$ and $f \in \mathcal{L}^1(m)$. It is then clear that $\mathcal{L}^\infty(m) = L^\infty([0,1]) \not\subseteq \mathcal{L}^1(m)$. Since Lebesgue measure $\lambda$ on $\Sigma$ satisfies $\langle m, \xi\rangle \ll \lambda$, for every $\xi \in Y'$, it follows that $m$ is a closed measure, [14; IV, Theorem 7.3]. We now establish that $m(\Sigma)$ is *not* relatively $w$-compact in $Y$ showing that (in general) it is not possible to omit the condition $\mathcal{L}^\infty(m) \subseteq \mathcal{L}^1(m)$ in Proposition 1.7.

Let $m_Z$ denote $m$ considered as a $Z$-valued measure. It is shown in [12; Example 4] that there exists a sequence $\{A_n\}_{n=1}^\infty \subseteq \Sigma$ such that $\chi_{A_n} \to \frac{1}{2}\chi_\Omega$ in $Z$. Accordingly, $\frac{1}{2}\chi_\Omega \in \overline{m_Z(\Sigma)}$ where the "bar" denotes closure in $Z$ (which already has its weak topology as a lcHs). Fix $E \in \Sigma$. Then for each $\varphi \in L^1([0,1])$ we have $\langle \chi_{A_n}, \chi_E\varphi\rangle \to \langle \frac{1}{2}\chi_\Omega, \chi_E\varphi\rangle$, that is, $\langle \chi_{E\cap A_n}, \varphi\rangle \to \langle \frac{1}{2}\chi_E, \varphi\rangle$. This shows that $\frac{1}{2}\chi_E \in \overline{m_Z(\Sigma)}$. Furthermore, $\frac{1}{2}\chi_{A_n} \to \frac{1}{2}(\frac{1}{2}\chi_\Omega) = \frac{1}{4}\chi_\Omega$ in $Z$ with $\{\frac{1}{2}\chi_{A_n}\}_{n=1}^\infty \subseteq \overline{m_Z(\Sigma)}$, which shows that $\frac{1}{4}\chi_\Omega \in \overline{m_Z(\Sigma)}$. Then a similar argument as above establishes that $\frac{1}{4}\chi_E \in \overline{m_Z(\Sigma)}$, for every $E \in \Sigma$. An inductive argument shows that $2^{-n}\chi_E \in \overline{m_Z(\Sigma)}$ for every $n \in \mathbb{N}$ and $E \in \Sigma$. Suppose that $f, g$ are elements of $\overline{m_Z(\Sigma)}$ whose supports $S(f)$ and $S(g)$ are *disjoint*. Choose nets $\{m_Z(E_\alpha)\}$ and $\{m_Z(F_\beta)\}$ from $m_Z(\Sigma)$ such that $m_Z(E_\alpha) \to f$ in $Z$ and $E_\alpha \subseteq S(f)$ and such that $m_Z(F_\beta) \to g$ in $Z$ and $F_\beta \subseteq S(g)$. Then, for any $\varphi \in L^1([0,1])$ we have

$$\int (f+g)\varphi\,d\lambda = \lim_{\alpha,\beta}\int (\chi_{E_\alpha}+\chi_{F_\beta})\varphi\,d\lambda = \lim_{\alpha,\beta}\int \chi_{E_\alpha\cup F_\beta}\varphi\,d\lambda,$$

which shows that $f+g \in \overline{m_Z(\Sigma)}$. For each $n \geq 1$ let $D(n) = \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ and $h_n = \sum_{k=1}^{n} \frac{1}{2^k}\chi_{D(k)}$. If $h \in Z$ is the function $\sum_{k=1}^{\infty} \frac{1}{2^k}\chi_{D(k)}$, then $\|h_n - h\|_\infty \to 0$, from which it follows that $h_n \to h$ in $Z$. Since each $h_n \in \overline{m_Z(\Sigma)}$, for $n \geq 1$, it follows that $\{h_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\overline{m_Z(\Sigma)}$. It is clear from the above discussion that each $h_n$ actually belongs to the closure $\overline{m(\Sigma)}^{Y} = \overline{m(\Sigma)} \cap Y$ of $m(\Sigma)$ in $Y$. Accordingly, $\{h_n\}$ is also a Cauchy sequence in $\overline{m(\Sigma)}^{Y}$ which has no limit in $\overline{m(\Sigma)}^{Y}$ because $h \in Z \setminus Y$. This shows that $\overline{m(\Sigma)}^{Y}$ cannot be $w$-compact. Indeed, we have shown that $m(\Sigma)$ is not even relatively $w$-sequentially compact. ∎

Let $\varrho$ be any lcH-topology on $Y$ consistent with the duality $(Y, Y')$. If $Y_\varrho$ denotes $Y$ equipped with the topology $\varrho$ and $m_\varrho : \Sigma \to Y_\sigma$ denotes $m$ considered as being $Y_\varrho$-valued, then the Orlicz–Pettis lemma implies that $m_\varrho$ is also $\sigma$-additive [14; I, Theorem 1.3]. Moreover, it is clear that $\mathcal{L}^\infty(m) = \mathcal{L}^\infty(m_\varrho)$ and $\mathcal{L}^1(m) = \mathcal{L}^1(m_\varrho)$ as vector spaces. So, if the inclusion $\mathcal{L}^\infty(m) \subset \mathcal{L}^1(m)$ is to be satisfied, then it suffices to establish that $\mathcal{L}^\infty(m_\varrho) \subseteq \mathcal{L}^\infty(m_\varrho)$ for any such topology $\varrho$. Accordingly, a sufficient condition for the inclusion $\mathcal{L}^\infty(m) \subseteq \mathcal{L}^1(m)$ to hold is that $Y_\varrho$ (or $[Y_\varrho]_{m_\varrho}$) is sequentially complete for *some* such topology $\varrho$. This condition is quite useful in practice. For instance, if $Y$ is the lcHs $c_0$ equipped with its weak topology $\sigma(c_0, \ell^1)$, then $Y$ is not sequentially complete. However, if $\varrho$ denotes the Mackey (= norm) topology in $Y$, then $Y_\varrho$ is complete. Accordingly, *every* vector measure $m : \Sigma \to (c_0)_\sigma$ has the property that $\mathcal{L}^\infty(m) \subseteq \mathcal{L}^1(m)$. Combining the above comments with the fact that a vector measure is closed iff it is closed with respect to any consistent lcH-topology (see [32; Proposition 2]) gives the following

Corollary 1.7.1. *Let $Y$ be a lcHs and $m : \Sigma \to Y$ be a closed vector measure. If there exists a lcH-topology $\varrho$ in $Y$, consistent with the duality $(Y, Y')$, such that $Y_\varrho$ (or $[Y_\varrho]_{m_\varrho}$) is sequentially complete, then $m(\Sigma)$ is a relatively $w$-compact subset of $Y$.*

Let $m : \Sigma \to Y$ be a vector measure. A set $E \in \Sigma$ is called an *atom* of $m$ if $m(E) \neq 0$ and, for each $F \in \Sigma$, either $m(E \cap F) = 0$ or $m(E \setminus F) = 0$. In this case, the range of $m$ restricted to $E \cap \Sigma = \{E \cap F : F \in \Sigma\}$ equals $\{0, m(E)\}$. The measure $m$ is called *$\sigma$-atomic* if there are countably many pairwise disjoint atoms $E_j \in \Sigma$, for $j \in \mathbb{N} = \{1, 2, \ldots\}$, such that $m(\bigcup_{j=1}^{\infty} E_j) = m(\Omega)$.

Proposition 1.9. *Let $Y$ be a lcHs and $m : \Sigma \to Y$ be a $\sigma$-atomic vector measure. Then $m$ is a closed measure.*

Proof. Let $E_j \in \Sigma$, for $j \in \mathbb{N}$, be atoms for $m$ as above. Then the set function $\nu_j : E_j \cap \Sigma \to [0, 2^{-j}]$ defined by

$$\nu_j(E_j \cap F) = \begin{cases} 2^{-j} & \text{if } m(E_j \cap F) = m(E_j), \\ 0 & \text{if } m(E_j \cap F) = 0, \end{cases}$$

is a measure which we extend to all of $\Sigma$ by declaring $\nu_j(F) = 0$ for every $F \in E_j^{c} \cap \Sigma$. Then $\nu = \sum_{j=1}^{\infty} \nu_j$ is a finite measure on $\Sigma$ with the property that $\langle m, \xi \rangle \ll \nu$ for every $\xi \in Y'$. Accordingly, $m$ is a closed measure [14; IV, Theorem 7.3]. ∎

The vector space of all continuous linear operators of a lcHs $X$ into itself is denoted by $L(X)$. The space $L(X)$ equipped with the strong operator topology (i.e. the topology of pointwise convergence on $X$) is denoted by $L_{\mathrm{s}}(X)$. It is a lcHs whose topology is generated by the family of seminorms $q_x : T \mapsto q(Tx)$, with $T \in L(X)$, for each $x \in X$ and $q \in \mathcal{P}(X)$. The dual space $(L_{\mathrm{s}}(X))'$ consists of all linear functionals $\xi : L_{\mathrm{s}}(X) \to \mathbb{C}$ of the form

$$\xi : T \mapsto \sum_{j=1}^{n} \langle Tx_j, x'_j \rangle, \qquad T \in L_{\mathrm{s}}(X),$$

for some finite subsets $\{x_j\}_{j=1}^n$ of $X$ and $\{x'_j\}_{j=1}^n$ of $X'$. In particular, the weak topology $\sigma(L_{\mathrm{s}}(X), (L_{\mathrm{s}}(X))')$ of $L_{\mathrm{s}}(X)$ is precisely the weak operator topology. The space $L(X)$ equipped with its weak operator topology is denoted by $L_{\mathrm{w}}(X)$. Finally, the set of all continuous projection operators in $X$ is denoted by $\mathbb{P}(X)$.

Since $L_{\mathrm{s}}(X)$ is a lcHs we may consider vector measures $P : \Sigma \to L_{\mathrm{s}}(X)$, which are usually referred to as *operator-valued measures*. For each $x \in X$, let $Px : \Sigma \to X$ denote the $X$-valued vector measure $E \mapsto P(E)x$, for $E \in \Sigma$. As noted before, the range $P(\Sigma) \subseteq L_{\mathrm{s}}(X)$ is always a bounded set. If $P(\Sigma)$ is an equicontinuous part of $L(X)$, then $P$ is said to be an *equicontinuous measure*. There exists a large class of lcH-spaces, including all metrizable and all bornological lcH-spaces and many more, in which all operator-valued measures are necessarily equicontinuous.

LEMMA 1.10 ([22; Proposition 2.5]). *Let $X$ be a quasibarrelled lcHs. Then every operator-valued measure $P : \Sigma \to L_{\mathrm{s}}(X)$ is equicontinuous.*

An operator-valued measure $P : \Sigma \to L_{\mathrm{s}}(X)$ which is multiplicative is called a *spectral measure*. For such measures a set $E \in \Sigma$ is *$P$-null* iff $P(E) = 0$. Moreover, the integrability of functions with respect to spectral measures is somewhat simpler than for general vector and operator-valued measures. Indeed, a $\Sigma$-measurable function $f : \Omega \to \mathbb{C}$ is $P$-integrable iff it is $\langle Px, x' \rangle$-integrable, for each $x \in X$ and $x' \in X'$, and there exists $T_f \in L(X)$ such that $\langle T_f x, x' \rangle = \int_{\Omega} f \, d\langle Px, x' \rangle$, for each $x \in X$ and $x' \in X'$ (see [22; Lemma 1.2]). In this case $T_f = \int_{\Omega} f \, dP$.

We always have $\mathrm{sim}(\Sigma) \subseteq \mathcal{L}(P)$. Under further assumptions $\mathcal{L}(P)$ contains elements other than just those from $\mathrm{sim}(\Sigma)$. For instance, if $L_{\mathrm{s}}(X)$ is sequentially complete, then all bounded $\Sigma$-measurable functions are $P$-integrable [14; II, Lemma 3.1]. Actually, it suffices for the smaller subspace $[L_{\mathrm{s}}(X)]_P$ to be sequentially complete [20; Proposition 2.2]. Or, if the spectral measure $P$ is also equicontinuous, then all bounded $\Sigma$-measurable functions will be $P$-integrable whenever $X$ is sequentially complete [34; p. 300]; this is weaker than requiring $L_{\mathrm{s}}(X)$ to be sequentially complete. It also suffices for $[X]_{Px}$ to be sequentially complete, for each $x \in X$, rather than $X$ itself.

The multiplicativity of a spectral measure $P : \Sigma \to L_{\mathrm{s}}(X)$ implies that the pointwise product $fg$ of two $P$-integrable functions $f$ and $g$ is again $P$-integrable. Under certain additional "completeness" assumptions on $X$ and $L_{\mathrm{s}}(X)$ this is recorded in [6; Lemma 1.3]. However, the result is valid in general; no such additional requirements are needed. See [23; Corollary 2.1]. Moreover,

$$\int_E fg\,dP = P(E)\cdot\Big(\int_\Omega f\,dP\Big)\int_\Omega g\,dP = P(E)\cdot\Big(\int_\Omega g\,dP\Big)\int_\Omega f\,dP,\qquad E\in\Sigma. \tag{2}$$

The *integration map* $I_P : \mathcal{L}^1(P) \to L_s(X)$ is defined by $I_P f = \int_\Omega f\,dP$, for $f \in \mathcal{L}^1(P)$. It is a continuous and linear bijection of $\mathcal{L}^1(P)$ onto its range $I_P(\mathcal{L}^1(P)) \subseteq L_s(X)$. Actually, the range of $I_P$ is contained in $[L_s(X)]_P$ and the sequential closure of $I_P(\mathcal{L}^1(P))$ in $L_s(X)$ coincides with $[L_s(X)]_P$; see [20; p. 347]. Moreover, (2) implies that $I_P$ is an algebra homomorphism and is a Boolean algebra isomorphism of $\Sigma(P)$ onto $P(\Sigma)$. Sometimes more is true.

LEMMA 1.11. *Let $P : \Sigma \to L_s(X)$ be an equicontinuous spectral measure. Then the integration map $I_P$ is a bicontinuous linear and algebra isomorphism of $\mathcal{L}^1(P)$ onto its range $I_P(\mathcal{L}^1(P)) \subseteq L_s(X)$. In particular, it is a bicontinuous topological and Boolean algebra isomorphism of $\Sigma(m)$ onto $P(\Sigma)$.*

Proof. This result is established in [24; Lemma 1] under the additional assumption that $X$ is quasicomplete. Since (2) is valid without this requirement on $X$ an examination of the proof given in [24] shows that it holds in general. ■

QUESTION 1. The integration map $I_P : \mathcal{L}^1(P) \to L_s(X)$ is always a continuous, linear and multiplicative bijection of $\mathcal{L}^1(P)$ onto its range in $L_s(X)$. It is shown in [29; pp. 297–298] that the inverse map $I_P^{-1} : I_P(\mathcal{L}^1(P)) \to \mathcal{L}^1(P)$ is not continuous in general. For the examples given there $P$ is not equicontinuous. Suppose that the inverse map $I_P^{-1}$ is continuous. Does it follow that $P$ is necessarily equicontinuous? ■

## 2. Relative weak compactness of the range

Let $X$ be a lcHs and $P : \Sigma \to L_s(X)$ be a spectral measure. One of the basic questions concerning $P$ (as with any vector measure) is whether or not $P(\Sigma)$ is a relatively $w$-compact subset of $L_s(X)$. We begin with an example to show that this is not always the case, even if $P$ is equicontinuous and $X$ is sequentially complete!

EXAMPLE 2.1. Let $X$ denote the space of all $\mathbb{C}$-valued Borel measurable functions on $\Omega = [0, 1]$ equipped with the topology of pointwise convergence on $\Omega$. Then $X$ is a sequentially complete lcHs with topology generated by the family of seminorms $q_w$, for $w \in \Omega$, where $q_w(f) = |f(w)|$, $f \in X$. Let $\Sigma$ be the $\sigma$-algebra of all Borel subsets of $\Omega$. Then the set function $P : \Sigma \to L_s(X)$ defined by $P(E)f = \chi_E f$, with $f \in X$, for each $E \in \Sigma$, is an equicontinuous spectral measure. Since $X$ has its weak topology it follows that $L_s(X) = L_w(X)$. Let $\mathbf{1} \in X$ denote the function constantly equal to 1 on $\Omega$. Suppose that $P(\Sigma)$ is relatively $w$-compact. Continuity of the linear map $T \mapsto T\mathbf{1}$ from $L_w(X)$ into $X = X_\sigma$ implies that $\mathcal{S} = \{\chi_E : E \in \Sigma\}$ is relatively $w$-compact in $X$. Since the weak completion of $X_\sigma = X$ is the lcHs $\overline{X}_\sigma$ consisting of *all* $\mathbb{C}$-valued functions on $\Omega$, equipped with the topology of pointwise convergence on $\Omega$, it follows that the closure of $\mathcal{S}$ in $\overline{X}_\sigma$ is contained in $X_\sigma = X$. But, if $E \subseteq \Omega$ is a non-Borel set and $\mathcal{F}$ is the family of all finite subsets of $E$ directed by inclusion, then the net $\{\chi_F : F \in \mathcal{F}\} \subseteq \mathcal{S}$ converges to $\chi_E$ in $\overline{X}_\sigma$. Since $\chi_E \notin X$ we have a contradiction. Accordingly, $P(\Sigma)$ is not relatively $w$-compact. ■

In view of Example 2.1 it is useful to have available conditions which are sufficient for large classes of spectral measures to have relatively $w$-compact range. Of course, if $L_s(X)$ is quasicomplete, then this is always the case; see Lemma 1.1. However, there exist complete and quasicomplete lcH-spaces $X$ such that $L_s(X)$ is not even sequentially complete [22 (§3); 26], and so this condition imposes some restrictions. The next result follows immediately from Lemma 1.3.

Proposition 2.2. *Let $X$ be a lcHs and $P : \Sigma \to L_s(X)$ be a spectral measure. If the lcHs $[L_s(X)]_P$ is quasicomplete, then $P$ has relatively $w$-compact range.*

There exist examples where Proposition 2.2 applies but $L_s(X)$ is not quasicomplete.

Example 2.3. Let $X = \ell^1$ equipped with its weak-$*$ topology $\sigma(\ell^1, c_0)$. Then $X$ is a quasicomplete lcHs but $L_s(X)$ is not even sequentially complete [26; Example 1]. Let $\Omega = \mathbb{N}$ and $\Sigma = 2^{\mathbb{N}}$ be the family of all subsets of $\mathbb{N}$. Define a spectral measure $P : \Sigma \to L_s(X)$ by $P(E)\varphi = \chi_E\varphi$ (coordinatewise multiplication) for each $\varphi \in X$ and $E \in \Sigma$. Then $L^1(P) = \ell^\infty$ as vector spaces and, for each $f \in \ell^\infty$, the operator $\int_\Omega f\,dP$ is the operator in $X = \ell^1$ of coordinatewise multiplication by $f$. Let $\{f_\alpha\}$ be a net from $\mathcal{L}^1(P)$ such that $\{\int_\Omega f_\alpha\,dP\}$ is a bounded Cauchy net in $L_s(X)$. For each $\varphi \in X$ the quasicompleteness of $X$ ensures the existence of an element $T\varphi \in X$ such that $f_\alpha\varphi \to T\varphi$ in $X$. By considering $\varphi = e_n$, the $n$th standard basis vector of $\ell^1$, it follows that $f_\alpha(n) \overset{\alpha}{\to} (Te_n)(n)$, for each $n \in \mathbb{N}$. Moreover, a uniform boundedness argument implies that $\sup_\alpha \|f_\alpha\|_\infty < \infty$. Accordingly, the element $f : \mathbb{N} \to \mathbb{C}$ defined by $f(n) = (Te_n)(n)$ belongs to $\ell^\infty$ and it follows that $T = \int_\Omega f\,dP$ belongs to $I_P(\mathcal{L}^1(P))$. Hence, $I_P(\mathcal{L}^1(P))$ is a quasicomplete subspace of $L_s(X)$ and so $[L_s(X)]_P = I_P(\mathcal{L}^1(P))$ is necessarily quasicomplete [20; p. 347]. ■

A result of A. Grothendieck [11; pp. 97–98] implies that if $X$ is a lcHs and $\mathcal{M} \subseteq L(X)$ is an equicontinuous family, then $\mathcal{M}$ is relatively $w$-compact iff $\mathcal{M}(x) = \{Tx : T \in \mathcal{M}\}$ is relatively $w$-compact in $X$, for each $x \in X$; see also [30; Proposition 1]. Combining this fact with Lemma 1.3 yields the following

Proposition 2.4. *Let $X$ be a lcHs and $P : \Sigma \to L_s(X)$ be a spectral measure such that*

(i) *$P$ is equicontinuous, and*
(ii) *$[X]_{Px}$ is quasicomplete, for each $x \in X$.*

*Then $P$ has relatively $w$-compact range in $L_s(X)$.*

Lemma 1.10 shows that Proposition 2.4 has some generality. For instance, (i) and (ii) are satisfied whenever $X$ is any quasicomplete, barrelled lcHs (e.g. Fréchet spaces and, in particular, Banach spaces).

Proposition 2.4 is quite different to Proposition 2.2. For instance, it follows from [19; Proposition 4] that the spectral measure of Example 2.3 is not equicontinuous and so Proposition 2.4 is not applicable, whereas Proposition 2.2 is. The following example shows that Proposition 2.4 may apply when Proposition 2.2 does not.

Example 2.5. Let $X$ denote the subspace $\ell^2_{00}$ of the Hilbert space $\ell^2$, i.e. elements of $X$ having only finitely many non-zero coordinates. Let $\Sigma = 2^{\mathbb{N}}$ and define a spectral measure $P : \Sigma \to L_s(X)$ by

$$P(E)(\{x_n\}_{n=1}^\infty) = \{\chi_E(n)x_n\}_{n=1}^\infty, \qquad \{x_n\}_{n=1}^\infty \in X, \tag{3}$$

for each $E \in \Sigma$. Then $\|P(E)\| \le 1$ for all $E \in \Sigma$ and so $P$ is equicontinuous. If $\varphi \in X$ is fixed, then $[X]_{P\varphi}$ is a finite-dimensional subspace of $X$ and so is complete. Accordingly, Proposition 2.4 applies to show that $P(\Sigma)$ is relatively $w$-compact. However, Proposition 2.2 does not apply. Indeed, $[L_s(X)]_P$ is not even sequentially complete. For if $f_n = (1, 2, \ldots, n, 0, 0, \ldots) \in \ell^\infty$, for each $n \in \mathbb{N}$, then $f_n \in \mathcal{L}^1(P)$ and $\int_\Omega f_n\, dP \in L(X)$ is the operator in $X = \ell^2_{00}$ of coordinatewise multiplication by $f_n$. Hence, $\{\int_\Omega f_n\, dP\}_{n=1}^\infty$ is a Cauchy sequence in $I_P(\mathcal{L}^1(P)) \subseteq [L_s(X)]_P$ with the property that $(\int_\Omega f_n\, dP)\varphi \to f\varphi$ in $X$, for each $\varphi \in X$, where $f : \mathbb{N} \to \mathbb{C}$ is given by $f(n) = n$ for $n \in \mathbb{N}$. Since the operator in $X$ of multiplication by $f$ is not continuous it follows that the Cauchy sequence $\{\int_\Omega f_n\, dP\}_{n=1}^\infty \subseteq [L_s(X)]_P$ has no limit in $L_s(X)$. ∎

As general as the sufficient conditions of Proposition 2.2 and 2.4 appear to be, there exist spectral measures with relatively $w$-compact range to which neither result applies.

Example 2.6. Let $Y = X_\sigma$ where $X$ is the normed space of Example 2.5. Let $Q$ denote the spectral measure given by (3) but now considered as being $L_s(Y)$-valued. Then $Q$ is no longer equicontinuous; see [22; Example 3.9]. Hence, Proposition 2.4 is not applicable. The same argument as in Example 2.5 shows that $[L_s(Y)]_Q$ is not even sequentially complete and so Proposition 2.2 also fails to apply. However, since $X$ has its Mackey topology it follows that $L(Y) = L(X)$ as vector spaces. Moreover, $L_s(Y)$ and $L_s(X)$ have the same dual space. Accordingly, the relative $w$-compactness of $Q(\Sigma)$ in $L_s(Y)$ follows from the relative $w$-compactness of $P(\Sigma)$ in $L_s(X)$; see Example 2.5. ∎

Question 2. Let $X$ be a quasicomplete lcHs. Does every spectral measure $P : \Sigma \to L_s(X)$ have relatively $w$-compact range? This is false if $P(\Sigma)$ is replaced by an arbitrary family of operators $\mathcal{M} \subseteq L(X)$ with the property that $\mathcal{M}(x) = \{Tx : T \in \mathcal{M}\}$ is relatively $w$-compact for each $x \in X$; see [30; pp. 218–219]. ∎

A counterexample to Question 2, if it exists, will probably lie within the class of non-closed measures, as seen by the next result, which follows immediately from Proposition 1.7. We remark that this is the case with Example 2.1; see also Example 3.7 below.

Proposition 2.7. *Let $X$ be a lcHs and $P : \Sigma \to L_s(X)$ be a closed spectral measure such that $\mathcal{L}^\infty(P) \subseteq \mathcal{L}^1(P)$. Then $P(\Sigma)$ is a relatively $w$-compact subset of $L_s(X)$.*

Remark 2.8. The relative $w$-compactness of $P(\Sigma)$ for spectral measures of the type given in Example 2.6 does *not* follow (in general) from Proposition 2.7 as they need not necessarily be closed measures; consider $X$ of Example 3.2 below, equipped with its weak topology. This shows, in particular, that the relative $w$-compactness of $P(\Sigma)$, even within the class of quasicomplete spaces $X$ (which is the case for $X$ with its weak topology in Example 3.2), does not characterize closedness of spectral measures. ∎

In Remark 3.12(c) an equicontinuous spectral measure $P$ is exhibited such that $P(\Sigma)$ is $w$-closed but not relatively $w$-compact. We conclude this section with a result about a class of spectral measures for which $P(\Sigma)$ is always $w$-compact.

PROPOSITION 2.9. *Let $X$ be a lcHs and $P : \Sigma \to L_s(X)$ be a $\sigma$-atomic spectral measure. Then $P(\Sigma)$ is a compact subset of $L_s(X)$ and hence, in particular, is also $w$-compact.*

Proof. This result is a particular case of [12; Theorem 10]. ∎

If $X$ is a Fréchet Montel space, the *every* spectral measure $P : \Sigma \to L_s(X)$ has $w$-compact range; this follows from Lemma 3.1 below, Proposition 2.9 and [34; Corollary 4.7]. A similar argument (using [35; p. 1270]) shows that the same conclusion is valid for *every* spectral measure $P : \Sigma \to L_s(\ell^1)$. *Particular* spectral measures with this property exist in other Banach spaces. Indeed, let $X$ be any separable, reflexive Banach space with an unconditional base, say $\{x_n\}$, and associated family of biorthogonal coefficient functionals, say $\{\xi_n\} \subseteq X'$. Then $P : 2^{\mathbb{N}} \to L_s(X)$ defined by $P(E) = \sum_{n\in E} P_n$, for $E \in 2^{\mathbb{N}}$, has the desired property, where $P_n$ is the projection given by $P_n : x \mapsto \langle x, \xi_n\rangle x_n$, for $x \in X$ and $n \in \mathbb{N}$. Finally, Proposition 2.9 fails in general if $P$ has uncountably many atoms; see Example 2.1.

## 3. Closed spectral measures

The aim of this section is to give a detailed description of some of the most important properties of closed spectral measures.

Let $X$ be a lcHs and $\mathcal{M} \subseteq L(X)$ be a Boolean algebra (briefly, B.a.) of commuting projections (with $I$ as the unit). $\mathcal{M}$ is partially ordered by requiring $P \le Q$ iff $PX \subseteq QX$. Recall that $\mathcal{M}$ is *Bade complete* (resp. *Bade $\sigma$-complete*) if

(i) $\mathcal{M}$ is complete (resp. $\sigma$-complete) as an abstract B.a., and

(ii) $(\bigwedge_\alpha P_\alpha)X = \bigcap_\alpha P_\alpha X$ and $(\bigvee_\alpha P_\alpha)X = \overline{\mathrm{sp}}\{\bigcup_\alpha P_\alpha X\}$ whenever $\{P_\alpha\}$ is a family (resp. countable family) of elements from $\mathcal{M}$. For Banach spaces we refer to [1] and for lcHs to [33].

The following result is a routine consequence of the multiplicativity and $\sigma$-additivity of spectral measures.

LEMMA 3.1. *Let $X$ be a lcHs and $P : \Sigma \to L_s(X)$ be a spectral measure. Then $P(\Sigma)$ is a Bade $\sigma$-complete B.a. in $L(X)$.*

Not every spectral measure is closed, even in a Hilbert space.

EXAMPLE 3.2. Let $\Omega = [0,1]$ and $X$ be the (non-separable) Hilbert space $\ell^2(\Omega)$. Let $\Sigma$ be the Borel subsets of $\Omega$ and define an equicontinuous spectral measure $P : \Sigma \to L_s(X)$ by $P(E)\psi = \chi_E\psi$, for each $\psi \in X$ and $E \in \Sigma$. If $P$ were a closed measure, then Lemma 1.11 would imply that $P(\Sigma)$ is a complete subset of $L_s(X)$. But this is not the case. Indeed, let $E \subseteq \Omega$ be a non-Borel set and $\mathcal{F}$ be the family of finite subsets of $E$ directed by inclusion. Then $\{P(F) : F \in \mathcal{F}\} \subseteq P(\Sigma)$ is a Cauchy net which converges in

$L_s(X)$ to the continuous projection operator $Q$ of multiplication by $\chi_E$. Since $Q \notin P(\Sigma)$ it follows that $P(\Sigma)$ is not complete. ■

A B.a. $\mathcal{M} \subseteq L(X)$ is said to have the *monotone* (resp. *$\sigma$-monotone*) *property*, [27], if $\lim_\alpha B_\alpha$ exists in $L_w(X)$ and is an element of $\mathcal{M}$ whenever $\{B_\alpha\} \subseteq \mathcal{M}$ is a monotonic net (resp. sequence).

LEMMA 3.3. *Let $X$ be a lcHs and $\mathcal{M} \subseteq L(X)$ be a B.a. with the monotone* (*resp. $\sigma$-monotone*) *property. Then $\mathcal{M}$ is Bade complete* (*resp. Bade $\sigma$-complete*).

Proof. An examination of the proof of [18; Theorem 1], with $G = X'$, shows that it can be adapted to the setting of non-normable spaces to establish the stated claim. ■

For closed spectral measures Lemma 3.1 can be strengthened.

PROPOSITION 3.4. *Let $X$ be a lcHs and $P : \Sigma \to L_s(X)$ be a closed spectral measure. Then $P(\Sigma)$ is a Bade complete B.a. in $L(X)$.*

Proof. Let $\{P(E_\alpha)\} \subseteq P(\Sigma)$ be an upwards directed family. Since $I_P$ is a B.a. isomorphism of $\Sigma(P)$ onto $P(\Sigma)$ it follows that $\{\chi_{E_\alpha}\}$ is upwards directed in $\Sigma(P)$. By Lemma 1.4 there is $\chi_E \in \Sigma(P)$ such that $\chi_{E_\alpha} \uparrow \chi_E$ in $\Sigma(P)$, and hence $\chi_{E\setminus E_\alpha} \downarrow 0$ in $\Sigma(P)$. Again by Lemma 1.4 it follows that $P(E_\alpha) \to P(E)$ in $L_s(X)$. Similarly, if $\{P(E_\alpha)\} \subseteq P(\Sigma)$ is downwards directed, then $\{\chi_{E_\alpha}\}$ is downwards directed in $\Sigma(P)$ and so, by abstract completeness of $\Sigma(P)$ again, there is $\chi_E \in \Sigma(P)$ such that $\chi_{E_\alpha} \downarrow \chi_E$, that is, $\chi_{E_\alpha\setminus E} \downarrow 0$ in $\Sigma(P)$. Lemma 1.4 implies that $P(E_\alpha) \to P(E)$ in $L_s(X)$. So, whenever $\{P(E_\alpha)\} \subseteq P(\Sigma)$ is a monotonic net, then $\lim_\alpha P(E_\alpha)$ exists in $L_s(X)$, hence also in $L_w(X)$, and is an element of $P(\Sigma)$. The conclusion follows from Lemma 3.3. ■

Under additional hypotheses more is true.

PROPOSITION 3.5. *Let $X$ be a lcHs and $P : S \to L_s(X)$ be an equicontinuous spectral measure. The following statements are equivalent*:

(i) *$P$ is a closed measure.*
(ii) *$P(\Sigma)$ is a complete subset of $L_s(X)$.*
(iii) *$P(\Sigma)$ is a Bade complete B.a.*
(iv) *$P(\Sigma)$ has the monotone property.*

Proof. (i)⇔(ii) follows from Lemma 1.11 and (i)⇒(iii) by Proposition 3.4.

(iii)⇒(i). Lemma 1.11 implies that $\Sigma(P)$ is complete as an abstract B.a. Let $\chi_{E_\alpha} \downarrow 0$ in $\Sigma(P)$. Then $P(E_\alpha) \downarrow 0$ in $P(\Sigma)$. By [34; Proposition 1.3] it follows that $P(E_\alpha) \to 0$ in $L_s(X)$. Then Lemma 1.4 shows that $P$ is a closed measure.

(iv)⇒(iii) by Lemma 3.3 and (iii)⇒(iv) by [34; Proposition 1.3]. ■

COROLLARY 3.5.1. *Let $X$ be a lcHs and $P : \Sigma \to L_s(X)$ be a $\sigma$-atomic spectral measure. Then $P$ is a closed measure.*

Corollary 3.5.1 is a special case of Proposition 1.9. If $P$ also happens to be equicontinuous, then another proof is possible. Indeed, by Proposition 2.9, $P(\Sigma)$ is a compact subset of $L_s(X)$. In particular, $P(\Sigma)$ is a complete subset of $L_s(X)$ and the result follows from Proposition 3.5. Example 3.2 shows that Corollary 3.5.1 may fail if $P$ has more than countably many atoms.

COROLLARY 3.5.2 (cf. [24; Proposition 3]). *Let $X$ be a quasicomplete lcHs and $P : \Sigma \to L_s(X)$ be an equicontinuous spectral measure. Then $P$ is a closed measure iff $P(\Sigma)$ is a closed subset of $L_s(X)$.*

Proof. The equicontinuity of $P(\Sigma)$ implies that $P(\Sigma)$ is a complete subset of $L_s(X)$ iff it is a closed subset; see Lemma 1.11. ■

We note that there exist classes of quasicomplete lcH-spaces for which the conclusion of Corollary 3.5.2 is valid without the equicontinuity requirement (and in which there actually exist non-trivial spectral measures which fail to be equicontinuous!). For instance, if $X = Y_\sigma$ where $Y$ is any Montel space, then $X$ is quasicomplete and spectral measures in $L_s(X)$ are typically not equicontinuous (e.g. consider the spectral measures in the Montel spaces given in Examples 3.2 and 3.4 of [31]). Nevertheless, a spectral measure $P : \Sigma \to L_s(X)$ is a closed measure iff $P(\Sigma)$ is a closed subset of $L_s(X)$ (see [31; Proposition 6.4]).

Remark 3.6. The quasicompleteness of $X$ in Corollary 3.5.2 can be relaxed somewhat. It suffices to have $[X]_{Px}$ quasicomplete for each $x \in X$. For, suppose that $P(\Sigma)$ is closed in $L_s(X)$. Let $\{P(E_\alpha)\}$ be a Cauchy net in $P(\Sigma)$. Given $x \in X$, the net $\{P(E_\alpha)x\}$ is contained in $[X]_{Px}$, is Cauchy and is bounded. So, there exists $Tx \in [X]_{Px}$ such that $P(E_\alpha)x \to Tx$. By equicontinuity of $P(\Sigma)$ the linear map $T : X \to X$ so defined is necessarily continuous and so $P(E_\alpha) \to T$ in $L_s(X)$. By the assumption on $P(\Sigma)$ we have $T \in P(\Sigma)$ and so $P(\Sigma)$ is a complete set in $L_s(X)$. The converse is always true, i.e. complete sets are always closed. ■

For instance, the hypotheses of Remark 3.6 apply to the spectral measure $P$ of Example 2.5 whereas Corollary 3.5.2 is not applicable. However, the quasicompleteness hypotheses of Remark 3.6 cannot be further weakened in general.

EXAMPLE 3.7. Let $X$ be the sequentially complete lcHs and $P$ the equicontinuous spectral measure of Example 2.1. Let $\{P(E_\alpha)\} \subseteq P(\Sigma)$ be a net convergent to $T$ in $L_s(X)$. For each $w \in [0,1]$ the function $\chi_{\{w\}} \in X$ and so $T\chi_{\{w\}} = \lim_\alpha \chi_{E_\alpha}\chi_{\{w\}}$ in $X$. If $E = \{w \in [0,1] : \lim_\alpha \chi_{E_\alpha}(w) = 1\}$, then it follows that $T$ is the operator of multiplication by $\chi_E$. But $T\mathbf{1} = \lim_\alpha P(E_\alpha)\mathbf{1} = \lim_\alpha \chi_{E_\alpha} = \chi_E$ shows that $\chi_E \in X$, that is, $E \in \Sigma$. Accordingly, $P(\Sigma)$ is closed in $L_s(X)$. However, $P$ is not a closed measure, that is, $P(\Sigma)$ is not a complete subset of $L_s(X)$. For, if $E$ is a non-Borel subset of $[0,1]$ and $\mathcal{F}$ is the family of finite subsets of $E$ directed by inclusion, then $\{P(F) : F \in \mathcal{F}\}$ is a Cauchy set from $P(\Sigma)$ which has no limit in $L_s(X)$. ■

The next example shows that the equicontinuity of $P$ in Proposition 3.5 and Corollary 3.5.2 cannot be omitted in general.

EXAMPLE 3.8. Let $X$ denote the Hilbert space $H = L^2([0,1])$ equipped with its weak topology, in which case $X$ is a quasicomplete lcHs. Let $\Sigma$ denote the Lebesgue measurable subsets of $\Omega = [0,1]$ and define a spectral measure $P : \Sigma \to L_s(X)$ by $P(E)\psi = \chi_E\psi$ for $E \in \Sigma$ and $\psi \in X$. Then $P$ is not equicontinuous [19; Proposition 4]. If $\lambda : \Sigma \to [0,1]$

is Lebesgue measure, then we note that $\langle P, \xi \rangle$ is absolutely continuous with respect to $\lambda$ (in symbols $\langle P, \xi \rangle \ll \lambda$) for each $\xi \in (L_{\mathrm{s}}(X))'$. Hence, $P$ is a closed measure (see [14; IV, Theorem 7.3]). However, since $P$ is a non-atomic, selfadjoint spectral measure it follows from a result of H. Dye [8; Lemma 2.3] that $P(\Sigma)$ is not closed for the weak operator topology in $L(H)$. That is, $P(\Sigma)$ is not a closed (hence, not a complete) subset of $L_{\mathrm{s}}(X) = L_{\mathrm{w}}(H)$. ∎

The following criterion, which was used in Example 3.8, is of practical importance for determining the closedness of a given spectral measure.

PROPOSITION 3.9. *Let $X$ be a lcHs and $P : \Sigma \to L_{\mathrm{s}}(X)$ be a spectral measure. Then $P$ is a closed measure iff there exists a localizable measure $\lambda : \Sigma \to [0, \infty]$ such that $\langle P, \xi \rangle \ll \lambda$ for each $\xi \in (L_{\mathrm{s}}(X))'$.*

Proof. If such a localizable measure exists, then $P$ is a closed measure [14; IV, Theorem 7.3]. The converse follows from [13; Corollary 13]. ∎

If $X$ is a lcHs, then the *sequential completion* $\widehat{X}$ of $X$ is defined to be the sequential closure of $X$ in its completion $\overline{X}$. For the definition of the *quasicompletion* $\widetilde{X}$ of $X$ we refer to [15; §23.1]. Then $X \subseteq \widehat{X} \subseteq \widetilde{X} \subset \overline{X}$ and $X$ is dense in each of $\widehat{X}, \widetilde{X}$ and $\overline{X}$. Moreover, each $T \in L(X)$ has unique extensions $\widehat{T} \in L(\widehat{X})$, $\widetilde{T} \in L(\widetilde{X})$ and $\overline{T} \in L(\overline{X})$; see [22; Lemma 1.6]. So, given a spectral measure $P : \Sigma \to L_{\mathrm{s}}(X)$ we can, for each $E \in \Sigma$, define continuous projections $\widehat{P}(E), \widetilde{P}(E)$ and $\overline{P}(E)$ in $\widehat{X}, \widetilde{X}$ and $\overline{X}$, respectively. The set functions $\widehat{P} : \Sigma \to L_{\mathrm{s}}(\widehat{X}), \widetilde{P} : \Sigma \to L_{\mathrm{s}}(\widetilde{X})$ and $\overline{P} : \Sigma \to L_{\mathrm{s}}(\overline{X})$ so defined are finitely additive and multiplicative. Whereas $\widehat{P}$ is *always* $\sigma$-additive (i.e. a spectral measure), this is *not* the case for $\widetilde{P}$ and $\overline{P}$ in general. However, if $P$ is equicontinuous, then also $\widetilde{P}$ and $\overline{P}$ are spectral measures. All of these facts concerning $\widehat{P}, \widetilde{P}$ and $\overline{P}$ can be found in [22; §2].

LEMMA 3.10. *Let $X$ be a lcHs and $P : \Sigma \to L_{\mathrm{s}}(X)$ be a spectral measure such that $\widehat{P}, \widetilde{P}$ and $\overline{P}$ are $\sigma$-additive. Then $P$ is a closed measure iff $\widehat{P}$ is a closed measure iff $\widetilde{P}$ is a closed measure iff $\overline{P}$ is a closed measure.*

Proof. If either of $\widehat{P}, \widetilde{P}$ or $\overline{P}$ is closed, then so is $P$ by Proposition 3.9.

Conversely, suppose $P$ is closed. By Proposition 3.9 there is a localizable measure $\lambda : \Sigma \to [0, \infty]$ such that $\langle P, \xi \rangle \ll \lambda$ for all $\xi \in (L_{\mathrm{s}}(X))'$. Let $\widehat{x} \in \widehat{X}$ and $\eta \in (\widehat{X})'$. Then $\varrho = \eta|_X$ belongs to $X'$. Moreover, there is a net $\{x_\alpha\} \subseteq X$ such that $x_\alpha \to \widehat{x}$ in $\widehat{X}$. Let $E \in \Sigma$ satisfy $\lambda(E) = 0$, in which case $\langle P(E)x_\alpha, \varrho \rangle = 0$ for all $\alpha$. By continuity of $\widehat{P}(E)$ we have $P(E)x_\alpha = \widehat{P}(E)x_\alpha \to \widehat{P}(E)\widehat{x}$ and so $\langle \widehat{P}(E)x_\alpha, \eta \rangle \to \langle \widehat{P}(E)\widehat{x}, \eta \rangle$. Since $\langle \widehat{P}(E)x_\alpha, \eta \rangle = \langle P(E)\widehat{x}_\alpha, \varrho \rangle = 0$, for all $\alpha$, it follows that $\langle \widehat{P}(E)\widehat{x}, \eta \rangle = 0$. Accordingly, $\langle \widehat{P}, \nu \rangle \ll \lambda$ for all $\nu \in (L_{\mathrm{s}}(\widehat{X}))'$ and so $\widehat{P}$ is closed by Proposition 3.9. The argument for $\widetilde{P}$ and $\overline{P}$ is similar. ∎

Given a lcHs $X$ and subset $\mathcal{M} \subseteq L(X)$ we denote by $\overline{\mathcal{M}}^{\mathrm{w}}$ (resp. $\overline{\mathcal{M}}^{\mathrm{s}}$) the closure of $\mathcal{M}$ in $L_{\mathrm{w}}(X)$ (resp. $L_{\mathrm{s}}(X)$). The sequential closure, in $L_{\mathrm{s}}(X)$, of the linear hull of $\mathcal{M}$ is denoted by $[L_{\mathrm{s}}(X)]_{\mathcal{M}}$. An important property is that if $X$ is quasicomplete and $P : \Sigma \to L_{\mathrm{s}}(X)$ is a closed, equicontinuous spectral measure, then $P(\Sigma)$ contains all projections in the weakly closed operator algebra generated by $P(\Sigma)$, that is, $\overline{P(\Sigma)}^{\mathrm{w}} \cap \mathbb{P}(X) = P(\Sigma)$

(see [3; Proposition 4.7]). See also [6; Corollary 4.22] under the additional hypothesis that $L_{\mathrm{s}}(X)$ is sequentially complete. The classical Banach space case can be found in [7; XVII, Corollary 3.7 and Theorem 3.27]. We now show that this fact holds without any completeness assumptions on $X$ or $L_{\mathrm{s}}(X)$ whatsoever.

PROPOSITION 3.11. *Let $X$ be a lcHs and $P : \Sigma \to L_{\mathrm{s}}(X)$ be a closed, equicontinuous spectral measure. Then*

$$\overline{P(\Sigma)}^{\mathrm{w}} \cap \mathbb{P}(X) = P(\Sigma). \tag{4}$$

Proof. By the remarks prior to Lemma 3.10 it follows that $\widetilde{P} : \Sigma \to L_{\mathrm{s}}(\widetilde{X})$ is also a spectral measure, necessarily equicontinuous [22; Lemma 1.8]. By Lemma 3.10, $\widetilde{P}$ is a closed measure, and hence $\widetilde{P}(\Sigma)$ is an equicontinuous, Bade complete B.a. in $L(\widetilde{X})$; see Proposition 3.4. Then [3; Proposition 4.7] implies that

$$\overline{\widetilde{P}(\Sigma)}^{\mathrm{w}} \cap \mathbb{P}(\widetilde{X}) = \widetilde{P}(\Sigma). \tag{5}$$

Let $T \in \overline{P(\Sigma)}^{\mathrm{w}} \cap \mathbb{P}(X)$. Choose a net $\{P(E_\alpha)\}$ converging to $T$ in $L_{\mathrm{w}}(X)$. With $\widetilde{T} \in L(\widetilde{X})$ being the (unique) continuous extension of $T$ to $\widetilde{X}$ it is routine to check that the equicontinuity of $\{P(E_\alpha)\}$ implies that $\widetilde{P}(E_\alpha) \to \widetilde{T}$ in $L_{\mathrm{w}}(\widetilde{X})$. Since $\widetilde{T} \circ \widetilde{T}$ coincides with $\widetilde{T}$ on $X$ (as $T \in \mathbb{P}(X)$) and $X$ is dense in $\widetilde{X}$ it follows that $\widetilde{T} \in \mathbb{P}(\widetilde{X})$. Then (5) implies that $\widetilde{T} = \widetilde{P}(E)$ for some $E \in \Sigma$, and hence $T = P(E) \in P(\Sigma)$. ■

Remark 3.12. (a) Under the assumptions of Proposition 3.11 it also follows (cf. Proposition 3.5) that $\overline{P(\Sigma)}^{\mathrm{s}} = P(\Sigma)$. This poses the question of whether $\mathbb{P}(X)$ can be omitted in (4). Unfortunately, this is not possible. For, if $H$ is the Hilbert space of Example 3.8, now considered with respect to its norm topology, and $P : \Sigma \to L_{\mathrm{s}}(H)$ is the spectral measure given there, then it is known [8; Lemma 2.3] that $\overline{P(\Sigma)}^{\mathrm{w}}$ is dense in the set of all (selfadjoint) operators of the form $\int_\Omega f\,dP$ for $\Sigma$-measurable functions $f$ satisfying $0 \le f \le \mathbf{1}$.

(b) For equicontinuous spectral measures $P$ which are not closed the equality (4) may fail, that is, the inclusion $P(\Sigma) \subseteq \overline{P(\Sigma)}^{\mathrm{w}} \cap \mathbb{P}(X)$ may be proper. Indeed, let $X$ and $P$ be as in Example 3.2, in which case $X$ is complete and $P$ is equicontinuous but not closed. It was shown in Example 3.2 that $P(\Sigma)$ is a proper subset of $\overline{P(\Sigma)}^{\mathrm{s}}$ (which equals $\overline{P(\Sigma)}^{\mathrm{s}} \cap \mathbb{P}(X)$ by equicontinuity) and hence, $P(\Sigma)$ is also a proper subset of $\overline{P(\Sigma)}^{\mathrm{w}} \cap \mathbb{P}(X)$.

(c) Proposition 3.11 does not characterize closed spectral measures (not even equicontinuous ones). For, let $X$ and $P$ be as in Example 2.1. Since $X = X_\sigma$ we have $L_{\mathrm{s}}(X) = L_{\mathrm{w}}(X)$, and hence Example 3.7 implies that $P(\Sigma)$ is a closed set in $L_{\mathrm{w}}(X)$. In particular, (4) is satisfied. However, it was shown in Example 3.7 that $P$ is not a closed measure.

This same example also shows that the equality $\overline{P(\Sigma)}^{\mathrm{w}} = P(\Sigma)$ does not imply $P(\Sigma)$ is relatively $w$-compact in $L_{\mathrm{s}}(X)$; see Example 2.1. ■

The critical feature of the example in Remark 3.12(c) is that $X$ is only sequentially complete. A slight strengthening on the completeness properties of $X$ gives the following characterization of closed spectral measures in terms of (4).

PROPOSITION 3.13. *Let $X$ be a quasicomplete lcHs and $P : \Sigma \to L_{\mathrm{s}}(X)$ be an equicontinuous spectral measure. Then $P$ is a closed measure iff (4) is satisfied.*

P r o o f. One direction is clear by Proposition 3.11. For the other, suppose that (4) holds. To establish that $P$ is closed it suffices to show that $P(\Sigma)$ is a closed subset of $L_{\mathrm{s}}(X)$; see Corollary 3.5.2. So, let $P(E_\alpha) \to T$ in $L_{\mathrm{s}}(X)$. The equicontinuity of $P(\Sigma)$ implies that the limit operator $T \in \mathbb{P}(X)$. Then $P(E_\alpha) \to T$ in $\mathbb{P}(X) \cap L_{\mathrm{w}}(X)$ and so (4) implies that $T \in P(\Sigma)$. ∎

R e m a r k 3.14. The quasicompleteness of $X$ in Proposition 3.13 can be weakened to the requirement that $[X]_{Px}$ is quasicomplete, for each $x \in X$; see Remark 3.6. ∎

The equicontinuity of $P$ in Proposition 3.13 can be removed if additional completeness assumptions are made on $L_{\mathrm{s}}(X)$.

PROPOSITION 3.15. *Let $X$ be a lcHs and $P : \Sigma \to L_{\mathrm{s}}(X)$ be a spectral measure. Suppose that either* $L_{\mathrm{s}}(X)$ *or* $[L_{\mathrm{s}}(X)]_P$ *is quasicomplete. Then $P$ is a closed measure iff* (4) *is satisfied.*

P r o o f. Lemma 1.3 implies that $P(\Sigma)$ is relatively $w$-compact, and hence so is its closed, convex hull $K = \overline{\mathrm{co}}(P(\Sigma))$ in $L_{\mathrm{s}}(X)$ (see [33; Theorem 1]). With respect to composition as multiplication $L_{\mathrm{w}}(X)$ is a locally convex algebra with unit the identity operator. Clearly, $K$ is commutative and, by using the multiplicativity of $P$, it can be checked that $\mathrm{co}(P(\Sigma))$ is closed under multiplication. Since multiplication is separately continuous it follows that $K$ is also closed under multiplication. Each element of $\mathrm{co}(P(\Sigma))$ can be written in the form $I_P(f)$ for a suitable $f \in \mathrm{sim}(\Sigma)$ with $0 \le f \le \mathbf{1}$. From this observation and the fact that $I_P$ is a linear and algebra homomorphism on $\mathcal{L}^1(P)$, it follows that $(S + T - ST) \in \mathrm{co}(P(\Sigma))$ whenever $T, S \in \mathrm{co}(P(\Sigma))$. Again by separate continuity of multiplication it follows that $(S+T-ST) \in K$ whenever $T, S \in K$. Hence, $K \subseteq L_{\mathrm{w}}(X)$ is a *spectral carrier* in the sense of [9; Definition 2.1].

Let $\partial_{\mathrm{e}} K$ denote the set of extreme points of $K$. It follows from [9; Theorem 2.10] that $\mathbb{P}(X) \cap K = \partial_{\mathrm{e}} K$. Since $\overline{P(\Sigma)}^{\mathrm{w}} \subseteq K$ it is clear that $\mathbb{P}(X) \cap \overline{P(\Sigma)}^{\mathrm{w}} \subseteq \partial_{\mathrm{e}} K$. But it is also known that $\partial_{\mathrm{e}} K \subseteq \overline{P(\Sigma)}^{\mathrm{s}}$; see [14; p. 113, Corollary 2], where a hypothesis is missing, namely the space $X$ there should be quasicomplete. So, $\partial_{\mathrm{e}} K \subseteq \overline{P(\Sigma)}^{\mathrm{s}} \subseteq \overline{P(\Sigma)}^{\mathrm{w}}$ and, since $\partial_{\mathrm{e}} K \subseteq \mathbb{P}(X)$, it follows that $\partial_{\mathrm{e}} K \subseteq \mathbb{P}(X) \cap \overline{P(\Sigma)}^{\mathrm{w}}$. Accordingly,

$$\partial_{\mathrm{e}} K = \mathbb{P}(X) \cap \overline{P(\Sigma)}^{\mathrm{w}}.$$

Assume now that (4) holds, in which case $P(\Sigma) = \partial_{\mathrm{e}} K$. Since $P : \Sigma(P) \to P(\Sigma)$ is a B.a. isomorphism and $P(\Sigma) = \partial_{\mathrm{e}} K$ is abstractly complete by [9; Theorem 2.11(a)] it follows that $\Sigma(P)$ is abstractly complete as a B.a. Suppose that $\chi_{E_\alpha} \downarrow 0$ in $\Sigma(P)$. Then $P(E_\alpha) \downarrow 0$ in $\partial_{\mathrm{e}} K$. Accordingly, $P(E_\alpha) \to \bigwedge_\alpha P(E_\alpha) = 0$ in $L_{\mathrm{w}}(X)$ [9; Theorem 2.11(c)]. Then Lemma 1.4 implies that $P$ is a closed measure when considered in $Y = L_{\mathrm{w}}(X)$. Since $Y$ is $L_{\mathrm{s}}(X)$ equipped with its weak topology it follows that also $P : \Sigma \to L_{\mathrm{s}}(X)$ is a closed measure (see [32; Proposition 2]).

Conversely, assume that $P$ is a closed measure. Since the integration map $I_P$ is injective it follows from [14; p. 113, Corollary 3], where again the quasicompleteness hypothesis is missing, that $P(\Sigma) = \partial_{\mathrm{e}} K$. Since $\partial_{\mathrm{e}} K = \mathbb{P}(X) \cap \overline{P(\Sigma)}^{\mathrm{w}}$ it is clear that (4) holds. ∎

We point out that the example in Remark 3.12(c), for which (4) holds but the spectral measure is not closed, shows that the completeness assumptions in Proposition 3.15 cannot be omitted in general.

It is worthwhile to collect some of the main results of §3 in a single statement.

PROPOSITION 3.16. *Let $X$ be a lcHs and $P : \Sigma \to L_s(X)$ be a spectral measure. The following statements are equivalent*:

(i) *$P$ is a closed measure.*

(ii) *There is a localizable measure $\lambda : \Sigma \to [0, \infty]$ such that $\langle P, \xi\rangle \ll \lambda$ for all $\xi \in (L_s(X))'$.*

*If $P$ is equicontinuous*, *then* (i)–(ii) *are equivalent with the following statements*:

(iii) *$P(\Sigma)$ is a complete subset of $L_s(X)$.*

(iv) *$P(\Sigma)$ is a Bade complete B.a. in $L(X)$.*

*If $P$ is equicontinuous and $[X]_{Px}$ is quasicomplete*, *for each $x \in X$*, *then* (i)–(iv) *are equivalent with the following statements*:

(v) $\overline{P(\Sigma)}^{\mathrm{w}} \cap \mathbb{P}(X) = P(\Sigma)$.

(vi) *$P(\Sigma)$ is a closed subset of $L_s(X)$.*

*If $[L_s(X)]_P$ is sequentially complete*, *then* (i)–(ii) *are equivalent with*

(vii) *$\mathcal{L}^1(P)$ is a complete lcHs.*

*If either $[L_s(X)]_P$ or $L_s(X)$ is quasicomplete*, *then* (i)–(ii) *are equivalent with* (v).

Proof. (i)⇔(ii) is Proposition 3.9. If $P$ is equicontinuous, then (i)⇔(iii) and (i)⇔(iv) are Proposition 3.5. Under the extra requirement (i.e. in addition to equicontinuity of $P$) of quasicompleteness of $[X]_{Px}$, for each $x \in X$, we have (i)⇔(v) by Proposition 3.13 and Remark 3.14, and (i)⇔(vi) by Remark 3.6. If $[L_s(X)]_P$ is sequentially complete, then (i)⇔(vii) is Lemma 1.5. Finally, if either of $L_s(X)$ or $[L_s(X)]_P$ is quasicomplete, then the equivalence of (i)–(ii) with (v) follows from Proposition 3.15. ∎

We remark that a sufficient condition for $[L_s(X)]_P$ to be sequentially complete is that $X$ is barrelled and $[X]_{Px}$ is sequentially complete, for each $x \in X$ (see [23; Proposition 10]). However, the space $[L_s(X)]_P$ can be sequentially complete without each space $[X]_{Px}$, $x \in X$, being sequentially complete [23; Remark 11(ii)]. In general, it is not possible (unfortunately) to relax the barrelledness of $X$ to quasibarrelledness [23; Remark 11(i)].

## 4. Spectral measures and B.a.'s of projections

In Section 3 it was noted that the range of any spectral measure is a Bade $\sigma$-complete B.a. and the range of any closed spectral measure is a Bade complete B.a. It is well known in the Banach space setting [7] and for certain situations in the lc-setting [34] that the Bade complete and $\sigma$-complete B.a.'s are precisely the ranges of spectral measures. The aim of this section is to further clarify this interplay.

A B.a. $\mathcal{M} \subseteq L(X)$ is said to have the *ordered* (resp. $\sigma$-*ordered*) *convergence property* if it is abstractly complete (resp. $\sigma$-complete) and $\lim_\alpha P_\alpha = \bigvee_\alpha P_\alpha$ (resp. $\bigwedge_\alpha P_\alpha$) in $L_s(X)$ whenever $\{P_\alpha\} \subseteq \mathcal{M}$ is an upwards (resp. downwards) directed family (resp. countable family).

PROPOSITION 4.1. (i) *Every Bade complete* (*resp. $\sigma$-complete*) *B.a. $\mathcal{M} \subseteq L(X)$ with the ordered* (*resp. $\sigma$-ordered*) *convergence property is the range of an $L_s(X)$-valued spectral measure defined on the Borel* (*resp. Baire*) *sets of the Stone space of $\mathcal{M}$.*

(ii) *Every equicontinuous Bade complete* (*resp. $\sigma$-complete*) *B.a. $\mathcal{M} \subseteq L(X)$ has the ordered* (*resp. $\sigma$-ordered*) *convergence property.*

(iii) *The range of any $L_s(X)$-valued spectral measure is a Bade $\sigma$-complete B.a. with the $\sigma$-ordered convergence property.*

For Banach spaces this result can be found in [7; XVII]. For more general spaces $X$ parts (i) & (ii) can be found in [34; p. 299]. Part (iii) follows from Lemma 3.1 and [18; Lemma 1] where the proof given does not depend on the normability of the underlying space.

The following result is well known in the Banach space setting [7; XVII, Lemma 3.23]; for lcH-spaces see [34; Proposition 3.17].

PROPOSITION 4.2. *Let $X$ be a quasicomplete lcHs and $P : \Sigma \to L_s(X)$ be an equicontinuous spectral measure. Then $\overline{P(\Sigma)}^{s}$ is a Bade complete B.a. in $L(X)$.*

Remark 4.3. (a) Since $\overline{P(\Sigma)}^{s}$ is again equicontinuous it follows from Proposition 4.1(ii) that $\overline{P(\Sigma)}^{s}$ has the ordered convergence property.

(b) In general it is not possible to remove the quasicompleteness of $X$. For, if $X$ and $P$ are as in Example 2.1, then $P$ is equicontinuous but $X$ is not quasicomplete. As noted in Example 3.7, $\overline{P(\Sigma)}^{s} = P(\Sigma)$ but $P$ is not a closed measure. Then Proposition 3.5 implies that $\overline{P(\Sigma)}^{s} = P(\Sigma)$ is not a Bade complete B.a.

(c) It is also not possible to remove the equicontinuity of $P$ in Proposition 4.2. Let $X$ and $P$ be as in Example 3.8. Then $X$ is quasicomplete, but $P$ is not equicontinuous. Since $\overline{P(\Sigma)}^{s} = \overline{P(\Sigma)}^{w}$ contains non-projection operators (cf. Remark 3.12(a)) it is not even a B.a. ∎

Despite the example of Remark 4.3(b) it is possible to extend Proposition 4.2 further, as seen by the following result. Example 2.5 shows that this is a genuine extension of Proposition 4.2.

PROPOSITION 4.4. *Let $X$ be a lcHs and $P : \Sigma \to L_s(X)$ be an equicontinuous spectral measure. If $[X]_{Px}$ is quasicomplete, for each $x \in X$, then $\overline{P(\Sigma)}^{s}$ is a Bade complete B.a. in $L_s(X)$.*

Proof. Let $\mathcal{M} = \overline{P(\Sigma)}^{s}$. The equicontinuity of $P$ implies that $\mathcal{M}$ is a B.a. in $L(X)$, is closed in $L_s(X)$ and is again equicontinuous. If $B \in \mathcal{M}$, then $B = \lim P_\alpha$ in $L_s(X)$ for some net $\{P_\alpha\} \subseteq P(\Sigma)$. Fix $x \in X$. Then $P_\alpha x \in [X]_{Px}$, for each $\alpha$, and $\{P_\alpha x\}$ is a bounded net converging to $Bx$. By quasicompleteness of $[X]_{Px}$ it follows that $Bx \in [X]_{Px}$. This shows that $\mathcal{M}(x) \subseteq [X]_{Px}$, for each $x \in X$, and it follows by an argument as in

Remark 3.6 that $\mathcal{M}$ is actually a complete subset of $L_s(X)$. So, by Proposition 3.5 it suffices to show that $\mathcal{M}$ is the range of some spectral measure, necessarily equicontinuous.

Let $\widetilde{X}$ be the quasicompletion of $X$. It was seen in Section 3 that $\widetilde{P}: \Sigma \to L_s(\widetilde{X})$ is again an equicontinuous spectral measure. By Proposition 4.2 it follows that $\mathcal{N} = \overline{\widetilde{P}(\Sigma)}^{\,s}$ is a Bade complete B.a. in $L(\widetilde{X})$. By Proposition 4.1 there is a $\sigma$-algebra $\Lambda$ and an equicontinuous spectral measure $Q : \Lambda \to L_s(\widetilde{X})$ such that $Q(\Lambda) = \mathcal{N}$. It is routine to check that if $\widetilde{\mathcal{M}} = \{\widetilde{B} : B \in \mathcal{M}\} \subseteq L(\widetilde{X})$, then $\widetilde{\mathcal{M}} = \mathcal{N}$. Since $X$ is invariant for all elements of $\mathcal{N}$ it follows that $R : \Lambda \to L_s(X)$ defined by $R(E) = Q(E)|_X$, for each $E \in \Lambda$, is a spectral measure with range $\mathcal{M}$. ■

A natural question is the converse of Proposition 4.4: Is every Bade complete B.a. $\mathcal{M} \subseteq L(X)$ actually a closed subset of $L_s(X)$? For $\mathcal{M}$ equicontinuous this is indeed the case by Propositions 3.5 and 4.1. The following example shows this is not the case in general.

Example 4.5. Let $X$ and $P$ be as in Example 3.8. It was shown there that $\mathcal{M} = P(\Sigma)$ is not a closed subset of $L_s(X)$. However, it follows from [29; Lemma 1] that $\mathcal{M}$ is a Bade complete B.a. ■

One of the desirable features of certain types of B.a.'s is their realization as the range of a spectral measure. Proposition 4.1 shows that this is true of all equicontinuous Bade $\sigma$-complete B.a.'s. However, there are plenty of non-trivial spectral measures which fail to be equicontinuous [19; Proposition 4]. By Lemma 3.1 the ranges of such spectral measures are still Bade $\sigma$-complete B.a.'s. So, it is useful to have available criteria without the equicontinuity condition which ensure that Bade $\sigma$-complete B.a.'s are the range of a spectral measure.

Proposition 4.6. *Let $X$ be a lcHs and $\mathcal{M} \subseteq L(X)$ be a B.a. with the $\sigma$-monotone property. Then $\mathcal{M}$ is the range of a spectral measure.*

We immediately state the following consequence.

Corollary 4.6.1. *Let $X$ be a lcHs and $\mathcal{M} \subseteq L(X)$ be a B.a. The following statements are equivalent*:

(i) *$\mathcal{M}$ is the range of a spectral measure.*
(ii) *$\mathcal{M}$ has the $\sigma$-monotone property.*
(iii) *$\mathcal{M}$ has the $\sigma$-ordered convergence property.*

Proof. (i)⇒(iii) by Proposition 4.1(iii) and (iii)⇒(ii) is clear from the two definitions concerned. Finally, (ii)⇒(i) by Proposition 4.6. ■

Proof of Proposition 4.6. The argument of Theorem 1(b) in [18] applies in a general lcHs and so, putting $G = X'$ there, it follows from the $\sigma$-monotone property that $\mathcal{M}$ is $X'$-$\sigma$-complete in the sense defined in [18]. Moreover, the $G$-operator topology defined in [18] is then precisely the weak operator topology. By Theorem 1(a) of [18], whose proof again carries over to the lc-setting, it follows that $\mathcal{M}$ has the property that $\lim_n P_n = \bigvee_n P_n$ (resp. $\bigwedge_n P_n$) in $L_w(X)$ whenever $\{P_n\} \subseteq \mathcal{M}$ is an increasing (resp. decreasing) sequence.

Since $L(X) \subseteq L(X_\sigma)$, as vector spaces, and convergence in $L_{\mathrm{w}}(X)$ implies convergence in $L_{\mathrm{s}}(X_\sigma)$, it follows that $\lim_n P_n = \bigvee_n P_n$ (resp. $\bigwedge_n P_n$) in $L_{\mathrm{s}}(X_\sigma)$ whenever $\{P_n\} \subseteq \mathcal{M}$ is an increasing (resp. decreasing) sequence. That is, $\mathcal{M}$ has the $\sigma$-ordered convergence property in $L_{\mathrm{s}}(X_\sigma)$. Moreover, $\mathcal{M}$ is Bade $\sigma$-complete by Lemma 3.3. Interpreting $\mathcal{M} \subseteq L_{\mathrm{s}}(X_\sigma)$ it is clear that $\mathcal{M}$ is still Bade $\sigma$-complete in $L_{\mathrm{s}}(X_\sigma)$. Applying Proposition 4.1 to $\mathcal{M}$ in $L_{\mathrm{s}}(X_\sigma)$ it follows that there exists a spectral measure $P : \Sigma \to L_{\mathrm{s}}(X_\sigma)$ with $P(\Sigma) = \mathcal{M}$. Since $\mathcal{M} \subseteq L(X)$ we see that $P$ is actually $L_{\mathrm{s}}(X)$-valued; it is still $\sigma$-additive in $L_{\mathrm{s}}(X)$ by the Orlicz–Pettis lemma. ∎

The above result occurs in [27] under the additional hypothesis that $X$ is quasicomplete and $L_{\mathrm{s}}(X)$ is sequentially complete. The proof given there is based on a version of Kluvánek's extension theorem for vector measures. This states that if $Y$ is a quasicomplete lcHs (actually, sequentially complete suffices) and $m : \Lambda \to Y$ is a $\sigma$-additive vector measure defined on an *algebra* of sets $\Lambda$ such that $\lim_{n\to\infty} m(E_n)$ exists in $Y$ whenever $\{E_n\}_{n=1}^\infty \subseteq \Lambda$ is a monotone sequence, then there exists a vector measure $m : \Sigma_\Lambda \to Y$ which extends $m$ from $\Lambda$ to the $\sigma$-algebra $\Sigma_\Lambda$ that it generates. This result was applied in [27] to the sequentially complete space $Y = L_{\mathrm{s}}(X)$. The following example shows that Kluvánek's extension theorem fails to hold without the sequential completeness assumption, and hence the argument of [27] is not applicable in the general setting of Proposition 4.6. So, the proof of Proposition 4.6, as given above, requires a different argument (by necessity).

Let $Z = \mathbb{R}^{[0,1]}$, equipped with the topology of pointwise convergence on $\Omega = [0,1]$, and $Z_1$ denote the subspace of $Z$ consisting of the Baire functions of class 1. That is, $f \in Z$ belongs to $Z_1$ iff there is a sequence of continuous functions $f_n : \Omega \to \mathbb{R}$, for $n \in \mathbb{N}$, such that $f_n \to f$ in $Z$. Let $\Lambda$ denote the family of all subsets $E \subseteq \Omega$ such that $\chi_E \in Z_1$. Then $\Lambda$ is an algebra of sets and contains all subintervals of $\Omega$. Accordingly, $\Sigma_\Lambda$ is the $\sigma$-algebra of all Borel subsets of $\Omega$. Let $X$ denote the subspace of $Z$ consisting of all Baire functions of class 2 on $\Omega$. That is, $f \in Z$ belongs to $X$ iff there is a sequence $\{f_n\}_{n=1}^\infty \subseteq Z_1$ such that $f_n \to f$ in $Z$. Define $P : \Lambda \to L_{\mathrm{s}}(X)$ by $P(E)f = \chi_E f$, for $f \in X$ and $E \in \Lambda$. Then $P$ is equicontinuous, multiplicative and $\sigma$-additive on $\Lambda$. Moreover, $\lim_{n\to\infty} P(E_n)$ exists in $L_{\mathrm{s}}(X)$ whenever $\{E_n\}_{n=1}^\infty \subseteq \Lambda$ is a monotone sequence. However, $P$ cannot be extended to an $L_{\mathrm{s}}(X)$-valued measure on $\Sigma_\Lambda$.

The next result gives characterizations of closed spectral measures additional to those already listed in Proposition 3.16. It also shows that equicontinuity is not needed for the equivalence of (i) and (iv) in Proposition 3.5.

PROPOSITION 4.7. *Let $X$ be a lcHs and $\mathcal{M} \subseteq L(X)$ be a B.a. The following statements are equivalent*:

(i) *$\mathcal{M}$ has the monotone property.*
(ii) *$\mathcal{M}$ has the ordered convergence property.*
(iii) *$\mathcal{M}$ coincides with the range of some closed*, *$L_{\mathrm{s}}(X)$-valued spectral measure.*

Proof. (ii)⇒(i) is clear from the definitions involved.

(i)⇒(iii). An analogous argument as in (part of) the proof of Proposition 4.6 shows that $\mathcal{M}$ is Bade complete and hence, in particular, is complete as an abstract B.a. By

Corollary 4.6.1 there is a spectral measure $P : \Sigma \to L_s(X)$ such that $\mathcal{M} = P(\Sigma)$. An argument as in the proof of Proposition 3.15 shows that $P$ is a closed measure.

(iii)⇒(ii). Suppose $P : \Sigma \to L_s(X)$ is a closed measure such that $\mathcal{M} = P(\Sigma)$. Then $\mathcal{M}$ is complete as an abstract B.a. by Proposition 3.4. Let $P(E_\alpha) \uparrow P(E)$ in $\mathcal{M}$. Since the integration map is always a B.a. isomorphism of $\Sigma(P)$ onto $P(\Sigma)$ it follows that $\chi_{E\setminus E_\alpha} \downarrow 0$ in $\Sigma(P)$. It follows from Lemma 1.4 that $P(E_\alpha) \to P(E)$ in $L_s(X)$. A similar argument shows that $P(E_\alpha) \to P(E)$ in $L_s(X)$ if $P(E_\alpha) \downarrow P(E)$ in $\mathcal{M}$. Accordingly, $\mathcal{M}$ has the ordered convergence property. ∎

Proposition 3.4 implies that any B.a. $\mathcal{M}$ satisfying either of (i)–(iii) in Proposition 4.7 is necessarily Bade complete. If, in addition, $\mathcal{M}$ is equicontinuous, then the converse is also true by Proposition 4.1.

Question 3. Does *every* Bade complete (resp. $\sigma$-complete) B.a. $\mathcal{M}$ satisfy (i)–(iii) of Proposition 4.7 (resp. Corollary 4.6.1) or is equicontinuity of $\mathcal{M}$ necessary for this to be the case? ∎

Given a lcHs $X$ let $L_b(X)$ denote $L(X)$ equipped with the topology of uniform convergence on the bounded subsets of $X$. The seminorms generating this topology are given by $q_B : T \mapsto \sup\{q(Tx) : x \in B\}$, for $T \in L(X)$, where $q \in \mathcal{P}(X)$ and $B \subseteq X$ is any bounded set.

Proposition 4.8. *Let $X$ be a lcHs and $\mathcal{M} \subseteq L(X)$ be a Bade $\sigma$-complete B.a. with the $\sigma$-ordered convergence property. Then $\mathcal{M}$ is a bounded subset of $L_b(X)$. If $X$ is quasibarrelled, then $\mathcal{M}$ is equicontinuous.*

Proof. By Proposition 4.6, $\mathcal{M}$ is the range of a spectral measure and so the boundedness of $\mathcal{M}$ in $L_b(X)$ follows from [22; Lemma 1.3]. The claim about $\mathcal{M}$ when $X$ is quasibarrelled follows from Lemma 1.10. ∎

The space $L_b(X)$ arises in other related contexts. Given a set $M \subseteq L_b(X)$ let $\overline{\mathcal{M}}^{\mathrm{b}}$ denote its closure in $L_b(X)$.

Lemma 4.9. *Let $X$ be a lcHs and $P : \Sigma \to L_s(X)$ be a spectral measure. Then $\overline{P(\Sigma)}^{\mathrm{b}}$ is a B.a. of projections satisfying*

$$\overline{P(\Sigma)}^{\mathrm{b}} \subseteq \overline{P(\Sigma)}^{\mathrm{s}}. \tag{6}$$

Proof. Since the topology of $L_b(X)$ is stronger than that of $L_s(X)$ the inclusion (6) is clear. Let $\{P_\alpha\} \subseteq P(\Sigma)$ be a net such that $P_\alpha \to T$ in $L_b(X)$. Fix $x \in X$ and $q \in \mathcal{P}(X)$. Since $T$ commutes with each $P_\alpha$ we have

$$q((T^2 - P_\alpha^2)x) = q((T - P_\alpha)(T + P_\alpha)x) \le \sup_{\xi \in B} q((T - P_\alpha)\xi)$$

where $B = \{(T + Q)x : Q \in P(\Sigma)\}$. Since $P(\Sigma)$ is a bounded subset of $L_s(X)$ it is clear that $B \subseteq X$ is a bounded set. It follows that $P_\alpha^2 \to T^2$ in $L_s(X)$. But also $P_\alpha^2 = P_\alpha \to T$ in $L_b(X)$, and hence in $L_s(X)$. Accordingly, $T^2 = T$ and so $\overline{P(\Sigma)}^{\mathrm{b}} \subseteq \mathbb{P}(X)$. Since multiplication in $L_b(X)$ is separately continuous it follows that $\overline{P(\Sigma)}^{\mathrm{b}}$ is actually a B.a. ∎

If $X$ is a Banach space, then $L_{\mathrm{b}}(X)$ is $L(X)$ equipped with the operator norm topology, and hence $\overline{P(\Sigma)}^{\mathrm{b}} = P(\Sigma)$. So, if $P(\Sigma)$ is not a closed set in $L_{\mathrm{s}}(X)$, such as in Example 3.2, then the inclusion (6) is necessarily strict (even for equicontinuous $P$). In certain situations this cannot happen.

PROPOSITION 4.10. *Let $X$ be a lcHs and $P : \Sigma \to L_{\mathrm{s}}(X)$ be a spectral measure which is closed and equicontinuous. Then $\overline{P(\Sigma)}^{\mathrm{b}} = \overline{P(\Sigma)}^{\mathrm{s}}$. In particular, $\overline{P(\Sigma)}^{\mathrm{b}}$ is a Bade complete B.a.*

Proof. Apply Lemma 4.9 and Propositions 3.5 & 4.2. ∎

A spectral measure $P : \Sigma \to L_{\mathrm{s}}(X)$ is called *boundedly $\sigma$-additive* if it is also $\sigma$-additive when considered as being $L_{\mathrm{b}}(X)$-valued, in which case it is denoted by $P_{\mathrm{b}}$. In Banach spaces only trivial spectral measures can be boundedly $\sigma$-additive but in non-normable spaces plenty of such measures exist [31]. Even though the topology of $L_{\mathrm{b}}(X)$ is *not* consistent with the duality $(L_{\mathrm{s}}(X), (L_{\mathrm{s}}(X))')$, it is known that if $L_{\mathrm{s}}(X)$ is sequentially complete and $P : \Sigma \to L_{\mathrm{s}}(X)$ is a boundedly $\sigma$-additive spectral measure, then $P$ is a closed measure iff $P_{\mathrm{b}}$ is a closed measure [31; Theorem 6.2]. Actually, the sequential completeness of $L_{\mathrm{s}}(X)$ is not needed. Indeed, noting that $\mathcal{L}^1(P_{\mathrm{b}}) \subseteq \mathcal{L}^1(P)$ and $\mathcal{L}^1(P_{\mathrm{b}})$ has a basis of neighbourhoods of zero consisting of closed sets with respect to the induced topology from $\mathcal{L}^1(P)$, the proof of [31; Theorem 6.2] can be modified to apply without sequential completeness of $L_{\mathrm{s}}(X)$.

Under certain conditions on $X$ it follows from Propositions 4.1, 4.2 and 4.4 that the closure $\overline{\mathcal{M}}^{\mathrm{s}}$ of an *equicontinuous*, Bade $\sigma$-complete B.a. is a Bade complete B.a. The example in Remark 4.3(c) shows that this fails in general without equicontinuity. That is, not every Bade $\sigma$-complete B.a. $\mathcal{M}$ can be embedded in a Bade complete B.a. by simply forming $\overline{\mathcal{M}}^{\mathrm{s}}$. The problem is that, without equicontinuity, $\overline{\mathcal{M}}^{\mathrm{s}}$ may contain elements other than projections. Under mild completeness assumptions the next result shows that this can be overcome by replacing $\overline{\mathcal{M}}^{\mathrm{s}}$ with $\overline{\mathcal{M}}^{\mathrm{s}} \cap \mathbb{P}(X)$.

PROPOSITION 4.11. *Let $X$ be a lcHs and $\mathcal{M} \subseteq L(X)$ be a B.a. with the $\sigma$-monotone property. If either $L_{\mathrm{s}}(X)$ or $[L_{\mathrm{s}}(X)]_{\mathcal{M}}$ is quasicomplete, then $\mathcal{M}$ can be embedded in a Bade complete B.a., namely $\overline{\mathcal{M}}^{\mathrm{s}} \cap \mathbb{P}(X)$.*

Proof. By Proposition 4.6 there is a spectral measure $P : \Sigma \to L_{\mathrm{s}}(X)$ such that $\mathcal{M} = P(\Sigma)$. The completeness assumptions imply that $K = \overline{\mathrm{co}}(\mathcal{M})$ is $w$-compact and, a spectral carrier in the lc-algebra $L_{\mathrm{w}}(X)$. It was established in the proof of Proposition 3.15 that $\partial_{\mathrm{e}} K \subseteq \overline{P(\Sigma)^{\mathrm{s}}} \cap \mathbb{P}(X) = \overline{\mathcal{M}}^{\mathrm{s}} \cap \mathbb{P}(X)$. Since $\overline{\mathcal{M}}^{\mathrm{s}} \subseteq \overline{P(\Sigma)}^{\mathrm{w}}$ it follows that $\overline{\mathcal{M}}^{\mathrm{s}} \cap \mathbb{P}(X) \subseteq \overline{P(\Sigma)}^{\mathrm{w}} \cap \mathbb{P}(X) = \partial_{\mathrm{e}} K$ (cf. the proof of Proposition 3.15). Accordingly, $\partial_{\mathrm{e}} K = \overline{\mathcal{M}}^{\mathrm{s}} \cap \mathbb{P}(X)$. Since $\partial_{\mathrm{e}} K$ is a B.a. with the monotone property [9; Theorem 2.11], it follows from Lemma 3.3. that $\overline{\mathcal{M}}^{\mathrm{s}} \cap \mathbb{P}(X)$ is a Bade complete B.a. ∎

Somewhat curiously, in certain situations the space $L_{\mathrm{b}}(X)$ is also useful for imbedding Bade $\sigma$-complete B.a.'s into Bade complete ones.

PROPOSITION 4.12 ([29; Proposition 1]). *Let $Y$ be a Fréchet lcHs and denote by $X$ the quasicomplete lcHs $Y'$ equipped with its weak-$*$ topology $\sigma(Y', Y)$. If $\mathcal{M} \subseteq L(X)$ is a Bade $\sigma$-complete B.a., then $\overline{\mathcal{M}}^{\mathrm{b}}$ is a Bade complete B.a. in $L(X)$.*

We remark that typically $\mathcal{M}$ is not equicontinuous and $\overline{\mathcal{M}}^{\mathrm{b}}$ is not a closed set in $L_{\mathrm{s}}(X)$. Moreover, Proposition 4.12 is not a consequence of Proposition 4.11 in general since $L_{\mathrm{s}}(Y'_{\sigma(Y',Y)})$ may fail to be quasicomplete.

PROPOSITION 4.13 ([34; Proposition 1.2]). *Let $X$ be a metrizable lcHs and $\mathcal{M} \subseteq L(X)$ be an abstract $\sigma$-complete B.a. Then $\mathcal{M}$ is equicontinuous.*

It is sometimes the case that results which hold for metrizable spaces continue to hold in spaces where bounded sets are metrizable. Unfortunately, Proposition 4.13 is not in this category.

EXAMPLE 4.14. Let $X$ be the Hilbert space $\ell^2$ equipped with its weak topology. Then $X$ is a separable, quasicomplete lcHs with the property that bounded sets are metrizable. For each $E \in \Sigma = 2^{\mathbb{N}}$ define $P(E) \in L(X)$ to be the projection $P(E)x = \chi_E x$, for $x \in X$. Then $\mathcal{M} = P(\Sigma)$ is a Bade complete B.a. which is not equicontinuous [19; Proposition 4]. ∎

By Proposition 4.13 and the discussion prior to Question 3 it follows that every Bade $\sigma$-complete B.a. in a metrizable lcHs has the $\sigma$-ordered convergence property. There exist other classes of (non-metrizable) lc-spaces with this property.

For instance, let $Y$ be a metrizable lcHs. Let $X = Y_\sigma$ and $\mathcal{M} \subseteq L(X)$ be a Bade $\sigma$-complete B.a. Since $Y$ has its Mackey topology [15; §21.5] we have $L(X) = L(Y)$ as vector spaces [15; (6) in §21.4]. It is clear that $\mathcal{M}$ is also Bade $\sigma$-complete considered as a subset of $L(Y)$. By the previous paragraph $\mathcal{M}$ has the $\sigma$-ordered convergence property in $L(Y)$. Since convergence of a net in $L_{\mathrm{w}}(Y)$ implies its convergence in $L_{\mathrm{s}}(X) = L_{\mathrm{w}}(X)$ it follows that $\mathcal{M} \subseteq L(X)$ has the $\sigma$-ordered convergence property. Or, let $Z = Y'$ equipped with the weak-$*$ topology $\sigma(Y', Y)$. Then any Bade $\sigma$-complete B.a. $\mathcal{N} \subseteq L(Z)$ has the $\sigma$-ordered convergence property. To see this, for each $T \in L(Z)$ let ${}^*T \in L(Y)$ denote the unique element whose adjoint operator is $T$; this is possible as $Y$ has its Mackey topology. By [29; Lemma 1] it follows that ${}^*\mathcal{N}$ is a Bade $\sigma$-complete B.a. in $L(Y)$, and hence has the $\sigma$-monotone property (cf. the previous paragraph). Since a monotone sequence $\{P_n\} \subseteq \mathcal{N}$ has the property that $\{{}^*P_n\}$ is monotone in ${}^*\mathcal{N}$ (see [29; p. 290]), and convergence of a sequence in $L_{\mathrm{w}}(Y)$ implies convergence in $L_{\mathrm{s}}(Z)$ of the sequence of adjoint operators, it follows that $\mathcal{N}$ has the $\sigma$-monotone property in $L(Z)$. Then Propositions 4.1(iii) and 4.6 give the conclusion.

It is known, even in the Banach space setting, that there exist examples $\mathcal{M}$ which fail to have the $\sigma$-ordered convergence property. Indeed, $\mathcal{M}$ can even be a complete subset of $L_{\mathrm{s}}(X)$ and fail to have this property.

EXAMPLE 4.15. Let $X = \ell^\infty$. For each subset $E \subseteq \mathbb{N}$ let $P(E)$ denote the projection operator in $\ell^\infty$ of coordinatewise multiplication by $\chi_E$. The collection $\mathcal{M}$ of all such projections is an abstractly complete B.a. which is a closed (hence, complete) subset of $L_{\mathrm{s}}(\ell^\infty)$. But $\mathcal{M}$ does not have the $\sigma$-ordered convergence property. ∎

QUESTION 4. (i) Does there exist a Bade $\sigma$-complete B.a. $\mathcal{M}$ in a lcHs $X$ such that $\mathcal{M}$ is not a bounded subset of $L_{\mathrm{s}}(X)$? In view of Proposition 4.1(i), $\mathcal{M}$ cannot have the $\sigma$-ordered convergence property and in view of the comments after Example 4.14 the space $X$ cannot be metrizable or be of the form $Y_\sigma$ or $(Y', \sigma(Y', Y))$ with $Y$ metrizable. In particular, such an $\mathcal{M}$ cannot be the range of a spectral measure.

(ii) Does there exist a Bade $\sigma$-complete B.a. in a quasibarrelled space which is not equicontinuous? ∎

Let $\mathcal{M} \subseteq L(X)$ be an abstractly $\sigma$-complete B.a. For each $x \in X$ and $x' \in X'$ we can consider the "measure" $\mu_{x,x'} : B \mapsto \langle Bx, x'\rangle$ defined on $\mathcal{M}$. If $\mathcal{M}$ has the $\sigma$-monotone property, then $\mu_{x,x'}$ is $\sigma$-additive, and hence $\sup\{|\langle Bx, x'\rangle| : B \in \mathcal{M}\} < \infty$ for each $x \in X$ and $x' \in X'$. This implies that $\mathcal{M}$ is a bounded subset of $L_s(X)$. Without the $\sigma$-monotone property is it still the case that $\mu_{x,x'}$ *always* has bounded range in $\mathbb{C}$? As seen by Example 4.15, for *particular* $\mathcal{M}$ this may be the case. Question 4 is asking whether this is the case for *every* Bade $\sigma$-complete B.a. The answer does not appear to come directly from general measure theory since there exist *finitely* additive set functions $\mu : \Sigma \to \mathbb{C}$ defined on a $\sigma$-*algebra* $\Sigma$ with $\mu(\Sigma)$ being an *unbounded* subset of $\mathbb{C}$. For example, let $e_n$, $n \in \mathbb{N}$, be the standard unit vector in $\ell^\infty$ (over $\mathbb{R}$) with 1 at position $n$ and 0 elsewhere. Extend $\{e_n : n \in \mathbb{N}\}$ to a Hamel base of $\ell^\infty$. For $x \in \ell^\infty$, let $f_n(x)$ be the $e_n$-coordinate of $x$ with respect to this Hamel base. Then $f_n : \ell^\infty \to \mathbb{R}$ is linear and $\{n : f_n(x) \neq 0\}$ is finite for each $x \in \ell^\infty$. For $E \in \Sigma = 2^{\mathbb{N}}$ define $\mu(E) = \sum_{n=1}^{\infty} f_n(\chi_E)$. Then $\mu : \Sigma \to \mathbb{R}$ is a finitely additive set function with unbounded range (since $\mu(\{1, \ldots, n\}) = n$ for each $n \in \mathbb{N}$).

The failure of a B.a. like that in Example 4.15 to be Bade $\sigma$-complete is typical in a certain sense. Namely, if $\mathcal{M}$ has "a bit of the $\sigma$-ordered convergence property" and $X$ is "not like an $\ell^\infty$-space", then it actually *has* the $\sigma$-ordered convergence property. This is made more precise in the following

PROPOSITION 4.16. *Let $X$ be a lcHs and $\mathcal{M} \subseteq L(X)$ be an equicontinuous B.a. such that $\mathcal{M} = P(\Sigma)$ where $P$ is a finitely additive, multiplicative, $L(X)$-valued set function defined on a $\sigma$-algebra of sets $\Sigma$. Suppose that*

(i) *$[X]_{Px}$ is sequentially complete, for each $x \in X$,*
(ii) *$[X]_{Px}$ does not contain a closed subspace isomorphic to $\ell^\infty$, for each $x \in X$, and*
(iii) *there is a total set $\Gamma \subseteq X'$ such that $\langle Px, x'\rangle$ is $\sigma$-additive, for $x \in X$ and $x' \in \Gamma$.*

*Then $P$ is a spectral measure (i.e. $\sigma$-additive in $L_s(X)$). In particular, $\mathcal{M}$ is a Bade $\sigma$-complete B.a. with the $\sigma$-ordered convergence property.*

Proof. This result is established in [25; Proposition 1(i)] under the additional hypothesis that $X$ is quasicomplete and does not contain a copy of $\ell^\infty$. An examination of the proof given in [25] shows that the weaker properties (i) and (ii) above can replace these assumptions. ∎

For Banach spaces the above result is due to T. A. Gillespie [10]. We point out that the total set of functionals $\Gamma$ in (iii) can even vary with respect to $x \in X$; see [25]. We now present two further results of a similar, but different, kind.

PROPOSITION 4.17. *Let $X$ be a lcHs and $\mathcal{M} \subseteq L(X)$ be B.a. such that $\mathcal{M} = P(\Sigma)$ where $P$ is a finitely additive, multiplicative, $L(X)$-valued set function defined on a $\sigma$-algebra of sets $\Sigma$. For each $x \in X$, suppose that*

(i) $[X]_{Px}$ *is a fully complete lcHs which does not contain a copy of* $\ell^\infty$, *and*

(ii) *there is a lcH-topology* $\varrho_x$ *in* $[X]_{Px}$, *weaker than the relative topology from* $X$, *such that* $Px : \Sigma \to [X]_{Px}$ *is* $\sigma$-*additive for the* $\varrho_x$-*topology.*

*Then* $P$ *is a spectral measure. In particular*, $\mathcal{M}$ *is a Bade* $\sigma$-*complete B.a. with the* $\sigma$-*ordered convergence property.*

Proof. It is to be shown that $P$ is $\sigma$-additive in $L_s(X)$. Fix $x \in X$. Since $Px : \Sigma \to X$ is the "same" set function when considered as being $[X]_{Px}$-valued, it suffices to show that $Px : \Sigma \to [X]_{Px}$ is $\sigma$-additive. But assumptions (i) & (ii) then imply, by a result of G. Bennett & N. Kalton [16; (4)], that this is indeed the case. ∎

Propositions 4.16 & 4.17 are different. In Proposition 4.17 it is not assumed that $\mathcal{M}$ is equicontinuous and in Proposition 4.16 the completeness requirements on $[X]_{Px}$ are weaker than those of Proposition 4.17.

PROPOSITION 4.18. *Let* $X$ *be a lcHs and* $\mathcal{M} \subseteq L(X)$ *be a B.a. such that* $\mathcal{M} = P(\Sigma)$ *where* $P$ *is a finitely additive*, *multiplicative*, $L(X)$-*valued set function defined on a* $\sigma$-*algebra of sets* $\Sigma$. *For each* $x \in X$, *suppose that*

(i) $[X]_{Px}$ *is sequentially complete and does not contain a copy of* $\ell^\infty$,

(ii) *there is a lcH-topology* $\alpha_x$ *in* $[X]_{Px}$ *such that the topology of* $X$ *restricted to* $[X]_{Px}$ *has a neighbourhood base at* 0 *consisting of sequentially* $\alpha_x$-*closed neighbourhoods of* 0, *and*

(iii) $Px : \Sigma \to [X]_{Px}$ *is* $\sigma$-*additive for the* $\alpha_x$-*topology.*

*Then* $P$ *is a spectral measure. In particular*, $\mathcal{M}$ *is a Bade* $\sigma$-*complete B.a. with the* $\sigma$-*ordered convergence property.*

Proof. The proof follows along the same lines as that of Proposition 4.17 with the Bennett–Kalton theorem replaced by a result of I. Labuda [16; (6)]. ∎

Given a lcHs $X$ and a B.a. $\mathcal{M} \subseteq L(X)$ let $\langle \mathcal{M} \rangle$ denote the algebra generated by $\mathcal{M}$ in $L(X)$. Elements $T$ of $\langle \mathcal{M} \rangle$ are of the form $T = \sum_{j=1}^n \lambda_j P_j$ for some finite set $\{\lambda_j\}_{j=1}^n \subseteq \mathbb{C}$ and $\{P_j\}_{j=1}^n \subseteq \mathcal{M}$. If $P_j P_k = 0$ whenever $j$ and $k$ are distinct, $\sum_{j=1}^n P_j = I$, all $P_j \neq 0$ and $\lambda_1, \ldots, \lambda_n$ are distinct, then $T$ is said to be *represented in standard form*; in this case we define $|T| = \sum_{j=1}^n |\lambda_j| P_j$. Then $\langle \mathcal{M} \rangle$ becomes a complex Riesz space. Suppose now that $\mathcal{M}$ is *equicontinuous.* For each $q \in \mathcal{P}(X)$ define the seminorm $\varrho_q$ in $X$ by

$$\varrho_q(x) = \sup\{q(Tx) : T \in \langle \mathcal{M} \rangle,\ |T| \leq I\}, \qquad x \in X.$$

Such seminorms were first introduced in [34; pp. 303–304] and further considered in [3, 5]. It turns out that $\{\varrho_q : q \in \mathcal{P}(X)\}$ generates an equivalent topology in $X$ and that the topology of $L_s(X)$ is generated by the seminorms $\{\varrho_{q,x} : q \in \mathcal{P}(X),\ x \in X\}$, where $\varrho_{q,x}(T) = \varrho_q(Tx)$ for $T \in L(X)$ (see [3; §3]). The point is that each $\varrho_{q,x}$ when restricted to $\langle \mathcal{M} \rangle$ is a Riesz space seminorm, for $x \in X$ and $q \in \mathcal{P}(X)$, and with respect to this topology $\langle \mathcal{M} \rangle$ is a complex, locally solid, lc-Riesz space [3; Lemma 3.2]. This fact allows the very general and powerful methods of lc-Riesz spaces to be introduced into the study of certain operator algebras generated by $\mathcal{M}$ in $L(X)$; see [3, 4, 5]. From the perspective of the present paper we present a final characterization of Bade complete B.a.'s in terms of these concepts. Recall that a locally solid Riesz space $L$ has a *Lebesgue topology* if, whenever $\{u_\alpha\} \subseteq L$ is downwards directed to 0 in the order of $L$, then $u_\alpha \to 0$ in the topology of $L$. For the notion of the *projection property* in lc-Riesz spaces we refer to [17]. Since the monotone property implies abstract completeness of a B.a. (cf. proof of Proposition 4.7) it is clear from Proposition 3.5 that the definition of Bade completeness adopted in [3; Definition 3.4(iii)] coincides with the usual definition (as given in Section 3). Accordingly, the following result is essentially known [3; Proposition 3.7].

PROPOSITION 4.19. *Let $X$ be a lcHs and $\mathcal{M} \subseteq L(X)$ be an equicontinuous B.a. Then $\mathcal{M}$ is Bade complete iff the lc-Riesz space $\langle\mathcal{M}\rangle$ has a Lebesgue topology.*

We point out that Proposition 4.19 is an improvement of [3; Corollary 3.8] as it shows that the projection property is not needed there.

There is a related result along these lines. Suppose that $X$ is a Dedekind complete, complex Riesz space with locally solid, Lebesgue topology and is quasicomplete as a lcHs. Then the B.a. $\mathcal{M} \subseteq L(X)$ of all *band projections* is always an equicontinuous, Bade complete B.a. [5; Proposition 2.4].

We now formulate a short summary of those results of this section directly related to closedness of spectral measures; the characterizations are additional to those of Proposition 3.16.

PROPOSITION 4.20. *Let $X$ be a lcHs and $P : \Sigma \to L_{\mathrm{s}}(X)$ be a spectral measure. The following statements are equivalent*:

(i) *$P$ is a closed measure.*
(ii) *The B.a. $P(\Sigma)$ has the monotone property.*
(iii) *The B.a. $P(\Sigma)$ has the ordered convergence property.*

*If $P$ is equicontinuous*, *then* (i)–(iii) *are equivalent to*:

(iv) *The lc-Riesz space $I_P(\mathrm{sim}(\Sigma)) \subseteq L_{\mathrm{s}}(X)$ has a Lebesgue topology.*

In the Banach space setting it is a classical result [1; Theorem 3.2] that if a net of projections in a Bade $\sigma$-complete B.a. converges to a projection in $L_{\mathrm{w}}(X)$, then it also converges to that projection in $L_{\mathrm{s}}(X)$. This result was extended to *equicontinuous*, Bade $\sigma$-complete B.a.'s acting in a quasicomplete space $X$ for which $L_{\mathrm{s}}(X)$ is sequentially complete [6; Proposition 4.4]. We end with a significant generalization of these classical results. Given a B.a. $\mathcal{M} \subseteq L(X)$ let $\langle\mathcal{M}\rangle_{\mathrm{s}}$ denote the closed subalgebra of $L_{\mathrm{s}}(X)$ generated by $\mathcal{M}$.

PROPOSITION 4.21. *Let $X$ be a lcHs and $\mathcal{M} \subseteq L_{\mathrm{s}}(X)$ be a B.a. with the $\sigma$-monotone property. Suppose that at least one of the spaces $[L_{\mathrm{s}}(X)]_{\mathcal{M}}, \langle\mathcal{M}\rangle_{\mathrm{s}}$ or $L_{\mathrm{s}}(X)$ is quasicomplete. If $\{B_\alpha\} \subseteq \mathcal{M}$ is a net which converges to $B \in \mathbb{P}(X)$ in $L_{\mathrm{w}}(X)$, then $B_\alpha \to B$ in $L_{\mathrm{s}}(X)$.*

Proof. Let $P : \Sigma \to L_{\mathrm{s}}(X)$ be a spectral measure such that $\mathcal{M} = P(\Sigma)$; see Proposition 4.6. The completeness assumption ensures that $K = \overline{\mathrm{co}}(\mathcal{M})$ is $w$-compact, and hence that $K$ is a spectral carrier in the lc-algebra $L_{\mathrm{w}}(X)$. It was established in the proof of Proposition 3.15 that $\partial_{\mathrm{e}} K = \overline{P(\Sigma)}^{\mathrm{w}} \cap \mathbb{P}(X)$ and so $B_\alpha \to B$ in $\partial_{\mathrm{e}} K$ with respect to the weak topology of $L_s(X)$, where $\partial_e K$ is the set of all extreme points of $K$. It is known [14; VI, Theorem 2.1] that if $m : \Sigma \to Y$ is any vector measure in a lcHs $Y$ which is *quasicomplete* (this hypothesis needs to be added to [14; VI, Theorem 2.1] to make it correct), then on the extreme points of $\overline{\mathrm{co}}(m(\Sigma))$ all topologies consistent with the duality $(Y, Y')$ coincide. Accordingly, $B_\alpha \to B$ in $\partial_e K$ with respect to the strong operator topology. ∎

We point out that if $X$ is quasicomplete, $L_s(X)$ is sequentially complete and $\mathcal{M} \subseteq L(X)$ is an equicontinuous, Bade $\sigma$-complete B.a., then $\langle \mathcal{M} \rangle_s$ is actually a complete lcHs and so [6; Proposition 4.4] is indeed a special case of Proposition 4.21. To see this, note that $\langle \mathcal{M} \rangle_s = \langle \overline{\mathcal{M}}^s \rangle_s$ and $\overline{\mathcal{M}}^s$ is a Bade complete B.a. (cf. Propositions 4.1 and 4.2). Then Propositions 4.1 and 4.7 imply that there exists a closed spectral measure $P : \Sigma \to L_s(X)$ with $P(\Sigma) = \overline{\mathcal{M}}^s$. Moreover, Proposition 3.16(vii) shows that $\mathcal{L}^1(P)$ is a complete lcHs. By Lemma 1.11 the integration map $I_P$ is a bicontinuous isomorphism of $\mathcal{L}^1(P)$ onto $I_P(\mathcal{L}^1(P)) = \langle \overline{\mathcal{M}}^s \rangle_s = \langle \mathcal{M} \rangle_s$ and so $\langle \mathcal{M} \rangle_s$ is complete.

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