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Containing spaces for planar rational compacta

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CONTENTS

1. Introduction

All spaces in this paper are bounded subsets of the plane E^2. A *compactum* is a compact metric space; a *continuum* is a connected compactum; a *Peano continuum* is a locally connected continuum. A space is said to be *rational* iff it has a basis of open sets with countable boundaries. Necessarily, rational spaces are one-dimensional. A space is said to be *scattered* iff every nonempty subset has an isolated point. Scattered separable spaces are countable, and are classified according to their *topological type* by the countable ordinals. Rational spaces which have a basis with scattered boundaries are called *rim-scattered spaces* and are classified according to their *rim-type* by the countable ordinals. (See Sections 2.1 and 2.3 below for definitions.)

A space X is said to be *universal* for a class $\mathscr{A}$ of spaces iff both of the following condition are satisfied: (1) $X \in \mathscr{A}$, and (2) for each $Y \in \mathscr{A}$, there is an embedding $e\colon Y \to X$. If only condition (2) is satisfied, we say that X is a *containing space* for the class $\mathscr{A}$.

It is known [I3] that for each countable ordinal $\alpha \geqslant 1$, there does not exist a universal space of rim-type $\leqslant \alpha$. Iliadis [I3, I4] and independently the authors [MT] proved that there exist universal spaces of rim-type α for each countable ordinal α, and that there exists a universal rational space. Iliadis' proof involves quotients of subsets of the Cantor set. Mayer and Tymchatyn use combinatorial partition-matching techniques first developed by Anderson [A1, A2] for the Menger Curve. The present work is an adaptation to the plane of the techniques of our earlier paper.

Our main result in this paper is the construction, for each countable ordinal α, of a planar Peano continuum S_α of rim-type $\alpha+1$ which, as we show in Theorem 4.2, is a containing space for the class of planar compacta of rim-type $\leqslant \alpha$. This provides an affirmative answer to a question of Iliadis [I3, Problem 5].

In Section 2 we prove some results on ordered scattered spaces. In Section 3 we study *disk partitions* of planar rational spaces. In Section 4 we prove the

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main theorem. Our containing spaces may be considered rim-scattered analogues of the Sierpiński universal plane curve. We conclude in Section 5 with some speculations about a universal planar rational space and a planar space universal for planar spaces of rim-type $\leqslant \alpha$.

2. Ordered scattered spaces

In this section we construct ordered universal spaces of type α and obtain some elementary properties of ordered spaces of type α.

Let A be a nonempty subset of a metric space (X, d). For $\varepsilon > 0$, we let $S(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\}$. By $\mathrm{Bd}(A)$ and $\mathrm{Cl}(A)$ we denote the boundary and closure of A, respectively. By $\mathrm{Int}(A)$ we denote the interior of A, namely $A - \mathrm{Bd}(A)$. By $|A|$ we denote the cardinality of A.

2.1. Topological type. Let A be a subset of X. Let $A^{(0)} = A$, let $A^{(1)}$ be the set of limit points of A in A, and for each countable ordinal $\alpha \geqslant 0$, define $A^{(\alpha+1)} = (A^{(\alpha)})^{(1)}$. If α is a limit ordinal, let $A^{(\alpha)} = \bigcap \{A^{(\beta)} \mid \beta < \alpha\}$. We call $A^{(\alpha)}$ the *derivative of A of order* α or the α*-derivative* of A.

If A is scattered, then for some countable ordinal $\alpha \geqslant 0$, $A^{(\alpha)} = \varnothing$. If $A^{(\alpha)} = \varnothing$, we say that A is of *type* $\leqslant \alpha$, and write type $(A) \leqslant \alpha$. If α is the least such ordinal, we say that A is of *type* α, and we write type $(A) = \alpha$. If $B \subset A$, then type $(B) \leqslant$ type (A). If A and B are closed sets of type $\leqslant \alpha$, then $A \cup B$ is of type $\leqslant \alpha$. A compactum is scattered iff it is countable. A scattered compactum has type α for some isolated ordinal α. There exist compacta of type α for every *isolated* countable ordinal α. A compactum has type 1 iff it is finite and nonempty. (See [K, MS] for proofs of some of the statements above.)

2.2. Ordered spaces. By an *ordered space* is meant an ordered pair $(X, \leqslant)$ where X is a space with the order topology induced by the linear order $\leqslant$ on X. If there is no risk of confusion, we shall often suppress the order $\leqslant$ and speak simply of the ordered space X. If α is a countable ordinal, then by saying X is an *ordered space of type* $\leqslant \alpha$ we mean that X is an ordered space and type $(X) \leqslant \alpha$. A subset A of an ordered space X is said to be *convex* iff for all $x < y \in A$, for all $z \in X$ with $x < z < y$, $z \in A$. If A and B are disjoint subsets of X and $x < y$ for all $x \in A$ and $y \in B$, we write $A < B$.

A *containing space* X for a class $\mathscr{A}$ of ordered spaces is an ordered space which admits an order-preserving embedding of every member of $\mathscr{A}$. If, in addition, $X \in \mathscr{A}$, we then say that X is *universal* for the class $\mathscr{A}$.

Let C denote Cantor's ternary set in the unit interval $[0, 1]$ with metric d and the standard order. One can prove that C is a containing space for every countable ordered space. In [MT] it is proved that every compactum of type $\leqslant \alpha$ embeds in every compactum of type $\alpha + 1$, and that for every countable ordinal α, there exists a universal space of type α. We prove next the appropriate versions of these theorems for ordered spaces.

2.2.1. THEOREM. *For each countable ordinal α, there exists an ordered compactum C_α such that $|C_\alpha^{(\alpha)}| = 1$ and C_α is a universal space for all ordered compacta Y with $|Y^{(\alpha)}| \leqslant 1$.*

Proof. Let $C_0 = \{0\}$. Let $C_1 = \{0\} \cup \{n^{-1} \mid n \in \boldsymbol{Z} - \{0\}\}$, where $\boldsymbol{Z}$ denotes the integers, topologized as a subset of $\boldsymbol{R}^1$. Every ordered compactum of type 1 is finite, and hence, can be embedded in an order-preserving fashion in C_1.

Let $\alpha > 1$ be a countable ordinal, and suppose that C_λ is defined for each $\lambda < \alpha$.

If α is an isolated ordinal, let

$$C_\alpha = (C_1 \times C_{\alpha-1})/(\{0\} \times C_{\alpha-1})$$

with the order induced by the lexicographic order on $C_1 \times C_{\alpha-1}$. Clearly, $C_\alpha^{(\alpha)} = \{\{0\} \times C_{\alpha-1}\}$.

Suppose that Y is an ordered compactum with $Y^{(\alpha)} = \{y\}$. Then $Y^{(\alpha-1)}$ is a sequence converging to y. At most one of the sets

$$\{z \in Y^{(\alpha-1)} \mid z < y\} \quad \text{and} \quad \{z \in Y^{(\alpha-1)} \mid z > y\}$$

is finite. We suppose that neither is finite. (The other case is handled similarly.)

We may suppose

$$Y^{(\alpha-1)} = \{y_{-1} < y_{-2} < \ldots < y_{-n} < \ldots < y < \ldots < y_n < \ldots < y_2 < y_1\}.$$

Then $Y = \{y\} \cup \bigcup_{n \in \boldsymbol{Z}-\{0\}} U_n$, where the U_n are pairwise disjoint convex open sets such that $\{y_n\} = U_n^{(\alpha-1)}$, for each $n \in \boldsymbol{Z} - \{0\}$. By the inductive hypothesis, there exists an order-preserving embedding $f_n\colon U_n \to \{n^{-1}\} \times C_{\alpha-1}$. Then $f\colon Y \to C_\alpha$ defined by

$$f = \bigcup_{n \in \boldsymbol{Z}-\{0\}} f_n \cup \{(y, \{0\} \times C_{\alpha-1})\}$$

is the desired order-preserving embedding of Y into C_α.

If Y is an ordered compactum with $Y^{(\alpha)} = \emptyset$, then $Y^{(\alpha-1)}$ is finite (or empty). It is easy, using methods similar to the above, to get an order-preserving embedding of Y into C_α.

If α is a limit ordinal, let $\alpha_1 < \alpha_2 < \ldots$ be a sequence of isolated ordinals converging to α. Let

$$C_\alpha = \{0\} \cup \bigcup_{n \in \boldsymbol{Z}-\{0\}} \{n^{-1}\} \times C_{\alpha_{|n|}}$$

with the order on C_α given by the linear order on $C_{\alpha_{|n|}}$ for each $\{n^{-1}\} \times C_{\alpha_{|n|}}$, and with $(q^{-1}, z_1) < (p^{-1}, z_2) < (s^{-1}, z_3) < (r^{-1}, z_4)$ for all $p < q < 0 < r < s$ and $z_i \in C_{\alpha_{|n_i|}}$. Then by an argument very similar to the one above, C_α is a universal space for ordered compacta Y with $|Y^{(\alpha)}| \leqslant 1$. ∎

By the proof of the above theorem, we have the following:

2.2.2. COROLLARY. *With C_α as in Theorem 2.2.1, if Y_1 and Y_2 are ordered compacta of type $\leqslant \alpha$, then there exist order-preserving embeddings e_1: $Y_1 \to C_\alpha$ and e_2: $Y_2 \to C_\alpha$ such that $e_1(Y_1) < C_\alpha^{(\alpha)}$ and $e_2(Y_2) > C_\alpha^{(\alpha)}$.*

For noncompact ordered, scattered spaces, by way of contrast, we have the following:

2.2.3. THEOREM. *For each countable ordinal α, there exists a universal space A_α for the class of ordered spaces of type $\leqslant \alpha$.*

Proof. Let $A_0 = \emptyset$ and let

$$A_1 = \{x \in [0, 1] \mid d(x, C) = n^{-1} \text{ for some } n \in \mathbf{Z}^+ = \{1, 2, \ldots\}\}.$$

Then type $(A_1) = 1$, since A_1 is discrete. Let $D = \{x_1, x_2, \ldots\}$ be any ordered space of type 1. We observe the following fact about A_1: between any pair S and T of disjoint, convex, monotone sequences in A_1 which have an infinite subset of A_1 between them, there are infinitely many other such sequences. Hence, there is an order-preserving one-to-one function f: $D \to A_1$ such that $f(D)$ meets each monotone, convex sequence in A_1 in at most one point. Since D is discrete, f is an embedding.

Let α be a countable ordinal with $\alpha > 1$ and suppose that A_λ has been constructed for each ordinal $\lambda < \alpha$ in such a way that $A_\lambda \subset (0, 1)$ and $0, 1 \in (A_\lambda \cup \{0, 1\})^{(\lambda)}$.

If α is an isolated ordinal, let

$$A_\alpha^{(\alpha-1)} = A_1$$

and let

$$A_\alpha = A_1 \cup \bigcup \{B_{a,b} \mid a < b \text{ in } A_1 \text{ and } (a, b) \cap A_1 = \emptyset\}$$

where $B_{a,b} \subset (a, b)$ is order isomorphic to $A_{\alpha-1}$ and $a, b \in (B_{a,b} \cup \{a, b\})^{(\alpha-1)}$.

Suppose Y is an ordered space of type $\leqslant \alpha$, and, without loss of generality, suppose $Y \subset [0, 1]$ and $Y^{(\alpha-1)} = \{x_1, x_2, \ldots\}$, a discrete set. For each i, let U_i be an open interval about x_i such that for $i \neq j$, $U_i \cap U_j = \emptyset$, and $U_i \cap Y$ is both open and closed in Y. Since Y is countable, we may suppose that $Y \subset \bigcup_{i=1}^{\infty} U_i$. Let f: $Y^{(\alpha-1)} \to A_1$ be an order-preserving embedding. Using the techniques of the proof of Theorem 2.2.1, we can extend f to an order-preserving embedding g: $Y \to A_\alpha$.

If α is a limit ordinal, we model A_α on A_1. Let $\alpha_1 < \alpha_2 < \ldots$ be a sequence of ordinals increasing to α. Let

$$A_\alpha = \bigcup \{A_1 \times \{n\} \times A_{\alpha_n} \mid n \in \mathbf{Z}^+\}$$

with the lexicographic order.

Suppose Y is an ordered space of type $\leqslant \alpha$. Since Y is countable and $y \notin Y^{(\alpha-1)}$ for each $y \in Y$, there exists a family $\{U_i\}_{i=1}^{\infty}$ of mutually disjoint, convex open and closed sets in Y such that type $(U_i) < \alpha$ and $Y = \bigcup_{i=1}^{\infty} U_i$. With the induced order, it is easy to see that the set $\{U_i\}_{i=1}^{\infty}$ is a discrete ordered space. So there exists an order-preserving embedding $f\colon \{U_i\}_{i=1}^{\infty} \to A_1$. This function can be used as a guide to construct an embedding $g\colon Y \to A_\alpha$ by letting $g(U_i)$ be contained in $\{f(U_i)\} \times \{n\} \times A_{\alpha_n}$ for some sufficiently large n. ∎

2.2.4. Remark. It is known [IT] that scattered spaces admit scattered compactifications. In the case of ordered scattered spaces, the situation is very different. The space A_1 constructed above has no order-preserving embedding into any ordered scattered compactum. For if Y is an ordered compactum which admits an order-preserving, dense embedding of A_1, then there is an order-preserving mapping of Y onto $C \cup A_1$ which is at most two-to-one. Hence, Y cannot be scattered.

2.3. Rim-type. A space X is said to be of *rim-type* $\leqslant \alpha$ iff X has a basis of open sets whose boundaries each have type $\leqslant \alpha$, and we write rim-type $(X) \leqslant \alpha$. If α is the least such ordinal, then we say that X is of *rim-type* α, and we write rim-type$(X) = \alpha$. If $B \subset A$, then rim-type$(B) \leqslant$ rim-type(A). A space has rim-type 0 iff it is zero-dimensional. A space has rim-type 1 iff it has a basis of open sets with discrete boundaries. A compactum has rim-type 1 iff it has a basis of open sets with finite boundaries. Hence, a compactum of rim-type $\leqslant 1$ is said to be *rim-finite*. There exist continua of rim-type α for each countable ordinal $\alpha > 0$.

2.4. Disk partitions. A *disk partition* $\mathcal{U}$ is a finite collection of closed topological disks in E^2 such that

(1) $U \neq V \in \mathcal{U}$ and $U \cap V \neq \emptyset$ implies $U \cap V$ is a union of finitely many pairwise disjoint arcs, and
(2) no 3 distinct members of $\mathcal{U}$ have a nonempty intersection.

If $\mathcal{U}$ is a disk partition and $X \subset \bigcup \mathcal{U}$ is a compactum such that

(3) if $x \in U \cap V \cap X$ for some $U \neq V \in \mathcal{U}$, then x is in the interior in $\mathrm{Bd}(U)$ of the arc of $U \cap V$ containing x,

then we say that $\mathcal{U}$ is a *disk partition* of X. If, in addition, for every $U \in \mathcal{U}$, type$(\mathrm{Bd}(U) \cap X) \leqslant \alpha$ for some countable ordinal α (respectively, $\mathrm{Bd}(U) \cap X$ is countable), then we say that $\mathcal{U}$ is an *α-disk partition* (respectively, *rational disk partition*) of X.

2.4.1. *Decreasing sequences of disk partitions.* A disk partition $\mathcal{U}$ is said to *refine* a disk partition $\mathcal{V}$ iff every element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. For any disk partition $\mathcal{W}$, we define mesh$(\mathcal{W}) = \max\{\mathrm{diam}(W) \mid W \in \mathcal{W}\}$. A sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of disk partitions is said to be a *decreasing sequence* of disk partitions iff, for all $i > 0$, $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$, and mesh$(\mathcal{U}_i)$ approaches 0 as i approaches ∞.

2.4.2. *Notation.* Let $\mathcal{U}$ refining $\mathcal{V}$ be disk partitions with $Y \subset \bigcup \mathcal{V}$ and $\mathcal{W} \subset \mathcal{U}$. We adopt the following notational conveniences:

(1) $\mathscr{W}(Y) = \{U \in \mathscr{W} \mid U \subset Y\}$.
(2) $\mathrm{Bd}(\mathscr{W}) = \bigcup \{\mathrm{Bd}(U) \mid U \in \mathscr{W}\}$.
(3) $\partial\mathscr{W} = \bigcup \{W_1 \cap W_2 \mid W_1 \neq W_2 \in \mathscr{W}\}$.
(4) The *star of Y in $\mathscr{W}$* is $\mathrm{Star}(Y, \mathscr{W}) = \{W \in \mathscr{W} \mid W \cap Y \neq \emptyset\}$.
(5) The *boundary of Y in $\mathscr{W}$* is $\mathrm{Bound}(Y, \mathscr{W}) = \{W \in \mathscr{W}(Y) \mid \text{for some } V \in \mathscr{W} - \mathscr{W}(Y),\ V \cap W \neq \emptyset\}$.
(6) The *core of Y in $\mathscr{W}$* is $\mathrm{Core}(Y, \mathscr{W}) = \mathscr{W}(Y) - \mathrm{Bound}(Y, \mathscr{W})$.

2.4.3. *Topological hull.* Let $X \subset E^2$ be a compactum. The *topological hull* of X, denoted $\mathrm{Hull}(X)$, is the union of X and its bounded complementary domains (the bounded components of $E^2 - X$).

2.4.4. *Amalgams.* A disk partition $\mathscr{V}$ is said to be an *amalgam* of a disk partition $\mathscr{U}$ iff each element of $\mathscr{V}$ is the topological hull of the union of a subcollection of $\mathscr{U}$. Note that an amalgam of a disk partition is again a disk partition. We usually denote an amalgam of $\mathscr{U}$ by $\hat{\mathscr{U}}$. A subcollection $\bar{\mathscr{U}}$ of $\hat{\mathscr{U}}$ is said to be a *partial amalgam* of $\mathscr{U}$.

2.4.5. THEOREM (see Theorem 2.4.1 of [MT]; also in [IT]). *Let X be a rim-scattered space and α a countable ordinal. Then the following are equivalent*:

(1) *rim-type*$(X) \leqslant \alpha$.
(2) *Every pair of disjoint closed subsets of X can be separated by a closed set of type $\leqslant \alpha$.*

2.4.6. THEOREM. *Let X be a planar compactum and α a countable ordinal. Then the following are equivalent*:

(1) *rim-type*$(X) \leqslant \alpha$.
(2) *For every α-disk partition $\mathscr{U}$ of X and every $\varepsilon > 0$, there exists an α-disk partition $\mathscr{V}$ of X such that $\mathscr{V}$ refines $\mathscr{U}$ and* $\mathrm{mesh}(\mathscr{V}) < \varepsilon$.

Proof. That (2) implies (1) is obvious. Assume (1). We will work within one element of $\mathscr{U}$ at a time. Let $U \in \mathscr{U}$. We may, without loss of generality, assume that U is the rectangular disk $[0, 1] \times [0, 1]$. Let $\mathscr{C}$ be a partition of U into rectangular disks of mesh $< \varepsilon/2$ such that the following conditions are satisfied:

(a) At most 3 distinct elements of $\mathscr{C} \cup \{\mathrm{Cl}(E^2 - U)\}$ have a nonempty intersection.
(b) No point of $X \cap \mathrm{Bd}(U)$ is a vertex of any element of $\mathscr{C}$.
(c) There are at most finitely many points $\{y_1, \ldots, y_k\}$ of U each of which lie in three elements of $\mathscr{C} \cup \{\mathrm{Cl}(E^2 - U)\}$.

Since X is one-dimensional, we may assume (by adjusting the cover $\mathscr{C}$ slightly so that rectangular disks become polygonal, but retain the other properties noted above) that $\{y_1, \ldots, y_k\} \subset U - X$.

Remove from each member of $\mathscr{C}$ a very small circular open disk E_i about each y_i such that $\mathrm{Cl}(E_i) \cap X = \emptyset$ and $D - \bigcup_{i=1}^{k} E_i$ is a disk for each $D \in \mathscr{C}$. See

Figure 2.1. Then

$$\mathscr{C}' = \{D - \bigcup_{i=1}^{k} E_i \mid D \in \mathscr{C} \text{ and } D \cap X \neq \emptyset\}$$

is a disk partition of $X \cap U$ of mesh $< \varepsilon/2$, but the boundaries need to be adjusted slightly. Note that $X \cap U \subset \text{Int}_U(\bigcup \text{Star}(X, \mathscr{C}'))$.

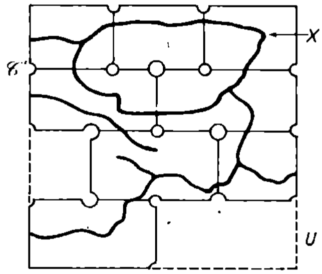

Fig. 2.1

Let $V \neq W \in \mathscr{C}'$ such that $V \cap W \neq \emptyset$. For the moment assume that we may replace the arc $V \cap W$ by an arc $A_{V,W}$ with the same endpoints as $V \cap W$ arbitrarily close to $V \cap W$ such that $\text{type}(X \cap A_{V,W}) \leqslant \alpha$. We can then replace V and W by two new disks whose intersection is $A_{V,W}$. By finitely many such modifications, we arrive at an α-disk partition $\mathscr{V}(U)$ of $X \cap U$ of mesh $< \varepsilon$. Then, using conditions (b) and (c) above, it is easy to see that $\mathscr{V} = \bigcup \{\mathscr{V}(U) \mid U \in \mathscr{U}\}$ is the required α-disk partition of X.

We now justify the assumption in the above paragraph. Let a and b be the endpoints of the arc $V \cap W$. Let E be an arc from a to b, contained in $\text{Int}(V)$ except for its endpoints, and let F be an arc from a to b, contained in $\text{Int}(W)$ except for its endpoints. It is clear we may take E and F as close to $V \cap W$ as we desire. Then $\{a, b\}$ separates the simple closed curve $E \cup F$ into two components. Without loss of generality, we may assume that $E \cap X \neq \emptyset \neq F \cap X$. Let D denote the closed disk whose boundary is $E \cup F$. Note that $E \cap X$ and $F \cap X$ are compact and miss $\{a, b\}$. By Theorem 2.4.5, there is a compact set $G \subset X \cap \text{Int}(D)$ so that $\text{type}(G) \leqslant \alpha$ and G separates $E \cap X$ from $F \cap X$ in $X \cap D$. Now

$$H = G \cup (\text{Int}(D) - X) \cup \{a, b\}$$

separates $E \cap X$ from $F \cap X$ in D. Since D is hereditarily normal, some closed subset H' of H separates $E \cap X$ from $F \cap X$ in D [K, II, p. 155]. Note that $a, b \in H'$ and $H' \cap X \subset G$. Since $H' - (X \cup \{a, b\}) \subset \text{Int}(D) - X$ and G is compact, we may replace H' by a closed set $H'' \subset (\text{Int}(D) - X) \cup \{a, b\} \cup G$ such that $H'' - G$ is a locally finite graph (use for H'' the boundary of the union of a locally finite cover by a null sequence of disks of $H' - (G \cup \{a, b\})$ in $\text{Int}(D) - X$).

Since D is locally connected and unicoherent, there exists a closed and connected subset A of H'' such that A irreducibly separates $E \cap X$ from $F \cap X$ in D [K, II, pp. 162 and 244]. Then A is locally connected except at points of G. Since a continuum cannot fail to be locally connected only at points of a zero-dimensional set, A is locally connected. Since A is an irreducible separator, $A = A_{V,W}$ is an arc from a to b. ∎

2.4.7. *Isomorphisms.* Let Y and X be compacta and $\mathscr{V}_1$ and $\mathscr{U}_1$ disk partitions of Y and X, resepctively. A one-to-one, onto function $\varphi_1\colon \mathscr{V}_1 \to \mathscr{U}_1$ such that for all $U, V \in \mathscr{V}_1$, $U \cap V \neq \emptyset$ iff $\varphi_1(U) \cap \varphi_1(V) \neq \emptyset$ is said to be an *isomorphism*. Suppose that $\partial\varphi_1\colon \mathrm{Bd}(\mathscr{V}_1) \to \mathrm{Bd}(\mathscr{U}_1)$ is a continuous function such that for all $V \in \mathscr{V}_1$, $\partial\varphi_1 | \mathrm{Bd}(V)\colon \mathrm{Bd}(V) \to \mathrm{Bd}(\varphi_1(V))$ is an orientation-preserving homeomorphism of $\mathrm{Bd}(V)$ onto $\mathrm{Bd}(\varphi_1(V))$ with the property that $\partial\varphi_1(Y \cap \mathrm{Bd}(V)) \subset X \cap \mathrm{Bd}(\varphi_1(V))$. Then we say that $\partial\varphi_1$ is a *boundary embedding corresponding to* φ_1.

Suppose further that $\mathscr{V}_2$ and $\mathscr{U}_2$ are disk partitions of Y and X, refining $\mathscr{V}_1$ and $\mathscr{U}_1$, respectively. Let $\varphi_2\colon \mathscr{V}_2 \to \mathscr{U}_2$ be an isomorphism such that for every $V_1 \in \mathscr{V}_1$ and $V_2 \in \mathscr{V}_2$, $V_2 \subset V_1$ iff $\varphi_2(V_2) \subset \varphi_1(V_1)$. Then we say that φ_2 is an *isomorphism with respect to* φ_1.

The following theorem, in various versions, is well-known:

2.4.8. Partition-Matching Theorem. *Let X and Y be compacta in E^2 with $\{\mathscr{U}_i\}_{i=1}^{\infty}$ and $\{\mathscr{V}_i\}_{i=1}^{\infty}$ decreasing sequences of disk partitions of X and Y, respectively. Suppose that for all i, $\varphi_i\colon \mathscr{V}_i \to \mathscr{U}_i$ is an isomorphism, and φ_{i+1} is an isomorphism with respect to φ_i. Then there exists an embedding $\varphi\colon Y \to X$ such that for every $y \in Y$, $\varphi(y) = \bigcap_{i=1}^{\infty}\left(\bigcup \varphi_i(\mathrm{Star}(y, \mathscr{V}_i))\right)$.*

2.4.9. *Chains, coherent collections, and core refinement.* A subcollection $\{U_1, \ldots, U_n\}$ of a disk partition $\mathscr{U}$ is called a *chain* provided that $U_i \cap U_j \neq \emptyset$ iff $|i-j| \leqslant 1$, and it is called a *circle-chain* provided that $U_i \cap U_j \neq \emptyset$ iff $|i-j| \leqslant 1$ or $\{i, j\} = \{1, n\}$. A subcollection $\mathscr{V}$ of a disk partition $\mathscr{U}$ is said to be *coherent* iff for any pair of elements $U, V \in \mathscr{V}$, there is a chain in $\mathscr{V}$ with U as the first element and V as the last element. A subcollection $\mathscr{V}$ of $\mathscr{U}$ is called a *graph-chain* provided that $\mathscr{V}$ is coherent, and it is called a *tree-chain* provided that in addition $\mathscr{V}$ contains no circle-chain.

If $\mathscr{V}$ is a graph-chain, we call the elements of $\mathscr{V}$ *links*, and we call those elements of $\mathscr{V}$ which meet at most one other element of $\mathscr{V}$ *endlinks*.

Let $\mathscr{U}$ refining $\mathscr{V}$ be partitions. We say that $\mathscr{U}$ *core refines* $\mathscr{V}$ iff for every $V \in \mathscr{V}$, $\mathrm{Core}(V, \mathscr{U})$ is a coherent nonempty subcollection of $\mathscr{U}(V)$ such that for every $U \in \mathrm{Bound}(V, \mathscr{U})$, $U \cap \left(\bigcup \mathrm{Core}(V, \mathscr{U})\right) \neq \emptyset$.

3. Partition theorems

In this section we prove the theorems which are the principal ingredients in our proof of the main theorem in Section 4, and we define the combinatorial properties that suffice to insure that a decreasing sequence of disk partitions define a containing space for compacta of rim-type $\leqslant \alpha$.

3.1. Partition Pullback Theorem. *Let Y be a planar compactum of rim-type $\leqslant \alpha$ and let D be a disk containing Y. Let $\mathscr{B} = \{B_1, \ldots, B_m\}$ for $m \geqslant 0$ be a (possibly empty, in which case $m = 0$) collection of disjoint subarcs of* $\mathrm{Bd}(D)$ *whose interiors in* $\mathrm{Bd}(D)$ *form a cover of $Y \cap \mathrm{Bd}(D)$. Let $\mathscr{U} = \{U_1, \ldots, U_n\}$ for $n \geqslant 1$ be a tree-chain disk partition such that $U = \bigcup \mathscr{U}$ is a disk. Let $\mathscr{C} = \{C_1, \ldots, C_m\}$ be a collection of pairwise disjoint subarcs of* $\mathrm{Bd}(\bigcup \mathscr{U})$ *such that for every i there exists a j such that $C_i \subset U_j - \partial \mathscr{U}$. Suppose also that there exists an orientation-preserving homeomorphism h:* $\mathrm{Bd}(D) \to \mathrm{Bd}(U)$ *such that $h(B_i) = C_i$. Then there exists an α-disk partition $\mathscr{V} = \{V_1, \ldots, V_n\}$ of Y such that $\bigcup \mathscr{V} \subset D$, $\partial \mathscr{V} \cap \mathrm{Bd}(D) = \varnothing$, $\bigcup \mathscr{B} \subset \bigcup \mathscr{V} - \partial \mathscr{V}$, and φ: $\mathscr{V} \to \mathscr{U}$ defined by $\varphi(\mathscr{V}_i) = U_i$ is an isomorphism of disk partitions with the property that $h(V_i \cap (\bigcup \mathscr{B})) = \varphi(V_i) \cap (\bigcup \mathscr{C})$.*

Proof. Since $\mathscr{U}$ is a tree-chain whose union $U = \bigcup \mathscr{U}$ is a disk, h extends to a homeomorphism H: $D \to U$. It follows that $\mathscr{V}' = \{H^{-1}(U_i)\}_{i=1}^{n}$ is a disk partition of D. See Figure 3.1. We modify the boundaries of the elements of $\mathscr{V}'$ as in the proof of Theorem 2.4.6 to obtain an α-disk partition $\mathscr{V}$ of Y satisfying the theorem. ∎

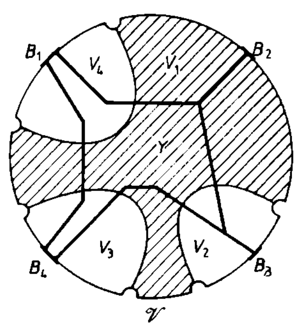

Fig. 3.1 (a). Disk partition $\mathscr{V}$ of Y contained in disk D

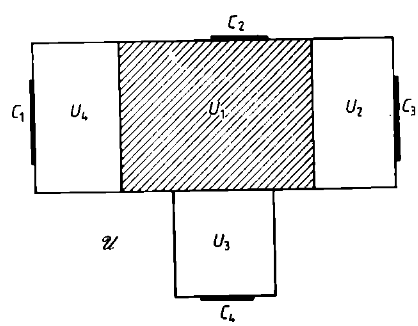

Fig. 3.1 (b). Disk $U = \bigcup \mathscr{U}$

3.2. Defining sequences of partitions. Let $\mathscr{U}$ refining $\mathscr{V}$ be $(\alpha+1)$-disk partitions of a compactum $X \subset E^2$ and let $\mathscr{W} \subset \mathscr{U}$.

3.2.1. *α-Connected.* We say that $\mathscr{W}$ is *α-connected* iff $\mathscr{W}$ is coherent and for all $U \neq V \in \mathscr{W}$, if $U \cap V \neq \varnothing$, then $U \cap V \cap X$ contains an order isomorphic copy of C_α (Theorem 2.2.1). If in addition $\mathscr{W}$ is a chain (respectively, tree-chain, graph-chain, circle-chain) we say that $\mathscr{W}$ is an *α-chain* (respectively, *α-tree-chain, α-graph-chain, α-circle-chain*). We say that $\mathscr{W}$ is *2-α-connected* iff $\mathscr{W}$ is coherent and for all $U \neq V \in \mathscr{W}$, if $U \cap V \neq \varnothing$, then $U \cap V \cap X$ contains two order isomorphic copies of C_α which lie in disjoint arcs of $U \cap V$. By a *2-α-chain* we mean a 2-α-connected chain.

3.2.2. *α-Circlelacing.* We say that $\mathscr{U}$ is *α-circlelaced* in $\mathscr{V}$ iff
(1) for every $V \in \mathscr{V}$, $\mathrm{Core}(V, \mathscr{U})$ contains an α-circle-chain $\mathscr{D}$,

(2) for every $B \in \text{Bound}(V, \mathcal{U})$, there exists a 2-$\alpha$-chain $\mathscr{C}_B$ in $\mathcal{U}(V)$ with one endlink being B and the other endlink in $\mathscr{D}$, and
(3) for $B \neq B' \in \text{Bound}(V, \mathcal{U})$, $(\bigcup \mathscr{C}_B) \cap (\bigcup \mathscr{C}_{B'}) = \emptyset$.

3.2.3. *α-Splitting.* We say that $\mathcal{U}$ *α-splits* $\mathscr{V}$ iff for all $U \neq V \in \mathscr{V}$ such that $U \cap V \neq \emptyset$ and $|(U \cap V \cap X)^{(\alpha)}| = 2$, there exist $W_1, W_2 \in \text{Bound}(U, \mathcal{U})$ and $W_3, W_4 \in \text{Bound}(V, \mathcal{U})$ such that $W_1 \cap W_2 = \emptyset = W_3 \cap W_4$ and

$$|(W_1 \cap W_3 \cap X)^{(\alpha)}| = 1 = |(W_2 \cap W_4 \cap X)^{(\alpha)}|.$$

3.2.4. *α-Defining sequence.* Let X be a continuum in E^2. We say that a decreasing sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of $(\alpha+1)$-disk partitions of X such that $X = \bigcap_{i=1}^{\infty} \bigcup \mathcal{U}_i$ is an *α-defining sequence* for X iff for all $i > 0$ the following conditions hold:

(1) $\mathcal{U}_{i+1}$ core refines $\mathcal{U}_i$.
(2) For all $U \in \mathcal{U}_i$, for all $V \neq W \in \text{Bound}(U, \mathcal{U}_{i+1})$, $V \cap W = \emptyset$.
(3) $\mathcal{U}_1$ is α-connected.
(4) For all $U \in \mathcal{U}_i$, $\mathcal{U}_{i+1}(U)$ is α-connected.
(5) $\mathcal{U}_{i+1}$ α-splits $\mathcal{U}_i$.
(6) $\mathcal{U}_{i+1}$ is α-circlelaced in $\mathcal{U}_i$.
(7) For all $U \neq V \in \mathcal{U}_i$, $U \cap V \neq \emptyset$ implies $U \cap V$ is an arc.

3.2.5. *α-Amalgam.* Let $\bar{\mathcal{U}}$ be a partial amalgam of an $(\alpha+1)$-disk partition $\mathcal{U}$. We say that $\bar{\mathcal{U}}$ is an *α-amalgam* of $\mathcal{U}$ iff for all $U \in \bar{\mathcal{U}}$, $\mathcal{U}(U)$ is α-connected.

3.3. α-Circlelacing lemmas.

3.3.1. Lemma. *Let X be a planar continuum with $\{\mathcal{U}_i\}_{i=1}^{\infty}$ an α-defining sequence for X. Then for each i, for each $j > i$, $\mathcal{U}_j$ is α-circlelaced in $\mathcal{U}_i$.*

Proof. We proceed by induction on $j-i$. Suppose that $j-i > 1$ and $\mathcal{U}_{j-1}$ is α-circlelaced in $\mathcal{U}_i$. Let $U \in \mathcal{U}_i$ and let $\text{Bound}(U, \mathcal{U}_{j-1}) = \{B_1, \ldots, B_n\}$. By induction there is a subcollection $\mathscr{W}$ of $\mathcal{U}_{j-1}(U)$ such that

$$\mathscr{W} = \mathscr{C} \cup \bigcup_{k=1}^{m} \mathscr{C}_k,$$

where $\mathscr{C}$ is an α-circle-chain and $\mathscr{C}_k$ is a 2-α-chain minimal with respect to having one endlink in $\mathscr{C}$ and the other endlink being B_k, and such that for $s \neq t$, $(\bigcup \mathscr{C}_s) \cap (\bigcup \mathscr{C}_t) = \emptyset$. See Figure 3.2.

Let $\{B_{k,m} \mid m = 1, \ldots, n_k\}$ be the links of $\text{Bound}(B_k, \mathcal{U}_j)$ which meet $\text{Bd}(U)$ for $k = 1, \ldots, n$. Using α-splitting and the fact that $\mathcal{U}_j$ is α-circlelaced in $\mathcal{U}_{j-1}$, there is a collection $\mathscr{W}' \subset \mathcal{U}_j(\bigcup \mathscr{W})$ such that

$$\mathscr{W}' = \mathscr{C}' \cup \bigcup \{\mathscr{C}_{k,m} \mid k = 1, \ldots, n \quad \text{and} \quad m = 1, \ldots, n_k\}$$

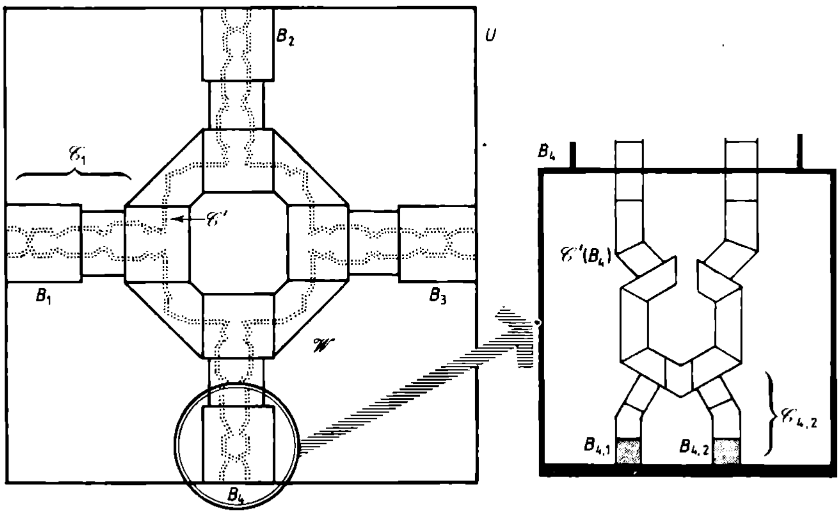

Fig. 3.2. $\mathscr{W}$ in U (left) and a magnified view of the part of $\mathscr{W}'$ in B_4 (right)

where $\mathscr{C}'$ is an α-circle-chain which runs through the length of each $\mathscr{C}_k$ exactly twice, and $\mathscr{C}_{k,m}$ is a 2-α-chain in $\mathscr{U}_j(B_k)$ minimal with respect to having one endlink in $\mathscr{C}'$ and the other endlink being $B_{k,m}$ and such that for $s \neq t$, $\bigcup \mathscr{C}_{k,s} \cap \bigcup \mathscr{C}_{k,t} = \emptyset$. This proves that $\mathscr{U}_j$ is α-circlelaced in $\mathscr{U}_i$. ∎

3.3.2. LEMMA. *Let X be a planar continuum with $\{\mathscr{U}_i\}_{i=1}^{\infty}$ an α-defining sequence for X. Let $j > i$ and let $U \in \mathscr{U}_j$. Suppose $\mathscr{D}_1$ and $\mathscr{D}_2$ are disjoint subcollections of the collection of arcs $\{B \cap \mathrm{Bd}(U) \mid B \in \mathrm{Bound}(U, \mathscr{U}_j)\}$ such that there are disjoint subarcs $C_1 \supset \bigcup \mathscr{D}_1$ and $C_2 \supset \bigcup \mathscr{D}_2$ in $\mathrm{Bd}(U)$. Then there are disjoint α-tree-chains $\mathscr{W}_1$ and $\mathscr{W}_2$ in $\mathscr{U}_j(U)$ such that $\bigcup \mathscr{D}_1 \subset \bigcup \mathscr{W}_1$, $\bigcup \mathscr{D}_2 \subset \bigcup \mathscr{W}_2$, and $\mathscr{W}_1 \cup \mathscr{W}_2$ is an α-connected tree-chain.*

Proof. The lemma follows immediately from Lemma 3.3.1. See Figure 3.3. ∎

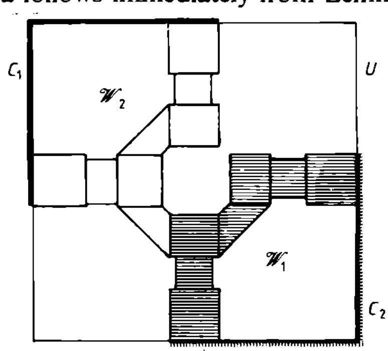

Fig. 3.3

3.4. BRICK PULVERIZING LEMMA. *Let Y be a planar compactum of rim-type $\leqslant \alpha$, $\mathscr{V} = \{V_1, V_2\}$ a coherent α-disk partition of Y, and $V = \mathrm{Hull}(V_1 \cup V_2)$. Let X be a plane continuum with $\{\mathscr{U}_i\}_{i=1}^{\infty}$ an*

α-defining sequence for X. Suppose that $\mathscr{U}_1$ is an α-tree-chain, and let $U = \bigcup \mathscr{U}_1$. (Note that U is a disk.) Let h: $\mathrm{Bd}(V) \to \mathrm{Bd}(U)$ be an orientation-preserving homeomorphism such that $h(\mathrm{Bd}(V) \cap Y) \subset \mathrm{Bd}(U) \cap X$. Then there exists

(a) *an α-disk partition $\mathscr{W}$ of Y refining $\mathscr{V}$,*
(b) *an integer $j \geqslant 1$,*
(c) *an α-amalgam $\bar{\mathscr{U}}_j$ of $\mathscr{U}_j$ such that $\bar{\mathscr{U}}_j$ refines $\mathscr{U}_1$,*
(d) *and an isomorphism φ: $\mathscr{W} \to \bar{\mathscr{U}}_j$*
(e) *with a corresponding boundary embedding*

$$\partial\varphi\colon\ \mathrm{Bd}(\mathscr{W}) \to \mathrm{Bd}(\bar{\mathscr{U}}_j)$$

such that the following conditions are satisfied:

(1) $\partial\mathscr{W} \cap \mathrm{Bd}(V) = \varnothing$.
(2) *For all $W \in \mathscr{W}$, $\varphi(W) \cap \mathrm{Bd}(U)$ contains $h(W \cap \mathrm{Bd}(V) \cap Y)$.*
(3) $\partial\varphi \mid \mathrm{Bd}(V) \cap Y = h \mid \mathrm{Bd}(V) \cap Y$.
(4) *$\varphi(\mathscr{W}(V_1))$ is α-connected.*

Proof. Since $\mathscr{U}_1$ is a tree-chain of disks any two of which meet, if at all, in an arc, it follows that for all $G \neq H \in \mathscr{U}_1$ with $G \cap H \neq \varnothing$, the endpoints of the arc $G \cap H$ are in $\mathrm{Bd}(U)$.

We may suppose that $\mathscr{U}_1$ is minimal with respect to containing

$$\mathscr{A} = \mathrm{Star}\big(h(\mathrm{Bd}(V) \cap Y),\ \mathscr{U}_1\big),$$

and that $\mathscr{A}$ consists entirely of endlinks of $\mathscr{U}_1$. (Otherwise, we may replace $\mathscr{U}_1$ by a partial amalgam of $\mathscr{U}_2$, by the properties of an α-defining sequence.) For $i = 1,\ 2$, let

$$\mathscr{A}_i = \mathrm{Star}\big(h(\mathrm{Bd}(V) \cap V_i \cap Y),\ \mathscr{U}_1\big).$$

Then $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$. Since h is an embedding of the compact set $\mathrm{Bd}(V) \cap Y$, and $h(\mathrm{Bd}(V) \cap Y)$ is a union of the disjoint closed sets $h(\mathrm{Bd}(V) \cap V_1 \cap Y)$ and $h(\mathrm{Bd}(V) \cap V_2 \cap Y)$, by replacing $\mathscr{U}_1$ by a $\mathscr{U}_k$ of sufficiently small mesh if necessary, we may assume that $\bigcup \mathscr{A}_1 \cap \bigcup \mathscr{A}_2 = \varnothing$.

There exist minimal disjoint arcs F_1 and F_2 in $\mathrm{Bd}(U)$ with $h(\mathrm{Bd}(V) \cap V_1 \cap Y) \subset F_1$ and $h(\mathrm{Bd}(V) \cap V_2 \cap Y) \subset F_2$, since h is an orientation-preserving homeomorphism of $\mathrm{Bd}(V)$ onto $\mathrm{Bd}(U)$. Consequently, for $i = 1, 2$, there exists an α-tree-chain $\mathscr{T}_i = \mathrm{Star}(F_i, \mathscr{U}_1)$ with F_i contained in the simple closed curve $\mathrm{Bd}(\bigcup \mathscr{T}_i)$, and $\mathscr{T}_i$ is the smallest tree-chain in $\mathscr{U}_1$ which contains $\mathscr{A}_1$. There are three cases to consider.

Case 1. $\mathscr{T}_1 \cap \mathscr{T}_2 = \varnothing$ and $(\bigcup \mathscr{T}_1) \cap (\bigcup \mathscr{T}_2) = T_1 \cap T_2 \neq \varnothing$, for some $T_1 \in \mathscr{T}_1$ and $T_2 \in \mathscr{T}_2$. See Figure 3.4(a). Let

$$h_i\colon\ \mathrm{Bd}(V_i) \to \mathrm{Bd}(\textstyle\bigcup \mathscr{T}_i)$$

be an orientation-preserving homeomorphism such that

$$h_i \mid \mathrm{Bd}(V) \cap V_i \cap Y = h \mid \mathrm{Bd}(V) \cap V_i \cap Y,$$

$$h_1 \mid V_1 \cap V_2 = h_2 \mid V_1 \cap V_2 \quad \text{and} \quad h_1(V_1 \cap V_2) \subset T_1 \cap T_2 - \mathrm{Bd}(U).$$

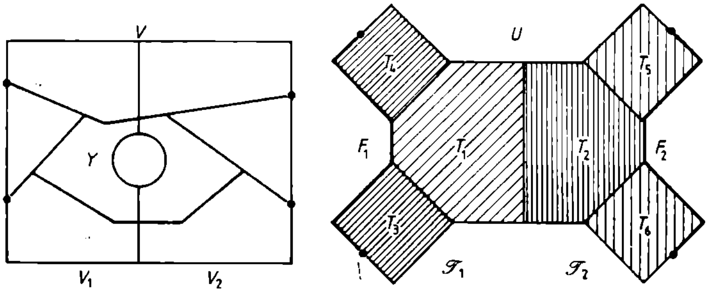

Fig. 3.4(a)

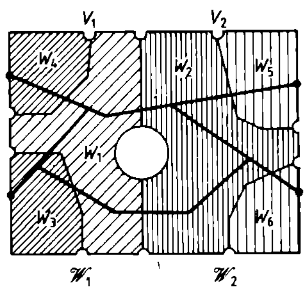

Fig. 3.4(b). $V_1 \cup V_2$ after pullback of $\mathcal{T}_1$ and $\mathcal{T}_2$

Let $\mathscr{E}$ denote the set of components of $V_1 \cap V_2$. For $i = 1, 2$, let $\mathscr{C}'_i$ be a minimal collection of disjoint arcs in $\mathrm{Bd}(U) - \partial\mathscr{U}_1$ whose interiors in $\mathrm{Bd}(U)$ cover $h_i(\mathrm{Bd}(V) \cap V_i \cap Y)$. We define the collections

$$\mathscr{C}_i = \mathscr{C}'_i \cup \{h(E) \mid E \in \mathscr{E}\} \quad \text{and} \quad \mathscr{B}_i = \{h_i^{-1}(C) \mid C \in \mathscr{C}_i\}.$$

One may check that $Y \cap V_i$, V_i, $\mathscr{B}_i$, $\mathcal{T}_i$, $\mathscr{C}_i$, and h_i satisfy the hypotheses of the Partition Pullback Theorem (3.1). Hence, for $i = 1, 2$, there exists an α-disk partition $\mathscr{W}_i$ of $Y \cap V_i$ and an isomorphism φ_i: $\mathscr{W}_i \to \mathcal{T}_i$ such that

$$\bigcup \mathscr{W}_i \subset V_i, \quad \partial\mathscr{W}_i \cap \mathrm{Bd}(V_i) = \emptyset, \quad \bigcup \mathscr{B}_i \subset \bigcup \mathscr{W}_i - \partial\mathscr{W}_i,$$
$$h\big(W \cap (\textstyle\bigcup \mathscr{B}_i)\big) = \varphi_i(W) \cap (\textstyle\bigcup \mathscr{C}_i), \quad \text{for all } W \in \mathscr{W}_i.$$

It follows that there is a unique link $W_i = \varphi_i^{-1}(T_i) \in \mathscr{W}_i$ which contains $V_1 \cap V_2 \cap Y$. Furthermore, we may assume that $W_1 \cap W_2 \cap \mathrm{Bd}(V) = \emptyset$, as in the proof of Theorem 2.4.6.

Let $\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$, $j = 1$, $\bar{\mathscr{U}}_j = \mathcal{T}_1 \cup \mathcal{T}_2$, and $\varphi = \varphi_1 \cup \varphi_2$. Since each of $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_1 \cup \mathcal{T}_2$ is α-connected, we can construct the required boundary embedding $\partial\varphi$: $\mathrm{Bd}(\mathscr{W}) \to \mathrm{Bd}(\bar{\mathscr{U}}_j)$ as follows: Apply Theorem 2.2.1 to each $G \cap H \cap Y$ for $G \neq H \in \mathscr{W}$ with $G \cap H \cap Y \neq \emptyset$ to obtain an order-preserving embedding into the copy of the universal space C_α in $\varphi(G) \cap \varphi(H) \cap X$ these order-preserving embeddings can then be simultaneously extended to obtain the orientation-preserving homeomorphism $\partial\varphi$. It is easy to check that (a)–(e) and (1)–(4) are satisfied.

Case 2. $(\bigcup \mathscr{T}_1) \cap (\bigcup \mathscr{T}_2) = \emptyset$. Since $\mathscr{U}_1 \supset \mathscr{T}_1 \cup \mathscr{T}_2$ is an α-tree-chain, we can adjoin a chain to $\mathscr{T}_1$ to obtain a tree $\mathscr{T}_1'$ such that the α-tree-chains $\mathscr{T}_1'$ and $\mathscr{T}_2$ satisfy the condition defining Case 1.

Case 3. $\mathscr{T}_1 \cap \mathscr{T}_2 \neq \emptyset$. Then $\mathscr{T}_1 \cap \mathscr{T}_2$ is coherent because $\mathscr{T}_1 \cup \mathscr{T}_2$ is a tree-chain. Let $\mathscr{H}_i$ be a chain in $\mathscr{T}_i$ between the endpoints of the arc F_i. Then $\mathscr{T}_1 \cap \mathscr{T}_2 = \mathscr{H}_1 \cap \mathscr{H}_2$, by planarity. See Figure 3.5 (a). Consequently, $\mathscr{T}_1 \cap \mathscr{T}_2$ is a chain. Let $\mathscr{T}_1 \cap \mathscr{T}_2 = \{H_0, \ldots, H_n\}$.

We claim that $(\mathrm{Bd}(U) \cap H_0) - (F_1 \cup F_2) \neq \emptyset$. Note that for $i = 1, 2$

$$\bigcup \mathscr{T}_i \subset \mathrm{Hull}(F_i \cup (\bigcup \mathscr{H}_i)).$$

Hence, there is a chain $\mathscr{G} \subset \mathscr{H}_1 \cup \mathscr{H}_2$ one of whose endlinks H_1' is an endlink of $\mathscr{H}_1$, the other of whose endlinks H_2' is an endlink of $\mathscr{H}_2$, and with $H_0 \in \mathscr{G}$. (In Figure 3.5 (a), $\mathscr{G}$ would be the chain $\{T_2, H_0, T_4\}$.)

One endpoint y_1 of F_1 is in H_1' and the other endpoint y_2 is in H_2'. The component F' of $\mathrm{Bd}(U) - (F_1 \cup F_2)$ with endpoints y_1 and y_2 meets H_0. Let C_0 be an arc in $F' \cap H_0$ which misses the endpoints of $F' \cap H_0$. Then C_0 is disjoint from H for every $H \in (\mathscr{T}_1 \cup \mathscr{T}_2) - \{H_0\}$. See Figure 3.5 (a).

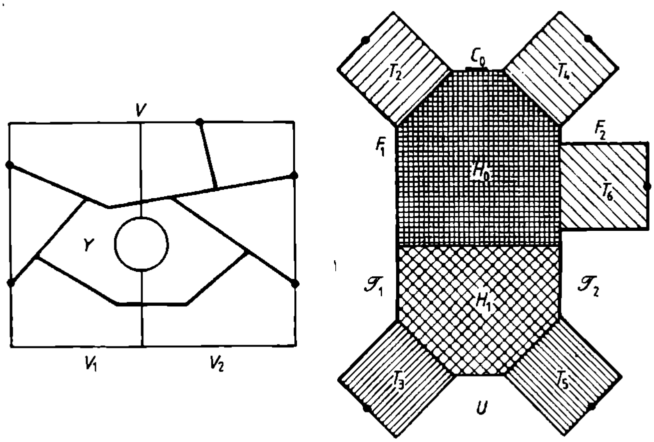

Fig. 3.5 (a). $\mathscr{T}_1$ and $\mathscr{T}_2$ overlap in $\mathscr{H} = \{H_0, H_1\}$.

As in Case 1, for $i = 1, 2$, let h_i: $\mathrm{Bd}(V_i) \to \mathrm{Bd}((\bigcup \mathscr{T}_i)$ be an orientation-preserving homeomorphism such that

$$h_i \mid \mathrm{Bd}(V) \cap V_i \cap Y = h \mid \mathrm{Bd}(V) \cap V_i \cap Y,$$

$$h_1 \mid V_1 \cap V_2 = h_2 \mid V_1 \cap V_2 \quad \text{and} \quad h_1(V_1 \cap V_2) \subset C_0.$$

Let $\mathscr{E}$ denote the set of components of $V_1 \cap V_2$. For $i = 1, 2$, let $\mathscr{C}_i'$ be a minimal collection of disjoint arcs in $\mathrm{Bd}(U) - \partial \mathscr{U}_1$ whose interiors in $\mathrm{Bd}(U)$ cover $h_i(\mathrm{Bd}(V) \cap V_i \cap Y)$. We define the collections

$$\mathscr{C}_i = \mathscr{C}_i' \cup \{h(E) \mid E \in \mathscr{E}\} \quad \text{and} \quad \mathscr{B}_i = \{h_i^{-1}(C) \mid C \in \mathscr{C}_i\}.$$

Then $Y \cap V_i$, V_i, $\mathscr{B}_i$, $\mathscr{T}_i$, $\mathscr{C}_i$, and h_i satisfy the hypotheses of the Partition Pullback Theorem (3.1). Hence, for $i = 1, 2$, there exists an α-disk partition $\mathscr{W}_i$ of $Y \cap V_i$ and an isomorphism $\varphi_i\colon \mathscr{W}_i \to \mathscr{T}_i$ such that

$$\bigcup \mathscr{W}_i \subset V_i, \quad \partial\mathscr{W}_i \cap \mathrm{Bd}(V_i) = \emptyset, \quad \bigcup \mathscr{B}_i \subset \bigcup \mathscr{W}_i - \partial\mathscr{W}_i,$$
$$h(W \cap (\bigcup \mathscr{B}_i)) = \varphi_i(W) \cap (\bigcup \mathscr{C}_i), \quad \text{for all } W \in \mathscr{W}_i.$$

It follows that there is a unique link $W_{i,0} = \varphi_i^{-1}(H_0) \in \mathscr{W}_i$ which contains $V_1 \cap V_2 \cap Y$. Furthermore, we may assume that $W_{1,0} \cap W_{2,0} \cap \mathrm{Bd}(V) = \emptyset$, as in the proof of Theorem 2.4.6. See Figure 3.5(b).

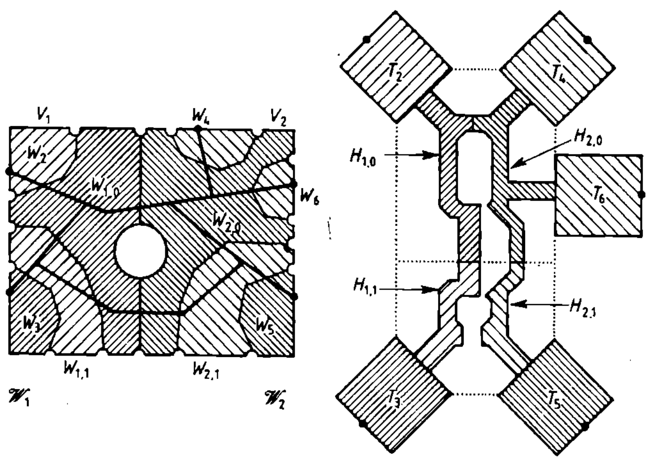

Fig. 3.5(b). After pullback and α-circlelacing

Let $\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$. Note that $\partial\mathscr{W} \cap \mathrm{Bd}(V) = \emptyset$. Note also that $\varphi_1 \cup \varphi_2$ is a function from $\mathscr{W}$ onto $\mathscr{T}_1 \cup \mathscr{T}_2$ which is two-to-one on $\mathscr{T}_1 \cap \mathscr{T}_2 = \{H_0, \ldots, H_n\}$ and one-to-one on $(\mathscr{T}_1 \cup \mathscr{T}_2) - (\mathscr{T}_1 \cap \mathscr{T}_2)$. We must modify the range of $\varphi_1 \cup \varphi_2$ to obtain an isomorphism.

For $i = 1, 2$, by the α-circlelacing Lemma 3.3.1 and Corollary 2.2.2 (to obtain conditions (iii) and (iv) below), there exists a $j > 1$ and chains $\mathscr{D}_i = \{H_{i,0}, \ldots, H_{i,n}\}$ in an amalgam $\mathscr{U}_j(\bigcup \mathscr{H})$ such that the following conditions are satisfied:

(i) $\mathscr{U}_j(H_{i,k})$ is an α-tree-chain.

(ii) $H_{1,0} \cap H_{2,0} \cap X$ contains an order-isomorphic copy of C_α.

(iii) $H_{1,k} \cap H_{1,k+1} \cap X$ contains an order-isomorphic copy of C_α, and hence, of $\varphi_1^{-1}(H_k) \cap \varphi_1^{-1}(H_{k+1}) \cap Y$.

(iv) $H_{2,k} \cap H_{2,k+1} \cap X$ contains an order-isomorphic copy of $\varphi_2^{-1}(H_k) \cap \varphi_2^{-1}(H_{k+1}) \cap Y$.

(v) If $H \in \mathscr{T}_i - \{H_0, \ldots, H_n\}$ and $H \cap H_k \neq \emptyset$, for some $k \in \{0, \ldots, n\}$, then $H \cap H_{i,k} \cap X$ contains an order-isomorphic copy of C_α.

(vi) $(\bigcup \mathscr{D}_1) \cap (\bigcup \mathscr{D}_2) \subset H_{1,0} \cap H_{2,0}$.

For $H_{i,k} \in \mathscr{D}_i$, $H_k = \varphi_i(W)$ for some unique $W \in \mathscr{W}_i$; define $\varphi(W) = H_{i,k}$. For each $T \in \mathscr{T}_i - (\mathscr{T}_1 \cap \mathscr{T}_2)$, we have $T = \varphi_i(W_T)$ for some unique $W_T \in \mathscr{W}_i$; define $\varphi(W_T)$ to be an α-tree-chain in $\mathscr{U}_j(T)$ such that if $T' \in \mathscr{T}_i - \{T\}$ and $T \cap T' \neq \emptyset$, then $\varphi(W_T) \cap T' \cap X$ contains an order-isomorphic copy of C_α. Let

$$\mathscr{T}'_i = \{\varphi(W_T) \mid T \in \mathscr{T}_i - (\mathscr{T}_1 \cap \mathscr{T}_2)\} \cup \mathscr{D}_i,$$

and let $\bar{\mathscr{U}}_j = \mathscr{T}'_1 \cup \mathscr{T}'_2$. Then $\varphi\colon \mathscr{W} \to \bar{\mathscr{U}}_j$ defined as above is an isomorphism.

We now apply Theorem 2.2.1 to construct $\partial\varphi$ as required. That we can embed $V_1 \cap V_2 \cap Y$ into $H_{1,0} \cap H_{2,0} \cap X$ by an orientation-preserving homeomorphism is a consequence of (ii). The existence of $\partial\varphi$ then follows from conditions (iii)–(v) and the α-connectedness of $\mathscr{T}_i$. ■

3.5. Construction of a containing space. We now construct a planar Peano continuum S_α with an α-defining sequence of partitions. In Section 4 we will prove that any such continuum is a containing space for all planar compacta of rim-type $\leqslant \alpha$.

3.5.1. *Standard copy of C_α.* Recall that C_α, constructed in Theorem 2.2.1, is a containing space (of type $\alpha+1$) for all ordered compacta of type $\leqslant \alpha$. We may suppose that C_α is contained as a subset of the Cantor set C in $[0, 1]$ missing 0 and 1. We will call any embedding of C_α into a straight line segment $[a, b]$ in E^2 by a linear rescaling of $[0, 1]$ onto $[a, b]$ a *standard copy of C_α*.

3.5.2. THEOREM. *There is a Peano continuum $S_\alpha \subset E^2$ with an α-defining sequence of disk partitions.*

Proof. We construct a sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ of disk partitions in E^2 so that $S_\alpha = \bigcap_{i=1}^{\infty} \bigcup \mathscr{U}_i$ is a Peano continuum of rim-type $\alpha+1$. See Figure 3.6 for a guide to the construction.

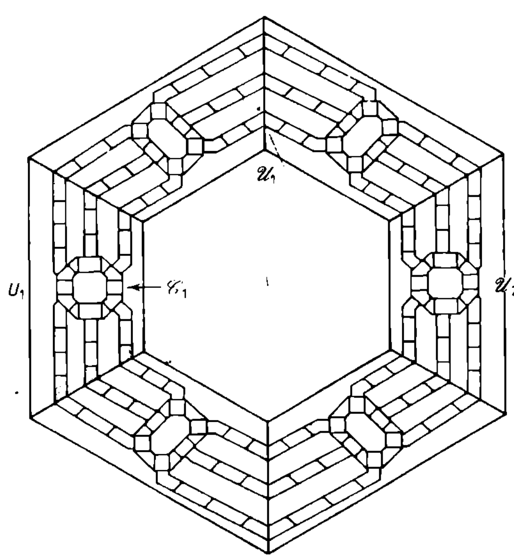

Fig. 3.6. Bound$(U_1, \mathscr{U}_2)$ shaded

Let $\mathscr{U}_1 = \{U_0, \ldots, U_5\}$ be a circle-chain of polygonal disks in E^2 such that $U_i \cap U_{i+1}$ (subscript addition modulo 6) is a line segment and $\operatorname{mesh}(\mathscr{U}_1) < 1$. In each line segment $U_i \cap U_{i+1}$ we embed a standard copy of C_α.

For each $U_i \in \mathscr{U}_1$, define $\operatorname{Bound}(U_i, \mathscr{U}_2)$ to be a collection of mutually disjoint polygonal disks in U_i such that each $B \in \operatorname{Bound}(U_i, \mathscr{U}_2)$ meets $\operatorname{Bd}(U_i)$ in an arc, $\bigcup \operatorname{Bound}(U_i, \mathscr{U}_2) \cap \operatorname{Bd}(U_i) \subset \partial \mathscr{U}_1$ and

$$\{B \cap \operatorname{Bd}(U_i) \mid B \in \operatorname{Bound}(U_i, \mathscr{U}_2)\}$$

is a collection of arcs whose interiors in $\operatorname{Bd}(U_i)$ irreducibly cover the standard copies of C_α in $U_{i-1} \cap U_i$ and in $U_i \cap U_{i+1}$. We also require that $\operatorname{mesh}(\operatorname{Bound}(U_i, \mathscr{U}_2)) < 2^{-1}$, and if $B \in \operatorname{Bound}(U_i, \mathscr{U}_2)$, $B' \in \operatorname{Bound}(U_j, \mathscr{U}_2)$, and $B \cap B' \neq \emptyset$, then $B \cap B' = B \cap \operatorname{Bd}(U_i) = B' \cap \operatorname{Bd}(U_j)$.

Let $\mathscr{C}_i$ be a circle-chain of polygonal disks in $\operatorname{Int}(U_i) - \bigcup \operatorname{Bound}(U_i, \mathscr{U}_2)$ such that $\operatorname{mesh}(\mathscr{C}_i) < 2^{-1}$, $|\mathscr{C}_i| \geqslant 2 \cdot |\operatorname{Bound}(U_i, \mathscr{U}_2)|$, and if $C \neq C' \in \mathscr{C}_i$ with $C \cap C' \neq \emptyset$, then $C \cap C'$ is a line segment. We embed a standard copy of C_α in $C \cap C'$ for each such pair. For each $B \in \operatorname{Bound}(U_i, \mathscr{U}_2)$, let $\mathscr{C}_{i,B}$ be a chain of polygonal disks in $\operatorname{Int}(U_i)$, one endlink of which is B and the other endlink of which is some $C \in \mathscr{C}_i$. We also require that $\operatorname{mesh}(\mathscr{C}_{i,B}) < 2^{-1}$, if $C \neq C' \in \mathscr{C}_{i,B}$ with $C \cap C' \neq \emptyset$, then $C \cap C'$ is a line segment, and for $B \neq B' \in \operatorname{Bound}(U_i, \mathscr{U}_2)$, $\bigcup \mathscr{C}_{i,B} \cap \bigcup \mathscr{C}_{i,B'} = \emptyset$. Let $C \neq C' \in \mathscr{C}_{i,B}$ as above, and suppose $C \cap C' = [a, b]$. We embed a standard copy of C_α in each of the intervals $[a, (a+b)/2]$ and $[(a+b)/2, b]$. Define

$$\mathscr{U}_2(U_i) = \mathscr{C}_i \cup \left(\bigcup \{\mathscr{C}_{i,B} \mid B \in \operatorname{Bound}(U_i, \mathscr{U}_2)\}\right) \quad \text{and} \quad \mathscr{U}_2 = \bigcup_{i=0}^{5} \mathscr{U}_2(U_i).$$

It is clear from the construction that the α-connected, α-splitting, and α-circlelacing conditions of Definition 3.2.4 will be satisfied by $\mathscr{U}_1$ and $\mathscr{U}_2$.

Assume that $\mathscr{U}_i$ has been constructed. We construct $\mathscr{U}_{i+1}$ inductively by working within one link $U \in \mathscr{U}_i$ at a time as we did above in constructing that part of $\mathscr{U}_2$ that lay in each element of $\mathscr{U}_1$. Some care must be taken to insure that the α-splitting condition is satisfied. It is easy to check that $S_\alpha = \bigcap_{i=1}^{\infty} \bigcup \mathscr{U}_i$ is a Peano continuum with $\{\mathscr{U}_i\}_{i=1}^{\infty}$ a sequence of $(\alpha+1)$-disk-partitions for S_α satisfying conditions (1)–(7) of Definition 3.2.4. Therefore, $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is an α-defining sequence for S_α. ∎

4. Embedding Theorem

In this section we prove that a planar continuum with an α-defining sequence of disk partitions is a containing space for all planar compacta of rim-type $\leqslant \alpha$. This provides an affirmative solution to Problem 5 of [I3]. The heart of the proof is the following lemma:

4.1. LEMMA. *Let $Y \subset E^2$ be a continuum of rim-type $\leqslant \alpha$. Let $\{\mathscr{U}_j\}_{j=1}^{\infty}$ be an α-defining sequence of disk partitions of a continuum $X \subset E^2$, $\mathscr{V}_1$ an α-disk* *partition of Y, $\bar{\mathscr{U}}_1$ an α-amalgam of $\mathscr{U}_1$, and φ: $\mathscr{V}_1 \to \bar{\mathscr{U}}_1$ an isomorphism with a corresponding boundary embedding $\partial\varphi$:* $\mathrm{Bd}(\mathscr{V}_1) \to \mathrm{Bd}(\bar{\mathscr{U}}_1)$. *Let $\mathscr{V}$ be an α-disk partition of Y refining $\mathscr{V}_1$. Then there exists*

(a) *an α-disk partition $\mathscr{W}$ of Y refining $\mathscr{V}$,*
(b) *an integer $j \geqslant 1$,*
(c) *an α-amalgam $\bar{\mathscr{U}}_j$* of *$\mathscr{U}_j$ such that $\bar{\mathscr{U}}_j$ refines $\mathscr{U}_1$ and $\bigcup \bar{\mathscr{U}}_j \subset \bar{\mathscr{U}}_1$.*

Moreover, there exists

(d) *an isomorphism θ: $\mathscr{W} \to \bar{\mathscr{U}}_j$ with respect to φ and*
(e) *a corresponding boundary embedding $\partial\theta$:* $\mathrm{Bd}(\mathscr{W}) \to \mathrm{Bd}(\bar{\mathscr{U}}_j)$ *such that $\partial\theta$ agrees with $\partial\varphi$ on* $\mathrm{Bd}(\mathscr{V}_1) \cap Y$.

Proof. We define $\mathscr{W}$ by working on one link of $\mathscr{V}_1$ at a time. Let $V \in \mathscr{V}_1$ and let $U = \varphi(V) \in \bar{\mathscr{U}}_1$. Suppose that $\mathscr{V}(V) = \{V_1, \ldots, V_n\}$. We will construct

(a_0) an α-disk partition $\mathscr{W}(V)$ of $V \cap Y$ refining $\mathscr{V}(V)$,
(b_0) an integer $j(V) \geqslant 1$,
(c_0) an α-amalgam $\bar{\mathscr{U}}_{j(V)}(U)$ of $\mathscr{U}_{j(V)}(U)$ refining $\mathscr{U}_1(U)$,
(d_0) and an isomorphism θ_V: $\mathscr{W}(V) \to \bar{\mathscr{U}}_{j(V)}(U)$ with
(e_0) a corresponding boundary embedding $\partial\theta_V$: $\mathrm{Bd}(\mathscr{W}(V)) \to \mathrm{Bd}(\bar{\mathscr{U}}_{j(V)}(U))$

such that the following conditions are satisfied:

(1) $\partial\mathscr{W}(V) \cap \mathrm{Bd}(V) = \emptyset$.
(2) For all $W \in \mathscr{W}(V)$, $\theta_V(W) \cap \mathrm{Bd}(U)$ contains $\partial\varphi(W \cap \mathrm{Bd}(V) \cap Y)$.
(3) $\partial\theta_V \mid \mathrm{Bd}(V) \cap Y = \partial\varphi \mid \mathrm{Bd}(V) \cap Y$.
(4) $\theta_V(\mathscr{W}(V_1))$ is α-connected.

We induct on n. We may assume that $n \geqslant 2$ by decomposing V, if necessary, into two links as in the proof of Theorem 2.4.6. The base case for $n = 2$ follows immediately from the Brick Pulverizing Lemma (3.4). It is condition (4) that is crucial to the induction.

Suppose $n > 2$ and that we can carry out the construction for all α-disk partitions of $Y \cap V$ with fewer than n elements. Since Y is connected and $\mathscr{V}(V)$ is finite, it follows by a simple induction that there exist $s \neq t \in \{1, \ldots, n\}$ such that $V_s \cap V_t \neq \emptyset$ and $\bigcup(\mathscr{V}(V) - \{V_s, V_t\})$ is contained in $\mathrm{Cl}(E^2 - \mathrm{Hull}(V_s \cup V_t))$. Without loss of generality, $s = 1$ and $t = 2$. Let $V' = \mathrm{Cl}(\mathrm{Hull}(V_1 \cup V_2))$, and let $\mathscr{V}'(V) = \{V', V_3, \ldots, V_n\}$.

By the induction hypothesis there exists

(a_1) an α-disk partition $\mathscr{W}'(V)$ of $V \cap Y$ refining $\mathscr{V}'(V)$,
(b_1) an integer $j'(V) \geqslant 1$,
(c_1) an α-amalgam $\bar{\mathscr{U}}_{j'(V)}(U)$ of $\mathscr{U}_{j'(V)}(U)$ refining $\mathscr{U}_1(U)$, and
(d_1) an isomorphism θ'_V: $\mathscr{W}'(V) \to \bar{\mathscr{U}}_{j'(V)}(U)$ with
(e_1) a corresponding boundary embedding

$$\partial\theta'_V\colon\ \mathrm{Bd}(\mathscr{W}'(V)) \to \mathrm{Bd}(\bar{\mathscr{U}}_{j'(V)}(U))$$

satisfying the analogs of conditions (1)–(4) above, namely

(1′) $\partial\mathscr{W}'(V) \cap \mathrm{Bd}(V) = \varnothing$.

(2′) For all $W \in \mathscr{W}'(V)$, $\theta'_V(W) \cap \mathrm{Bd}(U)$ contains $\partial\varphi(W \cap \mathrm{Bd}(V) \cap Y)$.

(3′) $\partial\theta'_V \mid \mathrm{Bd}(V) \cap Y = \partial\varphi \mid \mathrm{Bd}(V) \cap Y$.

(4′) $\theta'_V(\mathscr{W}'(V'))$ is α-connected.

Now $\mathscr{W}'(V)$ does not refine $\mathscr{V}(V)$. However, it fails to refine $\mathscr{V}(V)$ only on the pair $\{V_1, V_2\}$. Let $K = \bigcup \theta'_V(\mathscr{W}'(V'))$. Since by (4′), $\theta'_V(\mathscr{W}'(V')) = \bar{\mathscr{U}}_{j'(V)}(K)$ is α-connected, and since for each $W \in \mathscr{W}'(V')$, $\theta'_V(W)$ is an element of an α-amalgam $\bar{\mathscr{U}}_{j'(V)}(K)$ of $\mathscr{U}_{j'(V)}(K)$, we may assume that $\mathscr{U}_{j'(V)}(K)$ is an α-tree-chain. Thus, we apply the Brick Pulverizing Lemma (3.4) to $Y \cap V'$, $\{V_1, V_2\}$, K, and $h = \partial\theta'_V \mid \mathrm{Bd}(V')$: $\mathrm{Bd}(V') \to \mathrm{Bd}(K)$. We obtain

(a_2) an α-disk partition $\mathscr{W}(V')$ of $Y \cap V'$ refining $\{V_1, V_2\}$,

(b_2) an integer $j(V) \geqslant j'(V)$,

(c_2) an α-amalgam $\bar{\mathscr{U}}_{j(V)}(K)$ of $\mathscr{U}_{j(V)}(K)$ refining $\mathscr{U}_{j'(V)}(K)$, and

(d_2) an isomorphism $\theta_V \mid \mathscr{W}(V')$: $\mathscr{W}(V') \to \bar{\mathscr{U}}_{j(V)}(K)$

(e_2) with a corresponding boundary embedding

$$\partial\theta_V \mid \mathrm{Bd}(\mathscr{W}(V')):\ \mathrm{Bd}(\mathscr{W}(V')) \to \mathrm{Bd}(\bar{\mathscr{U}}_{j(V)}(K))$$

satisfying the analogs of conditions (1)–(4) above.

We can now define $\mathscr{W}(V)$ refining $\mathscr{V}(V)$. Let

(a_3) $\mathscr{W}(V) = (\mathscr{W}'(V) - \mathscr{W}'(V')) \cup \mathscr{W}(V')$, and

(c_3) $\bar{\mathscr{U}}_{j(V)}(U) = (\bar{\mathscr{U}}_{j'(V)}(U) - \bar{\mathscr{U}}_{j'(V)}(K)) \cup \bar{\mathscr{U}}_{j(V)}(K)$.

We can then define

(d_3) an isomorphism θ_V: $\mathscr{W}(V) \to \bar{\mathscr{U}}_{j(V)}(U)$ by setting

$$\theta_V = \theta'_V \mid (\mathscr{W}'(V) - \mathscr{W}'(V') \cup \theta_V \mid \mathscr{W}(V'),$$

(e_3) with a corresponding boundary embedding

$$\partial\theta_V:\ \mathrm{Bd}(\mathscr{W}(V)) \to \mathrm{Bd}(\bar{\mathscr{U}}_{j(V)}(U)),$$

defined by

$$\partial\theta_V = \partial\theta'_V \mid \mathrm{Bd}(\mathscr{W}'(V) - \mathscr{W}'(V')) \cup \partial\theta_V \mid \mathrm{Bd}(\mathscr{W}(V')).$$

Conditions (1)–(4) are satisfied, except technically $\bar{\mathscr{U}}_{j'(V)}(U) - \bar{\mathscr{U}}_{j'(V)}(K)$ is an α-amalgam of $\mathscr{U}_{j'(V)}(U)$ and not $\mathscr{U}_{j(V)}(U)$. This can be remedied by the technique used at the end of the proof of Theorem 3.4; θ_V and $\partial\theta_V$ must be modified accordingly, as well.

We conclude the proof by defining the requisite disk partitions, isomorphism, and boundary embedding. We obtain

(a_4) $\mathscr{W} = \bigcup \{\mathscr{W}(V) \mid V \in \mathscr{V}_1\}$,

(b_4) $j = \max \{j(V) \mid V \in \mathscr{V}_1\}$,

(c_4) $\bar{\mathscr{U}}_j = \bigcup \{\bar{\mathscr{U}}_{j(V)}(\varphi(V)) \mid V \in \mathscr{V}_1\}$,

(d_4) $\theta = \bigcup \{\theta_V \mid V \in \mathscr{V}_1\}$, and

(e_4) a boundary embedding $\partial\theta$: $\mathrm{Bd}(\mathscr{W}) \to \mathrm{Bd}(\bar{\mathscr{U}}_j)$ which is an extension of $\bigcup_{V \in \mathscr{V}_1} \partial\theta_V \mid (Y \cap \mathrm{Bd}(\mathscr{W}(V)))$.

Again, $\bar{\mathscr{U}}_j$, and consequently θ and $\partial\theta$, must be modified as in the preceding paragraph. Note that condition (1) implies that $\mathscr{W}$ is a disk partition (no three distinct elements meet). Conditions (2) and (3) imply that (d) and (e) are satisfied. Condition (4) is required only in the proof of 4.1. ∎

4.2. MAIN THEOREM. *Let X be a planar continuum with an α-defining sequence of disk partitions. Then X is a containing space for compacta of rim-type $\leqslant \alpha$.*

Proof. Suppose $\alpha = 0$. Every compactum of rim-type 0 is zero-dimensional, so embeds in an arc, and hence, in X. So suppose that $\alpha \geqslant 1$. Let $Y \subset E^2$ be a compactum of rim-type $\leqslant \alpha$ and $\{\mathscr{U}_j\}_{j=1}^{\infty}$ an α-defining sequence for X. By adjoining a null sequence of arcs to Y in the plane, disjoint except possibly for common endpoints, we may assume that Y is a continuum of rim-type $\leqslant \alpha$. Let V be a disk containing Y in its interior, $\mathscr{V}_1 = \{V\}$, and $\bar{\mathscr{U}}_1 = \{\mathrm{Hull}(\bigcup \mathscr{U}_1)\}$, φ_1: $\mathscr{V}_1 \to \bar{\mathscr{U}}_1$ the obvious isomorphism, and $\partial\varphi$ an orientation-preserving homeomorphism from $\mathrm{Bd}(V)$ onto $\mathrm{Bd}(\bar{\mathscr{U}}_1)$.

Suppose we have defined for all k, $i > k > 1$,

(1) $\mathscr{V}_k$ refining $\mathscr{V}_{k-1}$ as α-disk partitions of Y, with
(2) $\mathrm{mesh}(\mathscr{V}_k) < k^{-1}$,
(3) integers $j_k > j_{k-1} \geqslant 1$,
(4) α-amalgams $\bar{\mathscr{U}}_{j_k}$ refining $\mathscr{U}_{j_{k-1}}$ and $\bigcup \bar{\mathscr{U}}_{j_k} \subset \bigcup \bar{\mathscr{U}}_{j_{k-1}}$,
(5) isomorphisms φ_k: $\mathscr{V}_k \to \bar{\mathscr{U}}_{j_k}$, with
(6) φ_k an isomorphism with respect to φ_{k-1}, and
(7) corresponding boundary embeddings

$$\partial\varphi_k\colon \mathrm{Bd}(\mathscr{V}_k) \to \mathrm{Bd}(\bar{\mathscr{U}}_{j_k})$$

such that

$$\partial\varphi_k \mid \mathrm{Bd}(\mathscr{V}_{k-1}) \cap Y = \partial\varphi_{k-1} \mid \mathrm{Bd}(\mathscr{V}_{k-1}) \cap Y.$$

Let $\mathscr{V}$ be an α-disk partition of Y refining $\mathscr{V}_{i-1}$ of mesh $< i^{-1}$ by Theorem 2.4.6. That disk partitions, an isomorphism, and a corresponding boundary embedding exist satisfying conditions (1)–(2) and (4)–(7) for i follows from Lemma 4.1. We apply the technique at the end of the proof of Theorem 3.4, if necessary, in order to obtain the strict inequality of condition (3). Thus an infinite sequence satisfying conditions (1)–(7) exists. Condition (3) guarantees that $\mathrm{mesh}(\bar{\mathscr{U}}_{j_k}) \to 0$. The Theorem then follows from the Partition-Matching Theorem (2.4.8). ∎

5. Universal spaces

In [13] Iliadis asks whether there exists for each countable ordinal α, a universal planar space of rim-type $\leqslant \alpha$, and whether there exists a universal planar rational space. Such spaces are known to exist [MT, I3, I4] without the planar hypothesis.

In this section we will construct the Sierpiński universal plane curve as the intersection of a defining sequence of disk partitions, and we will conjecture that certain subspaces of it are, respectively, a universal planar rational space and a universal planar space of rim-type $\leqslant \alpha$. The main stumbling block in verifying the conjectures is that we are unable to prove the analogues of the Partition Pullback Theorem for noncompact planar spaces. In particular, we have been unable to answer the following question:

5.1. QUESTION. Let $D \subset E^2$ be a disk and $Y \subset \mathrm{Int}(D)$ a space of rim-type $\leqslant \alpha$ (respectively, rational space). Let $a \neq b \in Y$. Does there exist an arc $L \subset D$ such that $\mathrm{type}(L \cap Y) \leqslant \alpha$ (respectively, $L \cap Y$ is countable), and such that L separates a from b in D?

5.2. The Sierpiński curve. Let $\mathscr{U}$ refining $\mathscr{V}$ be disk partitions of a compactum $X \subset E^2$ and let $\mathscr{W} \subset \mathscr{U}$.

5.2.1. *Circlelacing.* We say that $\mathscr{U}$ is *circlelaced* in $\mathscr{V}$ iff

(1) for every $V \in \mathscr{V}$, $\mathrm{Core}(V, \mathscr{U})$ contains a circle-chain $\mathscr{D}$,
(2) for every $B \in \mathrm{Bound}(V, \mathscr{U})$, there exists a chain $\mathscr{C}_B$ in $\mathscr{U}(V)$ with one endlink being B and the other endlink in $\mathscr{D}$, and
(3) for $B \neq B' \in \mathrm{Bound}(V, \mathscr{U})$, $(\bigcup \mathscr{C}_B) \cap (\bigcup \mathscr{C}_{B'}) = \emptyset$.

5.2.2. *Splitting.* We say that $\mathscr{U}$ *splits* $\mathscr{V}$ iff for all $U \neq V \in \mathscr{V}$ such that $U \cap V \neq \emptyset$, there exist $W_1, W_2 \in \mathrm{Bound}(U, \mathscr{U})$ and $W_3, W_4 \in \mathrm{Bound}(V, \mathscr{U})$ such that $W_1 \cap W_2 = \emptyset = W_3 \cap W_4$ and $W_1 \cap W_3 \cap X \neq \emptyset \neq W_2 \cap W_4 \cap X$.

5.2.3. *S-defining sequence.* Let X be a continuum in E^2. We say that a decreasing sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ of disk partitions of X such that $X = \bigcap_{i=1}^{\infty} \bigcup \mathscr{U}_i$ is an *S-defining sequence* for X iff for all $i > 0$ the following conditions hold:

(1) $\mathscr{U}_{i+1}$ core refines $\mathscr{U}_i$.
(2) For all $U \in \mathscr{U}_i$, for all $V \neq W \in \mathrm{Bound}(U, \mathscr{U}_{i+1})$, $V \cap W = \emptyset$.
(3) $\mathscr{U}_i$ is coherent.
(4) $\mathrm{Bd}(U) \cap X$ is a Cantor set for each $U \in \mathscr{U}_i$.
(5) $\mathscr{U}_{i+1}$ splits $\mathscr{U}_i$.
(6) $\mathscr{U}_{i+1}$ is circlelaced in $\mathscr{U}_i$.
(7) For all $U \neq V \in \mathscr{U}_i$, $U \cap V \neq \emptyset$ implies $U \cap V$ is an arc.

There is some redundancy in these conditions for convenience.

5.2.4. THEOREM. *Let X be a planar continuum with an S-defining sequence of disk partitions. Then X is homeomorphic to the Sierpiński universal plane curve.*

Proof. Whyburn [W] characterizes the Sierpiński universal plane curve as a planar one-dimensional Peano continuum with no local separating points. Conditions (1)–(3) imply that X is a one-dimensional Peano continuum; conditions (5) and (6) imply that X contains no local separating points. ∎

5.3. A conjectured universal planar rational space. Note that the rationals Q can be embedded densely as a subset of Cantor's ternary set C by an order-preserving embedding. Let S be a plane continuum with $\{\mathscr{U}_i\}_{i=1}^{\infty}$ an S-defining sequence of disk partitions for S. Let $\mathscr{U}_0 = \{W\}$, where W is a disk containing S in its interior. For each $i \geqslant 1$, for each $U \neq V \in \mathscr{U}_i$ with $U \cap V \neq \emptyset$ and $U \cap V \cap \mathrm{Bd}(\mathscr{U}_{i-1}) = \emptyset$, choose an embedding $Q_{U,V}$ of Q as a dense subset of the Cantor set $U \cap V \cap S$. Otherwise, let $Q_{U,V} = \emptyset$. Define the space

$$X = S - \bigcup_{i=1}^{\infty} (\partial \mathscr{U}_i - \bigcup \{Q_{U,V} \mid U, V \in \mathscr{U}_i\}).$$

From each boundary set $U \cap V \cap S$, we have removed the complement of $Q_{U,V}$. Thus, the elements of the closed cover $\{X \cap U \mid U \in \mathscr{U}_i\}$ of X have countable boundaries. The resulting noncompact space X is easily seen to be a connected, locally connected planar rational space with no local separating points. We conjecture that it is universal for all planar rational spaces. In fact, one might conjecture that any connected, locally arcwise connected planar rational space with no local separating points is a universal space for all planar rational spaces.

5.4. A conjectured universal planar space of rim-type $\leqslant \alpha$. Note that the universal ordered space A_α of type α (Theorem 2.2.3) can be embedded as a subset of Cantor's ternary set $C \subset [0, 1]$ by an order-preserving embedding. Let S and $\{\mathscr{U}_i\}_{i=0}^{\infty}$ be as above. For each $i \geqslant 1$, for each $U \neq V \in \mathscr{U}_i$ with $U \cap V \neq \emptyset$ and $U \cap V \cap \mathrm{Bd}(\mathscr{U}_{i-1}) = \emptyset$, choose an order-preserving embedding $A_{\alpha,U,V}$ of A_α as a subset of the Cantor set $U \cap V \cap S$. Otherwise, let $A_{\alpha,U,V} = \emptyset$. Define the space

$$X_\alpha = S - \bigcup_{i=1}^{\infty} (\partial \mathscr{U}_i - \bigcup \{A_{\alpha,U,V} \mid U, V \in \mathscr{U}_i\}).$$

The resulting noncompact space X_α is easily seen to be a connected, locally connected planar space of rim-type α. We conjecture that it is universal for all planar spaces of rim-type $\leqslant \alpha$.

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