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BESSEL-CLIFFORD THIRD ORDER DIFFERENTIAL OPERATOR AND CORRESPONDING LAPLACE TYPE INTEGRAL TRANSFORM

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Abstract. The theory of hyper-Bessel differential operators of arbitrary order $m > 1$ has been shown to be closely related to the Meijer's G-functions ([9], [10], [18], [20]-[25], [27]). However, most of the operational calculi, integral transforms and solutions to the Bessel type differential equations developed by different authors concern special cases mainly of order $m = 2$ when the role of these special functions is not evident. Here, we give an example of a third order Bessel type operator and emphasize on the use of the generalized fractional calculus and G-functions. Main attention is paid to the corresponding Laplace–Obrechkoff type integral transform with examples of its applications for solving initial value problems for Bessel–Clifford differential equations of third order.

1. Introduction. Many initial and boundary value problems of mathematical physics are related to the class of so-called *Bessel type differential operators of arbitrary order*, or *hyper-Bessel operators*.

DEFINITION 1.1. By a *hyper-Bessel differential operator* of order $m \geq 1$ we

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mean each linear singular differential operator of the form:

$$B = x^{\alpha_0} D x^{\alpha_1} D x^{\alpha_2} \cdots \ D x^{\alpha_m}, \qquad D := \frac{d}{dx},\ 0 < x < \infty, \tag{1.1}$$

with real $\alpha_0, \alpha_1, \cdots, \alpha_m$ such that $\beta = m - (\alpha_0 + \alpha_1 + \cdots + \alpha_m) > 0$, or in terms of the parameters $\gamma_k = \frac{1}{\beta}(\alpha_k + \cdots + \alpha_m - m + k)$, $k = 1, \ldots, m$:

$$B = x^{-\beta} \prod_{k=1}^{m} (xD + \beta\gamma_k) = x^{-\beta}\ (xD + \beta\gamma_1) \cdots (xD + \beta\gamma_m). \tag{1.2}$$

Often, these m-th order differential operators with variable coefficients are encountered in the form:

$$B = x^k \frac{d^m}{dx^m} + a_1 x^{k-1} \frac{d^{m-1}}{dx^{m-1}} + \cdots + a_m x^{k-m}, \qquad k = m - \beta, \tag{1.3}$$

where

$$a_{m-k} = \sum_{j=0}^{k} \left[\frac{(-1)^j}{j!(k-j)!} \prod_{i=1}^{m} (\beta\gamma_i + k - j) \right], \qquad k = 0, 1, \ldots, m-1.$$

The best known example giving rise to the name Bessel type operators for differential operators (1.1)-(1.3), is the second order *Bessel differential operator* ($m = \beta = 2;\ \gamma_{1,2} = \pm\frac{\nu}{2}$):

$$B_\nu = x^{-2}\ (xD + \nu)(xD - \nu) = D^2 + \frac{1}{x} D - \frac{\nu^2}{x^2}. \tag{1.4}$$

On the other hand, the simplest hyper-Bessel differential operator of order $m \geq 1$ is the *m-fold differentiation*

$$\widetilde{B} = D^m = \left(\frac{d}{dx} \right)^m \qquad \text{with parameters} \tag{1.5}$$

$$\alpha_0 = \alpha_1 = \ldots = \alpha_m = 0;\ \beta = m \geq 1;\ \gamma_k = \frac{k}{m} - 1,\ k = 1, \ldots, m.$$

Bessel-type operators (1.1)-(1.3) of arbitrary order $m > 1$ were introduced by Dimovski [4], who in a series of papers [5]-[7] developed operational calculi, integral transforms and transmutation operators for them. Later, these investigations were extended by Dimovski and Kiryakova [9], [10], Kiryakova [20]-[23] (and related to a generalized fractional calculus), McBride [27], Betancor and Barrios [2], Hayek and Hernández [18], Kiryakova and McBride [24]. Operational calculi, integral transforms, representations of the solutions of the corresponding differential equations have been developed for different special cases of (1.1)-(1.3) by many authors, like: Ditkin and Prudnikov [11], Meller [28], Krätzel [26], Hayek [14], McBride [27], Koh [25], Rodriguez [34]-[36], Gonzalez [13], Betancor [1], Hayek and Hernández [15]-[17] etc.

It turned out that for the general case of arbitrary $m \geq 2$ all the above mentioned elements are closely related to the so-called G-functions, whose role has been emphasized first by Kiryakova [20], Dimovski and Kiryakova [9]-[10] and McBride [27]. Such functions are the kernels of the integral operators and transforms as well as the solutions of the *hyper-Bessel differential equations of arbitrary order* $m \geq 1$ of the form $By(x) = \lambda y(x) + f(x)$, $\lambda =$ const, $f(x)$ a given function. These *generalized hypergeometric functions* incorporate as particular cases the basic elementary functions and almost all the special functions of mathematical physics.

DEFINITION 1.2. By a *G-function of Meijer* we mean the analytic function of $\sigma \neq 0$, defined by means of the contour integral

$$(1.6)\qquad G^{m,n}_{p,q}\left[\sigma \,\middle|\, \begin{matrix} (a_k)_1^p \\ (b_l)_1^q \end{matrix}\right] = G^{m,n}_{p,q}\left[\sigma \,\middle|\, \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right] = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{\prod_{l=1}^{m} \Gamma(b_l - s) \prod_{k=1}^{n} \Gamma(1 - a_k + s)}{\prod_{l=m+1}^{q} \Gamma(1 - b_l + s) \prod_{k=n+1}^{p} \Gamma(a_k - s)} \sigma^s \, ds, \ \sigma \neq 0.$$

More details about the orders (integers) $0 \leq m \leq q$, $0 \leq n \leq p$, parameters $a_1, \cdots, a_p, b_1, \cdots, b_q$, contour $\mathcal{L}$ as well as the properties of the G-function, can be seen in [12, Vol. 1] and other recent handbooks on special functions, like [33].

Along with the G-functions, the basic techniques used here include the *generalized fractional calculus* [23] based on *generalized operators of fractional integration* involving *special functions* $\phi(\sigma)$ *in the kernels*:

$$(1.7)\qquad If(x) = \int\limits_0^1 \phi(\sigma)\sigma^\gamma \, f(x\sigma) \, d\sigma = x^{-\gamma-1} \int\limits_0^x \phi\left(\frac{t}{x}\right) t^\gamma f(t) \, dt.$$

These operators have been introduced by Kalla [19]. Various special cases involving e.g. the Bessel functions, Laguerre polynomials, Gauss hypergeometric functions, etc. have been studied in more detail by different authors. Naturally, the great generality is obtained if the kernel-function $\phi(\sigma)$ is taken to be a Meijer's G-function. However, a peculiar choice of such a function is more convenient for developing a detailed theory with practical applications.

DEFINITION 1.3. Let $m \geq 1$ be an integer, $\beta > 0$, $\gamma_1, \ldots, \gamma_m$ and $\delta_1 \geq 0, \ldots, \delta_m \geq 0$ be arbitrary real numbers. By a *multiple Erdélyi-Kober* (*multi-E.-K.*) *operator of fractional integration of multi-order* $\delta = (\delta_1, \ldots, \delta_m)$ we mean an integral operator

$$(1.8)\qquad I^{(\gamma_k),(\delta_k)}_{\beta,m} f(x) = \int\limits_0^1 G^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + \delta_k)_1^m \\ (\gamma_k)_1^m \end{matrix}\right] f(x\sigma^{\frac{1}{\beta}}) \, d\sigma.$$

Then, each operator of the form

$$(1.9)\qquad Rf(x) = x^{\beta\delta_0} \, I^{(\gamma_k),(\delta_k)}_{\beta,m} f(x) \quad \text{with some } \delta_0 \geq 0$$

is said to be a *generalized* (*m-tuple*) *operator of fractional integration of R.-L. type*, or briefly, a *generalized* (*R.-L.*) *fractional integral*.

The classical Riemann-Liouville (R.-L.) operators of integration of fractional order $\delta > 0$:

$$R^\delta f(x) = \int_0^x \frac{(x-t)^{\delta-1}}{\Gamma(\delta)} f(t)\,dt = x \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} f(x\sigma)\,d\sigma \tag{1.10}$$

and their generalizations, the so-called *Erdélyi-Kober* (*E.-K.*) *fractional integrals* (see [19], [23], [27], [37]-[38] etc.):

$$\begin{aligned} I_\beta^{\gamma,\delta} f(x) &= \int_0^1 \frac{(1-\sigma)^{\delta-1}\sigma^\gamma}{\Gamma(\delta)} f(x\sigma^{\frac{1}{\beta}})\,d\sigma \\ &= \frac{x^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^x (x^\beta - t^\beta)^{\delta-1} f(t)\,d(t^\beta), \qquad \delta > 0,\ \gamma \in \mathbb{R},\ \beta > 0 \end{aligned} \tag{1.11}$$

follow from operators (1.8)-(1.9) for $m = 1$, since

$$G_{1,1}^{1,0}\left[\sigma \,\middle|\, \begin{matrix} \gamma+\delta \\ \gamma \end{matrix}\right] = \frac{(1-\sigma)^{\delta-1}\sigma^\gamma}{\Gamma(\delta)} \quad \text{for } 0<\sigma<1; \quad G_{1,1}^{1,0}(\sigma) \equiv 0 \quad \text{for } \sigma > 1.$$

Namely,

$$I_\beta^{\gamma,\delta} = I_{\beta,1}^{\gamma,\delta} \quad \text{and} \quad R^\delta = x I_1^{0,\delta} = x I_{1,1}^{0,\delta}. \tag{1.12}$$

Also, for $\delta = 0$ and arbitrary γ, β, the identity operator I is obtained:

$$I_{\beta,1}^{\gamma,0} = I_\beta^{\gamma,0} = R^0 = I. \tag{1.13}$$

It is interesting to note also the case $m = 2$ when (1.8) reduce to the hypergeometric fractional integrals of Love, Saigo, McBride etc. (see e.g. [19], [23]):

$$\begin{aligned} & I_{\beta,2}^{(\gamma_k)_1^2,(\delta_k)_1^2} f(x) \\ &= \int_0^1 \frac{(1-\sigma)^{\delta_1+\delta_2-1}\sigma^{\gamma_2}}{\Gamma(\delta_1+\delta_2)}\, {}_2F_1(\gamma_2+\delta_2-\gamma_1, \delta_1; \delta_1+\delta_2; 1-\sigma) f(x\sigma^{\frac{1}{\beta}})\,d\sigma, \end{aligned} \tag{1.14}$$

due to the representation of the kernel $G_{2,2}^{2,0}$-function via the Gauss hypergeometric functions (see [12, Vol. 1], [23], [33]).

Operators (1.8) are studied by Kiryakova [21], [23] in several function spaces and their mapping, convolutional and operational properties are derived there by using the properties of the G-functions. Naturally, some of them look like those of the classical E.-K. operators, namely:

$$I_{\beta,m}^{(\gamma_1,\cdots,\gamma_m),(0,0,\cdots,0)} = I \qquad \text{(identity)}; \tag{1.15}$$

$$I_{\beta,m}^{(\gamma_k),(\delta_k)}\, I_{\beta,n}^{(\sigma_j),(\alpha_j)} = I_{\beta,n}^{(\sigma_j),(\alpha_j)}\, I_{\beta,m}^{(\gamma_k),(\delta_k)} = I_{\beta,m+n}^{((\gamma_k),(\sigma_j)),((\delta_k),(\alpha_j))}; \tag{1.16}$$

$$I_{\beta,m}^{(\gamma_k),(\delta_k)}\, x^{\beta\lambda} = x^{\beta\lambda}\, I_{\beta,m}^{(\gamma_k+\lambda),(\delta_k)}; \tag{1.17}$$

$$I_{\beta,m}^{(\gamma_k),(\delta_k)}\, I_{\beta,m}^{(\gamma_k+\delta_k),(\sigma_k)} = I_{\beta,m}^{(\gamma_k),(\delta_k+\sigma_k)} \qquad \text{(product rule)}; \tag{1.18}$$

$$(I_{\beta,m}^{(\gamma_k),(\delta_k)})^{-1} = I_{\beta,m}^{(\gamma_k+\delta_k),(-\delta_k)} := D_{\beta,m}^{(\gamma_k),(\delta_k)} \qquad \text{(inversion formula)}, \tag{1.19}$$

where $D_{\beta,m}^{(\gamma_k),(\delta_k)}$ is a properly and explicitly defined integro-differential operator, called a *generalized (E.-K.) fractional derivative* (see [23]):

$$(1.20)\qquad D_{\beta,m}^{(\gamma_k),(\delta_k)} = \left[\prod_{k=1}^{m}\prod_{j=1}^{\eta_k}\left(\frac{1}{\beta}x\frac{d}{dx}+\gamma_k+j\right)\right] I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}$$

$$= D_\eta \;\; I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}$$

$$\text{with integers } \eta_k = [\delta_k]+1 \quad \text{or } \eta_k = \delta_k, k=1,\dots,m.$$

One of the basic properties, justifying the name *multiple E.-K. operators* for operators (1.8) is their representation as *compositions of commuting clasical E.-K. operators* (1.12), namely:

$$(1.21)\qquad I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) := I_\beta^{\gamma_1,\delta_1}\{I_\beta^{\gamma_2,\delta_2}\cdots(I_\beta^{\gamma_m,\delta_m})\} = \prod_{k=1}^{m} I_\beta^{\gamma_k,\delta_k}.$$

Operators (1.8), (1.21) have arisen in Dimovski and Kiryakova [10] while studying the hyper-Bessel operators and their use has further simplified this theory and its applications. For instance, the right-inverse operators L of B $(BL=I)$, called *hyper-Bessel integral operators*, are shown ([10], [21], [23]) to be nothing but *generalized "fractional" integrals* (1.8) of multi-order $\delta=(1,1,\dots,1)$:

$$(1.22)\qquad Lf(x) = \frac{x^\beta}{\beta^m}\; I_{\beta,m}^{(\gamma_1,\dots,\gamma_m),(1,\dots,1)} f(x)$$

$$= \frac{x^\beta}{\beta^m}\int_0^1 G_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix}(\gamma_k+1)_1^m\\ (\gamma_k)_1^m\end{matrix}\right] f(x\sigma^{\frac{1}{\beta}})\, d\sigma$$

and then, the *Bessel type differential operators* B (1.1)-(1.3) are the corresponding *generalized "fractional" derivatives* of the form (1.20): $B = D_{\beta,m}^{(\gamma_k),(1,1,\dots,1)}\;\beta^m x^{-\beta}$. As established by Dimovski, there exist *transmutation operators* (*Poisson-Sonine type transformations*) between any two pairs of hyper-Bessel operators B, L and $\widetilde{B}, \widetilde{L}$ of the forms (1.2), (1.22) and order $m \geq 1$. One of the most useful transmutation operators (Dimovski [6], [7]) is that from the m-fold differentiation $\widetilde{B}$ (1.5) to the Bessel type differentiation B (1.2). This is the *generalized Poisson transformation*, represented in Dimovski and Kiryakova [10], Kiryakova [21] via the operators (1.8) as follows:

$$(1.23)\qquad Pf(x)$$

$$= \sqrt{\frac{m}{(2\pi)^{m-1}}}\prod_{k=1}^{m}\Gamma(\gamma_k+1)\left(\frac{x^\beta}{\beta^m}\right)^{-\gamma_m} I_{\beta,m-1}^{(k/m-1),(\gamma_k-\gamma_m+1-k/m)} f\left(\frac{m}{\beta}x^{\beta/m}\right).$$

The above operator is a similarity between the corresponding integral operators $\widetilde{L}=R^m$ (see (1.10)) and L (1.22), namely: $PR^m = LP$ and transforms $\widetilde{B}$ into B, by taking account of the initial value conditions at $x=0$. It is based on a generalization of the known *Poisson integral transform* $P=P_\nu$ relating the cosine function $\widetilde{y}(x) = \cos x$ and the Bessel function $y(x) = J_\nu(x)$, both considered as solutions of the corresponding Cauchy problems

$$\begin{aligned} &\widetilde{y}''(x) = -\widetilde{y}(x), \quad \widetilde{y}(0) = 1, \quad \widetilde{y}'(0) = 0; \\ &B_\nu y(x) = -y(x), \quad y(0) = 1, \quad y'(0) = 0, \end{aligned} \tag{1.24}$$

namely:

$$\begin{aligned} J_\nu(x) &= P_\nu\{\cos x\} \\ &= \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-\sigma^2)^{\nu-\frac{1}{2}} \cos(x\sigma)\, d\sigma, \qquad \nu > -\frac{1}{2}. \end{aligned} \tag{1.25}$$

The basic function spaces in which the Bessel type operators are usually considered are (see [4]-[8], [23]):

$$\begin{aligned} &C_\alpha^{(i)} := \{f(x) = x^p \widetilde{f}(x); p > \alpha, \widetilde{f} \in C^{(i)}[0,\infty)\}, \\ &\qquad\qquad\qquad i = 0,1,2,\dots;\ \alpha = \max_k[-\beta(\gamma_k+1)]; \\ &\operatorname{span}\{x^{-\beta\gamma_k}\}_1^m := \{c_1 x^{-\beta\gamma_1} + \ldots + c_m x^{-\beta\gamma_m}\} \subset C_\alpha^{(m)}; \\ &\mathcal{X} := \operatorname{span}\{x^{-\beta\gamma_k}\}_1^m \oplus C_{\alpha+\beta}^{(m)} \subset C_\alpha. \end{aligned} \tag{1.26}$$

2. Bessel-Clifford operators of third order in view of fractional calculus. In [14] Hayek studied in details the so-called *Bessel-Clifford differential equation of second order*

$$xy''(x) + (\nu+1)y'(x) + y(x) = 0, \quad \text{i.e.} \quad \widetilde{B}_\nu y(x) = -y(x) \tag{2.1}$$

with Bessel type differential operator $\widetilde{B}_\nu$, closely related to the Bessel operator B_ν (1.4):

$$\begin{aligned} \widetilde{B}_\nu &= x^{-\nu} D x^{\nu+1} D = x^{-1}(xD+\nu)xD \\ &= xD^2 + (\nu+1)D;\ m=2,\ \beta=1,\ \gamma_1=\nu, \gamma_2=0. \end{aligned}$$

One of its solutions, the *Bessel-Clifford function* (*of second order*):

$$C_\nu(x) = x^{-\frac{\nu}{2}}\, J_\nu(2\sqrt{x}) = \frac{1}{\Gamma(\nu+1)}\, {}_0F_1(\nu+1; -x) = G_{0,2}^{1,0}\left[x \middle| {}_{0,-\nu}\right] \tag{2.2}$$

has a number of advantages as an entire function with useful properties and applications (see [14]). Further, following the pattern of Humbert (1930) and Delerue [3], Hayek [15] introduced the *Bessel-Clifford functions of third order* depending on two indices and modifying the hyper-Bessel functions $J_{\mu,\nu}^{(2)}(x)$ ([3], [23]):

$$\begin{aligned} C_{\mu,\nu}(x) &= x^{-\frac{\mu+\nu}{3}}\, J_{\mu,\nu}^{(2)}(3\sqrt[3]{x}) \\ &= \frac{1}{\Gamma(\mu+1)\Gamma(\nu+1)}\, {}_0F_2(\mu+1,\nu+1;-x) = G_{0,3}^{1,0}\left[x \middle| {}_{0,-\mu,-\nu}\right]. \end{aligned} \tag{2.3}$$

These special functions are related to the following differential operator.

Definition 2.1. Let μ and ν be arbitrary real parameters. The Bessel type differential operator of order $m=3$:

$$\begin{aligned}B_{\mu,\nu} &= x^{-\mu}Dx^{\mu-\nu+1}Dx^{\nu+1}D \\ &= x^2D^3+(\mu+\nu+3)xD^2+(\mu+1)(\nu+1)D\end{aligned} \tag{2.4}$$

is said to be a *Bessel-Clifford third order differential operator*.

It is seen that the parameters of this operator of the form (1.1)-(1.3) are:

$$\begin{aligned} &m=3,\ \alpha_0=-\mu,\ \alpha_1=\mu-\nu+1,\ \alpha_2=\nu+1,\ \alpha_3=0 \\ &\Rightarrow \beta=1,\ \gamma_1=\mu,\ \gamma_2=\nu,\ \gamma_3=0, \\ &\text{resp. } a_1=\mu+\nu+3,\ a_2=(\mu+1)(\nu+1),\ a_3=0.\end{aligned} \tag{2.5}$$

Here we confine ourselves to the *nonlogarithmic case*, i.e. we suppose

$$\mu-\nu\neq 0,\pm 1,\pm 2,\dots \tag{2.6}$$

Now, $\alpha=\max\{-\mu,-\nu,0\}-1\geq -1$ and the function spaces (1.26) are:

$$\begin{aligned} &C_\alpha = C_{\max\{-\mu,-\nu,0\}-1}; \quad C^{(m)}_{\alpha+\beta}=C^{(3)}_{\alpha+1}; \\ &\operatorname{span}\{x^{-\beta\gamma_k}\}_1^3=\operatorname{span}\{x^{-\mu},x^{-\nu},1\}; \\ &\mathcal{X}=\operatorname{span}\{x^{-\mu},x^{-\nu},1\}\oplus C^{(3)}_{\max\{-\mu,-\nu,0\}}.\end{aligned} \tag{2.7}$$

Let us consider the initial value problem

$$\begin{cases} B_{\mu,\nu}y(x)=f(x), \\ \lim_{x\to+0} B_i y(x)=b_i, \quad i=1,2,3\end{cases} \tag{2.8}$$

with the "Bessel type" initial conditions

$$\begin{cases} B_1=x^{\mu-\nu+1}Dx^{\nu+1}D=x^\mu(xD+\nu)xD, \\ B_2=x^{\nu+1}D=x^\nu xD, \\ B_3=1, \quad \text{i.e.} \quad \lim_{x\to+0}B_3y(x)=f(+0).\end{cases} \tag{2.9}$$

It is known (Dimovski [4]-[7], Dimovski and Kiryakova [10]) that its solution $y(x)=L_{\mu,\nu}f(x)$ under zero initial conditions: $\lim_{x\to+0}B_iy(x)=0$, $i=1,2,3$, defines the *linear right inverse operator* $L_{\mu,\nu}$ of $B_{\mu,\nu}$, called also a *hyper-Bessel integral operator* (in particular, *Bessel-Clifford integral operator of* 3*rd order*): $B_{\mu,\nu}L_{\mu,\nu}=I$. It has the representations:

$$\begin{aligned} L_{\mu,\nu}f(x) &= \int_0^x x_1^{-\nu-1}\,dx_1\int_0^{x_1}x_2^{-\mu+\nu-1}\,dx_2\int_0^{x_2}x_3^\mu f(x_3)dx_3 \\ &= x\int_0^1\int_0^1\int_0^1 x_1^\mu x_2^\nu\ f(xx_1x_2x_3)\,dx_1\,dx_2\,dx_3 \\ &= x\int_0^1 G^{3,0}_{3,3}\left[\sigma\,\middle|\,\begin{matrix}\mu+1,\nu+1,1\\ \mu,\nu,0\end{matrix}\right]f(x\sigma)\,d\sigma.\end{aligned} \tag{2.10}$$

Evidently, the Bessel-Clifford operators (2.4), (2.10) are 3-*tuple generalized "fractional" derivatives/integrals* of the forms (1.20), (1.8) and multi-order (1,1,1):

$$B_{\mu,\nu} = D_{1,3}^{(\mu,\nu,0),(1,1,1)} x, \tag{2.11}$$
$$L_{\mu,\nu} = x I_{1,3}^{(\mu,\nu,0),(1,1,1)}; \quad L_{\mu,\nu} : C_\alpha \mapsto C_{\alpha+1}^{(3)} \subset C_\alpha.$$

In view of the above interpretation and the results from [23], one can find easily some images of functions under the Bessel-Clifford operators. For example, if $f(x) = x^{p-1} \in C_\alpha$, i.e. $p - 1 > \alpha \geq -1$ and $p \neq -\mu, -\nu$, then

$$L_{\mu,\nu}\{x^{p-1}\} = x^p \int_0^1 G_{3,3}^{3,0}\left[\sigma \,\middle|\, \begin{matrix} \mu+p, \nu+p, p \\ \mu+p-1, \nu+p-1, p-1 \end{matrix}\right] d\sigma$$
$$= \frac{1}{(\mu+p)(\nu+p)}\, x^p,$$

due to Lemma B.2, [23]. Applying $B_{\mu,\nu}$ to both sides, we obtain

$$B_{\mu,\nu}\{x^p\} = \begin{cases} (\mu+p)(\nu+p)\; x^{p-1} & \text{if } p \neq -\mu, -\nu, 0, \\ 0 & \text{if } p = -\mu, p = -\nu, p = 0. \end{cases}$$

This means that

$$y_1(x) = x^{-\mu}, \quad y_2(x) = x^{-\nu}, \quad y_3(x) = 1 \quad (\mu \neq \nu \neq 0) \tag{2.12}$$

are linearly independent solutions of $B_{\mu,\nu} y(x) = 0$. Also, the *defining projector* (*operator of initial conditions*) in the sense of Przeworska-Rolewicz [31] (see also [7]-[8]) of $L_{\mu,\nu}$ can be found to have the form (Th. 3.2.3, [23])

$$F_{\mu,\nu} f(x) = (I - L_{\mu,\nu} B_{\mu,\nu}) f(x) = c_1 x^{-\mu} + c_2 x^{-\nu} + c_3\ , \qquad \text{with} \tag{2.13}$$
$$\begin{cases} c_1 = \frac{1}{\mu(\mu-\nu)} \lim_{x\to+0} B_1 f(x), \\ c_2 = -\frac{1}{\nu} \lim_{x\to+0} B_2 f(x), \\ c_3 = \lim_{x\to+0} B_3 f(x) = f(+0), \end{cases}$$

and B_1, B_2, B_3 like in (2.9).

Analogously, by using (2.3), property $G_{3,3}^{3,0}(\sigma) = 0$, $\sigma > 1$ and the following *formula for integral of a product of two G-functions* ((A.29), [23], also (1), §2.24, [33]):

$$\int_0^\infty G_{p,q}^{m,n}\left[\eta x \,\middle|\, \begin{matrix} (a_i) \\ (b_j) \end{matrix}\right] G_{\sigma,\tau}^{\mu,\nu}\left[\omega x \,\middle|\, \begin{matrix} (c_\alpha) \\ (d_\beta) \end{matrix}\right] dx \tag{2.14}$$
$$= \frac{1}{\omega} G_{p+\tau,q+\sigma}^{m+\nu,n+\mu}\left[\frac{\eta}{\omega} \,\middle|\, \begin{matrix} a_1,\dots,a_n; (-d_\beta)_1^\tau; a_{n+1},\dots a_p \\ b_1,\dots,b_m; (-c_\alpha)_1^\sigma; b_{m+1},\dots,b_q \end{matrix}\right],$$

we find

$$L_{\mu,\nu}\{C_{\mu,\nu}(x)\} = x \int_0^\infty G_{3,3}^{3,0}\left[\sigma \,\middle|\, \begin{matrix} \mu+1, \nu+1, 1 \\ \mu, \nu, 0 \end{matrix}\right] G_{0,3}^{1,0}\left[x\sigma \,\middle|\, \begin{matrix} \\ 0, -\mu, -\nu \end{matrix}\right] d\sigma$$

$$= G^{1,1}_{1,4}\left[x \,\middle|\, \begin{matrix} 1 \\ 1, -\mu, -\nu, 0 \end{matrix}\right] = -G^{1,0}_{0,3}\left[x \,\middle|\, \begin{matrix} \\ 0, -\mu, -\nu \end{matrix}\right] = -C_{\mu,\nu}(x).$$

By applying $B_{\mu,\nu}$ again, this implies that the Bessel-Clifford function (2.3): $y(x) = C_{\mu,\nu}(x)$ satisfies the *Bessel-Clifford differential equation of third order*

$$\text{(2.15)} \qquad B_{\mu,\nu}\; y(x) = -y(x), \qquad \text{with } B_{\mu,\nu} \text{ as in (2.4), i.e.:}$$

$$x^2 y'''(x) + (\mu+\nu+3)xy''(x) + (\mu+1)(\nu+1)y'(x) + y(x) = 0.$$

Now, let us specify the *Poisson type integral transform* (1.23), transmuting the 3-fold differentiation/integration $\widetilde{B} = D^3,\ \widetilde{L} = R^3$ into the Bessel-Clifford operators $B = B_{\mu,\nu},\ L = L_{\mu,\nu}$. For example, if $\mu > \nu > 0$, i.e. $\alpha = -1,\ \gamma_3 = 0$, then $P = P_{\mu,\nu} : C_{-1} \mapsto C_{-1}$ has the form

$$\text{(2.16)} \qquad P_{\mu,\nu} f(x) = c\; I_{1,2}^{(-\frac{2}{3}, -\frac{1}{3}), (\mu+\frac{2}{3}, \nu+\frac{1}{3})} f(x)$$

$$= c \int\limits_0^1 \int\limits_0^1 \frac{(1-t_1)^{\mu-\frac{1}{3}}(1-t_2)^{\nu-\frac{2}{3}}}{\Gamma(\mu+\frac{2}{3})\Gamma(\nu+\frac{1}{3})}\; t_1^{-\frac{2}{3}} t_2^{-\frac{1}{3}} f\big(3\sqrt[3]{x t_1 t_2}\big)\, dt_1\, dt_2$$

$$= c \int\limits_0^1 G^{2,0}_{2,2}\left[\sigma \,\middle|\, \begin{matrix} \mu, \nu \\ -\frac{2}{3}, -\frac{1}{3} \end{matrix}\right] f\big(3\sqrt[3]{x\sigma}\big)\, d\sigma,$$

where $c = \sqrt{3}(2\pi)^{-1}\Gamma(\mu+1)\Gamma(\nu+1)$.

Unlike the case $m > 2$, when the kernel-functions $G^{m,0}_{m,m}(\sigma)$ of the integrals (1.8) cannot be identified with known special functions, the 2-tuple fractional integrals (1.14) can be represented via the Gauss hypergeometric functions. In particular, this is true for the operator $P_{\mu,\nu}$. In this case, formula (1.1.18), [23] (see also (22), §8.4.49, [33]) gives:

$$\text{(2.17)} \quad G^{2,0}_{2,2}\left[\sigma \,\middle|\, \begin{matrix} \mu, \nu \\ -\frac{2}{3}, -\frac{1}{3} \end{matrix}\right] = \frac{(1-\sigma)^{\mu+\nu}\sigma^{-\frac{1}{3}}}{\Gamma(\mu+\nu+1)}\; {}_2F_1\left(\mu + \frac{2}{3}, \nu + \frac{2}{3}; \mu+\nu+1; 1-\sigma\right)$$

and therefore,

$$\text{(2.18)} \qquad P_{\mu,\nu} f(x)$$

$$= c \int\limits_0^1 \frac{(1-\sigma)^{\mu+\nu}\sigma^{-\frac{1}{3}}}{\Gamma(\mu+\nu+1)}\; {}_2F_1\left(\mu + \frac{2}{3}, \nu + \frac{2}{3}; \mu+\nu+1; 1-\sigma\right) f\big(3\sqrt[3]{x\sigma}\big)\, d\sigma$$

$$= 3c \int\limits_0^1 \frac{(1-\zeta^3)^{\mu+\nu}\zeta}{\Gamma(\mu+\nu+1)}\; {}_2F_1\left(\mu + \frac{2}{3}, \nu + \frac{2}{3}; \mu+\nu+1; 1-\zeta^3\right) f\left(3\zeta\sqrt[3]{x}\right) d\zeta.$$

THEOREM 2.2. *The Poisson type integral transformation* (2.15), (2.18) *is a similarity from the* 3*-fold integration* $\widetilde{L} = R^3$ *to the third order Bessel-Clifford* *operator* $L = L_{\mu,\nu}$, *viz.*

$$P_{\mu,\nu} R^3 f(x) = P_{\mu,\nu} \left\{ \frac{x}{2} \int_0^1 (1-\sigma)^2 f(x\sigma)\, d\sigma \right\} = L_{\mu,\nu} P_{\mu,\nu} f(x)\ , \quad f \in C_\alpha. \tag{2.19}$$

P r o o f. This follows immediately using the rules (1.15)-(1.18) of the fractional integrals $P_{\mu,\nu}, L_{\mu,\nu}, R^3$. Then, the form of the defining projector (2.13) allows one to find the transmutation formula between the differential operators D^3 and $B_{\mu,\nu}$.

Representation (2.18) of $P_{\mu,\nu}$ can be used to write down *new forms of the Poisson integral representations of the Bessel-Clifford functions*, found earlier by Hayek and Hernández [16], [17]. For example, formula (2.6), [17] can be rewritten in the form

$$C_{\mu,\nu}(x) = \frac{\sqrt{3}}{2\pi} \int_0^1 G^{2,0}_{2,2}\left[\sigma \,\middle|\, \begin{matrix} \mu, \nu \\ -\frac{2}{3}, -\frac{1}{3} \end{matrix}\right] \cos_3\left(3\sqrt[3]{x\sigma}\right) d\sigma \tag{2.20}$$

$$= \frac{3\sqrt{3}}{2\pi} \int_0^1 \frac{(1-\zeta^3)^{\mu+\nu}\zeta}{\Gamma(\mu+\nu+1)}\, {}_2F_1\left(\mu+\frac{2}{3}, \nu+\frac{2}{3}; \mu+\nu+1; 1-\zeta^3\right) \cos_3\left(3\zeta\sqrt[3]{x}\right) d\zeta,$$

where $\cos_3(\zeta) = f_1(\zeta)$ stands for the *cosine function of* 3*rd order* ([12, Vol. 3], also [23]). Analogously, the other relationships between the Bessel-Clifford functions and the sin(cos)-functions of 3rd order ([16]-[17]) can also be put in integral forms involving the Gauss hypergeometric functions.

For the purposes of the operational calculi of Mikusiński type ([29]) it is important to know a convolution, or even families of convolutions, of a linear operator mapping a linear space into itself (the basic notions of the *convolutional calculus of Dimovski* [7]-[8]). In the general case of hyper-Bessel integral operators (1.22) of order $m \geq 2$ such convolutions were found by Dimovski [4]-[7] and later simplified by using the generalized fractional integrals (1.8) by Dimovski and Kiryakova [10]. The corresponding explicit result for the third order integral operator $L_{\mu,\nu}$ reads as follows.

THEOREM 2.3. *Let* $\lambda \geq \max\{\mu, \nu, 0\} \geq 0$ *and* $(\circ)$ *denote the auxiliary operation*

$$(f \circ g)(x) = x \int_0^1 \int_0^1 \int_0^1 f[xt_1t_2t_3] g[x(1-t_1)(1-t_2)(1-t_3)] \times [t_1(1-t_1)]^\mu [t_2(1-t_2)]^\nu\, dt_1\, dt_2\, dt_3. \tag{2.21}$$

Then the operations depending on the parameter λ:

$$(f \overset{\lambda}{*} g)(x) = x^\lambda\, I^{(2\mu,2\nu,0),(\lambda-\mu,\lambda-\nu,\lambda)}_{1,3} \{(f \circ g)(x)\}, \tag{2.22}$$

where

$$I^{(2\mu,2\nu,0),(\lambda-\mu,\lambda-\nu,\lambda)}_{1,3} F(x) = \int_0^1 G^{3,0}_{3,3}\left[\sigma \,\middle|\, \begin{matrix} \lambda+\mu, \lambda+\nu, \lambda \\ 2\mu, 2\nu, 0 \end{matrix}\right] F(x\sigma)\, d\sigma, \tag{2.23}$$

are 3*-tuple fractional integrals of the form* (1.8), *are convolutions of the linear operator* $L_{\mu,\nu}$ *in the linear space* C_α.

Proof. By definition ([7]-[8]), the operation $(\overset{\lambda}{*})$ will be a convolution of operator $L_{\mu,\nu}$ in C_α, if it satisfies the conditions: *bilinearity*, *associativity*, *commutativity* and $L_{\mu,\nu}(f \overset{\lambda}{*} g) = (L_{\mu,\nu}f) \overset{\lambda}{*} g$ for each $f, g \in C_\alpha$. These properties are easily verified for functions of the form: $f(x) = x^p,\ g(x) = x^q \in C_\alpha$ by using the *convolutional product*

$$x^p \overset{\lambda}{*} x^q = \Lambda x^{p+q+\lambda+1}, \quad p > \alpha,\ q > \alpha,\ \text{where} \tag{2.24}$$

$$\Lambda = \frac{\Gamma(\mu+p+1)\Gamma(\mu+q+1)\Gamma(\nu+p+1)\Gamma(\nu+q+1)\Gamma(p+1)\Gamma(q+1)}{\Gamma(\mu+\lambda+p+q+2)\Gamma(\nu+\lambda+p+q+2)\Gamma(\lambda+p+q+2)}.$$

Then, by virtue of the Weierstrass theorem, arbitrary functions from C_α can be approximated by functions of the form $f(x) = x^p P_n(x),\ g(x) = x^q Q_n(x)$, with $p, q > \alpha$ and polynomials $P(x), Q(x)$.

Note that if $\mu < \nu < 0$, one can take $\lambda = 0$ and then the 3-tuple fractional integrals (2.23) turn into 2-tuple integrals of the form (1.14).

Convolutions (2.22) can be used to find also explicit representations of the *fractional powers of the Bessel-Clifford operators*, namely:

$$L^\delta_{\mu,\nu} f(x) = x^\delta \int_0^1 G^{3,0}_{3,3}\left[\sigma \,\middle|\, \begin{matrix} \mu+\delta, \nu+\delta, \delta \\ \mu, \nu, 0 \end{matrix}\right] f(x\sigma)\, d\sigma, \qquad \delta \geq 0; \tag{2.25}$$

for more details see Kiryakova [23], compare also McBride [27].

3. Laplace type integral transform for the Bessel-Clifford operators. One of the approaches for justifying and developing the Heaviside operational calculus for the differentiation operator $D = \frac{d}{dx}$ is based on the *Laplace integral transform*

$$\mathcal{L}\{f(x); z\} = \int_0^\infty \exp(-zx) f(x)\, dx \tag{3.1}$$

and its well-known properties (differential law, convolution theorem, inversion formulas, tables of images). Incidentally, $D = \frac{d}{dx}$ is a Bessel type operator of order $m = 1$ and for the second order Bessel operator B_ν (1.4) there also exists a Laplace type integral transform. This is the so-called *Meijer transform*

$$\mathcal{K}_\nu\{f(x); z\} = \int_0^\infty \sqrt{zx} K_\nu(zx) f(x)\, dx, \tag{3.2}$$

where $K_\nu(z)$ stands for the McDonald function (Bessel function of third kind, see [12, Vol. 2]). Note that the kernel-functions of the above integral transform can be represented like Meijer's G-functions (1.6), namely (see [12, Vol. 1]):

$$\exp(-z) = G^{1,0}_{0,1}\left[z\,\middle|\,\begin{matrix} \\ 0\end{matrix}\right]; \qquad \sqrt{z}K_\nu(z) = \frac{\sqrt{2}}{2}G^{2,0}_{0,2}\left[\frac{z^2}{4}\,\middle|\,\begin{matrix} \\ \frac{\nu}{2}+\frac{1}{4}, -\frac{\nu}{2}+\frac{1}{4}\end{matrix}\right]. \tag{3.3}$$

Operational calculi for other Bessel type differential operators, based on integral transforms analogous to (3.1), (3.2) have been proposed by different authors, e.g.: Betancor [1], [2], Ditkin and Prudnikov [11], Krätzel [26], etc. In [5] Dimovski established that a modification of an integral transform introduced by Obrechkoff [30] can be used as a transform base of the operational calculus for the hyper-Bessel operators of arbitrary order $m > 1$. The *Obrechkoff transform* includes as special cases the above mentioned Laplace type transforms. Its original definition was proposed by Dimovski in the form:

$$\mathcal{O}\{f(x);z\} = \beta \int_0^\infty x^{\beta(\gamma_m+1)-1}K[(zx)^\beta]f(x)\,dx, \tag{3.4}$$

where

$$K(z) = \int_0^\infty \dots \int_0^\infty \exp\left(-u_1 - \dots - u_{m-1} - \frac{z}{u_1\dots u_{m-1}}\right)\prod_{k=1}^{m-1} u_k^{\gamma_m-\gamma_k-1}du_1\dots du_{m-1}.$$

Later, Kiryakova [20], [21] found a new representation of the kernel-function $K(z)$ as a $G^{m,0}_{0,m}$-function and, in these terms, the Obrechkoff transform (3.4) can be rewritten as a *G-transform*, namely:

$$\mathcal{O}\{f(x);z\} = \beta z^{-\beta(\gamma_m+1)+1}\int_0^\infty G^{m,0}_{0,m}\left[(zx)^\beta\,\middle|\,\begin{matrix} \\ (\gamma_k - \frac{1}{\beta}+1)_1^m\end{matrix}\right]f(x)\,dx. \tag{3.5}$$

Relationships to the hyper-Bessel operators B, L, basic operational rules, a number of inversion formulas, etc. were found for (3.4)-(3.5) by means of the G-functions technique ([9]-[10], [20]-[21]) and the convolutions found in [4]-[7] were analogously simplified in [10].

From the representation (3.5) it follows that in the particular case of the Bessel type differential operator of third order $B_{\mu,\nu}$ (2.4), the kernel-function of the corresponding integral transform should be a $G^{3,0}_{0,3}$-function. If the parameters of $B_{\mu,\nu}$ are arranged as in (2.5): $m = 3$, $\beta = 1$; $\gamma_1 = \mu$, $\gamma_2 = \nu$, $\gamma_3 = 0$, then

$$\begin{aligned} K(z) &= \int_0^\infty\int_0^\infty \exp\left(-u_1 - u_2 - \frac{z}{u_1u_2}\right)u_1^{-\mu-1}u_2^{-\nu-1}\,du_1\,du_2 \\ &= G^{3,0}_{0,3}\left[z\,\middle|\,\begin{matrix} \\ \mu,\nu,0\end{matrix}\right]. \end{aligned} \tag{3.6}$$

Unlike the cases $m = 1, m = 2$ when the kernel-functions (3.3) are expressible as known functions, the case $m = 3$ provides an example of an essential use of the Meijer G-functions.

DEFINITION 3.1. The integral transform of the form

$$\mathcal{O}_{\mu,\nu}\{f(x);z\} = \int_0^\infty G^{3,0}_{0,3}\left[zx \,\middle|\, {\atop \mu,\nu,0}\right] f(x)\,dx \tag{3.7}$$

is said to be a *Laplace-Obrechkoff integral transform, corresponding to the Bessel-Clifford differential operator* $B_{\mu,\nu}$ (2.4).

For a *transformable function* of the space

$$\Omega_{\mu,\nu} := \left\{f \in C_{\max\{-\mu,-\nu,0\}-1};\, f(x) = O\big(\exp\left(\rho\sqrt[3]{x}\right)\big),\ x\to\infty\right\} \tag{3.8}$$

the integral (3.7) defines an *analytic function* of z in the half-plane $\operatorname{Re} z > \rho$.

Using the properties of G-functions ([12, Vol. 1], [33]) and in particular, formula (2.14), one can easily calculate the $\mathcal{O}_{\mu,\nu}$-images of several basic functions $f \in \Omega_{\mu,\nu}$.

EXAMPLE 3.2. $f(x) = x^p = G^{1,0}_{1,1}\left[\,\middle|\, x {p+1 \atop p}\right]; p > \alpha \Rightarrow$

$$\mathcal{O}_{\mu,\nu}\{x^p, z\} = z^{-p-1}\ \Gamma(\mu+p+1)\Gamma(\nu+p+1)\Gamma(p+1). \tag{3.9}$$

EXAMPLE 3.3. $f(x) = x^p\exp(-\rho\sqrt[3]{x}) = \rho^{-3p}G^{1,0}_{0,1}\left[\rho\sqrt[3]{x}\,\middle|\, {}_{3p}\right] \Rightarrow$

$$\mathcal{O}_{\mu,\nu}\{x^p\exp(-\rho\sqrt[3]{x})\} = \frac{\sqrt{3}}{2\pi} z^{-p-1} G^{3,3}_{3,3}\left[\left(\frac{3}{\rho}\right)^3 z \,\middle|\, {\frac{2}{3},\frac{1}{3},0 \atop \mu+p+1,\nu+p+1,p+1}\right]. \tag{3.10}$$

EXAMPLE 3.4. $f(x) = C_{\mu,\nu}(x) = G^{1,0}_{0,3}\left[x\,\middle|\, {}_{0,-\mu,-\nu}\right] \Rightarrow$

$$\mathcal{O}_{\mu,\nu}\{C_{\mu,\nu}(x);z\} = G^{3,1}_{3,3}\left[z\,\middle|\, {0,\mu,\nu \atop \mu,\nu,0}\right] = G^{1,1}_{1,1}\left[z\,\middle|\, {0 \atop 0}\right] = \frac{1}{z+1} \tag{3.11}$$

and more generally,

$$\mathcal{O}_{\mu,\nu}\{x^{p-1}C_{\mu,\nu}(x);z\} = \frac{1}{(z+1)^p}, \qquad p > \alpha. \tag{3.12}$$

The following relationship between (3.7) and the third order Bessel-Clifford operators $B_{\mu,\nu}, L_{\mu,\nu}$ shows the possibilities for building an operational calculus via this integral transform.

THEOREM 3.5. *The Laplace-Obrechkoff integral transform* (3.7) *transfers the Bessel-Clifford integral and differential operators* (2.4), (2.10) *into algebraic operations. Namely, if* $f \in \Omega_{\mu,\nu}$:

$$\mathcal{O}_{\mu,\nu}\{L_{\mu,\nu}f(x);z\} = \frac{1}{z}\mathcal{O}_{\mu,\nu}\{f(x);z\}. \tag{3.13}$$

If, in addition to $f \in \Omega_{\mu,\nu}$, *we suppose* (2.6), *for example*: $\mu < \nu < 0 < \mu+1$ *and*

$$f \in \mathcal{X} = \operatorname{span}\{x^{-\mu}, x^{-\nu}, 1\} \oplus C^{(3)}_{\max\{-\mu,-\nu,0\}} \subset C_{\max\{-\mu,-\nu,0\}-1},$$

then

$$(3.14)\quad \mathcal{O}_{\mu,\nu}\{B_{\mu,\nu}f(x);z\} = z\mathcal{O}_{\mu,\nu}\{f(x);z\} - \{\Gamma(\mu+1)\Gamma(\nu+1)\dot{f}(+0) + z^{\nu}\ \Gamma(\mu-\nu+1)\Gamma(-\nu)\lim_{x\to+0}[x^{\nu+1}f'(x)] + z^{\mu}\Gamma(\nu-\mu)\Gamma(-\mu)\lim_{x\to+0}[x^{\mu-\nu+1}(x^{\nu+1}f'(x))']\}.$$

P r o o f. The operator (2.10) can also be put in the form

$$L_{\mu,\nu}f(x) = \int_0^x G^{3,0}_{3,3}\left[\frac{t}{x}\,\middle|\, \begin{matrix}\mu+1,\nu+1,1\\ \mu,\nu,0\end{matrix}\right] f(t)\,dt.$$

Then, by changing the order of the integrations, using the property $G^{3,0}_{3,3}\left[\frac{t}{x}\,\middle|\ \right] = 0,\ x < t$, and formula (2.14), we obtain subsequently:

$$\begin{aligned}
&\mathcal{O}_{\mu,\nu}\{L_{\mu,\nu}f(x);z\}\\
&\quad = \int_0^\infty G^{3,0}_{0,3}\left[zx\,\middle|\, \begin{matrix}\\ \mu,\nu,0\end{matrix}\right] dx \int_0^x G^{3,0}_{3,3}\left[\frac{t}{x}\,\middle|\, \begin{matrix}\mu+1,\nu+1,1\\ \mu,\nu,0\end{matrix}\right] f(t)\,dt\\
&\quad = \int_0^\infty f(t)\,dt \int_0^\infty G^{3,0}_{0,3}\left[zx\,\middle|\, \begin{matrix}\\ \mu,\nu,0\end{matrix}\right] G^{3,0}_{0,3}\left[\frac{t}{x}\,\middle|\, \begin{matrix}\mu+1,\nu+1,1\\ \mu,\nu,0\end{matrix}\right] dx\\
&\quad = \int_0^\infty tG^{6,0}_{3,6}\left[zt\,\middle|\, \begin{matrix}\mu,\nu,0\\ \mu,\nu,0,\mu-1,\nu-1,-1\end{matrix}\right] f(t)\,dt\\
&\quad = z^{-1}\int_0^\infty G^{3,0}_{0,3}\left[zt\,\middle|\, \begin{matrix}\\ \mu,\nu,0\end{matrix}\right] f(t)\,dt,
\end{aligned}$$

which proves (3.13) and shows that $L_{\mu,\nu}$ is transformed into algebraic multiplication by z^{-1}. Applying now (3.13) to a function $f_1(x) = Bf(x) \in \Omega_{\mu,\nu}$, we obtain

$$\begin{aligned}
\mathcal{O}_{\mu,\nu}\{B_{\mu,\nu}f(x);z\} &= z\mathcal{O}_{\mu,\nu}\{L_{\mu,\nu}f_1(x);z\} = z\mathcal{O}_{\mu,\nu}\{L_{\mu,\nu}B_{\mu,\nu}f(x);z\}\\
&= z\ \mathcal{O}_{\mu,\nu}\{(I-F_{\mu,\nu})f(x);z\}\\
&= z\mathcal{O}_{\mu,\nu}\{f(x);z\} - z\ \mathcal{O}_{\mu,\nu}\{F_{\mu,\nu}f(x);z\},
\end{aligned}$$

where $F_{\mu,\nu} = I - L_{\mu,\nu}B_{\mu,\nu}$ is the operator of initial conditions (2.13). Finally, the *differential law* (3.14) follows by replacing $F_{\mu,\nu}$ with its explicit form and using the result (3.9).

COROLLARY 3.6. *If* $f \in C^{(3)}_{\max\{-\mu,-\nu,0\}} = C^{(3)}_{-\mu}$, *i.e.* $f(x)$ *satisfies zero initial conditions* (2.9): $B_i f(x) = 0,\ i = 1,2,3,$ *then* $F_{\mu,\nu}f(x) = 0$ *and* (3.14) *simplifies to multiplication by* z, *namely*:

$$(3.15)\qquad \mathcal{O}_{\mu,\nu}\{B_{\mu,\nu}f(x);z\} = z\ \mathcal{O}_{\mu,\nu}\{f(x);z\}.$$

THEOREM 3.7. *The operations* (2.22) *are also convolutions for the Laplace-Obrechkoff integral transform* (3.7). *In particular*, *for* $\mu < \nu < 0$ *and* $\lambda = 0$, *the corresponding convolution is*

$$(f * g)(x) = I_{1,2}^{(2\mu,2\nu),(-\mu,-\nu)}\{(f \circ g)(x)\}, \tag{3.16}$$

namely: $f, g \in \Omega_{\mu,\nu} \Rightarrow (f * g) \in \Omega_{\mu,\nu}$ *and*

$$\mathcal{O}_{\mu,\nu}\{(f * g)(x); z\} = \mathcal{O}_{\mu,\nu}\{f(x); z\}\dot{\mathcal{O}}_{\mu,\nu}\{g(x); z\}. \tag{3.17}$$

Proof. The convolutional theorem (3.17) can be easily verified for functions $f(x) = x^p,\ g(x) = x^q,\ p, q > \alpha = -\mu - 1$ in view of (2.24) and (3.9) and then by approximating arbitrary functions f, g by polynomials like in Theorem 2.3. Alternatively, the same convolution (3.16) can be found by the *similarity method* (Meller [28], Dimovski [7]-[8]). We use the Sonine type transform which is a similarity from $L_{\mu,\nu}$ to R^3 and so, is in a sense converse to the Poisson type transform $P_{\mu,\nu}$ (compare with (2.19)). Its form follows from Dimovski [6]-[7], Dimovski and Kiryakova [10], $S_{\mu,\nu} : C_{-\mu-1} \mapsto C_{-1}$,

$$S_{\mu,\nu} f\big(\sqrt[3]{x}\big) = x^{\frac{2}{3}} I_{1,2}^{(\mu,\nu),(-\mu+\frac{1}{3},-\nu+\frac{2}{3})} f(x). \tag{3.18}$$

The same transmutation operator serves as a *relation between the Laplace and Laplace-Obrechkoff transforms*, given by the following assertion.

LEMMA 3.8. *If* $f \in \Omega_{\mu,\nu}$, *then*

$$\mathcal{O}_{\mu,\nu}\{f(x); (z/3)^3\} = 2\pi\sqrt{3}\,\mathcal{L}\{S_{\mu,\nu} f(x); z\}. \tag{3.19}$$

For an effective use of the Laplace type integral transform (3.7) one should dispose of *inversion formulas*. Here we list some of them, following from the general theory of the Obrechkoff transform (3.4)-(3.5) (compare with Theorems 3.9.10-3.9.12, [23]).

THEOREM 3.9. *If we denote by* $\mathcal{O}(z) = \mathcal{O}_{\mu,\nu}\{f(x; z)\}$ *the Laplace-Obrechkoff image of* $f \in \Omega_{\mu,\nu}$, *then the complex inversion formulas hold*:

$$f(x) = \frac{x^{-\frac{1}{3}(\mu+\nu)}}{(2\pi i)^3} \int_{\kappa-i\infty}^{\kappa+i\infty} \cdots \int_{\kappa-i\infty}^{\kappa+i\infty} \exp\big[\sqrt[3]{x}(z_1 + z_2 + z_3)\big] \times z_1^{\mu} z_2^{\nu} \mathcal{O}(z_1 z_2 z_3)\, dz_1\, dz_2\, dz_3, \qquad \text{with } \kappa > \frac{\rho}{3}, \rho \text{ as in (3.8)}. \tag{3.20}$$

Also,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{-p} dp}{\Gamma(\mu - p + 1)\Gamma(\nu - p + 1)\Gamma(1 - p)} \int_0^{\infty} z^{-p} \mathcal{O}(z) dz, \tag{3.21}$$

provided $c < -\alpha = 1 - \min\{\mu, \nu, 0\}$.

A *real inversion formula*, analogous to the Post-Widder formula, as well as *Abel type theorems* for the integral transform (3.7) can also be written as consequences of Ths. 3.9.13, 3.9.15-16 from [23]. Formula (3.21), however, seems the most useful. We illustrate some of its applications for finding originals and in this way, for interpreting some *partial fractions* arising in operational calculus. Till now, from Examples 3.2, 3.4 we know how to interpret the images z^{-p-1} and $\frac{1}{z+1}$, $\frac{1}{(z+1)^p}$, see (3.9),(3.11)-(3.12). Now we can go further by using (3.21) and formulas (12), §1.5, [12, Vol. 1], resp. (24), §2.2.9 of [32].

EXAMPLE 3.10. The fraction $\mathcal{O}(z) = \frac{z^q}{(z+a)^p}$ is the Laplace-Obrechkoff image (3.7) of the G-function

$$(3.22)\quad f(x) = \frac{x^{p-q-1}}{\Gamma(p)} G^{1,1}_{1,4}\left[ax \,\middle|\, \begin{matrix} 1-p \\ 0, -\mu-p+q+1, -\nu-p+q+1, -p+q+1 \end{matrix}\right],$$

in particular, the case $q = 0$, $a = 1$ gives (3.12).

EXAMPLE 3.11. The fraction $\mathcal{O}(z) = \frac{1}{z^2+a^2}$ has as original the G-function

$$(3.23)\qquad f(x) = \frac{\pi\sqrt{\pi}}{2^{\mu+\nu}a} G^{1,0}_{0,6}\left[\left(\frac{ax}{8}\right)^2 \,\middle|\, \begin{matrix} \\ \frac{1}{2}, \frac{1}{2}-\frac{\mu}{2}, -\frac{\mu}{2}, \frac{1}{2}-\frac{\nu}{2}, -\frac{\nu}{2}, 0 \end{matrix}\right]$$

and a more general result for $\mathcal{O}(z) = \frac{1}{(az^2+bz+c)^p}$ with $b^2 < 4ac$ can be obtained similarly by using formulas (7), (23), §2.2.9, [32].

4. Examples: applications to Bessel-Clifford differential equations. The techniques of the Laplace-Obrechkoff integral transform developed above can be used, for example, in solving initial value problems for differential equations of the form

$$P(B_{\mu,\nu})y(x) = f(x), \qquad P \text{ a polynomial with constant coefficients,}$$

involving the third order Bessel-Clifford operator (2.4).

EXAMPLE 4.1. Consider the problem

$$(4.1)\qquad B_{\mu,\nu}y(x) = -\lambda y(x), \qquad y(0) = y'(0) = y''(0) = 0.$$

Writing $\mathcal{O}_{\mu,\nu}\{f(x); z\} = Y(z)$ and applying (3.7) to (4.1) via (3.15), we find

$$Y(z) = \frac{1}{z+\lambda} \Rightarrow y(x) = C_{\mu,\nu}(\lambda x)$$

which confirms once again the result (2.15).

EXAMPLE 4.2. The problem

$$(4.3)\qquad B_{\mu,\nu}y(x) = f(x), \qquad y(0) = y'(0) = y''(0) = 0$$

is transformed into

$$(4.4)\qquad zY(z) = F(z) \Rightarrow y(x) = \mathcal{O}^{-1}(z^{-1}) * f(x) = \frac{1 * f(x)}{\Gamma(\mu+1)\Gamma(\nu+1)} = L_{\mu,\nu}f(x),$$

as found in (2.10).

Till now, the *nonhomogeneous Bessel type differential equation*

$$B_{\mu,\nu}y(x) = \lambda y(x) + f(x), \quad \lambda \neq 0,\ f \not\equiv 0 \tag{4.5}$$

has not been treated yet. Since the corresponding homogeneous equation from (4.1) under arbitrary initial conditions is satisfied by a linear combination of third order Bessel-Clifford functions (forming the f.s.s., see (4.3), Hayek and Hernández [16]), it is worth trying to find a particular solution to (4.5) under zero initial conditions. As shown in [24], such a problem can be treated by solving first a simpler one for the equation $\widetilde{y}'''(x) = \lambda\widetilde{y}(x) + \widetilde{f}(x)$ and then, transmuting its solution by means of the Poisson type transform (2.16). However, a direct approach via the Laplace-Obrechkoff integral transform is also successfully applicable and we illustrate it by the following example.

Example 4.3. Consider the initial value problem ($p > \alpha$)

$$B_{\mu,\nu}y(x) = \lambda y(x) + x^p, \quad y(0) = y'(0) = y''(0) = 0. \tag{4.6}$$

It is transformed into

$$zY(z) = \lambda Y(z) + z^{-p-1}\Gamma(\mu+p+1)\Gamma(\nu+p+1)\Gamma(p+1), \quad \text{i.e.}$$

$$Y(z) = \Gamma\frac{z^{-p-1}}{z-\lambda}, \qquad \Gamma := \Gamma(\mu+p+1)\Gamma(\nu+p+1)\Gamma(p+1).$$

Then,

$$y(x) = \Gamma\ \mathcal{O}^{-1}(z^{-p-1}) * \mathcal{O}^{-1}\left(\frac{1}{z-\lambda}\right) = x^p * C_{\mu,\nu}(-\lambda x).$$

Calculating this convolution, like in (2.21), (2.23) and (2.22), we find

$$\begin{aligned} y(x) &= \Gamma x^{p+1} G^{1,1}_{1,4}\left[-\lambda x \,\middle|\, \begin{matrix} 0 \\ 0, -\mu-p-1, -\nu-p-1, -p-1 \end{matrix}\right] \\ &= \frac{1}{(p+1)(\mu+p+1)(\nu+p+1)}\, {}_1F_3(1; \mu+p+2, \nu+p+2; p+2; \lambda x). \end{aligned} \tag{4.7}$$

Note also that the integral transform (3.7) incorporates some special cases of other Laplace type transforms for hyper-Bessel operators. For example, such transforms are: the Ditkin and Prudnikov integral transform [11] in the case $m = 3$, corresponding to the operator $B_3 = DxDxD$ and the Krätzel transform [26] ($m = 3$), related to the operator $B = D_{3,\nu} = Dx^{1-\nu}Dx^{\frac{2}{3}}Dx^{\nu+\frac{1}{3}}$, etc.

In a next paper we shall consider another integral transform related to the third order Bessel-Clifford operator, analogous to the Fourier and Hankel transforms.

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