TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 1
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

CHARACTERIZING RINGS BY THEIR MODULES

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1. Introduction

Throughout all rings have identity and all modules are unital. There are a number of well known theorems which characterize a ring in terms of its modules, or some of them at least, for example, the following result:

THEOREM 1.1. *The following statements are equivalent for a ring R.*

(i) *R is semiprime Artinian.*
(ii) *Every right R-module is projective.*
(iii) *Every cyclic right R-module is projective.*
(iv) *Every right R-module is injective.*
(v) *Every right ideal of R is an injective right R-module.*
(vi) *Every cyclic right R-module is injective.*

The equivalence of (i)–(v) can be found in [1, Corollaries 17.4 and 18.8], and (i)$\Leftrightarrow$(vi) is a theorem of Osofsky [16], [17].

It is not true that if the analogue of (v) in the above theorem holds for projective modules then R is semiprime Artinian. Rings with the property that every right ideal is projective are precisely right hereditary rings and this class of rings contains, for example, all commutative Dedekind domains. The following characterization of right hereditary rings is due to Cartan and Eilenberg [2, Theorem 5.4].

This paper is in final form and no version of it will be submitted for publication elsewhere.

THEOREM 1.2. *The following statements are equivalent for a ring R.*

(i) *R is right hereditary.*

(ii) *Every submodule of a projective right R-module is projective.*

(iii) *Every homomorphic image of an injective right R-module is injective.*

Theorem 1.2 is one of many examples of theorems which characterize a ring in terms of its injective modules. The following composite theorem, due to Bass and Papp (see [1, Proposition 18.13 and Exercise 25.5]), Faith and Walker (see [1, Theorem 25.8]), and Matlis (see [1, Theorem 25.6]), gives several characterizations of right Noetherian rings.

THEOREM 1.3. *The following statements are equivalent for a ring R.*

(i) *R is right Noetherian.*

(ii) *The direct sum of any (countable) collection of injective right R-modules is injective.*

(iii) *There exists a cardinal c such that every injective right R-module is a direct sum of c-generated modules.*

(iv) *Every injective right R-module is a direct sum of indecomposable injective R-modules.*

Right Noetherian rings can be characterized in other ways, for example a ring R is right Noetherian if and only if it does not contain a nonzero right ideal E which is the union of a chain of proper submodules of E. The analogue for right Artinian rings is the next result due to Vamos (see [1, Proposition 10.10]).

THEOREM 1.4. *A ring R is right Artinian if and only if every cyclic right R-module is finitely cogenerated.*

Finally, we mention two more recent theorems, the first due to Chatters [3] generalizing work of Smith [19], and the second due to van Huynh and Dung [9].

THEOREM 1.5. *A ring R is right Noetherian if and only if every cyclic right R-module is a direct sum of a projective module and a Noetherian module.*

THEOREM 1.6. *A ring R is right Artinian if and only if every cyclic right R-module is a direct sum of an injective module and a finitely cogenerated module.*

In the sequel we shall review a number of recent results similar in spirit to the above theorems.

2. Right Noetherian rings

Let R be a ring. By a *class* X of right R-modules we mean any collection of right R-modules which contains a zero module and is closed under isomorphisms, i.e. an R-module M' belongs to X whenever $M' \cong M \in X$. Given a class

X of R-modules, any module in X will be called an *X-module*. Furthermore, an R-module M is called a *locally X-module* provided every finitely generated submodule of M is an X-module. Let c be any cardinal. An R-module M will be called *c-limited* provided every direct sum of nonzero submodules of M contains at most c direct summands. The cardinality of any set X will be denoted $|X|$. Given an R-module M then $E(M)$ will denote the injective hull of M.

The following theorem is taken from [3, Theorem 3.1], [12, Theorems 4 and 12] and [9, Corollary 1.3].

THEOREM 2.1. *The following statements are equivalent for a ring R.*

(i) *R is right Noetherian.*

(ii) *Every cyclic right R-module is a direct sum of a projective module and a Noetherian module.*

(iii) *Every finitely generated right R-module is a direct sum of a projective module and a Noetherian module.*

(iv) *Every right R-module is a direct sum of an injective module and a locally Noetherian module.*

(v) *There exists a cardinal c such that every direct sum of injective right R-modules is a direct sum of an injective module and a c-limited module.*

(vi) *Every essential maximal right ideal is a direct sum of an injective right ideal and a Noetherian right ideal.*

Proof. (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii). See Theorem 1.5.

(i)$\Rightarrow$(iv), (i)$\Rightarrow$(vi). Clear.

(i)$\Rightarrow$(v). See Theorem 1.4.

(vi)$\Rightarrow$(i). Suppose that R satisfies (vi). Let N denote the sum of all Noetherian right ideals of R. Suppose $N \neq R$. Then there exists a maximal right ideal M such that $N \subseteq M$. Clearly M is essential in R, so that, by hypothesis, $M = I \oplus K$ for some injective right ideal I and Noetherian right ideal K. Now $M/K \cong I$ so that M/K is an injective submodule of R/K, and hence there exists a right ideal L containing K such that $R/K = (M/K) \oplus (L/K)$. Note that $L/K \cong R/M$, thus L/K is simple and L is Noetherian. But this implies $L \subseteq N \subseteq M$ and $R = M + L \subseteq M$, a contradiction. Thus $N = R$ and hence R is right Noetherian.

(v)$\Rightarrow$(i). Suppose that R satisfies (v). Let $\{S_\omega \colon \omega \in \Omega\}$ denote a collection of representatives of the isomorphism classes of simple right R-modules and let $X = \bigoplus_{\omega\in\Omega} S_\omega$. Let Λ be an index set with $|\Lambda| \geqslant c$, and for each λ in Λ let $X_\lambda = X$. Define $Y = \bigoplus_{\lambda\in\Lambda} X_\lambda$ and $k = |E(Y)|$. Let U_i $(i \geqslant 1)$ be simple right R-modules and $E_i = E(U_i)$ for each $i \geqslant 1$. Let $E = \bigoplus_{i\geqslant 1} E_i$. Let Π be any index set with $|\Pi| > k$. For each π in Π let $F_\pi = E$, and consider $F = \bigoplus_{\pi\in\Pi} F_\pi$. By hypothesis, $F = J \oplus A$ for some injective module J and c-limited module A.

Now $\operatorname{soc} A = A \cap \operatorname{soc} F$, $\operatorname{soc} A$ is an essential submodule of A and is the direct sum of at most c simple submodules. Thus there exists a monomorphism θ: $\operatorname{soc} A \to Y$ which can be lifted to a monomorphism φ: $A \to E(Y)$. Thus $|A| \leqslant |E(Y)| = k$. It follows that there exist disjoint subsets Π', Π'' of Π such that $\Pi = \Pi' \cup \Pi''$, $|\Pi'| \leqslant k$ and $A \subseteq \bigoplus_{\pi \in \Pi'} F_\pi$. Let $G = \bigoplus_{\pi \in \Pi'} F_\pi$ and $H = \bigoplus_{\pi \in \Pi''} F_\pi$. Thus $A \subseteq G$ and $F = G \oplus H$. Now $G = (J \cap G) \oplus A$ implies

$$F = J \oplus A = (J \cap G) \oplus A \oplus H.$$

It follows that $J \cong F/A \cong (J \cap G) \oplus H$, and hence H is injective. But Π'' is nonempty because $|\Pi'| \leqslant k < |\Pi|$. Thus E is injective. It follows that every countable direct sum of injective hulls of simple right R-modules is injective. By Theorem 1.3, R is right Noetherian.

(iv) $\Rightarrow$ (i). Suppose that R satisfies (iv). In particular, every cyclic right R-module is a direct sum of an injective module and a Noetherian module, and in the next section we shall prove that this implies that R has finite right uniform dimension. There exist a positive integer n and indecomposable right ideals C_i $(1 \leqslant i \leqslant n)$ such that $R = C_1 \oplus \ldots \oplus C_n$.

Let $1 \leqslant i \leqslant n$ and $C = C_i$. Note that C is a principal right ideal of R. Suppose C is not Noetherian. Let D be a maximal submodule of C. If $d \in D$ then $dR = I \oplus N$ for some injective right ideal I and Noetherian right ideal N. Since C is indecomposable it follows that $I = 0$ and hence dR is Noetherian. Thus every cyclic submodule of D is Noetherian and D is a sum of Noetherian submodules. If D' is a distinct maximal submodule of C then D' is also a sum of Noetherian submodules and $C = D + D'$, so that C, being finitely generated, is Noetherian, a contradiction. Thus D is the only maximal submodule of C.

Since every cyclic submodule of D is Noetherian it follows that every finitely generated submodule of D is also Noetherian. If H is a proper submodule of C then $H \subseteq D$, C/H is cyclic and D/H is the only maximal submodule of C/H. By hypothesis C/H is Noetherian or indecomposable injective. In particular, if H is finitely generated then C/H is uniform. Because C is not Noetherian, there exists a chain of right ideals

(1) $$0 = D_0 \subset D_1 \subset D_2 \subset \ldots \subseteq \bigcup_{i \geqslant 0} D_i \subseteq D \subset C,$$

such that D_i is finitely generated for each $i \geqslant 0$. By (iv),

$$\bigoplus_{i \geqslant 0} (C/D_i) = X \oplus Y,$$

for some injective R-module X and locally Noetherian R-module Y. Suppose $Y \neq 0$. Then $X \cap (C/D_j) = 0$ for some $j \geqslant 1$, and this implies C/D_j is isomorphic to a cyclic submodule of Y. But this means that C/D_j is Noetherian, a contradiction to (1). Thus $Y = 0$. It follows that $\bigoplus_{i \geqslant 0}(C/D_i)$ is injective and by a standard argument (see the proof of [1, Proposition 18.13]), $D_m = D_{m+1} = D_{m+2} = \ldots$, for some $m \geqslant 0$, again contradicting (1). Thus C is Noetherian. It follows that R is right Noetherian. This completes the proof of Theorem 2.1.

Theorem 2.1 raises several questions. For example, if R is a ring such that every cyclic (or finitely generated) right R-module is a direct sum of an injective module and a Noetherian module, is R right Noetherian? First of all, note that some properties that hold for cyclic modules hold for finitely generated modules also. Let R be a ring and

$$0 \to M' \to M \to M'' \to 0$$

a short exact sequence of R-modules. A class X of R-modules is called

S-closed provided $M' \in X$ whenever $M \in X$,

Q-closed provided $M'' \in X$ whenever $M \in X$, and

P-closed provided $M \in X$ whenever $M' \in X$, $M'' \in X$.

Moreover, X is called $\{P, Q\}$*-closed* provided it is both P-closed and Q-closed, and so on. With this notation we prove:

LEMMA 2.2. *Let R be a ring and X any $\{P, Q\}$-closed class of right R-modules. Then the following statements are equivalent.*

(i) *Every cyclic right R-module is a direct sum of an injective module and an X-module.*

(ii) *Every finitely generated right R-module is a direct sum of an injective module and an X-module.*

Proof. Since (ii) $\Rightarrow$ (i) is obvious, it remains to prove (i) $\Rightarrow$ (ii). Suppose that R satisfies (i). Let M be any finitely generated R-module. Then there exist a positive integer n and elements $m_i \in M$ $(1 \leqslant i \leqslant n)$ such that $M = m_1 R + \ldots + m_n R$. If $n = 1$ then there is nothing to prove. Suppose $n > 1$ and let $K = m_1 R + \ldots + m_{n-1} R$. By induction on n, $K = K_1 \oplus K_2$, where K_1 is an injective module and $K_2 \in X$. Now K/K_2 is an injective submodule of M/K_2, so $M/K_2 = (K/K_2) \oplus (L/K_2)$ for some submodule L of M. Note that $M = K_1 \oplus L$ and $L/K_2 \cong M/K$, so that L/K_2 is cyclic. There exists $x \in L$ such that $L = xR + K_2$. By (i), $xR = A \oplus B$ for some injective module A and X-module B. As before, there exists a submodule C of L, containing B, such that $L/B = (xR/B) \oplus (C/B)$. Then $L = A \oplus C$ and $C/B \cong L/(xR) \cong K_2/(xR \cap K_2)$. Thus $C/B \in X$ and hence $C \in X$. It follows that $M = K_1 \oplus A \oplus C$, the direct sum of an injective module $K_1 \oplus A$ and an X-module C.

In particular, Lemma 2.2 applies to the class X of Noetherian R-modules. An example of a ring R such that every cyclic (or finitely generated) module is a direct sum of an injective module and a Noetherian module, yet R is not right Noetherian, can be found in [12, Example 11]. The ring in question is defined as follows. Let F be any field, $S = F[[X]]$ the ring of formal power series in an indeterminate X over F, and K the field of fractions of S. Then R is the subring of the ring of 2×2 matrices with entries in K, consisting of all matrices

$$\begin{bmatrix} x & y \\ 0 & s \end{bmatrix}$$

where x, $y \in K$ and $s \in S$. It is rather easy to see that R is not right Noetherian, but checking that every cyclic R-module is a direct sum of an injective module and a Noetherian module requires a little work (see [12] for details). Note that if R is a ring, with prime radical N, such that every cyclic right R-module is a direct sum of an injective module and a Noetherian module then R is right Noetherian provided R is semiprime [10, Proposition 8] or commutative [12, Corollary 10] or R/N has zero socle or R contains only one minimal prime ideal [12, Proposition 8].

Another question raised by Theorem 2.1 is the following one: if R is a ring such that every right R-module is a direct sum of a projective (or injective) module and a Noetherian module then what can be said about R? It is clear that Noetherian modules are $\aleph_0$-limited, so that the answers to both versions of this question are contained in the next result.

THEOREM 2.3. *The following statements are equivalent for a ring R.*

(i) *R is semiprime Artinian.*

(ii) *There exists a cardinal c such that every right R-module is a direct sum of a projective module and a c-limited module.*

(iii) *There exists a cardinal c such that every right R-module is a direct sum of an injective module and a c-limited module.*

Proof. Theorem 1.1 shows that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). Conversely, suppose that R satisfies (ii). Let S denote the socle of the right R-module R and suppose that $S \neq R$. There exists a maximal right ideal M such that $S \subseteq M$. Let Λ be any index set with $|\Lambda| > c$. For each λ in Λ let $U_\lambda = R/M$ and let $U = \bigoplus_{\lambda\in\Lambda} U_\lambda$. By (ii) there exist a projective module P and a c-limited module X such that $U = P \oplus X$. Since U is not c-limited it follows that $P \neq 0$. Thus $P \cong \bigoplus_{\lambda\in\Lambda'} U_\lambda$ for some nonempty subset Λ' of Λ, by [1, Proposition 9.4]. It follows that R/M is projective and hence $R = M \oplus V$ for some minimal right ideal V of R. But this cannot be the case because $V \subseteq S \subseteq M$. Thus $S = R$ and R is semiprime Artinian. This proves (i).

Now suppose that R satisfies (iii). By Theorem 2.1 the ring R is right Noetherian. The argument used to prove (ii) $\Rightarrow$ (i) can easily be modified to prove that every simple right R-module is injective, i.e. R is a right V-ring. By [15, Corollary 2.2 and Lemma 3.1], the ring R is a finite direct sum of simple rings. Thus, without loss of generality we can suppose that R is a simple ring.

Again, let Λ be an index set with $|\Lambda| > c$. Let $M_\lambda = R_R$ ($\lambda \in \Lambda$) and $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$. By (iii), $M = I \oplus Y$ for some injective module I and c-limited module Y. Note that $I \neq 0$. By Goldie's Theorem (see [20, p. 56, Proposition 2.6, and p. 58, Proposition 3.8]) R has a simple Artinian classical right quotient ring Q which is the R-injective hull of R_R. Let $Q_\lambda = Q$ ($\lambda \in \Lambda$) and note that $Q_\lambda = E(M_\lambda)$ for each $\lambda \in \Lambda$. By Theorem 1.3, the module $\bigoplus_{\lambda\in\Lambda} Q_\lambda$ is injective and hence $\bigoplus_{\lambda\in\Lambda} Q_\lambda \cong I \oplus E(Y)$.

Let C be any nonzero indecomposable injective direct summand of I. By

Azumaya's Theorem (see [1, Theorems 12.6 and 25.6]), $C \cong K$ for some indecomposable injective direct summand K of Q. Now K is a right ideal of Q by [21, p. 58, Proposition 3.8], and $Q \cong K^{(n)}$ for some positive integer n. But I is a projective right R-module and hence so too are C, K and Q. By [22, Lemma 6.1 and Proposition 6.3] it follows that R is right Artinian. This completes the proof of (i).

3. A theorem

Let R be a ring and M a right R-module. If M has Krull dimension then $\operatorname{K dim} M$ will denote the Krull dimension of M. See [13] for the basic definitions and properties concerning Krull dimension. Note in particular that if M has Krull dimension then M has finite uniform dimension [13, Proposition 1.4]. The ring R *has right Krull dimension* α, for some ordinal α, provided the right R-module R has Krull dimension α. Furthermore, the right R-module M has *essential Krull dimension at most* α provided M contains an essential submodule N such that $\operatorname{K dim} N \leqslant \alpha$, and in this case we shall write $\operatorname{EK dim} M \leqslant \alpha$. We shall say that a module M *has essential Krull dimension* provided it contains an essential submodule which has Krull dimension. Note that $\operatorname{EK dim} M$, if it exists, is the least ordinal α such that M contains an essential submodule with Krull dimension α. Note further that if $\operatorname{K dim} M \leqslant \alpha$ then $\operatorname{EK dim} \leqslant \alpha$ [13, Lemma 1.1]. In the proof of Theorem 2.1(iv) $\Rightarrow$ (i), we did not prove that R has finite right uniform dimension. This is an immediate consequence of the following result, because Noetherian modules have Krull dimension [13, Proposition 1.3].

THEOREM 3.1. *The following statements are equivalent for a ring R and an ordinal* α.

(i) *R has right Krull dimension at most* α.

(ii) *Every cyclic right R-module is a direct sum of an injective module and a module with essential Krull dimension at most* α.

(iii) *Every finitely generated right R-module is a direct sum of an injective module and a module with essential Krull dimension at most* α.

In contrast, Theorem 2.3 shows that a ring R is semiprime Artinian if and only if *every* right R-module is a direct sum of an injective module and a module with essential Krull dimension. In Theorem 3.1, the implication (i) $\Rightarrow$ (iii) is a consequence of [13, Lemma 1.1] and (iii) $\Rightarrow$ (ii) is clear. It remains to prove (ii) $\Rightarrow$ (i). The proof is split into two cases. Firstly we prove the result when R is semiprime and then prove the general case. The rest of this section will be devoted to the proof of Theorem 3.1(ii) $\Rightarrow$ (i). For convenience, for any ordinal $\alpha \geqslant 0$ we shall say that R *satisfies* $P(\alpha)$ provided every cyclic right R-module is a direct sum of an injective module and a module with essential Krull dimension at most α.

Case 1. *R semiprime.* The following simple fact concerning uniform modules will be needed. It does not require R to be semiprime.

LEMMA 3.2. *Let* φ: $U \to V$ *be an* R*-homomorphism of a uniform* R*-module* U *to a nonsingular* R*-module* V. *Then* $\varphi = 0$ *or* φ *is a monomorphism.*

Proof. Let $K = \ker \varphi$. If $K \neq 0$ then $\operatorname{im} \varphi \cong U/K$ implies $\operatorname{im} \varphi$ is singular, so that $\varphi = 0$.

LEMMA 3.3. *Let* R *be a semiprime ring,* I *a nonsingular injective right ideal of* R *and* U *an* R*-module with finite uniform dimension such that* $\operatorname{Hom}_R(U, I) \neq 0$. *Then* U *contains a nonzero injective submodule.*

Proof. Let $0 \neq \varphi \in \operatorname{Hom}_R(U, I)$. There exist a positive integer n and uniform submodules $U_i (1 \leqslant i \leqslant n)$ of U such that $U_1 \oplus \ldots \oplus U_n$ is an essential submodule of U. If $\varphi(U_i) = 0$ $(1 \leqslant i \leqslant n)$ then $\operatorname{im} \varphi$ is singular, so that $\varphi = 0$. Without loss of generality we can suppose $\varphi(U_1) \neq 0$. Let $V = \varphi(U_1)$. By Lemma 3.2, $V \cong U_1$ and in particular V is uniform. Since I is injective it follows that $E = E(V) \subseteq I$. Note that E is uniform. Let $0 \neq e \in E$. Then $(eR)^2 \neq 0$, so that $erE \neq 0$ for some $r \in R$. Define θ: $E \to eR$ by $\theta(x) = erx$ $(x \in E)$. By Lemma 3.2, θ is a monomorphism, and hence erE is injective. This implies $E = erE \subseteq eR$. It follows that E is simple and hence $E = V$. Therefore U_1 is a nonzero injective submodule of U.

Lemma 3.3 allows us to prove the following first step in the proof of Theorem 3.1. For any element $a \in R$, the right annihilator of a will be denoted $r(a)$. The right singular ideal of R will be denoted $Z(R)$.

LEMMA 3.4. *Let* R *be a semiprime ring which satisfies* $P(\alpha)$. *Then* $R = S \oplus T$ *where* S *is a right nonsingular right self-injective ring which satisfies* $P(\alpha)$ *and* T *is a ring with right Krull dimension at most* α.

Proof. We prove first that R is right nonsingular. Let $a \in Z(R)$. By hypothesis $aR = I \oplus A$ where I is an injective right ideal of R and A a right ideal with $\operatorname{EK} \dim A \leqslant \alpha$. Since $Z(R)$ contains no nonzero injective right ideals it follows that $I = 0$. Suppose $A \neq 0$. Let C be a critical right ideal contained in A. Let $c \in C$. Then $c \in Z(R)$ implies $r(c) \cap C \neq 0$. Thus $\operatorname{K} \dim \{C/[r(c) \cap C]\} < \operatorname{K} \dim C$. But $cC \cong C/[r(c) \cap C]$ and hence $cC = 0$. It follows that $C^2 = 0$ and hence $C = 0$, a contradiction. Thus $A = 0$ and hence $Z(R) = 0$.

There exist an injective right ideal J and a right ideal T with $\operatorname{EK} \dim T \leqslant a$ such that $R = J \oplus T$. Because T has finite uniform dimension, we can suppose, without loss of generality, that T contains no nonzero injective submodule. By Lemma 3.3, $\operatorname{Hom}_R(T, J) = 0$. But for any $r \in R$, there exists a homomorphism θ: $T \to J$ defined by $\theta(t) = \pi(rt)$ $(t \in T)$, where π: $R \to J$ is the canonical projection. It follows that $\theta = 0$ and hence T is a two-sided ideal of R. But R is semiprime and hence $T = eR$ for some central idempotent e. Moreover, T is a semiprime right Goldie ring by [4, Lemma 1.14] and $\operatorname{EK} \dim T \leqslant \alpha$, so that

$K\dim T \leqslant \alpha$ by [4, Theorem 1.10]. Let $S = (1-e)R$. Then $R = S \oplus T$ where S, T are rings with the desired properties.

To complete Case 1 we shall show that (in the notation of Lemma 3.4) S is semiprime Artinian. To do so, we require the following result.

LEMMA 3.5. *Let M be a finitely generated injective right R-module and N a submodule of M such that N is an infinite direct sum of nonzero submodules. Then M/N does not have finite uniform dimension.*

Proof. Suppose M/N has finite uniform dimension k. There exist non-finitely generated submodules $N_i (1 \leqslant i \leqslant k+1)$ of N such that $N = N_1 \oplus \dots \oplus N_{k+1}$. Let $E_i = E(N_i)$ $(1 \leqslant i \leqslant k+1)$. Note that for each $1 \leqslant i \leqslant k+1$, E_i is a direct summand of M, thus is finitely generated and hence $E_i \neq N_i$. Now

$$(E_1 \oplus \dots \oplus E_{k+1})/N \cong (E_1/N_1) \oplus \dots \oplus (E_{k+1}/N_{k+1}),$$

so that M/N has uniform dimension greater than k, a contradiction.

We are now in a position to prove the following statement.

LEMMA 3.6. *Let R be a semiprime ring which satisfies $P(\alpha)$. Then R has right Krull dimension at most α.*

Proof. By Lemma 3.4 it is certainly sufficient to prove that if R is right nonsingular and right self-injective then R is semiprime Artinian. Suppose that R is right nonsingular right self-injective. By the proof of Lemma 3.3 every uniform right ideal of R is a simple injective R-module. Thus aR is injective for each $a \in R$, i.e. R is von Neumann regular. Let $\{e_i : i \geqslant 1\}$ be an infinite collection of nonzero orthogonal idempotents of R and let $A = \bigoplus_{i \geqslant 1} e_i R$. There exist right ideals B, C containing A such that $R/A = (B/A) \oplus (C/A)$, B/A is injective and $\mathrm{EK}\dim(C/A) \leqslant \alpha$. In particular $R/C \cong B/A$ implies R/C is injective. Now $C = cR + A$ for some $c \in C$ and $cR = f_0 R$ for some idempotent $f_0 \in R$. From the properties of von Neumann regular rings we can produce a family $\{f_i : i \geqslant 0\}$ of orthogonal idempotents of R such that

$$cR + e_1 R + \dots + e_n R = f_0 R + \dots + f_n R \qquad (n \geqslant 1).$$

Thus $C = \bigoplus_{i \geqslant 0} f_i R$ and, by [16, Lemma 5], R/C injective implies that there exists $k \geqslant 1$ such that $f_i = 0$ $(i > k)$. Thus C is injective and, by Lemma 3.5, C/A does not have finite uniform dimension, a contradiction. Thus R does not contain an infinite collection of nonzero orthogonal idempotents. It follows that R is semiprime Artinian.

Case 2. *R any ring.* We begin the general case with the following consequence of Lemma 3.6.

LEMMA 3.7. *Let R be any ring satisfying $P(\alpha)$. Then the prime radical N of R is nilpotent.*

Proof. Let N denote the prime radical of R. By Lemma 3.6 the ring R/N has right Krull dimension at most α. Let n be a positive integer and $\bar{e}_i = e_i + N$ $(1 \leqslant i \leqslant n)$ a complete set of nonzero orthogonal idempotents of R/N. Because N is a nil ideal of R, we can suppose without loss of generality that $R = e_1R \oplus \ldots \oplus e_nR$ and e_iR is indecomposable for $1 \leqslant i \leqslant n$ (see [1, Proposition 27.4]).

Let $1 \leqslant i \leqslant n$ and $e = e_i$. If $\operatorname{EK}\dim eR \leqslant \alpha$ then eR has finite uniform dimension. Otherwise eR is injective. If $eR \cap N = 0$ then $\operatorname{K}\dim eR \leqslant \alpha$. If $eR \cap N \neq 0$ and $0 \neq a \in eR \cap N$ then $aR = I \oplus A$ for some injective right ideal I and right ideal A with $\operatorname{EK}\dim A \leqslant \alpha$. Since I is nilpotent it follows that $I = 0$ and hence $aR = A$. Thus eR is uniform. In any case eR has finite uniform dimension. It follows that R has finite right uniform dimension. Thus every nonzero ring homomorphic image of R contains an essential right ideal with Krull dimension at most α. Combining this fact with [13, Corollary 5.10] it follows that N is nilpotent.

The next result is implicit in [13] (see pages 16 and 33), and a proof is given in [10, Lemma 6].

LEMMA 3.8. *A right R-module M has Krull dimension at most α if and only if* $\operatorname{EK}\dim(M/N) \leqslant \alpha$ *for every submodule N of M.*

With these preliminaries out of the way we now complete the proof of Theorem 3.1.

Proof of Theorem 3.1(ii) $\Rightarrow$ (i). Let R be a ring which satisfies $P(\alpha)$. Let N denote the prime radical of R. By Lemma 3.6, $\operatorname{K}\dim(R/N) \leqslant \alpha$ and by Lemma 3.7, $N^k = 0$ for some $k \geqslant 1$. By induction on k we can suppose $k > 1$ and $\operatorname{K}\dim(R/N^{k-1}) \leqslant \alpha$. To prove $\operatorname{K}\dim R \leqslant \alpha$ it is sufficient, in view of Lemma 3.8, to prove that $\operatorname{EK}\dim M \leqslant \alpha$ for every cyclic injective right R-module M.

Let M be a cyclic injective right R-module. Let $L = \{m \in M \colon mN = 0\}$. Since $N^k = 0$ it follows that L is an essential submodule of M. Moreover, M/L is a cyclic right (R/N^{k-1})-module, so that $\operatorname{K}\dim(M/L) \leqslant \alpha$. In particular, this implies that M/L has finite uniform dimension n (say) by [13, Proposition 1.4].

We claim that L has finite uniform dimension. Suppose not. (We modify the proof of Lemma 3.5.) There exists an essential submodule K of L which is an infinite direct sum of nonzero submodules. Thus $K = K_1 \oplus \ldots \oplus K_{n+1}$, where, for each $1 \leqslant i \leqslant n+1$, K_i is a submodule of K and K_i does not have finite uniform dimension. Since M is an injective R-module it follows that L is an injective (R/N)-module. For each $1 \leqslant j \leqslant n+1$ let E_j denote the (R/N)-injective hull of K_j in L. Note that $L = E_1 \oplus \ldots \oplus E_{n+1}$. Suppose E_j is finitely generated for some $1 \leqslant j \leqslant n+1$. Since $\operatorname{K}\dim(R/N) \leqslant \alpha$ it follows that $\operatorname{K}\dim E_j \leqslant \alpha$ and hence $\operatorname{K}\dim K_j \leqslant \alpha$. But this contradicts the fact that K_j does not have finite uniform dimension. Thus E_j is not finitely generated for each $1 \leqslant j \leqslant n+1$. For each $1 \leqslant j \leqslant n+1$, let F_j denote the R-injective hull of E_j in M. Note that M being cyclic implies F_j is cyclic and hence $F_j \neq E_j$ $(1 \leqslant j \leqslant n+1)$. Then we have $M = F_1 \oplus \ldots \oplus F_{n+1}$ and $M/L \cong (F_1/E_1) \oplus \ldots \oplus (F_{n+1}/E_{n+1})$, so that M/L has uniform dimension greater than n, a contradiction.

Thus L has finite uniform dimension. In particular, this means that L contains a finitely generated essential submodule X. Since $XN = 0$ it follows that $\mathrm{K}\dim X \leqslant \alpha$ by [13, Lemma 1.1]. But X is an essential submodule of M. Hence $\mathrm{EK}\dim M \leqslant \alpha$, as required. This completes the proof of Theorem 3.1.

4. Rings with Krull dimension

Chatters [3, Theorem 4.1] proved that if R is a ring and α an ordinal such that every cyclic right R-module is a direct sum of a projective module and a module with Krull dimension at most α then R has right Krull dimension at most $\alpha+1$. In fact, Chatters' theorem can be expressed in a slightly different way because of the following result.

LEMMA 4.1. *The following statements are equivalent for a ring R and an ordinal α.*

(i) *Every cyclic right R-module is a direct sum of a projective module and a module with Krull dimension at most α.*

(ii) *Every cyclic right R-module is a direct sum of a projective module and a module with essential Krull dimension at most α.*

Proof. (i) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (i). Suppose R satisfies (ii) and let M be any cyclic right R-module. Then M is a direct sum $P \oplus N$ of a projective module P and a module N with $\mathrm{EK}\dim N \leqslant \alpha$. Thus there exists an essential submodule L of N such that $\mathrm{K}\dim L \leqslant \alpha$. Let K be any submodule of N containing L. Then K is an essential submodule of N, so that N/K does not contain any nonzero projective direct summands. Since N/K is cyclic it follows that $\mathrm{EK}\dim (N/K) \leqslant \alpha$. Thus every homomorphic image of N/L has essential Krull dimension at most α. By Lemma 3.8, $\mathrm{K}\dim (N/L) \leqslant \alpha$, and, by [13, Lemma 1.1], $\mathrm{K}\dim N \leqslant \alpha$. This proves (i).

Combining [3, Theorem 4.1], [13, Lemma 1.1], Theorem 3.1 and Lemma 4.1 we have at once:

COROLLARY 4.2. *The following statements are equivalent for a ring R.*

(i) *R has right Krull dimension.*

(ii) *Every cyclic right R-module is a direct sum of a projective module and a module with essential Krull dimension.*

(iii) *Every finitely generated right R-module is a direct sum of a projective module and a module with essential Krull dimension.*

(iv) *Every cyclic right R-module is a direct sum of an injective module and a module with essential Krull dimension.*

(v) *Every finitely generated right R-module is a direct sum of an injective module and a module with essential Krull dimension.*

There is an analogue of Lemma 4.1 for finitely generated modules. Now consider the following three conditions for a ring R and an ordinal $\alpha \geqslant 0$:

$S_1(\alpha)$: every right R-module is a direct sum of a projective module and a module with Krull dimension at most α;

$S_2(\alpha)$: every finitely generated right R-module is a direct sum of a projective module and a module with Krull dimension at most α;

$S_3(\alpha)$: every cyclic right R-module is a direct sum of a projective module and a module with Krull dimension at most α.

Clearly $S_1(\alpha) \Rightarrow S_2(\alpha) \Rightarrow S_3(\alpha)$. Moreover, by [13, Proposition 1.4] and Theorem 2.3,

$$R \text{ satisfies } S_1(\alpha) \Leftrightarrow R \text{ is semiprime Artinian.}$$

Next, any nonsemiprime ring with right Krull dimension at most α satisfies $S_2(\alpha)$ but not $S_1(\alpha)$. On the other hand, let R be a principal right ideal domain with Krull dimension $\alpha+1$ which is not a left Ore domain (see [13, Example 10.3]). By [13, Proposition 6.1], R satisfies $S_3(\alpha)$. Now suppose that R satisfies $S_2(\alpha)$. Let M be any finitely generated nonsingular right R-module. Then $M = P \oplus K$ for some projective R-module P and R-module K with $\operatorname{K\,dim} K \leqslant \alpha$. Suppose $K \neq 0$ and let $0 \neq x \in K$. Then there exists a nonessential right ideal B of R such that $xR \cong R/B$. There exists a nonzero right ideal C such that $B \cap C = 0$. In this case, C embeds in R/B and hence $\operatorname{K\,dim} C \leqslant \alpha$. But this implies $\operatorname{K\,dim} R_R \leqslant \alpha$, a contradiction. Thus $K = 0$. It follows that every finitely generated nonsingular right R-module is projective. By [14, Theorem 5.3], R is a left Ore domain, another contradiction. Thus R satisfies $S_3(\alpha)$ but does not satisfy $S_2(\alpha)$.

THEOREM 4.3. *Let α be an ordinal. Then the following statements are equivalent for a ring R.*

(i) *R has right Krull dimension at most α.*

(ii) *Every right R-module is a direct sum of a projective module and a module with local essential Krull dimension at most α.*

Proof. (i) $\Rightarrow$ (ii). Clear by [13, Lemma 1.1].

(ii) $\Rightarrow$ (i). Suppose that R satisfies (ii). By Corollary 4.2, R has right Krull dimension. Let A denote the sum of all right ideals B of R with $\operatorname{K\,dim} B \leqslant \alpha$. Then A is a two-sided ideal of R and, as a right R-module, $\operatorname{K\,dim} A \leqslant \alpha$ by [13, Corollary 4.2].

It remains to prove that the ring R/A has Krull dimension at most α. Note first that the right R-module R/A does not contain any nonzero submodule with Krull dimension at most α, so that R/A is R-projective by (ii). Thus $A = eR$ for some idempotent e in R. Because $A = eR$, it is not difficult to prove that every cyclic right (R/A)-module is a direct sum of a projective (R/A)-module and an (R/A)-module with local essential Krull dimension at most α. Thus without loss of generality we can suppose that $A = 0$.

If C is a right ideal of R then C is projective by (ii). Thus the ring R is right hereditary. By [13, Proposition 1.4], R has finite right uniform dimension and by [18, Theorem 2.1, Corollary 2], R is right Noetherian. Furthermore, the injective hull E of R_R is a projective R-module by (ii). By [21, Lemma 6.1 and Proposition 6.3] it follows that R is right Artinian. This completes the proof of Theorem 4.3.

Note that the ring $\mathbf{Z}$ has the property that every finitely generated $\mathbf{Z}$-module is a direct sum of a projective module and a module of Krull dimension at most 0, but $\operatorname{K dim} \mathbf{Z} = 1$. Thus in Theorem 4.3 the condition (ii) cannot be replaced by "every finitely generated right R-module is a direct sum of a projective module and a module with Krull dimension at most α". The case $\alpha = 0$ in Theorems 3.1 and 4.3 gives most of the following result (see [9] and [11, Corollary 1.3]).

COROLLARY 4.4. *The following statements are equivalent for a ring R.*

(i) *R is right Artinian.*

(ii) *Every right R-module is a direct sum of a projective module and a locally finitely cogenerated module.*

(iii) *Every cyclic right R-module is a direct sum of an injective module and a finitely cogenerated module.*

(iv) *Every finitely generated right R-module is a direct sum of an injective module and a finitely cogenerated module.*

(v) *Every essential maximal right ideal of R is a direct sum of an injective right ideal and an Artinian right ideal.*

Note that in this corollary, (i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) follows by Theorem 3.1, (i)$\Leftrightarrow$(ii) by Theorem 4.3, and (i)$\Leftrightarrow$(v) by the proof of Theorem 2.1(i)$\Leftrightarrow$(vi). The next result is taken from [11, Theorem 2.6] and [8, Theorem 6].

THEOREM 4.5. *The following statements are equivalent for a ring R with Jacobson radical J.*

(i) *R is a (right and left) Artinian (right and left) serial ring with $J^2 = 0$.*

(ii) *R is a direct sum of minimal right ideals and injective right ideals of (composition) length 2.*

(iii) *Every right R-module is a direct sum of a projective module and a semisimple module.*

(iv) *Every right R-module is a direct sum of an injective module and a semisimple module.*

(v) *Every cyclic right R-module is a direct sum of an injective module and a semisimple module.*

(vi) *Every right ideal is a direct sum of an injective right ideal and a finitely generated semisimple right ideal.*

(vii) *Every essential right ideal is a direct sum of an injective right ideal and a finitely generated semisimple right ideal.*

(viii) *Any left-handed version of* (iii)–(vii).

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds and R is not semiprime Artinian. Let A be a nonsimple indecomposable right ideal of R. Let $E = E(A)$. Then E is indecomposable and, by [6, Theorem 25.4.2], E is uniserial and cyclic, say $E = xR$. Now $R = A_1 \oplus \ldots \oplus A_k$ for some positive integer k and right ideals A_i of length at most 2 for each $1 \leqslant i \leqslant k$. Thus

$$E = xA_1 + \ldots + xA_k = xA_j$$

for some $1 \leqslant j \leqslant k$ and hence E has length at most 2. It follows that $A = E$ and (ii) is proved.

(ii) $\Rightarrow$ (iii). Suppose $R = A_1 \oplus \ldots \oplus A_k$ for some positive integer k and right ideals A_i ($1 \leqslant i \leqslant k$), where A_i is minimal or injective of length 2 for each $1 \leqslant i \leqslant k$. Then R is right Artinian. Suppose that M is an R-module. By Zorn's Lemma, M contains a submodule P maximal among those submodules of the form $\bigoplus_{\lambda \in \Lambda} M_\lambda$, where for each λ in Λ, M_λ is isomorphic to A_i for some $1 \leqslant i \leqslant k$ such that A_i has length 2. By Theorem 1.6, P is injective and hence $M = P \oplus M'$ for some submodule M' of M. Suppose M' is not semisimple. Then there exists $m \in M'$ such that mR is not semisimple. Now $mR = mA_1 + \ldots + mA_k$, so that $A_i \cong mA_i$ for some $1 \leqslant i \leqslant k$ and A_i has length 2. This contradicts the choice of P. Thus M' is semisimple. Note that P is projective. This proves (iii).

(iii) $\Rightarrow$ (iv). Suppose that (iii) holds. By Corollary 4.4, R is right Artinian and, by [1, p. 204], (iv) follows.

(iv) $\Rightarrow$ (v). Obvious.

(v) $\Rightarrow$ (vi). Suppose that (v) holds. By Corollary 4.4, R is right Artinian. Let X be any right R-module. Because R is right Noetherian, X contains a maximal injective submodule I. Thus $X = I \oplus X'$ for some submodule X' of X. For any $y \in X'$ there exist an injective module I' and a semisimple module X'' such that $yR = I' \oplus X''$. By the choice of I it follows that $I' = 0$ and hence yR is semisimple. It follows that X' is semisimple. By Theorem 1.6 there exists an index set Λ and indecomposable injective modules I_λ ($\lambda \in \Lambda$) such that $I = \bigoplus_{\lambda \in \Lambda} I_\lambda$. Let $\lambda \in \Lambda$. Suppose I_λ is not simple. Let $z \in I_\lambda$ such that zR is not simple. Then, by (v), zR is injective and hence $I_\lambda = zR$. Let U be a maximal submodule of I_λ. Then U is semisimple by (v), and, because I_λ is uniform, U is simple. Thus I_λ has length 2. It follows that X is a direct sum of uniserial right R-modules. By [5, Theorem 1.3], R is an Artinian serial ring. Moreover, if S is the right socle of R then R/S is a semisimple right R-module, so that $J \subseteq S$ and $J^2 = 0$.

(iv) $\Rightarrow$ (vi) $\Rightarrow$ (vii). Obvious.

(vii) $\Rightarrow$ (ii). Suppose that (vii) holds. By Corollary 4.4, R is right Artinian. Any right ideal is a direct summand of an essential right ideal so is a direct sum of an injective right ideal and a semisimple right ideal by the Krull–Schmidt Theorem. Let C be a nonsimple indecomposable right ideal of R. Then C is injective and hence uniform. Let D be a maximal submodule of C. Then D is semisimple and hence simple. Thus C has length 2. Clearly (ii) follows. This completes the proof of Theorem 4.5.

A ring R is called a *right CS-ring* provided for each right ideal A there exists an idempotent e such that A is essential in eR. The final result is taken from [7, Theorem 4.1 and Corollary 4.4]. The proof can be found in [7].

PROPOSITION 4.6. *The following statements are equivalent for a ring R.*

(i) *Every cyclic right R-module is a direct sum of a projective module and a semisimple module.*

(ii) *Every right ideal of R is the intersection of a direct summand of R and finitely many maximal right ideals.*

(iii) *R is a right CS-ring such that R/E is semisimple for each essential right ideal E of R.*

Acknowledgement

The authors wish to thank the referee for several helpful suggestions. They also wish to thank the organizers of the Semester on Algebra 1988 at the Banach Center, Warsaw, and Dr Jan Krempa in particular.

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