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HOCHSCHILD COHOMOLOGY OF AUSLANDER ALGEBRAS

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Let k be an algebraically closed field of characteristic zero and let A be a finite-dimensional, basic and connected k-algebra. We denote by $\operatorname{mod} A$ the category of finitely generated left A-modules. Moreover, we assume that A is representation-finite. Let $M_1, \ldots, M_d$ be a complete set of representatives of the isomorphism classes of indecomposable left A-modules. Let $\Lambda = \operatorname{End}(\bigoplus_{i=1}^{d} M_i)$ be the associated Auslander algebra. We will concentrate on computing the Hochschild cohomology of Λ.

In Section 1 we recall the relevant information on Auslander algebras and say a few words about Hochschild cohomology. The remaining sections deal with the comparison of the cohomology of Λ with the cohomology of A.

1. Preliminaries

1.1. Let Λ be a finite-dimensional, basic and connected k-algebra. Then it is well known [A] that Λ is an Auslander algebra if and only if $\operatorname{gl.dim} \Lambda \leqslant 2$ and $\operatorname{dom.dim} \Lambda \geqslant 2$. In this case there exists a representation-finite, finite-dimensional, basic and connected k-algebra A such that Λ is the Auslander algebra of A. Let $\vec{\Gamma}(A)$ be the Auslander–Reiten quiver of A and denote by $m(\vec{\Gamma}(A))$ the mesh ideal of $\vec{\Gamma}(A)$ [G]. Let $k(\vec{\Gamma}(A))$ be the path algebra associated with $\vec{\Gamma}(A)$. Note that this algebra may be infinite-dimensional. Then it follows from [BGRS] that $\Lambda \simeq k(\vec{\Gamma}(A))/m(\vec{\Gamma}(A))$. (Note that our assumptions on k imply that A is standard.) This can be reformulated as follows. Let $\operatorname{gr} \Lambda$ be the associated graded algebra of Λ; then $\Lambda \simeq \operatorname{gr} \Lambda$.

Let $M_1, \ldots, M_d$ be a complete set of representatives of the isomorphism classes of indecomposable left A-modules. We may assume that $M_1, \ldots, M_n$ for

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$n \leqslant d$ is a complete set of representatives of the isomorphism classes of indecomposable projective A-modules. Then $\Lambda = \operatorname{End}(\bigoplus_{i=1}^{d} M_i)$.

This notation will be fixed throughout this article.

If X is an indecomposable A-module occurring in the list above we denote by e_X the primitive idempotent of Λ corresponding to X. Let $P_X = \Lambda e_X$ be the corresponding indecomposable projective Λ-module and let $S_X = \operatorname{top}(P_X)$ be the corresponding simple Λ-module. Clearly $P_{M_1}, \ldots, P_{M_d}$ is a complete list of isomorphism classes of indecomposable projective Λ-modules. It is well known and easy to see that $\operatorname{proj.dim} S_X \leqslant 1$ if and only if X is indecomposable projective. In this case let $\operatorname{rad} X = \bigoplus_{i=1}^{t} E_i$ with $E_1, \ldots, E_t$ indecomposable and pairwise nonisomorphic. Then

$$0 \to \bigoplus_{i=1}^{t} P_{E_i} \to P_X \to S_X \to 0$$

is a minimal projective resolution of S_X in $\operatorname{mod} \Lambda$.

Otherwise let

$$0 \to \tau X \to \bigoplus_{i=1}^{r} E_i \to X \to 0$$

be the Auslander–Reiten sequence ending with X, where $E_1, \ldots, E_r$ are indecomposable and pairwise nonisomorphic. Then

$$0 \to P_{\tau X} \to \bigoplus_{i=1}^{r} P_{E_i} \to P_X \to S_X \to 0$$

is a minimal projective resolution of S_X in $\operatorname{mod} \Lambda$.

In particular, we see that $\operatorname{Ext}^1_\Lambda(S_X, S_Y) \neq 0$ if and only if there exists an irreducible map $\alpha: X \to Y$ and that $\operatorname{Ext}^2_\Lambda(S_X, S_Y) \neq 0$ if and only if $Y \simeq \tau X$. We will use these observations in Section 3.

The algebra A is called *representation-directed* if $\vec{\Gamma}(A)$ is a directed quiver. In this case there exists a quiver $\vec{\mathcal{O}}(A)$, called the *orbit quiver* of $\vec{\Gamma}(A)$ [R], such that $\vec{\Gamma}(A)$ can be identified with a full subquiver of $\mathbf{Z}\vec{\mathcal{O}}(A)$. We say that A is *simply connected* if $\vec{\mathcal{O}}(A)$ is a tree.

For different characterizations using the topological realization of $\vec{\Gamma}(A)$ as a two-dimensional simplicial complex or the separation property of the radicals of indecomposable projective A-modules we refer to [BLS] and [BrG].

1.2. Let B be a finite-dimensional k-algebra and let ${}_BX_B$ be a finitely generated B-bimodule. Following [Ho], [CE] we define the *Hochschild cohomology groups* $H^i(B, X)$ of B with coefficients in X by $H^i(B, X) = \operatorname{Ext}^i_{B^e}(B, X)$ where we denote by $B^e = B \otimes_k B^*$ the enveloping algebra (B^* denotes the opposite algebra). Note that a B-bimodule ${}_BX_B$ can be considered in a natural way as a left B^e-module by $(a \otimes b')x = axb$, for $a, b \in B$ and $a \in X$, where b' denotes the element in B^* corresponding to $b \in B$.

Of particular interest to us is the example ${}_B X_B = {}_B B_B$. In this case $H^i(B, B)$ is simply denoted by $H^i(B)$. For some results on computations for $H^i(B)$ we refer to [H] and to the articles mentioned there.

We will need the fact that $\text{proj.dim}_{B^e} B = \text{gl.dim} B$.

For later purposes we recall that $H^0(B)$ is the center of B. The description of $H^1(B)$ is as follows.

Let $\text{Der}(B) = \{\delta \in \text{Hom}_k(B, B) \mid \delta(b_1 b_2) = b_1 \delta(b_2) + \delta(b_1) b_2\}$ be the space of derivations and let $\text{Der}^0(B) = \{\delta_b \colon B \to B \mid \delta_b(b') = b'b - bb'\}$ be the space of inner derivations. Then $H^1(B) \simeq \text{Der}(B)/\text{Der}^0(B)$.

A different description which is easier to use was obtained in [H]. Let $e_1, \ldots, e_n$ be a complete set of primitive orthogonal idempotents in B. Let $\text{Der}^n(B) = \{\delta \in \text{Der}(B) \mid \delta(e_i) = 0 \text{ for } 1 \leqslant i \leqslant n\}$ be the subspace of normalized derivations. By $\text{Der}^{n,0}(B)$ we denote the subspace $\text{Der}^n(B) \cap \text{Der}^0(B)$. Then $\text{Der}^n(B)/\text{Der}^{n,0}(B) \simeq H^1(B)$. Note that a normalized derivation preserves the two-sided Pierce decomposition of B with respect to the chosen set of idempotents.

Finally, recall that $H(B) = \bigoplus_{i \in \mathbf{Z}} H^i(B)$ is a $\mathbf{Z}$-graded algebra where the multiplication is given by the Yoneda product. $H(B)$ is called the *Hochschild cohomology algebra*.

2. The center of Λ

We keep the notation from Section 1.

PROPOSITION. *The canonical algebra map $\Lambda \to A$ induces an isomorphism Φ: $H^0(\Lambda) \to H^0(A)$.*

Proof. Let $f \in \Lambda$. Then $f = \sum_{i,j} f_{ij}$ where $f_{ij} \in \text{Hom}_A(M_i, M_j)$ for $1 \leqslant i, j \leqslant d$. If $f \in H^0(\Lambda)$, then clearly $f_{ij} = 0$ for $i \neq j$. Let $f_i = f_{ii}$. Recall that we have agreed that $M_1, \ldots, M_n$ are indecomposable projective A-modules. Then we clearly obtain a linear map Φ: $H^0(\Lambda) \to H^0(A)$ by setting $\Phi(f) = \sum_{i=1}^n f_i$, and Φ is induced by the canonical map $\Lambda \to A$.

Suppose that $\sum_{i=1}^n f_i = 0$. Let $n < j \leqslant d$. Then there exists $1 \leqslant i \leqslant n$ and a nonzero A-linear map π: $M_i \to M_j$. Since $f \in H^0(\Lambda)$ we infer that $0 = f_i = f\pi = \pi f = \pi f_j$. Thus $f_j = 0$, hence Φ is injective.

Conversely, let $f = \sum_{i=1}^n f_i \in H^0(A)$. Let $n < j \leqslant d$ and let $A^s \xrightarrow{\pi_j^1} A^r \xrightarrow{\pi_j^0} M_j$ be a free presentation of M_j. Then $\pi_j^1 f^r = f^s \pi_j^1$, where f^i: $A^i \to A^i$ is given by $f^i(a_1, \ldots, a_i) = (f(a_1), \ldots, f(a_i))$. So we obtain an induced map f_j: $M_j \to M_j$. Let $\tilde{f} = \sum_{i=1}^d f_i$. By the computation below we infer that $\tilde{f} \in H^0(\Lambda)$ and clearly $\Phi(\tilde{f}) = f$, thus Φ is also surjective.

Let $g \in \Lambda$ with decomposition $g = \sum_{i,j} g_{ij}$, where $g_{ij} \in \text{Hom}_A(M_i, M_j)$. Clearly it is enough to show that $g_{ij} f_j = f_i g_{ij}$.

In fact, $g\tilde{f} = \sum_{i,j} g_{ij} \sum_t f_t = \sum_{i,j} g_{ij} f_j = \sum_{i,j} f_i g_{ij} = \tilde{f} g$.
Let

$$A^s \xrightarrow{\pi_i^1} A^r \xrightarrow{\pi_i^0} M_i \quad \text{and} \quad A^{s'} \xrightarrow{\pi_j^1} A^{r'} \xrightarrow{\pi_j^0} M_j$$

be free presentations. So we obtain maps g_{ij}^1: $A^s \to A^{s'}$ and g_{ij}^0: $A^r \to A^{r'}$ such that $\pi_i^1 g_{ij}^0 = g_{ij}^1 \pi_j^1$ and $g_{ij}^1 \pi_i^0 = \pi_i^0 g_{ij}$. We claim that $f_i g_{ij} = g_{ij} f_j$. For this it is enough to show that $\pi_i^0 g_{ij} f_j = \pi_i^0 f_i g_{ij}$. But

$$\pi_i^0 f_i g_{ij} = f^r \pi_i^0 g_{ij} = f^r g_{ij}^0 \pi_j^0 = g_{ij}^0 f^{r'} \pi_j^0 = g_{ij}^0 \pi_j^0 f_j = \pi_i^0 g_{ij} f_j.$$

This finishes the proof.

3. A minimal projective resolution

Using the information collected in Section 1 and in 1.5 of [H] we immediately get the following statement.

LEMMA. *Let $0 \to R_2 \to R_1 \to R_0 \to \Lambda \to 0$ be a minimal projective resolution of Λ over Λ^e. Then*

$$\mathrm{Hom}_{\Lambda^e}(R_0, \Lambda) = \bigoplus_{i=1}^{d} e_{M_i} \Lambda e_{M_i}, \quad \mathrm{Hom}_{\Lambda^e}(R_1, \Lambda) = \bigoplus_{i \xrightarrow{\alpha} j} e_{M_i} \Lambda e_{M_j},$$

where α is an irreducible map from M_i to M_j, and

$$\mathrm{Hom}_{\Lambda^e}(R_2, \Lambda) = \bigoplus_{i=n+1}^{d} e_{\tau M_i} \Lambda e_{M_i}.$$

We want to write down the complex

$$\mathrm{Hom}_{\Lambda^e}(R_0, \Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(R_1, \Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(R_2, \Lambda)$$

more explicitly. For this we have to introduce some more notation.

Let α: $X \to Y$ be an irreducible map between the indecomposable A-modules X and Y. Then we denote by $\sigma^{-1}(\alpha)$ the irreducible map $\tau Y \to X$, which is uniquely determined up to scalar multiples, and by $\sigma(\alpha)$ the irreducible map $Y \to \tau^- X$, again uniquely determined up to scalar multiples. Moreover, for an indecomposable module X we denote by X^+ the set of isomorphism classes of indecomposable A-modules such that there exists an irreducible map from X to Y in X^+. Similarly we define for an indecomposable module X the set X^- of isomorphism classes of indecomposable A-modules such that there exists an irreducible map from Y in X^- to X.

Since A is standard we can choose for each pair of indecomposable A-modules M_i, M_j such that there exists an arrow α_{ij} from M_i to M_j in $\vec{\Gamma}(A)$ a morphism, again denoted by α_{ij}, satisfying the mesh relations. For simplicity we will assume that $\dim_k \mathrm{Hom}_A(M_i, M_j) \leqslant 1$ whenever there exists an irreducible map from M_i to M_j. It is easily seen that this is satisfied in our applications in Section 5.

Using this and the lemma above it is straighforward to see that δ^0 and δ^1 can be described as follows: Let $f \in e_{M_i} \Lambda e_{M_i}$; then

$$\delta^0(f) = \sum_{M_j \in M_i^+} f\alpha_{ij} - \sum_{M_j \in M_i^-} \alpha_{ji} f.$$

And let $\alpha_{ij} \in e_{M_i} \Lambda e_{M_j}$ be irreducible; then

$$\delta^1(\alpha_{ij}) = \alpha_{ij}\sigma(\alpha_{ij}) - \sigma^{-1}(\alpha_{ij})\alpha_{ij}.$$

We will show in Section 5 that this is very useful for direct computations.

4. Derivations of Λ

Let $\vec{\Delta}$ be a finite and connected quiver without oriented cycles and let $k\vec{\Delta}$ be the corresponding path algebra. Then $k\vec{\Delta}$ is a finite-dimensional hereditary and connected k-algebra. Note that all finite-dimensional hereditary, basic and connected k-algebras are of this form. Let $e_1, \ldots, e_n$ be the complete set of primitive orthogonal idempotents corresponding to the trivial paths. Let α be an arrow in $\vec{\Delta}$; then we denote by $s(\alpha)$ the starting point and by $e(\alpha)$ the endpoint of α. Moreover, by Δ_0 (resp. Δ_1) we denote the set of vertices (resp. of arrows) of $\vec{\Delta}$. We define $\nu\colon \Delta_1 \to \mathbb{N}$ by $\nu(\alpha) = \dim_k(e_{s(\alpha)} k\vec{\Delta} e_{e(\alpha)})$. Then it is easily seen (cf. [H]) that $H^0(k\vec{\Delta}) = k$, $\dim_k H^1(k\vec{\Delta}) = 1 - n + \sum_{\alpha \in \Delta_1} \nu(\alpha)$ and $H^i(k\vec{\Delta}) = 0$ for $i \geqslant 2$. In particular, $H^1(k\vec{\Delta}) = 0$ if and only if $\vec{\Delta}$ is a tree.

THEOREM. *Let Λ be the Auslander algebra of A. If $H^1(\Lambda) = 0$, then A is representation-directed.*

Proof. This was already outlined in [H]. For the convenience of the reader we recall the argument. Our general assumptions on k imply (see Section 1) that $\Lambda \simeq k(\vec{\Gamma}(A))/m(\vec{\Gamma}(A))$. Let w be a path in $\vec{\Gamma}(A)$. Then we denote by $l(w)$ the length of w. It is easily seen that $\delta\colon k(\vec{\Gamma}(A)) \to k(\vec{\Gamma}(A))$ defined by $\delta(w) = l(w)w$ is a normalized derivation having the property that $\delta\big(m(\vec{\Gamma}(A))\big) \subseteq m(\vec{\Gamma}(A))$. Thus δ induces a normalized derivation $\bar{\delta} \in \mathrm{Der}^n(\Lambda)$. By assumption there exists $\lambda \in \Lambda$ such that $\bar{\delta} = \delta_\lambda$. Since $\bar{\delta} \in \mathrm{Der}^n(\Lambda)$ we infer that

$$\lambda = \sum_{i=1}^{d} \mu_i e_{M_i} + \lambda' \quad \text{for } \mu_i \in k \text{ and } \lambda' \in \bigoplus_i e_{M_i}(\operatorname{rad}\Lambda)e_{M_i}.$$

Let α be an arrow from M_i to M_j in $\vec{\Gamma}(A)$ and $\bar{\alpha}$ the residue class of α in Λ. Then $\bar{\alpha} = \bar{\delta}(\bar{\alpha}) = \delta_\lambda(\bar{\alpha}) = \lambda\bar{\alpha} - \bar{\alpha}\lambda = \mu_i\bar{\alpha} - \bar{\alpha}\mu_j + \tilde{\lambda}$, where $\tilde{\lambda} \in \operatorname{rad}^2\Lambda$. Thus $\mu_i - \mu_j = 1$. Now suppose that

$$M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \to \ldots \to M_{r-1} \xrightarrow{\alpha_{r-1}} M_r = M_1$$

is an oriented cycle in $\vec{\Gamma}(A)$. Then we infer that $\mu_i - \mu_{i+1} = 1$ for $1 \leqslant i < r$ and

$\mu_r - \mu_1 = 1$. And it is easily seen that in char$k = 0$ this system of linear equations has no solutions, a contradiction.

THEOREM. *Let A be representation-directed and Λ its Auslander algebra. Moreover, let $\vec{\mathcal{O}}(A)$ be the orbit quiver of $\vec{\Gamma}(A)$. Then $H^1(\Lambda)$ and $H^1(k\vec{\mathcal{O}}(A))$ are isomorphic.*

Proof. Since $\vec{\mathcal{O}}(A)$ is $\tilde{\mathbf{A}}_{1,p}$-free [BB], we know that $\dim(e_X \Lambda e_Y) = 1$ whenever there exists an irreducible map $\alpha\colon X \to Y$ for X, Y indecomposable A-modules.

Let α be an arrow in $\vec{\Gamma}(A)$ which we may consider as an element of Λ and let $\delta \in \mathrm{Der}^n(\Lambda)$. Then there exists $\lambda_\alpha \in k$ such that $\delta(\alpha) = \lambda_\alpha \alpha$. By construction of $\vec{\mathcal{O}}(A)$ we can identify the arrows of $\vec{\mathcal{O}}(A)$ with the arrows of $\vec{\Gamma}(A)$ given by the inclusions $\mathrm{rad}\, P \rightarrowtail P$ for the indecomposable projective A-modules P. Let ψ: $\mathrm{Der}^n(\Lambda) \to \mathrm{Der}^n(k\vec{\mathcal{O}}(A))$ be defined by $\psi(\delta) = \bar{\delta}$, where $\bar{\delta}(\alpha) = \delta(\alpha)$ for an arrow α in $\vec{\mathcal{O}}(A)$.

We claim that ψ is surjective.

In fact, let $\delta \in \mathrm{Der}^n(k\vec{\mathcal{O}}(A))$. We construct $\tilde{\delta} \in \mathrm{Der}^n(\Lambda)$ as follows. Set $\tilde{\delta}(e_{M_i}) = 0$ for $1 \leqslant i \leqslant d$. Let $0 \to M_i \to \bigoplus_{j=1}^r E_j \to M_t \to 0$ be the Auslander–Reiten sequence starting with M_i where $E_1, \ldots, E_r$ are indecomposable and pairwise nonisomorphic. We denote the irreducible maps from M_i to E_j by α_{ij} and those from E_j to M_t by β_{jt} for $1 \leqslant j \leqslant r$. Inductively we may assume that $\tilde{\delta}$ is defined on α_{ij}, say $\tilde{\delta}(\alpha_{ij}) = \lambda_{ij}\alpha_{ij}$ for $\lambda_{ij} \in k$. Then set $\tilde{\delta}(\beta_{it}) = (\sum_{i \neq j} \lambda_{ij})\beta_{jt}$. Then $\tilde{\delta}$ extends to a well-defined normalized derivation, again denoted by $\tilde{\delta}$, of Λ such that the restriction $\psi(\tilde{\delta})$ coincides with δ.

Next suppose that $\delta \in \mathrm{Der}^n(\Lambda)$ is such that $\psi(\delta)$ is inner. We claim that $\delta \in \mathrm{Der}^{n,0}(\Lambda)$.

Clearly it is enough to show that if

$$0 \to M_i \xrightarrow{(\alpha_{ij})} \bigoplus_{j=1}^{r} E_j \xrightarrow{(\beta_{jt})} M_t \to 0$$

is an Auslander–Reiten sequence such that $\delta(\alpha_{ij}) = 0$ for $1 \leqslant j \leqslant r$, then there exists $\mu \in k$ such that $(\delta - \delta_{\mu e_{M_t}})(\beta_{jt}) = 0$ for $1 \leqslant j \leqslant r$. Let $\delta(\beta_{jt}) = \lambda_{jt}\beta_{jt}$ for $1 \leqslant j \leqslant r$. Then $\lambda_{1t} = \ldots = \lambda_{rt} = \lambda$, since

$$0 = \delta\Big(\sum_{j=1}^{r} \alpha_{ij}\beta_{jt}\Big) = \sum_{j=1}^{r} \delta(\alpha_{ij}\beta_{jt}) = \sum_{j=1}^{r} \lambda_{jt}\alpha_{ij}\beta_{jt}.$$

So choose $\mu = \lambda$. Then $(\delta - \delta_{\mu e_{M_t}})(\beta_{jt}) = \lambda_{jt}\beta_{jt} - \mu\beta_{jt} = 0$ for $1 \leqslant j \leqslant r$.

Let $\delta \in \mathrm{Der}^{n,0}(\Lambda)$. Then it is straightforward to see that $\psi(\delta) \in \mathrm{Der}^{n,0}(k\vec{\mathcal{O}}(A))$. In fact, let $\delta = \delta_\lambda$, where $\lambda = \sum_{i=1}^{d} \mu_i e_{M_i} \in \Lambda$. Let j be a vertex of $\vec{\mathcal{O}}(A)$. Let $\alpha_{1j}, \ldots, \alpha_{rj}$ (resp. $\beta_{j1}, \ldots, \beta_{js}$) be the arrows of $\vec{\mathcal{O}}(A)$ starting (resp. ending) at j. Let $M_{1j}, \ldots, M_{tj}$ (resp. $M_{j1}, \ldots, M_{js}$) be the indecomposable A-modules which are sources (resp. targets) of the irreducible maps corresponding to these arrows. Let $v_j = \sum_{i=1}^{r} \mu_{ij} + \sum_{i=1}^{s} \mu_{ji}$. Then $\psi(\delta) = \sum_j \delta_{v_j} \in \mathrm{Der}^{n,0}(k\vec{\mathcal{O}}(A))$.

Summarizing our calculations then shows the assertion.

COROLLARY. *Let A be representation-finite and Λ its Auslander algebra. Then $H^1(\Lambda) = 0$ if and only if A is simply connected.*

5. Computations for $H^2(\Lambda)$

In Section 3 we have constructed a minimal projective resolution of the Auslander algebra Λ over its enveloping algebra Λ^e.

THEOREM. *Let A be representation-finite and Λ its Auslander algebra. If A is representation-directed, then $H^2(\Lambda) = 0$.*

Proof. We consider the map

$$\delta^1 \colon \bigoplus_{i \xrightarrow{\alpha} j} e_{M_i} \Lambda e_{M_i} \to \bigoplus_{i=n+1}^{d} e_{\tau M_i} \Lambda e_{M_i}$$

defined in Section 3. We will show that δ^1 is surjective, hence $H^2(\Lambda) = 0$. Let $f \in e_{\tau M_i} \Lambda e_{M_i}$. We may assume that $f = \alpha_{tj} \beta_{ji}$ for some j, where

$$0 \to M_t \xrightarrow{(\alpha_{tj})} \bigoplus_{j=1}^{r} E_j \xrightarrow{(\beta_{ji})} M_i \to 0$$

is the Auslander–Reiten sequence ending with M_i. The σ-orbit of β_{ji} is finite. Then by construction we infer that $\delta^1(\sum_{r>0} \sigma^{-r}(\beta_{ji})) = \alpha_{tj} \beta_{ji} = f$.

COROLLARY. *Let A be representation-finite and Λ its Auslander algebra. Then the Hochschild cohomology algebra of Λ is trivial if and only if A is simply connected.*

This can be reformulated as follows. Let A be representation-finite. We denote by $n(A)$ the number of vertices of $\vec{\Gamma}(A)$ (thus the number of isomorphism classes of indecomposable A-modules), by $s(A)$ the number of arrows of $\vec{\Gamma}(A)$ and by $m(A)$ the sum $\sum_{i=n+1}^{d} r_i$ where $r_i + 1$ is the number of isomorphism classes of indecomposable direct summands of the middle term of the Auslander–Reiten sequence ending with M_i.

COROLLARY. *Suppose that A is representation-directed. Then A is simply connected if and only if $1 = n(A) - s(A) + m(A)$.*

Proof. With the notation above the complex

$$\mathrm{Hom}(R_0, \Lambda) \xrightarrow{\delta^0} \mathrm{Hom}(R_1, \Lambda) \xrightarrow{\delta^1} \mathrm{Hom}(R_2, \Lambda)$$

constructed in Section 3 reduces to

$$k^{n(A)} \xrightarrow{\delta^0} k^{s(A)} \xrightarrow{\delta^1} k^{m(A)}.$$

By the theorem above we know that δ^1 is surjective. Since $H^0(\Lambda) = \ker\delta^0 = k$ we infer that $\dim H^1(\Lambda) = s(A) - m(A) - n(A) + 1$. So the assertion follows from the corollary in Section 4.

We refer to [H] for some different examples for the computation of $H^2(\Lambda)$.

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