TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 2
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

JONES' TRACE FUNCTION ON THE HECKE ALGEBRA OF SYMMETRIC GROUPS

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This note is mainly expository. It discusses the trace function of the title (see [J]), as much as possible from the point of view offered by the general theory of Weyl groups and their Hecke algebras.

1.

Let $\Sigma_n = (s_i)_{1 \leqslant i \leqslant n-1}$ be the set of canonical generators of the symmetric group S_n. So s_i is the transposition $(i, i+1)$. Denote by l the length function on S_n defined by Σ_n. We view S_{n-1} as a subgroup of S_n, with generators $\Sigma_{n-1} = (s_i)_{1 \leqslant i \leqslant n-2}$.

There is a distinguished set of representatives D_n for the cosets wS_{n-1} consisting of the elements of minimal length in their coset (see [B, p. 37]). They are the elements $s_i \dots s_{n-1}$ $(1 \leqslant i \leqslant n-1)$ and the identity. There are only two distinct double cosets $S_{n-1} w S_{n-1}$, viz. S_{n-1} and $S_{n-1} s_{n-1} S_{n-1}$.

Let H_n be the (generic) Hecke algebra of the Coxeter group (S_n, Σ_n). It is an algebra over the polynomial ring $Q[q]$, with generators $(e_i)_{1 \leqslant i \leqslant n-1}$, subject to the relations

$$e_i^2 = (q-1)e_i + q,$$

$$e_i e_j = e_j e_i \quad \text{if } |i-j| \geqslant 2,$$

$$e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1} \quad \text{if } 1 \leqslant i \leqslant n-2.$$

(If one specializes q to 1 one gets the relations defining the group algebra $Q[S_n]$.)

If $w = s_{i_1} \dots s_{i_l}$ is a shortest expression for $w \in S_n$ (so $l(w) = l$) we put $e_w = e_{i_1} \dots e_{i_l}$, this is independent of the choice of the reduced expression. Then

This paper is in final form and no version of it will be submitted for publication elsewhere.

$(e_w)_{w\in S_n}$ is a basis of H_n over $Q[q]$. We view H_{n-1} as the subalgebra of H_n generated by $(e_i)_{1\leq i\leq n-2}$.

We have $e_{xy} = e_x e_y$ if $x, y\in S_n$, $l(xy) = l(x)+l(y)$.

Let M be the submodule of H_n spanned by the e_w with $w\in S_{n-1}s_{n-1}S_{n-1}$. It is an (H_{n-1}, H_{n-1})-bimodule. Also, H_{n-1} is an H_{n-2}-bimodule, in the obvious way.

LEMMA 1. *There is a map* $\varphi: H_{n-1}\otimes_{H_{n-2}} H_{n-1}\to M$ *with* $\varphi(u\otimes v) = ue_1 v$. *It is an isomorphism of* (H_{n-1}, H_{n-1})*-bimodules.*

This is an application of the following general result, proved in [C, p. 75]. Let (W, S) be a Coxeter group. If J is a subset of S, denote by W_J the subgroup of W generated by J. Each coset wW_J contains a unique element of minimal length, let $D_J = D_J^S$ be the set of these elements. Similarly, if $I, J\subset S$ each double coset $W_I wW_J$ contains a unique element of minimal length, let $D_{I,J}$ be the set of them. In these circumstances one has the following

LEMMA 2. (i) *If* $d\in D_{I,J}$ *then* $W_I\cap dW_Jd^{-1} = W_{I\cap dJd^{-1}}$.

(ii) *Any element of* W *can be uniquely written in the form* $w = d'dx$, *with* $x\in W_J$, $d\in D_{I,J}$, $d'\in D^I_{I\cap dJd^{-1}}$. *Moreover* $l(w) = l(d')+l(d)+l(x)$.

Lemma 1 follows by applying Lemma 2 in the case that $(W, S) = (S_n, \Sigma_n)$, $I = J = \Sigma_{n-1}$, $d = s_{n-1}$. By part (i) we have $S_{n-1}\cap s_{n-1}S_{n-1}s_{n-1} = S_{n-2}$, moreover S_{n-2} centralizes s_{n-1}. It then follows readily that there is a well-defined map φ as in Lemma 1. That it is an isomorphism follows by applying part (ii). We skip the details of the argument.

2. The trace function

Let z be another indeterminate. The following theorem establishes the existence of Jones' trace function on H_n.

THEOREM 1. *There exists a unique* $Q[q]$*-linear function* τ_n *on* H_n, *with values in* $Q[q, z]$, *such that for* $n\geqslant 2$

(a) $\tau_n(1) = 1$,

(b) $\tau_n|_{H_{n-1}} = \tau_{n-1}$,

(c) $\tau_n(vu) = \tau_n(uv)$ $(u, v\in H_n)$,

(d) $\tau_n(ue_{n-1}v) = z\tau_{n-1}(uv)$ $(u, v\in H_{n-1})$.

We prove the theorem by induction on n, starting with $n = 1$ (where $H_1 = Q[q]$). Properties (b) and (d) define τ_n uniquely on H_n, assuming τ_{n-1} to be known (here one uses that $|S_{n-1}\backslash S_n/S_{n-1}| = 2$, and also Lemma 1). It remains to prove property (c). Clearly, it suffices to prove that $\tau_n(e_i u) = \tau_n(ue_i)$ for $i = 1, \ldots, n-1$, $u\in H_n$. For $i < n-1$ this follows from the induction assumption and for $i = n-1$, $u\in H_{n-1}$ the same holds. So, finally, we have to prove that for $u, v \in H_{n-1}$

$$\tau_n(e_{n-1}ue_{n-1}v) = \tau_n(ue_{n-1}ve_{n-1}). \tag{1}$$

If $u \in H_{n-2}$ we have, since u and e_{n-1} commute,

$$\tau_n(e_{n-1}ue_{n-1}v) = \tau_n(ue_{n-1}^2v) = (q-1)\tau_n(ue_{n-1}v) + q\tau_n(uv),$$

whence

$$\tau_n(e_{n-1}ue_{n-1}v) = ((q-1)z+q)\tau_n(uv) \qquad (u \in H_{n-2},\ v \in H_{n-1}). \tag{2}$$

If $u = u'e_{n-2}u'' \in H_{n-2}e_{n-2}H_{n-2}$ then

$$\tau_n(e_{n-1}ue_{n-1}v) = \tau_n(u'e_{n-1}e_{n-2}e_{n-1}u''v) = \tau_n(u'e_{n-2}e_{n-1}e_{n-2}u''v),$$

whence

$$\tau_n(e_{n-1}ue_{n-1}v) = (q-1)z\tau_{n-1}(uv) + qz\tau_{n-1}(u'u''v), \tag{3}$$

for $u = u'e_{n-2}u''$ with $u', u'' \in H_{n-2}$ and $v \in H_{n-1}$.

Similarly,

$$\tau_n(ue_{n-1}ve_{n-1}) = ((q-1)z+q)\tau_n(uv) \qquad (u \in H_{n-1},\ v \in H_{n-2}), \tag{4}$$

$$\tau_n(ue_{n-1}ve_{n-1}) = (q-1)z\tau_{n-1}(uv) + qz\tau_{n-1}(uv'v''), \tag{5}$$

for $u \in H_{n-1}$ and $v = v'e_{n-2}v''$ with $v', v'' \in H_{n-2}$.

We see from (2) and (4) that (1) holds if $u, v \in H_{n-2}$. If $u \in H_{n-2}$, $v = v'e_{n-2}v'' \in H_{n-2}e_{n-2}H_{n-2}$ then (2) gives

$$\begin{aligned}\tau_n(e_{n-1}ue_{n-1}v) &= ((q-1)z+q)\tau_{n-1}(uv'e_{n-2}v'')\\ &= (q-1)z\tau_{n-1}(uv) + qz\tau_{n-2}(uv'v''),\end{aligned}$$

and by (5) we again have (1). Similarly in the case where $u \in H_{n-2}e_{n-2}H_{n-2}$, $v \in H_{n-2}$. The last case is that $u, v \in H_{n-2}e_{n-2}H_{n-2}$. By (3) we then have, with obvious notations,

$$\tau_n(e_{n-1}ue_{n-1}v) = (q-1)z\tau_{n-1}(uv) + qz^2\tau_{n-2}(u'u''v'v''),$$

and (5) implies that this also equals $\tau_n(ue_{n-1}ve_{n-1})$. The theorem is proved.

3. Some properties

We shall establish now some properties of τ_n, to be needed for its identification in Section 4. We put

$$c_n = \sum_{y \in S_n} \tau_n(e_y),$$

and we write $\zeta = 1 - z^{-1}(q-1)$.

LEMMA 3. *For all $n \geqslant 2$ we have*

$$c_n = z^n (q-1)^{-n} \prod_{i=1}^{n} (q^{i-1} - \zeta).$$

This is trivial for $n = 2$. The general case follows from the inductive formula

$$(1) \qquad c_n = \bigl(1 + z(1 + q + \dots + q^{n-2})\bigr) c_{n-1} \qquad (n \geqslant 3).$$

To prove (1), we first show that for $x, y \in S_n$

$$(2) \qquad \sum_{y \in S_n} \tau_n(e_x e_y) = q^{l(x)} c_n.$$

This is proved by induction on $l(x)$, starting with $x = 1$. Let $l(x) = m \geqslant 1$ and assume that $x = sw$ where $l(w) = m-1$. Then $e_x = e_s e_w$ and

$$\begin{aligned}\sum_{y \in S_n} \tau_n(e_x e_y) &= \sum_{y \in S_n} \tau_n(e_s e_w e_y) = \sum_{y \in S_n} \tau_n(e_w e_y e_s)\\ &= \sum_{\substack{y \in S_n\\ l(ys) > l(y)}} \tau_n(e_w e_{ys}) + (q-1) \sum_{\substack{y \in S_n\\ l(ys) < l(y)}} \tau_n(e_w e_y) + q \sum_{\substack{y \in S_n\\ l(ys) < l(y)}} \tau_n(e_w e_{ys})\\ &= q \sum_{y \in S_n} \tau_n(e_w e_y),\end{aligned}$$

from which (3) follows.

We now have, with the notation of Section 2,

$$\begin{aligned}c_n &= \sum_{y \in S_{n-1}} \tau_n(e_y) + \sum_{\substack{y \in S_{n-1}\\ d \in D_{n-1}}} \tau_n(e_d e_{n-1} e_y)\\ &= c_{n-1} + z \sum_{\substack{y \in S_{n-1}\\ d \in D_{n-1}}} \tau_{n-1}(e_d e_y)\\ &= c_{n-1} + z\Bigl(\sum_{d \in D_{n-1}} q^{l(d)}\Bigr) c_{n-1},\end{aligned}$$

by (2). Since the element $s_i \dots s_{n-2}$ of D_{n-1} (see Section 2) has length $n-1-i$ formula (1) holds. This proves Lemma 3.

For $1 \leqslant p \leqslant n-1$ define a homomorphism φ: $S_p \times S_{n-p} \to S_n$ by

$$\varphi(x, y) = \begin{cases} x.i & \text{if } 1 \leqslant i \leqslant p \\ y.(i-p)+p & \text{if } p+1 \leqslant i \leqslant n. \end{cases}$$

LEMMA 4. *For $x \in S_p$, $y \in S_{n-p}$ we have*

$$\tau_n(e_{\varphi(x,y)}) = \tau_p(e_x) \tau_{n-p}(e_y).$$

We use induction on n. If $y \in S_{n-p}$ has the form $y' s_{n-p-1} y''$ with $y', y'' \in S_{n-p-1}$, we see that

$$e_{\varphi(x,y)} = e_{\varphi(x,y')} e_{n-1} e_{\varphi(1,y'')},$$

and $\varphi(x, y')$, $\varphi(1, y'') \in S_{n-1}$. By property (d) of τ_n,

$$\tau_n(e_{\varphi(x,y)}) = z\tau_{n-1}(e_{\varphi(x,y')}e_{\varphi(1,y'')}),$$

which by induction may assumed to be equal to

$$z\tau_p(e_x)\tau_{n-p-1}(e_{y'}e_{y''}) = \tau_p(e_x)\tau_{n-p}(e_y).$$

This proves the lemma in that case. If $y \in S_{n-p-1}$ the proof is easier and may be omitted.

4. Identification of τ_n

The algebra $H_n \otimes Q(q)$ is semisimple. Its absolutely irreducible representations can be realized over the field $Q(q)$ (see [BC]). Their isomorphism classes can be parametrized by partitions λ of n, in such a way that under the specialization $q \to 1$ (suitably defined) one recovers the irreducible representations of $Q[S_n]$ parametrized by the same partition such that $\lambda = (n)$ corresponds to the trivial representation of S_n (see [M, Ch. I, no. 7]). The corresponding representation of H_n sends e_w to $q^{l(w)}$.

Let N_λ be an $H_n \otimes Q(q)$-module affording the representation parametrized by λ and define a linear function X_λ on H_n by

$$X_\lambda(e_w) = \mathrm{Tr}(e_w, N_\lambda).$$

Property (c) of τ_n implies, by generalities about semisimple algebras, the existence of elements $\alpha_\lambda \in Q(q, z)$ such that

$$\tau_n = \sum_{|\lambda| = n} \alpha_\lambda X_\lambda. \tag{1}$$

(For notations regarding partitions see [M].) The multiplicative property of Lemma 4 shows that for $1 \leqslant p \leqslant n-1$, $x \in S_p$, $y \in S_{n-p}$ we have

$$\sum_{\substack{|\mu| = p \\ |\nu| = n-p}} \alpha_\mu \alpha_\nu X_\mu(e_x) X_\nu(e_y) = \sum_{|\lambda| = n} \alpha_\lambda X_\lambda(e_{\varphi(x,y)}). \tag{2}$$

Denote by χ_λ the character of S_n corresponding to λ. Define the Littlewood–Richardson coefficients $c^\lambda_{\mu\nu}$ by

$$\chi_\lambda(\varphi(x, y)) = \sum_{\substack{|\mu| = p \\ |\nu| = n-p}} c^\lambda_{\mu\nu} \chi_\mu(x) \chi_\nu(y),$$

where $x \in S_p$, $y \in S_{n-p}$ (see [M, Ch. I, no. 9]). There is a "generization" of this formula, namely

$$X_\lambda(e_{\varphi(x,y)}) = \sum_{\substack{|\mu| = p \\ |\nu| = n-p}} c^\lambda_{\mu\nu} X_\mu(e_x) X_\nu(e_y),$$

which follows from the results of [BC]. Inserting this formula into (2) and using that the functions X_λ on $H_n \otimes Q(q)$ are linearly independent (which follows from their definition) we see that if $|\mu| = p$, $|\nu| = n-p$, we have

$$\alpha_\mu \alpha_\nu = \sum_{|\lambda|=n} c^\lambda_{\mu\nu} \alpha_\lambda.$$

Let S be the ring of symmetric functions (see [M, Ch. I]. The preceding formula shows that the linear map φ which sends the S-function s_λ of [loc. cit.] to α_λ is a homomorphism $S \to Q(q, z)$. Next we notice the following, where ζ is as in Lemma 3.

LEMMA 5.

$$\alpha_{(n)} = z^n \prod_{i=1}^{n} \left(\frac{q^{i-1} - \zeta}{q^i - 1} \right).$$

We have the orthogonality relations for the X_λ

$$\sum_{w \in S_n} q^{-l(w)} X_\lambda(e_w) X_{\lambda'}(e_{w^{-1}}) = 0 \qquad (\lambda \neq \lambda'),$$

see e.g. [L, p. 62]. We apply this for $\lambda' = (n)$. Since $X_{(n)}(e_{w^{-1}}) = q^{l(w)}$, we get

$$\sum_{w \in S_n} X_\lambda(e_w) = 0 \qquad \text{if } \lambda \neq (n),$$

and (1) shows that

$$c_n = \sum_{w \in S_n} \tau_n(e_w) = \alpha_{(n)}\Big(\sum_{w \in S_n} q^{l(w)}\Big).$$

Since, as is well-known,

$$\sum_{w \in S_n} q^{l(w)} = (q-1)^{-n} \prod_{i=1}^{n} (q^i - 1),$$

the asserted formula follows from Lemma 3.

Our homomorphism is completely determined by the $\varphi(s_{(n)}) = \alpha_{(n)}$. The results of [M, Ch. I, no. 3, no. 7] imply that if $|\lambda| = n$,

$$\alpha_\lambda = \det(\alpha_{(\lambda_i - i + j)})_{1 \leqslant i,j \leqslant n}.$$

There is a multiplicative formula for α_λ, which is perhaps more explicit. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition, let $n(\lambda) = \sum_{i \geqslant 1} (i-1)\lambda_i$. Viewing λ as a set of lattice points in the plane as in [M, Ch. I] define for $x = (i, j) \in \lambda$ the hook length by $h(x) = \lambda_i + \lambda'_j - i - j + 1$ (where λ' is the dual partition) and the content by $c(x) = j - i$. From [M, Ch. I, no. 2 ex. 5 and no. 3 ex. 3], together with Lemma 5, we then obtain

THEOREM 2.

$$\alpha_\lambda = q^{n(\lambda)} \prod_{x \in \lambda} \left(\frac{z(q^{c(x)} - 1) + (q-1)}{q^{h(x)} - 1} \right).$$

The formula is due to A. Ocneanu (see [J]).

5. Comments

It is natural to ask whether there exists an analogue of Theorem 1 for other classes of Weyl groups. The construction of τ_n uses that $|S_{n-1}\backslash S_n/S_{n-1}| = 2$. This property characterizes the pair (S_n, S_{n-1}), as the following result shows.

Let (W, S) be a Weyl group. We assume W to be irreducible. If $J \subset S$ we denote, as in no. 1, by W_J the parabolic subgroup generated by J.

PROPOSITION 1. *If $J \subset S$ is such that $|W_J\backslash W/W_J| = 2$ then the pair (W, W_J) is isomorphic to (S_n, S_{n-1}).*

It is immediate that if $|W_J\backslash W/W_J| = 2$, the parabolic group W_J is maximal, i.e. there is $s \in S$ such that $J = S - \{s\}$.

Assume that W is the Weyl group of a root system R in a real vector space V. For $\beta \in R$ denote by s_β the reflection in V defined by it. There is a basis B of R such that $S = \{s_\beta | \beta \in B\}$. So $s = s_\alpha$, with $\alpha \in B$. Denote by w_0 the longest element of W and by ι the opposition involution of B, i.e. $\iota\beta = -w_0\beta$ $(\beta \in B)$. The fundamental weights defined by B are denoted by π_β.

Since $|W_J\backslash W/W_J| = 2$, w_0 and s lie in the same double coset, so $w_0 = w'sw''$, with $w', w'' \in W_J$. Using that $W_J\pi_\alpha = \pi_\alpha$, $s\pi_\alpha = \pi_\alpha - \alpha$, we see that

$$-\pi_{\iota\alpha} = w_0\pi_\alpha = w'sw''\pi_\alpha = w'(\pi_\alpha - \alpha) = \pi_\alpha - \tilde{\alpha}$$

with $\tilde{\alpha} \in R$, i.e.

$$\tilde{\alpha} = \pi_\alpha + \pi_{\iota\alpha}, \tag{1}$$

and $\tilde{\alpha}$ is a dominant weight. If $\iota\alpha = \alpha$ then $\tilde{\alpha} = 2\pi_\alpha$, which can only be if R is of type C_n for some $n \geqslant 2$ and $\tilde{\alpha}$ is the longest root in R. But now w_0 is central and clearly $w_0 s \notin W_J w_0 W_J$, whence $|W_J\backslash W/W_J| \geqslant 3$.

If $\iota\alpha \neq \alpha$ all roots have the same length and $\tilde{\alpha}$ is the highest root. From (1) we see that the affine Dynkin graph defined by B is a cycle, hence it is of type A. The proposition then readily follows.

The proposition indicates that analogues of Theorem 1 for other Weyl groups could be somewhat more complicated to deal with. One should consider such analogues in the following framework.

Consider families F of triples (W, S, J), where (W, S) is a Weyl group and $J \subset S$, such that the following holds:

(a) if $(W, S, J) \in F$ and $s \in J$ then, putting $S' = S - \{s\}$, $W' = W_{S'}$, there is $J' \subset S'$ such that $(W', S', J') \in F$;

(b) if $(W, S, J) \in F$ there is a $Q[q]$-linear map τ_W of the Hecke algebra H_W of (W, S) to $Q[q, z]$ such that

$$\tau_W(uv) = \tau_W(vu) \quad (u, v \in H_W)$$

and that, with the notations of (a)

$$\tau_W|_{H_{W'}} = \tau_{W'}, \quad \tau_W(ue_s v) = z\tau_{W'}(uv) \quad (u, v \in H_{W'}).$$

(Here e_s is the generator of H_W defined by $s \in S$.)

The problem is to construct such families. Theorem 1 exhibits one, viz. $(S_n, \Sigma_n, \{s_{n-1}\})$ (notations of Section 1). One can deduce that $(S_n, \Sigma_n, \Sigma_n)$ is also one.

The question arises whether the family (W, S, S), where (W, S) is any Weyl group, has the properties of (b).

A more modest question is whether a family F exists whose Weyl groups are the ones of type B_n $(= C_n)$, resp. D_n $(n \geqslant 3)$. Notice that if W is a Weyl group of type B_n (resp. D_n) and W' the parabolic subgroup of type B_{n-1} (resp. D_{n-1}) we have $|W' \backslash W/W'| = 3$.

I do not know the answer to these questions.

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