PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 19
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1987

BOUNDARY INTEGRAL EQUATIONS AND MODIFIED GREEN'S FUNCTIONS

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§ 1. Preamble

This article is essentially a survey of recent work consisting of two main parts; in § 1 to § 5 the boundary integral equation method for solving elliptic boundary value problems is described in some detail as much to introduce the required notation as to highlight a number of fundamental difficulties associated with the method. The remaining sections of the article are concerned with a means of removing these difficulties, presenting a constructive method of solution and indicating areas of research activity.

§ 2. Introduction

Many physically significant problems can be represented in the form of a linear elliptic boundary value problem and as such they tend to fall into one or other of two main classes; those governed by Laplace (type) equations and those by Helmholtz (type) equations for which certain differences of approach are required. However, for both classes it is well known [5], [8], [22] that if the exact Green's function is known for the problem in question then that problem has a solution which can be represented quite simply as an integral taken over the region of interest. Unfortunately, however, the exact Green's function is not an easy quantity to obtain in practice.

An alternative approach is to use, instead of the exact Green's function, a fundamental solution of the equation in question. This two-point function of position, sometimes also called a "free space" Green's function, satisfies the same equation as the exact Green's function but is not required to satisfy any boundary conditions other than possibly some regularity condition at infinity. This extra freedom associated with the fundamental solution allows notions of volume and surface potentials to be introduced and these quanti-

ties in turn have properties which allow the given boundary value problem to be replaced by an equivalent integral equation.

When treating exterior problems, that is, problems defined in the unbounded region exterior to a closed bounded boundary surface, the volume potentials are defined over this unbounded region and often present considerable technical difficulties. The surface potentials are therefore much to be preferred for two main reasons. First, all integrals could now be defined over a bounded region, the boundary of the unbounded region of interest. Second, an elliptic boundary value problem can be given an equivalent representation as a Fredholm equation of the second kind and such equations are particularly attractive from the numerical analysis point of view. These surface potentials form the basis of the so-called boundary integral equation method.

Attractive as it may appear to be at first sight, the boundary integral equation method does present a number of intriguing problems which have variously either entertained or tantalized but which have certainly provided a good living for a number of mathematicians over the years.

In this article we shall give a flavour of the boundary integral equation method by considering exterior problems associated with the Helmholtz equation. Such problems arise in a number of areas of mathematical physics: notably acoustics, electromagnetism and elasticity. The reason for concentrating on the Helmholtz equation is that the associated boundary integral equation method has associated with it certain problems which simply do not present themselves in an equivalent discussion of the Laplace equation.

There are essentially two ways of obtaining the boundary integral equations equivalent to a given elliptic boundary value problem; by either the layer theoretic method or the Green's theorem method. In the layer theoretic method boundary integral equations of the second kind are derived on the basis of the assumption that the required solution of the given boundary value problem can be represented in the form of either a single layer or a double layer surface potential. This leads to boundary integral equations for the exterior Neumann (Dirichlet) and interior Dirichlet (Neumann) problems which are adjoint in an L_2 sense. Alternatively, boundary integral equations of the second kind can be derived from Green's theorem and these equations are related to but not identical with those derived via the layer approach. Whilst it might be argued that the Green's theorem approach is more general than the layer theoretic method in the sense that in the former no *a priori* assumptions regarding the form of the required solution are made, nevertheless, the resulting boundary integral equations exhibit in each case the same difficulties, particularly when subjected to numerical analysis. We shall discuss these difficulties in the following sections indicating how they arise and how they are being resolved. Before doing this, however, we shall describe in some detail how the various boundary integral equations are obtained by the two approaches.

§ 3. Formulation of boundary integral equations

Let B_- denote a bounded open domain in $\boldsymbol{R}^3$ with a smooth closed, bounded boundary ∂B. We denote by B_+ the exterior of ∂B, that is, the complement in $\boldsymbol{R}^3$ of $\bar{B} = B_- \cup \partial B$.

Let $R \equiv R(P, Q)$ denote the distance between two typical points $P, Q \in \boldsymbol{R}^3$. A fundamental solution of the Helmholtz equation

$$(\Delta + k^2)u = 0, \quad k^2 > 0,$$

is a two-point function of position in $\boldsymbol{R}^3$ denoted by $\gamma(P, Q)$ which for convenience we take in the form

$$\gamma(P, Q) = -\frac{e^{ikR}}{2\pi R}, \quad P, Q \in \boldsymbol{R}^3. \tag{3.1}$$

Typical of the exterior problems associated with the Helmholtz equation are the following. Determine $u(P)$ satisfying

$$(\Delta + k^2)u(P) = 0, \quad P \in B_+, \tag{3.2}$$

$$\lim_{r_P \to \infty} \left\{ r_P \left[\frac{\partial u}{\partial r_P}(P) - iku(P) \right] \right\} = 0 \tag{3.3}$$

and either

$$u(p) = f(p), \quad p \in \partial B \quad \text{(Dirichlet problem)} \tag{3.4}$$

or

$$\frac{\partial u}{\partial n_p}(p) = g(p), \quad p \in \partial B \quad \text{(Neumann problem)}, \tag{3.5}$$

$$\frac{\partial u}{\partial n_p}(p) + \sigma u(p) = h(p), \quad p \in \partial B \quad \text{(Robin problem)}. \tag{3.6}$$

Here $\partial/\partial n_p$ denotes differentiation in the direction of the unit normal $\hat{n}_p$ normal to ∂B at the point $p \in \partial B$ and $P \in B_+$ is assumed to have spherical polar coordinates $(r_P, \theta_P, \varphi_P)$ relative to a Cartesian coordinate system erected with origin in B_-. We emphasize that throughout we shall assume that $\hat{n}_p$ is the outward drawn normal with respect to B_-, that is, $\hat{n}_p$ is directed from ∂B into B_+. Further, we shall write $\partial/\partial n_p^-$ and $\partial/\partial n_p^+$ to denote the normal derivative in the limit as $P \to p \in \partial B$ from B_- and B_+, respectively.

We now define

$$(S\mu)(P) = \int_{\partial B} \gamma(P, q)\, \mu(q)\, dS_q, \quad P \in \boldsymbol{R}^3 \setminus \partial B, \tag{3.7}$$

$$(Dv)(P) = \int_{\partial B} \frac{\partial \gamma}{\partial n_q}(P, q)\, v(q)\, dS_q, \quad P \in \boldsymbol{R}^3 \setminus \partial B, \tag{3.8}$$

which we shall refer to as *single* and *double layer potentials*, respectively, with continuous densities μ and ν as indicated.

It can be shown [5], [8], [22] that as $P \to p \in \partial B$ the single and double layer potentials satisfy certain jump relations across the boundary ∂B. Introducing the notation

$$(K\mu)(p) = \int_{\partial B} \frac{\partial \gamma}{\partial n_p}(p, q)\,\mu(q)\,dS_q, \qquad p \in \partial B, \tag{3.9}$$

$$(K^*\mu)(p) = \int_{\partial B} \frac{\partial \bar{\gamma}}{\partial n_q}(p, q)\,\mu(q)\,dS_q, \qquad p \in \partial B, \tag{3.10}$$

$$(D_n\nu)(p) = \frac{\partial}{\partial n_p}\int_{\partial B} \frac{\partial \gamma}{\partial n_q}(p, q)\,\nu(q)\,dS_q, \qquad p \in \partial B, \tag{3.11}$$

the jump conditions can now be conveniently written in the form

$$\frac{\partial}{\partial n_p^{\pm}}(S\mu)(p) = (\pm I + K)\,\mu(p), \qquad p \in \partial B, \tag{3.12}$$

$$\lim_{P \to p^{\pm}} Dv(P) = (\mp I + \bar{K}^*)\,\nu(p), \qquad p \in \partial B. \tag{3.13}$$

Furthermore, we notice that

$$(Dv)(p) = (\bar{K}^*\nu)(p), \qquad p \in \partial B, \tag{3.14}$$

$$\frac{\partial}{\partial n_p^+}(Dv)(p) = \frac{\partial}{\partial n_p^-}(Dv)(p) = (D_n\nu)(p), \qquad p \in \partial B. \tag{3.15}$$

With this notation representations of solutions of the Helmholtz equation obtained by applying Green's Theorem (Green's second identity)

$$\int_B \{u\Delta v - v\Delta u\}\,d\tau_p = \int_{\partial B}\left\{u(p)\frac{\partial v}{\partial n_p}(p) - v(p)\frac{\partial u}{\partial n_p}\right\}(p)\,dS_p \tag{3.16}$$

have the following forms. For radiating wave functions, that is, functions φ_+ satisfying (3.2) and (3.3), we obtain

$$\left(S\frac{\partial \varphi_+}{\partial n}\right)(P) - (D\varphi_+)(P) = \begin{cases} 2\varphi_+(P), & P \in B_+, \\ \varphi_+(p), & p \in \partial B, \\ 0, & P \in B_-, \end{cases} \tag{3.17}$$

whilst for any solution φ_- of the Helmholtz equation in B_- we obtain

$$(D\varphi_-)(P) - \left(S\frac{\partial \varphi_-}{\partial n}\right)(P) = \begin{cases} 0, & P \in B_+, \\ \varphi_-(p), & p \in \partial B, \\ 2\varphi_-(P), & P \in B_-. \end{cases} \tag{3.18}$$

We remark that the form of the radiation condition given in (3.3) is satisfied

by $\gamma(P, Q)$ defined in (3.1) and thus it represents a point source of spherical waves which are outgoing at infinity. If k is replaced by $-k$ then $\gamma(P, Q)$ would have to be replaced by its complex conjugate $\bar{\gamma}(P, Q)$.

Now consider the exterior Neumann problem (3.2), (3.3) and (3.5). In the (usual) layer theoretic method we assume that the required solution has the form

$$u(P) = (Sw)(P), \quad P \in \mathbf{R}^3. \tag{3.19}$$

We then use the jump relation (3.12) together with the boundary condition (3.5) to obtain the boundary integral equation

$$(I+K)w(p) = g(p), \quad p \in \partial B. \tag{3.20}$$

for the unknown density w.

The alternative Green's theorem method involves applying the boundary condition (3.5) to the representation (3.17). Taking the limit $P \to p^+ \in \partial B$ of the first equation in (3.17) yields

$$(I+\bar{K}^*)w(p) = (Sg)(p), \quad p \in \partial B, \tag{3.21}$$

where $w = u|\partial B$. If (3.21) has a solution w then (3.17) implies that the required solution has the representation

$$u(P) = \tfrac{1}{2}\{(Sg)(P) - (Dw)(P)\}, \quad P \in B_+. \tag{3.22}$$

However, although there is no *a priori* guarantee that the function defined by (3.22) actually satisfies the boundary condition (3.5), nevertheless this can always be ensured by applying the expressions for the normal derivatives of single and double layer potentials (3.12) and (3.15) together with the boundary condition (3.5) to the representation (3.17) to obtain

$$D_n w(p) = (-I+K)g(p), \quad p \in \partial B. \tag{3.23}$$

Thus (3.22) gives a representation of a solution to the exterior Neumann problem (3.2), (3.3) and (3.5) provided w solves both (3.21) and (3.23). A more detailed discussion of the relevance of the addition equation (3.23) is given in [10].

A similar analysis for the other boundary value problems leads to similar boundary integral equations and these are given in Table I.

The boundary integral equations all have the typical form

$$(I+A)w = g,$$

where $A = K$ or $K+S$ and g is a known term. When dealing with the Helmholtz equation the quantity A is, strictly speaking, an operator-valued function of the frequency parameter k appearing in the Helmholtz equation and we should write $A \equiv A(k)$. As we shall see this dependence of A on k

Table I

Boundary condition	Representation in B_+	Boundary integral equations
Dirichlet		
$u = f$ on ∂B	$u = \frac{1}{2}(Sw - Df)$	$(I-K)w = -\frac{\partial}{\partial n}(Df)$
	$z = -Dw$	$(I-\bar{K}^*)w = f$
Neumann		
$\frac{\partial u}{\partial n} = f$ on ∂B	$u = \frac{1}{2}(Sf - Dw)$	$(I+\bar{K}^*)w = Sf$
	$u = Sw$	$(I+K)w = f$
Robin		
$\frac{\partial u}{\partial n} + u = f$ on ∂B	$u = \frac{1}{2}(Sf - \sigma Sw - Dw)$	$(I+\bar{K}^*+\sigma S)w = Sf$
	$u = Sw$	$(I+K+\sigma S)w = f$

will raise difficulties when we try to invert the operator $I+A$. We return to this later. A full discussion of the properties of these boundary integral equations is given in [10].

§ 4. Concerning existence

The question of existence of solution to the various boundary integral equations, and hence also of the boundary value problems which generate them, can be settled as a consequence of a few standard but independently derived results. For completeness we shall give these results here and indicate how they are used.

The integrals we have so far considered ((3.7) to (3.10)) all have the typical form:

$$\int_{\partial B} \frac{A(p, q)}{R^m} w(q)\, dS_q, \qquad R = |p-q|.$$

Although some of the properties of such integrals have already been mentioned more precise statements are summarized in the following theorems.

THEOREM 4.1 ([5], [18]). *If*

(i) *∂B is smooth,*

(ii) *w is a continuous density on ∂B,*

then

(iii) *Sw is continuous in $\boldsymbol{R}^3$,*

(iv) *Dw is continuous in $\boldsymbol{R}^3 \setminus \partial B$,*

(v) *on ∂B the following jump relations hold:*

$$\left(\frac{\partial}{\partial n_p^{\pm}} Sw\right)(p) = (\pm I + K)\, w(p), \qquad p \in \partial B,$$

$$\lim_{P \to p^{\pm}} \{(Dw)(P)\} = (\mp I + \bar{K}^*)\, w(p), \qquad p \in \partial B,$$

where

$$P \to p^+ \ (P \to p^-) \Rightarrow P \to p \in \partial B \text{ from } B_+ \ (B_-),$$

$\partial/\partial n_p^{\pm} \Rightarrow$ *differentiation in direction of normal to* ∂B *in the limit as* $P \to p^{\pm} \in \partial B$.

THEOREM 4.2 ([22]). *Let*

(i) $B \subset \boldsymbol{R}^n$ *be a bounded domain,*

(ii) $A(x, y)$ *be a kernel with the properties:*

(a) $A(x, y)$ *is bounded* $\forall x, y \in \bar{B}$,

(b) $A(x, y)$ *is continuous* $\forall x \neq y$.

Then

$$(Au)(x) = \int_B \frac{A(x, y)}{R^m} u(y)\, d\tau_y, \qquad R = |x-y|, \qquad x, y \in \boldsymbol{R}^n, \tag{4.1}$$

defines an operator

$$A\colon L_p(B) \to C_0(B) \tag{4.2}$$

which is compact for

$$m < n - \frac{n}{p}.$$

Furthermore, if

(iii) $n > m \geqslant n - n/p$,

(iv) $\exists$ *integer* $s \leqslant n$ *such that* $s > n-(n-m)\,p$,

(v) $B_s = s$*-dimensional section of* B,

(vi) $\exists$ q *such that*

$$q < \frac{sp}{n-(n-m)\,p}$$

then

$$A\colon L_p(B) \to L_q(B_s)$$

is compact.

THEOREM 4.3 ([22]). *Let*

(i) $B \subset \boldsymbol{R}^3$ *be a bounded domain with smooth boundary* ∂B,

(ii) $A(x, y)$ *be a* C^1 *kernel for* $x \in B$, $y \in \partial B$.

Then

$$(Au)(x) = \int_{\partial B} \frac{A(x, y)}{R} u(y)\, dS, \qquad R = |x-y|,$$

defines an operator

$$A\colon\ L_2(\partial B) \to L_2(B)$$

which is compact.

THEOREM 4.4 ([18]). *Let*

(i) $B \subset \boldsymbol{R}^n$ *be bounded,*

(ii) $A(x, y)$ *be bounded for* $x, y \in \bar{B}$,

(iii) $A(x, y)$ *be continuous for* $x \neq y$,

(iv) $u \in L_2(B)$ *solve*

$$u(x) = \int_{\partial B} \frac{A(x, y)}{\boldsymbol{R}^m} u(y)\, dS + f(x), \qquad m < n.$$

Then

$$f \in C(B) \quad \Rightarrow \quad u \in C(B).$$

To illustrate the use of these several results consider once again the exterior Neumann problem: find an element $u \in C^2(B_+) \cap C^1(\partial B)$ which satisfies (3.2), (3.3) and (3.5).

Adopting the layer theoretic approach we look for a solution in the form

$$u(P) = (Sw)(P), \qquad P \in \boldsymbol{R}^3,$$

and assume

A1: ∂B is a smooth, closed, bounded boundary surface.

A2: $w \in C(\partial B)$.

Consequently, Theorem 4.1 can be used to deduce that

$$u \in C(\boldsymbol{R}^3) \tag{4.3}$$

and

$$(I + K)w = g. \tag{4.4}$$

Furthermore, Theorem 4.3 implies that the operator

$$K\colon\ L_2(\partial B) \to L_2(\partial B)$$

is a compact operator. Consequently the Fredholm Alternative [18] can be applied to a discussion of (4.4) and the question of existence can be settled. Specifically, if it can be shown that the equation

$$(I + K)v = 0 \tag{4.5}$$

has only the trivial solution $v = 0$ then there exists a unique $w \in L_2(\partial B)$ which satisfies (4.4) for any given $g \in L_2(\partial B)$. As it happens this is not quite

good enough since we must satisfy A2. To ensure A2 we must first assume

A3: $g \in C(\partial B)$.

Theorem 4.4 then guarantees $w \in C(\partial B)$ as required.

Consequently, for the boundary integral equation method, as described here, to be a practical proposition we have to address two main problems:

P1: Ensure uniqueness results hold for equations of the form (4.4).

P2: Develop constructive methods of solutions.

We would remark that many of the assumptions made above can in fact be weakened quite considerably and in this connection we would refer to [1]–[3] and the references cited there.

§ 5. Irregular frequencies

Questions of uniqueness of solution to boundary value problems governed by the Laplace equation are innocent enough. However, this is not the case when dealing with the Helmholtz equation since the various boundary integral operators depend on k and this causes difficulties. To see how these difficulties arise recall

DEFINITION 5.1. k is a *characteristic value* of an operator $\pm A(k)$ if and only if there exists a nontrivial element w in the domain of $A(k)$ such that

$$(I \mp A(k))w = 0.$$

Furthermore, it follows that if k is a characteristic value of $\pm A(k)$ then it is also a characteristic value of $\pm \bar{A}(k)$, $\pm A^*(k)$ and $\pm \bar{A}^*(k)$. The following result can be obtained:

LEMMA 5.2 ([10]). *k is a characteristic value of $A(k)$ ($-A(k)$) if and only if k is an eigenvalue of the interior Neumann problem (interior Dirichlet problem).*

With this preparation we can illustrate the difficulties associated with the dependence of the boundary integral operators on the frequency related parameter k.

Consider the exterior Neumann problem (3.2), (3.3) and (3.5). When the data function g is taken to be zero then a uniqueness theorem can be proved [22]. This implies that for $g = 0$ the only solution of the exterior Neumann problem is $u = 0$ for all values of k. However, if we adopt a layer theoretic approach and assume that the exterior Neumann problem (3.2), (3.3) and (3.5) has a solution in the form $u = Sw$ then in the case when $g = 0$ this leads to the boundary integral equation

$$(I + K(k))w(p) = 0, \quad p \in \partial B. \tag{5.1}$$

Lemma 5.2 indicates that (5.1) has a nontrivial solution $w(p)$ whenever k is

an eigenvalue of the interior Dirichlet problem and this is turn implies that the exterior Neumann problem with $g = 0$ also has a nontrivial solution in contradiction to the independently obtained uniqueness theorem [22]. These values of k which are responsible for this contradiction are called *irregular frequencies* and, as can be seen, they give rise to a "pathological" nonuniqueness problem.

§ 6. Modified Green's functions

As we have seen, the boundary integral equations associated with the Helmholtz equation present a number of intriguing problems, mainly concerned with questions of uniqueness of solution and with difficulties near eigenvalues. Of course, these problems can be removed entirely if the exact Green's function for the problem is known. Unfortunately, the exact Green's function is only known for a few simple surfaces. Despite this, however, an attempt has been made [20], [23] to use, instead of the free space Green's function, a Green's function which is known exactly for some neighbouring region. This approximate Green's function technique leads to similar boundary integral equations to those obtained when the fundamental solution was used, which is not surprising since the approximate Green's function is itself a fundamental solution. Whilst the boundary integral equations which are obtained by using an approximate Green's function offer good prospects for numerical analysis, nevertheless a number of problems remain, in particular the pathological nonuniqueness problem. Recently Jones [7] introduced a theory of modified Green's functions in order to overcome the uniqueness problem arising in the boundary integral formulation of the exterior Dirichlet and Neumann problems for the Helmholtz equation. In this theory the fundamental solution, or free space Green's function, for the Helmholtz equation was modified by adding radiating spherical wave functions, that is, outgoing solutions of the Helmholtz equation, and the coefficients of these added terms were chosen so as to ensure that the boundary integral formulation of the problem was uniquely solvable for all real values of the wave number. Ursell [23] simplified the proof of a key theorem in [7] but confined his remarks to the exterior Neumann problem in two dimensions.

In [10] a systematic account is given of the boundary integral formulations of both the Dirichlet and Neumann problems together with a number of properties of the boundary integral operators arising in both the layer theoretic method and the Green's theorem method. In particular, it is shown that uniqueness of the boundary integral equation formulation of exterior problems can be retained even at eigenvalues of the corresponding adjoint interior problems by treating a pair of coupled equations. Similar results for the Robin problem are given in [1] where it is shown how classical problems for smooth boundaries may include boundary values in L_2.

In a recent paper Kleinman and Roach [11] have shown how Jones' modification can be incorporated into the boundary integral formulation presented in [10]. Furthermore, Ursell's simplification has been adapted to three dimensions and explicit results have been obtained for both the Dirichlet and Neumann problems. In particular, it is shown in [11] that a single boundary integral equation is uniquely solvable in each case even at interior eigenvalues of the adjoint problem by suitably modifying the Green's function in the way suggested by Jones [7]. Furthermore, it is shown in [11] that by abandoning the restriction to real coefficients in the modification which Jones and Ursell found sufficient to eliminate nonuniqueness of interior eigenvalues, the coefficients may be chosen to be optimal with respect to certain specific criteria. This was motivated by a desire not only to ensure unique solvability but also to provide a constructive method of solving the boundary integral equations. In particular, in [11] results are presented which show how to choose the coefficients so as to minimize the difference between modified and exact Green's functions for the Dirichlet and Neumann problems. In [12] it is shown that the coefficients can be chosen to minimize the norm of the modified boundary integral operator. This provides a bound on the spectral radius of the modified boundary integral operator and hence an indication when the associated boundary integral equation is solvable by iteration. In [12] an explicit definition of these "optimal" coefficients is given together with an alternative definition which, although not optimal, nevertheless simplifies the computation of the coefficients considerably, at the same time ensuring unique solvability. These results for the Dirichlet and Neumann problems have been extended to the Robin problem in [1].

Here we give a brief description of the modified Green's function technique together with a summary of results obtained so far.

The basic problems with which we are concerned are given by (3.2), (3.3) and one of either (3.4), (3.5) or (3.6).

We now change notation slightly in order to emphasize any modification we may introduce. Specifically, we denote a fundamental solution of the Helmholtz equation by

$$\gamma_0(P, Q; k) \equiv \gamma_0(P, Q) := -\frac{e^{ikR}}{2\pi R}, \qquad R = |P-Q|. \tag{6.1}$$

If now $g(P, Q)$ is a radiating solution of the Helmholtz equation in both P and Q then

$$\gamma_1(P, Q; k) \equiv \gamma_1(P, Q) := \gamma_0(P, Q)+g(P, Q) \tag{6.2}$$

is also a fundamental solution. We refer to γ_1 as the *modified (free space) Green's function.*

If $w \in L_2(\partial B)$ then we may obtain standard or modified forms of the

single and double layer distributions of density w according as γ_0 or γ_1 is used as a fundamental solution. Specifically, we have for $j = 0, 1$

$$(S_j w)(P) := \int_{\partial B} \gamma_j(P, q)\, w(q)\, dS_q, \qquad P \in \mathbf{R}^3, \tag{6.3}$$

$$(D_j w)(P) := \int_{\partial B} \frac{\partial \gamma_j}{\partial n_q}(P, q)\, w(q)\, dS_q, \qquad P \in \mathbf{R}^3. \tag{6.4}$$

Corresponding to (3.9) to (3.11) we also introduce the obvious notation K_j, K_j^*, D_{nj}, $j = 0, 1$. Consequently, we obtain the same form for the jump relations (3.12) to (3.15) and Green's Theorem (3.17) and (3.18), except that now all the integral operators have a subscript j to denote whichever fundamental solution is used. Similarly, we also recover Table I but with all the boundary integral operators suitably subscripted.

Turning now to the question of uniqueness of solution we first notice that all the boundary integral equations in Table I have the typical form

$$(I + A_j) w = g,$$

where A_j denotes K_j or $K_j + \sigma S_j$ and g the known terms. As we have already remarked, strictly speaking $A_j = A_j(k)$ is an operator-valued function of the frequency parameter k in the Helmholtz equation. Consequently, consideration must be given to the influence of the so-called characteristic or irregular values of $A_j(k)$ which are defined to be those real values of k for which $(I + A_j(k))^{-1}$ does not exist. It is these values of k which give rise to problems of nonuniqueness and difficulties near eigenvalues. Consequently, we now enquire into the possibility of modifying $\gamma_0(P, Q; k)$ in a systematic way so that the resulting modified fundamental solution $\gamma_1(P, Q; k)$ generates boundary integral equations for which there is no problem of nonuniqueness. In the modification (6.2) the choice of the function g is still at our disposal. Therefore we shall define

$$g(P, Q) = \sum_{l=0}^{\infty} \alpha_l v_l(P) v_l(Q), \tag{6.5}$$

where

$$v_l^{e,i} := \Lambda_{nm} z_n^{e,i}(kr) P_n^m(\cos\theta) \begin{cases} \cos m\varphi, & l \text{ even}, \\ \sin m\varphi, & l \text{ odd}, \end{cases} \tag{6.6}$$

with

$$z_n^e(kr) = h_n^{(1)}(kr), \qquad z_n^i(kr) = j_n(kr),$$

$$\Lambda_{nm} = \left\{ \frac{-ik}{2\pi} \varepsilon_m (2n+1) \cdot \frac{(n-m)!}{(n+m)!} \right\}^{1/2}$$

$$\varepsilon_0 = 1, \qquad \varepsilon_m = 2, \qquad m > 0,$$

$$l = \tfrac{1}{2} n(n+1) + m, \qquad 0 \leqslant m \leqslant n.$$

The object now is to choose the coefficients α_l in the definition of g so as to ensure that the boundary integral equations associated with the modified fundamental solution (6.2) are uniquely solvable.

We notice first that Lemma 5.2 does not generalize to the operator K_1 because g is not defined throughout B_-. Nevertheless, it is possible to establish the following [11].

THEOREM 6.1. *If $g(P, Q)$ is defined by (6.5) so that*

$$|2\alpha_l + 1| < 1 \quad \textit{for all } l,$$

then K_1 and $-K_1 - \sigma S_1$ have no characteristic values.

Thus the modification of the fundamental solution as indicated by (6.5) removes the characteristic values of the associated boundary integral operators and in doing so also removes the nonuniqueness problem. Furthermore, the following useful results can be established [11].

THEOREM 6.2. *If k is an eigenvalue of multiplicity m of the interior Neumann (Dirichlet) problem then there exists a modification $g(P, Q)$ with only m nonzero coefficients such that k is not a characteristic value of K_1 $(-K_1)$.*

THEOREM 6.3. *If $k_1, k_2, \ldots, k_N$ are eigenvalues (not necessarily ordered) of multiplicity $m_1, m_2, \ldots, m_N$ respectively of the interior Neumann (Dirichlet) problem then there exists a modification $g(P, Q)$ with $\sum_{i=1}^{N} m_i$ nonzero coefficients such that $k_1, k_2, \ldots, k_N$ are not characteristic values of K_1 $(-K_1)$ and hence also not characteristic values of $\bar{K}_1$ $(-\bar{K}_1)$, K_1^* $(-K_1^*)$ and $\bar{K}_1^*$ $(-\bar{K}_1^*)$.*

Thus, we have indicated here that the familiar boundary value problems for the Helmholtz equation can be replaced by a single boundary integral equation which is uniquely solvable, even at eigenvalues of the interior adjoint problem, by using a suitably modified Green's function.

§ 7. Optimal modifications

In this section we present results based on different criteria for choosing the coefficients in the modification (6.5). These are obtained by a desire not only to ensure unique solvability but also to provide constructive methods of solving the integral equations. For instance, we might wish to choose the coefficients so as to minimize the difference between the modified and exact Green's functions for the Dirichlet and Neumann problems. Alternatively, we might take as a desideratum the minimization of the norm of the modified integral operator as this will provide a bound on the spectral radius of the operator and so give an indication of when the modified boundary integral equation can be solved by iteration. The derivation of the results which are simply presented in Table II can be found in references [1], [11] and [12].

In Table II the symbol S_A denotes an auxiliary sphere completely

Table II

Coefficient choices for optimal modifications

Problem	Quantity to be minimized	Optimal coefficients	
		General	Sphere
Dirichlet			
$\gamma_1^{\mathrm{D}} = \gamma_0 + \sum_l \alpha_l^{\mathrm{D}} v_l(P) v_l(Q)$	$\int_{S_A}\int_{\partial B} \lvert\gamma_1^{\mathrm{D}}(P, q)\rvert^2 \, dS_q \, dS_P$	$\alpha_l^{\mathrm{D}} = -\dfrac{(v_l^{\mathrm{i}}, v_l^{\mathrm{e}})}{\lVert v_l^{\mathrm{e}}\rVert^2}$	$\alpha_{2l}^{\mathrm{D}} = \alpha_{2l+1}^{\mathrm{D}} = \dfrac{-j_n(ka)}{h_n^{(1)}(ka)}$
Neumann			
$\gamma_1^{\mathrm{N}}, \alpha_l^{\mathrm{N}}$	$\int_{S_A}\int_{\partial B} \lvert\partial\gamma_1^{\mathrm{N}}/\partial n\rvert^2 \, dS_q \, dS_P$	$\alpha_l^{\mathrm{N}} = -\dfrac{\left(\dfrac{\partial v_l^{\mathrm{i}}}{\partial n}, \dfrac{\partial v_l^{\mathrm{e}}}{\partial n}\right)}{\left\lVert \dfrac{\partial v_l^{\mathrm{e}}}{\partial n} \right\rVert^2}$	$\alpha_{2l}^{\mathrm{N}} = \alpha_{2l+1}^{\mathrm{N}} = \dfrac{-j_n'(ka)}{h_n^{(1)'}(ka)}$
Robin			
$\gamma_1^{\mathrm{R}}, \alpha_l^{\mathrm{R}}$	$\int_{S_A}\int_{\partial B} \left\lvert \dfrac{\partial\gamma_1^{\mathrm{R}}}{\partial n} + \sigma(q)\gamma_1^{\mathrm{R}} \right\rvert^2 dS_q \, dS_P$	$\alpha_l^{\mathrm{R}} = -\dfrac{\left(\dfrac{\partial v_l^{\mathrm{i}}}{\partial n} + \sigma v_l^{\mathrm{i}}, \dfrac{\partial v_l^{\mathrm{e}}}{\partial n} + \sigma v_l^{\mathrm{e}}\right)}{\left\lVert \dfrac{\partial v_l^{\mathrm{e}}}{\partial n} + \sigma v_l^{\mathrm{e}} \right\rVert^2}$	
Minimum norm			
$\gamma_1^{\mathrm{D}}, \alpha_l^0$	$\lVert K_1\rVert$	$\alpha_l^0 = \dfrac{\left(K_0 v_l^{\perp}, \dfrac{\partial v_l^{\mathrm{e}}}{\partial n}\right)}{\left\lVert \dfrac{\partial v_l^{\mathrm{e}}}{\partial n} \right\rVert^2}$	$\alpha_l^0 = \frac{1}{2}(\alpha_l^{\mathrm{D}} + \alpha_l^{\mathrm{N}})$

containing the given surface ∂B; furthermore, $\{v_l^{\perp}\}$ denotes a dual basis to $\{v_m\}$ with the property that $(v_l^{\perp}, v_m) = \delta_{lm}$. Here $(\cdot, \cdot)$ and $\|\cdot\|$ denote the usual $L_2(\partial B)$ inner product and norm.

We remark that in this notation [11], Lemma 2.1, ensures that $\{v_l\}_{l=0}^{\infty}$ and $\{\partial v_l/\partial n\}_{l=0}^{\infty}$ are linearly independent and complete on $L_2(\partial B)$.

Obviously any subset of these functions will also be linearly independent though not complete.

Since $\{v_l\}_{l=0}^{N}$ are linearly independent on ∂B, though not orthogonal, there does exist a dual basis of the span of $\{v_l\}$ denoted by $\{v_l^{\perp}\}_{l=0}^{N}$ with the property that

$$(v_l, v_m^{\perp}) = \delta_{lm}.$$

In fact the functions v_m may be represented in terms of v_l by

$$v_m^{\perp}(P) = \sum_{j=0}^{N} C_{mj} v_j(P),$$

where the coefficients C_{mj} are solutions of the equations

$$\sum_{j=0}^{N} \bar{C}_{mj}(v_l, v_j) = \sum_{j=0}^{N} C_{mj}(v_j, v_l) = \delta_{lm}.$$

For each m, the set of $N+1$ equations is uniquely solvable because the linear independence of $\{v_l\}$ implies that the coefficient matrix with elements (v_l, v_j) is nonsingular.

The linear independence of the expansion functions and the question of invertibility of associated matrices is treated in detail in [11], [12].

8. A numerical example

If we restrict our attention to two dimensions the various quantities introduced in the previous sections assume the following particular form:

$$v_{2l}^{e}(P) := \sqrt{-i\varepsilon_l/2}\, H_l^{(1)}(kr)\cos l\theta, \qquad v_{2l+1}^{e}(P) := \sqrt{-i\varepsilon_l/2}\, H_l^{(1)}(kr)\sin l\theta,$$

$$v_{2l}^{i}(P) := \sqrt{-i\varepsilon_l/2}\, J_l(kr)\cos l\theta, \qquad v_{2l+1}^{i}(P) := \sqrt{-i\varepsilon_l/2}\, J_l(kr)\sin l\theta,$$

$$\gamma_0(P, q) = -\frac{i}{2} H_0^{(1)}(kR(P, q)) = \sum_{l=0}^{\infty} v_l^{i}(P_<) v_l^{e}(P_>),$$

where

$$P_< := \begin{cases} P, & r_P < r_q, \\ q, & r_q < r_P, \end{cases} \qquad P_> := \begin{cases} P, & r_P > r_q, \\ q, & r_q > r_P, \end{cases}$$

$$\gamma_l(P, q) = \gamma_0(P, q) + \sum_{l=0}^{\infty} \alpha_l v_l^{e}(P) v_l^{e}(q) = \sum_{l=0}^{\infty} v_l^{e}(P_>)\{v_l^{i}(P_<) + \alpha_l(v_l^{e}(P_<)\}.$$

We now consider the question of how to define the coefficients α_l in the modifications of the Green's function so as to minimize $\|K_1\|$, because if we can ensure that $\|K_1\| < 1$ then the associated boundary integral equations will be solvable by iteration. A particularly useful result in this direction is provided by the following [12].

THEOREM 8.1. *If*

$$\partial B := \{p \in \mathbf{R}^2 : r_p = a + \varepsilon f(\theta_p)\},$$

$$\alpha_l := -\tfrac{1}{2}\left\{\frac{(v_l^i, v_l^e)}{\|v_l^e\|^2} + \frac{(\partial v_l^i/\partial n, \partial v_l^e/\partial n)}{\|\partial v_l^e/\partial n\|^2}\right\}$$

then $\|K_1\| = O(\varepsilon)$.

We remark that this theorem provides a relatively simple definition for the coefficients in the modification which makes the operator norm small for small perturbations of a sphere; this same definition also ensures that there are no characteristic values of the modified operator for all Lyapunov surfaces without restriction. Indeed, the following result can be obtained.

THEOREM 8.2. [11]. *If* α_l *are defined as in Theorem* 8.1 *for all* l, *then* $I \pm K_1$ *is invertible for all real* k.

It is now natural to enquire if $\|K_1\|$ will be reduced if the modification contains only a finite number of nonzero coefficients. To demonstrate that this is indeed the case we shall assume that the coefficients are chosen as in Theorem 8.1 for $l \leqslant 2N+1$ and zero otherwise, and present numerical results corresponding to the case when ∂B is a sphere of radius a.

The spectral radius of the operator K_1 is denoted by

$$\sigma_{K_1}(k) := \sup_n \{\lambda_n(k) : (\lambda_n(k) I - K_1) w = 0\} \leqslant \|K_1\|.$$

In the particular case of a sphere of radius a it is known that

$$\lambda_n(k) = 1 - ika\pi J_n'(ka) H_n^{(1)}(ka) = -1 - ika\pi J_n(ka) H_n^{(1)'}(ka).$$

Furthermore, if

$$J_n'(ka) = 0 \quad \text{then} \quad \lambda_n = 1$$

and k is an eigenvalue of the interior Neumann problem, and if

$$J_n(ka) = 0 \quad \text{then} \quad \lambda_n = -1$$

and k is an eigenvalue of the interior Dirichlet problem.

If now we choose α_l as in Theorem 8.1 for $l \leqslant 2N+1$ and zero otherwise then

$$\sigma_{K_1}(k) = \sup_{n>N} \{|\lambda_n| : (\lambda_n I - K_1) w = 0\}.$$

The manner in which $\sigma_{K_1}(k)$ varies with N and k is illustrated in Table III.

Table III

Number of terms	Spectral radius $\sigma_{K_1}(k)$										
	$ka = 0.1$	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0	1.01	1.20	1.35	1.31	1.14	1.20	1.21	1.13	1.18	1.14	1.16
1	0.11	0.18	0.49	0.82	1.07	1.20	1.21	1.13	1.18	1.14	1.16
3	0.11	0.15	0.23	0.33	0.51	0.76	0.98	1.13	1.18	1.14	1.16
5	0.11	0.15	0.23	0.33	0.42	0.50	0.54	0.74	0.93	1.10	1.16
7	0.11	0.15	0.23	0.32	0.42	0.50	0.54	0.55	0.53	0.72	0.90
9	0.11	0.15	0.23	0.32	0.42	0.50	0.53	0.55	0.53	0.48	0.54
11	0.11	0.15	0.23	0.32	0.42	0.42	0.53	0.55	0.53	0.47	0.41

These results clearly indicate that the number of terms in the modification required to reduce significantly the spectral radius increases with ka. Nevertheless, we note that even for $ka = 5$ (well into the so-called resonance region) only 11 terms are required in the modification to reduce the spectral radius below 0.5. Results for noncircular boundaries are given in [9].

§ 9. The null field equations

The null field method (also known as the T-matrix, extended boundary or Waterman method) has been extensively used as a constructive method for solving the Dirichlet and Neumann problems for the Helmholtz equation in an exterior domain. In this and the following section we shall show that there is a close connection between the null field method and the modified Green's function technique described here.

Assume u is a solution of either the exterior Dirichlet or exterior Neumann problem for the Helmholtz equation. Applying Green's Theorem in $\bar{B}_+$ to u and a radiating wave function $v_l^e(P)$ we obtain

$$\int_{\partial B} \left\{ \frac{\partial u(p)}{\partial n_p} v_l^e(p) - u(p) \frac{\partial v_l^e}{\partial n_p}(p) \right\} dS_p = 0, \qquad \forall l. \tag{9.1}$$

Now employing the boundary conditions we obtain, for Dirichlet data,

$$\int_{\partial B} \frac{\partial u(p)}{\partial n_p} v_l^e(p)\, dS_p = \int_{\partial B} f(p) \frac{\partial v_l^e}{\partial n_p} dS_p, \qquad \forall l, \tag{9.2}$$

and for Neumann data

$$\int_{\partial B} u(p) \frac{\partial v_l^e(p)}{\partial n_p} dS_p = \int_{\partial B} f(p)\, v_l^e(p)\, dS_p, \qquad \forall l. \tag{9.3}$$

Equations (9.2) and (9.3) are the *null field equations* for the Dirichlet and Neumann problems respectively. Once $\partial u/\partial n$ for the Dirichlet problem and u

in the Neumann problem are found on ∂B, the Green representation theorem may be employed to represent the solution in B_+.

In the null field method $\partial u(p)/\partial n_p$ and $u(p)$, $p \in \partial B$, are expanded in terms of nonorthogonal bases for $L_2(\partial B)$. In [19] it is shown that the system of "projected" spherical harmonics $Y_{nm}(w)$, $w = f^{-1}(p)$, $p \in \partial B$, form a basis and also a Riesz basis for $L_2(\partial B)$, where f defines ∂B.

Often the outgoing cylindrical or spherical functions $\{v_l^e\}$ and $\{\partial v_l^e/\partial n\}$ are employed as expansion functions. They are known to be complete and linearly independent in $L_2(\partial B)$ for any k [11], [17], as are $\{v_l^i\}$ and $\{\partial v_l^i/\partial n\}$ when k is not an eigenvalue of either the interior Dirichlet problem or the interior Neumann problem, respectively. Therefore they form a closed minimal system (or Hamel basis) for $L_2(\partial B)$. Unfortunately, they do not necessarily also form a (Schauder) basis for $L_2(\partial B)$. When they do, the *Rayleigh hypothesis* is said to be satisfied.

It may be noted here that if one uses spherical wave expansions on ∂B for impenetrable scatterers (i.e. for Dirichlet, Neumann, or Robin boundary conditions), the k value in these expansions need not coincide with the given exterior k value in the problem. Therefore, if the given exterior k value happens to coincide with an interior eigenvalue, of the Dirichlet or Neumann problem as the case may be, one can choose a slightly different k value for the expansions on ∂B. However, when the given exterior k value can be used also for the expansions on ∂B, useful simplifications and interesting relations are obtained. Technically, the simplifications usually result from an application of the formulas

$$\int_{\partial B} \left[v_m^e \frac{\partial v_l^e}{\partial n} - v_l^e \frac{\partial v_m^e}{\partial n} \right] dS = 0, \tag{9.4a}$$

$$\int_{\partial B} \left[v_m^i \frac{\partial v_l^i}{\partial n} - v_l^i \frac{\partial v_m^i}{\partial n} \right] dS = 0, \tag{9.4b}$$

$$\int_{\partial B} \left[v_m^i \frac{\partial v_l^e}{\partial n} - v_l^e \frac{\partial v_m^i}{\partial n} \right] dS = 2\delta_{lm} \tag{9.4c}$$

(where the k value is the same everywhere!). It can be shown [13], [14] that when v_l^e, v_l^i, $\partial v_l^e/\partial n$, $\partial v_l^i/\partial n$ are bases, the various "Q-matrices" that occur in the null field method are in fact invertible.

As an illustration of these general remarks, we show that by a suitable choice of expansion for u and $\partial u/\partial n$ on ∂B, an interesting relation between the Dirichlet and Neumann problems can be obtained.

Assuming that both $\{v_l^e\}$ and $\{\partial v_l^e/\partial n\}$ are bases for $L_2(\partial B)$, introduce the expansions

$$u(p) = \sum_{m=0}^{\infty} c_m v_m^e(p), \qquad p \in \partial B, \tag{9.5}$$

and

$$\frac{\partial u(p)}{\partial n_p} = \sum_{m=0}^{\infty} d_m \frac{\partial v_m^e(p)}{\partial n_p}, \qquad p \in \partial B. \tag{9.6}$$

Observe that we do not assume that $c_m = d_m$. Nevertheless, when we substitute (9.4) and (9.5) in (9.6) we obtain

$$\int_{\partial B} \left\{ \sum_{m=0}^{\infty} d_m \frac{\partial v_m^e}{\partial n} v_l^e - \sum_{m=0}^{\infty} c_m v_m^e \frac{\partial v_l^e}{\partial n} \right\} dS = 0, \qquad \forall l. \tag{9.7}$$

By taking (9.4a) into account, (9.7) may be written, provided that the interchange of integration and summation can be justified, as

$$\sum_{m=0}^{\infty} (d_m - c_m) \int_{\partial B} \frac{\partial v_m^e}{\partial n} v_l^e \, dS = 0, \qquad \forall l. \tag{9.8}$$

Furthermore, it can be shown [13] that the matrix

$$\left[\int_{\partial B} \partial v_m^e / \partial n \cdot v_l^e \, dS \right]$$

is invertible. Therefore we find that

$$c_m = d_m, \qquad \forall m. \tag{9.9}$$

This says that the coefficients of an expansion of u in terms of $\{v_l^e\}$ on ∂B are the same as the coefficients of an expansion of $\partial u/\partial n$ in terms of $\{\partial v_l^e/\partial n\}$ on ∂B. In a sense this provides a solution of the boundary value problems because it has these implications: for the Dirichlet problem the coefficients of an expansion of the unknown function $\partial u/\partial n$ in terms of $\{\partial v_l^e/\partial n\}$ on ∂B are the same as the coefficients of an expansion of the data, f, in terms of $\{v_l^e\}$; and for the Neumann problem the coefficients of an expansion of the unknown function u in terms of $\{v_l^e\}$ on ∂B are the same as the coefficients of an expansion of the data, f, in terms of $\{\partial v_l^e/\partial n\}$. Moreover, if this result is incorporated into the Green's representation of the solution we find

$$u(P) = \tfrac{1}{2} \int_{\partial B} \left\{ \gamma_0(P, q) \sum_{l=0}^{\infty} c_l \frac{\partial v_l^e(q)}{\partial n_q} - \frac{\partial \gamma_0}{\partial n_q}(P, q) \sum_{l=0}^{\infty} c_l v_l^e(q) \right\} dS_q, \qquad P \in B_+, \tag{9.10}$$

and upon interchanging summation and integration

$$u(P) = \sum_{l=0}^{\infty} \tfrac{1}{2} c_l \int_{\partial B} \left\{ \gamma_0(P, q) \frac{\partial v_l^e(q)}{\partial n_q} - \frac{\partial \gamma_0}{\partial n_q}(P, q) v_l^e(q) \right\} dS_q, \qquad P \in B_+. \tag{9.11}$$

But since v_l^e is a radiating solution of the Helmholtz equation, Green's theorem implies that (9.11) becomes

$$u(P) = \sum_{l=0}^{\infty} c_l v_l^e(P), \qquad P \in B_+, \tag{9.12}$$

where we repeat that the coefficients are those in an expansion of the given data on ∂B, i.e.

$$f(p) = \sum_{l=0}^{\infty} c_l v_l^e(p), \quad p \in \partial B, \quad \text{Dirichlet case,} \tag{9.13}$$

$$f(p) = \sum_{l=0}^{\infty} c_l \frac{\partial v_l^e}{\partial n_p}(p), \quad p \in \partial B, \quad \text{Neumann case.} \tag{9.14}$$

The series (9.11) or (9.12) must converge uniformly with respect to P in order to justify the interchange of summation and integration in (9.10). This is known to be true for all $P \in B_+$ when the Rayleigh criterion is fulfilled (see, for example, [17]).

The solution of these scattering problems is thus seen to depend solely upon the ability to find the Fourier coefficients of a given function f on ∂B with respect to the, in general nonorthogonal, bases $\{v_l^e\}$ and $\{\partial v_l^e/\partial n\}$. In general, in acoustic problems we have either

$$f(p) = -u^i(p) \quad \text{(Dirichlet case),} \tag{9.15}$$

or

$$f(p) = -\partial u^i(p)/\partial n_p \quad \text{(Neumann case),} \tag{9.16}$$

where u^i is an incident field. The incident field is usually conveniently represented as an expansion, not in outgoing waves $\{v_l^e\}$ but in standing waves $\{v_l^i\}$. So, for instance, if the sources of u^i lie outside a circumscribing sphere of ∂B one can usually readily find a representation of f in the Dirichlet case as

$$f(p) = \sum_{l=0}^{\infty} a_l v_l^i(p), \quad p \in \partial B. \tag{9.17}$$

To obtain the result (9.13) an expansion of the form

$$v_l^i(p) = \sum_{m=0}^{\infty} \alpha_{ml} v_m^e(p), \quad p \in \partial B, \tag{9.18}$$

is required. Assuming that such an expansion is available we obtain by (9.17), (9.18), (9.13)

$$f(p) = \sum_{l=0}^{\infty} a_l v_l^i(p) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} a_l \alpha_{ml} v_m^e(p) = \sum_{m=0}^{\infty} c_m v_m^e(p). \tag{9.19}$$

The linear independence of the radiating wave functions on ∂B enables us to write

$$c_m = \sum_{l=0}^{\infty} a_l \alpha_{ml}, \tag{9.20a}$$

which can be written in matrix form as

$$c = T_D, \qquad T_D = (\alpha_{ml}). \tag{9.20b}$$

Here T_D is known as the *transition matrix for the Dirichlet problem.* (The result (9.20b) that the transition matrix coincides with the matrix that relates v_l^i and v_m^e on ∂B was noted by A. T. de Hoop and P. van den Berg (personal communication).)

We see that the given problem has now been reduced, at least formally, to the problem of determining the elements of T_D.

For the Neumann problem, the data f is usually most conveniently expressible as

$$f(p) = \sum_{l=0}^{\infty} b_l \, \partial v_l^i(p)/\partial n_p, \qquad p \in \partial B. \tag{9.21}$$

Now to obtain the result (9.14) an expansion of the form

$$\frac{\partial v_l^i(p)}{\partial n_p} = \sum_{m=0}^{\infty} \beta_{ml} \frac{\partial v_m^e(p)}{\partial n_p}, \qquad p \in \partial B, \tag{9.22}$$

is required. Assuming that such an expansion is available we obtain, from (9.21), (9.22) and (9.14)

$$f(p) = \sum_{l=0}^{\infty} b_l \frac{\partial v_l^i(p)}{\partial n_p} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} b_l \beta_{ml} \frac{\partial v_m^e(p)}{\partial n_p} = \sum_{m=0}^{\infty} c_m \frac{\partial v_m^e(p)}{\partial n_p}. \tag{9.23}$$

As before, the linear independence of $\{\partial v_m^e/\partial n\}$ on ∂B enables us to write

$$c_m = \sum_{l=0}^{\infty} b_l \beta_{ml}, \tag{9.24}$$

which may be written in matrix form as

$$c = T_N b, \qquad T_N = (\beta_{ml}), \tag{9.25}$$

and T_N is the *transition matrix for the Neumann problem.* We see that the Neumann problem has been reduced, at least formally, to the problem of determining the elements of T_N.

Actual construction of the elements of the transition matrices may be accomplished in a variety of ways. For example, we can obtain from (9.18)

$$(v_l^i, v_n^e) = \sum_{m=0}^{\infty} \alpha_{ml} (v_m^e, v_n^e), \tag{9.26}$$

where $(\ ,\ )$ denotes the usual $L_2(\partial B)$ inner product. Equation (9.26) can be written in matrix form as

$$Q_D^i = Q_D T_D, \tag{9.27}$$

where

$$Q_{\mathrm{D}}^{\mathrm{i}} = [(v_l^{\mathrm{i}}, v_n^{\mathrm{e}})] \quad \text{and} \quad Q_{\mathrm{D}} = \overline{[(v_n^{\mathrm{e}}, v_m^{\mathrm{e}})]}, \tag{9.28}$$

and since it can be shown [13] that Q_{D}^{-1} exists,

$$T_{\mathrm{D}} = Q_{\mathrm{D}}^{-1} Q_{\mathrm{D}}^{\mathrm{i}}. \tag{9.29}$$

A similar treatment is possible for the exterior Neumann problem [13].

§ 10. Connection with the modified Green's function technique

We now consider a more general form for the modification of the Green's function than that used in § 6. Specifically, we assume a modified Green's function of the form

$$\gamma_1^{\mathrm{D}}(P, Q) = \gamma_0(P, Q) - \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \alpha_{ml} v_l^{\mathrm{e}}(P) v_m^{\mathrm{e}}(Q), \qquad P, Q \in \bar{B}_+, \tag{10.1}$$

for the Dirichlet problem and

$$\gamma_1^{\mathrm{N}}(P, Q) = \gamma_0(P, Q) - \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \beta_{ml} v_l^{\mathrm{e}}(P) v_m^{\mathrm{e}}(Q), \qquad P, Q \in \bar{B}_+, \tag{10.2}$$

for the Neumann problem. We should remark that the minus sign is introduced simply for convenience later on. If we try to choose the coefficients in the modification so that the modified Green function is the best least squares approximation to the actual Green function we will want to choose α_{ml} so as to minimize

$$\int_{S_A} \int_{\partial B} |\gamma_1^{\mathrm{D}}(P, Q)|^2 \, dS_q \, dS_P \tag{10.3}$$

in the Dirichlet case and choose β_{ml} to minimize

$$\int_{S_A} \int_{\partial B} \left| \frac{\partial \gamma_1^{\mathrm{N}}}{\partial n_q} \right|^2 dS_q \, dS_P \tag{10.4}$$

in the Neumann case, where S_A is a circle (sphere) of radius A which contains $\bar{B}_-$ in its interior. Clearly the quantities (10.3) and (10.4) would vanish if γ_1^{D} and γ_1^{N} were the exact Green functions for the Dirichlet and Neumann problems respectively. We notice that (10.3) may be written in the form

$$\int_{S_A} \int_{\partial B} \Big| \sum_{l=0}^{\infty} v_l^{\mathrm{e}}(P) \Big\{ v_l^{\mathrm{i}}(q) - \sum_{m=0}^{\infty} \alpha_{ml} v_m^{\mathrm{e}}(q) \Big\} \Big|^2 dS_q \, dS_P, \tag{10.5}$$

whilst (10.4) becomes

$$\int_{S_A} \int_{\partial B} \left| \sum_{l=0}^{\infty} v_l^{\mathrm{e}}(P) \left\{ \frac{\partial v_l^{\mathrm{i}}(q)}{\partial n_q} - \sum_{m=0}^{\infty} \beta_{ml} \frac{\partial v_m^{\mathrm{e}}(q)}{\partial n_q} \right\} \right|^2 dS_q \, dS_P. \tag{10.6}$$

In this form it is clear that the expression (10.5) hence (10.3) will vanish, and hence be minimized, if α_{ml} are chosen so that

$$v_l^i(q) - \sum_{m=0}^{\infty} \alpha_{ml} v_m^e(q) = 0, \qquad q \in \partial B. \tag{10.7}$$

(This is always possible when the Rayleigh criterion is satisfied.) But (10.7) is precisely the same as (9.18). Therefore, the coefficients that cause γ_1^D to best approximate the Dirichlet Green function are precisely the elements of the transition matrix T_D; see (9.20b). In fact we may now express the Green's function explicitly.

$$\gamma_1^D(P, q) = \gamma_0(P, q) - \bar{v}^T T_D \bar{v}, \qquad P \in B_+, q \in \partial B, \tag{10.8}$$

where $\bar{v}$ is the vector with elements (v_l^e).

Similarly, (10.6) and hence (10.4) will vanish if β_{ml} are chosen so that

$$\frac{\partial v_l^i(q)}{\partial n_q} - \sum_{m=0}^{\infty} \beta_{ml} \frac{\partial v_m^e(q)}{\partial n_q} = 0, \qquad q \in \partial B, \tag{10.9}$$

which is precisely the same as (9.22). Hence the coefficients that cause γ_1^N to best approximate the Neumann Green function are the elements of the transition matrix T_N (see (9.25)) and

$$\gamma_1^N(P, q) = \gamma_0(P, q) - \bar{v}^T T_N \bar{v}, \qquad P \in B_+, q \in \partial B. \tag{10.10}$$

Finally, we exhibit the solutions of the Dirichlet and Neumann problems in terms of the modified Green functions. For this we employ Green's theorem, which takes the form

$$\int \left\{ \gamma_1(P, q) \frac{\partial u(q)}{\partial n_q} - \frac{\partial \gamma_1}{\partial n_q}(P, q) u(q) \right\} dS_q = \begin{cases} 2u(P), & P \in B_+, \\ u(p), & p \in \partial B, \end{cases} \tag{10.11}$$

for any radiating wave function u. The fact that $\gamma_1(P, q)$ has singularities in B_- prevents the integral in (10.11) from vanishing when $P \in B_-$. Now if we employ the Dirichlet modified Green function (10.1) and seek the solution of the Dirichlet problem we find

$$u(P) = \tfrac{1}{2} \int_{\partial B} \left\{ \gamma_1^D(P, q) \frac{\partial u(q)}{\partial n_q} - \frac{\partial \gamma_1^D}{\partial n_q}(P, q) f(q) \right\} dS_q, \qquad P \in B_+. \tag{10.12}$$

Moreover, if $P \in S_A$, where S_A is any circumscribing circle (sphere), then we obtain an expansion of the Green function in the form

$$\gamma_1^D(P, q) = \sum_{l=0}^{\infty} v_l^e(P) \{ v_l^i(q) - \sum_{m=0}^{\infty} \alpha_{ml} v_m^e(q) \}, \qquad P \in S_A, q \in \partial B. \tag{10.13}$$

If α_{ml} are chosen to satisfy (10.7) then $\gamma_1^D(P, q)$, given by (10.8), vanishes for

$q \in \partial B$ and

$$u(P) = -\tfrac{1}{2} \int_{\partial B} \frac{\partial \gamma_1^D}{\partial n_q}(P, q) f(q)\, dS_q, \qquad r_P \geqslant \max_{q \in \partial B} r_q. \tag{10.14}$$

But, expansion of f in terms of $\{v_l^e\}$, as in (9.13), yields

$$u(P) = -\tfrac{1}{2} \sum_{l=0}^{\infty} c_l \int_{\partial B} \frac{\partial \gamma_1^D}{\partial n_q}(P, q) v_l^e(q)\, dS_q, \qquad r_P \geqslant \max_{q \in \partial B} r_q. \tag{10.15}$$

Since v_l^e is a radiating wave function, (10.12) yields

$$v_l^e(P) = -\tfrac{1}{2} \int_{\partial B} \frac{\partial \gamma_1^D}{\partial n_q}(P, q) v_l^e(q)\, dS_q, \tag{10.16}$$

which when substituted in (10.15) gives

$$u(P) = \sum_{l=0}^{\infty} c_l v_l^e(P), \qquad r_P \geqslant \max_{q \in \partial B} r_q, \tag{10.17}$$

in agreement with (9.12), except that here there is an explicit restriction on P.

Similar results are obtained for the Neumann problem [13].

§ 11. Some final remarks

In the previous sections we have given a flavour of the boundary integral equation method for solving elliptic boundary value problems, indicated some of the inherent difficulties and described a method, the modified Green's function method, for overcoming these difficulties which offers good prospects for numerical work. The promise of the modified Green's function approach has also been further emphasized by demonstrating its connection with the T-matrix method.

Although the boundary integral equation method is not at all new, nevertheless it is still an active area of research. In recent years, apart from the concept of a modified Green's function, it has given rise to a number of other techniques for solving boundary value problems such as boundary element methods [2] and coupled boundary integral and finite element methods [6]. At the moment perhaps the majority of research effort in this area is being devoted towards developing boundary integral equation methods, with the aid of the theory of pseudodifferential operators, so that problems involving nonsmooth boundaries and more complicated boundary conditions can be handled constructively.

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