TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 1
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

ON PURE SEMISIMPLICITY AND REPRESENTATION-FINITE PIECEWISE PRIME RINGS

BOGUMIŁA KLEMP and DANIEL SIMSON

Institute of Mathematics, Nicholas Copernicus University
Toruń, Poland

It is proved that a schurian artinian right peak PI-ring R is right pure sp-semisimple if and only if R is sp-representation finite. A diagrammatic characterization of representation-finite piecewise prime artinian PI-rings (defined in Section 3) is given.

1. Introduction

We recall from [25] that a semiperfect ring R is a *right peak ring* if R is a generalized matrix ring of the form

$$(1.1)\qquad R = \begin{bmatrix} F_1 & {}_1M_2 & & {}_1M_n & {}_1M_* \\ {}_2M_1 & F_2 & & {}_2M_n & {}_2M_* \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ {}_nM_1 & {}_nM_2 & & F_n & {}_nM_* \\ 0 & 0 & & 0 & F \end{bmatrix} = \begin{matrix} P_1 \\ \oplus \\ P_2 \\ \oplus \\ \vdots \\ \oplus \\ P_n \\ \oplus \\ P_* \end{matrix}$$

such that $\mathrm{soc}(R_R)$ is an essential right ideal in R isomorphic to a direct sum of finitely many copies of P_* (P_* is called the *right peak* of R). Here $F_1, \ldots, F_n$ are local rings, $F = F_*$ is a division ring, ${}_iM_j$ is an F_i-F_j-bimodule and the multiplication in R is given by F_i-F_t-bimodule maps c_{ijt}: ${}_iM_j \otimes {}_jM_t \to {}_iM_t$

This paper is in final form and no version of it will be submitted for publication elsewhere.

satisfying the natural associativity conditions. We denote by $P_1, \ldots, P_n, P_*$ the right indecomposable row ideals of R.

We call R *schurian* if $F_1, \ldots, F_n$ are division rings. We denote by $\mathrm{mod}_{\mathrm{sp}}(R)$ the category of finitely generated socle projective right R-modules. R is called *sp-representation-finite* if the number of isomorphism classes of indecomposable modules in $\mathrm{mod}_{\mathrm{sp}}(R)$ is finite.

A ring A is called *right pure semisimple* [18] if every right A-module is a direct sum of finitely generated modules. Every such ring is right artinian. We call a right peak ring R *right pure sp-semisimple* if every socle projective right R-module is a direct sum of finitely presented modules.

We recall that the following theorem holds:

(pss$_A$) *If the ring A is right pure semisimple then A is representation-finite*

provided A is either an Artin algebra [2], or A is an l-hereditary PI-ring [21, 24], or A is a local PI-ring [23]. However, it is still an open problem if **(pss$_A$)** is true for an arbitrary artinian ring A.

It follows from [25], [26] and [28; 7.3] that under some assumptions on A the proof of **(pss$_A$)** can be reduced to the proof of

(pss$_R^*$) *If R is a right pure sp-semisimple artinian right peak PI-ring then R is sp-representation finite.*

In the present note **(pss$_R^*$)** is proved for schurian right peak PI-rings R (Theorem 2.7) and **(pss$_A$)** is proved for *piecewise prime PI-rings* defined in Section 3. Moreover, a diagrammatic characterization of representation-finite piecewise prime PI-rings is given (Theorem 3.7).

Throughout this paper we freely use the terminology and notation introduced in [19, 21, 25]. In particular, we denote by $J(A)$ the Jacobson radical of the ring A. If R is schurian of the form (1.1) then $(I_R, \boldsymbol{d})$ denotes the *value scheme* of R [25; p. 536] consisting of the set I_R of points $1, 2, \ldots, n+1 = *$ connected by valued dashed arrows $i \xrightarrow{(d_{ij}, d'_{ij})} j$, $i \neq j$, whenever $d_{ij} = \dim({}_iM_j)_{F_j}$ and $d'_{ij} = \dim_{F_i}({}_iM_j)$ are nonzero. We recall from [25; Proposition 2.3] that if all bimodules ${}_iM_*$ are simple then $(I_R, \boldsymbol{d})$ is a valued poset having a unique maximal element $*$ with respect to the relation $i \prec j \Leftrightarrow {}_iM_j \neq 0$. We draw a solid arrow from i to j if there is no t such that $i \prec t \prec j$.

2. Pure sp-semisimple rings

We begin by recalling a few facts on pure semisimple rings.

THEOREM 2.1 [2, 20]. *A right artinian ring B is right pure semisimple if and only if given any sequence*

$$X_1 \xrightarrow{f_1} X_2 \to \ldots \to X_n \xrightarrow{f_n} X_{n+1} \to \ldots$$

of monomorphisms between indecomposable modules in mod(B) *there is an integer* n *such that* f_j *is an isomorphism for* $j \geqslant n$.

THEOREM 2.2 [21]. *A hereditary artinian PI-ring B is right pure semisimple if and only if B is of finite representation type. In this case the valued graph of B is a disjoint union of Dynkin diagrams.*

For artinian right peak rings there is a counterpart of Theorem 2.1.

THEOREM 2.3. *Let R be an artinian right peak ring. Then the following conditions are equivalent:*

(a) *R is right pure sp-semisimple.*

(b) *If* $X_1 \xrightarrow{f_1} X_2 \to \dots \to X_n \xrightarrow{f_n} X_{n+1} \to \dots$ *consists of indecomposable modules in* $\mathrm{mod}_{sp}(R)$ *and* $f_1, f_2, \dots$ *are monomorphisms then there is m such that* f_j *is an isomorphism for* $j \geqslant m$.

(c) *Every indecomposable module in* $\mathrm{Mod}_{sp}(R)$ *is of finite length.*

(d) *Every additive functor* H: $\mathrm{mod}_{sp}(R) \to \mathcal{A}b$ *has a nonzero simple subfunctor.*

(e) *The Jacobson radical of the category* $\mathrm{mod}_{sp}(R)$ *is right T-nilpotent in the sense that for any sequence* $Y_1 \xrightarrow{g_1} Y_2 \to \dots \to Y_n \xrightarrow{g_n} Y_{n+1} \to \dots$ *of nonisomorphisms between indecomposable modules in* $\mathrm{mod}_{sp}(R)$ *there is m such that* $g_m g_{m-1} \cdots g_1 = 0$.

Proof. (a) $\Rightarrow$ (c) is obvious.

(a) $\Rightarrow$ (b). Since $X = \mathrm{colim}\, X_i$ is a socle projective module, by (a) X has a nonzero summand Y of finite length. Let m be such that $Y \subseteq \mathrm{Im}(g_j: X_j \to X)$ for $j \geqslant m$, where g_j is the natural colimit monomorphism. Since obviously Y is a summand of $\mathrm{Im}\, g_j$, $X_j \cong Y$ for all $j \geqslant m$ because X_j is indecomposable. It follows that f_j is an isomorphism for $j \geqslant m$.

The implications (b) $\Rightarrow$ (d) $\Leftarrow$ (c) can be proved by the method of Auslander [2]. For the convenience of the reader we outline the proof.

Suppose the converse of (d) and let H: $\mathrm{mod}_{sp}(R) \to \mathcal{A}b$ be an additive functor having no nonzero simple subfunctors. By arguments used in the proof of [2; Proposition 2.9(a)] for any module X in $\mathrm{mod}_{sp}(R)$ and a nonzero element x in $H(X)$ there is a homomorphism f: $X \to X'$ in $\mathrm{mod}_{sp}(R)$ which is not a splittable monomorphism and satisfies $H(f)x \neq 0$. Since modules in $\mathrm{mod}_{sp}(R)$ are of finite length, using the same type of argument as in the proof of [2; Theorem 1.5] one can construct a sequence

$$X_1 \xrightarrow{f_1} X_2 \to \dots \to X_n \xrightarrow{f_n} X_{n+1} \to \dots$$

of indecomposable modules in $\mathrm{mod}_{sp}(R)$ and proper monomorphisms f_j such that $\mathrm{colim}\, X_i$ is indecomposable of infinite length. This contradiction finishes the proof of (b) $\Rightarrow$ (d) $\Leftarrow$ (c).

By the well-known arguments of Bass (see [0; p. 317]), (d) is equivalent to the fact that every flat functor in $\mathrm{Add}(\mathrm{mod}_{\mathrm{sp}}(R)^{\mathrm{op}}, \mathscr{A}b)$ is projective, which is equivalent to (e) ([17; Theorem 2.4], [19; Lemma 5.3]). In order to prove (e) ⇒ (a), given a module M in $\mathrm{Mod}_{\mathrm{sp}}(R)$ we consider the functor

$$\boldsymbol{h}_M = \mathrm{Hom}_R(-, M) \colon \mathrm{mod}_{\mathrm{sp}}(R) \to \mathscr{A}b.$$

Since M is a directed union of submodules M_s of finite length, $\boldsymbol{h}_M = \mathrm{colim}\, \boldsymbol{h}_{M_s}$ is flat. By our remark above $\boldsymbol{h}_M$ is projective. Hence $\boldsymbol{h}_M \cong \bigoplus_t \boldsymbol{h}_{N_t} \cong \boldsymbol{h}_{(\oplus N_t)}$ for some modules N_t in $\mathrm{mod}_{\mathrm{sp}}(R)$. It follows that $M \cong \bigoplus_t N_t$ and the proof is complete.

We recall from [10, 26] that given a subset $J \subseteq I_R$ such that $* \in J$ we denote by R_J the ring obtained from the matrix form (1.1) by omitting all rows and columns with indices $t \in I_R - J$. We have a pair of functors [26; (1.14)]

$$\mathrm{mod}_{\mathrm{sp}}(R_J) \underset{r_J}{\overset{T_J}{\rightleftarrows}} \mathrm{mod}_{\mathrm{sp}}(R) \tag{2.4}$$

having the following properties proved in [26; Corollary 1.16].

LEMMA 2.5. *T_J is full, faithful, $r_J T_J \simeq \mathbf{id}$ and $\mathrm{Im}\, T_J$ is the full subcategory $\mathrm{mod}_{\mathrm{sp}}(R)|_J$ of $\mathrm{mod}_{\mathrm{sp}}(R)$ consisting of modules X such that $P(X) \simeq \bigoplus_{j \in J} P_j^{s_j}$. Moreover, T_J preserves monomorphisms and epimorphisms.*

COROLLARY 2.6. *If R is a schurian artinian right pure sp-semisimple PI-ring then $d_{j*} d'_{j*} \leqslant 3$ for all $j \in I_R$ and $(I_R, \boldsymbol{d})$ is a valued poset.*

Proof. If $J = \{j, *\}$ then $R_J = \begin{bmatrix} F_j & {}_jM \\ 0 & F \end{bmatrix}$ is hereditary and in view of Lemma 2.5 and (a) ⇔ (b) in Theorem 2.3, R_J is right pure semisimple because $\mathrm{mod}_{\mathrm{sp}}(R)$ is cofinite in $\mathrm{mod}(R)$. Then Theorem 2.2 yields $d_{j*} d'_{j*} \leqslant 3$ and by [25; Prop. 2.3], $(I_R, \boldsymbol{d})$ is a valued poset.

THEOREM 2.7. *Let R be a schurian artinian right peak PI-ring. Then the following statements are equivalent*:

(a) *R is right pure sp-semisimple.*

(b) *R is sp-representation-finite.*

(c) *The width $w(R)$ of R* [10, 11] *is $\leqslant 3$ and $(I_R, \boldsymbol{d})$ is a valued poset which does not contain critical peak valued posets* (2,2,2)*, (1,3,3)*, (N,4)*, (1,2,5)*, $\tilde{F}'_{41}$, $\tilde{F}''_{41}$, $\tilde{F}'_{42}$, $\tilde{F}''_{42}$, $\tilde{G}'_2$, $\tilde{G}''_2$ (see [10, Theorem 2], [11, Theorem A]) *as full peak subposets.*

Proof. (b) ⇔ (c) is proved in [11].

(a) ⇒ (c). We know from Corollary 2.6 that $(I_R, \boldsymbol{d})$ is a valued poset and $d_{ij} d'_{ij} \leqslant 3$ for all $i, j \in I_R$. It follows that $w(R) > 3$ if and only if $(I_R, \boldsymbol{d})$ contains as a full peak subposet one of the posets of Fig. 1. Since the functor T_J ((2.4)) preserves monomorphisms and indecomposability (Lemma 2.5), in view of

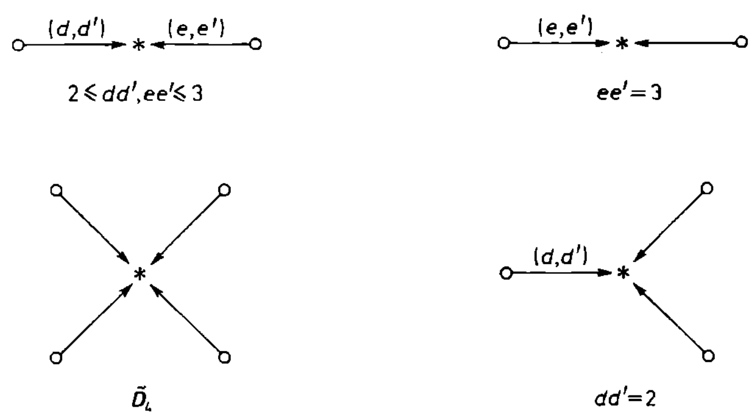

Fig. 1

Theorem 2.3 it is sufficient to prove that if $(I_R, \boldsymbol{d})$ is either one of the valued posets above or one of the critical valued posets then R is not right pure sp-semisimple.

First we consider the case when $(I_R, \boldsymbol{d})$ is the poset $(N,4)^*$. By [11; Lemma 2.13], $\mathrm{mod}_{\mathrm{sp}}(R) \cong (N,4)$-sp and therefore it is of infinite type by [9]. In order to prove that R is not right pure sp-semisimple we proceed as follows. By applying the Nazarova–Roiter differentiation with respect to maximal elements in the poset $(N,4)$ we get in a finite number of steps a poset I' of width 4. It follows from [22; Corollary 6.9] that there exists a full and dense functor ∂: $(N,4)\text{-sp} \to I'\text{-sp}$ such that $\mathrm{Ker}\,\partial$ is generated by finitely many modules (see also [25; Section 5]). Then ∂ induces a representation equivalence ∂: $\mathscr{A} \to I_0\text{-sp} \subseteq I'\text{-sp}$, where I_0 is a subposet of I' consisting of four incomparable elements and $\mathscr{A}$ is a full subcategory of $(N,4)$-sp. Since FI_0^* is a hereditary PI-ring of type $\tilde{\boldsymbol{D}}_4$ and I_0-sp is obviously cofinite in $\mathrm{mod}(FI_0^*)$, Theorem 2.2 shows that FI_0^* is not right pure semisimple. Therefore Theorem 2.1 and the functor ∂ allow us to construct a sequence

$$N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} \cdots \to N_n \xrightarrow{g_n} N_{n+1} \to \cdots$$

of nonzero nonisomorphisms between indecomposable modules in $\mathscr{A}$ such that $g_j g_{j-1} \cdots g_1 \neq 0$ for all $j \geqslant 1$. It follows from Theorem 2.3 that R is not right pure sp-semisimple and (a) $\Rightarrow$ (c) is proved in the case where R is of type $(N,4)^*$.

Now we suppose that R is such that $(I_R, \boldsymbol{d})$ is of one of the forms above or $(I_R, \boldsymbol{d})$ is a critical valued poset different from $(N,4)^*$. Then R is hereditary and a case by case inspection shows that $\mathrm{mod}_{\mathrm{sp}}(R)$ is cofinite in $\mathrm{mod}(R)$ in the sense that all but a finite number of indecomposable modules in $\mathrm{mod}(R)$ are in $\mathrm{mod}_{\mathrm{sp}}(R)$. By Theorem 2.2, R is not of finite representation type and R is not right pure semisimple. Then from Theorems 2.1 and 2.3 we conclude that R is not right pure sp-semisimple, which is a contradiction. This finishes the proof of (a) $\Rightarrow$ (c).

Since (b) $\Rightarrow$ (a) follows from Theorem 2.3, the proof is complete.

3. Representation-finite piecewise prime PI-rings

DEFINITION 3.1. A is a *piecewise prime ring* if A is a semiperfect ring with the property that if $e, f, g \in A$ are primitive orthogonal idempotents such that $eJfJg = 0$, $J = J(A)$, then either $eJf = 0$ or $fJg = 0$ (cf. [8]).

It is clear that the definition is left and right symmetric.

LEMMA 3.2 [27]. *A basic semiprimary ring A is piecewise prime if and only if A has (up to isomorphism) a triangular form*

$$A = \begin{bmatrix} D_1 & {}_1N_2 \cdots {}_1N_m \\ & D_2 \cdots {}_2N_m \\ & \ddots \quad \vdots \\ 0 & D_m \end{bmatrix} \qquad (3.2')$$

where $D_1, \ldots, D_m$ are division rings, ${}_iN_j$ are D_i-D_j-bimodules and the multiplication in A is defined by D_i-D_t-bilinear maps c_{ijt}: ${}_iN_j \otimes {}_jN_t \to {}_iN_t$ satisfying the obvious associativity conditions and $c_{ijt} = 0$ if and only if ${}_iN_j = 0$ or ${}_jN_t = 0$.

Proof. Let $e_1, \ldots, e_m$ be a complete set of primitive orthogonal idempotents ordered in such a way that $e_iJe_j \neq 0$ implies $i < j$. If we put $D_j = e_jAe_j$, ${}_iN_j = e_iAe_j$ and $c_{ijk}(x \otimes y) = x \cdot y$ then we get the triangular form (3.2′) satisfying the required conditions. The proof of the converse implication is left to the reader.

Throughout we denote by $e_1, \ldots, e_m$ the standard set of primitive orthogonal matrix idempotents in the form (3.2′) of A.

Note that if I is a poset and D is a division ring then the path algebra DI is piecewise prime. If R is a right peak ring (1.1) having all bimodules ${}_jM_*$ simple then $A = (1-e_*)R(1-e_*)$ is piecewise prime, where e_* is the peak idempotent. Finally, semiperfect piecewise domains [8, 24] are piecewise prime.

If A is an artinian piecewise prime ring of the form (3.2′) then we associate to A a valued poset $(I_A, \boldsymbol{d})$ in the same way as we did it for right peak rings. We are going to give a characterization of representation-finite piecewise prime rings in terms of $(I, \boldsymbol{d})$.

We call A *homogeneous* if $d_{ij}d'_{ij} \leqslant 1$ for all i, j. A map f: $(I, \boldsymbol{d}) \to (\bar{I}, \bar{\boldsymbol{d}})$ is a *contraction* if $f^{-1}(j)$ is homogeneous and connected for any $j \in \bar{I}$.

The following result follows from [3; Proposition 3.2] and its proof.

LEMMA 3.3. *Let A be a homogeneous piecewise prime ring and let I be the poset I_A with $i \prec j \Leftrightarrow {}_iN_j \neq 0$. If $(I_A, \boldsymbol{d})$ or its contraction does not contain the poset of Fig. 2 as a full subposet then $A \simeq DI$, where $D = D_1 \simeq \ldots \simeq D_m$.*

LEMMA 3.4. *Let A be a basic artinian piecewise prime PI-ring and let $(I_A, \boldsymbol{d})$ be the valued poset of A.*

(a) *If A is right pure semisimple then $d_{ij}d'_{ij} \leqslant 3$ for all $i, j \in I_A$.*

(b) *Suppose that $d_{ij}d'_{ij} \leqslant 3$ for all $i, j \in I_A$ and given $s \in I_A$ put $\hat{e}_s = \sum_{j \preccurlyeq s} e_j$.*

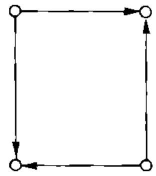

Fig. 2

Then:

(i) $\hat{e}_s A \hat{e}_s$ *is a right peak ring with peak idempotent* e_s.

(ii) *The Cartan matrix of* A (see [11; Section 3])

$$(3.5)\qquad \mathbf{C}(A) = \begin{bmatrix} 1 & d_{12} & & d_{1m} \\ d'_{12} & 1 & & d_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ d'_{1m} & d'_{2m} & & 1 \end{bmatrix}$$

is symmetrizable and if $d_{ij} \neq 0$, $d_{jt} \neq 0$ *then* d'_{ij}, d'_{jt}, d_{it}, d'_{it} *are nonzero and*

$$(\mathbf{pp}_1)\qquad d_{it} = d'_{it} = 1 \quad \textit{iff} \quad d_{ij} = d'_{jt} \ \textit{and} \ d'_{ij} = d_{jt};$$

$$(\mathbf{pp}_2)\qquad (d_{it}, d'_{it}) = (d_{ij}, d'_{ij}) \quad \textit{iff} \quad d_{jt} = d'_{jt} = 1,$$
$$= (d_{jt}, d'_{jt}) \quad \textit{iff} \quad d_{ij} = d'_{ij} = 1;$$

$(\mathbf{pp}_3)$ *if, in addition,* $d_{tk} \neq 0$ *and* $d_{jt}d'_{jt} \geqslant 2$ *then either* $d_{ij}d'_{ij} = 1$ *or* $d_{tk}d'_{tk} = 1$.

Proof. (a) follows by the arguments in the proof of Corollary 2.6 with J and $\{i, j\}$ interchanged.

(b) Since A is piecewise prime, the D_i-D_j-bimodule map $\bar{c}_{ijs}$: ${}_iN_j \to \mathrm{Hom}_{D_s}({}_jN_s, {}_iN_s)$ adjoint to c_{ijs} is nonzero. Then $\bar{c}_{ijs}$ is injective because by our assumptions ${}_iN_j$ is a simple bimodule. It follows that $\hat{e}_s A \hat{e}_s$ is a right peak ring. The remaining part follows from the results in [11; Section 3].

Let us consider the class of all matrices C of the form (3.5) with natural entries satisfying the following conditions:

1° If $d_{ij} \neq 0$ and $d_{jt} \neq 0$ then $d_{it} \neq 0$ for $1 \leqslant i, j, t \leqslant m$.

2° C is symmetrizable, i.e. there exist nonzero natural numbers $f_1, \ldots, f_m$ such that $d_{ij}f_j = f_i d'_{ij}$ for all $1 \leqslant i, j \leqslant m$.

3° $d_{ij}d'_{ij} \leqslant 3$ for all $1 \leqslant i, j \leqslant m$.

4° The rules $(\mathbf{pp}_1)$–$(\mathbf{pp}_3)$ above are valid.

Applying the same type of arguments as in the proof of Theorem 3.8 in [11] one can prove the following realization result.

PROPOSITION 3.6. *Let* C *be a matrix of the form* (3.5) *with natural entries satisfying conditions* **1°**–**4°** *above. Then there exists a finite-dimensional piecewise prime algebra* A *over a field* k *such that* $C = \mathbf{C}(A)$.

Let $C = \mathbf{C}(A)$ be a matrix as above and let C' be the upper-triangular matrix obtained from C by replacing all d'_{ij}'s by zeros. Let F be the diagonal matrix with entries $f_1, \ldots, f_m$ satisfying **2°** for C. Set $D = C' \cdot F$. The matrix D is invertible and D^{-T} defines the bilinear form

$$\langle -, - \rangle : Q^m \times Q^m \to Q, \quad \langle x, y \rangle = xD^{-T}y^T,$$

and the quadratic form

$$\chi_A : Q^m \to Q, \quad \chi_A(x) = \langle x, x \rangle = xD^{-T}x^T. \tag{3.7}$$

Note that if A is a finite-dimensional piecewise prime algebra over a field k and $C = \mathbf{C}(A)$, then $D = \mathbf{D}(A)$ is the Cartan matrix of A in the sense of [16; 2.4]. Moreover, similarly to [4], [16; 2.4], it follows that if X, Y are A-modules with proj.dim $X < \infty$ or inj.dim $Y < \infty$, then

$$\langle \dim X, \dim Y \rangle = \sum_{t \geq 0} (-1)^t \dim \operatorname{Ext}^t_A(X, Y).$$

It is easy to see that if B is a basic artinian piecewise prime PI-ring such that $\mathbf{C}(B) = \mathbf{C}(A)^T$ and $f = f_1 \cdot \ldots \cdot f_m$, then

$$\chi_B(x) = x\left(f \cdot F^{-1} \cdot \mathbf{C}(A)'\right)^{-T} x^T. \tag{3.8}$$

Now we are able to prove the main result of this section.

THEOREM 3.9. *Let A be a basic artinian piecewise prime PI-ring of the form* (3.2′) *and suppose that the valued poset $(I_A, \mathbf{d})$ is connected. The following statements are equivalent*:

(1) *A is representation-finite.*

(2) *A is right pure semisimple.*

(3) *$(I_A, \mathbf{d})$ is symmetrizable and the quadratic form* (3.7) *is weakly positive.*

(4) *$d_{ij}d'_{ij} \leq 3$ for all $i, j \in I_A$ and $(I_A, \mathbf{d})$, $(I_A, \mathbf{d})^{\mathrm{op}}$ have no contraction containing as a full valued subposet one of the following critical PP-posets*:

(i) *the extended Dynkin diagrams* [6];

(ii) *the minimal wild graphs* $\circ \xrightarrow{(d,d')} \circ \xleftarrow{(e,e')} \circ$, $2 \leq dd' < ee' \leq 3$ *or* $dd' = ee' = 3$;

(iii) *the crucial posets of Fig.* 3 (see [12, 13, 3, 24, 27]).

(5) *$(I_A, \mathbf{d})$ is a valued full subposet of one of the following forms or its dual*:

(i) *Dynkin diagrams* [6];

(ii) *Loupias posets of finite representation type* [12, 13];

(iii) *nonhomogeneous representation-finite valued PP-posets of Fig.* 4.

(In Figs. 3 and 4, $d_{st} = d'_{st} = 1$ if s and t are black points and $\circ\!-\!\!-\!\circ$ means either $\circ \!\longrightarrow\! \circ$ or $\circ \!\longleftarrow\! \circ$.)

Proof. (1) ⇒ (2) follows from Theorem 2.1.

(2) ⇒ (4). It follows from Corollary 2.6 that $d_{ij}d'_{ij} \leq 3$ and since the rings

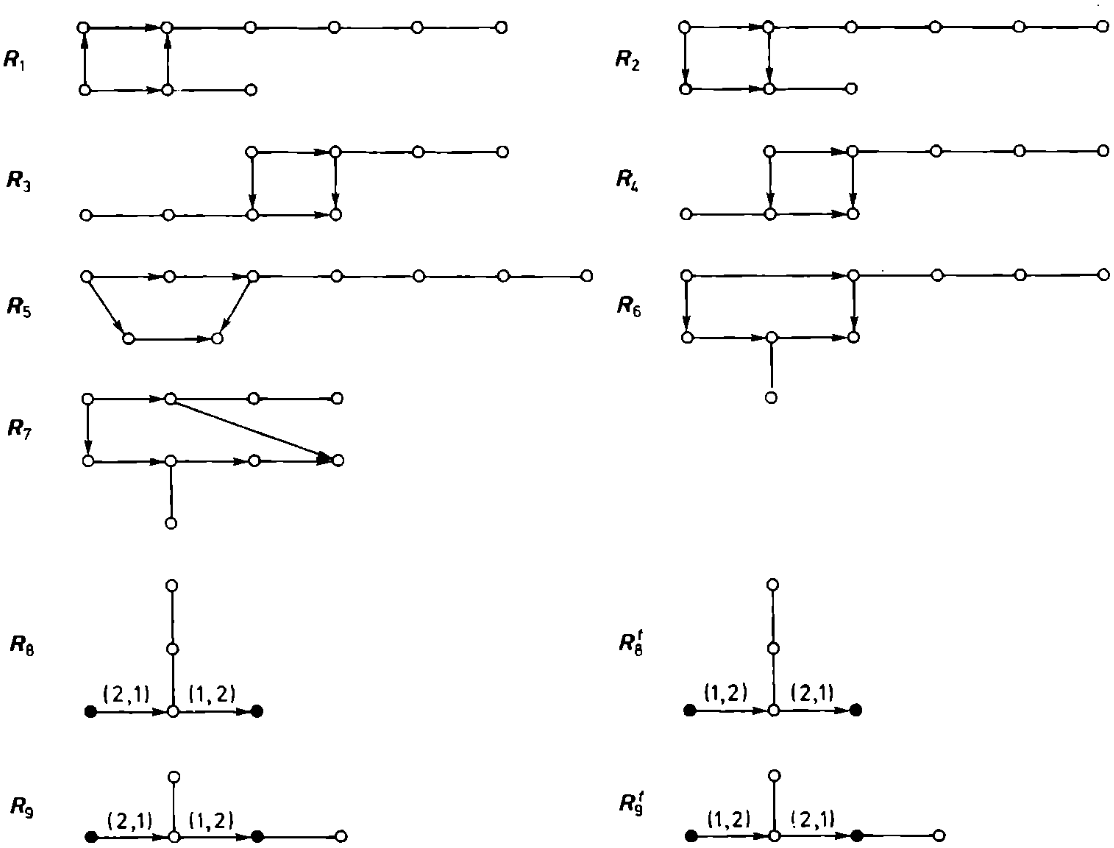

Fig. 3

B corresponding to the crucial posets R_1–R_7 and R_8, R_9 are l-hereditary, it follows that if (I_A, d) has a contraction containing these crucial posets then by [24; Theorem 2.5], A is not right pure semisimple because there is a full and faithful embedding $\mathrm{mod}(B) \to \mathrm{mod}(A)$. If B is of the type R_8^t, R_9^t then applying the triangular reduction [25; Theorem 4.1] we show similarly to [24; p. 171] that there are a ring epimorphism $A \to S$, a schurian artinian right peak PI-ring R and a full dense functor G_+: $\mathrm{mod}(A) \to \mathrm{mod}_{sp}(R)$ such that $\mathrm{Ker}\, G_+ = [\mathrm{mod}(S)]$ and (I_R, d) contains an extended Dynkin diagram. It follows from Theorem 2.7 that R is not right pure sp-semisimple and by Theorem 2.3, A is not right pure semisimple; a contradiction.

(4) $\Rightarrow$ (5). If A is homogeneous then in view of Lemma 3.3, $A \simeq DI_A$ and (5) follows from [12, 13]. If A is not homogeneous then we are in the situation of Lemma 3.4(b) and a simple combinatorial analysis involving the rules (pp_1)–(pp_3) shows that (I_A, d) or its dual is either a Dynkin diagram or a full valued subposet of one of the PP-posets of Fig. 4.

(5) $\Rightarrow$ (1). If either A is homogeneous or (I_A, d) is one of the forms $PP_1 - PP_{10}$ (see Fig. 4) then A is l-hereditary and by [24; Theorem 2.5], A is representation-finite. There remains the case when (I_A, d) is of one of the forms $PP_1' - PP_{10}'$. Similarly to [24] we can proceed by induction on $|I_A|$ and apply to

A the triangular reduction [25]. In each case we get an sp-representation-finite schurian right peak PI-ring R, a representation-finite piecewise prime factor ring S of A and an equivalence of categories $\operatorname{mod}(A)/[\operatorname{mod}(S)] \cong \operatorname{mod}_{sp}(R)$. Hence we conclude that A is representation-finite.

(5) $\Rightarrow$ (3). Suppose that $(I_A, \mathbf{d})$ is of one of the types (i)–(iii) in (5). By a simple analysis of each of the possible finite type poset forms of $(I_A, \mathbf{d})$ presented in [13] and all possible PP-forms for $(I_A, \mathbf{d})$ in Fig. 4 one can show that $\operatorname{gl.dim} A \leqslant 2$ and that the Auslander–Reiten valued translation quiver (Γ_A, τ) has a complete directed preprojective component [16] because of the separation property for radicals of indecomposable projective A-modules. Since A is of finite type, every indecomposable A-module is directing [16]. Furthermore, looking at all possible shapes of $(I_A, \mathbf{d})$ it is easy to check that there exists a finite-dimensional algebra B such that $(I_A, \mathbf{d}) = (I_B, \mathbf{d})$ and $\mathbf{C}(A) = \mathbf{C}(B)$. Hence B has the properties mentioned above for A. Now using the same type of argument as in [4] or in [16; 2.4] we get (3).

(3) $\Rightarrow$ (4). It is easy to check that if either $d_{ij}d'_{ij} \geqslant 4$ for some i, j, or $(I_A, \mathbf{d})$ or $(I_A, \mathbf{d})^{\text{op}}$ has one of the forms (i)–(iii) in (4) then the form (3.7) is not weakly positive. This finishes the proof.

By the discussion in the proof of (5) $\Rightarrow$ (3) and the results in [16; 2.4] we get

COROLLARY 3.10. *If A is a representation-finite piecewise prime PI-ring then*:

(a) $\operatorname{gl.dim} A \leqslant 2$.

(b) *The Auslander–Reiten valued translation quiver Γ_A of A has a complete preprojective component which is simply connected in the sense of* [5, 15].

(c) *If X is an indecomposable A-module then X is directing in the sense of Ringel* [16; 2.4], $\operatorname{Ext}^1_A(X, X) = 0$ *and* $\operatorname{End}(X) \cong D_j$ *for some j. Moreover, X is uniquely determined by its composition factors.*

Remark 3.11. Suppose that A, B are basic representation-finite piecewise prime artinian PI-rings such that the Cartan matrix $\mathbf{C}(A)$ is the transpose $\mathbf{C}(B)^T$ of $\mathbf{C}(B)$. Denote by $\tilde{A}$ and $\tilde{B}$ the Auslander rings of A and B, respectively. Since by Corollary 3.10 the Auslander–Reiten valued translation quivers Γ_A and Γ_B are simply connected, they can be constructed by the well-known cokernel procedure starting from hereditary projective modules and radicals of indecomposable projective modules. An analysis of this construction shows that the translation quivers obtained from Γ_A and Γ_B by forgetting the values over edges are isomorphic. Moreover, we have $\mathbf{C}(\tilde{A}) = \mathbf{C}(\tilde{B})^T$. It would be interesting to give a more conceptual explanation of this phenomenon which has an analogue for sp-representation-finite schurian right peak PI-rings studied in [11].

In connection with this problem we have the following result which is a simple consequence of the criterion in [29] and the formula (3.8).

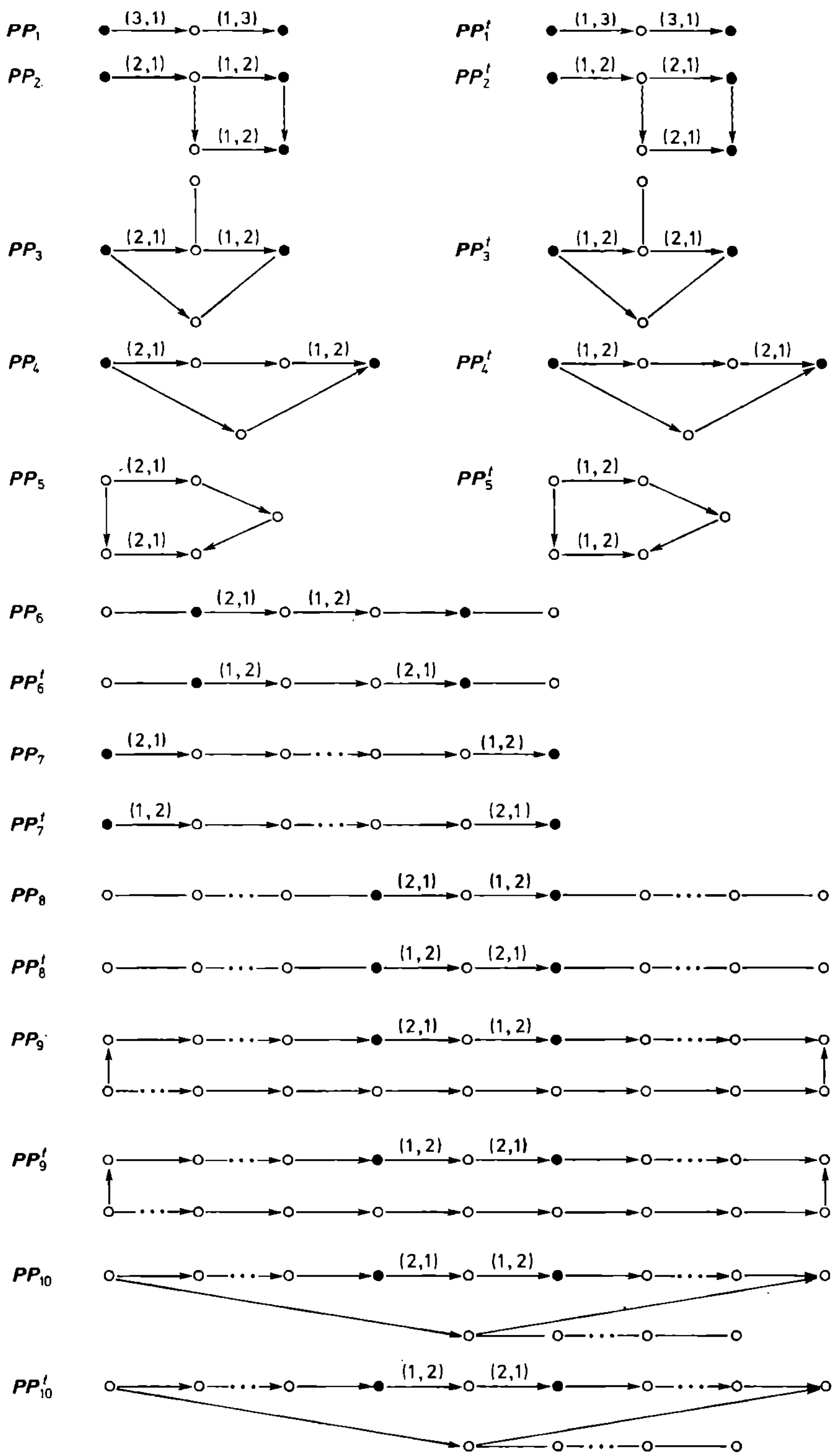

Fig. 4. Nonhomogeneous representation-finite valued PP-posets

LEMMA 3.12. *Let $C = C(A)$ be a matrix of the form* (3.5) *satisfying condition* $2°$ *and let* $C(B) = C^T$. *Then* χ_A *is weakly positive (weakly nonnegative) if and only if so is* χ_B.

References

[0] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Math. 13, Springer, 1973.

[1] M. Auslander, *Representation theory of artin algebras II*, Comm. Algebra 1 (1974), 269–310.

[2] —, *Large modules over artin algebras*, in: Algebra, Topology and Category Theory, Academic Press, New York 1976, 3–17.

[3] R. Bautista, *On algebras close to hereditary Artin algebras*, An. Inst. Mat. Univ. Nac. Autónoma México 21 (1) (1981), 21–104.

[4] K. Bongartz, *Algebras and quadratic forms*, J. London Math. Soc. 28 (1983), 461–469.

[5] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Invent. Math. 65 (1982), 331–378.

[6] V. Dlab and C. M. Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. 173 (1976).

[7] P. Gabriel, *Indecomposable representations II*, in: Sympos. Math. Ist. Naz. Alta Math. 11, Academic Press, 1973, 81–104.

[8] R. Gordon and L. W. Small, *Piecewise domains*, J. Algebra 23 (1972), 553–564.

[9] M. M. Kleiner, *Partially ordered sets of finite type*, Zap. Nauchn. Sem. LOMI 28 (1972), 32–41 (in Russian).

[10] B. Klemp and D. Simson, *A diagrammatic characterization of schurian vector space PI-categories of finite type*, Bull. Polish Acad. Sci. Math. 32 (1984), 385–396.

[11] —, —, *Schurian sp-representation-finite right peak PI-rings and their indecomposable socle projective modules*, J. Algebra (1990), to appear.

[12] M. Loupias, *Représentations indécomposables des ensembles ordonnés finis*, Thèse, Université François Rabelais de Tours, 1975.

[13] —, *Indecomposable representations of finite ordered sets*, in: Lecture Notes in Math. 488, Springer, 1975, 201–209.

[14] L. A. Nazarova and A. V. Roiter, *Representations of partially ordered sets*, Zap. Nauchn. Sem. LOMI 28 (1972), 5–31 (in Russian).

[15] C. M. Ringel, *Kawada's theorem*, in: Abelian Group Theory, Lecture Notes in Math. 874, Springer, 1981, 431–447.

[16] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.

[17] D. Simson, *Functor categories in which every flat object is projective*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 22 (1974), 375–380.

[18] —, *Pure semisimple categories and rings of finite representation type*, J. Algebra 48 (1977), 290–296.

[19] —, *On pure global dimension of locally finitely presented Grothendieck categories*, Fund. Math. 96 (1977), 91–116.

[20] —, *On pure semisimple Grothendieck categories, II*, ibid. 110 (1980), 107–116.

[21] —, *Partial Coxeter functors and right pure semisimple hereditary rings*, J. Algebra 71 (1981), 195–218.

[22] —, *Representations of partially ordered sets, vector space categories and socle projective modules*, Paderborn 1983, 141 pp.

[23] —, *Indecomposable modules over one-sided serial local rings and right pure semisimple rings*, Tsukuba J. Math. 7 (1983), 87–103.

[24] —, *Right pure semisimple l-hereditary PI-rings*, Rend. Sem. Mat. Univ. Padova 71 (1984), 141–175.
[25] —, *Vector space categories, right peak rings and their socle projective modules*, J. Algebra 92 (1985), 532–571.
[26] —, *Socle reductions and socle projective modules*, ibid. 103 (1986), 18–68.
[27] —, *On methods for the computation of indecomposable modules over artinian rings*, in: Reports of 28th Symp. on Algebra, Ring Theory and Algebraic Geometry, University of Chiba, Japan, 26–29 July 1982. 143–170.
[28] —, *Moduled categories and adjusted modules over traced rings*, Dissertationes Math. 269 (1990).
[29] M. V. Zel'dich, *A criterion for weak positivity of quadratic forms*, in: Linear Algebra and Representation Theory, Inst. Math. Acad. Sci. Ukrain. SSR, Kiev 1983, 135–137 (in Russian).