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DIFFERENTIABLE SEMIGROUPS

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Suppose S is a topological semigroup, X is a Banach space and, for each x in S, there is a homeomorphism g_x defined from a neighborhood of x in S onto X. In other words, S, with the set of charts $\{g_x\colon x$ is in $S\}$, is a manifold based on the space X. We say that S is *differentiable* if, for each pair (x, y) in $S\times S$, the function

$$V_{(x,y)} = \Big\{\big((a, b), g_{xy}(g_x^{-1}(a)g_y^{-1}(b))\big)\colon (a, b) \text{ is in } X\times X \text{ and } g_x^{-1}(a)g_y^{-1}(b) \text{ is in } \operatorname{dom}(g_{xy})\Big\}$$

is continuously Fréchet differentiable on a neighborhood of $(0, 0)$.

We show that if S is a differentiable semigroup and S contains an idempotent e, then the maximal subgroup $H(e)$ of S containing e is a differentiable group and is an open subset of eSe. Moreover, there is an open subsemigroup of Se containing e which is, topologically and algebraically, the product of a left trivial subsemigroup L of S with $H(e)$. Each component of L is a submanifold of S, and dual results are given for eS.

Note that, in the definition of a differentiable semigroup, S is not assumed to be a differentiable manifold. If S contains a left or right identity, then the compositions $g_x\circ g_y^{-1}$ for appropriate x and y in S are differentiable and S is a differentiable manifold.

Definition 1. Suppose S is a differentiable semigroup and f is a continuous function from an open subset of S into S. Then f is *differentiable at an element x of* $\operatorname{dom}(f)$ if

$$[g_{f(x)}\circ f\circ g_x^{-1} \,|\, g_x(f^{-1}(\operatorname{dom} g_x))]$$

is continuously Fréchet differentiable on a neighborhood of 0 in X.

Here $[h\,|\,A]$ means the restriction of the function h to the set A.

Definition 2. If S is a differentiable semigroup and M is a subset of S we say that M is a *submanifold of* S if there is a linear continuous idempotent mapping p from X into X so that each point of M has a neighborhood in M homeomorphic to $p(X)$.

In the following, S denotes a differentiable semigroup based on the Banach space X with charts $\{g_x: x \text{ is in } S\}$ which contains at least one idempotent. $E(S)$ denotes the set of all idempotents of S and if e is in $E(S)$, then $H(e)$ denotes the maximal subgroup of S containing e.

The first theorem uses only the manifold structure of S. It is the vehicle for deducing the topological structure of eS, Se and eSe when e is in $E(S)$.

THEOREM 1. *Suppose S is a differentiable semigroup and G is a connected open subset of S. Suppose p is a function from G into G which is differentiable at each point of G and satisfies $p(p(x)) = p(x)$ for each x in G. Then $p(G)$ is a submanifold of S.*

Proof. Suppose x is in $p(G)$ and define r_x by $r_x(a) = g_x(p(g_x^{-1}(a)))$ for each a in X, so that $(g_x^{-1}(a))$ is in the domain of p and $p(g_x^{-1}(a))$ is in the domain of g_x. Since p is differentiable and $p(x) = x$, r_x is differentiable on a neighborhood of 0. Moreover, $r_x(r_x(a)) = r_x(a)$ for each a, so that $r_x(a)$ is in the domain of r_x. In particular, $r_x(0) = 0$.

From the proof of Theorem 1 in [3], there are neighborhoods U and V of 0 in $r_x'(0)(X)$ and $\mathrm{im}(r_x)$, respectively, so that $r_x'(0)\circ[r_x \mid U]$ is a homeomorphism onto a neighborhood of 0 in $r_x'(0)(X)$, $(r_x'(0)\circ[r_x\mid U])^{-1}$ is continuously differentiable on its domain, and $[r_x \mid U]$ is a homeomorphism onto V.

$g_x^{-1}\circ[r_x \mid U]$ is a homeomorphism onto a neighborhood of x in $p(G)$. Thus, for each x in $p(G)$, there is a homeomorphism h_x from a neighborhood of x in $p(G)$ onto a neighborhood of 0 in $r_x'(0)(X)$. This implies, since $p(G)$ is connected, that if each of x and y is in $p(G)$, then $r_x'(0)(X)$ is homeomorphic to $r_y'(0)(X)$, and hence that $p(G)$ is a manifold.

Note from the chain rule that, for each x in $p(G)$, $r_x'(0)$ is an idempotent linear transformation on X, and hence that $p(G)$ is a submanifold of S.

COROLLARY 1.1. *If S is a differentiable semigroup and e is in $E(S)$, then each component of each of eS, Se and eSe is a submanifold of S.*

Proof. Define f on S by $f(x) = ex$ for each x in S. If x is in S, then

$$g_{f(x)}(f(g_x^{-1}(a))) = g_{ex}(g_e^{-1}(0)g_x^{-1}(a)) = V_{ex}(0, a)$$

for each a satisfying $f(g_x^{-1}(a))$ is in $\mathrm{dom}(g_{f(x)})$. Thus f is differentiable. Clearly, $f(f(x)) = f(x)$ for each x in S.

Suppose C is a component of $f(S) = eS$. Now, C is contained in some component D of S, and D is open in S since S is locally connected and $f(D) = C$. Thus, by Theorem 1, C is a submanifold of S.

Define g on S by $g(x) = xe$, and p_e on S by $p_e(x) = g(f(x)) = exe$. Each of g and p_e is differentiable, and an analogous argument to the one just given yields our conclusion for Se and eSe.

Corollary 1.1 allows us to see that not every semigroup on a manifold is differentiable. Denote by R the Banach space of real numbers and define the continuous associative multiplication m on R by $m(x, y) = \min\{x, y\}$. Then $m(0, 0) = 0$ and $m(\{0\} \times S) = (-\infty, 0]$. This is not a manifold, so (R, m) is not differentiable.

THEOREM 2. *Suppose S is a differentiable semigroup and e is in $E(S)$. Then $H(e)$ is an open subset of eSe and $H(e)$ is a topological group.*

Proof. Let $p(a) = g_e\big(p_e(g_e^{-1}(a))\big)$ for each a in X such that $p_e(g_e^{-1}(a))$ is in $\mathrm{dom}(g_e)$. Then there is a connected neighborhood G of 0 in X, so that $p(G)$ is contained in G, $p\big(p(x)\big) = p(x)$ for each x in G, and $g_e^{-1}\big(p(G)\big) = p\big(g_e^{-1}(G)\big)$ is a neighborhood of e in $p_e(S)$.

As in the proof of Theorem 1, choose U to be a neighborhood of 0 in $Y = p'(0)(X)$, so that $[p \mid U]$ is a homeomorphism onto a neighborhood of 0 in the image of p and $p'(0) \circ [p \mid U]$ is a reversibly continuously differentiable homeomorphism onto a neighborhood of 0 in Y. Define W by

$$W(a, b) = [p \mid U]^{-1}\big(V_{ee}(p(a), p(b))\big)$$

whenever each of a and b is in U and $V_{ee}\big(p(a), p(b)\big)$ is in the domain of $[p \mid U]^{-1}$. We see that W, as in the proof of Theorem 2 in [3], is associative and continuously differentiable on a neighborhood of $(0, 0)$ in $Y \times Y$. Moreover, $W(0, x) = W(x, 0) = x$ for each appropriate x in Y.

$W'(0, 0)(x, y) = x + y$ for each (x, y) in $Y \times Y$, so, by the implicit function theorem ([2], p. 270), there is a neighborhood A of 0 in Y contained in U and a continuously differentiable homeomorphism h from A into A such that $W\big(x, h(x)\big) = W\big(h(x), x\big) = 0$ for each x in A.

$g_e^{-1}\big(p(A)\big)$ is a neighborhood of e in $p_e(S)$, since $p(A)$ is a neighborhood of 0 in the image of p. If x is in $g_e^{-1}\big(p(A)\big)$, then

$$e = x g_e^{-1}\Big(p\Big(h\big([p \mid U]^{-1}(g_e(x))\big)\Big)\Big).$$

Thus

$$x^{-1} = g_e^{-1}\Big(p\Big(h\big([p \mid U]^{-1}(g_e(x))\big)\Big)\Big).$$

Hence $H(e)$ is a neighborhood of e in eSe and the inversion is continuous on a neighborhood of e in $H(e)$. The rest of the argument is standard.

Note that this shows $H(e)$ is an analytical group in the sense of Birkhoff [1].

COROLLARY 2.1. *If S is a compact connected differentiable semigroup, then $E(S)$ is contained in the minimal ideal K of S.*

Proof. Suppose e is in $E(S)$. By Theorem 2, $H(e)$ is open in eSe. Since S is compact, $H(e)$ is closed in S, and hence in eSe. Thus $H(e) = eSe$, since eSe is connected.

Suppose x is in K. Since K is an ideal, exe is in K. Similarly, $exe(exe)^{-1} = e$ is in K, so $E(S)$ is contained in K.

LEMMA. *If each of p and U is as in the proof of Theorem 2, then $[p \mid U]^{-1} \circ p$ is continuously differentiable on a neighborhood of 0 in X.*

Proof. Recall from the proof of Theorem 2 that $(p'(0) \circ [p \mid U])^{-1}$ is continuously differentiable on $p'(0)(p(U))$. Now,

$$(p'(0) \circ [p \mid U])^{-1} = [p \mid U]^{-1} \circ [p'(0) \mid p(U)]^{-1},$$

and

$$[[p \mid U]^{-1} \circ p \mid p^{-1}(p(U))] = [[p \mid U]^{-1} \circ [p'(0) \mid p(U)]^{-1} \circ p'(0) \circ p \mid p^{-1}(p(U))].$$

Hence $[p \mid U]^{-1} \circ p$ is the composition of continuously differentiable functions and we are done.

Let $p_e^{-1}(H(e)) = S^e$. By Theorem 2, S^e is open in S. Theorem 3 begins the study of the structure of $Se \cap S^e$.

THEOREM 3. *If S is a differentiable semigroup and e is in $E(S)$, then $L = E(S) \cap Se \cap S^e$ is a left trivial subsemigroup of S and each component of L is a manifold. Moreover, if f is in L, then $H(f)$ is topologically isomorphic to $H(e)$.*

Proof. If x is in S^e, then exe is in $H(e)$ and hence $(exe)^{-1}$ (the inversion relative to $H(e)$) exists. Define K on S^e by $K(x) = x(exe)^{-1}$. Then $K(x)$ is in Se for each x in S^e,

$$eK(x)e = eK(x) = ex[e(exe)^{-1}] = (exe)(exe)^{-1} = e$$

is in $H(e)$, and

$$K(x)K(x) = x(exe)^{-1}x(exe)^{-1} = x(exe)^{-1}(exe)(exe)^{-1} = x(exe)^{-1}e = K(x).$$

Thus $K(x)$ is in L for each x in S^e. If f is in L, then

$$(efe)(efe) = (ef)(ef) = eff = ef = efe,$$

so $K(f) = f$. Hence $L = K(S^e)$.

If each of x and y is in S^e, then

$$K(x)K(y) = x(exe)^{-1}y(eye)^{-1} = x(exe)^{-1}(eye)(eye)^{-1} = K(x),$$

so L is left trivial.

If f is in L and x is in Sf, then $x = xf = xfe$, so $xe = x$ and x is in Se. Similarly, Se is contained in Sf, so $Se = Sf$.

Suppose f is in L. Define h_1 on $H(f)$ by $h_1(x) = ex$ and h_2 on $H(e)$ by $h_2(x) = fx$. If each of x and y is in $H(f)$, then

$$h_1(xy) = exy = (exe)y = h_1(x)h_1(y),$$

so h_1 is a homomorphism. If x is in $H(f)$ and y is chosen in $H(f)$ so that $xy = yx = f$, then $h_1(x)h_1(y) = exy = ef = e$, so $h_1(x)$ is in $H(e)$. Similarly, h_2 is a homomorphism from $H(e)$ into $H(f)$.

If x is in $H(e)$, then $h_1(h_2(x)) = h_1(fx) = efx = ex = x$, and if x is in $H(f)$, then $h_2(h_1(x)) = x$, so $H(f)$ and $H(e)$ are topologically isomorphic.

Define q on S^e by $q(x) = (exe)^{-1}$. From the proof of Theorem 2 there is a neighborhood B of e in S so that, for each x in B,

$$q(x) = g_e^{-1}\Big(p\Big(h\Big([p \mid U]^{-1}\big(g_e(p_e(x))\big)\Big)\Big)\Big),$$

where p, U and h are as in the proof of Theorem 2.

Now,

$$g_e\big(q(g_e^{-1}(x))\big) = p\Big(h\Big([p \mid U]^{-1}\big(g_e(p_e(g_e^{-1}(x)))\big)\Big)\Big)$$

for each x in $g_e(B)$. Thus

$$\begin{aligned}[g_e \circ q \circ g_e^{-1} \mid g_e(B)] &= [p \circ h \circ [p \mid U]^{-1} \circ g_e \circ p_e \circ g_e^{-1} \mid g_e(B)] \\ &= [p \circ h \circ ([p \mid U]^{-1} \circ p) \mid g_e(B)].\end{aligned}$$

For the last equality recall the definition of p. By the Lemma, $[p \mid U]^{-1} \circ p$ is continuously differentiable on a neighborhood of 0. Each of p and h is continuously differentiable, so $g_e \circ q \circ g_e^{-1}$ is continuously differentiable on a neighborhood of 0.

Thus, since the multiplication of S is differentiable, $g_e \circ K \circ g_e^{-1}$ is continuously differentiable on a neighborhood of 0. Now,

$$K(K(x)) = x(exe)^{-1}[ex(exe)^{-1}]^{-1} = x(exe)^{-1}e = K(x)$$

for each x in S^e. Thus, by Theorem 1 of [3], the image of $g_e \circ K \circ g_e^{-1}$ is locally homeomorphic to $(g_e \circ K \circ g_e^{-1})'(0)(X)$. Hence there is a neighborhood of e in L which is homeomorphic to $(g_e \circ K \circ g_e^{-1})'(0)(X)$.

Suppose f is in L and define u on $S^f = \{x \text{ in } S\colon fxf \text{ is in } H(f)\}$ by $u(x) = x(fxf)^{-1}$ (here the inversion is relative to $H(f)$). As in the beginning of this proof, $L_f = E(S) \cap Sf \cap S^f$ is a left trivial subsemigroup of S which is the image of u.

Suppose x is in L_f. Now L_f is contained in Sf and $Sf = Se$, so x is in Se. Since $ex = (ef)x = e(fx) = ef = e$, we have

$$K(x) = x(exe)^{-1} = x(ex)^{-1} = x(e)^{-1} = xe = x$$

and x is in L. Similarly, L is contained in L_f, so $L = L_f$.

By the argument just given for e, there is a neighborhood of f in $L_f = L$ homeomorphic to $(g_f \circ u \circ g_f^{-1})'(0)(X)$. Thus each point of L has a neighborhood homeomorphic to a Banach space, and hence each component of L is a manifold.

THEOREM 4. *If each of S and e is as above, then $Se \cap S^e$ is topologically isomorphic to $L \times H(e)$.*

Proof. Define i on $Se \cap S^e$ by $i(x) = (K(x), p_e(x))$; i is continuous and into $L \times H(e)$. Define j on $L \times H(e)$ by $j(f, g) = fg$; j is continuous and into $Se \cap S^e$.

If (f, g) is in $L \times H(e)$, then

$$i(fg) = (fg(efge)^{-1}, efge) = (fgg^{-1}, g) = (f, g),$$

and if x is in $Se \cap S^e$, then

$$j(i(x)) = x(exe)^{-1}(exe) = xe = x,$$

so $i = j^{-1}$ is a homeomorphism.

If each of (f, g) and (f', g') is in $L \times H(e)$, then

$$j((f, g)(f', g')) = j(ff', gg') = ff'gg' = fgg' = f(gf')g' = (fg)(f'g') = j(f, g)j(f', g'),$$

so j is an isomorphism.

We will now state the dual result for $eS \cap S^e$.

THEOREM 4′. *Suppose S is a differentiable semigroup with idempotent e. Then*

$$R = \{(exe)^{-1}x\colon x \text{ is in } S^e\} = E(S) \cap eS \cap S^e$$

is a right trivial subsemigroup of S and each component of R is a manifold. Moreover, $eS \cap S^e$ is a subsemigroup of S and is topologically isomorphic to the product semigroup $H(e) \times R$.

Theorems 4 and 4′ are characterizations, on an open subsemigroup containing e, of differentiable semigroups with right (left) identity e.

COROLLARY 4.1. *If S is a differentiable semigroup, e is in $E(S)$, and e is isolated in each of R and L, then there is a neighborhood V of e so that $ex = xe$ for x in V.*

Proof. Choose N to be a neighborhood of e in S which contains no member of $R \cup L$, and let V be a neighborhood of e in S^e so that if x is in V, then each of $x(exe)^{-1}$ and $(exe)^{-1}x$ is in N.

If x is in V, then $x(exe)^{-1} = e = (exe)^{-1}x$. Hence

$$xe = x(exe)^{-1}(exe) = exe = (exe)(exe)^{-1}x = ex.$$

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