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Spaces of Lipschitz type, embeddings and entropy numbers

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Abstract

We establish the sharpness of the embedding of certain Besov and Triebel–Lizorkin spaces in spaces of Lipschitz type. In particular, this proves the sharpness of the Brézis–Wainger result concerning the "almost" Lipschitz continuity of elements of the Sobolev space $H_p^{1+n/p}(\mathbb{R}^n)$, where $1 < p < \infty$. Upper and lower estimates are obtained for the entropy numbers of related embeddings of Besov spaces on bounded domains.

Introduction

It is well known (though not very well documented in the literature) that functions in the (fractional) Sobolev space $H_p^{1+n/p}(\mathbb{R}^n)$, when $1 < p < \infty$, are Hölder-continuous with exponent α for any $\alpha \in (0,1)$ but need not be Lipschitz-continuous. This limiting situation was clarified in an important paper by Brézis and Wainger [3] in which it was shown that every function u in $H_p^{1+n/p}(\mathbb{R}^n)$ is "almost" Lipschitz-continuous, in the sense that for all $x, y \in \mathbb{R}^n$, $x \neq y$, $|x-y| < 1/2$,

$$|u(x) - u(y)| \leq c|x-y|\, |\log |x-y||^{1/p'} \|u \,|\, H_p^{1+n/p}(\mathbb{R}^n)\|. \tag{0.1}$$

Here c is a constant independent of x, y and u, and $1/p' + 1/p = 1$. We are thus led to define a function space of Lipschitz type, $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$, with $\alpha \geq 0$, to be the space of all functions $f \in C(\mathbb{R}^n)$ such that

$$\|f \,|\, \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| := \|f \,|\, L_\infty(\mathbb{R}^n)\| + \sup_{\substack{x,y \in \mathbb{R}^n \\ 0<|x-y|<1/2}} \frac{|f(x) - f(y)|}{|x-y|\, |\log |x-y||^\alpha} \tag{0.2}$$

is finite. With this definition, (0.1) simply means that $H_p^{1+n/p}(\mathbb{R}^n)$ is continuously embedded in $\mathrm{Lip}^{(1,-1/p')}(\mathbb{R}^n)$; we write this as

$$H_p^{1+n/p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-1/p')}(\mathbb{R}^n). \tag{0.3}$$

The present paper has two main objectives. Firstly, we show that the embedding (0.3) is sharp in the sense that

$$H_p^{1+n/p}(\mathbb{R}^n) \not\hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$$

if $\alpha < 1/p'$. In fact, considerably more general sharpness assertions are established, dealing with embeddings of Besov and Triebel–Lizorkin spaces in spaces of Lipschitz type. The proofs are based on unpublished notes of H. Triebel: we are indebted to Professor Triebel for his encouragement to publish them here. For other work on sharpness of related embeddings see also [7], [8] and [9].

Secondly, we discuss the compactness of embeddings into spaces of Lipschitz type and analyse these embeddings from the standpoint of entropy numbers. The study of entropy numbers of embeddings between function spaces is closely related to the distribution of eigenvalues of (degenerate) elliptic operators, as the books [12] and [24] show. We are motivated by this and also by the fact that nothing appears to have been proved before now about entropy numbers of the embeddings which we consider.

To explain the results in a little more detail, let Ω be a bounded domain in $\mathbb{R}^n$ and define spaces $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$ by a natural adaptation of (0.2). We consider the embedding

$$\mathrm{id} : B^{1+n/p}_{p,q}(\Omega) \to \mathrm{Lip}^{(1,-\alpha)}(\Omega),$$

where $0 < p \le \infty$, $0 < q \le \infty$, $\alpha > \max(1 - 1/q, 0)$ and $B^{1+n/p}_{p,q}(\Omega)$ is the usual Besov space. It is shown that there are positive numbers c_1 and c_2 such that for all $k \in \mathbb{N}$ with $k > 1$ the entropy numbers $e_k(\mathrm{id})$ of this embedding satisfy

$$c_1 k^{-1/p} (\log k)^{-\alpha} \le e_k(\mathrm{id}) \le c_2 (\log k)^{-\alpha+\max(1-1/q,0)}.$$

In particular, when $0 < q \le 1$, $p = \infty$ and $\alpha > 0$, then

$$e_k(\mathrm{id} : B^1_{\infty,q}(\Omega) \to \mathrm{Lip}^{(1,-\alpha)}(\Omega)) \sim (\log k)^{-\alpha}, \quad k \in \mathbb{N},\ k > 1,$$

in the above sense. This partly confirms an unpublished conjecture of Triebel.

The logarithmic decay and dependence upon the parameter q (for $q > 1$) are especially interesting. Atomic representations of elements of Besov spaces are needed in the proof; a major technical difficulty to be surmounted arises from the need to construct a commutative diagram which reduces certain considerations to the estimation of entropy numbers of embeddings between certain types of weighted sequence spaces.

We also link this work with that of Leopold [16] on the entropy numbers of embeddings between spaces of type $B^{(s,b)}_{p,q}(\Omega)$, which are similar to the usual Besov spaces $B^s_{p,q}(\Omega)$, the additional parameter b providing a finer tuning of smoothness.

Section 1 contains the basic definitions which are needed, together with some prerequisites. The main theorems are given in Sections 2 and 3, and the final section contains a comparison with related results as well as some extension of these.

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1. Preliminaries

Let $\mathbb{R}^n$ be Euclidean n-space and $\langle x \rangle = (2 + |x|^2)^{1/2}$, $x \in \mathbb{R}^n$. In a slight abuse of notation we also use $\langle k \rangle$ to stand for $(2 + k^2)^{1/2}$ when $k \in \mathbb{N}$.

Given two (quasi-)Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. For non-negative functions $f, g : \mathbb{N} \to \mathbb{R}$, the symbol $f(k) \sim g(k)$ means that there are positive numbers c_1, c_2 such that for all $k \in \mathbb{N}$,

$$c_1 f(k) \le g(k) \le c_2 f(k).$$

All unimportant positive constants are denoted by c, occasionally with subscripts.

Spaces on $\mathbb{R}^n$. Let $C(\mathbb{R}^n)$ be the space of all complex-valued bounded continuous functions on $\mathbb{R}^n$, equipped with the sup-norm as usual. If $m \in \mathbb{N}$, we define

$$C^m(\mathbb{R}^n) = \{f : D^\alpha f \in C(\mathbb{R}^n) \text{ for all } |\alpha| \le m\}.$$

Here D^α are classical derivatives and $C^m(\mathbb{R}^n)$ is endowed with the norm

$$\|f\,|\,C^m(\mathbb{R}^n)\| = \sum_{|\alpha|\le m} \|D^\alpha f\,|\,L_\infty(\mathbb{R}^n)\|.$$

Definition 1.1. Let $\alpha \ge 0$. Then the space $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ is defined as the set of all $f \in C(\mathbb{R}^n)$ such that

$$\|f\,|\,\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| = \|f\,|\,L_\infty(\mathbb{R}^n)\| + \sup_{\substack{x,y\in\mathbb{R}^n\\ 0<|x-y|<1/2}} \frac{|f(x)-f(y)|}{|x-y|\,|\log|x-y||^\alpha} \tag{1.1}$$

is finite.

This definition was first suggested by Triebel in his unpublished notes in 1997. Note that $\mathrm{Lip}^{(1,0)}(\mathbb{R}^n)$ is just the usual space of Lipschitz-continuous functions on $\mathbb{R}^n$, and $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ would consist only of constants were α allowed to be negative.

Remark 1.2. The somehow unusual notation using $-\alpha$ instead of α in (1.1) is simply due to the fact that we want to emphasise that the additional smoothness parameter α acts in such a way that the usual spaces $\mathrm{Lip}^1(\mathbb{R}^n)$ are extended: $\mathrm{Lip}^1(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ for all $\alpha \ge 0$, i.e. the spaces become larger when less smoothness is assumed—as it should be in some reasonable notation.

We may rewrite (1.1) using the difference operator Δ_h^m, $m \in \mathbb{N}_0$, $h \in \mathbb{R}^n$. Let $f(x)$ be an arbitrary function on $\mathbb{R}^n$. Then

$$(\Delta_h^1 f)(x) = f(x+h) - f(x), \quad (\Delta_h^{m+1} f)(x) = \Delta_h^1(\Delta_h^m f)(x), \tag{1.2}$$

where $x, h \in \mathbb{R}^n$. For convenience we may write Δ_h instead of Δ_h^1. Then (1.1) reads as

$$\|f\,|\,\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| = \|f\,|\,L_\infty(\mathbb{R}^n)\| + \sup_{\substack{x,h\in\mathbb{R}^n\\ 0<|h|<1/2}} \frac{|(\Delta_h f)(x)|}{|h|\,|\log|h||^\alpha}. \tag{1.3}$$

We briefly recall the basic ingredients needed to introduce spaces of type $B^s_{p,q}$ and $F^s_{p,q}$, the latter being only needed in Section 2. Furthermore, in Section 4 we compare our results (achieved in Section 3) with the corresponding ones of Leopold [16]. He studied spaces of type $B^{(s,b)}_{p,q}$, $b \in \mathbb{R}$, which extend the scale of usual B-spaces in terms of smoothness. Thus we give here the more general definition of $B^{(s,b)}_{p,q}$ instead of $B^s_{p,q}$. The Schwartz space $S(\mathbb{R}^n)$ and its dual $S'(\mathbb{R}^n)$ of all complex-valued tempered distributions have their usual meaning here. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$ is the usual quasi-Banach space with respect to Lebesgue measure. Let $\varphi \in S(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\varphi \subset \{y \in \mathbb{R}^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if } |x| \le 1,$$

put $\varphi_0 = \varphi$ and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then since $1 = \sum_{j=0}^\infty \varphi_j(x)$ for all $x \in \mathbb{R}^n$, the $\{\varphi_j\}_{j=0}^\infty$ form a *dyadic partition of unity.* Given any $f \in S'(\mathbb{R}^n)$, we denote by $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively.

Definition 1.3. Let $s\in \mathbb{R}$, $b\in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$, and let $\{\varphi_j\}$ be the above dyadic partition of unity. The space $B^{(s,b)}_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \mid B^{(s,b)}_{p,q}(\mathbb{R}^n)\| = \Big(\sum_{j=0}^{\infty} 2^{jsq}(1+j)^{bq}\|\mathcal{F}^{-1}\varphi_j\mathcal{F}f \mid L_p(\mathbb{R}^n)\|^q\Big)^{1/q} \tag{1.4}$$

(with the usual modification if $q=\infty$) is finite.

When $b=0$ this definition coincides with the usual one (see [21, Def. 2.3.1/2, p. 45]). For later reasons (see Section 2), it appears useful to recall the definition of $F^s_{p,q}$ spaces as well.

Definition 1.4. Let $s\in\mathbb{R}$, $0<p<\infty$, $0<q\le\infty$, and let $\{\varphi_j\}$ be the above dyadic partition of unity. The space $F^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f\in S'(\mathbb{R}^n)$ such that

$$\|f \mid F^s_{p,q}(\mathbb{R}^n)\| = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|\mathcal{F}^{-1}\varphi_j\mathcal{F}f(\cdot)|^q\Big)^{1/q} \Big| L_p(\mathbb{R}^n)\Big\| \tag{1.5}$$

(with the usual modification if $q=\infty$) is finite.

Remark 1.5. The theory of the spaces $B^s_{p,q}$ $(b=0)$ and $F^s_{p,q}$ has been developed in detail in [21] and [22]. Recall that these two scales $B^s_{p,q}$ and $F^s_{p,q}$ cover (fractional) Sobolev spaces, Hölder–Zygmund spaces, local Hardy spaces, and classical Besov spaces—characterised via derivatives and differences. The spaces $B^{(s,b)}_{p,q}$, $b\in\mathbb{R}$, were introduced in [16, Def. 1].

Atomic decompositions. We closely follow [24, Sect. 13]. Let $Q_{\nu m}$, $\nu\in\mathbb{N}_0$, $m\in\mathbb{Z}^n$, denote a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-\nu}m$, and with side length $2^{-\nu}$. For a cube Q in $\mathbb{R}^n$ and $r>0$ we mean by rQ the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q.

Definition 1.6. (i) Let $K\in\mathbb{N}_0$ and $d>1$. A K times differentiable complex-valued function a on $\mathbb{R}^n$ (continuous if $K=0$) is called a 1_K-*atom* if

$$\operatorname{supp} a \subset dQ_{0m} \quad \text{for some } m\in\mathbb{Z}^n \tag{1.6}$$

and

$$|D^\alpha a(x)| \le 1 \quad \text{for } |\alpha|\le K. \tag{1.7}$$

(ii) Let $s\in\mathbb{R}$, $0<p\le\infty$, $K\in\mathbb{N}_0$, $L+1\in\mathbb{N}_0$, and $d>1$. A K times differentiable complex-valued function a on $\mathbb{R}^n$ (continuous if $K=0$) is called an $(s,p)_{K,L}$-*atom* if for some $\nu\in\mathbb{N}_0$,

$$\operatorname{supp} a \subset dQ_{\nu m} \quad \text{for some } m\in\mathbb{Z}^n, \tag{1.8}$$

$$|D^\alpha a(x)| \le 2^{-\nu(s-n/p)+|\alpha|\nu} \quad \text{for } |\alpha|\le K \tag{1.9}$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x)\,dx = 0 \quad \text{if } |\beta|\le L. \tag{1.10}$$

This definition coincides with [24, Def. 13.3, p. 73]. The number d in (1.6) and (1.8) is unimportant insofar as it simply makes it clear that at the level ν some controlled overlapping of the supports of $a_{\nu m}$ must be allowed. Assumption (1.10) is called a *moment condition*, where $L=-1$ means that there are no moment conditions. It is convenient to write $a_{\nu m}(x)$ instead of $a(x)$ if this atom is located at $Q_{\nu m}$ according to (1.6) and (1.8). Furthermore, we denote by $\chi^{(p)}_{\nu m}$ the p-normalised characteristic function of the cube $Q_{\nu m}$, that is,

$$\chi^{(p)}_{\nu m}(x) = 2^{\nu n/p} \quad \text{if } x \in Q_{\nu m} \quad \text{and} \quad \chi^{(p)}_{\nu m}(x) = 0 \quad \text{if } x \notin Q_{\nu m},$$

where $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and $0 < p \le \infty$. Plainly, $\|\chi^{(p)}_{\nu m} \,|\, L_p(\mathbb{R}^n)\| = 1$.

Definition 1.7. Let $0 < p \le \infty$, $0 < q \le \infty$, and

$$\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}.$$

Then

$$b_{pq} = \Big\{\lambda : \|\lambda \,|\, b_{pq}\| = \Big(\sum_{\nu=0}^{\infty}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p\Big)^{q/p}\Big)^{1/q} < \infty\Big\}$$

and

$$f_{pq} = \Big\{\lambda : \|\lambda \,|\, f_{pq}\| = \Big\|\Big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}\chi^{(p)}_{\nu m}(\cdot)|^q\Big)^{1/q} \Big| L_p(\mathbb{R}^n)\Big\| < \infty\Big\}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$).

This definition is a modification of the related one in [13] and is given in the above version in [24, Def. 13.5, p. 74]. Recall that

$$\sigma_p = n\Big(\frac{1}{p} - 1\Big)_+ \quad \text{and} \quad \sigma_{pq} = n\Big(\frac{1}{\min(p,q)} - 1\Big)_+ \tag{1.11}$$

where $0 < p \le \infty$ and $0 < q \le \infty$. As usual, let

$$\varkappa_+ = \max(\varkappa, 0) \quad \text{and} \quad [\varkappa] = \max\{k \in \mathbb{Z} : k \le \varkappa\} \quad \text{if } \varkappa \in \mathbb{R}. \tag{1.12}$$

We come to the main theorem now, the atomic characterisation of function spaces of type $B^s_{p,q}$ and $F^s_{p,q}$, respectively.

Theorem 1.8 (Triebel '97). (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, *and* $s \in \mathbb{R}$. *Fix* $K \in \mathbb{N}_0$ *and* $L + 1 \in \mathbb{N}_0$ *with*

$$K \ge (1 + [s])_+ \quad \textit{and} \quad L \ge \max(-1, [\sigma_p - s]). \tag{1.13}$$

Then $f \in S'(\mathbb{R}^n)$ *belongs to* $B^s_{p,q}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x), \quad \textit{convergence being in } S'(\mathbb{R}^n), \tag{1.14}$$

where the $a_{\nu m}$ *are* 1_K*-atoms* ($\nu = 0$) *or* $(s,p)_{K,L}$*-atoms* ($\nu \in \mathbb{N}$) *in the sense of Definition* 1.6, *with*

$$\operatorname{supp} a_{\nu m} \subset dQ_{\nu m}, \quad \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ d > 1, \tag{1.15}$$

and $\lambda \in b_{pq}$. *Furthermore,*

$$\inf \|\lambda \,|\, b_{pq}\|, \tag{1.16}$$

where the infimum is taken over all admissible representations (1.14), *is an equivalent quasi-norm in* $B^s_{p,q}(\mathbb{R}^n)$.

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, *and* $s \in \mathbb{R}$. *Fix* $K \in \mathbb{N}_0$ *and* $L + 1 \in \mathbb{N}_0$ *with*

$$K \ge (1 + [s])_+ \quad \textit{and} \quad L \ge \max(-1, [\sigma_{pq} - s]). \tag{1.17}$$

Then $f \in S'(\mathbb{R}^n)$ *belongs to* $F^s_{p,q}(\mathbb{R}^n)$ *if, and only if, it can be represented by* (1.14), *where the atoms* $a_{\nu m}$ *have the same meaning as in part* (i) (*now perhaps with a different value of* L) *and* $\lambda \in f_{pq}$. *Furthermore,*

$$\inf \|\lambda \mid f_{pq}\|, \tag{1.18}$$

where the infimum is taken over all admissible representations (1.14), *is an equivalent quasi-norm in* $F^s_{p,q}(\mathbb{R}^n)$.

For the proof as well as further remarks and consequences we refer to [24, Thm. 13.8, p. 75].

Spaces on domains. We complement Definition 1.1 by its counterpart for a bounded domain Ω in $\mathbb{R}^n$. Note that in the sequel we always understand a bounded C^∞ domain in $\mathbb{R}^n$ in the sense of [6, Def. V.4.1, p. 244].

Definition 1.9. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, let $\alpha \geq 0$. Then the space $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$ is defined as the set of all $f \in C(\overline{\Omega})$ such that

$$\|f \mid \mathrm{Lip}^{(1,-\alpha)}(\Omega)\| = \|f \mid L_\infty(\Omega)\| + \sup_{\substack{x,y\in\Omega \\ 0<|x-y|<1/2}} \frac{|f(x)-f(y)|}{|x-y|\,|\log|x-y||^\alpha} \tag{1.19}$$

is finite.

Remark 1.10. Note that standard procedures (see, for example, [6, pp. 250–251]) show that there is a bounded extension map from $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$ to $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$.

For later reasons it also appears useful to introduce the spaces $\widetilde{\mathrm{Lip}}^{(1,-\alpha)}(\Omega)$, $\alpha \geq 0$, as subspaces of $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$:

$$\widetilde{\mathrm{Lip}}^{(1,-\alpha)}(\Omega) = \{f \in \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) : \operatorname{supp} f \subset \Omega\}. \tag{1.20}$$

We give the definition for the spaces $B^{(s,b)}_{p,q}(\Omega)$ parallel to [16, p. 8].

Definition 1.11. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $s \in \mathbb{R}$, $b \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Then $B^{(s,b)}_{p,q}(\Omega)$ is the restriction of $B^{(s,b)}_{p,q}(\mathbb{R}^n)$ to Ω, i.e.

$$B^{(s,b)}_{p,q}(\Omega) = \{f \in D'(\Omega) : \exists g \in B^{(s,b)}_{p,q}(\mathbb{R}^n),\ g|_\Omega = f\}.$$

Furthermore,

$$\|f \mid B^{(s,b)}_{p,q}(\Omega)\| = \inf \|g \mid B^{(s,b)}_{p,q}(\mathbb{R}^n)\|$$

where the infimum is taken over all $g \in B^{(s,b)}_{p,q}(\mathbb{R}^n)$ such that $g_{|\Omega} = f$.

We again complement this definition by that for $\widetilde{B}^{(s,b)}_{p,q}(\Omega)$, $s, b \in \mathbb{R}$, $0 < p, q \leq \infty$:

$$\widetilde{B}^{(s,b)}_{p,q}(\Omega) = \{f \in B^{(s,b)}_{p,q}(\mathbb{R}^n) : \operatorname{supp} f \subset \Omega\}. \tag{1.21}$$

The comparison between Definitions 1.9 and 1.11 shows that the approach to spaces on Ω (corresponding to spaces on $\mathbb{R}^n$) is quite different in the two cases: we have adapted the intrinsic characterisation to the bounded domain situation in the case of $\mathrm{Lip}^{(1,-\alpha)}$, whereas the spaces $B^{(s,b)}_{p,q}(\Omega)$ were introduced by restriction. To deal first with spaces of type $\widetilde{\mathrm{Lip}}^{(1,-\alpha)}(\Omega)$ and $\widetilde{B}^{(s,b)}_{p,q}(\Omega)$ causes some problems afterwards. However, in view of Remark 1.10 and following the proofs in detail we can cope with that technicality. One could obviously avoid it from the very beginning by introducing both spaces on Ω in the same way (either by restriction or by intrinsic characterisation) but the above definitions are more natural in our opinion.

Embeddings. We briefly collect what is already known about these spaces, in particular, about (natural) embeddings of these spaces into each other.

Proposition 1.12. *Let* $0 < p \le \infty$, $1 < q \le \infty$, $1/q' = 1 - 1/q$, *as usual. There is some* $c > 0$ *such that for all* $x, y \in \mathbb{R}^n$, $0 < |x-y| < 1/2$, *and all* $f \in B^{1+n/p}_{p,q}(\mathbb{R}^n)$,

$$|f(x) - f(y)| \le c|x-y|\,|\log|x-y||^{1/q'}\,\|f \mid B^{1+n/p}_{p,q}(\mathbb{R}^n)\|.$$

The above proposition is part of Theorem 2.1 which is given in Section 2 in its full extent and proved there. As we already mentioned at the beginning we are indebted to H. Triebel for this result. He stated it together with a sketch of proof in some unpublished notes recently and encouraged us to publish it here.

Together with the already known embedding

$$B^{1+n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow C^1(\mathbb{R}^n) \quad \text{if, and only if, } 0 < q \le 1,\ 0 < p \le \infty \tag{1.22}$$

(cf. [12, (2.3.3/10)]), the above theorem implies that

$$B^{1+n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) \quad \text{if } \alpha \ge (1-1/q)_+, \tag{1.23}$$

where $0 < p \le \infty$, $0 < q \le \infty$. Recall our notation (1.12).

Proposition 1.13 (Leopold '98). *Let* $-\infty < s_2 \le s_1 < \infty$, $b_1, b_2 \in \mathbb{R}$, $0 < p_1 \le p_2 \le \infty$, $0 < q_1, q_2 \le \infty$ *and assume* $s_1 - n/p_1 = s_2 - n/p_2$. *Then*

$$B^{(s_1,b_1)}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{(s_2,b_2)}_{p_2,q_2}(\mathbb{R}^n) \tag{1.24}$$

if, and only if,

$$\begin{aligned} &\textit{either } q_1 \le q_2 \textit{ and } b_1 - b_2 \ge 0 \\ &\quad \textit{or } q_1 > q_2 \textit{ and } b_1 - b_2 > 1/q_2 - 1/q_1. \end{aligned} \tag{1.25}$$

Remark 1.14. The proposition is proved in [16, Thm. 1]. In Section 4 we study the relation between spaces of type $\mathrm{Lip}^{(1,-\alpha)}$ and $B^{(s,b)}_{p,q}$. Then Proposition 1.13 together with these results gives another proof of Proposition 1.12.

Entropy numbers. Note that by Definition 1.3,

$$B^{s+\varepsilon}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{(s,b)}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{s-\varepsilon}_{p,q}(\mathbb{R}^n),$$

where $s, b \in \mathbb{R}$, $0 < p, q \le \infty$, and $\varepsilon > 0$. Thus there cannot be a compact embedding in case of spaces on $\mathbb{R}^n$,

$$\begin{aligned} &\mathrm{id} : B^{(s_1,b_1)}_{p_1,q_1}(\mathbb{R}^n) \to B^{(s_2,b_2)}_{p_2,q_2}(\mathbb{R}^n), \\ &\qquad \text{with } s_1 > s_2,\ 0 < p_1 < p_2 \le \infty, \text{ and } s_1 - n/p_1 = s_2 - n/p_2, \end{aligned} \tag{1.26}$$

for any b_i, q_i, $i = 1, 2$. Otherwise the compactness of (1.26) would imply that of

$$\mathrm{id} : B^{\sigma_1}_{p_1,q_1}(\mathbb{R}^n) \to B^{\sigma_2}_{p_2,q_2}(\mathbb{R}^n),$$

$\sigma_1 > \sigma_2$, $0 < p_1 < p_2 \le \infty$, $\sigma_1 - n/p_1 > \sigma_2 - n/p_2$, $0 < q_1, q_2 \le \infty$, which contradicts, for instance, [14, Thm. 2.3, Rem. 2.3/2].

However, dealing with embeddings of spaces (of the above type) on domains we obtain compactness under certain assumptions. Then it becomes important to characterise the "quality" of that compactness in terms of entropy numbers. This concept has proved very useful in connection with applications to spectral theory: we only mention the distribution of eigenvalues of (degenerate) elliptic operators (via Carl's inequality) and the study of "negative" spectra (via the Birman–Schwinger principle). Though these possible applications of our results are out of the scope of the present paper, this emphasises the interest in the asymptotic behaviour of entropy numbers of certain compact embeddings.

Let us briefly recall the definition of entropy numbers. Let A_1 and A_2 be two complex quasi-Banach spaces and let T be a linear and continuous operator from A_1 into A_2. If T is compact then for any given $\varepsilon > 0$ there are finitely many balls in A_2 of radius ε which cover the image TU_1 of the unit ball $U_1 = \{a \in A_1 : \|a \mid A_1\| \le 1\}$.

Definition 1.15. Let $k \in \mathbb{N}$ and let $T : A_1 \to A_2$ be the above continuous operator. The kth *entropy number* e_k of T is the infimum of all numbers $\varepsilon > 0$ such that there exist 2^{k-1} balls in A_2 of radius ε which cover TU_1.

For details and properties of entropy numbers we refer to [4], [6], [15] and [17] (always restricted to the case of Banach spaces). The extension of these properties to quasi-Banach spaces causes no problems (see [12]). Among other features we only want to mention the multiplicativity of entropy numbers: let A_1, A_2 and A_3 be complex (quasi-)Banach spaces and $T_1 : A_1 \to A_2$, $T_2 : A_2 \to A_3$ two operators as in Definition 1.15. Then

$$e_{k_1+k_2-1}(T_2 \circ T_1) \le e_{k_1}(T_1) e_{k_2}(T_2), \quad k_1, k_2 \in \mathbb{N}. \tag{1.27}$$

Studying entropy numbers we shall be concerned with spaces $B^{(s,b)}_{p,q}(\Omega)$, $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$ in the sequel, where $\Omega \subset \mathbb{R}^n$ is bounded. We give Leopold's recent result [16, Thm. 2] first.

Theorem 1.16 (Leopold '98). *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $s, b \in \mathbb{R}$, $0 < p, q_1, q_2 \le \infty$. Let*

$$\mathrm{id}_\Omega = \mathrm{id} : B^{(s,b)}_{p,q_1}(\Omega) \to B^s_{p,q_2}(\Omega).$$

(i) *If $q_1 \le q_2$ and $b > 0$, then* id_Ω *is compact and*

$$e_k(\mathrm{id}_\Omega) \sim (\log\langle k\rangle)^{-b}. \tag{1.28}$$

(ii) *If $q_1 > q_2$ and $b > 1/q_2 - 1/q_1$, then the embedding* id_Ω *is compact and there exist positive constants c_1 and c_2 such that for all $k \in \mathbb{N}$,*

$$c_1(\log\langle k\rangle)^{-b} \le e_k(\mathrm{id}_\Omega) \le c_2(\log\langle k\rangle)^{-b+1/q_2-1/q_1}. \tag{1.29}$$

Remark 1.17. For convenience let us briefly mention the corresponding result in the non-limiting situation studied by Edmunds and Triebel in [10], [11]: let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain and $-\infty < s_2 < s_1 < \infty$, $0 < p_1, p_2 \le \infty$, $0 < q_1, q_2 \le \infty$, $s_1 - s_2 > n(1/p_1 - 1/p_2)_+$. Then

$$e_k(\mathrm{id}_\Omega : B^{s_1}_{p_1,q_1}(\Omega) \to B^{s_2}_{p_2,q_2}(\Omega)) \sim k^{-(s_1-s_2)/n}, \quad k \in \mathbb{N}.$$

2. Sharpness

It is a standard fact that if $1 < p < \infty$, then every element of the Sobolev space $H_p^{1+n/p}(\mathbb{R}^n)$ is Hölder-continuous with exponent α for every $\alpha \in (0,1)$. It is also true, though a proof is harder to find, that not every such element is Lipschitz-continuous. In [3] Brézis and Wainger showed that every $u \in H_p^{1+n/p}(\mathbb{R}^n)$ is "almost" Lipschitz-continuous in the sense that for all $x, y \in \mathbb{R}^n$ with $x \neq y$, $|x-y| < 1/2$,

$$|u(x) - u(y)| \le c|x-y|\,|\log|x-y||^{1/p'}\|u \mid H_p^{1+n/p}(\mathbb{R}^n)\|,$$

where c is a constant independent of u, x and y, and $1/p' + 1/p = 1$ as usual. This celebrated result raises the question about the optimality of the exponent $1/p'$ in the logarithmic term above. Here we give results which, when specialised to the situation of Brézis and Wainger, show that the exponent is indeed optimal.

Theorem 2.1. (i) *Let $1 < p < \infty$, $0 < q \le \infty$, $1/p' = 1 - 1/p$, as usual. There is some $c > 0$ such that for all $x, y \in \mathbb{R}^n$, $0 < |x-y| < 1/2$, and all $f \in F_{p,q}^{1+n/p}(\mathbb{R}^n)$,*

$$|f(x) - f(y)| \le c|x-y|\,|\log|x-y||^{1/p'}\|f \mid F_{p,q}^{1+n/p}(\mathbb{R}^n)\|. \tag{2.1}$$

The exponent $1/p'$ is sharp.

(ii) *Let $0 < p \le \infty$, $1 < q \le \infty$, $1/q' = 1 - 1/q$, as usual. There is some $c > 0$ such that for all $x, y \in \mathbb{R}^n$, $0 < |x-y| < 1/2$, and all $f \in B_{p,q}^{1+n/p}(\mathbb{R}^n)$,*

$$|f(x) - f(y)| \le c|x-y|\,|\log|x-y||^{1/q'}\|f \mid B_{p,q}^{1+n/p}(\mathbb{R}^n)\|. \tag{2.2}$$

The exponent $1/q'$ is sharp.

One immediately recognises that Proposition 1.12 is covered by assertion (ii) of the above theorem. As we already pointed out several times, the theorem as well as the idea of its proof relies on some recent unpublished notes of H. Triebel.

We call an exponent α *sharp* if there is some $c > 0$ such that for all $f \in B_{p,q}^{1+n/p}(\mathbb{R}^n)$ and all $x, y \in \mathbb{R}^n$, $0 < |x-y| < 1/2$,

$$|f(x) - f(y)| \le c|x-y|\,|\log|x-y||^{\alpha}\|f \mid B_{p,q}^{1+n/p}(\mathbb{R}^n)\|, \tag{2.3}$$

but there is no $c > 0$ such that (2.3) holds with α replaced by β, $0 < \beta < \alpha$. Similarly for F-spaces.

In order to split the long proof into more handy pieces as well as to emphasise the importance of the logarithmic term, we first give a proposition. It obviously contributes to the sharpness assertion later.

Proposition 2.2. *Let $1 < p < \infty$ and $\sigma > 1/p$. There is a function $g_{p\sigma}$ with*

$$g_{p\sigma} \in B_{p,p}^{1+n/p}(\mathbb{R}^n), \quad g_{p\sigma}(0) = 0,$$
$$|g_{p\sigma}(x)| \ge c|x|\,|\log|x||^{1/p'}(\log|\log\varepsilon|x||)^{-\sigma}$$

for some $c > 0$, small $\varepsilon > 0$ and $x = (x_1, 0, \dots, 0)$, $0 < x_1 < \delta$, $\delta > 0$ small.

Proof. Step 1. Let $n = 1$. Put

$$f_{p\sigma}(x) = |\log|x||^{1/p'}(\log|\log\varepsilon|x||)^{-\sigma}\psi(x),$$

where $\psi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\psi \subset (-\delta,\delta)$, $\delta > 0$, $\psi(x) = 1$ if $|x| \le \delta/2$. Then $f_{p\sigma} \in B_{pp}^{1/p}(\mathbb{R})$ by [23, Thms. 3.1.2, 4.2.2] (see also [12, Thm. 2.7.1, p. 82]). Let

$$g_{p\sigma}(x) := \int_0^x f_{p\sigma}(y)\,dy, \quad x \in (-1,1);$$

then $g_{p\sigma}$ belongs (locally) to $B_{pp}^{1+1/p}(\mathbb{R})$, and

$$g_{p\sigma}(x) \ge \int_{x/2}^x f_{p\sigma}(y)\,dy \ge cxf_{p\sigma}(x) \quad \text{if } 0 < x < \delta$$

for some $c > 0$. Plainly, $g_{p\sigma}(0) = 0$, so the proposition is proved for $n = 1$.

Step 2. We extend the result to $n \ge 2$. In fact, we use the atomic characterisation of $g_{p\sigma}$ from Step 1 and construct $\mathbb{R}^n$-atoms from the $\mathbb{R}^1$-atoms afterwards. In order to indicate our extension of the one-dimensional result from Step 1, we write $g_{p\sigma}^1(x_1) := g_{p\sigma}(x_1)$, $x_1 \in \mathbb{R}$. Thus $g_{p\sigma}^1 \in B_{pp}^{1+1/p}(\mathbb{R})$ and we have, by Theorem 1.8,

$$g_{p\sigma}^1(x_1) = \sum_{j=0}^\infty \sum_{m=-\infty}^\infty \lambda_{jm} a_{jm}^1(x_1),$$

where the a_{jm}^1 are $(1+1/p,p)_{K,L}$ -atoms (in $\mathbb{R}^1$) in the sense of Definition 1.6, and $\{\lambda_{jm}\} \in b_{pp}$ (see Definition 1.7). In view of Theorem 1.8, in particular (1.13), we may choose $K \ge 2$ and $L = -1$ (no moment conditions). The atoms a_{jm}^1 thus have to satisfy a support condition in the sense of (1.15) as well as

$$\left|\frac{d^l}{dy^l} a_{jm}^1(y)\right| \le 2^{-j(s-1/p)+jl} = 2^{-j+jl}, \quad 0 \le l \le K \tag{2.4}$$

(see (1.9)). By Theorem 1.8(i) there exist atoms a_{jm}^1 and a sequence $\{\lambda_{jm}\} \in b_{pp}$ of complex numbers such that

$$\Big(\sum_{j=0}^\infty \sum_{m=-\infty}^\infty |\lambda_{jm}|^p\Big)^{1/p} \sim \|g_{p\sigma}^1 \mid B_{pp}^{1+1/p}(\mathbb{R})\|.$$

Let $\chi \in C_0^\infty(\mathbb{R}^{n-1})$ be a function with $\operatorname{supp}\chi \subset dQ_{00}$, where $d > 1$ is the number given by (1.15) and Q_{00} the cube in $\mathbb{R}^{n-1}$ centred at the origin with side length 1. Let $x = (x_1, x_2, \ldots, x_n) = (x_1, x')$ where $x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ and assume $|D^\beta\chi(x')| \le 1$, $|\beta| \le K$, K being the same number as above, and $\chi(0) = 1$. Then a_{jm}, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, given by

$$a_{jm}(x) = \begin{cases} a_{jm_1}^1(x_1)\chi(2^j x'), & m = (m_1, 0, \ldots, 0) \in \mathbb{Z}^n, \\ 0, & \text{otherwise}, \end{cases}$$

are $(1 + n/p, p)_{K,-1}$-atoms in $\mathbb{R}^n$: the support condition (1.15) is clear by our above assumptions and there are no moment conditions needed (as in the one-dimensional case); finally, by (2.4) with $l = 0$ and $|\chi(x')| \le 1$ we obtain

$$|a_{jm}(x)| \le 2^{-j} = 2^{-j(s-n/p)},$$

with similar results for the derivatives $|D^\gamma a_{jm}(x)|$, $|\gamma| \le K$. Theorem 1.8 shows that

$$g_{p\sigma}(x) = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm}(x) = \sum_{j=0}^{\infty} \sum_{m_1=-\infty}^{\infty} \lambda_{jm} a^1_{jm_1}(x_1)\chi(2^j x')$$

belongs to $B^{1+n/p}_{pp}(\mathbb{R}^n)$,

$$\|g_{p\sigma} \,|\, B^{1+n/p}_{pp}(\mathbb{R}^n)\| \le c\|\lambda \,|\, b_{pp}\| \le c'\|g^1_{p\sigma} \,|\, B^{1+1/p}_{pp}(\mathbb{R})\|,$$

and $g_{p\sigma}(x_1, 0, \ldots, 0) = g^1_{p\sigma}(x_1)$. Now the proposition follows from Step 1. ■

We now come to the proof of Theorem 2.1. It is based on ideas of H. Triebel, as was the preceding proof.

Proof. Step 1. We show that

$$\sup_{\substack{x,y\in\mathbb{R}^n\\ 0<|x-y|<1/2}} \frac{|f(x)-f(y)|}{|x-y|\,|\log|x-y||^{1/q'}} \le c\|f \,|\, B^{1+n/p}_{p,q}(\mathbb{R}^n)\|, \tag{2.5}$$

which implies (2.2). For convenience assume $y = 0$. Then $|x| \sim 2^{-j}$, $j \in \mathbb{N}$. By Theorem 1.8 we have

$$f = \sum_{k=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{km} a_{km}(x), \tag{2.6}$$

where the a_{km} are 1_K-atoms ($k = 0$) or $(n/p+1, p)_{K,L}$-atoms ($k \in \mathbb{N}$), with the support condition (1.15), and by (1.9),

$$|D^\gamma a_{km}(x)| \le 2^{-k+k|\gamma|}, \quad k \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ \gamma \in \mathbb{N}^n_0,\ |\gamma| \le K. \tag{2.7}$$

By (1.13) we may assume $K \ge (2+[n/p])$ and $L = -1$ (no moment conditions). Moreover, as (2.2) describes the local behaviour of some function $f \in B^{n/p+1}_{p,q}(\mathbb{R}^n)$, we may assume $\operatorname{supp} f \subset U$, the unit ball in $\mathbb{R}^n$, and thus $f \in \widetilde{B}^{n/p+1}_{p,q}(U)$. Hence Theorem 1.8(i) shows that there is an (optimal) atomic characterisation in the above sense with

$$\|f \,|\, \widetilde{B}^{n/p+1}_{p,q}(U)\| \sim \Big(\sum_{k=0}^{\infty}\Big(\sum_{|m|\le c2^k} |\lambda_{km}|^p\Big)^{q/p}\Big)^{1/q}. \tag{2.8}$$

Let x be fixed for the moment, $|x| \sim 2^{-j}$, $j \in \mathbb{N}$. We split (2.6) into different sums, in particular

$$\begin{aligned} f(x) - f(0) &= \sum_{k=0}^{\infty} \sum_{|m|\le c2^k} \lambda_{km}(a_{km}(x) - a_{km}(0)) \\ &= \underbrace{\sum_{k=0}^{j} \sum_{|m|\le c2^k} \lambda_{km}(a_{km}(x) - a_{km}(0))}_{f_j(x)} + \underbrace{\sum_{k=j+1}^{\infty} \sum_{|m|\le c2^k} \lambda_{km}(a_{km}(x) - a_{km}(0))}_{f^j(x)} \\ &= f_j(x) + f^j(x). \end{aligned} \tag{2.9}$$

Note that by the support condition on the atoms a_{km} and the fact that $|x| \sim 2^{-j}$, the sums over $m \in \mathbb{Z}^n$ in (2.9) reduce to a few terms only where $x \in \operatorname{supp} a_{km}$ with

$|m| \sim 2^{k-j}$. Thus it is sufficient to consider

$$f_j^1(x) = \sum_{k=0}^{j} \lambda_{k0}\big(a_{k0}(x) - a_{k0}(0)\big) \tag{2.10}$$

($m = 0 \in \mathbb{Z}^n$) instead of $f_j(x)$, and likewise

$$f_j^2(x) = \sum_{k=j+1}^{\infty} \lambda_{km_k^x}(a_{km_k^x}(x) - a_{km_k^x}(0)),$$

where $m_k^x = m(k,x) \in \mathbb{Z}^n$, $|m_k^x| \sim 2^{k-j}$, replacing $f^j(x)$ in (2.9). The few other terms are similar to handle. Using (2.7) with $\gamma = (0,\ldots,0)$ we find that

$$\begin{aligned} 2^j|f_j^2(x)| &\le c_1 \sum_{k=j+1}^{\infty} |\lambda_{km_k^x}| 2^{-(k-j)} \le c_2 \sum_{k=j+1}^{\infty} 2^{-(k-j)} \Big(\sum_{|m|\le c2^k} |\lambda_{km}|^p \Big)^{1/p} \\ &\le c_3 \Big(\sum_{k=j+1}^{\infty} \Big(\sum_{|m|\le c2^k} |\lambda_{km}|^p \Big)^{q/p} \Big)^{1/q} \le c_4 \|f \mid \widetilde{B}^{n/p+1}_{p,q}(U)\| \end{aligned} \tag{2.11}$$

(see (2.8)), where the constant $c_4 > 0$ does not depend on $j \in \mathbb{N}$. Hence, with $|x| \sim 2^{-j}$,

$$|f_j^2(x)| \le c|x|\, \|f \mid \widetilde{B}^{n/p+1}_{p,q}(U)\|. \tag{2.12}$$

We now deal with $f_j^1(x)$. We apply the mean value theorem to $(a_{k0}(x) - a_{k0}(0))$ to obtain, by (2.7) with $|\gamma| = 1$,

$$\begin{aligned} 2^j|f_j^1(x)| &\le c_1 \sum_{k=0}^{j} |\lambda_{k0}| \le c_2 \Big(\sum_{k=0}^{j} |\lambda_{k0}|^q \Big)^{1/q} \langle j \rangle^{1/q'} \\ &\le c_3 |\log|x||^{1/q'} \|f \mid \widetilde{B}^{n/p+1}_{p,q}(U)\|, \end{aligned} \tag{2.13}$$

where we applied Hölder's inequality for $1 = 1/q + 1/q'$. Recall $|x| \sim 2^{-j}$. Then (2.12) and (2.13) imply that

$$\frac{|f_j^1(x)| + |f_j^2(x)|}{|x|\, |\log|x||^{1/q'}} \le c\|f \mid B^{1+n/p}_{p,q}(\mathbb{R}^n)\|,$$

which, together with our above remarks about the similar terms and with finally taking the supremum over all $x \in U$, $x \neq 0$, proves (2.5).

Step 2. We show the counterpart of (2.5) for F-spaces:

$$\sup_{\substack{x,y\in\mathbb{R}^n \\ 0<|x-y|<1/2}} \frac{|f(x) - f(y)|}{|x-y|\, |\log|x-y||^{1/p'}} \le c\|f \mid F^{1+n/p}_{p,q}(\mathbb{R}^n)\|. \tag{2.14}$$

We proceed in the same way as in Step 1. Recall that

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n) \tag{2.15}$$

(cf. [21, Prop. 2.3.2/2(iii), p. 47]). Repetition of Step 1 for $B^s_{p,u}(\mathbb{R}^n)$, $u = \max(p,q)$, yields the F-counterpart of (2.12) by monotonical embeddings:

$$|f_j^2(x)| \le c|x|\, \|f \mid \widetilde{F}^{n/p+1}_{p,q}(U)\|. \tag{2.16}$$

As for (2.13) we arrive at

$$2^j|f_j^1(x)| \le c_1 \sum_{k=0}^{j} |\lambda_{k0}| \le c_2 \Big(\sum_{k=0}^{j} |\lambda_{k0}|^p\Big)^{1/p} \langle j\rangle^{1/p'}. \tag{2.17}$$

We make use of a trick by Frazier and Jawerth (see [13, §11]), which applies in our situation as well, to obtain

$$\Big(\sum_{k=0}^{j} |\lambda_{k0}|^p\Big)^{1/p} \sim \Big\| \Big(\sum_{k=0}^{j} |\lambda_{k0}\chi_{k,0}^{(p)}(\cdot)|^q\Big)^{1/q} \Big| L_p(\mathbb{R}^n)\Big\|,$$

where q, $0 < q \le \infty$, is arbitrary. Consequently, (2.17) becomes

$$2^j|f_j^1(x)| \le c|\log|x||^{1/p'} \|f \mid \widetilde{F}^{n/p+1}_{p,q}(U)\|,$$

which in a completely parallel way to that presented in Step 1 proves (2.14).

Step 3. We prove the sharpness now. For B-spaces, the sharpness for $p=q$, $1<p<\infty$, follows from Proposition 2.2 and Step 1. To extend it to all admissible p and q we use (complex) interpolation. Introduce, for the sake of this proof only, the spaces $C(U_\delta, \alpha)$, $\delta > 0$ small, $\alpha > 0$, of all functions $f \in C(\mathbb{R}^n)$ such that

$$\|f \mid C(U_\delta,\alpha)\| = \sup_{0<|x|<\delta} \frac{|f(x)|}{|x|\,|\log|x||^\alpha} < \infty;$$

these are weighted C-spaces near the origin. Let $\delta > 0$ be fixed in the following. Denote by $\dot{B}^{1+n/p}_{p,q}(\mathbb{R}^n)$ the subspace of $B^{1+n/p}_{p,q}(\mathbb{R}^n)$ defined via

$$\dot{B}^{1+n/p}_{p,q}(\mathbb{R}^n) = \{f \in B^{1+n/p}_{p,q}(\mathbb{R}^n) : f(0) = 0\}, \quad 1 < q \le \infty.$$

Then by Step 1,

$$\|f \mid C(U_\delta, 1/q')\| \le c\|f \mid B^{1+n/p}_{p,q}(\mathbb{R}^n)\| \quad \text{if } f \in \dot{B}^{1+n/p}_{p,q}(\mathbb{R}^n). \tag{2.18}$$

Let $1 \le p < \infty$, $1 < q_0, q_1 \le \infty$, $\alpha_0, \alpha_1 > 0$, $0 < \theta < 1$, and

$$\alpha_\theta = (1-\theta)\alpha_0 + \theta\alpha_1, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Now complex interpolation yields

$$[C(U_\delta,\alpha_0), C(U_\delta,\alpha_1)]_\theta = C(U_\delta,\alpha_\theta) \tag{2.19}$$

and

$$[\dot{B}^{1+n/p}_{p,q_0}(\mathbb{R}^n), \dot{B}^{1+n/p}_{p,q_1}(\mathbb{R}^n)]_\theta = \dot{B}^{1+n/p}_{p,q_\theta}(\mathbb{R}^n). \tag{2.20}$$

Concerning (2.19), see [19, 1.18.5, p. 130]; one can strengthen the same argument as in Step 2 on p. 131 in [19]. For (2.20), the parallel result for the B-spaces (instead of $\dot{B}$) is given in [19, 2.4.1, p. 182] and [21, Rem. 2.4.7/2, p. 73]. But the interpolation of complemented subspaces holds as well (cf. [19, 1.17.1, p. 118]).

In order to prove the sharpness of $1/q'$ in Theorem 2.1(ii) we proceed by contradiction. Let p, $1 < p < \infty$, be fixed for the moment and assume that there is some $q_0 \ne p$, $1 < q_0 \le \infty$, and an $\varepsilon > 0$ such that there is a number $c > 0$ with

$$\|f \mid C(U_\delta, 1/q_0' - \varepsilon)\| \le c\|f \mid B^{1+n/p}_{p,q_0}(\mathbb{R}^n)\| \quad \text{for all } f \in \dot{B}^{1+n/p}_{p,q_0}(\mathbb{R}^n) \tag{2.21}$$

(this would mean a smaller exponent $\beta = 1/q_0' - \varepsilon < 1/q_0'$ and would thus disprove the sharpness of $1/q_0'$). Without loss of generality we may assume $q_0 < p$ (otherwise one just has to modify the subsequent consideration slightly). Choose $q_1 > p$ and $0 < \theta < 1$ such that

$$\frac{1}{p} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

We have (2.18) for q_1. Now complex interpolation (2.19), (2.20) yields that for all $f \in \dot{B}^{1+n/p}_{p,p}(\mathbb{R}^n)$,

$$\|f \mid C(U_\delta, 1/p' - (1-\theta)\varepsilon)\| \le c\|f \mid B^{1+n/p}_{p,p}(\mathbb{R}^n)\|,$$

which contradicts Step 1. Hence, for every q_0, $1 < q_0 \le \infty$, the exponent $1/q_0'$ is sharp.

Now assume that (2.21) is true for some p_0, $0 < p_0 < \infty$, and

$$\|f \mid C(U_\delta, 1/q_0' - \varepsilon)\| \le c\|f \mid B^{1+n/p_0}_{p_0,q_0}(\mathbb{R}^n)\| \tag{2.22}$$

for all $f \in \dot{B}^{1+n/p_0}_{p_0,q_0}(\mathbb{R}^n)$. Let $p_1 \in (1,\infty)$ and $0 < \theta < 1$ such that $1/p = (1-\theta)/p_0 + \theta/p_1 < 1$. By complex interpolation (2.19), and the counterpart of (2.20), i.e. [21, Thm. 2.4.7, p. 69], we again obtain, now applying (2.18) with $q = q_0$, $p = p_1$, and (2.22),

$$\|f \mid C(U_\delta, 1/q_0' - (1-\theta)\varepsilon)\| \le c\|f \mid B^{1+n/p}_{p,q_0}(\mathbb{R}^n)\| \quad \text{for all } f \in \dot{B}^{1+n/p}_{p,q_0}(\mathbb{R}^n).$$

By setting $\varepsilon' = (1-\theta)\varepsilon$ we thus get (2.21) which was disproved above.

So the case $0 < p < \infty$, $1 < q \le \infty$ is covered by the above arguments, and it remains to handle the case $p = \infty$. However, this simply follows from the embedding

$$B^{1+n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow B^1_{\infty,q}(\mathbb{R}^n), \tag{2.23}$$

$0 < p \le \infty$, i.e. if there were a smaller exponent than $1/q'$ in the case of $B^1_{\infty,q}(\mathbb{R}^n)$, then there would be one for $B^{1+n/p}_{p,q}(\mathbb{R}^n)$ as well. This was disproved above. Hence we have verified the sharpness of $1/q'$ in (2.2).

Step 4. We have to complete the proof of Theorem 2.1(i), i.e. to verify the sharpness of $1/p'$ in the F-case. Assume that for some q_0, $0 < q_0 \le \infty$, and some $\varepsilon > 0$ there exists $c > 0$ such that

$$|f(x)| \le c|x||\log|x||^{1/p'-\varepsilon}\|f \mid F^{1+n/p}_{p,q_0}(\mathbb{R}^n)\| \tag{2.24}$$

for all x, $0 < |x| < 1/2$, and all $f \in F^{1+n/p}_{p,q_0}(\mathbb{R}^n)$, $f(0) = 0$. Let $g \in F^{1+n/p}_{p,p}(\mathbb{R}^n)$, $g(0) = 0$. Let $|x| \sim 2^{-j}$. Recall the decomposition (2.9), i.e.

$$g(x) = g^1_j(x) + g^2_j(x) + \text{similar terms} \tag{2.25}$$

with

$$\|g^1_j \mid F^{1+n/p}_{p,q_0}(\mathbb{R}^n)\| \sim \|g^1_j \mid F^{1+n/p}_{p,p}(\mathbb{R}^n)\| \le c\|g \mid F^{1+n/p}_{p,p}(\mathbb{R}^n)\| \tag{2.26}$$

(see also the end of Step 2),

$$\|g^2_j \mid F^{1+n/p}_{p,p}(\mathbb{R}^n)\| \le c\|g \mid F^{1+n/p}_{p,p}(\mathbb{R}^n)\|,$$

and

$$|g^2_j(x)| \le c|x|\,\|g \mid F^{1+n/p}_{p,p}(\mathbb{R}^n)\| \tag{2.27}$$

(see (2.16)). Now (2.24) and (2.26) imply that

$$|g^1_j(x)| \le c|x|\,|\log|x||^{1/p'-\varepsilon}\|g \mid F^{1+n/p}_{p,p}(\mathbb{R}^n)\|. \tag{2.28}$$

Hence (2.25), (2.27) and (2.28) contradict the sharpness of $1/p'$ when $q = p$, which has been proved in Step 3. ■

Remark 2.3. Theorem 2.1(ii) implies for $p = \infty$ that there is some $c > 0$ such that for all $f \in B^1_{\infty,q}(\mathbb{R}^n)$, $0 < |x| < 1/2$,

$$|f(x) - f(0)| \le c|x|\,|\log|x||^{1/q'}\,\|f \mid B^1_{\infty,q}(\mathbb{R}^n)\|,$$

where $1 < q \le \infty$, $1/q + 1/q' = 1$. In particular,

$$|f(x) - f(0)| \le c|x|\,|\log|x||\,\|f \mid \mathcal{C}^1(\mathbb{R}^n)\| \tag{2.29}$$

for all x, $0 < |x| < 1/2$, and all f belonging to the Hölder–Zygmund space $\mathcal{C}^1(\mathbb{R}^n) = B^1_{\infty,\infty}(\mathbb{R}^n)$ (cf. [21, Thm. 2.5.7, p. 90]). The exponent 1 of $|\log|x||$ in (2.29) is sharp.

Remark 2.4. We want to briefly mention another approach which yields (1.23) and thus Theorem 2.1(ii) when $p = \infty$ and $1 \le q \le \infty$ (apart from the sharpness assertion). This can be obtained using Marchaud's inequality as well as equivalent characterisations of $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$, $B^s_{p,q}(\mathbb{R}^n)$, via the modulus of smoothness. We indicate the idea of this different approach.

Recall the definition of the difference operator Δ^m_h, $m \in \mathbb{N}_0$, $h \in \mathbb{R}^n$, in (1.2). The *rth modulus of smoothness* (or *rth order modulus of continuity*) of a function $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, is defined by

$$\omega_r(f,t)_p = \sup_{|h|\le t} \|\Delta^r_h f \mid L_p(\mathbb{R}^n)\|, \quad t > 0$$

(see [1, Ch. 5, Def. 4.2, p. 332] or [5, Ch. 2, §7, pp. 44–46]). Let $s > 0$, $1 \le p, q \le \infty$ and $r \in \mathbb{N}$ with $r > s$. Then

$$\|f \mid B^s_{p,q}(\mathbb{R}^n)\| \sim \|f \mid L_p(\mathbb{R}^n)\| + \Big(\int_0^\infty [t^{-s}\omega_r(f,t)_p]^q \,\frac{dt}{t}\Big)^{1/q} \tag{2.30}$$

(with the usual modification if $q = \infty$); see [1, Ch. 5, Def. 4.3, p. 332], [5, Ch. 2, §10, pp. 54–56] (where the Besov spaces are defined in that way) and [21, Thm. 2.5.12, p. 110] for what concerns the equivalence of Definition 1.3 (with $b = 0$) and the characterisation (2.30).

There are extensions to $0 < p, q \le 1$, but it will be clear from what follows that we are eventually only interested in the case $p = \infty$ and $1 \le q \le \infty$. Similarly, it is clear by (1.3) that in the above notation,

$$\|f \mid \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| \sim \|f \mid L_\infty(\mathbb{R}^n)\| + \sup_{0<t<1/2} \frac{\omega_1(f,t)_\infty}{t|\log t|^\alpha}. \tag{2.31}$$

Now Marchaud's inequality states the following: let $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, $t > 0$, and $k \in \mathbb{N}$; then

$$\omega_k(f,t)_p \le \frac{k}{\log 2}\, t^k \int_t^\infty \frac{\omega_{k+1}(f,u)_p}{u^k}\,\frac{du}{u} \tag{2.32}$$

(see [1, Ch. 5, (4.11), p. 334] or [5, Ch. 2, Thm. 8.1, p. 47], the latter dealing with the one-dimensional case).

Assume $k=1$ and $p=\infty$; then (2.32) implies that there is some $c>0$ such that

$$\omega_1(f,t)_\infty \le ct \int_t^\infty \frac{\omega_2(f,u)_\infty}{u}\,\frac{du}{u} \tag{2.33}$$

for all $f\in L_\infty(\mathbb{R}^n)$ and $t>0$. On the other hand, from (2.30) we obtain

$$\|f\,|\,B^1_{\infty,q}(\mathbb{R}^n)\| \sim \|f\,|\,L_\infty(\mathbb{R}^n)\| + \bigg(\int_0^\infty \bigg[\frac{\omega_2(f,t)_\infty}{t}\bigg]^q \frac{dt}{t}\bigg)^{1/q}. \tag{2.34}$$

Now (2.31) together with Marchaud's inequality (2.33) lead to

$$\begin{aligned}
&\|f\,|\,\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| \\
&\quad\le c\bigg\{\|f\,|\,L_\infty\| + \sup_{0<t<1/2}\frac{1}{|\log t|^\alpha}\int_t^\infty \frac{\omega_2(f,u)_\infty}{u}\,\frac{du}{u}\bigg\}\\
&\quad\le c'\bigg\{\|f\,|\,L_\infty\| + \sup_{0<t<1/2}\frac{1}{|\log t|^\alpha}\bigg(\int_t^1 \bigg[\frac{\omega_2(f,u)_\infty}{u}\bigg]^q\frac{du}{u}\bigg)^{1/q}\bigg(\int_t^1\frac{du}{u}\bigg)^{1/q'}\bigg\}\\
&\quad\le c''\bigg\{\|f\,|\,L_\infty\| + \bigg(\int_0^\infty \bigg[\frac{\omega_2(f,u)_\infty}{u}\bigg]^q\frac{du}{u}\bigg)^{1/q}\sup_{0<t<1/2}\frac{1}{|\log t|^\alpha}\bigg(\int_t^1\frac{du}{u}\bigg)^{1/q'}\bigg\}
\end{aligned}$$

where the second estimate comes from Hölder's inequality for $1\le q\le\infty$ and $1=1/q+1/q'$. Moreover,

$$\frac{1}{|\log t|^\alpha}\bigg(\int_t^1\frac{du}{u}\bigg)^{1/q'} \xrightarrow{t\downarrow 0} 0 \quad \text{when } \alpha\ge 1/q',$$

and so the relevant supremum over all small t, $0<t<1/2$, is bounded from above by some constant. Thus we finally arrive by (2.34) at

$$\|f\,|\,\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| \le C\|f\,|\,B^1_{\infty,q}(\mathbb{R}^n)\| \quad \text{if } \alpha\ge 1/q',$$

which yields (1.23) for $p=\infty$ and $1\le q\le\infty$. The extension to $0<p<\infty$ then comes from the elementary embedding (2.23).

Remark 2.5. Let

$$(\mathrm{bmo})^1(\mathbb{R}^n) = (\mathrm{id}-\Delta)^{-1/2}\mathrm{bmo}(\mathbb{R}^n) = F^1_{\infty,2}(\mathbb{R}^n)$$

(see [21, Thm. 2.5.8/2, p. 93] for $\mathrm{bmo}(\mathbb{R}^n)=F^0_{\infty,2}(\mathbb{R}^n)$). Then

$$B^1_{\infty,2}(\mathbb{R}^n) \hookrightarrow (\mathrm{bmo})^1(\mathbb{R}^n) \hookrightarrow B^1_{\infty,\infty}(\mathbb{R}^n) = \mathcal{C}^1(\mathbb{R}^n)$$

(see (2.15)). Hence, by (2.29),

$$|f(x)-f(0)| \le c|x|\,|\log|x||\,\|f\,|\,(\mathrm{bmo})^1(\mathbb{R}^n)\|,$$

for all x, $0<|x|<1/2$, and all $f\in(\mathrm{bmo})^1(\mathbb{R}^n)$, and the exponent 1 of $|\log|x||$ is again sharp: one can use the above technique and the complex interpolation

$$[(\mathrm{bmo})^1(\mathbb{R}^n), H_p^{1+n/p}(\mathbb{R}^n)]_\theta = H_r^{1+n/r}(\mathbb{R}^n),$$

where $1<p<\infty$, $0<\theta<1$, and $1/r=\theta/p$. This follows by the lifting arguments of formula (45) on p. 87 in [12].

REMARK 2.6. The estimate (2.1) was already known when $1 < p < \infty$, $q = 2$, i.e. $F^{1+n/p}_{p,2}(\mathbb{R}^n) = H^{1+n/p}_p(\mathbb{R}^n)$ (see [3], [9]). However, the sharpness result seems new (see also [8]).

3. Lipschitz embedding, entropy numbers

The main result of this section is Theorem 3.5. It provides an upper estimate for the entropy numbers of the embedding

$$\mathrm{id} : \widetilde{B}^{1+n/p}_{p,q}(U) \to \widetilde{\mathrm{Lip}}^{(1,-\alpha)}(U)$$

and of the embedding

$$\mathrm{id} : B^{1+n/p}_{p,q}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U),$$

where $0 < p \le \infty$, $0 < q \le \infty$, $\alpha > (1 - 1/q)_+$ and U is the unit ball in $\mathbb{R}^n$.

As in [12] and [24], our estimation of the entropy numbers of embedding maps involves a reduction of the problem to the study of maps between finite-dimensional sequence spaces. Accordingly, we introduce the sequence spaces ℓ^M_p, $M \in \mathbb{N}$, $0 < p \le \infty$, following [12, 3.2.1, p. 97]. These are the linear spaces of all complex M-tuples $y = (y_j)$ endowed with the quasi-norm

$$\|y \,|\, \ell^M_p\| = \Big(\sum_{j=1}^{M} |y_j|^p\Big)^{1/p}, \quad 0 < p < \infty,$$

with the usual modification if $p = \infty$. Moreover, we also need weighted ℓ_p-spaces in the following sense: Let $(M_j)_{j\in\mathbb{N}_0}$ be a sequence of natural numbers with $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$. Let $0 < p \le \infty$ and $0 < q \le \infty$. Let $(w_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers (weights), mainly of the type

$$w_j = 2^{j\delta} \quad \text{or} \quad w_j = \langle j\rangle^{\varkappa}, \quad j \in \mathbb{N}_0,\ \delta > 0,\ \varkappa \in \mathbb{R}.$$

We extend Triebel's definition given in [24, 8.1, p. 38]: $\ell_q(w_j\ell^{M_j}_p)$ is the linear space of all complex sequences $x = (x_{j,l} : j \in \mathbb{N}_0;\ l = 1, \ldots, M_j)$ endowed with the quasi-norm

$$\|x \,|\, \ell_q(w_j\ell^{M_j}_p)\| = \Big(\sum_{j=0}^{\infty} w_j^q \Big(\sum_{l=1}^{M_j} |x_{j,l}|^p\Big)^{q/p}\Big)^{1/q} \tag{3.1}$$

with the obvious modifications if $p = \infty$ or $q = \infty$. For $w_j \equiv 1$ we write $\ell_q(\ell^{M_j}_p)$. The above notation coincides with [24, (8.2), p. 38] when $w_j = 2^{j\delta}$, $\delta > 0$. Concerning entropy numbers of the embedding map $\mathrm{id} : \ell^M_{p_1} \to \ell^M_{p_2}$, $0 < p_1 \le p_2 \le \infty$, we make use of [12, Prop. 3.2.2, p. 98] as well as [24, Prop. 7.2, p. 36].

PROPOSITION 3.1. *Let $\varkappa > 0$, $0 < p \le \infty$, $0 < q \le \infty$, $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$. Then*

$$e_k(\mathrm{id} : \ell_q(\ell^{M_j}_p) \to \ell_q(\langle j\rangle^{-\varkappa}\ell^{M_j}_p)) \sim (\log\langle k\rangle)^{-\varkappa}, \quad k \in \mathbb{N}. \tag{3.2}$$

PROOF. The proof is essentially based on [24, 8.1, pp. 39–43]. We first verify that

$$\mathrm{id} : \ell_q(\ell^{M_j}_p) \to \ell_q(\langle j\rangle^{-\varkappa}\ell^{M_j}_p) \tag{3.3}$$

is compact. Let

$$\mathrm{id} = \sum_{j=0}^{\infty} \mathrm{id}_j, \tag{3.4}$$

where

$$\mathrm{id}_j x = (\delta_{jk} x_{k,l} : k \in \mathbb{N}_0,\ l = 1, \ldots, M_k) = (0, \ldots, 0, x_{j,1}, \ldots, x_{j,M_j}, 0, \ldots). \tag{3.5}$$

Then

$$\begin{aligned} \|\mathrm{id}_j x \,|\, \ell_q(\langle k\rangle^{-\varkappa} \ell_p^{M_k})\| &= \Big(\sum_{k=0}^{\infty} \langle k\rangle^{-\varkappa q} \Big(\sum_{l=0}^{M_k} \delta_{jk} |x_{k,l}|^p\Big)^{q/p}\Big)^{1/q} \\ &= \langle j\rangle^{-\varkappa} \Big(\sum_{l=0}^{M_j} |x_{j,l}|^p\Big)^{1/p} \le \langle j\rangle^{-\varkappa} \|x \,|\, \ell_q(\ell_p^{M_k})\|. \end{aligned} \tag{3.6}$$

Hence id from (3.3) is compact for $\varkappa > 0$.

To prove the assertion about the entropy numbers we use the following commutative diagram:

$$\begin{array}{ccc} \ell_p^{M_j} & \xrightarrow{\ \mathrm{id}^j\ } & \ell_q(\ell_p^{M_j}) \\ \Big\downarrow \mathrm{id} & & \Big\downarrow \mathrm{id} \\ \ell_p^{M_j} & \xleftarrow{\ \mathrm{id}_j\ } & \ell_q(\langle j\rangle^{-\varkappa} \ell_p^{M_j}) \end{array}$$

where id_j is given by (3.5) (acting in the slightly modified way as indicated above) and id^j maps $\ell_p^{M_j}$ identically onto $\ell_q(\ell_p^{M_j})$ interpreted as dyadic blocks. Obviously,

$$\mathrm{id}(\ell_p^{M_j} \to \ell_p^{M_j}) = \mathrm{id}_j \circ \mathrm{id} \circ \mathrm{id}^j,$$

id being always given by (3.3) in the sequel. Moreover, we use [24, Thm. 7.3, p. 37] with $p_1 = p_2$, $M = M_j \sim 2^{jn}$ and $k = 2M \sim 2^{jn}$ to obtain

$$e_{2^{jn}}(\mathrm{id} : \ell_p^{M_j} \to \ell_p^{M_j}) \ge c.$$

Then $\|\mathrm{id}_j\| = \langle j\rangle^{\varkappa}$ by (3.6), $\|\mathrm{id}^j\| = 1$, so the multiplicativity (1.27) of entropy numbers yields

$$e_{2^{jn}}(\mathrm{id}) \ge c\langle j\rangle^{-\varkappa}.$$

It remains to show the upper estimate. As $p_1 = p_2 = p$, the proof becomes much easier than its counterpart in [24, pp. 40–41]. Let $J \in \mathbb{N}$. We split the sum in (3.4) into two parts,

$$\mathrm{id} = \sum_{j=0}^{J} \mathrm{id}_j + \sum_{j=J+1}^{\infty} \mathrm{id}_j,$$

and obtain

$$\Big\| \sum_{j=J+1}^{\infty} \mathrm{id}_j \Big\| \le c\langle J\rangle^{-\varkappa}. \tag{3.7}$$

Let $\varrho = \min(1, p, q)$; then the additivity of entropy numbers (in ϱ-Banach spaces) leads to

$$e_k^{\varrho}(\mathrm{id}) \le c\Big(\langle J\rangle^{-\varkappa\varrho} + \sum_{j=0}^{J} e_{k_j}^{\varrho}(\mathrm{id}_j)\Big), \quad k = \sum_{j=0}^{J} k_j, \tag{3.8}$$

where we applied (3.7). On the other hand, another application of (3.6) yields

$$e_{k_j}(\mathrm{id}_j) = \langle j\rangle^{-\varkappa} e_{k_j}(\mathrm{id} : \ell_p^{M_j} \to \ell_p^{M_j}). \tag{3.9}$$

Note that from [24, Thm. 7.3, p. 37] we have

$$e_{k_j}(\mathrm{id} : \ell_p^{M_j} \to \ell_p^{M_j}) \le c2^{-k_j/(2M_j)}, \quad k_j > 2M_j,$$

which together with (3.9) gives

$$e_{k_j}(\mathrm{id}_j) \le c\langle j\rangle^{-\varkappa} 2^{-k_j/(2M_j)}, \quad k_j > 2M_j. \tag{3.10}$$

Let

$$\begin{aligned} k_j &= 2M_j(\log\langle J\rangle^{\varkappa} + \log\langle j\rangle^{-\varkappa+(1+\varepsilon)/\varrho}) \\ &\sim 2^{jn}\bigg(\varkappa\log\langle J\rangle + \Big(\frac{1+\varepsilon}{\varrho} - \varkappa\Big)\log\langle j\rangle\bigg), \end{aligned} \tag{3.11}$$

where $\varepsilon > 0$ is small. Rewriting (3.11) as

$$k_j = 2M_j\bigg(\varkappa\log\frac{\langle J\rangle}{\langle j\rangle} + \frac{1+\varepsilon}{\varrho}\log\langle j\rangle\bigg)$$

one recognises that $k_j > 2M_j$ and thus by (3.10),

$$e_{k_j}(\mathrm{id}_j) \le c_1\langle j\rangle^{-\varkappa}2^{-\log(\langle J\rangle^{\varkappa}\langle j\rangle^{-\varkappa+(1+\varepsilon)/\varrho})} \le c_2\langle J\rangle^{-\varkappa}\langle j\rangle^{-(1+\varepsilon)/\varrho}. \tag{3.12}$$

Replacing $e_{k_j}(\mathrm{id}_j)$ in (3.8) by (3.12) we arrive at

$$e_k^{\varrho}(\mathrm{id}) \le c\langle J\rangle^{-\varkappa\varrho}, \quad k = \sum_{j=0}^{J} k_j. \tag{3.13}$$

Calculating $\sum_{j=0}^{J} k_j$ we obtain $k \le c2^{Jn}\log\langle J\rangle^{\varkappa+(1+\varepsilon)/\varrho}$.

Let $L \sim 2^{Jn}\log\langle J\rangle^{\varkappa+(1+\varepsilon)/\varrho}$; thus $\log\langle L\rangle \sim J$, and finally (3.13) results in

$$e_{cL}(\mathrm{id}) \le c'(\log\langle L\rangle)^{-\varkappa},$$

which gives the upper estimate in (3.2) and ends the proof. ■

Remark 3.2. As we mentioned above, when $w_j = 2^{j\delta}$, $\delta > 0$, our notation (3.1) coincides with [24, (8.2)]. The result parallel to Proposition 3.1, assuming $0 < p \le \infty$, $0 < q \le \infty$, $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$, is then a special case of [24, Thm. 8.2, p. 39] and reads

$$e_k(\mathrm{id} : \ell_q(2^{j\delta}\ell_p^{M_j}) \to \ell_q(\ell_p^{M_j})) \sim k^{-\delta/n}, \quad k \in \mathbb{N}.$$

It turns out later on that we do also need some generalisation of Proposition 3.1. But in some sense this only concerns technicalities, thus we give the result and a brief idea of its proof only. In addition to the above notion of the spaces $\ell_q(w_j\ell_p^{M_j})$ endowed with the quasi-norm (3.1) we have to introduce spaces $\ell_u[2^{\mu m}\ell_q(w_j\ell_p^{M_j})]$, $0 < u \le \infty$, $\mu > 0$, as the linear spaces of all $\ell_q(w_j\ell_p^{M_j})$-valued sequences $x = (x^m)_{m\in\mathbb{N}_0}$ such that the quasi-norm

$$\|x \mid \ell_u[2^{\mu m}\ell_q(w_j\ell_p^{M_j})]\| = \Big(\sum_{m=0}^{\infty} 2^{\mu m u}\|x^m \mid \ell_q(w_j\ell_p^{M_j})\|^u\Big)^{1/u} \tag{3.14}$$

(with the obvious modification if $u = \infty$) is finite. For $w_j \equiv 1$ and $\mu = 0$ we write $\ell_u[\ell_q(\ell_p^{M_j})]$. The above notation coincides with [24, (9.1)] when $w_j = 2^{j\delta}$, $\delta > 0$.

We are interested in an extension of Proposition 3.1, i.e. when $w_j = \langle j\rangle^{\varkappa}$. Let $\mu > 0$, $\varkappa > 0$, $0 < p \le \infty$, $0 < q \le \infty$, $0 < u_1, u_2 \le \infty$, $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$. Then the identity map

$$\mathrm{id} : \ell_{u_1}[2^{\mu m}\ell_q(\ell_p^{M_j})] \to \ell_{u_2}[\ell_q(\langle j\rangle^{-\varkappa}\ell_p^{M_j})] \tag{3.15}$$

is compact. This is simply the extension of what has been said above for the scalar case to the ℓ_u-valued case. It immediately results from our assumption $\mu > 0$.

Corollary 3.3. *Let $\varkappa > 0$, $\mu > 0$, $0 < p \le \infty$, $0 < q \le \infty$, $0 < u_1, u_2 \le \infty$, $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$. Then for all $k \in \mathbb{N}$,*

$$e_k(\ell_{u_1}[2^{\mu m}\ell_q(\ell_p^{M_j})] \to \ell_{u_2}[\ell_q(\langle j\rangle^{-\varkappa}\ell_p^{M_j})]) \sim (\log\langle k\rangle)^{-\varkappa}. \tag{3.16}$$

Proof. The estimate from below comes from Proposition 3.1. Concerning the estimate from above, we proceed in the same way as in [24, Thm. 9.2, p. 47] with modifications parallel to those in the proof of Proposition 3.1. ■

Remark 3.4. The parallel result with $w_j = 2^{j\delta}$, $\delta > 0$, is given in [24, Thm. 9.2, p. 47]. We will apply Corollary 3.3 when $u_1 = u_2 = \infty$ and with the embedding (3.15) replaced by the very similar

$$\mathrm{id} : \ell_\infty[2^{\varrho_1 m}\ell_q(\ell_p^{M_j})] \to \ell_\infty[2^{\varrho_2 m}\ell_q(\langle j\rangle^{-\varkappa}\ell_p^{M_j})],$$

where $\varrho_1 > \varrho_2$.

Recall that U is the unit ball in $\mathbb{R}^n$, i.e. $U = \{x \in \mathbb{R}^n : |x| < 1\}$.

Theorem 3.5. *Let $0 < p \le \infty$, $0 < q \le \infty$, $\alpha > (1-1/q)_+$. Then there is some $c > 0$ such that for all $k \in \mathbb{N}$,*

$$e_k(\mathrm{id} : B^{n/p+1}_{p,q}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U)) \le c(\log\langle k\rangle)^{-\alpha+(1-1/q)_+}. \tag{3.17}$$

The same result is true with $\mathrm{id} : B^{n/p+1}_{p,q}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U)$ *replaced by* $\mathrm{id} : \widetilde{B}^{n/p+1}_{p,q}(U) \to \widetilde{\mathrm{Lip}}^{(1,-\alpha)}(U)$.

Proof. Note that (3.17) implies that $\mathrm{id} : B^{n/p+1}_{p,q}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U)$ is compact. For convenience we want to use the following convention: when $1 \le q \le \infty$ the number q' is defined by $1 = 1/q + 1/q'$, as usual. Otherwise, when $0 < q \le 1$, we put $q' := \infty$ (i.e. $1/q' := 0$) in the sequel. Thus (3.17) can be rewritten as

$$e_k(\mathrm{id} : B^{n/p+1}_{p,q}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U)) \le c(\log\langle k\rangle)^{-(\alpha-1/q')}.$$

Step 1. We explain the last statement in Theorem 3.5. Spaces of type $B^s_{p,q}(\Omega)$ and $\widetilde{B}^s_{p,q}(\Omega)$ (recall Definition 1.11 and (1.21), respectively, with $b = 0$) do not coincide in general; in particular, they are different in some cases of interest for us: when $s = 1$ and $p = \infty$ (see [19, Thm. 4.3.2/1, Rem. 4.3.2/2, pp. 317–318]). Likewise for the Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$ and $\widetilde{\mathrm{Lip}}^{(1,-\alpha)}(\Omega)$, given by Definition 1.9 and (1.20), respectively.

However, it turns out that in terms of entropy numbers no distinction has to be made (apart from constants), because via the usual extension-restriction procedures we may always deal with the more convenient form for our purposes—at the expense of some constants only. This follows from the multiplicativity (1.27) of entropy numbers when the compact embedding we study is combined with some linear and bounded mappings. Recall Remark 1.10 as well as [22, Thm. 5.1.3, p. 239], the latter concerning the extension operator for B-spaces.

For convenience, we always use subspaces of $\mathbb{R}^n$ in the sequel and only study the embedding

$$\mathrm{id} : \widetilde{B}^{n/p+1}_{p,q}(U) \to \widetilde{\mathrm{Lip}}^{(1,-\alpha)}(U),$$

but from the above considerations it is clear that this completely covers all assertions in Theorem 3.5.

Step 2. Let $f \in \widetilde{B}^{n/p+1}_{p,q}(U)$; then by Theorem 1.8(i) there is an atomic representation

$$f = \sum_{k=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{km} a_{km}(x), \tag{3.18}$$

where the a_{km} are 1_K-atoms ($k = 0$) or $(n/p + 1, p)_{K,L}$-atoms ($k \in \mathbb{N}$) in the sense of Definition 1.6, with

$$\|f \mid \widetilde{B}^{n/p+1}_{p,q}(U)\| \sim \Big(\sum_{k=0}^{\infty} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{km}|^p \Big)^{q/p} \Big)^{1/q}. \tag{3.19}$$

By (1.13) we may assume $K \geq \big(2 + \big[n/p\big]\big)$ and $L = -1$. We decompose

$$\mathrm{id} : \widetilde{B}^{n/p+1}_{p,q}(U) \to \widetilde{\mathrm{Lip}}^{(1,-\alpha)}(U)$$

into the following maps:

$$\begin{array}{ccc} \widetilde{B}^{1+n/p}_{p,q}(U) & \xrightarrow{\quad S \quad} & \ell_q(\ell_p^{M_k}) \\ \Big\downarrow \mathrm{id} & & \Big\downarrow \mathrm{id} \\ \widetilde{\mathrm{Lip}}^{(1,-\alpha)}(U) & \xleftarrow{\quad T \quad} & \ell_q(\langle k\rangle^{-(\alpha-1/q')} \ell_p^{M_k}) \end{array} \tag{3.20}$$

where $S : f \mapsto \{\lambda_{km}\}_{k\in\mathbb{N}_0,\, |m|\leq c2^k}$ is given by Theorem 1.8, ensuring the existence of such an (optimal) atomic representation. By (3.19) we have

$$\|S\| \leq c. \tag{3.21}$$

Step 3. We have to include at this point a brief digression concerning quarkonial (subatomic) decompositions of function spaces. We give the main idea only, to explain the subsequent steps of our proof; for all details we refer to [24, Sect. 14]. The idea is to decompose the atoms (recall Definition 1.6) even further to obtain a decomposition of some $f \in S'(\mathbb{R}^n)$ in the form

$$f = \sum_{\gamma\in\mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^{\gamma}_{\nu m} (\gamma q u)_{\nu m}(x), \tag{3.22}$$

where $\lambda^\gamma_{\nu m} \in \mathbb{C}$, and the $(\gamma qu)_{\nu m}$ are so-called $(s,p)_{-1}$-γ-quarks: using our notation from Section 1, let $\psi \in S(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset dQ$ for some $d > 1$ and

$$\sum_{m\in\mathbb{Z}^n} \psi(x-m) = 1 \quad \text{for } x \in \mathbb{R}^n.$$

Let $s \in \mathbb{R}$, $0 < p \le \infty$, $(L+1)/2 \in \mathbb{N}_0$, $\gamma \in \mathbb{N}_0^n$ and define $\psi^\gamma(x) := x^\gamma \psi(x)$. Then for $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$,

$$(\gamma qu)^L_{\nu m}(x) = 2^{-\nu(s-n/p)}((-\Delta)^{(L+1)/2}\psi^\gamma)(2^\nu x - m) \tag{3.23}$$

is called an $(s,p)_L$-γ-*quark* related to $Q_{\nu m}$. Let $(\gamma qu)_{\nu m} = (\gamma qu)^L_{\nu m}$ if $L = -1$ (cf. [24, Def. 14.2, p. 92]). We denote by λ^γ the set $\{\lambda^\gamma_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$, $\gamma \in \mathbb{N}_0^n$. We use a result of Triebel, [24, Cor. 14.7, p. 97], in the following form:

Let $n/(n+1) < p \le \infty$, $0 < q \le \infty$, and $\mu > \mu_0$, with μ_0 sufficiently large. Then $f \in S'(\mathbb{R}^n)$ belongs to $B^{1+n/p}_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{\gamma\in\mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^\gamma_{\nu m} (\gamma qu)_{\nu m}(x), \tag{3.24}$$

convergence being in $S'(\mathbb{R}^n)$, where $(\gamma qu)_{\nu m}$ are $(1+n/p,p)_{-1}$-γ-quarks, and

$$\sup_\gamma 2^{\mu|\gamma|} \|\lambda^\gamma \mid b_{pq}\| < \infty. \tag{3.25}$$

Furthermore, the infimum in (3.25) *over all admissible representations* (3.24) *is an equivalent quasi-norm in $B^{1+n/p}_{p,q}(\mathbb{R}^n)$.*

Recall Definition 1.7 of b_{pq}. Note that the restriction $p > n/(n+1)$ is in our context not substantial: on the one hand, there is also some version of quarkonial decompositions for the case $0 < p \le n/(n+1)$ when f has compact support (cf. [24, Cor. 14.11, p. 99]); on the other hand, we immediately obtain our result (3.17) for all smaller p by the continuous embedding $B^{1+n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{1+n/r}_{r,q}(\mathbb{R}^n)$ if $p \le r$ and the multiplicativity of entropy numbers (having proved the upper estimate for some $r > n/(n+1)$).

The advantage of this further decomposition (3.24) compared with (3.18) is that these γ-quarks (3.23) are "universal" elementary building blocks in the sense that they do not depend upon the function f (unlike the case of atoms where one usually constructs an "optimal" atomic decomposition involving $f \in S'(\mathbb{R}^n)$ itself; see [24, 13.2, Thm. 13.8, pp. 71–75]).

Replace diagram (3.20) by

$$\begin{array}{ccc} \widetilde{B}^{1+n/p}_{p,q}(U) & \xrightarrow{\;S_\gamma\;} & \ell_\infty[2^{\varrho_1 m}\ell_q(\ell_p^{M_k})] \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}_\gamma} \\ \widetilde{\mathrm{Lip}}^{(1,-\alpha)}(U) & \xleftarrow{\;T_\gamma\;} & \ell_\infty[2^{\varrho_2 m}\ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_p^{M_k})] \end{array} \tag{3.26}$$

where $\varrho_1 > \varrho_2$, both ϱ_1 and ϱ_2 being sufficiently large. We thus have to estimate $\|S_\gamma\|$ rather than $\|S\|$ and $\|T_\gamma\|$ instead of $\|T\|$. Likewise, we have to apply Corollary 3.3 instead of Proposition 3.1 to estimate the relevant entropy numbers. Now the above-mentioned result [24, Cor. 14.7, p. 97] (together with our subsequent remark concerning p) implies that S_γ (acting as indicated in (3.26)) is bounded as well (where ϱ_1 is assumed to be sufficiently large). Concerning the boundedness of T_γ it suffices to deal with the less complicated operator T from (3.20), as the γ-quarks are special atoms (up to normalising constants)—and this turns out to be the essential point in the following steps. We refer to [24, Prop. 20.5, pp. 162–165] where these technicalities ("translation" of atomic into quarkonial situation and vice versa) has been performed in detail. This machinery works in a completely parallel way in our case.

STEP 4. We show the boundedness of T from (3.20),

$$T : \{\varkappa_{km}\}_{k\in\mathbb{N}_0,|m|\le c2^k} \mapsto g = \sum_{k=0}^{\infty} \sum_{|m|\le c2^k} \varkappa_{km} a_{km}(x). \tag{3.27}$$

By Step 3 we know that the atoms in (3.27) are very special ones, namely γ-quarks, but we only need the fact that the operator T in (3.27) does not depend upon $f \in S'(\mathbb{R}^n)$. To prove

$$\|T\{\varkappa_{km}\} \,|\, \widetilde{\mathrm{Lip}}^{\,(1,-\alpha)}(U)\| = \|g \,|\, \widetilde{\mathrm{Lip}}^{\,(1,-\alpha)}(U)\| \le c\|\varkappa \,|\, \ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_p^{M_k})\|,$$

we have to show that

$$\sup_{\substack{x,y\in\mathbb{R}^n\\ 0<|x-y|<1/2}} \frac{|g(x)-g(y)|}{|x-y|\,|\log|x-y||^\alpha} \le c\|\varkappa \,|\, \ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_p^{M_k})\|, \tag{3.28}$$

as well as

$$\|g \,|\, L_\infty(U)\| \le c\|\varkappa \,|\, \ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_p^{M_k})\|, \tag{3.29}$$

where $g \in \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$, $\operatorname{supp} g \subset U$ (see (1.20) and Definition 1.1). Thus on the left-hand side of (3.28) we have to take the supremum over those $x, y \in \mathbb{R}^n$ only where $|x|, |y| < 3/2$, $0 < |x-y| < 1/2$. For convenience assume $x, y \in \overline{U}$ (neglecting constants).

We deal with (3.28) first. Here we use a modified version of Step 1 of the proof of Theorem 2.1. Note that for the atoms a_{km} in (3.18) and thus also in (3.27), we have

$$|D^\gamma a_{km}(x)| \le 2^{-k+k|\gamma|}, \quad k\in\mathbb{N}_0,\ m\in\mathbb{Z}^n,\ \gamma\in\mathbb{N}_0^n,\ |\gamma|\le K \tag{3.30}$$

(see (1.9)). We may assume $y = 0$ in (3.28) to simplify the following considerations technically, but one may easily check that everything works for arbitrary $y \in \overline{U}$ as well. Let $x \in U$, $|x| \sim 2^{-j}$, $j \in \mathbb{N}$. In dependence upon x, taken fixed for the moment, we proceed parallel to (2.9) and split (3.27) into different sums; in particular,

$$\begin{aligned} g(x) - g(0) &= \sum_{k=0}^{\infty} \sum_{|m|\le c2^k} \varkappa_{km}(a_{km}(x) - a_{km}(0)) \\ &= \underbrace{\sum_{k=0}^{j} \sum_{|m|\le c2^k} \varkappa_{km}(a_{km}(x) - a_{km}(0))}_{g_j(x)} + \underbrace{\sum_{k=j+1}^{\infty} \sum_{|m|\le c2^k} \varkappa_{km}(a_{km}(x) - a_{km}(0))}_{g^j(x)} \\ &= g_j(x) + g^j(x). \end{aligned} \tag{3.31}$$

By the same arguments as just before (2.10) it is sufficient to consider

$$g_j^1(x) = \sum_{k=0}^{j} \varkappa_{k0}(a_{k0}(x) - a_{k0}(0)) \tag{3.32}$$

($m = 0 \in \mathbb{Z}^n$) instead of $g_j(x)$, and likewise

$$g_j^2(x) = \sum_{k=j+1}^{\infty} \varkappa_{km_k^x}(a_{km_k^x}(x) - a_{km_k^x}(0)), \tag{3.33}$$

where $m_k^x = m(k,x) \in \mathbb{Z}^n$, $|m_k^x| \sim 2^{k-j}$, replacing $g^j(x)$ in (3.31). The few other terms are similar to handle.

Using (3.30) with $\gamma = (0, \ldots, 0)$ we find that

$$\begin{aligned} 2^j(1+j)^{-\alpha}|g_j^2(x)| &\le c_1 \sum_{k=j+1}^{\infty} |\varkappa_{km_k^x}| 2^{-(k-j)}\langle j\rangle^{-\alpha} \\ &\le c_2 \sum_{k=j+1}^{\infty} \langle k\rangle^{-\alpha}\Big(\sum_{|m|\le c2^k} |\varkappa_{km}|^p\Big)^{1/p} 2^{-(k-j)}\left(\frac{\langle j\rangle}{\langle k\rangle}\right)^{-\alpha} \\ &\le c_3\Big(\sum_{k=j+1}^{\infty} \langle k\rangle^{-\alpha q}\Big(\sum_{|m|\le c2^k} |\varkappa_{km}|^p\Big)^{q/p}\Big)^{1/q} \\ &\le c_4\|\varkappa \mid \ell_q(\langle k\rangle^{-\alpha}\ell_p^{M_k})\|, \end{aligned} \tag{3.34}$$

where $0 < q \le \infty$ and the constant $c_4 > 0$ in (3.34) depends upon α, q, but not on $j \in \mathbb{N}$.

We now deal with $g_j^1(x)$. We apply the mean value theorem to $a_{k0}(x) - a_{k0}(0)$ to obtain, by (3.30) with $|\gamma| = 1$,

$$\begin{aligned} 2^j(1+j)^{-\alpha}|g_j^1(x)| &\le c_1\langle j\rangle^{-\alpha}\sum_{k=0}^{j}|\varkappa_{k0}| \\ &\le c_2\langle j\rangle^{-\alpha}\sum_{k=0}^{j}\langle k\rangle^{-(\alpha-1/q')}|\varkappa_{k0}|\langle k\rangle^{\alpha-1/q'} \\ &\le c_3\langle j\rangle^{-\alpha}\Big(\sum_{k=0}^{j}\langle k\rangle^{-(\alpha-1/q')q}|\varkappa_{k0}|^q\Big)^{1/q}\langle j\rangle^{\alpha} \\ &\le c_4\|\varkappa \mid \ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_p^{M_k})\|, \end{aligned} \tag{3.35}$$

where we applied Hölder's inequality for $1 = 1/q + 1/q'$ when $1 \le q \le \infty$. Otherwise we obtain

$$2^j(1+j)^{-\alpha}|g_j^1(x)| \le c\sum_{k=0}^{j}\langle k\rangle^{-\alpha}|\varkappa_{k0}| \le c'\|\varkappa \mid \ell_q(\langle k\rangle^{-\alpha}\ell_p^{M_k})\|,$$

by the monotonicity of ℓ_r-spaces, $\ell_q \hookrightarrow \ell_1$ for $0 < q \le 1$. Using our convention $1/q' = 0$ when $0 < q \le 1$ we thus obtain (3.35) for all q, $0 < q \le \infty$.

Recall that $|x| \sim 2^{-j}$. Then (3.34) and (3.35) imply

$$\frac{|g_j^1(x)| + |g_j^2(x)|}{|x|\,|\log|x||^{\alpha}} \le c\|\varkappa \mid \ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_p^{M_k})\|,$$

which together with our above remarks about the similar terms leads to

$$\frac{|g(x)-g(0)|}{|x|\,|\log|x||^{\alpha}} \le c\|\varkappa\,|\,\ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_p^{M_k})\|, \quad 0<|x|<\frac{1}{2}. \tag{3.36}$$

One verifies that the constant $c>0$ in (3.36) does not depend upon $x\in U$ (or, similarly, $j\in\mathbb{N}$, since $|x|\sim 2^{-j}$), but may depend upon α, q. Taking the supremum in (3.36) over all $x\in U$, $x\neq 0$, we have proved (3.28).

It remains to establish (3.29). Let $x\in U$; then

$$\begin{aligned}|g(x)| &\le |g(x)-g(x/2)|+|g(x/2)-g(0)|+|g(0)| \\ &\le c_1\left\{\frac{|g(x)-g(x/2)|}{|x/2|\,|\log|x/2||^{\alpha}}+\frac{|g(x/2)-g(0)|}{|x/2|\,|\log|x/2||^{\alpha}}\right\}+|g(0)| \\ &\le c_2\|\varkappa\,|\,\ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_p^{M_k})\|+|g(0)|\end{aligned} \tag{3.37}$$

where we applied (3.28). We estimate $|g(0)|$ further by applying (3.30) with $|\gamma|=0$:

$$|g(0)| \le \sum_{k=0}^{\infty}\sum_{|m|\le c2^k}|\varkappa_{km}|2^{-k}.$$

We proceed in the same way as we did before several times (see Step 1 of the proof of Theorem 2.1 as well as the arguments following (3.32)), to conclude that we need only consider

$$\sum_{k=0}^{\infty}|\varkappa_{k0}|2^{-k},$$

because there are only few non-vanishing terms in the sum over all $m\in\mathbb{Z}^n$ and they are similar to handle. We apply Hölder's inequality when $1\le q\le\infty$, and $\ell_q\hookrightarrow\ell_1$ if $0<q\le 1$, to conclude that

$$\sum_{k=0}^{\infty}|\varkappa_{k0}|2^{-k} \le c\|\varkappa\,|\,\ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_p^{M_k})\|,$$

which together with (3.37) yields

$$|g(x)| \le c\|\varkappa\,|\,\ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_p^{M_k})\|$$

for all $x\in U$. Hence (3.29) is verified and so, finally, T is bounded:

$$\|T\| \le c. \tag{3.38}$$

In view of our remarks concerning T from (3.20) and T_γ from (3.26) it is obvious that one can show the boundedness of T_γ using an analogous argument.

Step 5. We deduce from (3.26) that

$$\mathrm{id}(\widetilde{B}^{n/p+1}_{p,q}(U)\to\widetilde{\mathrm{Lip}}^{(1,-\alpha)}(U)) = T_\gamma\circ\mathrm{id}_\gamma\circ S_\gamma,$$

or likewise from (3.20) that

$$\mathrm{id}(\widetilde{B}^{n/p+1}_{p,q}(U)\to\widetilde{\mathrm{Lip}}^{(1,-\alpha)}(U)) = T\circ\mathrm{id}(\ell_q(\ell_p^{M_k})\to\ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_p^{M_k}))\circ S.$$

By our remarks in Step 3 we have to apply Corollary 3.3 (or its scalar version, Proposition 3.1) with $\varkappa=\alpha-1/q'>0$. Note that $1/q'=0$ if $0<q\le 1$. Now (3.21) and (3.38) as well as the multiplicativity of entropy numbers (1.27) yield the desired estimate (3.17). ∎

We postpone the counterpart of (3.17), i.e. some estimate from below, until the next section.

Remark 3.6. Examining the above proof of Theorem 3.5 in detail one verifies that it can be transferred without difficulty to bounded C^∞ domains Ω in $\mathbb{R}^n$ (always in the sense of [6, Def. V.4.1, p. 244]). This can be done at the expense of some constants, but we do not care about (domain-dependent) constants at all.

4. Comparison with related results

Recall the definitions of $B^{(s,b)}_{p,q}(\Omega)$ and $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, as given in Section 1. We introduce the counterparts of the spaces $\mathrm{Lip}^{(1,-\alpha)}$ on $\mathbb{R}^n$, the Zygmund spaces $\mathcal{C}^{(1,-\alpha)}(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$. Recall the difference operator Δ^m_h, $m \in \mathbb{N}_0$, $h \in \mathbb{R}^n$, introduced in (1.2).

Definition 4.1. Let $\alpha \in \mathbb{R}$. Then the space $\mathcal{C}^{(1,-\alpha)}(\mathbb{R}^n)$ is defined as the set of all $f \in C(\mathbb{R}^n)$ such that

$$\|f \,|\, \mathcal{C}^{(1,-\alpha)}(\mathbb{R}^n)\| = \|f \,|\, L_\infty(\mathbb{R}^n)\| + \sup_{\substack{x,h\in\mathbb{R}^n \\ 0<|h|<1/2}} \frac{|(\Delta^2_h f)(x)|}{|h|\,|\log|h||^\alpha} < \infty. \tag{4.1}$$

This definition was suggested by Triebel in some unpublished notes.

Embeddings. We study the relation between $\mathrm{Lip}^{(1,-\alpha)}$, $\mathcal{C}^{(1,-\alpha)}$ and $B^{(s,b)}_{p,q}$ on $\mathbb{R}^n$. We are able to give natural counterparts of the facts already known in case $\alpha = b = 0$ (cf. [21, Prop. 2.5.7, p. 89; and Cor. 2.5.12, p. 113]).

Proposition 4.2. *Let* $\alpha \geq 0$. *Then*

$$B^{(1,-\alpha)}_{\infty,1}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^{(1,-\alpha)}(\mathbb{R}^n) = B^{(1,-\alpha)}_{\infty,\infty}(\mathbb{R}^n). \tag{4.2}$$

Proof. The second inclusion is obvious by the respective definitions, thus we have to show the left-hand and right-hand ones. As already mentioned above, the case $\alpha = 0$ is known: see [21, (2.5.7/11), p. 90] for the first embedding, and [21, Cor. 2.5.12(ii), p. 113] for the other one. We adapt the proof of Thm. 2.5.12 in [21, pp. 110–112] to our situation. Let all the spaces be defined on $\mathbb{R}^n$ unless otherwise stated. We use the notation $\widehat{g} = \mathcal{F}g$ for the Fourier transform of $g \in S'$ and $g^\vee = \mathcal{F}^{-1}g$ for the inverse Fourier transform.

Step 1. Let $\{\varphi_j\}_{j=0}^\infty$ be a smooth partition of unity (as introduced before Definition 1.3):

$$f = \sum_{j=0}^\infty (\varphi_j \widehat{f}\,)^\vee. \tag{4.3}$$

Using the notation of the above-mentioned Theorem 2.5.12 in [21] we put $s = 1$, $p = q = \infty$ and $M = 1, 2$ for $\mathrm{Lip}^{(1,-\alpha)}$ and $\mathcal{C}^{(1,-\alpha)}$, respectively. Proceeding in the same way, we obtain the counterpart of [21, (2.5.12/8)],

$$(4.4)\qquad \sup_{|\varrho|\le 2^{-k}} \|\Delta_\varrho^M(\varphi_j\widehat{f}\,)^\vee \,|\, L_\infty\| \le c\min(1,2^{(j-k)M})\|\varphi_j^* f \,|\, L_\infty\| = c\min(1,2^{(j-k)M})\|(\varphi_j\widehat{f}\,)^\vee \,|\, L_\infty\|,$$

where the maximal function $\varphi_j^* f$ is given by

$$(\varphi_j^* f)(x) = \sup_{y\in\mathbb{R}^n} \frac{|(\varphi_j\widehat{f}\,)^\vee(x-y)|}{1+|2^j y|^a}, \quad x\in\mathbb{R}^n,\ a>0$$

(see [21, Def. 2.3.6/2, p. 53]). Thus, with (4.4) and $M=1$, we arrive at

$$(4.5)\qquad \|f\,|\,\mathrm{Lip}^{(1,-\alpha)}\| \le c_1\Big(\|f\,|\,L_\infty\| + \sup_{k\in\mathbb{N}} 2^k k^{-\alpha}\sum_{j=0}^\infty \sup_{|\varrho|\le 2^{-k}} \|\Delta_\varrho^1(\varphi_j\widehat{f})^\vee \,|\, L_\infty\|\Big)$$
$$\le c_2\Big(\|f\,|\,L_\infty\| + \sup_k 2^k k^{-\alpha}\sum_{j=0}^k 2^{j-k}\|(\varphi_j\widehat{f}\,)^\vee \,|\, L_\infty\| + \sup_k 2^k k^{-\alpha}\sum_{j=k+1}^\infty \|(\varphi_j\widehat{f}\,)^\vee \,|\, L_\infty\|\Big).$$

Plainly, $\|f\,|\,L_\infty\| \le c\|f\,|\,B^{(1,-\alpha)}_{\infty,1}\|$ because of (4.3) and $2^j(1+j)^{-\alpha}\ge c'$. Now

$$(4.6)\qquad \sup_k 2^k k^{-\alpha}\sum_{j=k+1}^\infty \|(\varphi_j\widehat{f}\,)^\vee \,|\, L_\infty\| \le c\sup_k \sum_{j=k+1}^\infty 2^j j^{-\alpha}\|(\varphi_j\widehat{f}\,)^\vee \,|\, L_\infty\| \le c\|f\,|\,B^{(1,-\alpha)}_{\infty,1}\|,$$

and

$$(4.7)\qquad \sup_k 2^k k^{-\alpha}\sum_{j=0}^k 2^{j-k}\|(\varphi_j\widehat{f}\,)^\vee \,|\, L_\infty\| = \sup_k k^{-\alpha}\sum_{j=0}^k 2^j\|(\varphi_j\widehat{f}\,)^\vee \,|\, L_\infty\|$$
$$\le c\sup_k \sum_{j=0}^k 2^j(1+j)^{-\alpha}\|(\varphi_j\widehat{f}\,)^\vee \,|\, L_\infty\| = c\|f\,|\,B^{(1,-\alpha)}_{\infty,1}\|,$$

where we used $\alpha\ge 0$ in the last estimate. Combining (4.5)–(4.7) we obtain

$$\|f\,|\,\mathrm{Lip}^{(1,-\alpha)}\| \le c\|f\,|\,B^{(1,-\alpha)}_{\infty,1}\|,$$

which proves the left-hand embedding in (4.2).

Assume now $M=2$ in (4.4). Then the counterpart of (4.5) reads

$$\|f\,|\,\mathcal{C}^{(1,-\alpha)}\| \le c_1\Big(\|f\,|\,L_\infty\| + \sup_{k\in\mathbb{N}} 2^k k^{-\alpha}\sum_{j=0}^\infty \sup_{|\varrho|\le 2^{-k}} \|\Delta_\varrho^2(\varphi_j\widehat{f}\,)^\vee \,|\, L_\infty\|\Big)$$
$$\le c_2\Big(\|f\,|\,L_\infty\| + \sup_k 2^k k^{-\alpha}\sum_{j=0}^k 2^{(j-k)2}\|(\varphi_j\widehat{f}\,)^\vee \,|\, L_\infty\| + \sup_k 2^k k^{-\alpha}\sum_{j=k+1}^\infty \|(\varphi_j\widehat{f}\,)^\vee \,|\, L_\infty\|\Big),$$

upon applying (4.3) and (4.4). Again, $\|f \mid L_\infty\| \le c\|f \mid B^{(1,-\alpha)}_{\infty,\infty}\|$. Furthermore, (4.6) is replaced by

$$\sup_k 2^k k^{-\alpha} \sum_{j=k+1}^{\infty} \|(\varphi_j \widehat{f})^\vee \mid L_\infty\| \le c_1 \sup_{j\in\mathbb{N}} 2^j j^{-\alpha} \|(\varphi_j \widehat{f})^\vee \mid L_\infty\| \le c_2 \|f \mid B^{(1,-\alpha)}_{\infty,\infty}\|$$

and likewise (4.7) by

$$\sup_k 2^k k^{-\alpha} \sum_{j=0}^{k} 2^{(j-k)2} \|(\varphi_j \widehat{f})^\vee \mid L_\infty\| = \sup_k k^{-\alpha} \sum_{j=0}^{k} 2^{2j-k} \|(\varphi_j \widehat{f})^\vee \mid L_\infty\| \le c \sup_j 2^j (1+j)^{-\alpha} \|(\varphi_j \widehat{f})^\vee \mid L_\infty\| = c\|f \mid B^{(1,-\alpha)}_{\infty,\infty}\|.$$

Consequently, we have proved that

$$\|f \mid \mathcal{C}^{(1,-\alpha)}\| \le c\|f \mid B^{(1,-\alpha)}_{\infty,\infty}\|, \tag{4.8}$$

i.e. $B^{(1,-\alpha)}_{\infty,\infty} \hookrightarrow \mathcal{C}^{(1,-\alpha)}$. It remains to prove the converse.

Step 2. Assume $f \in \mathcal{C}^{(1,-\alpha)}$. One checks that Theorem 2.5.3(i) in [21, p. 81] remains valid with $B^s_{\infty,\infty}(\mathbb{R}^n)$ replaced by $B^{(s,-\alpha)}_{\infty,\infty}(\mathbb{R}^n)$, $\alpha > 0$, $s > 0$. In particular, we obtain

$$\|f \mid B^{(1,-\alpha)}_{\infty,\infty}\| \le c \inf(\|a_0 \mid L_\infty\| + \sup_k 2^k (1+k)^{-\alpha} \|f - a_k \mid L_\infty\|), \tag{4.9}$$

where the infimum is taken over all $\{a_j(x)\}_{j=0}^\infty$ such that $a_j \in S' \cap L_\infty$, $\operatorname{supp} \widehat{a_j} \subset \{y \in \mathbb{R}^n : |y| \le 2^{j+1}\}$, and $f = \lim_{k\to\infty} a_k$ in S' (see [21, 2.5.3, pp. 80–83] for details). Hence [21, (2.5.12/1), (2.5.12/2), pp. 109–110] yield

$$\inf \|f - a_k \mid L_\infty\| \le c \sup_{|\varrho| \le 2^{-k}} \|\Delta^M_\varrho f \mid L_\infty\|, \tag{4.10}$$

where $M \in \mathbb{N}$, and $c > 0$ may depend upon M, but not on f. Replacing the latter expression in (4.9) by (4.10) we arrive at

$$\|f \mid B^{(1,-\alpha)}_{\infty,\infty}\| \le c(\|f \mid L_\infty\| + \sup_k 2^k (1+k)^{-\alpha} \sup_{|\varrho| \le 2^{-k}} \|\Delta^M_\varrho f \mid L_\infty\|).$$

In view of (4.1) it remains to show that

$$\sup_{0<|h|<1/2} |h|^{-1} |\log|h||^{-\alpha} \sup_{|\varrho| \le |h|} \|\Delta^M_\varrho f \mid L_\infty\| \le c \sup_{0<|h|<1/2} |h|^{-1} |\log|h||^{-\alpha} \|\Delta^2_h f \mid L_\infty\|, \tag{4.11}$$

for some suitable $M \in \mathbb{N}$. We modify Step 3 of the proof of Theorem 2.5.12 in [21] and estimate the left-hand side of (4.11) from above by

$$\begin{aligned} \sup_{0<|h|<1/2} |h|^{-1} |\log|h||^{-\alpha} \sup_{|\varrho| \le |h|} \|\Delta^M_\varrho f \mid L_\infty\| &\le \sup_{0<|h|<1/2} |h|^{-1} |\log|h||^{-\alpha} \sup_{0 \le |\varrho| \le |h|/2} \|\Delta^M_\varrho f \mid L_\infty\| \\ &\quad + \sup_{0<|h|<1/2} |h|^{-1} |\log|h||^{-\alpha} \sup_{|h|/2 \le |\varrho| \le |h|} \|\Delta^M_\varrho f \mid L_\infty\|. \end{aligned} \tag{4.12}$$

We substitute $\eta = h/2$ in the first term on the right-hand side of (4.12) and obtain

$$\sup_{0<|h|<1/2} |h|^{-1}|\log|h||^{-\alpha} \sup_{0\le|\varrho|\le|h|/2} \|\Delta_\varrho^M f \mid L_\infty\|$$
$$= \sup_{0<|\eta|<1/4} 2^{-1}|\eta|^{-1}|\log 2+\log|\eta||^{-\alpha} \sup_{0\le|\varrho|\le|\eta|} \|\Delta_\varrho^M f \mid L_\infty\|$$
$$\le 2^{\alpha-1} \sup_{0<|\eta|<1/4} |\eta|^{-1}|\log|\eta||^{-\alpha} \sup_{0\le|\varrho|\le|\eta|} \|\Delta_\varrho^M f \mid L_\infty\|.$$

When $\alpha < 1$ we can rewrite (4.12) as

$$\sup_{0<|h|<1/2} |h|^{-1}|\log|h||^{-\alpha} \sup_{|\varrho|\le|h|} \|\Delta_\varrho^M f \mid L_\infty\|$$
$$\le \frac{1}{1-2^{\alpha-1}} \sup_{0<|h|<1/2} |h|^{-1}|\log|h||^{-\alpha} \sup_{|h|/2\le|\varrho|\le|h|} \|\Delta_\varrho^M f \mid L_\infty\|$$

and hence it is sufficient to prove that

$$\sup_{0<|h|<1/2} |h|^{-1}|\log|h||^{-\alpha} \sup_{|h|/2\le|\varrho|\le|h|} \|\Delta_\varrho^M f \mid L_\infty\| \le c \sup_{0<|h|<1/2} |h|^{-1}|\log|h||^{-\alpha}\|\Delta_h^2 f \mid L_\infty\| \tag{4.13}$$

instead of (4.11). If $\alpha \ge 1$ we have slightly to refine the above argument.

Let $K_\alpha < 1/\alpha$ be some constant; then

$$|\log 2+\log|\eta||^{-\alpha} \le 2^{\alpha K_\alpha}|\log|\eta||^{-\alpha} \quad \text{if } |\eta| \le 2^{-(1-2^{-K_\alpha})^{-1}}.$$

If we take the supremum over those h for which $0 < |h| \le 2^{-(2^{K_\alpha}-1)^{-1}} =: h_\alpha$, the above argument works as $1 - 2^{\alpha K_\alpha - 1} > 0$. On the other hand, only small values of h are of interest, as we may estimate $\|\Delta_h^M f \mid L_\infty\| \le c\|f \mid L_\infty\|$ and thus

$$\sup_{h_\alpha\le|h|<1/2} |h|^{-1}|\log|h||^{-\alpha}\|\Delta_h^M f \mid L_\infty\| \le c_\alpha\|f \mid L_\infty\|.$$

(For $\alpha < 1$ we may choose $K_\alpha = 1$ implying $h_\alpha = 1/2$ so that the last-mentioned term disappears.) Thus whenever $\alpha \ge 0$ we arrive at (4.13), possibly at the expense of some unimportant constants depending upon α. By [21, (2.5.12/17)] for ϱ, $|h|/2 \le |\varrho| \le |h|$, we have

$$\|\Delta_\varrho^4 f \mid L_\infty\| \le c \mathop{T}_{1/8\le|\lambda|\le1} \|\Delta_{\lambda|h|}^2 f \mid L_\infty\|\, d\lambda,$$

leading to

$$\sup_{0<|h|\le h_\alpha} |h|^{-1}|\log|h||^{-\alpha} \sup_{|h|/2\le|\varrho|\le|h|} \|\Delta_\varrho^4 f \mid L_\infty\| \le c \sup_{0<|h|\le h_\alpha} |h|^{-1}|\log|h||^{-\alpha}\|\Delta_h^2 f \mid L_\infty\|,$$

which by our remarks above finishes the proof. ∎

Remark 4.3. In view of Leopold's Proposition 1.13, the above proposition implies that

$$B^1_{\infty,q}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) \quad \text{if } \alpha \ge (1-1/q)_+;$$

see also (1.23) with $p = \infty$.

Proposition 4.4. *Let $\alpha \ge 0$. Then*

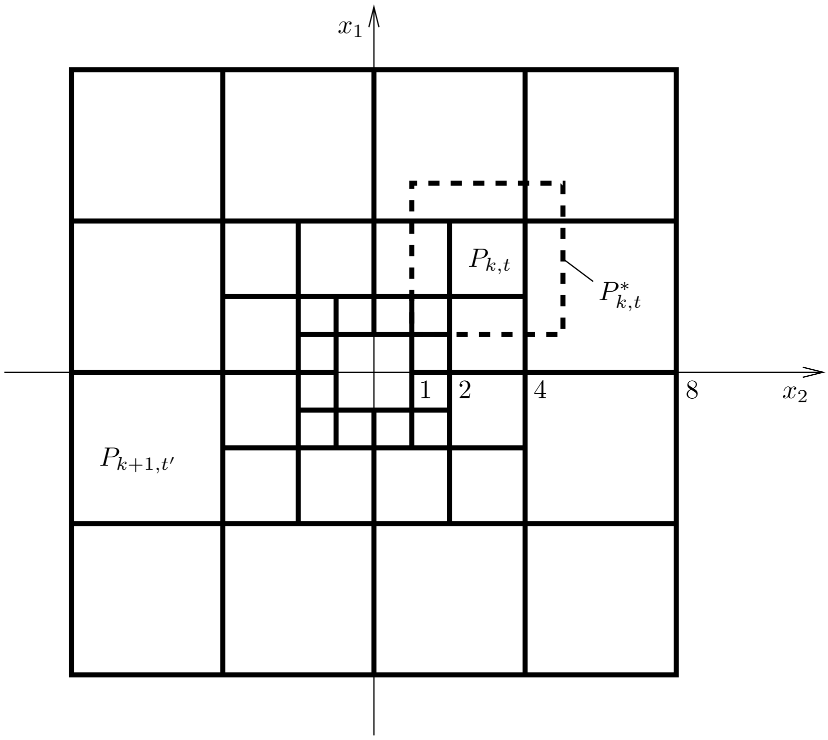
$$B^{(1,-\alpha)}_{\infty,q}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) \quad \textit{if, and only if}, \ 0 < q \le 1.$$

Proof. Let again all spaces be defined on $\mathbb{R}^n$ unless otherwise stated. In view of Proposition 4.2 it is sufficient to give, for each $q \in (1,\infty)$, a function $\psi \in B^{(1,-\alpha)}_{\infty,q}$ such that

$$\sup_{\substack{x\in\mathbb{R}^n\\ 0<|x|<1/2}} \frac{|\psi(x)-\psi(0)|}{|x|\,|\log|x||^{\alpha}}$$

is not finite. We modify the construction given in [20, pp. 134–135].

Assume $1<q<\infty$. Let $\varphi \in S$ with $\operatorname{supp}\widehat{\varphi} \subset P_{1,1}$, where the cubes $P_{k,t}$, $k \in \mathbb{N}_0$, $t = 1,\dots,4^n-2^n$, are introduced in [20, pp. 24–25] as follows:



The corridors K_k, $k \in \mathbb{N}$, given by

$$K_k = \{x \in \mathbb{R}^n : |x_j| \le 2^k,\ j=1,\dots,n\} \setminus \{x \in \mathbb{R}^n : |x_j| < 2^{k-1},\ j=1,\dots,n\},$$

are subdivided by the $3n$ hyperplanes

$$\{x \in \mathbb{R}^n : x_l = 0\}, \quad \{x \in \mathbb{R}^n : x_l = \pm 2^{k-1}\}, \quad l = 1,\dots,n,$$

in congruent (closed) cubes $P_{k,t}$. Here $t = 1,\dots,4^n-2^n$ numbers arbitrarily the cubes with fixed $k \in \mathbb{N}$.

Let $T = 4^n - 2^n$; then with

$$K_0 = P_0 = \{x \in \mathbb{R}^n : |x_j| \le 1,\ j=1,\dots,n\}, \quad P_{0,t} = P_0$$

we have

$$\mathbb{R}^n = \bigcup_{\substack{k=0,1,\ldots \\ t=1,\ldots,T}} P_{k,t}.$$

Moreover, let $P^*_{k,t}$ be the double of $P_{k,t}$, i.e. having the same centre, but its side-length doubled (see the picture opposite). We may assume

$$P_{1,1} = \{x \in \mathbb{R}^n : 1 \le |x_1| \le 2,\ 0 \le |x_j| \le 1,\ j = 2, \ldots, n\}.$$

We put

$$\psi(x) = \sum_{k=1}^{\infty} 2^{-k} k^{\alpha-1} \varphi(2^k x)$$

and thus obtain, for some $c > 0$,

$$\|(\varphi_{k,t}\widehat{\psi})^\vee \mid L_\infty\| = c\delta_{t,1} 2^{-k} k^{\alpha-1}, \quad k = 2, 3, \ldots, \tag{4.14}$$

where the $\{\varphi_{k,t}\}_{k\in\mathbb{N}_0,\, t=1,\ldots,T}$ form a smooth partition of unity adapted to the cubes $P_{k,t}$ (see [20, p. 25]). In particular, $\{\varphi_{k,t}\}_{k\in\mathbb{N}_0,\, t=1,\ldots,T} \subset S$, with

$$\operatorname{supp} \varphi_{k,t} \subset P^*_{k,t}, \quad k \in \mathbb{N}_0,\ t = 1, \ldots, T,$$
$$\sup_{k,t,\, |\alpha| \le L} \sup_{x\in\mathbb{R}^n} \langle x\rangle^{|\alpha|} |D^\alpha \varphi_{k,t}(x)| = C_\varphi < \infty$$

for some $L \in \mathbb{N}$, and

$$\sum_{k=0}^{\infty} \sum_{t=1}^{T} \varphi_{k,t}(x) = 1, \quad x \in \mathbb{R}^n$$

(cf. [20, 2.1.1] for details).

Note that $\operatorname{supp} \widehat{\varphi} \subset P_{1,1}$ implies $\operatorname{supp} \widehat{\varphi}(2^k \cdot) \subset P_{k+1,1}$ and thus (4.14). Hence,

$$\begin{aligned} \|\psi \mid B^{(1,-\alpha)}_{\infty,q}\| &\sim \Big(\sum_{k=2}^{\infty} \sum_{t=1}^{T} 2^{kq} k^{-\alpha q} \|(\varphi_{k,t}\widehat{\psi})^\vee \mid L_\infty\|^q \Big)^{1/q} \\ &= c\Big(\sum_{k=2}^{\infty} 2^{kq} k^{-\alpha q} 2^{-kq} k^{(\alpha-1)q} \Big)^{1/q} = c\Big(\sum_{k=2}^{\infty} k^{-q} \Big)^{1/q} < \infty \end{aligned} \tag{4.15}$$

because $q > 1$. Note that using a partition of unity adapted to the cubes $P_{k,t}$ (instead of the annuli $\{A_l\}_{l\in\mathbb{N}_0}$ as usual) only contributes to some constants. Now (4.15) implies $\psi \in B^{(1,-\alpha)}_{\infty,q}$ if $q > 1$.

On the other hand, we may assume $\varphi(0) = 1$ and $\varphi(x) = 1 + \gamma(x)$ with $|\gamma(x)| < 1/2$ if $|x| \le 2^{-N}$, where N is an appropriate number. Let $j > N$, and assume $|x| \sim 2^{-j}$; then

$$\begin{aligned} \frac{|\psi(x) - \psi(0)|}{|x|\, |\log |x||^\alpha} &\sim 2^j j^{-\alpha} |\psi(x) - \psi(0)| \\ &\ge 2^j j^{-\alpha} \Big| \sum_{k=1}^{j-N} \frac{\varphi(2^k x) - 1}{2^k k^{-\alpha+1}} \Big| - 2^j j^{-\alpha} \sum_{k=j-N+1}^{\infty} \frac{|\varphi(2^k x) - 1|}{2^k k^{-\alpha+1}} \\ &\ge 2^j j^{-\alpha} \frac{1}{2} \sum_{k=1}^{j-N} 2^{-k} k^{\alpha-1} \end{aligned}$$

$$
\begin{aligned}
&\quad - c_1 2^N \left(1-\frac{N}{j}\right)^{\alpha} \sum_{k=j-N+1}^{\infty} 2^{-(k-j+N)} \left(\frac{k}{j-N}\right)^{\alpha} \frac{1}{k} \\
&\geq c_2 2^N \left(1-\frac{N}{j}\right)^{\alpha} \left(\sum_{k=1}^{j-N} \frac{1}{k} - c_3\right) \\
&\geq c_4 2^N \left(1-\frac{N}{j}\right)^{\alpha} (\log j - c_5) \geq c_{6,N} \log j \\
&\geq c_{7,N} \log |\log |x||
\end{aligned}
$$

where $c_{7,N}$ does not depend upon $j \gg N$ if j is sufficiently large. Consequently,

$$\sup_{\substack{x\in\mathbb{R}^n \\ 0<|x|<1/2}} \frac{|\psi(x)-\psi(0)|}{|x|\,|\log|x||^{\alpha}}$$

is not finite and thus $\psi \notin \mathrm{Lip}^{(1,-\alpha)}$. ∎

REMARK 4.5. We give an alternative proof of Proposition 4.4 proceeding by contradiction. We apply Leopold's Proposition 1.13 as well as Theorem 2.1(ii), especially the sharpness assertion. By Proposition 4.2 it suffices to show the necessity in Proposition 4.4.

Let $\alpha \geq 0$ and assume that there is some $v > 1$ such that

$$B^{(1,-\alpha)}_{\infty,v}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n).$$

By Proposition 1.13 this implies that

$$B^{1}_{\infty,q}(\mathbb{R}^n) \hookrightarrow B^{(1,-\alpha)}_{\infty,v}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$$

if $q > v$, $\alpha > 1/v - 1/q > 0$. In other words,

$$B^{1}_{\infty,q}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) \tag{4.16}$$

if $\alpha > 1/v-1/q > 0$. Recall $v > 1$; thus we may choose α such that $1/v-1/q < \alpha < 1-1/q$ and hence (4.16) contradicts the sharpness assertion in Theorem 2.1(ii).

So both the constructive proof above as well as this consideration verify Proposition 4.4.

Entropy numbers. As we already mentioned in Section 1, there cannot be a compact embedding in the case of spaces on $\mathbb{R}^n$, unlike the situation for spaces defined on domains. Recall that in spite of the diversity of spaces of type $B^{(s,b)}_{p,q}(\Omega)$, $\widetilde{B}^{(s,b)}_{p,q}(\Omega)$ and $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$, $\widetilde{\mathrm{Lip}}^{(1,-\alpha)}(\Omega)$, in general, we can handle the entropy numbers of corresponding (compact) embeddings simultaneously (neglecting constants).

We essentially rely on Leopold's Theorem 1.16. We are able to extend his results in some cases. On the other hand, we use these outcomes together with Proposition 4.2 to obtain some counterpart of Theorem 3.5.

The main results in what follows are Corollary 4.9 and Theorem 4.10, which contain optimal estimates of the entropy numbers of embeddings between spaces of the type considered above.

In the sequel we make use several times of the interpolation property of entropy numbers:

Let A be a quasi-Banach space and $\{B_1, B_2\}$ an interpolation couple of λ-Banach spaces. Let $0 < \theta < 1$ and let B_θ be a quasi-Banach space such that $B_1 \cap B_2 \subset B_\theta \subset B_1 + B_2$ and

$$\|b \,|\, B_\theta\| \le \|b \,|\, B_1\|^{1-\theta} \|b \,|\, B_2\|^{\theta} \quad \text{if } b \in B_1 \cap B_2.$$

Let T be a linear and continuous operator from A into $B_1 \cap B_2$. Then for all $k_1, k_2 \in \mathbb{N}$,

$$e^{\lambda}_{k_1+k_2-1}(T : A \to B_\theta) \le 2^{1/\lambda} e^{1-\theta}_{k_1}(T : A \to B_1) e^{\theta}_{k_2}(T : A \to B_2). \tag{4.17}$$

See [14, Thm. 3.2(i)] for a proof and further details. We start our considerations with the following lemma. All spaces are defined on $\mathbb{R}^n$ unless otherwise stated.

Lemma 4.6. *Let $\sigma_i \in \mathbb{R}$, $0 < r_i \le \infty$, $a_i \in \mathbb{R}$, $0 < v_i \le \infty$, $i = 1, 2$, and $0 < \theta < 1$. Let*

$$\begin{gathered} s = \theta\sigma_1 + (1-\theta)\sigma_2, \quad a = \theta a_1 (1-\theta) a_2, \\ \frac{1}{p} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}, \quad \frac{1}{u} = \frac{\theta}{v_1} + \frac{1-\theta}{v_2}. \end{gathered} \tag{4.18}$$

Then for all $f \in B^{(\sigma_1,a_1)}_{r_1,v_1} \cap B^{(\sigma_2,a_2)}_{r_2,v_2}$,

$$\|f \,|\, B^{(s,a)}_{p,u}\| \le \|f \,|\, B^{(\sigma_1,a_1)}_{r_1,v_1}\|^{\theta} \|f \,|\, B^{(\sigma_2,a_2)}_{r_2,v_2}\|^{1-\theta}.$$

Proof. In view of Definition 1.3 this is an easy consequence of Hölder's inequality (applied twice) and (4.18). ■

Estimate from above. We study the embedding

$$\mathrm{id}_\Omega : B^{(s_1,b_1)}_{p_1,q_1}(\Omega) \to B^{(s_2,b_2)}_{p_2,q_2}(\Omega), \tag{4.19}$$

where now Ω is always assumed to be a bounded C^∞ domain in $\mathbb{R}^n$.

Proposition 4.7. *Let $-\infty < s_2 \le s_1 < \infty$, $0 < p_1 \le p_2 \le \infty$ with $s_1 - n/p_1 = s_2 - n/p_2$. Let $1 \le q_1, q_2 \le \infty$ and $b_1, b_2 \in \mathbb{R}$ with $b_1 - b_2 > (1/q_2 - 1/q_1)_+$. Then the embedding (4.19) is compact. Denote its entropy numbers by $e_k(\mathrm{id}_\Omega)$; then there is a constant $c > 0$ such that for all $k \in \mathbb{N}$,*

$$e_k(\mathrm{id}_\Omega) \le c(\log\langle k\rangle)^{-(b_1-b_2)+(1/q_2-1/q_1)_+}. \tag{4.20}$$

Proof. Step 1. The case $s_1 = s_2$, $p_1 = p_2$ and $b_2 = 0$ is covered by Theorem 1.16. In order to extend it to (4.20) we proceed as follows. We first deal with the case $s_1 = s_2$, $p_1 = p_2$, $b_1 = 0$ (and hence $-b_2 > (1/q_2 - 1/q_1)_+$), which then—by composition arguments—yields (4.20) in the case of $b_1 \ge 0$, $b_2 \le 0$ with $b_1 - b_2 > (1/q_2 - 1/q_1)_+$. Finally, to include the cases $b_1 > b_2 > 0$ and $b_2 < b_1 < 0$, we involve interpolation as well as duality arguments. Only that very last part of the proof, concerning duality, uses the restriction $q_1, q_2 \ge 1$. Hence one can follow the proof to verify that (4.20) remains true for $0 < q_1, q_2 < 1$ in all cases apart from $b_2 < b_1 < 0$ (where we have not been able to prove it yet).

We want to use Lemma 4.6 several times in the proof below but it is only given for the spaces $B^{(s,b)}_{p,q}(\mathbb{R}^n)$. Dealing with spaces on domains, defined via restriction, one has to be careful with their norm definition; see Definition 1.11. However, in view of our remarks before Lemma 4.6, it turns out very convenient again that we might restrict ourselves to considering subspaces of $\mathbb{R}^n$ only, i.e. $\widetilde{B}^{(s,b)}_{p,q}(\Omega)$ as well as $\widetilde{\mathrm{Lip}}^{(1,-\alpha)}(\Omega)$.

Step 2. We show

$$e_k(\mathrm{id}_\Omega : B^s_{p,q_1}(\Omega) \to B^{(s,-b)}_{p,q_2}(\Omega)) \le c(\log\langle k\rangle)^{-b+(1/q_2-1/q_1)_+}, \tag{4.21}$$

where $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q_1, q_2 \le \infty$ and $b > (1/q_2 - 1/q_1)_+$. We closely follow the proof in [16, pp. 9–13] and give the necessary modifications, with an outline of the essential ideas and steps for the reader not familiar with Leopold's proof (which, in fact, is a modified version of [10] in the case of $b_1 = b_2 = 0$ and $s_1 - n/p_1 > s_2 - n/p_2$).

Note that $g \in \mathcal{C}^\varrho(\mathbb{R}^n)$ is a pointwise multiplier for $B^{(s,b)}_{p,q}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$, $b \in \mathbb{R}$, if $\varrho > \max(s, n/p - s)$. In particular, there is some $c > 0$ such that for all $g \in \mathcal{C}^\varrho(\mathbb{R}^n)$, $f \in B^{(s,b)}_{p,q}(\mathbb{R}^n)$,

$$\|gf \mid B^{(s,b)}_{p,q}(\mathbb{R}^n)\| \le c\|g \mid \mathcal{C}^\varrho(\mathbb{R}^n)\| \, \|f \mid B^{(s,b)}_{p,q}(\mathbb{R}^n)\|. \tag{4.22}$$

To see this, one checks the proof of Theorem 2.8.2 in [21, p. 140] given for $b = 0$. One verifies that all the arguments work, mutatis mutandis, in our modified version.

The idea of Leopold's proof is as follows. Let Q_r, $r > 0$, be given by

$$Q_r = \{x \in \mathbb{R}^n : |x_j| \le r,\ j = 1, \ldots, n\}.$$

We may assume that $\Omega \subset Q_1$ and we have some function $\psi \in S(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset Q_2$, $\psi(x) = 1$, $x \in Q_1$. (In fact, ψ has to satisfy some more properties: see Leopold's proof.) Now one has the mappings

$$B^s_{p,q_1}(\Omega) \xrightarrow{E} \widetilde{B}^s_{p,q_1}(Q_1) \xrightarrow{\mathrm{id}} \widetilde{B}^{(s,-b)}_{p,q_2}(Q_1) \hookrightarrow B^{(s,-b)}_{p,q_2}(\mathbb{R}^n) \xrightarrow{R} B^{(s,-b)}_{p,q_2}(\Omega), \tag{4.23}$$

where E is the extension operator given by [22, Thm. 5.1.3, p. 239] and R the usual restriction operator. From (4.23) we thus get

$$e_k(\mathrm{id}_\Omega) \le c e_k(\mathrm{id} : \widetilde{B}^s_{p,q_1}(Q_1) \to \widetilde{B}^{(s,-b)}_{p,q_2}(Q_1)).$$

Next one splits $f \in \widetilde{B}^s_{p,q_1}(Q_1)$, $\|f \mid \widetilde{B}^s_{p,q_1}(Q_1)\| \le 1$, into

$$f = \psi \sum_{j=0}^{N} (\varphi_j \widehat{f})^\vee + \psi \sum_{j=N+1}^{\infty} (\varphi_j \widehat{f})^\vee =: f_N + f^N, \tag{4.24}$$

where $\{\varphi_j\}_{j=0}^\infty$ is a smooth dyadic partition of unity. Parallel to [16], now using (4.22) additionally, we estimate

$$\|f^N \mid B^{(s,-b)}_{p,q_2}(\mathbb{R}^n)\| \le c(1+N)^{-b+(1/q_2-1/q_1)_+}. \tag{4.25}$$

Considering f_N from (4.24) one expands $\varphi_j \widehat{f}$ in a trigonometric series in the cube $Q_{2^j\pi}$,

$$(\varphi_j \widehat{f})(\xi) = \sum_{m\in\mathbb{Z}^n} a_m e^{-2^{-j} i m\xi}, \quad \xi \in Q_{2^j\pi},$$

and arrives at

$$\Big(\sum_{m\in\mathbb{Z}^n} |(\varphi_j \widehat{f})^\vee (2^{-j}m)|^p\Big)^{1/p} \le c 2^{jn/p} \|(\varphi_j \widehat{f})^\vee \mid L_p(\mathbb{R}^n)\|.$$

Defining $\psi_\lambda(\xi) = \psi(2^\lambda \xi)$ where λ is chosen such that

$$(\psi - \psi_\lambda)(2^{-j-1}\xi)\varphi_j(\xi) = \varphi_j(\xi), \quad \xi \in \mathbb{R}^n,\ j \ge 1,$$

we split f_N further into $f_{N,1}$ and $f_{N,2}$:

$$f_N = \sum_{j=0}^{N} f_N^j + f_{N,2} =: f_{N,1} + f_{N,2}$$

with

$$f_N^j(x) = c\psi(x) \sum_{|m|\le N_j} (\varphi_j \widehat{f})^\vee (2^{-j}m)(\psi - \psi_\lambda)^\vee (2^{j+1}x - 2m)$$

and

$$\begin{aligned} f_{N,2}(x) = c\psi(x) \sum_{j=1}^{N} \sum_{|m|>N_j} (\varphi_j \widehat{f})^\vee (2^{-j}m)(\psi - \psi_\lambda)^\vee (2^{j+1}x - 2m) \\ + c\psi(x) \sum_{|m|>N_0} (\varphi_0 \widehat{f})^\vee (m)\psi^\vee(2x - 2m), \end{aligned}$$

where $N_j = \max(B_N^2, 2^{j+2}\sqrt{n})$ and $B_N = \log(1+N)^{b-(1/q_2-1/q_1)_+}$. Arguing as in [16, Step 4, p. 11] and taking (4.22) into account again, we obtain

$$\|f_{N,2} \,|\, B^{(s,-b)}_{p,q_2}(\mathbb{R}^n)\| \le c(1+N)^{-b+(1/q_2-1/q_1)_+}. \tag{4.26}$$

Choosing K such that $2^{K+1}\sqrt{n} < B_N^2 < 2^{K+2}\sqrt{n}$, one splits $f_{N,1}$ further into

$$f_{N,1} = \sum_{j=0}^{K} f_N^j + \sum_{j=K+1}^{N} f_N^j =: f_{N,3} + f_{N,4}.$$

Repeating the arguments of [16, Steps 5, 6, p. 12] one finds that $f_{N,3}$ and $f_{N,4}$ can be covered by at most $c2^{c'2^{nj}}$ balls of radius $C(1+N)^{-b}$ such that also

$$\{f_{N,1} : \|f \,|\, B^s_{p,q_1}(\mathbb{R}^n)\| \le 1,\ \operatorname{supp} f \subset Q_1\} \tag{4.27}$$

can be covered in $B^{(s,-b)}_{p,q_2}(\mathbb{R}^n)$ by this number of balls with radius $C(1+N)^{-b}$. Hence (4.25)–(4.27) prove (4.21).

Step 3. We extend (4.21) to the case $s_1 \ge s_2$, $0 < p_1 \le p_2 \le \infty$, $s_1 - n/p_1 = s_2 - n/p_2$, $0 < q_1, q_2 \le \infty$ and $b_1 \ge 0$, $b_2 \le 0$ with $b_1 - b_2 > (1/q_2 - 1/q_1)_+$. We use decomposition techniques as well as the multiplicativity (1.27) of entropy numbers to obtain, for id_Ω given by (4.19),

$$e_{3k}(\mathrm{id}_\Omega) \le c e_k(\mathrm{id}_1) e_k(\mathrm{id}_2) e_k(\mathrm{id}_3) \tag{4.28}$$

with

$$B^{(s_1,b_1)}_{p_1,q_1}(\Omega) \xrightarrow{\mathrm{id}_1} B^{s_1}_{p_1,q}(\Omega) \xrightarrow{\mathrm{id}_2} B^{s_2}_{p_2,q}(\Omega) \xrightarrow{\mathrm{id}_3} B^{(s_2,b_2)}_{p_2,q_2}(\Omega),$$

where $q \in (0,\infty]$ will be suitably chosen later. Plainly, id_2 is continuous in our case, thus $e_k(\mathrm{id}_2) \le c$. We apply Theorem 1.16 to id_1 and (4.21) to id_3 where q now satisfies $\min(q_1, q_2) \le q \le \max(q_1, q_2)$. Now (4.28) yields (4.20) for $b_1 \ge 0$, $b_2 \le 0$.

Step 4. We finally want to remove the restriction $b_1 \ge 0$, $b_2 \le 0$ (always assuming $b_1 - b_2 > (1/q_2 - 1/q_1)_+$) and first deal with the case $b_1 > b_2 > 0$. We apply Lemma 4.6 with $s_2 = s = \sigma_1 = \sigma_2$, $p_2 = p = r_1 = r_2$, $a = b_2$, $a_1 = b_1$ and $a_2 = b_1 - (b_1 - b_2)/(1-\theta) < 0$ (i.e. one chooses $\theta \in (0,1)$ such that $b_2/b_1 < \theta < 1$).

First let $q_1 \le q_2$, then put $q_2 = u = v_1 = v_2$. Now Lemma 4.6 together with the interpolation property of entropy numbers (4.17) yield

$$\begin{aligned}(4.29)\qquad e_k(\mathrm{id}_\Omega) &\le c_1 e_k(B^{(s_1,b_1)}_{p_1,q_1}(\Omega) \hookrightarrow B^{(s_2,b_1)}_{p_2,q_2}(\Omega))^\theta e_k(B^{(s_1,b_1)}_{p_1,q_1}(\Omega) \hookrightarrow B^{(s_2,a_2)}_{p_2,q_2}(\Omega))^{1-\theta}\\ &\le c_2 e_k(B^{(s_1,b_1)}_{p_1,q_1}(\Omega) \hookrightarrow B^{(s_2,a_2)}_{p_2,q_2}(\Omega))^{1-\theta}\\ &\le c_3(\log\langle k\rangle)^{-(b_1-a_2)(1-\theta)} = c_3(\log\langle k\rangle)^{-(b_1-b_2)}\end{aligned}$$

as desired. The second estimate relies on the continuity of $B^{(s_1,b_1)}_{p_1,q_1}(\Omega) \hookrightarrow B^{(s_2,b_1)}_{p_2,q_2}(\Omega)$ for $s_1 - n/p_1 = s_2 - n/p_2$, $q_1 \le q_2$, whereas the third one comes from Step 3 ($a_2 < 0$).

Assume $q_1 > q_2$; in that case we apply Lemma 4.6 with $u = q_2$, $v_1 = q_1$ and $1/v_2 = 1/q_1 + (1/q_2 - 1/q_1)/(1-\theta)$ (the other parameters as above). Note that

$$b_1 - a_2 = \frac{b_1 - b_2}{1-\theta} > \frac{1}{1-\theta}\left(\frac{1}{q_2} - \frac{1}{q_1}\right) = \frac{1}{v_2} - \frac{1}{q_1}.$$

We may conclude, in analogy to (4.29), that

$$\begin{aligned}e_k(\mathrm{id}_\Omega) &\le c_1 e_k(B^{(s_1,b_1)}_{p_1,q_1}(\Omega) \hookrightarrow B^{(s_2,b_1)}_{p_2,q_1}(\Omega))^\theta e_k(B^{(s_1,b_1)}_{p_1,q_1}(\Omega) \hookrightarrow B^{(s_2,a_2)}_{p_2,v_2}(\Omega))^{1-\theta}\\ &\le c_2(\log\langle k\rangle)^{-(b_1-a_2)(1-\theta)+(1/v_2-1/q_1)(1-\theta)}\\ &= c_2(\log\langle k\rangle)^{-(b_1-b_2)+1/q_2-1/q_1}.\end{aligned}$$

It remains to handle the case $b_2 < b_1 < 0$. Note that the whole proof until this point worked for all $q_1, q_2 \in (0,\infty]$. Only now, involving duality arguments, does the restriction $q_1, q_2 \ge 1$ come in. Recall that

$$(4.30)\qquad [B^s_{p,q}(\mathbb{R}^n)]' = B^{-s}_{p',q'}(\mathbb{R}^n), \quad s \in \mathbb{R},\ 1 \le p < \infty,\ 0 < q < \infty,$$

and $1/q + 1/q' = 1$ if $1 < q < \infty$, $q' = \infty$ if $0 < q \le 1$. In view of (4.23) we may deal with the spaces $\widetilde{B}^s_{p,q}(\Omega)$ instead of $B^s_{p,q}(\Omega)$. We use a result of Bourgain *et al.* (cf. [2]), stating the following:

Let A be a uniformly convex Banach space and B a Banach space, $T \in \mathcal{L}(A,B)$ compact, T^* its adjoint. Then there is some $c = c(A) > 0$ such that for all $m \in \mathbb{N}$ and all r, $0 < r < \infty$,

$$(4.31)\qquad c^{-1}\sup_{k=1,\ldots,m} k^{1/r} e_k(T^*) \le \sup_{k=1,\ldots,m} k^{1/r} e_k(T) \le c \sup_{k=1,\ldots,m} k^{1/r} e_k(T^*).$$

One can extend (4.30) to spaces $B^{(s,b)}_{p,q}(\mathbb{R}^n)$ as well; the proof is essentially the same as for $b = 0$ (see [21, Thm. 2.11.2, p. 178]). Moreover, $B^{(s,b)}_{p,q}(\mathbb{R}^n)$ is uniformly convex if $1 < p, q < \infty$. Then the result for $b_1 > b_2 > 0$ studied above together with (4.31) implies (4.20) when $1 \le p_i, q_i \le \infty$ (by (4.30) and (4.31) only one of the spaces has to be uniformly convex; thus we may also admit 1 and ∞ in certain cases, the rest comes from monotonicity).

The extension to $0 < p_1, p_2 < 1$ is due to monotonicity (in the case of p_1) and another application of Lemma 4.6: let $0 < p_1, p_2 < 1$ and choose $r \ge 1$ such that $1/p_2 = \theta/p_1 + (1-\theta)/r$, $s_2 = \theta s_1 + (1-\theta)\sigma$ with $\sigma - n/r = s_1 - n/p_1 = s_2 - n/p_2$. Now the argument is similar to the earlier one and is not given here. This ends the whole proof. ∎

Corollary 4.8. *Let the assumptions of Proposition* 4.7 *be satisfied. Then*

$$\mathrm{id} : \widetilde{B}^{(s_1,b_1)}_{p_1,q_1}(\Omega) \to \widetilde{B}^{(s_2,b_2)}_{p_2,q_2}(\Omega)$$

is compact. Furthermore, there exists some $c > 0$ *such that for all* $k \in \mathbb{N}$,

$$e_k(\mathrm{id} : \widetilde{B}^{(s_1,b_1)}_{p_1,q_1}(\Omega) \to \widetilde{B}^{(s_2,b_2)}_{p_2,q_2}(\Omega)) \le c(\log\langle k\rangle)^{-(b_1-b_2)+(1/q_2-1/q_1)_+}.$$

Proof. As we already mentioned in the proof of Proposition 4.7, the result of Leopold as well as the adapted arguments presented in Step 2 hold for the spaces $\widetilde{B}^{(s,b)}_{p,q}(\Omega)$ instead of $B^{(s,b)}_{p,q}(\Omega)$ as well. Recall (4.23). The subsequent considerations given in Steps 3 and 4 of the above proof can be transferred without difficulty. ■

Estimate from below. We deal with the entropy numbers of the compact embedding (4.19) where Ω is always assumed to be a bounded C^∞ domain in $\mathbb{R}^n$. One is looking for (better) counterparts of Proposition 4.7 and Corollary 4.8, but we could not so far extend Leopold's result (see [16, Rem. 2])

$$e_k(\mathrm{id}_\Omega) \ge ck^{-(s_1-s_2)/n}(\log\langle k\rangle)^{-(b_1-b_2)}. \tag{4.32}$$

We may, however, give some consequences of (4.32) and our results.

Corollary 4.9. *Let* $s \in \mathbb{R}$, $0 < p \le \infty$, $1 \le q_1, q_2 \le \infty$, *and* $b_1, b_2 \in \mathbb{R}$ *with* $b_1 - b_2 > (1/q_2 - 1/q_1)_+$. *There are numbers* $c_1, c_2 > 0$ *such that for all* $k \in \mathbb{N}$,

$$c_1(\log\langle k\rangle)^{-(b_1-b_2)} \le e_k(B^{(s,b_1)}_{p,q_1}(\Omega) \hookrightarrow B^{(s,b_2)}_{p,q_2}(\Omega)) \le c_2(\log\langle k\rangle)^{-(b_1-b_2)+(1/q_2-1/q_1)_+}.$$

In particular, *if* $1 \le q_1 \le q_2 \le \infty$, *then*

$$e_k(B^{(s,b_1)}_{p,q_1}(\Omega) \hookrightarrow B^{(s,b_2)}_{p,q_2}(\Omega)) \sim (\log\langle k\rangle)^{-(b_1-b_2)}.$$

Proof. This follows from Proposition 4.7 plus (4.32); see [16, Rem. 2]. Moreover, as mentioned several times in the proof of Proposition 4.7, there are extensions to $0 < q_1, q_2 < 1$ as well (apart from the case $b_2 < b_1 < 0$). ■

Another consequence of Proposition 4.2 and Leopold's result (4.32) is a counterpart of Theorem 3.5.

Theorem 4.10. *Let* $0 < p \le \infty$, $0 < q \le \infty$, $\alpha > (1 - 1/q)_+$. *Then there are positive numbers* c_1 *and* c_2 *such that for all* $k \in \mathbb{N}$,

$$c_1 k^{-1/p}(\log\langle k\rangle)^{-\alpha} \le e_k(\mathrm{id} : B^{n/p+1}_{p,q}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U)) \le c_2(\log\langle k\rangle)^{-\alpha+(1-1/q)_+}.$$

In particular, *when* $p = \infty$, $0 < q \le 1$ *and thus* $\alpha > 0$, *we obtain*

$$e_k(\mathrm{id} : B^1_{\infty,q}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U)) \sim (\log\langle k\rangle)^{-\alpha}. \tag{4.33}$$

Proof. The upper estimate comes from Theorem 3.5 whereas the lower follows from (4.32) together with Proposition 4.2. ■

Remark 4.11. An adapted version of Theorem 4.10 is true with

$$\mathrm{id} : B^{n/p+1}_{p,q}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U)$$

replaced by

$$\mathrm{id} : \widetilde{B}^{n/p+1}_{p,q}(U) \to \widetilde{\mathrm{Lip}}^{(1,-\alpha)}(U).$$

Moreover, in analogy to Remark 3.6 we can replace the unit ball $U \subset \mathbb{R}^n$ by any bounded C^∞ domain Ω in $\mathbb{R}^n$ (always neglecting constants).

Remark 4.12. We want to mention some (in our opinion) peculiar and very interesting consequences which might shed some light on the place of Lipschitz spaces among the Fourier-analytically based B-spaces (see Propositions 4.2 and 4.4). When $\alpha = 0$ one additionally knows that $C^1(\mathbb{R}^n) \hookrightarrow B^1_{\infty,v}(\mathbb{R}^n)$ if, and only if, $v = \infty$ (see [18, Thm. 3.1.1(iv)]). Though we have not yet been able to prove a similar assertion for the $\mathrm{Lip}^{(1,-\alpha)}$ and $B^{(s,b)}_{p,q}$ spaces on $\mathbb{R}^n$, one has somehow the feeling that the spaces of Lipschitz type $\mathrm{Lip}^{(1,-\alpha)}$ are "in between" the scale of spaces $B^{(1,-\alpha)}_{\infty,v}$, $1 \le v \le \infty$. In terms of entropy numbers of certain compact embeddings the situation is as follows: let $0 < q \le 1$, $\alpha > 0$; then

$$
\begin{array}{ccccc}
 & & B^1_{\infty,q}(\Omega) & & \\
 & \swarrow & \downarrow & \searrow & \\
\underbrace{B^{(1,-\alpha)}_{\infty,1}(\Omega) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\Omega) \hookrightarrow B^{1,-\alpha)}_{\infty,\infty}(\Omega)}_{e_k \sim (\log\langle k\rangle)^{-\alpha}} & & & &
\end{array}
$$

in view of Theorem 4.10, i.e. (4.33), and Corollary 4.9. Unfortunately, we have not yet been able to give complete results concerning (the asymptotic behaviour of the relevant) entropy numbers, i.e. for all $p, q \in (0, \infty]$. Our hope is that this might give some new and interesting information about the "place" where the (Fourier-unfriendly) Lipschitz spaces are within the (suitable) scale of Besov spaces.

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