POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

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Some fixed point theorems for multifunctions with applications in game theory

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PAŃSTWOWE WYDAWNICTWO NAUKOWE

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WROCŁAWSKA DRUKARNIA NAUKOWA

CONTENTS

Introduction

The main result of this paper is concerned with the conditions which guarantee that a multifunction $f\colon C \to 2^X$ defined on an arbitrary subset C of a topological vector space X admits a point x of C such that $x \in f(x)$.

First, we give some definitions and propositions which are associated with semicontinuous multifunctions (Part 1).

Next, in Part 2, we present a global convergence criterion on variable dimension algorithms for finding an approximate solution of the equation $x \in f(x)$, and then we consider some fixed point theorems for multifunctions defined in finite-dimensional spaces.

Part 3 contains fixed point theorems for quasi upper semicontinuous multifunctions defined on arbitrary domains of topological vector spaces which generalize the theorems with boundary conditions.

Part 4 is devoted to some fixed point theorems for strongly lower semicontinuous multifunctions and thus here we are first concerned with fixed point theorems under boundary conditions for this class of multifunctions.

The last part shows how we can apply the results obtained to existence problem of equilibrium situations in the theory of non-cooperative games.

1. Some classes of semicontinuous multifunctions

In this part we give some definitions and propositions which are associated with semicontinuous multifunctions.

All topological spaces considered in this paper are Hausdorff spaces.

Let X, Y be topological spaces, let 2^Y be the set of all subsets of Y and let A be a nonempty set in X. Let $f\colon A \to 2^Y$ be a multifunction defined on A such that, for every $x \in A$, $f(x)$ is a nonempty subset of Y. We write $f(B) = \bigcup \{f(u) \mid u \in B\}$ for any subset $B \subset A$, and $\bar{B}$ denotes the closure of the set B.

1.1. Weak upper semicontinuous multifunctions. First we recall that f is said to be *upper semicontinuous* (u.s.c.) *at* x if, for each neighbourhood $V \subset Y$ of $f(x)$, there exists a neighbourhood $U \subset A$ of x such that $f(U) \subset V$.

By $\mathscr{B}(x)$ we denote the set of neighbourhoods of x in A. We define $(\mathrm{Ls}\, f)$: $A \to 2^Y$ as follows:

$$(\mathrm{Ls}\, f)(x) = \bigcap \{\overline{f(B)} \mid B \in \mathscr{B}(x)\}.$$

DEFINITION 1. The multifunction f is *weak upper semicontinuous* (w.u.s.c.) *at* x if $(\mathrm{Ls}\, f)(x) \subset \overline{f(x)}$. The multifunction f is w.u.s.c. on A if f is w.u.s.c. at each point of A.

The multifunction f is *quasi upper semicontinuous* (q.u.s.c.) *at* x if $(\mathrm{Ls}\, f)(x) \subset f(x)$.

Clearly, if f is q.u.s.c. at x then f is w.u.s.c. at x.

Let us remark that in some cases the notions of w.u.s.c. multifunctions and q.u.s.c. multifunctions reveal some advantages over the usual notion of u.s.c. multifunctions. For instance, as usual, it is difficult to verify directly the upper semicontinuity of the generalized budget constraint multifunctions used in mathematical economics. However, in many cases it is easy enough to show that the generalized budget constraint multifunctions are w.u.s.c. (or q.u.s.c.) on its domains. We shall consider the following

EXAMPLE 1. Let $T = T_1 \cup T_2 \cup T_3$ be nonempty set such that $T_1 \cap T_3 = \emptyset$ and, for any $t \in T$, let a real-valued function g_t: $X \times Y \to R$ be defined. We define q: $X \to 2^Y$ as follows:

$$q(x) = \left\{ y \in Y\colon \begin{array}{l} g_t(x, y) < 0 \text{ for all } t \in T_1 \\ g_t(x, y) \leqslant 0 \text{ for all } t \in T_2 \\ g_t(x, y) = 0 \text{ for all } t \in T_3 \end{array} \right\}.$$

If each g_t is lower semicontinuous for $t \in T_1 \cup T_2$ and continuous for $t \in T_3$ on $X \times Y$, then q is w.u.s.c. on its domain. Moreover, if $T_1 = \emptyset$, then q is q.u.s.c. on its domain.

For any multifunction f: $A \to 2^Y$ we define the multifunction $\bar{f}$: $A \to 2^Y$ by $\bar{f}(u) = \overline{f(u)}$ for each $u \in A$ and Graph $f = \{(x, y) \mid x \in A, y \in f(x)\}$.

EXAMPLE 2. Let f: $A \to 2^Y$ be a multifunction such that the set Graph $\bar{f}$ is a closed subset of $A \times Y$, then f is w.u.s.c. on its domain.

We give some propositions.

PROPOSITION 1. *For Y being a regular space we have the following relations*:

(a) *if f is u.s.c. at $x \in A$, then f is w.u.s.c. at x,*
if f is u.s.c. at x and $f(x)$ is closed, then f is q.u.s.c. at x,

(b) *if f is w.u.s.c. at x, then $\bar{f}$ is q.u.s.c. at x,*

(c) *if $\bar{f}$ is w.u.s.c. at x and $f(x)$ is closed, then f is q.u.s.c. at x.*

For Y being a topological vector space and for any $B \subset Y$, $\operatorname{conv} B$ denotes the convex hull of the set B.

PROPOSITION 2. *If Y is a locally convex topological vector space and if f is* *w.u.s.c. at x and there exists a nonempty compact subset K of Y such that* $\operatorname{conv} f(x) \subset K$ *and* $f(U) \subset K$ *for some neighbourhood U of x in A, then the multifunction $f'\colon A \to 2^Y$ defined by $f'(u) = \operatorname{conv} f(u)$ for every $u \in A$ is w.u.s.c. at x and hence the multifunction $\overline{f'}$ defined by $\overline{f'}(u) = \overline{\operatorname{conv}}\, f(u)$ for $u \in A$, is q.u.s.c. at x.*

Indeed, suppose that there exists a $y \in (\operatorname{Ls} f')(x) \setminus \overline{f'(x)}$. Since $\overline{f'(x)}$ is compact and convex by the Hahn–Banach Theorem, there is a nontrivial linear continuous function a on Y and a number α such that

$$a(y) < \alpha < a(y') \quad \text{for all } y' \in \overline{f'(x)}.$$

Moreover, K is compact, $\bar{f}$ is q.u.s.c. at x, and thus there is a neighbourhood U' of x in A such that $a(y') > \alpha$ for all $y' \in \overline{f(U')}$; hence, since a is a linear function we have $a(y') > \alpha$ for all $y' \in f'(U')$. Then, by $y \in \overline{f'(U)}$, $a(y) \geqslant \alpha$, which is a contradiction.

For Y being a topological vector space Fan Ky (see [10], [11]) introduced the following notion:

Definition 2. The multifunction $f\colon A \to 2^Y$ is – by definition – *upper demicontinuous at x* if, for every open halfspace H in Y containing $f(x)$, there is a neighbourhood U of x in A such that $f(U) \subset H$.

Using the above argument, we have the following

Proposition 3. *If Y is a locally convex topological vector space, f is upper demicontinuous at x and $f(x)$ is convex, then f is w.u.s.c. at x.*

1.2. Strongly lower semicontinuous multifunctions. We recall that a multifunction $h\colon A \to 2^Y$ is said to be *lower semicontinuous* (l.s.c.) *at x* if, for every open subset V of Y such that $V \cap h(x) \neq \emptyset$, there exists a neighbourhood U of x in A such that $V \cap h(x') \neq \emptyset$ for every $x' \in U$.

Definition 3. The multifunction h is *strongly lower semicontinuous* (s.l.s.c.) *at x* if, for every $y \in h(x)$, the set $h^{-1}(y) = \{x' \in A \mid y \in h(x')\}$ is relatively open in A.

Obviously, if h is s.l.s.c. at x, then h is l.s.c. at x.

Directly from the definitions we have

Proposition 4. *If the multifunction $f\colon A \to 2^Y$ is q.u.s.c. at $x \in A$ and K is a subset of Y such that $f(A) \subset K$, then the multifunction $h'\colon A \to 2^Y$ defined by $h'(u) = K \setminus f(u)$ for every $u \in A$, is s.l.s.c. at x.*

Let Z be a topological space.

Proposition 5. *If h is l.s.c. at x and if a multifunction $l\colon Y \to 2^Z$ is s.l.s.c. at every point y of $h(x)$, then the multifunction $lh\colon A \to 2^Z$ defined by $lh(u) = l(h(u))$ for every $u \in A$ is s.l.s.c. at x.*

Proposition 6. *If Y is a topological vector space and h is s.l.s.c. at x, then* *the multifunction* $h'\colon A\to 2^Y$ *defined by* $h'(u)=\operatorname{conv} h(u)$ *for every* $u\in A$, *is s.l.s.c. at* x.

A class of s.l.s.c. multifunctions is given in the following

EXAMPLE 3. Let T be a finite set and, for each $t\in T$ and for every fixed point $y\in Y$, let $g_t\colon A\times Y\to R^1$ be a real-valued function which is u.s.c. at x; then the multifunction $h'\colon A\to 2^Y$ defined by $h'(u)=\{y\in Y|\ g_t(u, y)<0$ for all $t\in T\}$ for every $u\in A$, is s.l.s.c. at x.

1.3. A class of lower semicontinuous multifunctions. Let $h\colon A\to 2^Y$ be a multifunction defined on A such that for every $u\in A$, $h(u)$ is a nonempty subset of Y. Then we define $h^{-1}\colon Y\to 2^A$ by $h^{-1}(y)=\{u\in A|\ y\in h(u)\}$ for each $y\in Y$ and $h^{-}\colon 2^Y\to 2^A$ by $h^{-}(E)=\{u\in A|\ h(u)\cap E\neq\emptyset\}$ for each $E\in 2^Y$. Let $v\in h(A)$.

DEFINITION 4. The multifunction h is said to be *v-open* if for every relatively open subset B of X in A such that $v\in h(B)$ we have $v\in\operatorname{int} h(B)$.

Then we obtain the following

PROPOSITION 7. *Let* $x\in A$. *The multifunction* h *is v-open if the multifunction* h^{-1} *is l.s.c. at* v. *The multifunction* h *is l.s.c. at* x *iff the multifunction* h^{-} *is x-open.*

Let P be a nonempty set and let $y_0\in E\subset Y$. Now we need the following

DEFINITION 5. A real function $\varphi\colon E\times P\to R^1$ is *y_0-quasi convex on P* if for each $y\in E$ and for each neighbourhood V_y of y there is a point $y'\in V_y\cap E$ and a function $\alpha_y\colon P\to (0, 1]$ such that $\varphi(y', p)\leqslant(1-\alpha_y(p))\varphi(y, p)+$ $+\alpha_y(p)\varphi(y_0, p)$ holds for all $p\in P$.

For any pair y_1, y_2 of a vector space, the segment $[y_1, y_2]$ denotes the set $\{\alpha y_1+(1-\alpha)y_2|\ 0\leqslant\alpha\leqslant 1\}$.

EXAMPLE 4. Assume that E is a subset of a topological vector space and is *y_0-starshaped*, i.e., that E is a set such that, for any $y\in E$, the segment $[y_0, y]$ is a subset of E. If $\varphi\colon E\times P\to R^1$ is a function such that for each $p\in P$ and for each $y\in E$, the restriction of φ on $[y_0, y]$, $\varphi(\cdot, p)\colon [y_0, y]\to R^1$ is convex in $[y_0, y]$, then φ is y_0-quasi convex on P.

Now we show the lower semicontinuity of a class of multifunctions usually given in nonlinear programming.

PROPOSITION 8. *Assume that P is a compact space and that*

(i) *for each* $y\in E$, *the function* $\varphi\colon E\times P\times A\to R^{-1}$ *is u.s.c. on* $P\times A$;

(ii) $g\colon P\times A\to R^1$ *is a function l.s.c. on* $P\times A$;

(iii) *for each* $u\in A$, *there exists* $y_u\in E$ *such that* $\varphi(\cdot,\cdot, u)$ *is* y_u-*quasi convex on P and* $\varphi(y_u, p, u)<g(p, u)$ *for all* $p\in P$.

Then the multifunction $h'\colon A\to 2^Y$ *defined by letting*

$$h'(u)=\{y\in E|\ \varphi(y, p, u)\leqslant g(p, u)\ \text{for all}\ p\in P\}\quad \text{for each}\ u\in A$$

is lower semicontinuous on A.

Proof. Let $u\in A$ and let y_u be the point defined in condition (iii). By the compactness of P, the upper semicontinuity of the function $\varphi - g$ on P and condition (iii)

$$\varepsilon(u) = \max_{p \in P}(\varphi(y_u, p, u) - g(p, u)) < 0.$$

Let $x \in A$, $y \in h'(x)$ and let U be some neighbourhood of y. Since φ is y_x-quasi convex on P, there exists a $y' \in U \cap E$ and a function α: $P \to (0, 1]$ such that

$$\varphi(y', p, x) \leqslant (1 - \alpha(p))\varphi(y, p, u) + \alpha(p)\varphi(y_x, p, x)$$

for all $p \in P$. Then $\varphi(y', p, x) - g(p, x) \leqslant \alpha(p)\varepsilon(x) < 0$. Thus, for $y' \in E$ and for any $p \in P$, by the upper semicontinuity of $\varphi - g$ at (p, x) there exists a neighbourhood $V(p)$ of x and a neighbourhood $V(p, x)$ of p such that $\varphi(y', p', u) - g(p', u) < 0$ for all $p' \in V(p, x)$ and for all $u \in V(p)$.

Hence, since P is a compact space, there exists a finite subset $\{p_1, p_2, \ldots, p_m\}$ of P such that $P = \bigcup \{V(p_i, x) |\ i = 1, \ldots, m\}$.

Let $V(x) = \bigcap_{i=1}^{m} V(p_i)$; then $\varphi(y', p', u') - g(p', x) < 0$ for all $u' \in V(x)$ and for all $p' \in P$. It means that, for any $u' \in V(x)$, $y' \in h'(u')$, thus $h'(u') \cap U \neq \emptyset$ for any $u' \in V(x)$ as was required.

We note that in this proposition X, Y, P are *not* assumed to be vector spaces.

As a direct corollary, we have:

If P is a compact space, E is a nonempty convex set of a topological vector space Y and a real function φ: $E \times P \times A \to R^1$ satisfies condition (i) *and conditions*

(ii′) *for any $(u, p) \in A \times P$, $\varphi(\cdot, p, u)$: $E \to R^1$ is convex on E,*

(iii′) *for each $u \in A$, there exists a $y_u \in E$ such that*

$$\varphi(y_u, p, u) < 0 \quad \textit{for all } p \in P,$$

then the multifunction h': $A \to 2^Y$ defined by

$$h'(u) = \{y \in E|\ \varphi(y, p, u) \leqslant 0 \textit{ for all } p \in P\} \quad \textit{for} \quad u \in A,$$

is lower semicontinuous on A.

We can observe that condition (iii′) is similar to Slater's well-known condition in convex programming (see [22]); hence condition (iii) really is a weaker one.

2. A remark on the convergence of variable dimension algorithms

The variable dimension algorithms, first given by G. Vander Laan and A. J. J. Talman for finding an approximate solution of the equation $x \in f(x)$ of a multifunction from R^n into R^n, have been extensively developed in recent times (see, for example, [13], [16]).

First we describe the algorithms (Section (2.1)

In Section 2.2 we give a new global convergence criterion for the algorithms. This criterion shows that the convergence of the algorithms depends upon the relation between the starting vectors of the simplex procedure, the conical subdivision of the space containing the domain of the multifunction and the information received about the multifunction.

Section 2.3 is devoted to some fixed point theorems for quasi-upper semicontinuous multifunctions defined in finite-dimensional spaces. Moreover, in theses theorems solutions of the equation $x \in f(x)$ can be worked out by some algorithm procedures.

2.1. Variable dimension algorithms. First we describe variable dimension algorithms for computing an approximate zero of a multifunction for R^n into R^n (see [16], [19], [20]).

Let w be a preselected point of R^n (a first guess of the solution of the equation $0 \in f(x)$ to be sought) and $n+1$ arbitrary points $\{a^1, a^2, \ldots, a^{n+1}\}$ in R^n spanning an n-simplex containing w in its interior.

For each $I \subset \{1, 2, \ldots, n+1\}$, let P_I be the convex polyhedral cone with vertex at w. generated by all halflines from w through a^j with $j \notin I$. For each $i \in I$, P_i stands for $P_{\{i\}}$.

The system of cones

$$\text{(P)} \qquad P_I, \quad I \subset \{1, 2, \ldots, n+1\}, \quad I \neq \emptyset$$

determines a subdivision of R^n such that

(i) $\dim P_I = n+1-|I|$,

(ii) each maximal proper face of P_I is a cone P_J with $J = I \cup \{y\}$, $y \notin I$.

Let T be a triangulation of R^n consistent with the conical subdivision P, i.e., such that for every simplex t of T and every $I \subset \{1, 2, \ldots, n+1\}$ the intersection $t \cap P_I$ is a face (possibly empty) of t. Then the restriction of T in P_I, i.e., $T_I = \{t \cap P_I | \ t \in T\}$ triangulates P_I.

Definition 6. A set U of $n+2$ distinct elements of $Q \cup \{1, 2, \ldots, n+1\}$ is called a *P-set* if $U \cap Q$ is the vertex set of a maximal simplex (i.e., a simplex of maximal dimension of T). A subset W of a P-set U is called a *facet* of it if $W = U \setminus \{x\}$ for some $x \in U$.

The basic property of P-set is the following:

Lemma 1. *If U is a P-set and x is an element of U such that $1 \leqslant |(U \setminus \{x\}) \cap Q| \leqslant n$, then there exists exactly one $y \notin U$ for which $(U \setminus \{x\}) \cup \{y\}$ is also a P-set.*

Let $f\colon R^n \to 2^{R^n}$ be a multifunction such that $f(x)$ is nonempty for each $x \in R^n$. We wish to compute an approximate solution of the equation $0 \in f(x)$. Take a selection l of f, i.e., a single-valued function such that $l(x) \in f(x)$ for every x and $n+1$ vectors $b^1, b^2, \ldots, b^{n+1}$ of R^n such that for all numbers

$\theta > 0$ small enough

$$[\theta] \in \operatorname{conv}\{b^1, b^2, \dots, b^{n+1}\} \tag{2.1}$$

where $[\theta] = (\theta^1, \theta^2, \dots, \theta^n)$.

DEFINITION 7. A facet W of a P-set is said to be *complete* if for all $\theta > 0$ small enough we have

$$[\theta] \in \operatorname{conv}\{l(W \cap Q),\ b^i (i \in W \setminus Q)\}.$$

Obviously, if W is complete and $W \subset Q$ then

$$0 \in \operatorname{conv} l(W) \subset \operatorname{conv} f(W),$$

so that W will be an approximate zero of f.

We need the following lemmas:

LEMMA 2. *Let U be a P-set. If W is a complete facet of U such that $W = U \setminus \{x\}$ for some $x \in U$, then there exists exactly one $y \in W$ such that $(W \setminus \{y\}) \cup \{x\}$ is another complete facet.*

LEMMA 3. *$U_0 = \{w, 1, 2, \dots, n+1\}$ is the only P-set of a complete facet $W_0 = \{1, 2, \dots, n+1\}$ which does not contain any vertice of T.*

These lemmas taken together justify the following pivoting procedure:

Algorithm (w, P, T, b, l)

1. Start with $U_0 = \{w, 1, 2, \dots, n+1\}$, $W_0 = \{1, 2, \dots, n+1\}$ $x^0 = w$.

a) Find $y^1 \in W_0$ such that $W_1 = (W_0 \setminus \{y^1\}) \cup \{x^0\}$ is complete (by the Simplex procedure),

b) Then find x^1 such that $U_1 = (U_0 \setminus \{y^1\}) \cup \{x^1\}$ is a P-set (by triangulation).

2. Repeat the procedure with U_1, W_1, x^1 replacing U_0, W_0, x^0. And so on.

Stop when a P-set U is reached which contains a simplex t such that $0 \in \operatorname{conv} l(t)$, in particular if U contains a complete facet $W \subset Q$.

By normal argument one can show that the above algorithm never repeats.

2.2. Global convergence. Clearly, the criterion $W \subset Q$ is concrete and would be verified simultaneously, whereas the criterion $0 \in \operatorname{conv} l(t)$ for some simplex t is not concrete. Some of the more concrete stop criteria were given in [13], [16], [20]. Now we give the following

THEOREM 1. *Given (w, P, T). Assume that there exist $w_0 \in R^n$, $\alpha > 0$, $\beta > 0$ and a selection l of f such that to each $t \in T$ corresponds a nondegenerate linear operator $A_t \colon R^n \to R^n$ such that:*

(i) *for every $x \in t$,* if $\|x - w_0\| > \alpha$, *then*

$$\langle A_t l(x), u - w_0 \rangle > 0 \quad \textit{for all } u \in t, \tag{2.2}$$

(ii) $\{b^1, b^2, \ldots, b^{n+1}\}$ *are chosen so that*

(2.3) $\langle A_w b^i, z-w\rangle > 0$ *for all* $i = 1, 2, \ldots, n$ *and for all* $z \in P_i \setminus \{w\}$,

and for every $t \neq w$,

(2.4) $\langle A_t b^i, x-w\rangle > 0$ *for all* $x \in t\setminus\{w\}$ *and for all i such that* $t \subset P_i$.

Then if mesh $T < \beta/2$ the algorithm (w, P, T, b, l) *will terminate in a ball of radius* $\alpha+\beta$ *around* w_0.

Proof. First, we remark that if $0 \notin \operatorname{int} \operatorname{conv}(b^1, b^2, \ldots b^{n+1})$ then, since A_w is nondegenerate, $0 \notin \operatorname{int} \operatorname{conv}(A_w b^1, A_w b^2, \ldots, A_w b^{n+2})$, and hence, by the Separate Theorem, there exists a vector $z_0 \neq 0$ such that $\langle A_w b^i, z_0\rangle \leqslant 0$ for all $i = 1, 2, \ldots, n+1$. But this contradics (2.3) since $z_0 + w \in P_i \setminus \{w\}$ for some i; therefore

$$0 \in \operatorname{int} \operatorname{conv}(b^1, b^2, \ldots, b^{n+1}),$$

i.e., (2.1) holds, and thus the algorithm (w, P, T, b, l) applies.

Now we take a ball C around w_0 with radius $r+\beta$, where r is large enough, so that $r > \alpha$, $w \in C$, and such that, for every u, provided $\|u-w_0\| > r$, by (2.4), we have

$$\langle A_t b^i, u-w_0\rangle > 0 \quad \text{if} \quad t \subset P_i,\ u \in t. \tag{2.5}$$

Let mesh $T < \beta/2$ and let t' be a maximal simplex such that $t' \cap \partial C \neq \emptyset$. Then $t' \in T_I$ for some $I \subset \{1, 2, \ldots, n+1\}$, and for every $x \in t'$, $\|x-w_0\| > \alpha$; hence, for a fixed point $u' \in t'$, we get $\|u'-w_0\| > r$; hence by (2.2)

$$\langle A_{t'} l(x), u'-w_0\rangle > 0 \quad \text{for all } x \in t'$$

which together with (2.5) holds for $t = t'$ and for all $i \in I$ since $I \subset \{i|\ u' \in P_i\}$. Thus

$$0 \notin \operatorname{conv}\{A_{t'} l(x), A_{t'} b(t')\} \quad \text{for all } x \in t',$$

where $b(t') = \{b^i | i \text{ such that } t' \subset P_i\}$ and we have

$$0 \notin \operatorname{conv}\{A_{t'} l(t'), A_{t'} b(t')\}. \tag{2.6}$$

Moreover, if $w \in \partial C$, then $0 \in \operatorname{conv} b(w)$, or, if the algorithm does not terminate in C, then by the algorithm there exists a t'' such that $t'' \cap \partial C \neq \emptyset$ and $0 \in \operatorname{conv}(l(t''), b(t''))$. These two cases, by (2.6), are impossible, therefore the algorithm terminates in C.

Finally, by t_0 we denote the terminal simplex generated by the algorithm; then $0 \in \operatorname{conv}(l(t_0))$. If for some $u \in t_0$ $\|u-w_0\| > \alpha+\beta$, then, for all $x \in t_0$, by $\|u-x\| < \beta/2$, $\|x-w_0\| > \alpha$ and, by (2.2), we get

$$\langle A_{t_0} l(x), u-w_0\rangle > 0, \quad \text{i.e.,} \quad 0 \notin \operatorname{conv} A_{t_0}(l(t_0));$$

hence $0 \notin \operatorname{conv} l(t_0)$. This contradiction shows that $\|u-w_0\| < \alpha+\beta$ for all $u \in t_0$, as was required.

Remark 1. Condition (2.2) is a modification of Merrill's condition (see [20]). If, for all $t \in T$, $A_t = -I$, where I is the identity operator, then (2.2) gives some form of an 'inward' condition. But if, for all $t \in T$, $A_t = I$, from (2.2) one can follow some form of an 'outward' condition.

Remark 2. We shall discuss a variant of V. Laan and Tallman's algorithms. In [16] R^n is embedded in R^{n+1} so that the ith vertex of the simplex $S \subset R^n$ is identified with the ith unit vector e^i of R^{n+1} and

$$a^i = w + e^{i+1} - e^i, \quad b^i = (n+1)(e^i - w).$$

Clearly, condition (1.9) in [20] is not satisfied, i.e., $\langle b^i, x-w\rangle < 0$ does not hold for any $x \in P_i \setminus \{w\}$ and for any $i = 1, 2, \ldots, n+1$.

For the basic system $\{e^i\}$ of R^{n+1}, we have

$$w + \theta \in \operatorname{int} \operatorname{conv}\{b^i, \; i = 1, 2, \ldots, n+1\}.$$

On the other hand, since, each $i = 1, 2, \ldots, n+1$, P_i is a pointed cone, the set $P_i^+ = \{q \in R^{n+1} | \; \langle q, z-w\rangle > 0$ for all $z \in P_i \setminus \{w\}\}$ is nonempty, we can choose for each $i = 1, 2, \ldots, n+1$, a vector $q_i \in P_i^+$. But $\bigcup_{i=1}^{n+1} P_i$ is the space containing the vectors $\{e^i - w, \; i = 1, 2, \ldots, n+1\}$, then, by the Separate Theorem, we get $0 \in \operatorname{int} \operatorname{conv}\{q_i | \; i = 1, 2, \ldots, n+1\}$ and hence the system $\{q_i | \; i = 1, 2, \ldots, n+1\}$ is affine independent. Thus there exists a nongenerated linear operator A: $R^{n+1} \to R^{n+1}$ such that $Ab = q$. This implies $\langle Ab^i, z-w\rangle = \langle q_i, z-w\rangle > 0$ for all $z \in P_i \setminus \{w\}$, i.e., the system $\langle b^i | \; i = 1, \ldots, n+1\}$ satisfies condition (2.3) of Theorem 1.

2.3. Some results in finite-dimensional spaces. First, we give the following

Definition 6. We say that the equation $0 \in f(x)$ has an *approximate solution* in C if there exists an algorithm (w, P, T, b, l) such that when the mesh of the triangulation T is fine enough the algorithm will terminate in C. A simplex t of T is an approximate solution of the equation $0 \in f(x)$ if $0 \in \operatorname{conv} f(t)$.

Let $B(d) = \{x \in R^n | \; \|x\| < d\}$.

Theorem 2. *Given (w, P, T, b). Let f: $R^n \to 2^{R^n}$ be a multifunction such that, for every $x \in R^n$, $f(x)$ is nonempty. Suppose that there is compact subset C of R^n containing w and such that* int $C \neq \emptyset$, *a number $d > 0$ and a bounded subset D of R^n such that*:

(i) *for every $x \in C + B(d)$, $f(x) \cap D \neq \emptyset$,*

(ii) *f is q.u.s.c. on C and, for every $x \in C$, $f(x)$ is convex,*

(iii) *for every $x \in \partial C$, $0 \notin \operatorname{conv}(f(x) \cap \bar{D}, b(x))$, where $b(x) = \{b^i | \; x \in P_i\}$.*

Then there exists an $x_0 \in C$ such that $0 \in f(x_0)$ and the equation $0 \in f(x)$ has an approximate solution in C.

Proof. By condition (iii) $w \in \operatorname{int} C$. Let d_k be a positive number such that $d_k \to 0 (k \to \infty)$. Thus we can use the algorithm (w, P, T, b, l), T_k being a triangulation of T with mesh $T_k = d_k$ and $l(x) \in f(x) \cap D$ for all $x \in C + B(d)$ and $l(x) \in f(x)$ for all other x.

First, suppose that there is a sequence d_k, $d_k \to 0$ such that the algorithm terminates at t_k in C with $t_k \in T_k$. Then

$$0 \in \operatorname{conv} l(t_k) \subset \operatorname{conv}(f(t_k)).$$

Let $t_k = \{x_k^1, x_k^2, \ldots, x_k^{n+1}\}$. Since C is compact, there exists a subsequence, say also t_k, and a point $x_0 \in C$ such that $x_k^j \to x_0$ for all $j = 1, 2, \ldots, n+1$ $k \to \infty$. Since D is bounded, f is q.u.s.c. at x_0, it follows that for all k, $l(x_k^j)$ converges to some point $z_0^j \in (\mathrm{Ls}\, f)(x_0) \cap \bar{D}$, and thus $0 \in \operatorname{conv}(\mathrm{Ls}\, f)(x_0)$, hence $0 \in f(x_0)$.

On the other hand, if d_k is a subsequence such that as $k \to \infty$ the algorithm does not terminate in C, i.e., for each d_k, there exists a simplex $t_k \in T_k$ with mesh $t_k \leqslant d_k$ such that $t_k \cap \partial C \neq \emptyset$ and $0 \in \operatorname{conv}\ (l(t_k), b(t_k))$. By an analogous argument there is a point $x_0' \in \partial C$ such that for all $j \in J_k$, where $t_k = \{x_k^j |\ j \in J_k\}$, $J_k \subset \{1, \ldots, n+1\}$, $x_k^j \to x_0'$ and hence $b(t_k) \subset b(x_0')$, k being large enough, where $b(t_k) = \{b^i | i$ such that $t_k \subset P_i\}$. Now by the q.u. semicontinuity of f on C we get $0 \in \operatorname{conv}(f(x_0') \cap \bar{D}, b(x_0'))$, contradicting condition (iii) since $x_0' \in \partial C$.

Corollary 2.1. *Given $w \in R^n$. Let $f\colon R^n \to 2^{R^n}$ be a multifunction such that:*

(i) *there is a bounded subset D of R^n such that f is q.u.s.c. on some simplex S containing a neighbourhood of* $\operatorname{conv}(D, w)$,

(ii) *for each $x \in S$, $(f(x) + x) \cap D \neq \emptyset$.*

Then we can choose (P, T, b, l) such that whenever the mesh of T is fine enough the algorithm will terminate in S, yielding an approximate solution of $0 \in f(x)$ in S. Moreover, if $f(x)$ is convex for each $x \in S$, the system $x \in S$, $0 \in f(x)$ has a solution.

Proof. By condition (i), there is a number $d > 0$, and a simplex S' such that

$$\overline{\operatorname{conv}(D, w) + B(d/2)} \subset S' \subset S' + B(d/4) \subset S.$$

For each $i = 1, 2, \ldots, n+1$, and a^i be the i-vertex of S' and S_i' be the face of S' opposite to vertex a^i. Then we define b^i as the inward orthogonal vector to the face S_i' such that $\langle b^i, z - x \rangle > 0$ for all $z \in \operatorname{int} S'$, $x \in S_i'$. Let $P_i = \{w + \sum_{j \neq i} \alpha_j (a^j - w) |\ \alpha_j \geqslant 0, j \neq i\}$ and choose T as some triangulation according to the system (P); then we can take $l(x) \in f(x) \cap (D - x)$ for each $x \in S$, and $l(x) \in f(x)$ for each $x \notin S$. Thus, since D is a bounded subset, to apply Theorem 2 we have to verify the condition for each $x \in \partial S'$.

Suppose that, for some $x' \in \partial S'$, $0 \in \operatorname{conv}\ (f(x') \cap \overline{(D - x')}, b(x'))$. Then we have $u \in b(x')$ and $v \in f(x') \cap \overline{(D - x')}$ such that $0 = \alpha u + (1 - \alpha) v$ for $0 \leqslant \alpha \leqslant 1$.

Since, for $x \in P_i$, $0 \notin \operatorname{conv}(b^j, j \neq i')$ for some $i' \neq i$, if follows that $i' \in b(x')$, $u \neq 0$ and $\alpha \neq 1$, i.e., $(1-\alpha) > 0$ and $(1-\alpha)v = -\alpha u$. Moreover, $v + x' \in \bar{D} \subset \operatorname{int} S'$, $\langle u, -\alpha u \rangle = (1-\alpha) \sum_{j \neq i} \langle \alpha_j b^j, (v+x') - x' \rangle > 0$ since $b(x') = J(x') \subset J \setminus \{i'\}$, $\alpha_j = 0$ for $j \notin J(x')$. This contradiction shows that we can apply Theorem 2 and completes the proof.

Let D be a bounded subset of R^n and let $d > 0$. We denote by $S(D, d)$ some minimal simplex of R^n containing $D + B(d)$.

THEOREM 3. *Let C be a nonempty subset of R^n and let $f\colon C \to 2^{R^n}$ be a multifunction q.u.s.c. on C. Suppose that there exists a bounded subset D of R^n and a multifunction $g\colon S(D, d) \to 2^C$ u.s.c. on $S(D, d)$ for some $d > 0$ such that, for every $x \in g(S(D, d))$, $f(x) \cap D \neq \emptyset$. Then the system*

$$u \in S(D, d), \qquad u \in \operatorname{conv}(\overline{fg}(u))$$

has an approximate solution.

Moreover, if, for every $u \in S(D, d) \setminus C$, $u \notin \operatorname{conv}(\overline{fg}(u) \cap \bar{D})$ and if, for every $u \in S(D, d) \cap C$, $g(u) = u$ and $f(u)$ is convex, then f has a fixed point in C and the system

$$x \in C, \qquad x \in f(x)$$

has an approximate solution.

Proof. We define a multifunction $\varphi_1\colon S(D, d) \to 2^{R^n}$ by $\varphi_1(u) = \overline{fg}(u) \cap \bar{D}$ for each $u \in S(D, d)$. Clearly, φ_1 is q.u.s.c. on $S(D, d)$ and hence the multifunction $\varphi_2\colon S(D, d) \to 2^{R^n}$ by $\varphi_2(u) = \operatorname{conv} \varphi_1(u)$ for every $u \in S(D, d)$ is also q.u.s.c. on $S(D, d)$.

Now we take $w \in \operatorname{int} S(D, d)$ and can apply Corollary 2.1 with the multifunction $\varphi\colon R^n \to 2^{R^n}$ defined by

$$\varphi(u) = \begin{cases} \varphi_2(u) - u & \text{for every } u \in S(D, d) \\ R^n & \text{for every } u \notin S(D, d), \end{cases}$$

and we obtain the approximate solution of the system

$$u \in S(D, d), \qquad 0 \in \operatorname{conv} \overline{fg}(u) - u.$$

We infer the existence of a fixed point of the multifunction f in C and the existence of an approximate solution of the system $x \in C$, $x \in f(x)$ from the last condition in the theorem.

3. Some fixed point theorems

In this part of the paper we are concerned with the existence of a fixed point of a multifunction f from a subset C of a topological vector space X into X.

To ensure the existence of a fixed point in C, besides making the usual assumptions about convexity, the compactness of the domain C and the upper semicontinuity of the multifunction f, it was necessary to impose certain conditions upon the values of f on the boundary of C. Some of the well-known boundary conditions are the Leray–Schauder condition and the inward conditions (see [2], [3], [10], [17]). When C is a closed subset of the Euclidean n-space R^n, a new boundary condition was developed in [19, Theorem 2.1], providing a unification of many different conditions.

In this part we assume that C is a nonempty arbitrary subset of X and hence it is necessary to choose some suitable local boundary conditions; in this way we shall obtain some new fixed point theorems which generalize well-known results.

3.1. Local boundary mapping. Throughout this part X denotes a Hausdorff topological vector space.

Let E, D be nonempty subsets of X. By L_D we denote the smallest linear manifold of X containing D.

DEFINITION 9. A single-valued mapping $g\colon L_D \to E$ is said to be a *local boundary mapping* if g is continuous on L_D and $g(u) = u$ for every $u \in \overline{\text{conv}}\, D \cap E$.

EXAMPLE 5. Let us recall that a continuous mapping $r\colon X \to E$ with $r|E = \text{id}_E$ is called a *retraction* of X onto E. Clearly, if r is a retraction of X onto E, then, for every nonempty subset D' of X, $r|L_{D'}$ is a local boundary mapping (w.r.t. E, D').

By $\text{int}_{L_D}(E \cap L_D)$ we denote the relative interior of $E \cap L_D$ in L_D and $\partial_{L_D}(E \cap L_D) = (E \cap L_D) \setminus \text{int}_{L_D}(E \cap L_D)$. Another example is the following:

EXAMPLE 6. Assume that E is closed and convex with $\text{int}_{L_D}(E \cap L_D) \neq \emptyset$ and that there exists a continuous mapping $\varphi\colon L_D \setminus E \to \text{int}_{L_D}(E \cap L_D)$ such that $\overline{\varphi(L_D \setminus E)}$ is a compact subset of $\text{int}_{L_D}(E \cap L_D)$. Then we can define a local boundary mapping $g\colon L_D \to E$ as follows:

$$g(u) = \begin{cases} [u, \varphi(u)] \cap \partial_{L_D}(E \cap L_D) & \text{for } u \in L_D \setminus E, \\ u & \text{for } u \in L_D \cap E. \end{cases}$$

DEFINITION 10. A multifunction $g\colon L_D \to 2^E$ is called a *local boundary multifunction* if g is u.s.c. on L_D and $g(u) = \{u\}$ for every $u \in \overline{\text{conv}}\, D \cap E$.

Let $p \in A \subset X$. We recall that a subset A of X is *p-starshaped* if, for every $x \in A$, the segment $[p, x]$ is a subset of A.

EXAMPLE 7. Assume that $\text{int}_{L_D}(E \cap L_D) \neq \emptyset$ and $p_i \in \text{int}_{L_D}(E \cap L_D)$ for each $i = 1, 2, \ldots, n$. Moreover, if E is a subset of X such that $E \cap L_D$ is closed and, for each $i = 1, 2, \ldots, n$ is p_i-starshaped, then one can define a multifunction $g\colon L_D \to 2^E$ by

$$g(u) = \begin{cases} \bigcup \{\varphi_i(u) |\ i = 1, 2, \ldots, n\} & \text{for } u \in L_D \setminus E, \\ u & \text{for } u \in L_D \cap E, \end{cases}$$

where $\varphi_i(u) = \{x' \in [p_i, u] \cap E |\ [p_i, x'] \supset [p_i, x] \text{ for all } x \in [p_i, u] \cap E\}$.

Now we give some other classes of local boundary multifunctions used in fixed point theorems.

EXAMPLE 8. Assume that E is a compact subset and $d\colon L_D \times E \to R$ is a non-negative continuous real function such that $d(u, x) = 0$ iff $u = x \in E$. Then the multifunction g defined by

$$g(u) = \{x' \in E |\ d(u, x') \leqslant d(u, x) \text{ for all } x \in E\} \text{ for every } u \in L_D$$

is also a local boundary multifunction.

EXAMPLE 9. The following notion is defined in [19]: For a given closed subset E of R^n, a u.s.c. multifunction r from R^n into E is called a *retraction* (in the extended sense) if $r(u)$ is nonempty and compact for every $u \in R^n$ and $r(u) = u$ for every $u \in E$.

Clearly, if r is a retraction in the extended sense, then for every subset D' of R^n, $r|L_{D'}$ is a local boundary multifunction (w.r.t. E, D').

Remark 3. Let $\mathscr{U}(D)$ be the set of all finite subsets U of D. For each U, let $S(U)$ denote some fixed minimal simplex of the finite dimensional veriety L_U containing $\operatorname{conv}(U)$ and $S(D) = \bigcup \{S(U) |\ U \in \mathscr{U}(D)\}$.

As we shall show in Theorem 4′ below, to establish fixed point theorems one can use the following definitions by relaxing continuous assumptions:

A single-valued mapping $g\colon L_D \to E$ is a local boundary mapping provided g is continuous on $S(D)$ and $g = \mathrm{id}$ on $\overline{\operatorname{conv}}(D) \cap E$.

A multifunction $g\colon L_D \to 2^E$ is a local boundary multifunction if g is u.s.c. on $S(D)$ and $g = \mathrm{id}$ on $\overline{\operatorname{conv}}(D) \cap E$.

3.2. Some fixed point theorems. Throughout this section X denotes a locally convex topological vector space.

THEOREM 4. *Let C be a nonempty subset of X and let $f\colon C \to 2^X$ be a q.u.s.c. multifunction on C. Suppose that there exists a nonempty subset D of X and a local boundary mapping $g\colon L_D \to C$ such that:*

(i) *for each $x \in g(L_D)$, $f(x)$ is convex and $\overline{f(x) \cap D} \neq \emptyset$,*

(ii) *'local boundary condition': for every $u \in \operatorname{conv} D \setminus C$,*

(3.1) $$u \notin fg(u)$$

(iii) *$\overline{fg}(L_D) \cap \overline{\operatorname{conv}}\ D$ is compact.*

Then f has a fixed point in C.

Remark 4. C is not assumed to be closed or convex.

PROOF. Let $A = \overline{fg(L_D) \cap \overline{\text{conv}}\, D}$; then A is nonempty compact subset of X. Let $\{V_t\}$ be a convex neighbourhood basic for 0.

Now we fix a neighbourhood V_t of 0. Since A is compact, there exists a finite set $U = \{u_j |\ j \in J\}$ where $J = \{1, 2, \ldots, m\}$ (in fact, U and m depend on V_t, which we omit for simplicity) such that $\{u_j + V_t |\ j \in J\}$ is an open cover of A. Let $\{\beta_j |\ j \in J\}$ denote a partition of unity subordinate to this cover. We define $p\colon A \to L_U$ as follows: $p(u) = \sum_{j \in J} \beta_j(u) u_j$ for each $u \in A$. Since $u - u_j \in V_t$ if $\beta_j(u) > 0$, hence $u - p(u) \in V_t$. Continuing, we define $\varphi\colon L_U \to 2^{L_U}$ by

$$\varphi(u) = p\big(fg(u) \cap D\big) - u \quad \text{for every } u \in L_U; \tag{3.2}$$

hence $\varphi(u) + u$ is a subset of the convex, closed set conv U in the finite dimensional linear variety L_U. Without loss of generality, we suppose that L_U is a subspace and $0 \in \text{int}\, S(U)$, where $S(U)$ is some minimal simplex containing conv U.

By an analogous argument used in the proof of Corollary 2.1. (Section 2.3, Part 2) for each $k = 1, 2, \ldots, d_k > 0$ such that $d_k \to 0 (k \to \infty)$ we apply the algorithm (w, P, T, b, l) to the multifunction φ given in (3.2) and obtain a simplex $t_k = \{z_{k,j} |\ j \in J\}$ of T_k, where mesh $T_k \leqslant d_k$ and

$$0 \in \text{conv}\,\{l(z_{k,j}) |\ j \in J\}; \tag{3.3}$$

hence, there exists for each $j \in J$,

$$w_{k,j} \in fg(z_{k,j}) \cap D \quad \text{such that} \quad p(w_{k,j}) = l(z_{k,j}) - z_{k,j}.$$

This implies that, since A and $S(U)$ are compact, for each $j \in J$ there exist a $w_{t,j} \in A$, $z_t \in S(U)$ such that, as $k \to \infty$, $z_{k,j} \to z_t$ for all $j \in J$ and $w_{k,j} \to w_{t,j}$, and hence, by the quasi upper semicontinuity of fg on $S(U)$, we get $w_{t,j} \in fg(z_t)$ for all $j \in J$.

Further, we have, by (3.3),

$$0 = \sum_{j \in J} \alpha_{k,j} \big(p(w_{k,j}) - z_{k,j}\big)$$

for some $\alpha_{k,j} \geqslant 0$, for all $j \in J$ such that $\sum_{j \in J} \alpha_{k,J} = 1$.

This implies that there exists, for each $j \in J$, a number $\alpha_{t,j} \geqslant 0$ such that $\sum_{j \in J} \alpha_{t,j} = 1$ and $\alpha_{k,j} \to \alpha_{t,j}$ $(k \to \infty)$. Therefore, $0 = \sum_{j \in J} \alpha_{t,J}\big(p(w_{t,j}) - z_t\big)$, i.e., $z_t = \sum_{j \in J} \alpha_{t,J}\, p(w_{t,j})$.

Let $z'_t = \sum_{j \in J} \alpha_{t,j} w_{t,j}$. Since $fg(z_t)$ is convex and $w_{t,j} \in A$ for all $j \in J$, we get $z'_t \in fg(z_t) \cap \overline{\text{conv}}\, D \subset A$. Moreover, since V_t is convex, $z'_t - z_t \in V_t$. By the compactness of A, z'_t converges to point $z_0 \in \overline{\text{conv}}\, D$ and hence z_t converges also to z_0. Thus, since fg is q.u.s.c. on $\overline{\text{conv}}\, D$, we have $z_0 \in fg(z_0) \cap \overline{\text{conv}}\, D$.

Finally, the local boundary condition (3.1) implies that $z_0 \in C$ and $z_0 \in f(z_0)$.

COROLLARY 4.1. *Let C be a nonempty subset of a Fréchet space X and let $f\colon C \to 2^X$ be a q.u.s.c. multifunction on C such that there exists a nonempty compact subset D of X such that $\overline{\text{conv}}\, D \subset C$ and, for every $x \in C$, $f(x)$ is convex and $f(x) \cap D \neq \emptyset$; then f has a fixed point in C.*

Proof. We define $f'\colon \bar{C} \to 2^C$ as follows:

$$f'(x) = \begin{cases} f(x) \cap \overline{\text{conv}}\, D & \text{for } x \in C, \\ \bigcap \{\overline{f(V \cap C)} \mid V \text{ is a neighbourhood of } x \text{ in } \bar{C}\} \cap \overline{\text{conv}}\, D & \text{for } x \in \bar{C} \setminus C. \end{cases}$$

and $f_1(x) = \overline{\text{conv}}\,(f'(x))$ for every $x \in \bar{C}$. Then, by Proposition 2 the multifunction $f_1\colon \bar{C} \to 2^C$ is q.u.s.c. on $\bar{C}$. Now we can apply Theorem 4 to C_1, D_1, f_1 with $C_1 = \bar{C}$, $D_1 = \overline{\text{conv}}\, D$ and the local boundary mapping could be defined, since X is a Frechét space, as follows:

$$g(u) = \{y' \in D_1 \mid d(x, y') \leqslant d(x, y) \text{ for all } y \in D_1\} \quad \text{for each } u \in X,$$

where d is the metric of X. Then there exists an $x_0 \in \bar{C}$ such that $x_0 \in f_1(x_0) \subset C$; therefore, $x_0 \in C$ and $x_0 \in f(x_0)$.

COROLLARY 4.2. *Let C be a nonempty subset of X and let $f\colon C \to 2^X$ be a q.u.s.c. multifunction on C. Assume that there exists a nonempty subset D of X such that $\text{int}_{L_D}(C \cap L_D) \neq \emptyset$, $C \cap L_D$ is closed, convex and such that:*

(i) *for every $x \in C \cap L_D$, $f(x)$ is convex and $f(x) \cap D \neq \emptyset$,*

(ii) *$\overline{f(C \cap L_D)} \cap \overline{\text{conv}}\, D$ is compact,*

(iii) *there exists a continuous mapping $\varphi\colon L_D \setminus C \to \text{int}_{L_D}(C \cap L_D)$ such that $\overline{\varphi(L_D \setminus C)}$ is a comapct subset of $\text{int}_{L_D}(C \cap L_D)$ and such that for every $x \in B$,*

(3.4) $f(x) \cap \{x + t(x - \varphi(u)) \mid t > 0 \text{ and } u \in L_D \setminus C \text{ such that } x \in [u, \varphi(u)]\} = \emptyset,$

where

$$B = \{x \in \partial_{L_D}(C \cap L_D) \mid \text{there exists } u \in L_D \setminus C \text{ such that } x \in [u, \varphi(u)]\}.$$

Then f has a fixed point in C.

Proof. Indeed, we can define the local boundary mapping as follows:

$$g(u) = \begin{cases} \partial_{L_D}(C \cap L_D) \cap [u, \varphi(u)] & \text{for } u \in L_D \setminus C, \\ u & \text{for } u \in L_D \cap C. \end{cases}$$

Condition (3.4) applies to each $u \in L_D \setminus C$, $u \notin fg(u)$ i.e., the local boundary condition is satisfied. The result is obtained from Theorem 4.

Remark 5. If $z' \in \text{int}_{L_D}(C \cap L_D)$ and $\varphi(u) = z'$ for every $u \in L_D \setminus C$, then condition (iii) in this corollary reduces to the Leray–Schauder boundary condition ([17]) imposed by F. E. Browder in [3]:

A mapping $T: C \to X$ is said to satisfy *Leray–Schauder boundary condition* if there exists a point $z \in \operatorname{int} C$ such that, for every $x \in \partial C$ and $t > 1$, $T(x) - z \neq t(x-z)$.

We recall that a multifunction φ: $X \to 2^Y$ is said to be *closed* if its graph

$$\operatorname{Graph} \varphi = \{(x, y) |\ x \in X,\ y \in \varphi(x)\}$$

is a nonempty closed subset of $X \times Y$.

Using Remark 3, we can obtain the following modification of Theorem 4.

THEOREM 4'. *Let C be a nonempty subset of X and let f: $C \to 2^X$ be a q.u.s.c. multifunction on C. Suppose that there exists a nonempty subset D of X and a local boundary mapping* (in the sense given in Remark 3) g: $L_D \to C$ such that:

(i) *for every $x \in \underline{g(S(D))}$, $f(x)$ is convex and $f(x) \cap D \neq \emptyset$,*

(ii) *for each $u \in \overline{\operatorname{conv}}\, D \setminus C$, $u \notin \overline{fg}(u)$,*

(iii) *$\overline{fg}(S(D)) \cap \overline{\operatorname{conv}}\, D$ is compact.*

Then f has a fixed point in C.

THEOREM 5. *Let C be a nonempty subset of X and let f: $C \to 2^X$ be a q.u.s.c. multifunction on C. Assume that there exist a closed multifunction φ: $X \to 2^X$, a nonempty subset D of X and a local boundary multifunction g: $L_D \to 2^C$ such that:*

(i) *for every $x \in g(L_D)$, $f(x)$ is convex and $\overline{f(x)} \cap \varphi(x) \cap D \neq \emptyset$,*

(ii) *'local boundary condition': for every $u \in \overline{\operatorname{conv}}\, D \setminus C$,*

$$u \notin \overline{\operatorname{conv}}\,(fg(u) \cap \varphi(u)), \tag{3.5}$$

(iii) *$\overline{\operatorname{conv}}\,([fg \cap \varphi](L_D)) \cap \overline{\operatorname{conv}}\, D$ is compact.*

Then there exists an $x_0 \in C$ such that $x_0 \in f(x_0)$.

Proof. We denote $A = \overline{\operatorname{conv}}\,([fg \cap \varphi](L_D)) \cap \overline{\operatorname{conv}}\, D$. By (iii) A is compact. We can verifty that the multifunction f': $L_D \to 2^X$ defined for each $u \in L_D$ by

$$f'(u) = \overline{fg}\,(u) \cap \varphi(u) \cap \overline{\operatorname{conv}}\, D$$

is q.u.s.c. on L_D.

By the arguments used in Corollary 2.1. (Section 3.2, Part 2) and in Theorem 4, corresponding to each convex neighbourhood V_t of X there exist $z_t \in \overline{\operatorname{conv}}\, D$ and, for each $j \in J = \{1, 2, \ldots, m_t\}$, $w_{t,j} \in \overline{fg}\,(z_t) \cap \varphi(z_t) \cap \overline{\operatorname{conv}}\, D$, $\alpha_{t,j} \geqslant 0$ such that $\sum_{j \in J} \alpha_{t,j} = 1$, $z'_t = \sum_{j \in J} \alpha_{t,j} w_{t,j}$ and $z_t - z'_t \in V_t$. Hence $z'_t \in \overline{\operatorname{conv}}\,(\overline{fg}\,(zt) \cap \varphi(z_t)) \cap \overline{\operatorname{conv}}\, D$.

Noting that the multifunction f'': $L_D \to 2^X$ defined by

$$f''(u) = \overline{\operatorname{conv}}\,(fg\,(u) \cap \varphi(u)) \cap \overline{\operatorname{conv}}\, D \quad \text{for} \quad u \in L_D$$

is q.u.s.c. and A is compact, we have a subsequence of $\{z_t'\}$, say also z_t', which converges to a point $z_0 \in \overline{\text{conv}}\, D$. This implies that

$$z_0 \in \overline{\text{conv}}\,(fg(z_0) \cap \varphi(z_0)) \cap \overline{\text{conv}}\, D.$$

Hence, by the 'local boundary condition', $z_0 \in C$ and

$$z_0 \in \overline{\text{conv}}\,(f(z_0) \cap \varphi(z_0)) \subset \overline{\text{conv}}\,(f(z_0)) = f(z_0).$$

COROLLARY 5.1. (a generalization of Theorem 2.1. in [19]). *Let C be a nonempty subset of the Euclidean n-space R^n, and let $f\colon C \to 2^{R^n}$ be a q.u.s.c. multifunction on C such that $f(x)$ is nonempty convex for every $x \in C$ and $f(C)$ is bounded. Assume that there exists a u.s.c. multifunction $r\colon R^n \to 2^C$ such that $r(y) \neq \emptyset$ for every $y \in R^n$ and $r(y) = \{y\}$ for every $y \in C$ and a closed multifunction $s\colon R^n \to 2^{R^n}$ such that*

$$y \notin \overline{\text{conv}}\,(fr(y) \cap s(y)) \neq \emptyset \quad \text{for every } y \notin C.$$

Then f has a fixed point in C and the system $x \in C$, $x \in f(x)$ has an approximate solution.

Proof. We omit it.

THEOREM 6. *Let C be a nonempty subset of X and let $f\colon C \to 2^X$ be a q.u.s.c. multifunction on C. Suppose that D is a nonempty compact subset of X and $g\colon$* conv *$D \to 2^C$ is a u.s.c. multifunction on* conv *D such that $g = \text{id}$ on $C \cap D$ and that:*

(i) *for each $x \in g(\text{conv}\, D)$, $f(x)$ is a nonempty convex subset of D and* $\text{conv}(\overline{fg}(u)) \subset D$ *for each $u \in \overline{\text{conv}}\, D$,*

(ii) *for each $u \in D \setminus C$, $u \notin \overline{\text{conv}}\, fg(u)$.*

Then there exists an $x_0 \in C \cap D$ such that $x_0 \in f(x_0)$.

Proof. We shall give two proofs of this theorem.

First proof. Corresponding to a convex open neighbourhood V_t of 0, since D is compact, there is a finite subset $U_t = \{u_1, u_2, \ldots, u_m\}$ of D such that $\{u_i + V_t |\ i = 1, \ldots, m\}$ is an open cover of D. Let $\beta_1, \ldots, \beta_m$ be a partition of unity subordinate to this cover and let us define the projection $p\colon D \to \text{conv}\, U_t$ by $p(u) = \sum_{i=1}^{m} \beta_i(u) u_i$ for each $u \in D$; hence $u - p(u) \in V_t$.

Let $\varphi_t(u) = \text{conv}\, p(\overline{fg}(u))$ for every $u \in \text{conv}\, U_t$. By Proposition 2 (Section 1.1, Part 1), the multifunction $\varphi_t\colon \text{conv}\, U_t \to 2^{\text{conv}\, U_t}$ is q.u.s.c. on conv U_t and hence, by Corollary 4.1., there exists $u_t \in \text{conv}\, U_t$ such that $u_t \in \varphi_t(u_t) = \text{conv}\, p(\overline{fg}(u_t))$; thus $u_t = \sum_{i=1}^{m} \alpha_i w_i$ for suitable α_i and $w_i \in p(z_i)$, $z_i \in \overline{fg}(u_t)$ for each $i = 1, \ldots, m$.

Let $u'_t = \sum_{i=1}^{m} \alpha_i z_i$; then $u'_t \in \overline{\mathrm{conv}\, fg}(u_t) \subset D$, and hence, since D is compact, u'_t converges to a point $u_0 \in D$; therefore, by $u_t - u'_t \in V_t$ for all neighbourhood V_t, u_t converges also to $u_0 \in D$. Using the quasi upper semicontinuity of the multifunction $\overline{\mathrm{conv}}(fg)$ on conv D and condition (ii), we complete the proof.

Second proof. Corresponding to a closed, convex, circled neighbourhood V_t of 0, since D is compact, there exists a finite subset U_t of D such that $D \subset \bigcup \{u' + V_t | \; u' \in U_t\} \subset \mathrm{conv}\; U_t + V_t$. For each $u \in \mathrm{conv}\; U_t$ we set $\varphi_t(u) = (\mathrm{conv}(fg(u)) + V_t) \cap \mathrm{conv}\; U_t$. Using the same above argument, by φ_t is q.u.s.c. on conv U_t, we show that there exists a $u_t \in \mathrm{conv}\; U_t$ such that $u_t \in \varphi_t(u_t)$. This implies the existence of $v_t \in \overline{\mathrm{conv}}\, fg(u_t)$ such that $u_t - v_t \in V_t$; thus there is a $v'_t \in \mathrm{conv}(fg(u_t)) \subset D$ such that $u_t - v'_t \in 2V_t$. Analogously, since D is compact, we get a fixed point of f in $C \cap D$.

COROLLARY 6.1. *Let C be a nonempty subset of X and let f: $C \to 2^X$ be a q.u.s.c. multifunction on C. Suppose that D is a nonempty compact subset of X and g:* conv $D \to C$ *is a continuous mapping on* conv D *such that $g = \mathrm{id}$ on $C \cap D$ and, for every $x \in g(\mathrm{conv}\; D)$, $f(x)$ is a nonempty convex subset of D. Moreover, if, for each $u \in D \setminus C$, $u \notin fg(u)$, then f has a fixed point in C.*

COROLLARY 6.2 (a generalization of Theorem 11.3 in [7]). *Let C be a nonempty convex subset of X, and let f: $C \to 2^C$ be a q.u.s.c. multifunction on C such that, for each $x \in C$, $f(x)$ is nonempty convex and $\overline{f(C)}$ is a compact subset of C. Then f has a fixed point in C.*

A modification of this Corollary is the following

COROLLARY 6.3. *Let C be a nonempty subset of X and let f: $C \to 2^X$ be a q.u.s.c. multifunction on C such that there exists a convex compact subset D of C such that, for every $x \in C$, $f(x)$ is a convex subset of X and $f(x) \cap D \neq \emptyset$. Then f has a fixed point in C.*

3.3. Some fixed point theorems with special boundary conditions. Throughout this section X denotes a locally convex topological vector space

Let A be a nonempty subset of X.

DEFINITION 11 ([2]). The *algebraic boundary dA* of the set A is the set of points $x \in A$ such that there exists a $y \in X$ such that $x + ty \notin A$ for all $t > 0$. (See also [10]).

First we give a direct generalization of Theorem 3 in [2].

THEOREM 7. *Let C be a nonempty subset of X and let f: $C \to 2^X$ be a u.s.c. multifunction on C. Assume that there exist a nonempty convex compact subset D of X and a u.s.c. multifunction g: $D \to 2^C$ such that $g = \mathrm{id}$ on $C \cap D$ and such that:*

(i) *for each $u \in D$, $fg(u)$ is convex and, for each $x \in g(D)$, $f(x)$ is nonempty, closed and convex,*

(ii) *for each* $u \in dD$, *there exist a* $v \in D$, $w \in \overline{fg}(u)$ *and a number* $\alpha > 0$ *such that* $w-u = \alpha(v-u)$,

(iii) *for each* $u \in D \setminus C$, $u \notin \overline{fg}(u)$.

Then f *has a fixed point in* C.

Proof. One can verify that the multifunction $\overline{fg}: D \to 2^X$ satisfies all the conditions of Theorem 3 in [2]; hence there exists a point $u_0 \in D$ such that $u_0 \in \overline{fg}(u_0)$. This point is a fixed point of the multifunction f in C.

Another generalization of the fixed point theorem with the Leray–Schauder condition is the following

THEOREM 8. *Let* C *be a nonempty subset of* X *and let* $f\colon C \to 2^X$ *be a q.u.s.c. multifunction on* C. *Assume that there exists a nonempty compact subset* D *of* X *such that* L_D *is complete,* $L_D \cap C$ *is closed with* $\operatorname{int}_{L_D}(L_D \cap C) \neq \emptyset$ and such that:

(i) *for every* $x \in L_D \cap C$, $f(x)$ *is convex and* $f(x) \cap D \neq \emptyset$,

(ii) *there exist a nonempty compact set* $C_0 \subset \operatorname{int}_{L_D}(L_D \cap C)$ *and a continuous mapping* $\varphi\colon L_D \setminus \operatorname{int}_{L_D}(L_D \cap C) \to C_0$ *such that, for each* $x \in \partial_{L_D}(L_D \cap C) \cap \operatorname{conv}(D \cup C_0)$,

$$f(x) \cap \{u|\ u = x+t(x-\varphi(x)),\ t \geqslant 0\} = \emptyset.$$

Then f *has a fixed point in* C.

Proof. Let $D_1 = \operatorname{conv}(D \cup C_0)$. We define a multifunction $f_1\colon D_1 \to 2^X$ by given

$$f_1(u) = \begin{cases} f(u) \cap D_1 & \text{for} \quad u \in \operatorname{int}_{L_D}(C \cap L_D), \\ \varphi(u) & \text{for} \quad u \in D_1 \setminus C, \\ \operatorname{conv}(f(u) \cap D_1, \varphi(u)) & \text{for} \quad u \in \partial_{L_D}(C \cap L_D) \cap D_1. \end{cases}$$

The set D_1 is compact since L_D is a locally convex complete space (see [23]). Since f is *q.u.s.c.* on C and φ is continuous, we can immediately verify that f_1 is *q.u.s.c.* on D_1. Then by Corollary 6.2. we get a point $u_0 \in D_1$ such that $u_0 \in f_1(u_0)$.

However, by the definition of the multifunction f_1, $u_0 \notin D_1 \setminus C$. If $u_0 \in \partial_{L_D}(C \cap L_D) \cap D_1$ and if there exist a number α such that $0 < \alpha < 1$ and a point $v_0 \in f(u_0)$ such that $u_0 = \alpha\varphi(u_0)+(1-\alpha)v_0$, then for some $\alpha' > \alpha$ $u_0+\alpha'(u_0-\varphi(u_0)) = v_0 \in f(u_0)$, which contradicts condition (ii). Therefore, $u_0 \in C$ and $u_0 \in f_1(u_0) \subset f(u_0)$.

COROLLARY 8.1. *Let* C, f, D *be as in Theorem* 8. *Suppose that condition* (ii) *is replaced by the following: there is a point* $w_0 \in \operatorname{int}_{L_D}(L_D \cap C)$ *such that, for each* $x \in \partial_{L_D}(L_D \cap C) \cap \operatorname{conv}(D, w_0)$,

$$f(x) \cap \{x+t(x-w_0)|\ t \geqslant 0\} = \emptyset.$$

Then f *has a fixed point in* C.

Remark 6. Let $f\colon C \to 2^X$ be a w.u.s.c. multifunction on C. A point $x \in C$ is said to be a *quasi fixed point* of the multifunction f if $x \in \overline{f(x)}$. Using Proposition 1 (Section 1.1, Part 1) and the above fixed point theorems, we can directly obtain results about the existence of quasi fixed points of weak upper semicontinuous multifunctions. Further results concerning this problem will be presented in another paper.

4. Fixed point theorems for strongly lower semicontinuous multifunctions

In this part we consider some fixed point theorems for strongly lower semicontinuous multifunctions. First we generalize the well-known theorem of F. E. Browder ([2], Theorem 1) and then we give a generalization of a theorem of Fan Ky ([12], Theorem 10); hence we can consider fixed point theorems with boundary conditions for strongly lower semicontinuous multifunctions.

Throughout this part X denotes a topological vector space (not necessarily locally convex).

4.1. A remark on Fixed Point Theorem. First we prove the following

THEOREM 9. *Let C be a nonempty subset of X and let $h\colon C \to 2^X$ be a multifunction. Assume that there exist a nonempty convex compact subset D of X and a multifunction $g\colon D \to 2^C$ l.s.c. on D such that h is s.l.s.c. on $g(D)$. Moreover, if, for every $u \in D$, $\operatorname{conv} hg(u) \cap D \neq \emptyset$ and if, for every $u \in D \setminus C$, $u \notin \operatorname{conv} hg(u)$, then there exists a point $x_0 \in C \cap D$ such that $x_0 \in \operatorname{conv} hg(x_0)$.*

Proof. First, we remark that the multifunction $hg\colon D \to 2^X$, by Proposition 5 (Section 1.2, Part 1), is s.l.s.c. on D. Now we define the multifunction $\varphi\colon D \to 2^X$ by $\varphi(u) = \operatorname{conv} hg(u)$ for every $u \in D$. Then φ is also s.l.s.c. on D. Indeed, let $u \in D$, $z \in \varphi(u)$; then $z = \sum_{i=1}^{n} \alpha_i z_i$ for some suitable $z_i \in hg(u)$, $\alpha_i \geqslant 0$ for each $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i = 1$. However, by the strongly lower semicontinuity of the multifunction hg at u, for each $i = 1, 2, \ldots, n$ there exists a neighbourhood V_i of u in D such that $z_i \in hg(u')$ for all $u' \in V_i$. Thus, for every $u' \in V = \bigcap \{V^i \mid i = 1, \ldots, n\}$, $z_i \in hg(u')$ for all $i = 1, 2, \ldots, n$; hence, $z \in \varphi(u')$ for every $u' \in V$, as was required.

Now, since $\varphi(u) \cap D \neq \emptyset$ for every $u \in D$, we use the open cover $\{\varphi^-(z) \mid z \in D\}$ of the set D; then, since D is compact, there exists a finite cover $\{\varphi^-(z_j) \mid j = 1, \ldots, m\}$ of D. Let $\beta_1, \ldots, \beta_m$ be a partition of unity subordinate to the above cover. From the continuity of the projection

p: conv $U \to$ conv U defined by $p(u) = \sum_{j=1}^{m} \beta_j(u) z_j$ for each $u \in$ conv U, where $U = \{z_j |\ j = 1, \ldots, m\}$, we get a point $x_0 \in$ conv U such that $x_0 \in p(x_0)$.

Since if $\beta_j(x_0) > 0$ then $x_0 \in \varphi^-(z_j)$, i.e., $z_j \in \varphi(x_0)$, we have $x_0 = p(x_0) = \sum_{j=1}^{m} \beta_j(x_0) z_j \in$ conv $hg(x_0)$ and complete the proof by using the last condition in the theorem.

Corollary 9.1 (a generalization of Theorem 1 in [2]). *Let C, h, D, g be as in Theorem 9. Moreover, assume that $g = \mathrm{id}$ on $D \cap C$ and, for every $u \in D \cap C$, $h(u)$ is convex; then the multifunction h has a fixed point in C.*

Now we need the following

Definition 12. A multifunction ψ: $E \to 2^X$ defined on a convex set is said to be *convex* if its graph is nonempty convex subset of $E \times X$.

Corollary 9.2. *Let C be a nonempty subset of X and let h: $C \to 2^X$. Assume that there exist a nonempty compact subset D of X and a multifunction g: conv $D \to 2^C$ l.s.c. on conv D such that $g = \mathrm{id}$ on conv $D \cap C$ and h is s.l.s.c. on g(conv D). Moreover, if hg is a convex multifunction and $hg(u) \cap D \neq \emptyset$ for every $u \in D$ and if $u \notin hg(u)$ for every $u \in$ conv $D \setminus C$, then h has a fixed point in C.*

Proof. It is analogous to the argument in the proof of Theorem 9: there exists a finite subset $U = \{z_j |\ j = 1, \ldots, m\}$ of D such that $D \subset \bigcup \{g^-(h^-(z_j)) |\ j = 1, \ldots, m\}$. Then for each $z_j \in U$ there is a $z'_j \in U$ such that $z'_j \in hg(z_j)$; hence, for each $u \in$ conv U, $u = \sum_{j=1}^{m} \alpha_j z_j$ for suitable $\alpha_j \geqslant 0$, by the convexity of the multifunction hg, we have $\sum_{j=1}^{m} \alpha_j hg(z_j) \subset hg(u)$. This implies that $\{\sum_{j=1}^{m} \alpha_j z'_j | z_j$ such that $z_j \in g^-(h^-(z'_j))\} \subset hg(u)$ and hence $hg(u) \cap$ conv $U \neq \emptyset$. Finally, we apply Theorem 9 with C, h, $D' =$ conv U and g and complete the proof.

4.2. Some results related to Knaster–Kuratowski–Mazurkiewicz Theorem. Among the results equivalent to Browder's Fixed Point Theorem, the Knaster–Kuratowski–Mazurkiewicz Theorem occupies a special place: the infinite-dimensional form given in [9], [7] is particularly suitable for applications (see for example [7], [11]).

Now we give some fixed point theorems with boundary conditions for strongly lower semicontinuous multifunctions closely related to a generalized Knaster–Kuratowski–Mazurkiewicz Theorem.

Let A be a nonempty subset of X.

DEFINITION 13 ([14]). A multifunction $f\colon A \to 2^X$ is called a *Knaster–Kuratowski–Mazurkiewicz multifunction* (or simply a *KKM-multifunction*) provided

$$\operatorname{conv}\{u_1, u_2, \ldots, u_m\} \subset \bigcup \{f(u_i) \mid i = 1, 2, \ldots, m\}$$

for each finite subset $\{u_1, u_2, \ldots, u_m\}$ of A (see also [7], [11]).

We recall that a family of sets $\{B_t \mid t \in T\}$ of X is said to have the *finite intersection property* if the intersection of each finite subfamily of $\{B_t \mid t \in T\}$ is not empty.

The basic results are well known.

THEOREM 10 (The KKM-multifunction principle [14]). *If $f\colon A \to 2^X$ is a KKM-multifunction such that each $f(u)$ is closed, then the set $\{f(u) \mid u \in A\}$ has the finite intersection property.*

Proof. See [7, 9].

An immediate consequence is the following

THEOREM 11 ([9]). *Let $f\colon A \to 2^X$ be a KKM-multifunction. If all the sets $\{f(u)\}$ are closed, and if one of them is compact, then*

$$\bigcap \{f(u) \mid u \in A\} \neq \emptyset.$$

The following generalization of Theorem 11 is given in [12, Theorem 1] without proof.

THEOREM 12 ([12]). *Let A be a nonempty subset of X and let D be a convex set such that $A \subset D$. Assume that $f\colon A \to 2^D$ is a KKM-multifunction such that $f(u)$ is relatively closed in D for each $u \in A$. If there exists a nonempty subset A' of A such that $\bigcap \{f(u) \mid u \in A'\}$ is compact and A' is contained in a compact convex subset of D, then $\bigcap \{f(u) \mid u \in A\} \neq \emptyset$.*

Proof. Denote by K a compact convex subset of D such that $A' \subset K$. Now we define the multifunction $g\colon A' \to 2^K$ by $g(u) = f(u) \cap K$ for each $u \in A'$. Then, for every $u \in A'$, $g(u) \neq \emptyset$ and hence, since f is a KKM-multifunction and K is convex, the multifunction g is also a KKM-multifunction.

Now assume $\bigcap \{g(u) \mid u \in A'\} = \emptyset$. Since K is compact and $K \setminus g(u)$ is relatively open in K,

$$K = \bigcup \{K \setminus g(u_i) \mid u_i \in A', i = 1, \ldots, m\}$$

for some finite subset $\{u_1, u_2, \ldots, u_m\}$ of A'. Thus, $K = K \setminus \bigcap \{g(u_i) \mid i = 1, \ldots, m\} \neq K$, since g is a KKM-multifunction on A'. This contradiction shows that $\bigcap \{g(u) \mid u \in A'\}$ is a nonempty subset of K. Clearly, $\bigcap \{g(u) \mid u \in A'\}$ is compact and hence $\bigcap \{f(u) \mid u \in A'\} \neq \emptyset$.

Moreover, if $u' \in A \setminus A'$, then $K' = \operatorname{conv}(K, u')$ is compact and convex, since K is compact, convex and $A' \cup u' \subset K'$; hence we can apply the above argument to f, $A' \cup u'$ and K' and get $f(u') \cap \bigcap \{f(u) \mid u \in A'\} \neq \emptyset$, which implies $f(u') \cap K \neq \emptyset$.

So, we can define $g\colon A \to 2^K$ by $g(u) = f(u) \cap K$ for each $u \in A$. Then g is a KKM-multifunction where all sets $g(u)$ are nonempty. By a similar argument, if $\bigcap \{g(u) \mid u \in A\} = \emptyset$, then $K \neq K$, and this completes the proof.

Now we generalize the theorem of Fan Ky ([12, Theorem 10]).

THEOREM 13. *Let C be a nonempty subset of X, let D be a nonempty convex subset of X and let $h\colon C \to 2^X$ be a multifunction such that there exists an l.s.c. multifunction $g\colon D \to 2^C$ such that $g = \mathrm{id}$ on $D \cap C$. Moreover, if*

(i) *for each $u \in D$, $hg(u)$ is convex, $hg(u) \cap D \neq \emptyset$ and h is s.l.s.c. at $g(u)$,*

(ii) *there is a nonempty compact convex subset D' of D such that $hg(u) \cap D' \neq \emptyset$ for each $u \in D \setminus D'$,*

(iii) *for each $u \in D \setminus C$, $u \notin hg(u)$,*

then there exists a point $x_0 \in C$ such that $x_0 \in h(x_0)$.

Proof. First we define $h'\colon D \to 2^D$ by $h'(u) = hg(u) \cap D$ for each $u \in D$. By Proposition 5 (Section 1.2, Part 1) h' is *s.l.s.c. on D* where all sets $h'(u)$ are nonempty and convex.

Let $E = \{(v, u) \in D \times D \mid u \in D,\ v \in h'(u)\}$ and define $f\colon D \to 2^D$ by

$$f(v) = \{u \in D \mid (v, u) \notin E\} = \{u \in D \mid v \notin h'(u)\}$$

for each $v \in D$. Clearly, $f(v)$ is relatively closed in D.

Further, we assume that $(u, u) \notin E$ for all $u \in D$ and show that under this assumption f is a KKM-multifunction. Indeed, if for some finite subset $\{u_1, \ldots, u_n\}$ of D

$$\mathrm{conv}(u_1, u_2, \ldots, u_n) \setminus \bigcup_{i=1}^{n} \{f(u_i)\} \neq \emptyset,$$

then

$$\mathrm{conv}(u_1, u_2, \ldots, u_n) \cap \bigcap_{i=1}^{n} (D \setminus f(u_i)) \neq \emptyset,$$

which means that there is a $y_0 \in \mathrm{conv}(u_1, u_2, \ldots, u_n$ such that for all $i = 1, 2, \ldots, n$, $(u_i, y_0) \in E$, i.e., $u_i \in h'(y_0)$. Hence, since $h'(y_0)$ is convex, $y_0 \in h'(y_0)$ or $(y_0, y_0) \in E$ for $y_0 \in D$. This contradiction shows that f is a KKM-multifunction.

On the other hand, if $\bigcap \{f(v) \mid v \in D\} \neq \emptyset$, then there exists a $y' \in D$ such that $(v, y') \notin E$ for all $v \in D$ i.e., $h(y') = \emptyset$, and this contradiction implies the existence of $y_1 \in \bigcap \{f(v) \mid v \in D'\} \setminus D'$ by using Theorem 12 for D, f and D'. Hence, $h'(y_1) \cap D' = \emptyset$, which contradicts condition (ii) of the Theorem 13. This contradiction shows that the assumption $(u, u) \notin E$ for all $u \in D$ is not true and implies the existence of $u_0 \in D$ such that $u_0 \in h'(u_0)$. By the assumptions of the Theorem we have $u_0 \in C$ and $u_0 \in h'(u_0) \subset hg(u_0) = h(u_0)$, as was required.

COROLLARY 13.1. *Clearly, Theorem* 10 *in* [12] *is a special case of Theorem* 13. *From a modification of this theorem we obtain again Theorem* 9.

COROLLARY 13.2 (a direct generalization of Theorem 10 in [12]). *Let D be a nonempty convex subset of X and let $B \subset D \times D$ be such that*

(i) *for each $v \in D$, $b(v) = \{u \in D, (u, v) \in B\}$ is nonempty and convex,*

(ii) *for each $u \in b(D)$, $b^-(u)$ is relatively open in D,*

(iii) *there is a nonempty compact convex $D_0 \subset D$ such that $D_0 \cap b(v) \neq \emptyset$ for every $v \in D \setminus D_0$.*

Then there exists a $v_0 \in D$ such that $(v_0, v_0) \in B$.

Thus by choosing suitable pairs (D, g) we shall obtain fixed point theorems for strongly lower semicontinuous multifunctions under various boundary conditions. The following theorems are examples.

THEOREM 14. *Let C be a nonempty subset of X and let $h\colon C \to 2^X$ be a s.l.s.c. multifunction on C. Assume that there exists a nonempty convex subset D of X such that $\mathrm{int}_{L_D}(C \cap \bar{D}) \neq \emptyset$, $C \cap \bar{D}$, is convex, closed and such that*

(i) *for each $u \in C \cap \bar{D}$, $h(u)$ is convex and $h(u) \cap D \neq \emptyset$,*

(ii) $\overline{\mathrm{conv}}\,(h(C \cap \bar{D})) \cap \bar{D}$ *is compact,*

(iii) *there exists a continuous mapping $\varphi\colon \bar{D} \setminus C \to \mathrm{int}_{L_D}(C \cap \bar{D})$ such that $\overline{\varphi(\bar{D} \setminus C)}$ is a compact subset of $\mathrm{int}_{L_D}(C \cap \bar{D})$ and such that for every $x \in A$,*

$$h(x) \cap \{x + t(x - \varphi(u))|\ t > 0,\ u \in \bar{D} \setminus C \text{ such that } x \in [u, \varphi(u)]\} = \emptyset,$$

where

$$A = \{x \in \partial_{L_D}(C \cap \bar{D})|\ \text{there is a } u \in \bar{D} \setminus C \text{ such that } x \in [u, \varphi(u)]\}.$$

Then h has a fixed point in C.

Proof. We use Theorem 9 with the mapping defined analogously to the mapping used in the proof of Corollary 4.2. (Section 3.2. Part 3).

We can also obtain the following theorem by using Theorem 13:

DEFINITION 14. Let C be a nonempty subset of X. Two nonempty convex subsets D, D_1 of X are called *suitable* for C if D_1 is compact and if:

(i) $D_1 \subset D$, $C \cap \bar{D}$ is closed and convex and $\mathrm{int}_{L_D}(C \cap \bar{D}) \neq \emptyset$,

(ii) there exists a continuous mapping $\varphi\colon \bar{D} \setminus C \to \mathrm{int}_{L_D}(C \cap \bar{D})$ such that $\overline{\varphi(\bar{D} \setminus C)}$ is a compact subset of $\mathrm{int}_{L_D}(C \cap \bar{D})$.

For this pair we shall use the following subsets of D:

$$A_1 = \{x \in \partial_{L_D}(C \cap \bar{D})|\ \text{there is a } u \in \bar{D} \setminus C \text{ such that } x \in [u, \varphi(u)]\},$$

$$A_2 = \{x \in A_1|\ \text{there exists a } u \in \bar{D} \setminus (C \cup D_1) \text{ such that } x \in [u, \varphi(u)]\},$$

$$A_3 = (\bar{D} \setminus D_1) \cap \mathrm{int}_{L_D}(C \cap \bar{D}).$$

THEOREM 15. *Let C be a nonempty subset of X and let $h\colon C \to 2^X$ be a* *multifunction. If one can choose a pair of subsets of* X (D, D_1) *suitable for* C *such that*:

(i) *for every* $u \in C \cap \bar{D}$, h *is s.l.s.c. at* u, $h(u)$ *is convex and* $h(u) \cap D \neq \emptyset$,

(ii) *for every* $x \in A_2 \cup A_3$, $h(x) \cap D_1 \neq \emptyset$,

(iii) *for every* $x \in A_1$,

$$h(x) \cap \{x + t(x - \varphi(u)) \mid t > 0 \text{ and } u \in \bar{D} \backslash C \text{ such that } x \in [u, \varphi(u)]\} = \emptyset,$$

then h *has a fixed point in* C.

Remark 7. Many of the results obtained above can be generalized by using the following fact (which is called *an alternative theorem*):

Let D be a nonempty convex subset of a vector space Z and let $h\colon D \to 2^Z$ be a multifunction. A multifunction $h\colon D \to 2^Z$ has the *KKM-property* if $\operatorname{conv}\{u_1, \ldots, u_m\} \subset \bigcup \{h(u_i) \mid i = 1, 2, \ldots, m\}$ for any finite subset $\{u_1, u_2, \ldots, u_m\}$ of D. A multifunction $f\colon D \to 2^D$ associated with h is defined by $f(u) = D \backslash \{v \in D \mid u \in h(v)\}$ for each $u \in D$.

If h *is a multifunction with convex values (possibly empty) then either* h *has a fixed point in* D *or* f *has the KKM-property.*

Further results concerning this remark will be presented in another paper.

5. Some applications in game theory

It is not very difficult to show that the fixed point theorems obtained above be applied in the study of game theory, of economic equilibrium and some other fields.

We shall first give applications in game theory and a related problem.

5.1. A non-cooperative game model. We give a simple *non-cooperative game*. There are n agents numbered by $i \in I = \{1, 2, \ldots, n\}$. For each i, X_i is a nonempty set (action space of i-agent) and given a multifunction

$$\varphi_i\colon \underset{j \neq i}{X} X_j \to 2^{X_i} \backslash \{\emptyset\}.$$

We denote

$$X = \underset{i \in I}{X} X_i, \quad x = (x_1, x_2, \ldots, x_n) \in X,$$

and

$$(x \| x_i) = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

$$(x \| x_i, x_i') = (x_1, x_2, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n),$$

$$A_i = \operatorname{Graph} \varphi_i = \{(x \| x_i, x_i') \mid x_i' \in \varphi_i(x \| x_i),\ x \| x_i \in \underset{j \neq i}{X} X_j\}.$$

Let C be a nonempty subset of X (one can call C an *exgenous constraint* on the space of situations of the game. It is possible to interpret the set C as a 'budget constraint' in economic theory or the first constraint domain in mathematical programming).

DEFINITION 15. A point $x \in C \cap \bigcap \{A_i | \ i \in I\}$ is an *equilibrium situation* of a non-cooperative game.

We now consider some examples.

EXAMPLE 10. For each $i \in I$, let the payoff function of the i-agent f_i: $X \to R^1$ be a real function. Let ε be a positive number sufficiently small. Then we define the multifunction φ_i: $\underset{j \neq i}{X} X_j \to 2^{X_i}$ by

$$\varphi_i(x \| x_i) = \{x_i' \in X_i | \ f_i(x \| x_i, x_i') > \sup_{x_i \in X_i} f_i(x \| x_i, x_i') - \varepsilon\}.$$

Clearly, an equilibrium situation is a generalized equilibrium point of Nash in non-cooperative game (for example, see [4], [24]).

EXAMPLE 11. For each $i \in I$, let f_i: $X \to R^{n_i}$ be a multifunction (multipayoff $f_i(x)$ of the i-agent at situation x). Then we define the i-relation P_i on $X \times X$ as follows: for every pair $(x, x') \in X \times X$ $xP_i x'$ iff $f_{ij}(x) \geqslant f_{ij}(x')$ for all j and $f_{ij_0}(x) > f_{ij_0}(x')$ for some $j_0 (1 \leqslant j_0 \leqslant n_i)$. Hence one can use the multifunction φ_i: $\underset{j \neq i}{X} X_j \to 2^{X_i}$ defined by

$$\varphi_i(x \| x_i) = \left\{ \begin{matrix} u_i \in X_i\text{: there is not any } x_i' \in X_i \text{ such that} \\ (x \| x_i, x_i') P_i (x \| x_i, u_i) \end{matrix} \right\}$$

for each $(x \| x_i) \in \underset{j \neq i}{X} X_j$.

With these multifunctions each equilibrium situation is a generalized Nash–Pareto equilibrium point (see [5], [21]).

Thus, according to the multifunctions $\{\varphi_i | \ i \in I\}$ we can define

$$\varphi\colon X \to 2^X \text{ by } \varphi(x) = \underset{i \in I}{X} \varphi_i(x \| x_i) \quad \text{for each } x \in X.$$

Then by using Theorem 13 the following can be proved:

THEOREM 16. *For each* $i \in I$ *let* X_i *be a nonempty convex subset of a topological vector space* Z_i *and let* C *be a nonempty subset of* X. *Assume that there exists an l.s.c. multifunction* g: $X \to 2^C$ *such that* $g = \mathrm{id}$ *on* C *and such that*:

(i) *for every* $x \in X$, $\varphi(g(x))$ *is nonempty and convex, and* φ *is s.l.s.c. on* $g(x)$,

(ii) *there is a compact convex subset* D_1 *of* X *such that*

$$\varphi(g(x)) \cap D_1 \neq \emptyset \quad \textit{for each } x \in X \setminus D_1,$$

(iii) *for each* $x \in X \setminus C$, $x \notin \varphi(g(x))$.

Then the non-cooperative game has an equilibrium situation.

To establish the exsistence of a generalized Nash–Pareto equilibrium we can use multifunctions φ_i defined as follows:

Let us fix $i \in \{1, 2, \ldots, n\}$. Then P_i denotes the standard simplex of R^{n_i}, i.e., P_i is the set of vectors $p_i = (p_{ij}, j = 1, \ldots, n_i)$ such that $p_{ij} \geqslant 0$ and $\sum_j p_{ij} = 1$.

For any $p_i \in P_i$ and for any vector-value function f_i: $X \to R^{n_i}$ we define a single-value function $p_i f_i$ by putting

$$p_i f_i(x) = \sum_j p_{ij} f_{ij}(x)$$

for each $x \in X$; then a multifunction φ_i: $\underset{j \neq i}{\times} X_j \to 2^{X_i}$ is defined by

$$\varphi_i(x\|x_i) = \{x_i' \in X_i | \; p_i f_i(x\|x_i, x_i') > \sup_{x_i \in X_i} p_i f_i(x\|x_i, x_i') - \varepsilon\}$$

for any $(x\|x_i) \in \underset{j \neq i}{\times} X_j$, and as above the multifunction φ is defined by $\varphi(x) = \underset{i \in I}{\times} \varphi_i(x\|x_i)$ for each $x \in X$.

Let C_i denote the projection of the set C into the space $\underset{j \neq i}{\times} X_j$. We recall that a real-valued function h_i defined on a convex set X_i is quasi-concave if for any real number α the set $\{u \in X_i | \; h(u) < \alpha\}$ is convex.

THEOREM 17. *For each $i \in I$ let X_i be a nonempty convex subset of a topological vector space Z_i. Assume that C is a nonempty subset of X, that there exists a retraction r: $X \to C$ and that, for each $i \in I$, there is a $p_i \in P_i$ such that:*

(i) *for each fixed $x\|x_i \in C_i$, the function $p_i f_i(x\|x_i, \cdot)$ is bounded quasi--concave continuous on X_i,*

(ii) *there is a compact convex subset D of X such that, for each*

$$x \in X \setminus D, \quad \varphi(r(x)) \cap D \neq \emptyset,$$

(iii) *for each $x \in X \setminus C$, $x \notin \varphi(r(x))$.*

Then the non-cooperative game has a generalized Nash–Pareto equilibrium situation.

Clearly, the existence theorems of equilibrium situations of non--cooperative games given in [4], [5], [21] are corollaries to this theorem.

The following modification of the above model can be useful.

For each $i \in I$ let X_i be a nonempty set (action space of i-agent), $X = \underset{i \in I}{\times} X_i$ (set of situations), $C \subset X$ (an exgenous constraint)–a nonempty subset of X and φ_i: $C \to 2^{X_i}$ a multifunction (alternative multifunction of i-agent). Then a point $x \in C$ such that $x_i \in \varphi_i(x)$ for all $i \in I$ is an equilibrium situation of the non-cooperative game.

With the same argument we have the following

THEOREM 18. *For each* $i \in I$ *let* X_i *be a nonempty convex subset of a topological vector space* Z_i. *Assume that* C *is a nonempty subset of* X *and* φ_i *is a s.l.s.c. multifunction on* C *for each* $i \in I$. *If there exists an l.s.c. multifunction,* $g\colon X \to 2^C$ *is an l.s.c. multifunction such that* $g = \mathrm{id}$ *on* C *and such that*:

(i) *for every* $x \in X$, $\varphi_i(g(x))$ *is nonempty and convex for all* $i \in I$,

(ii) *there exists a nonempty compact convex subset* D *of* X *such that* $\big(\mathop{X}_{i \in I} \varphi_i(g(x))\big) \cap D \neq \emptyset$ *for each* $x \in X \setminus D$,

(iii) *for each* $x \in X \setminus C$, $x_i \notin \varphi_i(g(x))$ *for some* $i \in I$,

then the non-cooperative game has an equilibrium situation.

To choose suitable conditions for the pair (C, g) means to impose conditions on the alternative multifunctions of agents at the boundary of the exgenous constraint set, and hence we obtain the existence theorems under various boundary conditions of non-cooperative game theory.

5.2. A result in variation inequalities. Finnally, in this section we give a generalization of the existence theorem in variational inequalities (Theorem 8 in [12] and Theorem 1 in [1]).

THEOREM 19. *Let* D *be a nonempty convex subset of a Hausdorff topological vector space and let* C *be a nonempty arbitrary subset of* D. *Let* $f\colon D \times C \to R$ *be a real function such that, for each* $x \in D$, $f(x, \cdot)$ *is lower semicontinuous on* C *and, for each* $u \in C$, $f(\cdot, u)$ *is quasi-concave in* x *on* D. *Moreover, if there exists a retraction* g *from* D *onto* C *and a nonempty compact convex subset* D_1 *of* D *such that*:

(i) $f(x, g(x)) \leqslant 0$ *for all* $x \in D$,

(ii) *for each* $u \in D_1 \setminus C$, *there is an* $x \in D$ *such that*

$$f(x, g(u)) > 0,$$

(iii) *for each* $v \in D \setminus D_1$, *there is an* $x \in D_1$ *such that* $f(x, g(v)) > 0$,

then there exists a $u_0 \in C$ *such that* $f(x, u_0) \leqslant 0$ *for all* $x \in D$.

Proof. For each $x \in D$ let $h(x) = \{u \in D \mid f(x, g(u)) \leqslant 0\}$. Since $f(x, \cdot)$ is lower semicontinuous on C, $h(x)$ is relatively closed in D. Moreover, the multifunction $h\colon D \to 2^D$ is a KKM-multifunction. Indeed, suppose that there exists a finite subset $\{x_1, x_2, \ldots, x_m\}$ of D such that

$$\operatorname{conv}\{x_1, x_2, \ldots, x_m\} \setminus \bigcup_{i=1}^{m} h(x_i) \neq \emptyset, \text{ i.e.,}$$

there is a point $u \in \operatorname{conv}\{x_1, x_2, \ldots, x_m\}$ such that $f(x_i, g(u)) > 0$ for all $i = 1, 2, \ldots, m$, and hence, since $f(\cdot, g(u))$ is quasi-concave on D, we have $f(u, g(u)) > 0$, which is a contradiction.

Using condition (iii) and the argument used in the proof of Theorem 13, we have a point $u_0 \in D$ such that $u_0 \in \bigcap \{h(x)|\ x \in D\}$.However, by condition (ii), $u_0 \in C$ and thus $f(x, u_0) \leqslant 0$ for all $x \in D$.

Some results of this paper were announced without proof in [6].

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References

[1] G. Allen, *Variational inequalities, complementarity problems, and duality theorems*, J. Math. Anal. App. 58, 1977, 1–10.

[2] F. E. Browder, *The fixed point theory of multivalued mappings in topological vector spaces*, Math. Ann. 117, 1968, 283–301.

[3] – *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. 74, 1968, 660–665.

[4] Bũi Cong Cuõng, *Nash equilibrium point in n-persons games and in economics*, Mathematics, Inst. of Math., Hanoi, 1976, 10–16 (in Vietnamese).

[5] – *Some classes of games with multipayoffs*, Scientific Bulletin Sci. Resear. Center of Vietnam, Hanoi, 1981, No. 2, 1–7 (in Vietnamese).

[6] – *Some fixed point theorems for multifunctions in topological vector spaces* (announcement of results), Bull. Pol. Acad. Sci., Math. (to appear).

[7] J. Dungundji, A. Granas, *Fixed point theory*, Vol. I, Monografie Matematyczne, Polish Scientific Publishers, Warsaw 1982.

[8] R. Engelking, *General Topology*, Monografie Matematyczne, Tom 60, Polish Scientific Publishers, Warsaw 1977.

[9] Fan Ky, *A generalizataion of Tychonoff's fixed point theorem*, Math. Ann. 142, 1961, 305–310.

[10] – *Extensions of two fixed point theorems of F. E. Browder*, Math. Z. 112, 1969, 234–230.

[11] – *A minimax inequality and applications, Inequalities III*, O. Shioha, Ed., Academic Press, 1972, 103–113.

[12] – *Fixed point and related theorems for non-compact convex sets, Game Theory and Related Topics*, O. Moeschlin, D. Pallaschke, Eds., North-Holland Publ. Comp., 1979, 151–156.

[13] W. Forster, Ed., *Numerical Solution of Highly Nonlinear Problems*, North-Holland Pub. Comp., 1980.

[14] B. Knaster, K. Kuratowski and S. Mazurkiewicz, *Ein Beweis des Fispunktsatzes für n-dimensionale Simplexe*, Fund. Math. 14, 1929, 132–137.

[15] K. Kuratowski, *Topologie I et II*, Monografie Matematyczne, Polish Scientific Publishers, Warsaw 1958, 1961.

[16] V. Laan, A. J. J. Talman, *Convergence and properties of recent variable dimension algorithms*, in [13], W. Forster, Ed., 1980, 3–36.

[17] J. Leray, J. Schauder, *Topologie et equations fonctionelles*, Ann. Sci. Ecole Nor. Sup., Paris 51, 1934, 45–78.

[18] B. Halpern, *Fixed point theorems for set-valued mappings in infinite-dimensional space*, Math. Ann. 189, 87–98.

[19] Hoãng Tuy, *Solving equation* $0 \in f(x)$ *under general boundary conditions*, in [13], W. Forster, Ed., 1980, 271–298.

[20] – *On variable dimension algorithms and algorithms using primitive sets*, Math. Operation. Statist., Ser. Optimization, 12, 3, 1981, 361–381.

[21] L. S. Shapley, *Equilibrium points in games with vector payoffs*, Naval Res. Log. Quart. 6, 1, 57–62.

[22] R. T. Rockaffeller, *Convex analysis*, Princeton, New York 1969.

[23] N. Bourbaki, *Espaces vectoriels topologiques*, Chapitre 1 et 2 Deuxième edition, Hermann, Paris 1966.

[24] G. Owen, *Game Theory*, Philadelphia, Saunders 1968.

INSTITUTE OF COMPUTER SCIENCE, POLISH ACADEMY OF SCIENCES, POLAND
INSTITUTE OF MATHEMATICS, HANOI, S. R VIETNAM