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Dilations of Hilbert space operators
(General theory)

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CONTENTS

Introduction

The purpose of the present paper is to give a brief and unified pre-sentation of basic theorems of general dilation theory in Hilbert spaces. Our basic subjects are operator functions on involution semi-groups. Within these general frames we follow here the main ideas developed by Sz.-Nagy in [34] and suitably adapted concepts of Arveson [1], concerning minimality properties in absence of units, and intertwining operators.

After proving general existence theorems for dilations, which are due to Sz.-Nagy [34], Mlak–Szymański [18] and Arveson (unpublished), we derive all the other known dilation theorems for functions on special involution semi-groups, namely set fields, groups and several types of star·algebras.

There are given some applications of general dilations theorems to group representations and operator Schwarz inequalities.

Except the Naimark theorem [21] on spectral dilations of semi-spec-tral measures, we do not discuss here the questions, which concrete and natural conditions for sets of operators guarantee they have dilations of some form. Problems of this type will be discussed in detail in our next paper, on reasonably dilatable families of operators. The model theorem within this circle of ideas is the Sz.-Nagy theorem [32], which states that every contraction in a Hilbert space has a unitary power dilation. This theorem is basic for the Fourier analysis of Hilbert space operators developed in [36], with all its geometrical and analytic conse-quences. Involved herewith is the von Neumann inequality [23]

$$\|p(T)\| \leqslant \sup_{|z|=1} |p(z)|$$

which holds true for every contraction T in a Hilbert space and analy-tic polynomials $p(z)$. This inequality is a model theorem for the general theory of operator representations of unstarred algebras, the topic which we will discuss in a third paper on dilations of Hilbert space operators.

Let $\{T_a\}$ and $\{S_a\}$ be two families of operators (linear, bounded) in Hilbert spaces H and K respectively and let $R: H \to K$ be a linear bounded operator and R^* its adjoint. The family $\{S_a\}$ is called an *R-dilation of $\{T_a\}$* if $T_a = R^* S_a R$ for every index a.

Roughly speaking, the main idea of dilation theory is to characterize those $\{T_a\}$ which have some dilations $\{S_a\}$ with reasonable and nice properties, usually not shared by $\{T_a\}$. Given such dilations, one can hope to get some new information about the original $\{T_a\}$, when using these nice properties. We require that dilations be more "regular" than the dilated operators. The "regularity" of dilations will be expressed in terms of star representations of some algebraical structures with involution. More precisely, dilation should be a star representation, and the point is that it is not required that the dilated operators form such a representation.

We deal in this paper with necessary and sufficient conditions for operator functions to have dilations which are star representations, first for functions on involution semi-groups and later, with respect to special cases, for functions on groups, star algebras, etc. There are enclosed several simplifications and extensions of known results as well some new unpublished ones.

The recent development of dilation theory shows that it enters in an essential manner into several branches of operator theory in Hilbert spaces, in probability (prediction theory), the theory of electric networks, quantum field theory, the theory of several complex variables and others. The objective of the present paper is among others to give a theoretical background for understanding general dilation phenomena having in view their applications and interpretations in fields which we mentioned above.

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1. Notation and definitions

Let H be a complex Hilbert space. We denote by $f, g, h, \ldots$ the vectors of H. The inner product of f and g is (f, g), the norm $\|f\| = \sqrt{(f, f)}$. Given an other Hilbert space K, we denote by $L(H, K)$ the space of all linear bounded operators defined in H and having values in K. We write $L(H) = L(H, H)$.

The norm of $A \in L(H, K)$ is denoted by $\|A\|$, the adjoint of A by A^*. The restriction of A to $M \subset H$ is denoted by $A \mid M$. I_H is the identity operator in H. The range of $A \in L(H, K)$ is denoted by $\mathscr{R}(X)$, the null space $\{f: Af = 0\}$ by $\mathrm{Ker}(A)$.

Scalars are usually denoted by $a, \beta, \lambda, \ldots$

The orthogonal sum of Hilbert spaces H_a is denoted by $\bigoplus_a H_a$. The orthogonal sum of the sequence $H_1, H_2, \ldots$ of Hilbert spaces is denoted by $\bigoplus_{n/1}^{\infty} H_n$. Similar notation applies to orthogonal sums of operators.

The operator $P \in L(H)$ is called a *projection* if $P = P^2$, an orthogonal projection if additionally $P = P^*$.

By a subspace of a Hilbert space H we always mean a closed subspace. The set in H closed under forming linear combinations of its elements will be called a *manifold*. Subspaces are closed manifolds.

The orthogonal complement of the subspace $M \subset N \subset H$ (N — a subspace) is denoted by $N \ominus M$. If $H = N$, $M^\perp$ stands for $H \ominus M$.

The subspace $M \subset H$ reduces the operator $A \in L(H)$ if $AM \subset M$ and $A^* M \subset M$. This happens if and only if the orthogonal projection P_M commutes with A, that is $P_M A = A P_M$.

Suppose we are given a family $\{Z_a\}$ of subsets of the Hilbert space H. The smallest subspace which includes the union $\bigcup Z_a$ is denoted by $\bigvee Z_a$. If Z_a are subspaces, then $\bigwedge Z_a \overset{\mathrm{df}}{=} \bigcap_a Z_a$. We also use the notation $[Z]$ for the subspace spanned by the set $Z \subset H$.

If $\mathscr{F}$ is a family of operators in $L(H, K)$ and $Z \subset H$, then $[\mathscr{F} Z]$ stands for $\bigvee_{F \in \mathscr{F}} FZ$.

We write $\mathscr{F}^* = \{E: E = F^*, F \in \mathscr{F}\}$ for $\mathscr{F} \subset L(H, K)$. $\mathscr{F} \subset L(H)$ is called *symmetric* if $\mathscr{F} = \mathscr{F}^*$.

The commutant $\mathscr{F}'$ of $\mathscr{F} \subset L(H)$ is the totality of all operators which

commute with every operator in $\mathscr{F}$. The double commutant $\mathscr{F}''$ equals $(\mathscr{F}')'$. If $\mathscr{F}$ is symmetric, so is $\mathscr{F}'$. Every commutant is closed in the weak operator topology.

An operator algebra is called *symmetric* if it is a symmetric set of operators. A symmetric algebra of operators is called a *von Neumann algebra* if it is closed in the weak operator topology. A symmetric operator algebra is a von Neumann algebra if and only if it is closed in the strong operator topology.

2. Elementary properties of dilations

Let H, K be complex Hilbert spaces and let $R \in L(H, K)$. The operator $S \in L(K)$ is called an R-*dilation of* $T \in L(H)$ if

$$T = R^* S R,$$

that is

$$Tf = R^* S R f$$

for $f \in H$. We then write $T = \Delta_R S$ or $T = \Delta_R(S)$. T is called an R-*compression of* S.

Let $\mathscr{T} = \{T_a\}$ be a family of operators in H and let $\mathscr{S} = \{S_a\} \subset L(K)$ be a family of operators indexed by the same $a - s$ as are the elements of $\mathscr{T}$. We say that $\mathscr{S}$ is an R-*dilation of* $\mathscr{T}$ if $T_a = \Delta_R S_a$ for every a. The space H is called the *initial space*, the space K the *dilation space*.

It is an elementary exercise to prove the following properties:

(1) *If* $T = \Delta_{R_1} S_1$ *and* $S_1 = \Delta_{R_2} S_2$, *then* $T = \Delta_{R_2 R_1} S_2$.

(2) $\Delta_R(a_1 S_1 + a_2 S_2) = a_1 \Delta_R S_1 + a_2 \Delta_R S_2$.

(3) *If* $T = \Delta_R S$, *then* $T^* = \Delta_R S^*$,

(4) *If* $T_a = \Delta_{R_a} S_a$, $\sup \|R_a\| < +\infty$, $\sup \|S_a\| < +\infty$, *then* $\oplus T_a = \Delta_{\oplus R_a}(\oplus S_a)$.

(5) *If the net* $S_a \in L(K)$ *converges weakly (strongly, in operator norm) to* S *and* $T_a = \Delta_R S_a$, *then* T_a *converges weakly (strongly, in operator norm) to* $T = \Delta_R S$.

Suppose that $T = \Delta_R S$ and $R^* R = I_H$. Then R is an isometry. Now, interpreting R as an isometric embedding of H into K and identifying Rf with f, we treat H as a subspace of K. Then T is interpreted as an operator in $H \subset K$ and the adjoint R^* as the orthogonal projection P of K onto H. The equality $T = \Delta_R S$ may then be written in an equivalent form as

(6) $TP = PSP$

or explicitly

(7) $Tf = PSf, f \in H$.

We say in this case that T is a *projection* of S; S is called *simply a dilation of* T and we write $T = \operatorname{pr} S$. It is plain that (7) implies $T = R^* SR$, R being the identity embedding of H into K.

A similar terminology applies to families of operators T_a, S_a related by equalities $T_a = \Lambda_R S_a$ provided $R^* R = I_H$.

Notice that if $H \subset K$, $T \in L(H)$, $S \in L(K)$ and S is an extension of T, i.e., $SH \subset H$ and $Sf = Tf$ for $f \in H$, then $T = \operatorname{pr} S$ in a trivial way. This implies that $T^* = \operatorname{pr} S^*$ and it is more or less obvious that S^* need not be an extension of T^*.

It is not true in general that the operation Λ_R is multiplicative, i.e., that $\Lambda_R S_1 S_2 = \Lambda_R S_1 \cdot \Lambda_R S_2$, even if $R^*_? R = I_H$, that is when Λ_R reduces to taking projections of operators. Here is an example. We take the space H and $H_1 = H_2 = H$ and $K = H_1 \oplus H_2$. Let $S \in L(K)$ be the operator defined by the matrix

$$ S = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, $$

where $A, B, C \in L(H)$ and let P be the projection on the first coordinate space H_1, that is $P\{f_1, f_2\} = \{f_1, 0\}$ for $\{f_1, f_2\} \in K$ $(f_i \in H)$. We then have

$$ S\{f_1, f_2\} = \{Af_1 + Bf_2, Cf_1\} $$

which shows that

$$ PS\{f_1, 0\} = \{Af_1, 0\}. $$

On the other hand

$$ S^2\{f_1, f_2\} = \{A^2f_1 + ABf_2 + BCf_1, CAf_1 + CBf_2 + C^2f_1\}, $$

which proves that for some f_1

$$ PS^2\{f_1, 0\} = \{A^2f_1 + BCf_1, 0\} \neq PS^2\{f_1, 0\} = \{A^2f_1, 0\} $$

if $BC \neq 0$. This proves our claim.

Notes. Most of the material in this section follows the paper of Sz.-Nagy [34] and that of Stinespring [26], where the R-dilations have been introduced. The term "dilation" was introduced by Halmos in [8], the term "projection" (of an operator) by Sz.-Nagy in [32].

3. Semi-groups and their representations

Let $\mathscr{S}$ be a set and let u: $(a, b) \to ab$ be the mapping of $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}$. $\mathscr{S}$ together with this mapping is called a *semi-group* if

$$ (ab)c = a(bc) $$

for all $a, b, c \in \mathscr{S}$. In other words, the mapping u, which we call *multiplication*, is associative. The element $e \in \mathscr{S}$ of the semi-group $\mathscr{S}$ is called a *unit* if

$$ae = ea = a$$

for all $a \in \mathscr{S}$. It is obvious that $\mathscr{S}$ can have at most one unit.

The mapping $a \to a^*$ of the semi-group $\mathscr{S}$ into itself is called an *involution* if

$$(a^*)^* = a, \qquad (ab)^* = b^*a^*$$

for all $a, b \in \mathscr{S}$. The semi-group $\mathscr{S}$ with an involution is called an *involution semi-group* or simply a *-semi-group*. If e is a unit of the *-semi-group $\mathscr{S}$, then $ae^* = (ea^*)^* = a = (a^*e)^* = e^*a$ for all $a \in \mathscr{S}$, which proves that e^* is a unit. Consequently $e^* = e$.

If $\mathscr{S}$ is a comutative semi-group, i.e., $ab = ba$ for $a, b \in \mathscr{S}$, then the identity map in an involution. A commutative $\mathscr{S}$ with such an involution is called *hermitian*.

A few examples are now in order.

EXAMPLE 1. Let $\mathscr{F}$ be a family of subsets of the space X. If for σ_1, $\sigma_2 \in \mathscr{F}$ also $\sigma_1 \cap \sigma_2 \in \mathscr{F}$, then $\mathscr{F}$ becomes a semi-group with multiplication defined by the formula $\sigma_1 \sigma_2 = \sigma_1 \cap \sigma_2$. If $X \in \mathscr{F}$, then X is a unit in this semi-group. $\mathscr{F}$ becomes hermitian if involution is defined by formula $\sigma^* = \sigma$.

EXAMPLE 2. Every group is a semi-group with a unit the group multiplication being the semi-group operation. A group becomes an involution semi-group if involution is just taking the inverse — $a^* = a^{-1}$.

Suppose we are given the semi-group $\mathscr{S}$ and a Hilbert space K. The mapping $\pi \colon \mathscr{S} \to L(K)$ is called a *representation of $\mathscr{S}$* if

(1) $\pi(ab) = \pi(a)\pi(b)$ for $a, b \in \mathscr{S}$.

K is called the *representation space* and we say that π is a representation on K. Condition (1) means that π is a semi-group homomorphism of $\mathscr{S}$ into the semi-group of operators in K, the multiplication of operators being the composition of operators.

Two representations $\pi_i \colon \mathscr{S} \to L(K_i)$ $(i = 1, 2)$ of the semi-group $\mathscr{S}$ are called *unitarily equivalent*, if there is a unitary map $U \colon K_1 \to K_2$ such that $U\pi_1(a) = \pi_2(a) U$ for every $a \in \mathscr{S}$.

Let $\mathscr{S}$ be a *-semi-group. We say that the representation π is a *-representation (star representation)* if

(2) $\pi(a^*) = \pi(a)^*$ for $a \in \mathscr{S}$.

A *-representation is an involution preserving homomorphism, the involution in $L(K)$ being the operation of taking adjoints.

If the subspace $M \subset K$ is invariant under the representation π, that is $\pi(a) M \subset M$ for $a \in \mathscr{S}$, then $\pi_M(a) = \pi(a)| M$ is a representation on M. If M reduces π, i.e., P_M commutes with all $\pi(a)$ so does $K \ominus M$ and

$$\pi(a) = \pi_M(a) \oplus \pi_{K \ominus M}(a) \quad \text{for } a \in \mathscr{S},$$

which means that π is an orthogonal sum of its parts in M and $M^\perp$. If $\mathscr{S}$ is a *-semi-group and π a *-representation, then π_M and $\pi_{M^\perp}$ are *-representations.

Suppose $\pi_\eta\colon \mathscr{S} \to L(K_\eta)$ is a family of representations and $\sup_\eta \|\pi_\eta(a)\| < +\infty$ for every $a \in \mathscr{S}$. Then the orthogonal sum

$$\pi(a) = \bigoplus_\eta \pi_\eta(a)$$

is a representation in $\oplus K_\eta$. If $\mathscr{S}$ is a *-semi-group and π_η are *-representation, then π is a *-representation. π is called the *orthogonal sum of* π_η and we denote it by $\oplus \pi_\eta$.

PROPOSITION 1. *Let* $\mathscr{S}$ *be a* *-semi-group *and* $\varphi\colon \mathscr{S} \to L(H)$ *an operator function such that* $\varphi(a^*) = \varphi(a)^*$ *for every* $a \in \mathscr{S}$. *Then* H *can be decomposed in a unique way as* $H = H_r \oplus H_p$, *where* H_r, H_p *satisfy the following conditions*:

(a) H_r *and* H_p *reduce every operator* $\varphi(a)$ $(a \in \mathscr{S})$.

(b) *The part* φ_r *of* φ *in* H_r *is a* *-representation *of* $\mathscr{S}$.

(c) *There is no non-zero subspace of* H_p *which reduces all* $\varphi(a)$ *in such a way that the part* φ *in this subspace is a* *-representation.

Proof. We define

$$H_r = \{f \in H\colon \varphi(ab)f = \varphi(a)\varphi(b)f \text{ for every } a, b \in \mathscr{S}\}.$$

Plainly H_r is a subspace. Suppose $f \in H_r$ and $c \in \mathscr{S}$. Let $g = \varphi(c)f$. We wish to show that $g \in H_r$. Since $f \in H_r$, we have $\varphi(ab)g = \varphi(ab)\varphi(c)f = \varphi((ab)c)f = \varphi(a(bc))f = \varphi(a)\varphi(bc)f = \varphi(a)\varphi(b)\varphi(c)f = \varphi(a)\varphi(b)g$. Hence H_r is invariant under $\varphi(c)$, and replacing c by c^* we conclude that H_r reduces φ, because $\varphi(c^*) = \varphi(c)^*$. It is clear that the part φ_r of φ in H_r is a *-representation in H_r. Now, if $H_p \overset{\text{df}}{=} H \ominus H_r$ and $f \in M$, where M reduces φ to a *-representation, then $\varphi(a\,b)f = (\varphi(a\,b)| M)f = (\varphi(a)| M) (\varphi(b)| M)f = \varphi(a)\varphi(b)f$, which proves that $f \in H_r$ and consequently $f \perp f$, that is $f = 0$. This completes the proof of (a), (b) and (c). The uniqueness of the decomposition $H = H_r \oplus H_p$ follows from the property that H_r is the largest subspace reducing φ to a *-representation.

The decomposition described in Proposition 1 is called the *canonical decomposition* of φ. The part φ_r is called the $\mathscr{S}$-part of φ and the part φ_p — the purely non-$\mathscr{S}$-part of φ; similarly, the part H_r is called the $\mathscr{S}$-part of H and the part H_p — the *purely non-$\mathscr{S}$-part of* H.

If $\pi\colon \mathscr{S} \to L(K)$ is a representation of the semi-group $\mathscr{S}$ and e is the unit of $\mathscr{S}$, then $\pi(e)$ is a projection and the subspace $\pi(e)K = K_e$ is invariant under π. The complementary subspace $(I_K - \pi(e))K$ anihilates all $\pi(a)$ and $\pi_{K_e}(e) = I_{K_e}$. It follows that π is a direct sum of a zero representation and a representation which maps e onto the idehtity operator. A representation which shares this last property will be called *unital*. Notice that if $\mathscr{S}$ is a $*$-semi-group and π is a $*$-representation, then $\pi(e)$ is an orthogonal projection which implies that π is an orthogonal sum of a zero representation and a unital one.

Going back to Example 1, we see that if $\mathscr{F}$ is considered as a semi-group, then a representation π of $\mathscr{F}$ is just a function which satisfies the condition

$$\pi(\sigma_1 \cap \sigma_2) = \pi(\sigma_1)\pi(\sigma_2)$$

for σ_1, $\sigma_2 \in \mathscr{F}$. Consequently $\pi(\sigma)$ is a projection for every σ. If $\mathscr{F}$ is treated as a hermitian $*$-semi-group, then $\pi(\sigma^*) = \pi(\sigma)^* = \pi(\sigma)$ which shows that the values of π are an orthogonal projection; π is a $*$-representation.

If G is a group treated as a $*$-semi-group as in Example 2, then π is a unital $*$-representation of G if and only if π is a unitary representation of G. Indeed, since $\pi(e) = I_K$, we have $I_K = \pi(aa^{-1}) = \pi(aa^*) = \pi(a)\pi(a)^*$ for all $a \in G$ which proves the claim.

If $\mathscr{S}$ is a semi-group and $\varphi(a) = \Delta_R\pi(a)$ for every $a \in \mathscr{S}$, then we write shortly $\varphi = \Delta_R\pi$. If $\varphi(a) = \mathrm{pr}\,\pi(a)$ for $a \in \mathscr{S}$, then we write simply $\varphi = \mathrm{pr}\,\pi$.

PROPOSITION 2. *Let $\mathscr{S}$ be a $*$-semi-group and $\varphi\colon \mathscr{S} \to L(H)$ an operator function such that $\varphi = \mathrm{pr}\,\pi$, where $\pi\colon \mathscr{S} \to L(K)$ $(H \subset K)$ is a $*$-representation of $\mathscr{S}$. Then the set $\mathscr{S}' = \{a \in \mathscr{S}\colon \varphi(a^*a) = \varphi(a)^*\varphi(a)\}$ is a subsemigroup of $\mathscr{S}$ and $\varphi(a)f = \pi(a)f$ for $f \in H$, $a \in \mathscr{S}'$. Consequently, φ restricted to $\mathscr{S}'$ is a representation of $\mathscr{S}'$. Moreover, $\varphi(ba) = \varphi(b)\varphi(a)$ for all $b \in \mathscr{S}$ and all $a \in \mathscr{S}'$. If φ is itself a $*$-representation, then $\varphi(a)f = \pi(a)f$ for $a \in \mathscr{S}$ and $f \in H$.*

Proof. If $a \in \mathscr{S}'$ and $f \in H$, then $\|\varphi(a)f - \pi(a)f\|^2 = (\varphi(a)^*\varphi(a)f, f) + (\pi(a^*a)f, Pf) - 2\,\mathrm{Re}\,(\varphi(a)f, P\pi(a)f) = 2\,(\varphi(a^*a)f, f) - 2\,\mathrm{Re}\,(\varphi(a)^*\varphi(a)f, f) = 0$.

Suppose that $a \in \mathscr{S}'$, $b \in \mathscr{S}$ and $f \in H$. Then, since $\varphi(a)f = \pi(a)f$, we have $\varphi(b)\varphi(a)f = P\pi(ba)f$ and consequently $\|\varphi(b)\varphi(a)f - \varphi(ba)f\|^2 = 0$.

To prove that $\mathscr{S}'$ is a subsemi-group we take $a, b \in \mathscr{S}'$ and then, by the previous property, we successively obtain

$$\varphi((ab)^*ab) = \varphi(((ab)^*a)b) = \varphi((ab)^*a)\varphi(b) = \varphi((ab)^*)\varphi(a) \upharpoonright (b)$$
$$= \varphi(ab)^*\varphi(ab).$$

q.e.d. It follows that $\varphi|\mathscr{S}'$ is a representation.

Now if φ is a *-representation, then $\mathscr{S}'^* = \mathscr{S}' = \mathscr{S}$ and consequently $\varphi(a)f = \pi(a)f$ for all $a \in \mathscr{S}$, $f \in H$.

Notes. The *-semi-groups and their *-representations have been introduced by Sz.-Nagy [34]. Proposition 2 is due to Paschke [25]; see also Mlak–Ryll-Nardzewski [16]. Proposition 2 under weaker assumptions for linear maps on C^*-algebras is discussed by Choi — see [3], [28].

4. Minimality and uniqueness of dilations

Let K be a complex Hilbert space and $\mathscr{F} \subset L(K)$ a symmetric family of operators. The null space of $\mathscr{F}$ is the subspace of all $f \in K$ such that $Sf = 0$ for every $S \in \mathscr{F}$. Denote by $E_{\mathscr{F}}$ the orthogonal projection on the orthogonal complement of the null space of $\mathscr{F}$. Then

(a) $E_{\mathscr{F}}S = SE_{\mathscr{F}} = S$ for every $S \in \mathscr{F}$,

and the von Neumann double commutant theorem says that:

(b) The von Neumann algebra generated by $\mathscr{F}$ equals $\{A \in \mathscr{F}'' : AE_{\mathscr{F}} = E_{\mathscr{F}}A = A\}$ and $E_{\mathscr{F}}$ belongs to this algebra.

If $\mathscr{F}$ is a symmetric algebra, then the von Neumann algebra generated by $\mathscr{F}$ equals its closure in the strong operator topology.

If $\mathscr{F}$ is symmetric and $K = [\mathscr{F}M]$ for some subspace $M \subset K$, then $\mathscr{F}$ has a trivial null space. Indeed, if $Sf = 0$ for $S \in \mathscr{F}$, then $f \perp S\,M$ for such S, which implies that $f = 0$. Next, the equality $K = [\mathscr{F}M]$ implies that if the subspace $N \subset K \ominus M$ reduces $\mathscr{F}$, then $N = \{0\}$. To see this, note that $SN \subset N \perp M$ implies $N \perp SM$; S being an arbitrary element of $\mathscr{F}$, we get the assertion. The last property we have proved may be formulated as follows:

(1) If $K = [\mathscr{F}M]$ $(\mathscr{F} = \mathscr{F}^*!)$ and Q is an orthogonal projection in $\mathscr{F}'$ such that $Q \leqslant I_K - P_M$, then $Q = 0$.

The family $\mathscr{S} \subset L(K)$ is called a *-semi-group of operators if $\mathscr{S} = \mathscr{S}^*$ and $S_1, S_2 \in \mathscr{S}$ implies $S_1 S_2 \in \mathscr{S}$.

LEMMA 1. If $\mathscr{S} \subset L(K)$ is a *-semi-group of operators, M is a subspace of K and the only $\mathscr{S}$-reducing subspace of $K \ominus M$ is the zero subspace, then $E_{\mathscr{S}}K = [\mathscr{S}E_{\mathscr{S}}M]$.

Proof. Since $E_{\mathscr{S}}S = SE_{\mathscr{S}}$ for $S \in \mathscr{S}$, we have $[\mathscr{S}E_{\mathscr{S}}M] \subset E_{\mathscr{S}}K$. If $E_{\mathscr{S}}f = f \perp [\mathscr{S}E_{\mathscr{S}}M]$, then, by the symmetry of $\mathscr{S}$, $[\mathscr{S}f] \perp M$. But $\mathscr{S}$ is a *-semi-group of operators. Hence $[\mathscr{S}f]$ reduces $\mathscr{S}$, which by our assumption proves that $[\mathscr{S}f] = 0$. This means that f is in the null space of $\mathscr{S}$ and consequently $f \perp E_{\mathscr{S}}f = f$, that is $f = 0$. Notice that by (b) $E_{\mathscr{S}} = I_K$ if and only if $\mathscr{S}$ has a trivial null space.

Summing everything up, we get the following

LEMMA 2. *Let M be a subspace of K and let $\mathscr{S} \subset L(K)$ be a $*$-semigroup of operators. Then the following conditions are equivalent:*

(2) $K = [\mathscr{S}M]$.

(3) $\mathscr{S}$ *has a trivial null space and the only $\mathscr{S}$-reducing subspace of $K \ominus M$ is the zero subspace.*

Suppose we are given the Hilbert spaces H, K, an operator $R \in L(H, K)$, and the families $Q = \{Q_\eta\} \subset L(H)$, $\mathscr{F} = \{F_\eta\} \subset L(K)$. Let $\mathscr{F}$ be an R-dilation of Q, i.e., $Q_\eta = \varDelta_R(F_\eta)$ for every η. The dilation $\mathscr{F}$ is called *minimal* if $K = [\mathscr{F}RH]$ and K is then called a *minimal dilation space*.

In general we can enlarge both $\mathscr{F}$ and Q so as to have to do with dilations of $*$-semi-groups of operators. Namely we take, the $*$-semi-group $\mathscr{S}$ generated by $\mathscr{F}$. $\mathscr{S}$ is the collection of all finite products of operators from $\mathscr{F} \cup \mathscr{F}^*$. Next we take operators of the form $T = \varDelta_R(S)$ with S varying over $\mathscr{S}$. The totality $\mathscr{T}$ of all such T has $\mathscr{S}$ as an R-dilation and is a symmetric family of operators in $L(H)$. The space K can be cut down to the $\mathscr{S}$-reducing subspace $E_{\mathscr{S}}K$, the operators of $\mathscr{T}$ remaining unchanged. More precisely, the obvious formulae $(\varDelta_R(S)f, g) = (SE_{\mathscr{S}}Rf, E_{\mathscr{S}}Rg)$ $(f, g \in H$, $S \in \mathscr{S})$, show that $\varDelta_R S = \varDelta_{\tilde{R}}\tilde{S}$, where $\tilde{S} = S|E_{\mathscr{S}}K$, $\tilde{R} = E_{\mathscr{S}}R$. Then the dilation space $E_{\mathscr{S}}[\mathscr{S}RH] = [\mathscr{S}E_{\mathscr{S}}RH] = \bigvee_{S \in \mathscr{S}} \tilde{S}\tilde{R}H$ is minimal and $\tilde{\mathscr{S}}$ $= \{\tilde{S} : S \in \mathscr{S}\}$ is a minimal $\tilde{R}$-dilation of $\mathscr{T}$. Since the operation $\varDelta_{\tilde{R}}$ is linear and continuous in the weak operator topology, we can enlarge the $*$-semi-group $\tilde{\mathscr{S}}$ to the von Neumann algebra which it generates. This algebra is generated by operators $F_\eta|E_{\mathscr{S}}K$. It is a minimal R-dilation of a symmetric subspace of $L(H)$, and has a trivial null space. Hence $I_{E_{\mathscr{S}}K}$ is in the algebra.

If $\mathscr{S}_1$, $\mathscr{S}_2$ are $*$-semi-groups of operators in $L(K_1)$ and $L(K_2)$ respectively, then $\mathscr{S}_2$ is $*$-morphic to $\mathscr{S}$, as a $*$-semi-group if there is a mapping of $\mathscr{S}_1$ onto $\mathscr{S}_2$ which preserves products and involution. The double commutant theorem and such $*$-morphisms are involved in the next theorem.

THEOREM 1. *Let H, K_1, K_2 be Hilbert spaces and let $R_i \in L(H, K_i)$. Suppose that $\mathscr{S}_i \subset L(K_i)$ $(i = 1, 2)$ are $*$-semi-groups of operators such that $\mathscr{S}_2$ is $*$-morphic to $\mathscr{S}_1$ and $K_i = [\mathscr{S}_i R_i H]$ for $i = 1, 2$. We assume that for $S_1 \in \mathscr{S}_1$ and $S_2 \in \mathscr{S}_2$, which correspond through the $*$-morphism the equality $\varDelta_{R_1}(S_1) = \varDelta_{R_2}(S_2)$ holds true. Then there is a unitary isomorphism U such that:*

(4) $\qquad\qquad US_1 = S_2 U \quad$ *for S_2 corresponding to S_1,*

and

(5) $\qquad\qquad\qquad\qquad UR_1 = R_2.$

Proof. We write $\mathscr{S}_1 \ni S_1 \to S_2 \in \mathscr{S}_2$ for S-es corresponding through the $*$-morphism. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in H$ and suppose that

$$\mathscr{S}_1 \ni S_1^{(i)} \to S_2^{(i)} \in \mathscr{S}_2, \quad i = 1, \ldots, n,$$

$$\mathscr{S}_1 \ni Q_1^{(j)} \to Q_2^{(j)} \in \mathscr{S}_2, \quad j = 1, \ldots, m.$$

Then

$$Q_1^{(j)*} S_1^{(i)} \to Q_2^{(j)*} S_2^{(i)}$$

which implies that

$$\varDelta_{R_1}(Q_1^{(j)*} S_1^{(i)}) = \varDelta_{R_2}(Q_2^{(j)*} S_2^{(i)}).$$

It follows that

$$\left(\sum_i S_1^{(i)} R_1 f_i, \sum_j Q_1^{(j)} R_1 g_j \right) = \sum_{i,j} (R_1^* Q_1^{(j)*} S_1^{(i)} R_1 f_i, g_j)$$

$$= \sum_{i,j} (R_2^* Q_2^{(j)*} S_2^{(i)} R_2 f_i, g_j) = \left(\sum_i S_2^{(i)} R_2 f_i, \sum_j Q_2^{(j)} R_2 g_j \right).$$

Since $K_i = [\mathscr{S}_i R_i H]$, the correspondence

$$(6) \qquad U: \sum_i S_1^{(i)} R_1 f_i \to \sum_i S_2^{(i)} R_2 f_i$$

extends by the preceding equalities to a unitary isomorphism of K_1 and K_2. If $\mathscr{S}_1 \ni S_1 \to S_2 \in \mathscr{S}_2$, then $S_1 S_1^{(i)} \to S_2 S_2^{(i)}$ and $U S_1 \sum_i S_1^{(i)} R_1 f_i$

$= U \sum_i S_1 S_1^{(i)} R_1 f_i = \sum_i S_2 S_2^{(i)} R_2 f_i = S_2 \sum_i S_2^{(i)} R_2 f_i = S_2 U \sum_i S_1^{(i)} R_1 f_i$, which proves (4).

To prove (5) we proceed as follows: since $K_1 = [\mathscr{S}_1 R_1 H]$ and $\mathscr{S}_1$ is a $*$-semi-group of operators, by Lemma 2 $\mathscr{S}_1$ has a trivial null space. By the double commutant theorem the identity operator I_{K_1} is in the von Neumann algebra generated by linear combinations of operators from $\mathscr{S}_1$. Since the set of such combinations is a symmetric algebra $\mathscr{A}_1$, this von Neumann algebra is just the closure of $\mathscr{A}_1$ in the strong operator topology. Hence, given $\varepsilon > 0$ and $f \in H$ there is an $A \in \mathscr{A}_1$ such that

$$\|UAR_1 f - UR_1 f\| = \|AR_1 f - R_1 f\| < \varepsilon/2$$

and

$$\|UAU^* R_2 f - R_2 f\| = \|AU^* R_2 f - U^* R_2 f\| < \varepsilon/2.$$

Since by (4) and (6) $UAR_1 f = UAU^* R_2 f$, we conclude that $\|UR_1 f - R_2 f\| < \varepsilon$, which completes the proof of (5).

Suppose we are given a family $\mathscr{T} = \{T_\eta\} \subset L(H)$ and two families $\mathscr{S}_i = \{S_\eta^{(i)}\} \subset L(K_i)$ $(i = 1, 2)$, indexed by the same $\eta - s$ as the elements of $\mathscr{T}$ are. Let $\mathscr{S}_i$ be an R_i-dilation of $\mathscr{T}$, that is $T_\eta = \varDelta_{R_i}(S_\eta^{(i)})$, for every

η, $i = 1, 2$. We say that the dilations $\mathscr{S}_1$ and $\mathscr{S}_2$ are *unitarily isomorphic* under the correspondence $S_\eta^{(1)} \to S_\eta^{(2)}$ if there is a unitary map $U: K_1 \to K_2$ and $US_\eta^{(1)} = S_\eta^{(2)} U$ for every η. If additionaly $\mathscr{S}_1$ and $\mathscr{S}_2$ are *symmetric* and the correspondence $S_\eta^{(1)} \to S_\eta^{(2)}$ preserves adjoints, i.e., $S_\eta^{(1)*} \to S_\eta^{(2)*}$ if $S_\eta^{(1)} \to S_\eta^{(2)}$, then $\mathscr{S}_1$ and $\mathscr{S}_2$ are called *unitarily *-isomorphic* or simply equivalent with respect to the mapping "$\to$". Roughly speaking, Theorem 1 states that minimal dilations which are *-semi-groups of operators are unitarily *-isomorphic with respect to their *-morphism.

If $\mathscr{S}$ is a *-semi-group and π_1, π_2 are two *-representations of $\mathscr{S}$, then we have a *-morphism for $\mathscr{S}_i = \pi_i(\mathscr{S})$, namely $\pi_1(a) \to \pi_2(a)$ $(a \in \mathscr{S})$. We call this correspondence a natural one and get an equivalent formulation of Theorem 1 in terms of *-representations.

THEOREM 2. *Let* π_i: $\mathscr{S} \to L(K_i)$ $(i = 1, 2)$ *be a *-representation of the *-semi-group* $\mathscr{S}$ *and let* $R_i \in L(H, K_i)$ $(i = 1, 2)$. *Suppose* φ: $\mathscr{S} \to L(H)$ *and* $\Delta_{R_1} \pi_1(a) = \varphi(a) = \Delta_{R_2} \pi_2(a)$ *for* $a \in \mathscr{S}$. *If* π_1, π_2 *are minimal dilations* [1], *then they are unitarily *-isomorphic with respect to the natural correspondence.*

We can now say that an operator function on a *-semi-group can have at most one, up to a unitary *-isomorphism, minimal dilation which is a *-representation of the *-semi-group in question.

The proof of Theorem 2 is elementary and does not require the use of double commutant theorem if the involved *-semi-group $\mathscr{S}$ has a unit e. For if this is the case and π_1, π_2 are minimal, and hence unital *-representations such that $\Delta_{R_1} \pi_1(a) = \Delta_{R_2} \pi_2(a)$ for $a \in \mathscr{S}$, then the mapping

$$\tilde{U}: \quad \sum_i \pi_1(a_i) R_1 f_i \to \sum_i \pi_2(a_i) R_2 f_i$$

extends to a unitary map U of $K_1 = [\pi_1(\mathscr{S}) R_1 H]$ onto $K_2 = [\pi_2(\mathscr{S}) R_2 H]$ which sends $\pi_1(e) R_1 f = R_1 f$ onto $R_2 f$ for f varying over the initial space. Consequently $UR_1 = R_2$. The equality $U\pi_1(a) = \pi_2(a) U$ $(a \in \mathscr{S})$ is obvious.

Let φ: $\mathscr{S} \to L(H)$ be an operator function. If $\varphi(a) = \Delta_R \pi(a)$ $(a \in \mathscr{S})$, where π is a *-representation of the *-semi-group $\mathscr{S}$, then we say that φ is *R-dilatable* to π. If π is minimal, then the expression $\varphi(a) = \Delta_R \pi(a)$, usually written shortly as $\varphi = \Delta_R \pi$ will be called a *canonical form of* φ. It follows from Theorem 2 that π in the canonical form is unique up to a unitary *-isomorphism. Hence, when identifying unitarily *-isomorphic $\pi - s$, if φ has a canonical expression, this expression is unique.

Suppose now that $\varphi = \mathrm{pr}\,\pi_i$ that is $\varphi(a) f = P_i \pi_i(a) f$ for $f \in H$ and $a \in \mathscr{S}$, and $H \subset K_i = [\pi_i(\mathscr{S}) H]$, P_i being the orthogonal projection of

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[1] I.e., $K_i = [\pi_i(\mathscr{S}) R_i H]$.

K_i onto H. This means that we identify $R_1 f = f = R_2 f$. The unitary isomorphism U of Theorem 1 leaves then the vectors $f \in H$ invariant because $U R_1 = R_2$. The suitable version of Theorem 2 reads now as follows:

THEOREM 3. *Let* $\pi_i \colon \mathscr{S} \to L(K_i)$ $(i = 1, 2)$ *be* *$*$-representations of the* *$*$-semi-group $\mathscr{S}$. If $\varphi \colon \mathscr{S} \to L(H)$, $H \subset K_i$ $(i = 1, 2)$ and $\mathrm{pr}\,\pi_1 = \varphi = \mathrm{pr}\,\pi_2$ and $K_i = [\pi_i(\mathscr{S})H]$ $(i = 1, 2)$, then there is a unitary map $U \colon K_1 \to K_2$ such that $U\pi_1(a) = \pi_2(a)U$ for $a \in \mathscr{S}$ and $Uf = f$ for $f \in H$.*

In connection with Theorem 3 we notice what follows: if $\varphi = \mathrm{pr}\,\pi_1$, $H \subset K_1 = [\pi_1(\mathscr{S})H]$ and $\varphi = \varDelta_{R_2}\pi_2$, $K_2 = [\pi_2(\mathscr{S})R_2 H]$, then by Theorem 2 there is a unitary map $U \colon K_1 \to K_2$ such that $Uf = R_2 f$ for $f \in H$ and $U\pi_1 = \pi_2 U$. Consequently R_2 is an isometric operator and when identifying $R_2 f$ with f $(f \in H)$ we arrive at the formula $\varphi = \mathrm{pr}\,\pi_2$ and just have the assumptions of Theorem 3.

Notes. The minimality condition and the suitable uniqueness properties appear first in Naimark [20] and Halmos [8]. For von Neumann algebras we refer here to Topping [38] and Naimark [22]. Theorem 1 and its consequences and especially the use of double commutant theorem is essentialy due to Arveson [1]. Theorem 1 and 2 is simply an application of his method in the general $*$-semi-group setting — see Mlak–Szymański [18].

5. Positive definite operator valued functions

Let H be a complex Hilbert space and let X be an arbitrary set. Suppose we are given an operator function $C(\cdot, \cdot) \colon X \times X \to L(H)$. We say that C is *n-positive definite* $(n = 1, 2, 3, \ldots)$ if for every sequence $f_1, \ldots, f_n \in H$ and for arbitrary $x_1, \ldots, x_n \in X$ the following inequality holds true:

(1)
$$\sum_{i,k/1}^{n} \left(C(x_k, x_i)f_i, f_k \right) \geqslant 0.$$

Define $\mathscr{F}_n(X, H)$ as the set of all functions $f(\cdot) \colon X \to H$ such that $f(x) \neq 0$ for at most n elements $x \in X$. Then C is n-positive definite if and only if

(2)
$$\sum_{x,y} \left(C(x, y)f(y), f(x) \right) \geqslant 0$$

for every $f(\cdot) \in \mathscr{F}_n(X, H)$.

We write $C \gg_n 0$ if C is n-positive definite.

It is plain that $C \gg_n 0$ if and only if for every sequence $x_1, \ldots, x_n \in X$ the operator matrix

$$\begin{pmatrix} C(x_1, x_1) & C(x_1, x_2) & \ldots & C(x_1, x_n) \\ C(x_2, x_1) & C(x_2, x_2) & \ldots & C(x_2, x_n) \\ \vdots & \vdots & & \vdots \\ C(x_n, x_1) & C(x_n, x_2) & \ldots & C(x_n, x_n) \end{pmatrix}$$

defines on $\underbrace{H \oplus H \oplus \ldots \oplus H}_{n \text{ times}}$ a positive operator.

We say that $C \colon X \times X \to L(H)$ is *weakly n-positive definite* if for every $f \in H$, every sequence $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in C^1$ the inequality

$$(3) \qquad \sum_{i,k/1}^{n} (C(x_i, x_k)f, f)\lambda_i \bar{\lambda}_k \geqslant 0$$

holds true.

The function $C \colon X \times X \to L(H)$ is called *positive definite* (*weakly positive definite*) if it is n-positive (weakly n-positive) definite for every $n = 1, 2, 3, \ldots$

It is plain that if $C \gg_n 0$, then C is weakly n-positive definite. Define now $\mathscr{F}(X, H) = \bigcup_{n/1}^{\infty} \mathscr{F}_n(X, H)$. Then C is positive definite if and only if

$$\sum_{x,y} (C(x, y)f(y), f(x)) \geqslant 0$$

for every $f(\cdot) \in \mathscr{F}(X, H)$. We write $C \gg 0$ if C is positive definite.

If $C \gg 0$, then the bilinear functional

$$\langle f(\cdot), g(\cdot) \rangle = \sum_{x,y} (C(x, y)f(y), g(x))$$

is a semi-inner product on the linear space $\mathscr{F}(X, H)$. The Schwarz inequality for this product reads as follows:

$$(4) \qquad \left| \sum_{i,k/1}^{n} (C(x_i, x_k)f_k, g_i) \right|^2 \leqslant \sum_{i,k/1}^{n} (C(x_i, x_k)f_k, f_i) \cdot \sum_{i,k/1}^{n} (C(x_i, x_k)g_k, g_i).$$

A few direct consequences of previous definitions are now in order. First, C is weakly 1-positive definite if and only if $C(x, x)$ is a positive operator for every $x \in X$.

If C is weakly 2-positive definite, then the corresponding matrix in (3) is hermitian symmetric, which proves that $(C(x_1, x_2)f, f) = \overline{(C(x_2, x_1)f, f)}$ and consequently

$$(5) \qquad C(x, y) = C(y, x)^* \quad \text{for } x, y \in X.$$

Since the determinant of this matrix is non-negative, we conclude that for $x, y \in X, f \in H$

(6) $$|(C(x, y)f, f)|^2 \leqslant (C(x, x)f, f)(C(y, y)f, f).$$

Suppose now that C is 2-positive definite. Then taking in (1) $x_1 = x$, $x_2 = y$, $f_1 = f$, $f_2 = g$, we infer by the positivity of the corresponding determinant that

(7) $$|C(x, y)g, f)|^2 \leqslant (C(x, x)f, f)(C(y, y)g, g).$$

If $C \gg_3 0$, then replacing f_i by $\lambda_i f_i$ $(\lambda_i \in C^1)$; $i = 1, 2, 3$ in (1) and taking $\lambda_1 = -\lambda_2$, we infer by elementary computations that

(8) $$|(C(x_1, x_3)f_3, f_1) - (C(x_2, x_3)f_3, f_2)|^2$$

$$\leqslant (C(x_3, x_3)f_3, f_3)\{(C(x_1, x_1)f_1, f_1) + (C(x_2, x_2)f_2, f_2) - 2\,\mathrm{Re}\,(C(x_1, x_2)f_2, f_1)\}.$$

If X is a topological space, then (8) can be used to prove some continuity properties.

Suppose that $C(\cdot, \cdot)\colon X \times X \to L(H)$ is positive definite. Let us fix $f_1, \ldots, f_n \in H$ and consider the matrix

$$\tilde{C}(x, y) = \{(C(x, y)f_k, f_i)\}_{i,k=1,\ldots,n}.$$

This matrix will be treated as an operator on the Hilbert space C^n with the usual inner product. We have thus to do with an operator function $\tilde{C}(\cdot, \cdot)\colon X \times X \to L(C^n)$. Since $C(\cdot, \cdot)$ is positive definite, the function $\tilde{C}(\cdot, \cdot)$ is a positive definite operator function. Indeed if $x_1, \ldots, x_m \in X$ and $a_r = \{a_{1r}, a_{2r}, \ldots, a_{nr}\} \in C^n$, then defining the vector $u^{pq} = \{u_1^{pq}, \ldots, u_n^{pq}\} \in C^n$ $(p, q = 1, \ldots, m)$ by the formula

$$u_i^{pq} = \sum_k (C(x_p, x_q)f_k, f_i)a_{kq},$$

we get

$$u^{pq} = \tilde{C}(x_p, x_q)a_q.$$

Then

$$(\tilde{C}(x_p, x_q)a_q, a_p)_{C^n} = \sum_{i,k} (C(x_p, x_q)a_{kq}f_k, a_{ip}f_i).$$

Summing over p and q and defining

$$v_r = \sum_j a_{jr}f_j$$

we get

$$\sum_{p,q} (\tilde{C}(x_p, x_q)a_q, a_p) = \sum_{p,q} (C(x_p, x_q)v_q, v_p) \geqslant 0$$

as was to be proved.

With every positive definite operator function $C\colon X \times X \to L(H)$ we can associate a Hilbert space. Namely, consider $\mathscr{F}(X, H)$ as a linear

space with coordinatewise linear operations. If $C \gg 0$, then the functional $\{f(\cdot), g(\cdot) \in \mathscr{F}(X, H)\} \langle f(\cdot), g(\cdot) \rangle = \sum_{x,y} \{C(x,y)f(y), g(x)\}$ is a semi-inner product in $\mathscr{F}(X, H)$ and the Schwarz inequality (4) implies that the set $N_0 = \{f(\cdot) \in \mathscr{F}(X, H): \langle f(\cdot), f(\cdot) \rangle = 0\}$ is a linear subspace. It follows that the quotient space $\mathscr{F}(X, H)/N_0$ is a unitary space with the inner product induced by $\langle \cdot, \cdot \rangle$. Let H_C be its completion to a Hilbert space. We can now state the following

PROPOSITION 1. *Suppose that the function* $C: X \times X \to L(H)$ *is positive definite. Then there is a Hilbert space* H_C *and a mapping* $h: X \times H \to H_C$, *linear in the second variable and such that the linear span of* $h(X, H)$ *is dense in* H_C *and* $\{C(x, y)f, g\} = \{h(x,f), h(y, g)\}_{H_C}$ *for every* $x, y \in X$ *and* $f, g \in H$.

Proof. To complete our arguments we define

$$h(x, f) = \text{the coset of } \mathscr{F}(X, H)/N_0 \text{ corresponding to } f(y) = \delta_x(y)f$$

$$(\delta_x \text{ stands for } \delta\text{-measure at } x).$$

Since positive definite scalar functions are those which are positive definite operator functions on the one-dimensional space $H = C^1$ we get

COROLLARY 1. *Let* $C: X \times X \to C^1$ *be a positive definite scalar valued function. Then there is a Hilbert space* L *and a mapping* $h: X \to L$ *such that the linear span of* $h(X)$ *is dense in* L *and* $C(x, y) = \{h(x), h(y)\}_L$ *for every* $x, y \in X$.

Let $\mathscr{S}$ be a $*$-semigroup and let $\varphi: \mathscr{S} \to L(H)$ be an operator function. We say that φ is *n-positive definite* (weakly n-positive definite, weakly positive definite, positive definite) if the function $K(x, y) = \varphi(x^*y)$ $(x, y \in \mathscr{S})$ is n-positive definite (weakly n-positive definite, ... etc., respectively). We write $\varphi \gg_n 0$ if is n-positive definite and $\varphi \gg 0$ if φ is positive definite.

The positive definiteness of the function φ is a necessary condition of its dilatability to a $*$-representation of $\mathscr{S}$. Indeed, if $\varphi = \Delta_R \pi$, where π is a $*$-representation of $\mathscr{S}$, then for $f(\cdot) \in \mathscr{F}(\mathscr{S}, H)$

$$\sum_{a,b} \{\varphi(a^*b)f(b), f(a)\} = \sum_{a,b} \{\pi(b)Rf(b), \pi(a)Rf(a)\}$$

$$= \Big\| \sum_c \pi(c)Rf(c) \Big\|^2 \geq 0.$$

Formula (5) shows that a weakly positive function $\varphi: \mathscr{S} \to L(H)$ on the $*$-semi-group $\mathscr{S}$ satisfies the condition

(10) $\varphi(a^*b)^* = \varphi(b^*a)$

for $a, b \in \mathscr{S}$. If $\mathscr{S}$ has a unit, then

(11) $\varphi(a^*) = \varphi(a)^*$ for $a \in \mathscr{S}$.

It is plain that if φ is weakly positive definite, then

(12) $\varphi(a^*a) \geqslant 0$ for every $a \in \mathscr{S}$.

The weak positive definiteness on $*$-semi-groups does not imply positive definiteness.

EXAMPLE. Let $\mathscr{S}$ be the algebra of 2×2 complex matrices with involution consisting in the taking of hermitian adjoints. $\mathscr{S}$ may be identified in an obvious fashion with $L(C^2)$. We define the map $\varphi\colon \mathscr{S} \to L(C^2)$ by the formula

$$\varphi(a) = (\text{Trace}\, a)e - a,$$

where $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Since φ is linear and $\|a^*a\| \leqslant \text{Trace}\, a^*a$, φ is weakly positive definite. Indeed, if $a_1, \ldots, a_n \in \mathscr{S}$ and $\lambda_1, \ldots, \lambda_n \in C^1, f \in C^2$, then

$$\sum_{i,k/1}^{n} (\varphi(a_i^* a_k)f, f)\lambda_k \bar\lambda_i = \left(\varphi\left(\left(\sum_k a_k \lambda_k\right)^* \sum_i a_i \lambda_i\right)f, f\right) \geqslant 0.$$

We define now

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to check that $\text{Trace}\,(a_i^* a_k) = \delta_{ik}$. We take vectors $f_1 = \{1, 0\}, f_2 = \{0, 1\}, f_3 = \{0, 1\}, f_4 = \{1, 0\}$. Then

$$\Delta = \sum_{i,k/1}^{4} (\varphi(a_i^* a_k)f_k, f_i)$$

$$= \sum_{i,k} \text{Trace}\,(a_i^* a_k)(f_k, f_i) - \left\|\sum_{j/1}^{4} a_j f_j\right\|^2$$

$$= \sum_{j/1}^{4} \|f_j\|^2 - \left\|\sum_{j/1}^{4} a_j f_j\right\|^2.$$

Since $\|f_j\|^2 = 1$ for $j = 1, 2, 3, 4$ and $\sum_{j/1}^{4} a_j f_j = \{2, 2\}$ we obtain $\Delta = 4 - 8 = -4$, which shows that φ is not 2-positive definite.

We will now prove a proposition giving some sufficient conditions for the positive definiteness of weakly positive functions. Namely, suppose that $\mathscr{S}$ is a $*$-semi-group and $\varphi\colon \mathscr{S} \to L(H)$ an operator valued function. We say that φ is pseudomultiplicative if for every $a \in \mathscr{S}$ there is an $a' \in \mathscr{S}$ such that $\varphi(s\, a') = \varphi(s)\varphi(a)$ for every $s \in \mathscr{S}$.

PROPOSITION 2. Let $\mathscr{S}$ be a $*$-semi-group with unit e and $\varphi\colon \mathscr{S} \to L(H)$ a pseudomultiplicative mapping. Then, if $\varphi(e) = I_H$ and φ is weakly positive definite, then φ is positive definite.

Proof. Suppose $f \in H$ and define $H_f = [\varphi(\mathscr{S})f]$. H_f reduces φ. Indeed if $g = \varphi(a)f$, then $\varphi(s)g = \varphi(s)\varphi(a)f = \varphi(sa')f$ by pseudomultiplicativity. The symmetry of φ proves the claim $(\varphi(a^*) = \varphi(a)^*$ because $\mathscr{S}$ has a unit and φ is weakly positive definite).

It is now plain that it is sufficient to prove that the part of φ in H_f is positive definite.

Let $s_{ik} \in \mathscr{S}$ $(i = 1, \ldots, n; k = 1, \ldots, m)$ and define the vectors

$$f_p = \sum_{k/1}^{m} \lambda_{pk}\varphi(s_{pk})f,$$

where $\lambda_{ik} \in C^1$. Such vectors form a dense linear manifold in H_f. It follows that it is sufficient to prove that

$$\Delta = \sum_{p,q} \big(\varphi(u_p^* u_q)f_q, f_p\big) \geqslant 0$$

for $u_1, \ldots, u_r \in \mathscr{S}$.

The pseudomultiplicativity of φ proves that $\varphi(a') = \varphi(ea') = \varphi(e)\varphi(a) = \varphi(a)$. It follows that

$$\Delta = \sum_{p,q} \sum_{j,k} a_{qj}\overline{a_{pk}}\big(\varphi(u_p^* u_q)\varphi(s_{qj})f, \varphi(s_{pk})f\big)$$

$$= \sum_{p,q,j,k} a_{qj}\overline{a_{pk}}\big(\varphi(u_p^* u_q s_{qj}')f, \varphi(s_{pk}')f\big)$$

$$= \sum a_{qj}\overline{a_{pk}}\big(\varphi(s_{pk}')^*\varphi(u_p^* u_q s_{qj}')f, f\big).$$

By pseudomultiplicativity we also have for $a, s \in \mathscr{S}$

$$\varphi\big((a')^*s\big) = \varphi(s^*a')^* = \big(\varphi(s^*)\varphi(a)\big)^*$$
$$= \big(\varphi(s^*)\varphi(a')\big)^* = \varphi(a')^*\varphi(s).$$

Hence

$$\Delta = \sum \big(\varphi\big((s_{pk}')^* u_p^* u_q s_{qj}'\big)f, f\big)\overline{a_{qj}a_{pk}}.$$

By Corollary 1 there is a Hilbert space K and a map $h\colon \mathscr{S}\to K$ such that

$$\big(\varphi(a^*b)f, f\big) = \big(h(b), h(a)\big)_K$$

for $a, b \in \mathscr{S}$. Consequently

$$\Delta = \sum \big(\varphi\big((u_p s_{pk}')^* u_q s_{qj}'\big)f, f\big)\overline{a_{qj}a_{pk}}$$

$$= \sum_{p,q,j,k} \big(h(u_q s_{qj}'), h(u_p s_{pk}')\big)_K \overline{a_{qj}a_{pk}}.$$

We define

$$g_i = \sum_k a_{ik} h(u_i s'_{ik})$$

and just see that

$$\Delta = \sum_{p,q} (g_p, g_q)_K = \left\| \sum_s g_s \right\|_K^2 \geqslant 0. \quad \blacksquare$$

Notes. Positive definite operator functions on groups and set fields were first considered by Naimark [21], [20]. For general definitions and Proposition 1, Corollary 1 see Sz.-Nagy–Koranyi [37]. Positive definite operator functions have been considered by Halmos in connection with subnormal operators [8], and in a general *-semi-group setting by Sz.-Nagy [34].

The inequalities related to several types positive definiteness appear in several branches of analysis and probability theory. Inequality (8) is due to M. G. Krein. Proposition 2 is due to Choi, see Størmer [28]. Both authors consider operator maps of C^*-algebras.

Example 1 is due to Choi — see [28], where one considers more general examples. For other examples see Arveson [1] and Stinespring [27].

6. General existence theorems

As already noticed, if $\varphi: \mathscr{S} \to L(H)$ is an operator function on a *-semigroup $\mathscr{S}$ and $\varphi = \Delta_R \pi$, where π is a *-representation of $\mathscr{S}$, then $\varphi \geqslant 0$. Another necessary condition for dilatable φ will now be described. For if $\varphi = \Delta_R \pi$, then for $f(\cdot) \in \mathscr{F}(\mathscr{S}, H)$ we have for every $a \in \mathscr{S}$

$$\sum_{c,b} (\varphi(c^* a^* ab) f(b), f(c)) = \sum_{c,b} (\pi(c^* a^* ab) Rf(b), Rf(c))$$

$$= \sum_{c,b} (\pi(a)\pi(b) Rf(b), \pi(a)\pi(c) Rf(c))$$

$$= \left\| \pi(a) \sum_d \pi(d) Rf(d) \right\|^2 \leqslant \|\pi(a)\|^2 \sum_{c,b} (\varphi(c^* b) f(b), f(c)).$$

The above inequality gives rise to the following condition, which plays a crucial role in existence theorems for dilations:

(1)   There is a finite function $\varrho: \mathscr{S} \to R^+$ such that for every $a \in \mathscr{S}$ and every $f(\cdot) \in \mathscr{F}(\mathscr{S}, H)$

$$\sum_{c,b} (\varphi(c^* a^* ab) f(b), f(c)) \leqslant \varrho(a) \sum_{c,b} (\varphi(c^* b) f(b), f(c)).$$

Condition (1) will be called the *boundedness condition* ([1]). Positive definite functions which satisfy this condition are characterized in the following basic proposition.

PROPOSITION 1. *Let* φ: $\mathscr{S} \to L(H)$ *be a positive definite operator function on the* *-semi-group $\mathscr{S}$ satisfying the boundedness condition (1). Then there is a Hilbert space K, a *-representation π of $\mathscr{S}$ on K and a mapping τ: $\mathscr{F}(\mathscr{S}, H) \to K$, $\tau(f) = \hat{f}$, onto a dense linear manifold of K such that for every $f(\cdot)$, $g(\cdot) \in \mathscr{F}(\mathscr{S}, H)$ and $u \in \mathscr{S}$ the equality $\sum_{c,b} \left(\varphi(c^*ub)f(b), g(c)\right) = \left(\pi(u)\hat{f}, \hat{g}\right)_K$ holds true.*

Proof. We define for $g(\cdot) \in \mathscr{F}(\mathscr{S}, H)$ the function $\hat{g}(s) = \sum_b \varphi(s^*b)g(b)$.

The functions $\hat{g}$ with $g(\cdot)$ varying over $\mathscr{F}(\mathscr{S}, H)$ form a linear space M with pointwise defined linear operations and the map $g(\cdot) \to \hat{g}(\cdot)$ is linear. We define in M the hermitian bilinear form $\langle \cdot, \cdot \rangle$ as follows:

$$\langle \hat{g}, \hat{f} \rangle = \sum_s \left(\hat{g}(s), f(s)\right)$$

$$= \sum_s \sum_r \left(\varphi(s^*r)g(r), f(s)\right)$$

$$= \sum_r \sum_s \left(g(r), \varphi(r^*s)f(s)\right)$$

$$= \sum_s \left(g(s), \hat{f}(s)\right).$$

We have used here the equality $\varphi(s^*r) = \varphi(r^*s)^*$, which follows from the positive definiteness of φ, and just conclude that $\langle \cdot, \cdot \rangle$ is well defined, i. e., if $\hat{g} = \hat{g}_1$ and $\hat{f} = \hat{f}_1$, then $\langle \hat{g}, \hat{f} \rangle = \langle \hat{g}_1, \hat{f}_1 \rangle$. Since $\varphi \geqslant 0$ we have

$$\langle \hat{g}, \hat{g} \rangle = \sum_{s,r} \left(\varphi(s^*r)g(r), g(s)\right) \geqslant 0,$$

which implies the Schwarz inequality

$$|\langle \hat{f}, \hat{g} \rangle|^2 \leqslant \langle \hat{f}, \hat{f} \rangle \langle \hat{g}, \hat{g} \rangle.$$

It follows that if $\langle \hat{g}, \hat{g} \rangle = 0$, then $\langle \hat{g}, \hat{f} \rangle = 0$ for every $\hat{f}$, i.e.,

$$\sum_s \left(\hat{g}(s), f(s)\right) = 0$$

([1]) Added in proof. P. Masani noticed (Conf. Prob. in linear spaces, Trzebieszowice, Sept. 1–7, 1977) that using results of Szafraniec [30] one gets that (1) is equivalent to the following condition: there is $\varrho(\cdot)$: $S \to R^+$ such that for every a, $b \in \mathscr{S}$, $f \in H$

$$\left(\varphi(b^*a^*ab)f, f\right) \leqslant \varrho(a)\left(\varphi(b^*b)f, f\right).$$

Consequently, proofs in Section 9 may be pretly simplified.

for every $f(\cdot) \in \mathscr{F}(\mathscr{S}, H)$. Let $f \in H$ and define $f(s) = f$ if $s = s_0$ and $f(s) = 0$ if $s \neq s_0$. Then $(\hat{g}(s_0), f) = 0$. Since s_0 and f are arbitrary, we conclude that $\hat{g}(s) \equiv 0$. Hence $\langle \cdot, \cdot \rangle$ is an inner product.

For $\hat{g}$ and $a \in \mathscr{S}$ we define g_a as

$$g_a(s) = \hat{g}(a^*s) = \sum_b \varphi(s^*ab)g(b).$$

We will show that g_a is in M. Indeed, if $g_i = g(b_i)$ $(i = 1, \ldots, n)$ are all non-zero values of $g(\cdot)$, then

$$g_a(s) = \sum_{i/1}^{n} \varphi(s^*a\, b_i)g_i.$$

Let us define the function $\tilde{g}_a$ as follows:

$$\tilde{g}_a(z) = \sum_{ab_j = z} g_j \quad \text{if } ab_k = z \text{ for some } k,$$

$$\tilde{g}_a(z) = 0 \quad \text{if } ab_j \neq z \quad \text{for } j = 1, 2, \ldots, n.$$

It is plain that $\tilde{g}_a \in \mathscr{F}(\mathscr{S}, H)$. We now have

$$g_a(s) = \sum_{k/1}^{p} \varphi(s^*z_k)\tilde{g}_a(z_k),$$

where $\{z_1, \ldots, z_p\} = \{ab_1, \ldots, ab_n\}$ and $z_i \neq z_k$ for $i \neq k$. Since $\tilde{g}_a(\cdot) \in \mathscr{F}(\mathscr{S}, H)$, $g_a(\cdot) \in M$, as was to be proved.

Let us define for $a \in \mathscr{S}$ a linear map $\pi\langle a \rangle$ by the formula

$$(\pi\langle a \rangle \hat{g})(s) = g_a(s) = \hat{g}(a^*s), \quad \hat{g} \in M.$$

$\pi\langle a \rangle$ is a linear map of M into M and since $(\pi\langle a\, b \rangle \hat{g})(s) = \hat{g}(b^*a^*s) = (\pi\langle a \rangle f)(s)$ with $f(s) = \hat{g}(b^*s)$ we obtain

$$\pi\langle ab \rangle = \pi\langle a \rangle \pi\langle b \rangle.$$

On the other hand, for $\hat{f}(b) = \sum_d \varphi(b^*d)f(d)$ $(f(\cdot) \in \mathscr{F}(\mathscr{S}, H))$ we have

$$\langle \pi\langle a \rangle \hat{f}, \hat{f} \rangle = \sum_b (\hat{f}(a^*b), f(b)) = \sum_{b,c} (\varphi(b^*a\, c)f(c), f(b))$$

$$= \sum_{b,c} (f(c), \varphi(c^*a^*b)f(b)) = \sum_c (f(c), \hat{f}(a\, c)) = \langle \hat{f}, \pi\langle a^* \rangle \hat{f} \rangle,$$

which, by the polarization formula proves that

(2) $$\langle \pi\langle a \rangle \hat{f}, \hat{g} \rangle = \langle \hat{f}, \pi\langle a^* \rangle \hat{g} \rangle$$

for $\hat{f}, \hat{g} \in M$, $a \in \mathscr{S}$. We can now conclude that

$$\langle \pi\langle a \rangle \hat{f}, \pi\langle a \rangle \hat{f} \rangle = \langle \pi\langle a^* \rangle \pi\langle a \rangle \hat{f}, \hat{f} \rangle = \langle \pi\langle a^*a \rangle \hat{f}, \hat{f} \rangle.$$

Since

$$\left(\pi\langle a^* a\rangle \hat{f}\right)(s) = \hat{f}(a^* as) = \sum_c \varphi(s^* a^* ac)f(c)$$

we have

$$\left\langle \pi\langle a^* a\rangle \hat{f}, \hat{f}\right\rangle = \sum_{s,c} \left(\varphi(s^* a^* ac)f(c), f(s)\right).$$

By the boundedness condition (1) the right-hand side of the last equality is less than or equal to

$$\varrho(a) \sum_{s,c} \left(\varphi(s^* c)f(c), f(s)\right) = \varrho(a)\langle \hat{f}, \hat{f}\rangle.$$

Consequently

(3) $\langle \pi\langle a\rangle \hat{f}, \pi\langle a\rangle \hat{f}\rangle \leqslant \varrho(a)\langle \hat{f}, \hat{f}\rangle.$

Let K be the completion of M with respect to the inner product $\langle \cdot, \cdot \rangle$. It follows from (3) that $\pi\langle a\rangle$ extends in a unique way to a linear bounded operator on K, which we denote by $\pi(a)$. Since $\pi\langle ab\rangle = \pi\langle a\rangle \pi\langle b\rangle$, the function $\pi\colon \mathscr{S}\to L(K)$ is a representation of $\mathscr{S}$. By (2) $\pi(a^*) = \pi(a)^*$, which proves that π is a $*$-representation. Now, if we define $\tau(f) = \hat{f}$ for $f(\cdot) \in \mathscr{F}(\mathscr{S}, H)$, then by the equalities preceding (2) and by the polarization formula we easily obtain the final equality of the assertion.

We will establish some relationships between the space H and K of Proposition 1. Using the previous notation, we define $f_a(\cdot) \in \mathscr{F}(\mathscr{S}, H)$ for $a \in \mathscr{S}, f \in H$ as follows:

$$f_a(s) = \begin{cases} f & \text{if } s = a, \\ 0 & \text{if } s \neq a. \end{cases}$$

The operators $R_c\colon H\to K$ $(c \in \mathscr{S})$ defined by the equalities

$$(R_c f)(s) = \hat{f}_c(s) = \sum_b \varphi(s^* b)f_c(b) = \varphi(s^* c)f$$

satisfy the condition

(4) $R_b^* \pi(u) R_a = \varphi(b^* ua)$ $(u, a, b \in \mathscr{S}).$

Indeed, if $f \in H$, then $\left(R_b^* \pi(u) R_a f, f\right) = \langle \pi(u) R_a f, R_b f\rangle = \langle \pi(u)\hat{f}_a, \hat{f}_b\rangle.$ But $\left(\pi(u)\hat{f}_a\right)(s) = \varphi(s^* ua)f$ which proves that

$$\langle \pi(u)\hat{f}_a, \hat{f}_b\rangle = \sum_s \left(\varphi(s^* ua)f, f_b(s)\right) = \left(\varphi(b^* ua)f, f\right).$$

Taking $a = b$ in (4), we obtain

(5) $\varphi(a^* ua) = \varDelta_{R_a} \pi(u)$ $(a, u \in \mathscr{S}),$

which simply means that the operator function $\varphi_a(u) = \varphi(a^*ua)$ has $\pi(u)$ as an R_a-dilation. The question is what about the dilation of the function φ itself? So far we have had to do only with operator functions φ_a indexed by $a \in \mathscr{S}$ and having a common dilation with varying R_a operators acting from the initial space into the dilation space. The situation is simple if $\mathscr{S}$ has a unit, say e. Then, taking $a = e$ in (5), we have

$$\varphi(u) = \varDelta_{R_e}\pi(u) \quad (u \in \mathscr{S}),$$

and just get a dilation of φ which is a $*$-representation. This leads us to the following theorem:

THEOREM 1. *Assume that the positive definite function* $\varphi\colon \mathscr{S} \to L(H)$ *on the* $*$-*semi-group* $\mathscr{S}$ *satisfies the boundedness condition. If* $\mathscr{S}$ *has a unit* e, *then* $\varphi = \varDelta_R\pi$, *where* π *is a* $*$-*representation of* $\mathscr{S}$. *If additionally* $\varphi(e) = I_H$, *then* $\varphi = \mathrm{pr}\,\pi$.

Proof. Using Proposition 1. we get the formula $\varphi = \varDelta_R\pi$ simply by taking $R = R_e$. If $\varphi(e) = I_H$, then R becomes an isometric embbeding of H into K, which proves that $\varphi = \mathrm{pr}\,\pi$.

COROLLARY 1. *Suppose that* $\mathscr{S}$ *has a unit* e, $\varphi \geqslant 0$ *and* (1) *holds true and* $\varphi(e) = I_H$. *Then* $\varphi(a)^*\varphi(a) \leqslant \varphi(a^*a)$ *for every* $a \in \mathscr{S}$. *Hence, if* $a = a^*$, *then* $\varphi(a)^2 \leqslant \varphi(a^2)$.

Proof. Since $\varphi = \mathrm{pr}\,\pi$, where π is a unital $*$-representation of $\mathscr{S}$, for every vector f in the initial space $(\varphi(a)^*\varphi(a)f, f) = \|\varphi(a)f\|^2 \leqslant \|\pi(a)f\|^2 = (\pi(a^*a)f, f) = (\varphi(a^*a)f, f)$.

If $\mathscr{S}$ has no unit, then one can ask whether a certain limit passage in (4) with R_a, R_b is available. The following simple condition shows that this is possible:

(6)   There is a net $e_\eta \in \mathscr{S}$ such that $\gamma = \sup\|\varphi(e_\eta^*e_\eta)\| < +\infty$ and $\varphi(se_\eta) \to \varphi(s)$ weakly for every $s \in \mathscr{S}$.

If (6) holds true, then by the definition of $R_{e_\eta}(\overset{\mathrm{df}}{=} R_\eta)$ we have $\langle R_\eta f, R_\eta f\rangle = (\varphi(e_\eta^*e_\eta)f, f) \leqslant \gamma\|f\|^2$, which proves that R_η are equibounded. On the other hand,

$$\langle R_{\eta'}f - R_{\eta''}f, \hat{g}\rangle = \sum_c \left([\varphi(c^*e_{\eta'}) - \varphi(c^*e_{\eta''})]f, g(c) \right) \to 0$$

for $g(\cdot) \in \mathscr{F}(\mathscr{S}, H)$ and η', η'' varying. Since $\hat{g} - s$ are dense in K and R_η are equibounded, $\{R_\eta\}$ is a weak Cauchy net. Let $R = \lim R_\eta$. Formula (4) now implies that $R^*\pi(u)R = \varphi(u)$ $(u \in \mathscr{S})$. We have just proved the following theorem:

THEOREM 2. *Suppose that the positive definite function* $\varphi\colon \mathscr{S} \to L(H)$ *on the* $*$-*semi-group satisfies the boundedness condition and condition* (6). *Then* $\varphi = \varDelta_R\pi$, *where* π *is a* $*$-*representation*.

Notice that, taking $e_\eta = e$ if e is a unit of $\mathscr{S}$ we get Theorem 1 from Theorem 2 in a fairly simple way. Next, the construction performed in the proof of Proposition 1 guarantees that the dilation space K is minimal if (6) holds true. Indeed, notice that since

$$(Rf,\, \hat{g}) = \lim \langle R_\eta f,\, \hat{g}\rangle = \sum_c \big(\varphi(c^*)f,\, g(c)\big)$$

we have $(Rf)(s) = \varphi(s^*)f$ which implies that $\big(\pi(u)Rf\big)(s) = \varphi(s^*\,u)\,f$. Hence, if $h(\cdot) \in \mathscr{F}(\mathscr{S}, H)$, then

$$\hat{h}(s) = \sum_c \varphi(s^*c)h(c) = \sum_c \big(\pi(c)Rh(c)\big)(s).$$

But the totality of all $\hat{h}$ with $h(\cdot)$ varying over $\mathscr{F}(\mathscr{S}, H)$ is dense in K by the construction given in the proof of Proposition 1. So the linear span of manifolds $\pi(u)RH$ is dense in K, as was to be shown.

The boundedness condition (1) is a consequence of positive definiteness if φ is norm-bounded, that is, if $\sup\limits_{\mathscr{S}}\|\varphi(a)\| < +\infty$. To see this we start with the following lemma, which is interesting in itself.

LEMMA 1. *Suppose that $\mathscr{S}$ is a $*$-semi-group and $\varphi\colon \mathscr{S}\to L(H)$ is a positive definite operator function. Then for every $u \in \mathscr{S}$ and $f(\cdot) \in \mathscr{F}(\mathscr{S}, H)$*

$$(7)\qquad \sum_{a,b} \big(\varphi(a^*u^*uab)f(b),\, f(a)\big)$$

$$\leqslant \Big(\sum_{a,b}\big(\varphi(a^*b)f(b),\, f(a)\big)\Big)^{1-1/2^k}\cdot\Big(\sum_{a,b}\big(\varphi\big(a^*(u^*u)^{2^k}b\big)f(b),\, f(a)\big)\Big)^{1/2^k}$$

for $k = 1, 2, 3, \ldots$

Proof. We fix $u \in \mathscr{S}$ and $f_1, \ldots, f_n \in H$ and $x_1, \ldots, x_n \in \mathscr{S}$ and write $w = u^*u$. By the Schwarz inequality (4) of Section 5 we get

$$\Big(\sum_{i,k/1}^n \big(\varphi\big(x_i^*(u^*ux_k)f_k,\, f_i\big)\Big)^2 \leqslant \Big(\sum_{i,k/1}^n \big(\varphi(x_i^*x_k)f_k, f_i\big)\Big)\Big(\sum_{i,k/1}^n \big(\varphi(x_i^*w^2x_k)f_k, f_i\big)\Big),$$

which proves that for $p(z) \overset{\mathrm{df}}{=} \sum\limits_{i,k/1}^n \big(\varphi(x_i^*zx_k)f_k,\, f_i\big)$

we have

$$|p(w)|^2 \leqslant \Big(\sum_{i,k/1}^n \big(\varphi(x_i^*x_k)f_k,\, f_i\big)\Big)\cdot p(w^2).$$

Since w^{2^k} is of the form s^*s with a suitable s, we can apply the last inequality to w^{2^k} in place of w and, using simple induction, just obtain (7).

We are now in a position to prove the following

THEOREM 3. *Let $\mathscr{S}$ be a unital $*$-semi-group and φ: $\mathscr{S} \to L(H)$ a positive definite and bounded operator function. Then $\varphi = \varDelta_R \pi$, where π is a $*$-representation of $\mathscr{S}$.*

Proof. Since φ is bounded, the second factor on the right-hand side of (7) tends to 1 which implies by a limit passage that φ satisfies the boundedness condition (1) with $\varrho(a) = 1$. Now we apply Theorem 1.

The final properties of dilations which complete Theorem 1 and Theorem 2, are enclosed in the following

PROPOSITION 2. *Let $\mathscr{S}$ be a $*$-semi-group and φ: $\mathscr{S} \to L(H)$ an operator function. Suppose that π: $\mathscr{S} \to L(K)$ is an $*$-representation of $\mathscr{S}$ which is a minimal R-dilation of φ. Then the following conditions hold true:*

(a) *If for some $u_1, u_2, u_3 \in \mathscr{S}$ and all $a, b \in \mathscr{S}$ $\varphi(au_1b) + \varphi(au_2b)$ $= \varphi(au_3b)$, then $\pi(u_1) + \pi(u_2) = \pi(u_3)$.*

(b) *If $u_a \in \mathscr{S}$ is a net such that $\sup \|\pi(u_a)\| < +\infty$ and for some $u \in \mathscr{S}$ $\varphi(au_ab) \to \varphi(aub)$ weakly for all $a, b \in \mathscr{S}$, then*

$$\pi(u_a) \to \pi(u) \quad weakly.$$

Proof. To prove (a) we take $f, g \in H$ and $a, b \in \mathscr{S}$ and just check that

$$\left([\pi(u_1) + \pi(u_2)] \pi(b) Rf, \pi(a^*) Rg \right) = \left([\varphi(au_1b) + \varphi(au_2b)] f, g \right)$$

$$= \left(\varphi(au_3b) f, g \right) = \left(R^* \pi(au_3b) Rf, g \right)$$

$$= \left(\pi(u_3) \pi(b) Rf, \pi(a^*) Rg \right),$$

which by the minimality of π proves the claim.

For the proof of (b) we notice that

$$\left(\pi(u_a) \pi(b) Rf, \pi(a^*) Rg \right) = \left(\varphi(au_ab) f, g \right) \to \left(\varphi(aub) f, g \right)$$

$$= \left(\pi(u) \pi(b) Rf, \pi(a^*) Rg \right),$$

which by the minimality property and the equiboundedness of norms $\|\pi(u_a)\|$ proves the claim.

One of most important consequences of Theorem 1 is the theorem below, which concerns spectral dilations of semi-spectral measures.

Let K be a complex Hilbert space. Suppose we are given the set X and a σ-field $\mathscr{S}$ of subsets of X. As noticed in Example 1, Section 3, $\mathscr{S}$ becomes a $*$-semi-group with a unit equal to X if the semi-group operation consists in just taking the set theoretical joints and the identity operation stands for the involution. The $*$-representation E of $\mathscr{S}$ is a map of $\mathscr{S}$ into $L(K)$ such that for $\sigma_1, \sigma_2 \in \mathscr{S}$ $E(\sigma_1 \cap \sigma_2) = E(\sigma_1) E(\sigma_2)$. It follows that

$E(\sigma)$ is an orthogonal projection for every $\sigma \in \mathscr{S}$. E is unital if $E(X) = I_K$. We call the unital E a *spectral measure* if, additionally, $(E(\sigma)f, f)$ is a measure on $\mathscr{S}$ for every $f \in K$, that is the function $\mu_f(\sigma) = (E(\sigma)f, f)$ is a σ-additive measure on $\mathscr{S}$ for $f \in K$.

The function $F\colon \mathscr{S} \to L(H)$ is called a *semi-spectral measure* if only the map $\sigma \to (F(\sigma)f, f)$ $(f \in H)$ is a positive, σ-additive measure on $\mathscr{S}$. F is called *normalized* if $F(X) = I_H$. Every spectral measure is a normalized semi-spectral measure.

If $R\colon H \to K$ and $E\colon \mathscr{S} \to L(K)$ is a spectral measure, then $F(\sigma) = \Delta_R E(\sigma)$ $(\sigma \in \mathscr{S})$ is a semi-spectral measure and E is called a *spectral R-dilation of E*. By the results of Section 4 the minimality condition $K = \bigvee\limits_{\sigma \in \mathscr{S}} E(\sigma)RH$ determines E up to a unitary isomorphism and E is then called the *minimal spectral dilation of F*. We will prove the following

THEOREM 4. *Every semi-spectral measure has a spectral dilation. The spectral dilation is uniquely determined up to unitary equivalence by the minimality condition. If the semi-spectral measure is normalized, then it is a projection of its spectral dilation.*

Proof. The first thing we have to show is that the given semi-spectral measure $F\colon \mathscr{S} \to L(H)$ has a spectral dilation. We first check the positive definiteness condition. Suppose that $\sigma_1, \ldots, \sigma_n \in \mathscr{S}$ and $f_1, \ldots, f_n \in H$. We must show that

$$\sum_{i,k/1}^{n} (F(\sigma_i \cap \sigma_k)f_i, f_k) \geqslant 0.$$

Notice that $\sigma_j^* = \sigma_j$, $\sigma_i \cdot \sigma_k = \sigma_i \cap \sigma_k$ by the definition of $\mathscr{S}$ as a $*$-semi-group. Let p be a positive finite scalar measure on the σ-field $\mathscr{S}$ such that every measure $\mu_{ik}(\sigma) = (F(\sigma)f_i, f_k)$ is absolutely continuous with respect to p and let h_{ik} stand for the Radon–Nikodym derivative of μ_{ik} with regard to p. Since $F(\sigma) \geqslant 0$ for every $\sigma \in \mathscr{S}$, we have for $\lambda_1, \ldots, \lambda_n \in C^1$

$$\left(F(\sigma)\left(\sum_i \lambda_i f_i\right), \left(\sum_k \lambda_k f_k\right)\right) = \sum_{i,k} (F(\sigma)f_i, f_k)\lambda_i \bar{\lambda}_k = \sum_{i,k} \left(\int_\sigma h_{ik} dp\right)\lambda_i \bar{\lambda}_k \geqslant 0.$$

Keeping $\lambda = (\lambda_1, \ldots, \lambda_n)$ fixed, we can arbitrarily vary σ and just conclude that

$$(8) \qquad\qquad \sum_{i,k/1}^{n} h_{ik}(x)\lambda_i \bar{\lambda}_k \geqslant 0$$

for $x \in X - \sigma_\lambda$, where $p(\sigma_\lambda) = 0$. If we allow λ to range over a dense countable set in C^n, then we easily infer that (8) holds for all $\lambda \in C^n$ and $x \in X - \sigma_0$, where $p(\sigma_0) = 0$. On the other hand,

$$\left| \sum_{i/1}^{n} \lambda_i \chi_{\sigma_i} \right|^2 = \sum_{i,k} \chi_{\sigma_i \cap \sigma_k} \lambda_i \bar{\lambda}_k \geqslant 0$$

(χ_σ stands for the characteristic function of σ) all over X. The well-known lemma of Schur implies now by (8) that

$$\sum_{i,k/1}^{n} \chi_{\sigma_i \cap \sigma_k} h_{ik}(x) \geqslant 0 \qquad \text{for } x \in X - \sigma_0,$$

which by integrating proves that

(9) $$\sum_{i,k/1}^{n} (F(\sigma_i \cap \sigma_k) f_i, f_k) = \sum_{i,k/1}^{n} \int_{\sigma_i \cap \sigma_k} h_{ik} dp \geqslant 0.$$

We have shown that F, as a function on the $*$-semi-group $\mathscr{S}$, is positive definite. Since $0 \leqslant F(\sigma) \leqslant F(X)$, the function F is bounded and we can conclude the basic part of our proof simply by applying Theorem 3. However, the boundedness condition (1) can be verified for F easily, which makes it possible to apply Theorem 1. Indeed, applying (9) to $\sigma_i \cap \sigma$, $\sigma_k \cap \sigma$ in place of σ_i, σ_k, respectively, we conclude that $q(\sigma) = \sum_{i,k} (F(\sigma_i \cap \sigma \cap \sigma_k) f_i, f_k)$ is a positive measure on $\mathscr{S}$, which shows that, since $q(\sigma) \leqslant q(X)$, we have

$$\sum_{i,k/1}^{n} (F(\sigma_i \cap \sigma \cap \sigma_k) f_i, f_k) \leqslant \sum_{i,k/1}^{n} (F(\sigma_i \cap \sigma_k) f_i, f_k).$$

Hence the boundedness condition holds true with $\varrho(\sigma) = 1$. We conclude that there is a map $E \colon \mathscr{S} \to L(K)$ such that $\varDelta_R E = F$ and $E(\sigma_1 \cap \sigma_2) = E(\sigma_1) E(\sigma_2)$. Since $F(\sigma)$ is additive in σ so is E, provided a minimal E is chosen, by Proposition 2, (a). Now if $\sigma_n \in \mathscr{S}$, $\sigma_n \subset \sigma_{n+1}$ and $\sigma = \bigcup_{n/1}^{\infty} \sigma_n$, then by part (b) of Proposition 2 and weak σ-additivity of F we conclude that $\lim E(\sigma_n) = E(\sigma)$ weakly, which shows that E is weakly σ-additive, and this completes the proof.

Notes. The boundedness condition (1) has been introduced by Sz.-Nagy [34]. Proposition 1 is just a part of the original proof, given in [34], of Theorem 1, which belongs to Sz.-Nagy. Theorem 2 will be found in Mlak–Szymański [16]. Theorem 3 is due to Arveson (private communication, 1973). The enclosed proof of Theorem 3 as well Lemma 1 is due to Szafraniec [30]. Proposition 2 belongs to Sz.-Nagy [34]. Corollary 1, which concerns the so-called Schwarz operator inequalities, is due to Davis [4] — the first inequality, and Kadison [9] — the second inequality for $a = a^*$. Both authors considered positive linear maps φ on C^*-algebras; see Section 9 for a further discussion in this subject.

THEOREM 4 is the celebrated theorem of Naimark [21], which appears as one of the first theorems concerning the dilatability of *families* of operators to more *regular* ones, namely to spectral measures.

Going back to Proposition 1, one can give a simple proof after Masani's technique [13] as follows: By Proposition 1, Section 5 there is an operator function

$$\psi: \mathscr{S} \to L(H, H_\varphi) \quad (H_\varphi = H_C \text{ with } C(x, y) = \varphi(x^*y))$$

such that

$$\big(\varphi(y^*x)f, g\big) = \big(\psi(x)f, \psi(y)g\big)$$

for $x, y \in \mathscr{S}, f, g \in H$; and vectors $\sum \psi(x)f(x)$ $(f(\cdot) \in \mathscr{F}(\mathscr{S}, H))$ span H_φ. We define $\tau(f) = \sum \psi(x)f(x)$ and $\pi(u)\sum\psi(x)f(x) = \sum\psi(ux)f(x)$ and then check directly that π extends to a *-representation of $\mathscr{S}$. Taking $R_c f = \psi(c)f$, we easily verify formula (4) and then we are able to give the rest of the arguments to prove Theorems 1 and 2.

Added in proof. More general dilation theorems will be found in [16] and [19].

7. Positive definite functions on groups

Let G be a group with a unit e. The inner group operation in G is written multiplicatively. As was noticed already in Section 3 Example 2, G may be treated as a *-semi-group with involution $a \to a^{-1} = a^*$. The *-representation (unital!) of G is then simply a unitary representation. Suppose we are given a positive definite function $T(\cdot): G \to L(H)$ on G as a such *-semi-group. This means that

(1) $$\sum_{c,b} \big(T(c^{-1}b)f(b), f(c)\big) \geqslant 0$$

for every $f(\cdot) \in \mathscr{F}(G, H)$. Since

$$\sum_{c,b} \big(T(c^{-1}a^{-1}ab)f(b), f(c)\big) = \sum_{c,b} \big(T(c^{-1}b)f(b), f(c)\big),$$

the boundedness condition (1) of the previous section holds true with $\varrho(a) \equiv 1$. We are in a position to apply Theorem 1 of Section 6 and obtain the following theorem:

THEOREM 1. *Let G be a group and $T(\cdot): G \to L(H)$ a positive definite operator function, i.e., satisfying* (1). *Then there is a Hilbert space K and*

a unitary representation $U(\cdot)$: $G \to L(K)$ *and an operator* R: $H \to K$ *such that*

$$T(g) = \Delta_R U(g)$$

for every $g \in G$. *The minimality condition* $K = [U(G)RH]$ *determines* K *and* $U(\cdot)$ *up to a unitary isomorphism. If* $T(e) = I_H$, *then* $T(g) = \text{pr}\, U(g)$, $g \in G$.

The representation $U(\cdot)$ is called a *unitary dilation of* $T(\cdot)$. If K is minimal, then $U(\cdot)$ is called *minimal*. The minimal $U(\cdot)$ is unique up to unitary equivalence in the sense described in general in Section 4.

If G is a topological group and $T(\cdot)$ is weakly continuous on G, then by (b) of Proposition 2 of the previous section $U(\cdot)$ is a weakly, hence a strongly continuous unitary representation of G ($U(\cdot)$ minimal).

Krein's inequality (8) of Section 5 shows that it is sufficient to assume that $T(\cdot)$ is weakly continuous at the unit of G.

A complex function φ on G may be considered as an operator function in $L(H)$ with $H = C^1$. This function is positive definite if and only if

$$\sum_{i,k} \varphi(c_i^{-1} c_k) \lambda_k \bar{\lambda}_i \geqslant 0$$

for every finite choice of $c_1, \ldots, c_n \in G$ and $\lambda_1, \ldots, \lambda_n \in C^1$. We see that the positive definiteness of φ coincides with that considered in the theory of representations of groups. Now, if $\varphi \geqslant 0$, then by Theorem 1

$$\varphi(g) = \Delta_R U(g),$$

where $U(\cdot)$ is a unitary representation of G into $L(K)$. We take the vector $f = 1$, $C^1 = H$ and conclude that

$$\varphi(g) = (\varphi(g)f, f) = (U(g)Rf, Rf) \quad \text{for } g \in G.$$

If a minimal $U(\cdot)$ is chosen, then $K = \bigvee_G U(g)Rf$ which simply means that the vector $h = Rf$ is cyclic for the unitary representation $U(\cdot)$; $U(\cdot)$ is then unique up to unitary equivalence. In this way we arrive via Theorem 1 at the classical result which states that every scalar $\varphi \geqslant 0$ on G can be written in the form

$$\varphi(g) = (U(g)h, h), \quad g \in G,$$

where $U(\cdot)$ is a unitary representation of G with a cyclic vector h. If G is a topological group and φ is continuous, then $U(\cdot)$ is a strongly continuous representation.

One interesting consequence of Theorem 1 is at hand.

PROPOSITION 1. *Suppose that. the operator function* $T(\cdot)$: $G \to L(H)$ *is positive definite on the group* G. *If for every* $g \in G$ *the operator* $T(g)$ *is an isometry, then* $T(\cdot)$ *is a unitary representation of* G.

Proof. It follows from Theorem 1 that

$$T(g) \ = \ \varDelta_R U(g) \qquad (g \in G),$$

where $U(g)$ is a unitary representation of G. Let $U(\cdot)$ be minimal. We have $T(e)^* = T(e) = R^*R \geqslant 0$; since $T(e)$ is an isometry, we have $I_H = T(e)^* T(e) = T(e)^2$. But the positive square root of a positive operator is unique. It follows that $T(e) = I_H$, which implies that $T(\cdot)$ is a projection of $U(\cdot) - T(g) = \mathrm{pr}\, U(g)$ for $g \in G$. Let P be the orthogonal projection of the dilation space K on the initial one H. Since $T(g)$ is an isometry for every g, we have for $f \in H$

$$\|PU(g)f\| \ = \ \|T(g)f\| \ = \ \|f\| \ = \ \|U(g)f\|$$

which implies that $PU(g)f = U(g)f$. Hence $T(g)f = U(g)f$ (all f, all g), which by the minimality of U proves that $K = H$ and $T(\cdot) = U(\cdot)$.

To have an example related to the above proposition we take a complex positive definite function φ on G. The space H is identified with C^1. The requirement that the values of φ should be isometries means simply that $|\varphi(g)| = 1$ for all $g \in G$. So by Proposition 1 a positive definite complex valued function on a group with a modulus identically equal to one is necessarily a character of this group. In particular, if G is the additive group Z of integers and $\varphi(n)$ $(n \in Z)$ is a positive definite function on Z, such that $|\varphi(n)| = 1$ for all n, then there is a $z \in C^1$, $|z| = 1$ such that $\varphi(n) = z^n$; $n = 0, \pm 1, \pm 2, \ldots$

Let $T(\cdot)$: $G \to L(H)$ be a positive definite operator function on the group G. Then $T(g^{-1}) = T(g)^*$ for $g \in G$ (see Section 5). We are in a position to apply Proposition 1 of Section 3 and obtain the first part of the following

PROPOSITION 2. *Let the function $T(\cdot)$: $G \to L(H)$ be positive definite. Then there is a unique decomposition $H = H_K \oplus H_C$ such that:*

(a) H_K *and H_C reduce all $T(g)$, $g \in G$.*

(b) *The part $T_K(\cdot)$ of $T(\cdot)$ in H_K is a unitary representation of G.*

(c) *No non-zero subspace of H_C reduces $T_C(\cdot)$ $(= H_C$ part of T) to a unitary representation of G.*

If $U(\cdot)$: $G \to L(K)$ is a unitary representation of G and $T(g) = \mathrm{pr}\, U(g)$ for all $g \in G$, then

$$H_K \ = \ \bigcap_{g \in G} U(g)H \ = \ \bigcap_g \{f: \ \|T(g)f\| \ = \ \|f\|\}.$$

Proof. Only the final part of the assertion requires a proof. The statement $T(g) = \mathrm{pr}\, U(g)$ means that $H \subset K$ and $PU(g)f = T(g)f$ for $f \in H$, P being the orthogonal projection of K onto H. If $f \in H_K$, then

$T(gg_2)f = T(g_1)T(g_2)f$ $(g_1, g_2 \in G)$ and $T(g^{-1})f = T(g)^*f$ by the proof of Proposition 1, Section 3. It follows that since $T(e) = I_H$, we have

$$T(g^{-1}g)f = T(g)^*T(g)f = f,$$

which shows that $\|T(g)f\| = \|f\|$. Hence $f \in \bigcap_g \{f : \|T(g)f\| = \|f\|\}$. But since $T(g)f = PU(g)f$, the equality $\|T(g)f\| = \|f\|$ implies that $T(g)f = U(g)f \in H$ which finaly proves that

$$H_K \subset \bigcap_g \{f : \|T(g)f\| = \|f\|\} \subset \bigcap_g U(g)H.$$

Suppose that $f \in \bigcap_g U(g)H$. Then $f \in H$ and $U(g)f \in H$ for every $g \in G$. Consequently $T(g)f = U(g)f (g \in G)$ and

$$T(g_1g_2)f = U(g_1g_2)f = U(g_1)U(g_2)f$$
$$= T(g_1)T(g_2)f \quad \text{(all } g_1, g_2 \in G).$$

By the proof of Proposition 1, Section 3 we infer that $f \in H_K$ which proves the claim.

The part $T_K(\cdot)$ is called the *unitary part* of $T(\cdot)$, and $T_C(\cdot)$ the *completely non-unitary part of $T(\cdot)$*. If $H = H_C$, that is $T(\cdot) = T_C(\cdot)$, then $T(\cdot)$ is called simply a *completely non-unitary positive definite function*.

The next proposition concerns operator functions on products of groups. For the sake of simplicity we deal with finite products. We are given groups $G_1, \ldots, G_n$. Let G be the product $G_1 \times G_2 \times \ldots \times G_n$ = group which consist of sequences $(g_1, \ldots, g_n)$ $(g_i \in G_i)$ with coordinatewise defined group operation. We denote by e_i the unit of G_i.

PROPOSITION 3. *Suppose that the function $T(\cdot) : (G_1 \times G_2 \times \ldots \times G_n) \rightarrow L(H)$ is positive definite. If for every i $(i = 1, 2, \ldots, n)$ the section*

$$T_i(g_i) = T(e_1, \ldots, e_{i-1}, g_i, e_{i+1}, \ldots, e_n)$$

$(g_i \in G_i)$ *is a unitary representation of G_i, then $T(\cdot)$ is a unitary representation of $G = G_1 \times \ldots \times G_n$.*

Proof. Since $T_1(\cdot)$ is a unitary representation we have $T_1(e_1) = T(e_1, \ldots, e_n) = I_H$. By Theorem 1, $T(g) = \text{pr} U(g)$. In other words there is a space $K \supset H$ and a unitary representation $U(\cdot) : G \rightarrow L(K)$ such that $T(g)f = PU(g)f$ for $f \in H, g \in G, P$ being the orthogonal projection of K onto H. It follows that $T_i(g_i)f = PU_i(g_i)f$, where $U_i(g_i) = U(e_1 \ldots \ldots e_{i-1}, g_i, e_{i+1} \ldots e_n)$. Applying Proposition 2, Section 3 to $T_i(\cdot)$, which by assumption is a unitary representation of G_i, we conclude that $T_i(g_i)f = U_i(g_i)f$ for $f \in H, g_i \in G_i$. On the other hand,

$$U(g_1 \ldots g_n) = U_1(g_1)U_2(g_2) \ldots U_n(g_n).$$

It follows that for $f \in H$

$$U(g_1 \ldots g_n)f = U_1(g_1)U_2(g_2) \ldots U_n(g_n)f$$
$$= U_1(g_1)U_2(g_2) \ldots U_{n-1}(g_{n-1})T_n(g_n)f$$
$$= U_1(g_1)U_2(g_2) \ldots U_{n-2}(g_{n-2})T_{n-1}(g_{n-1})T_n(g_n)f$$
$$= \ldots = T_1(g_1)T_2(g_2) \ldots T_n(g_n)f.$$

Hence H is invariant under $U(\cdot)$. Since $T(g)$ is a projection of $U(g)$, $T(g)f = U(g)f$ for $f \in H$, $g \in G$, which completes the proof.

Dilations find a simple interpretation if the group in question is the additive group of integers Z. An operator function $T(n)$ on Z is simply a doubly infinite sequence $\{T_n\}$ ($n = 0, \pm 1, \pm 2, \ldots$) of operators. Such a sequence is a positive definite function on Z if and only if for every finite sequence of vectors $f_{-n}, f_{-n-1}, \ldots, f_{-1}, f_0, f_1, \ldots, f_n$

$$\sum_{i,k/-n}^{n} (T_{i-k}f_i, f_k) \geqslant 0.$$

We then simply say that $\{T_n\}$ is a positive definite sequence. In terms of dilations the positive definite sequences are characterized by the following

PROPOSITION 4. *Let* $T_n \in L(H)$ ($n = 0, \pm 1, \pm 2, \ldots$) *be a sequence. Then the following conditions are equivalent:*

(a) $\{T_n\}$ *is positive definite;*

(b) *There is a Hilbert* K, *a unitary operator* U *in* K *and an operator* $R: H \to K$ *such that*

$$T_n = \Delta_R U^n \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Proof. The equivalence of (a) and (b) follows from Theorem 1 and the fact that unitary representation of Z are generated by powers of single unitary operator.

The operator U of (b) is called the *unitary dilation* of $\{T_n\}$.

Notes. Positive definite operator functions on groups were first considered by Naimark [20], who proved Theorem 1 for abelian groups. It was noticed by Sz.-Nagy in [33] that this theorem holds for arbitrary groups. Both Naimark and Sz.-Nagy considered unital functions, i. e., such that $T(e) = I$. Proposition 1 appears in Mlak–Ryll-Nardzewski [16]. For the canonical decomposition described in Proposition 2 see Langer [10], Sz.-Nagy–Foiaş [35] and Mlak [15]. Proposition 3 will be found in Mlak–Ryll-Nardzewski [16]. Some general similar properties of positive definite functions on products of ∗-semi-groups will be found also in [16]. A typical property is the following one:

If $F: \mathscr{S} \to L(H)$ is a semi-spectral normalized measure on the product σ-field $\mathscr{S}$ of σ-fields $\mathscr{S}_1, \ldots, \mathscr{S}_n$ and if for every i the "section" $F_i(\sigma_i)$

$= F(X_1, X_2, \ldots, X_{i-1}, \sigma_i, X_{i+1}, \ldots, X_n)$ $(X_i =$ Polish space, is a basic space for $\mathscr{S}_i)$ is a spectral measure, then F is itself a spectral measure.

The problems discussed ∙in [17] originated from some questions of quantum field theory related to joint probability distributions. As to Proposition 4 see Naimark [20] and Sz.-Nagy–Koranyi [37].

The relationship between positive definite operator functions on groups and related (as in classical theory — see [5], [22]) functions on group algebras (see Section 9 for dilations on *-algebras) is discussed in Umegaki [39].

8. Intertwining operators for dilatable operator functions

Suppose we are given two Hilbert spaces H_1 and H_2 and operators $S_i \in L(H_i)$. We say that the operator $X \in L(H_1, H_2)$ intertwines S_1 and S_2 if $XS_1 = S_2 X$. If the operator functions $S_i(\cdot)$ are defined on the set Ω and $S_i(\omega) \in L(H_i)$, then the operator $X \in L(H_1, H_2)$ intertwines these functions if $XS_1(\omega) = S_2(\omega) X$ for every ω. If $H_1 = H_2 = H$ and $S_1(\cdot) = S_2(\cdot) = S(\cdot)$, then the operators intertwining $S_1(\cdot)$ and $S_2(\cdot)$ are just the operators in the commutant of $S(\cdot)$.

It is plain that the set of linear operators which intertwine two operator functions form a linear space which becomes an algebra, if these functions coincide.

Two operators $S_1 \in L(H_1)$, $S_2 \in L(H_2)$ are called *disjoint* if the only operator which intertwines S_1 and S_2 is a zero operator. A similar terminology applies to operator functions. The operator $X \in L(H_1, H_2)$ is called a *quasi-similarity* (similarity) if X is one-to-one and $\mathscr{R}(X)$ is dense $(S^{-1} \in L(H_2, H_1)$ resp.). It is plain that X is a quasi-similarity iff X^* is a quasi-similarity. The operator functions $S_1(\cdot)$, $S_2(\cdot)$ (values of $S_i(\cdot)$ are in $L(H_i)$) are called *quasi-similar* (similar) if there are quasi-similarities X_1, X_2 (there is a similarity X) such that $X_1 S_1(x) = S_2(x) X_1$ and $X_2 S_2(x) = S_1(x) X_2$ for all x $(XS_1(x) = S_2(x) X$ for all x resp.).

In what follows we are interested in the lifting properties of operators intertwining dilatable operator functions on *-semi-groups. To begin with we will prove a proposition on operators which intertwine *-representations of a *-semi-group.

PROPOSITION 1. *Let $\mathscr{S}$ be a *-semi-group and let π_i: $\mathscr{S} \to L(K_i)$ $(i = 1, 2)$ be *-representations of $\mathscr{S}$. Suppose that the operator $X \in L(K_1, K_2)$ intertwines π_1 and π_2, that is $X\pi_1(a) = \pi_2(a) X$ for $a \in \mathscr{S}$. We define $M = (\operatorname{Ker} X)^\perp$ and $N = \mathscr{R}(X)$. Then M reduces π_1, N reduces π_2 and the parts π_1' of π_1 in M, π_2' of π_2 in N are unitarily equivalent as *-representations.*

Proof. If $f \in M^{\perp}$, then $Xf = 0$. Hence $X\pi_1(a)f = \pi_2(a)Xf = 0$ for $a \in \mathscr{S}$, which, since π_1 is a $*$-homomorphism proves that $M^{\perp}$, and consequently M reduces π_1. If $f \in N$, then $f = \lim Xf_n$, which implies that $\pi_2(a)f = \lim X \pi_1(a)f_n \in \mathscr{R}(X)$ for every $a \in \mathscr{S}$, which proves that N reduces π_2.

We now define $Y \in L(M, N)$ by the formula $Yf = Xf$ ($f \in M$). For $f \in M$ we have

$$Y\pi_1'(a)f = X\pi_1(a)f = \pi_2(a)Xf = \pi_2'(a) Yf.$$

Let $Y = U|Y|$ ($\operatorname{Ker} U = \operatorname{Ker}|Y|$) be the polar decomposition of Y. Since $Y^*\pi_2'(a) = \pi_1'(a) Y^*$ we have, for $a \in \mathscr{S}$, $Y^* Y\pi_1'(a) = Y^* \pi_2'(a) Y = \pi_1'(a) Y^* Y$ and consequently $|Y| = (Y^* Y)^{1/2}$ commutes with all $\pi_1'(a)$. It follows that $U\pi_1'(a)|Y| = U|Y|\pi_1'(a) = \pi_2'(a) U|Y|$. But $\mathscr{R}(|Y|)$ $= (\operatorname{Ker}|Y|)^{\perp} = (\operatorname{Ker} Y)^{\perp} = M$ because if $Yf = 0$, then $Xf = 0$ and $f \perp \operatorname{Ker} X$, i.e., $f \perp f$, that is $f = 0$. It follows that $U\pi_1'(a) = \pi_2'(a) U$. On the other hand, $\mathscr{R}(Y) = N$, which since $\operatorname{Ker} Y = \{0\}$ implies that U is a unitary map. This completes the proof.

COROLLARY 1. *If π_1, π_2 are $*$-representations of a $*$-semi-group and there is a quasi-similarity which intertwines them, then π_1 and π_2 are unitarily equivalent.*

Proof. Let X be the quasi-similarity intertwining π_1 and π_2. Then $M = K_1$ and $N = K_2$.

COROLLARY 2. *Using the Fuglede–Putnam theorem, one easily derives from Corollary 1 that if there is a quasi-similarity which intertwines two normal operators, then they are unitarily equivalent.*

Suppose now that $\psi_i \colon \mathscr{S} \to L(H)$ ($i = 1, 2$) are positive definite operator functions on the $*$-semi-group $\mathscr{S}$. We write $\psi_2 \ll \psi_1$ if $\psi_1 - \psi_2 \gg 0$, i.e., if the function $u \to [\psi_1(u) - \psi_2(u)]$ is positive definite.

If $X \in L(H_1, H_2)$ and the functions $\varphi_i \colon \mathscr{S} \to L(H_i)$ ($i = 1, 2$) are positive definite, then denoting by $X^* \varphi_2 X$ the function $u \to X^* \varphi_2(u) X$ we see that it is positive definite and the inequality $\varphi_1 - X^* \varphi_2 X \gg 0$ makes sense.

The fundamental lemma for our purposes is the following one:

LEMMA 1. *Let $\varphi_i \colon \mathscr{S} \to L(H_i)$ ($i = 1, 2$) have canonical expressions $\varphi_i = \Delta_{R_i} \pi_i$, where $\pi_i \colon \mathscr{S} \to L(K_i)$ are $*$-representations of the $*$-semi-group $\mathscr{S}$. Suppose $X \in L(H_1, H_2)$. Then the following conditions are equivalent:*

(a) *There is a unique operator $\tilde{X} \in L(K_1, K_2)$ which intertwines π_1 and π_2 and is such that $\tilde{X} R_1 = R_2 X$,*

(b) $X^* \varphi_2 X \ll c^2 \varphi_1$ *for some c.*

Proof. Suppose (a) holds true and let $f_1, \ldots, f_n \in H_1$ and $s_1, \ldots, s_n \in \mathscr{S}$. Then

$$\sum_{i,k} \left(X^* \varphi_2(s_i^* s_k) X f_k, f_i \right) = \sum_{i,k} \left(\pi_2(s_i^* s_k) R_2 X f_k, R_2 X f_i \right),$$

$$= \left\| \sum_j \pi_2(s_j) R_2 X f_j \right\|^2 = \left\| \sum_j \pi_2(s_j) \tilde{X} R_1 f_j \right\|^2$$

$$= \left\| \sum_j \tilde{X} \pi_1(s_j) R_1 f_j \right\|^2 \leqslant \|\tilde{X}\|^2 \sum_{i,k} \left(\varphi_1(s_i^* s_k) f_k, f_i \right),$$

which proves (b) with $c = \|\tilde{X}\|$.

Assume now that (b) holds true. Then by similar computations as above we infer that

$$\left\| \sum_j \pi_2(s_j) R_2 X f_j \right\|^2 \leqslant c^2 \left\| \sum_j \pi_1(s_j) R_1 f_j \right\|^2,$$

which proves that there is a unique operator $\tilde{X} \colon K_1 \to K_2$ such that $\tilde{X}\pi_1(s) R_1 = \pi_2(s) R_2 X$ for all $s \in \mathscr{S}$. It follows that for $a, b \in \mathscr{S}$ and $f \in H_1$

$$\tilde{X}\pi_1(a)\pi_1(b) R_1 f = \tilde{X}\pi_1(ab) R_1 f = \pi_2(ab) R_2 X f = \pi_2(a)\tilde{X}\pi_1(b) R_1 f.$$

Since the elements $\pi_1(b) R_1 f$ span K_1, we conclude that $\tilde{X}$ intertwines π_1 and π_2. The rest of the proof uses the double commutant theorem in the same way as it was done in Section 4. Namely, consider the algebra $\mathscr{B}$ of diagonal matrices

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$$

on $K_1 \oplus K_2$, where $S_i = \sum_k a_k \pi_i(a_k)$. $\mathscr{B}$ is a symmetric subalgebra of $L(K_1 \oplus \oplus K_2)$. If $S\tilde{h} = 0$ ($\tilde{h} = (h_1, h_2)$, $h_i \in K_i$) for all $S \in \mathscr{B}$, then $\pi_i(a) h_i = 0$ for $a \in \mathscr{S}$, $i = 1, 2$. Hence $h_i \perp \pi_i(a) R_i f_i$ ($f_i \in H_i$) for all a, which by the minimality of K_i proves that $h_i = 0$. It follows that $\mathscr{B}$ has a trivial null space. On the other hand, $\tilde{X}S_1 R_1 f = S_2 R_2 X f$ for $f \in H$. Now allowing S to approach $I_{K_1 \oplus K_2}$ we get $\tilde{X}R_1 = R_2 X$.

Before we state the main theorem we will apply Lemma 1 to semi-spectral measures and obtain the following

PROPOSITION 2. *Let* $F_i \colon \mathscr{S} \to L(H_i)$ *be two semi-spectral measures on the σ-field $\mathscr{S}$. Then the following conditions are equivalent for $X \in L(H_1, H_2)$:*

(a) *There is* $\tilde{X} \in L(K_1, K_2)$ *such that* $\tilde{X}E_1(\sigma) = E_2(\sigma)\tilde{X}$ *for every* $\sigma \in \mathscr{S}$ *and* $\tilde{X}R_1 = R_2 X$ *($F_i = \Delta_{R_i} E_i$ are canonical expressions of F_i by Naimark's Theorem);*

(b) *There is a finite* $c > 0$ *such that*

$$\left(F_2(\sigma) X f, X f \right) \leqslant c^2 \left(F_1(\sigma) f, f \right)$$

for every $f \in H$ *and every* $\sigma \in \mathscr{S}$.

Proof. It is easy to show that (a) implies (b). Suppose that (b) holds true. It follows that $F(\sigma) = cF_1(\sigma) - X^*F_2(\sigma)X$ is a semi-spectral measure on $\mathscr{S}$. Appealing now to the proof of Theorem 4.6 or to Theorem 4.6 itself, we conclude that for the corresponding positive definite maps condition (b) of Lemma 1 holds true. This completes the proof.

Finally let us formulate the main theorem.

THEOREM 1. *Let $\mathscr{S}$ be a $*$-semi-group and let π_i: $\mathscr{S} \to L(K_i)$ $(i = 1, 2)$ be $*$-representations of $\mathscr{S}$. Suppose $\varphi_i = \Delta_{R_i}\pi_i$ $(i = 1, 2)$ and $K_i = [\pi_i(\mathscr{S})R_i H_i]$ for $i = 1, 2$. If $X \in L(H_1, H_2)$ intertwines φ_1 and φ_2, that is $X\varphi_1(a) = \varphi_2(a)X$ for $a \in \mathscr{S}$, then there is a unique $\tilde{X} \in L(K_1, K_2)$ such that:*

(a) $\tilde{X}\pi_1(a) = \pi_2(a)\tilde{X}$ *for $a \in \mathscr{S}$, i.e., $\tilde{X}$ intertwines π_1 and π_2.*

(b) $\tilde{X}R_1 = R_2 X$.

Moreover,

(c) $\|\tilde{X}\| \leqslant \|X\|$.

Proof. Since $X\varphi_1(a) = \varphi_2(a)X$ and $\varphi_i(b^*) = \varphi_i(b)^*$ $(i = 1, 2; b \in \mathscr{S})$, we have $X^*\varphi_2(a) = \varphi_1(a)X^*$, which implies that $X^*X\varphi_1(a) = \varphi_1(a)X^*X$. Let $K = X^*X$ and $f_1, \ldots, f_n \in H_1$. Then

$$\left\| \sum_i \pi_2(a_i)R_2 X f_i \right\|^2 = \sum_{i,k} \left(\pi_2(a_i^* a_k)R_2 X f_k, R_2 X f_i \right)$$

$$= \sum_{i,k} \left(\varphi_2(a_i^* a_k) X f_k, X f_i \right) = \sum_{i,k} \left(K\varphi_1(a_i^* a_k)f_k, f_i \right) \quad \text{for } a_1 \ldots a_n \in \mathscr{S}.$$

Let us define $H_n = \underbrace{H_1 \oplus H_1 \oplus \cdots \oplus H_1}_{n \text{ times}}$, $\hat{f} = (f_1, \ldots, f_n) \in H_n$ and an operator $\mathscr{K} \in L(H_n)$ corresponding to the matrix

$$\mathscr{K} = \begin{pmatrix} K & 0 & 0 & 0 & \ldots & 0 \\ 0 & K & 0 & 0 & \ldots & 0 \\ 0 & 0 & K & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & 0 & \ldots & K \end{pmatrix}$$

and the operator $S = \left(\varphi_1(a_i^* a_k) \right)_{i,k=1,\ldots,n} \in L(H_n)$. Since K commutes with every $\varphi_1(a_i^* a_k)$, we have $\mathscr{K}S = S\mathscr{K}$. But S is a positive operator. Hence

$$\Delta = (\mathscr{K}S\hat{f}, \hat{f}) = (\mathscr{K}S^{1/2}\hat{f}, S^{1/2}\hat{f}) \leqslant \|\mathscr{K}\|(S\hat{f}, \hat{f}) \leqslant \|X^*X\|(S\hat{f}, \hat{f})$$

$$\leqslant \|X\|^2 \sum_{i,k} \left(\varphi_1(a_i^* a_k)f_k, f_i \right).$$

But $\varDelta = \sum_{i,k} \left(\varphi_2(a_i^* a_k) X f_k, X f_i\right)$. It follows that (b) with $c = \|X\|$ of Lemma 1 holds true and (a), (b) and (c) of our theorem follow. The uniqueness of $\tilde{X}$ is obvious.

COROLLARY 3. *If π_1 and π_2 are disjoint, then φ_1 and φ_2 are disjoint, if $[\varphi_2(\mathscr{S})H_2] = H_2$.*

Assume that the assumptions of Theorem 1 hold true. Denote by J_{ik} the space of all operators which intertwine φ_i and φ_k and by J^{ik} the space of operators intertwining π_i and π_k. It follows that there is a lifting map $\varPsi_{12}$: $X \to \tilde{X}$ of J_{12} into J^{12} defined by $\varPsi_{12}(X) = \tilde{X}$. Changing the roles of φ_1 and φ_2 we get $\varPsi_{21}$ which to every $Y \in J_{21}$ assigns a unique $\tilde{Y} \in J^{21}$ according to Theorem 1. $\varPsi_{11}, \varPsi_{22}$ are the corresponding maps $X \to \tilde{X}$ of $X - s$, which commute with φ_1 and φ_2, respectively, on $\tilde{X} - s$, commuting with π_1 and π_2. By Theorem 1:

(1) $$\varPsi_{ik}(X)\pi_i(a) = \pi_k(a)\varPsi_{ik}(X),$$

(2) $$\varPsi_{ik}(X)R_i = R_k X,$$

(3) $$\|\varPsi_{ik}(X)\| \leqslant \|X\|$$

for $X \in J_{ik}$, $i, k = 1, 2$.

Notice that by (2) if $\varPsi_{ik}(X) = 0$, then $R_k X = 0$. Conversely, if $R_k X = 0$, then $0 = \pi_k(a)\varPsi_{ik}(X)R_i = \varPsi_{ik}(X)\pi_i(a)R_i$ which by the minimality of K_i proves that $\varPsi_{ik}(X) = 0$. Hence the kernel

$$\operatorname{Ker}\varPsi_{ik} = \{X \in J_{ik}: R_k X = 0\}.$$

Observe that if $\varphi_{i,k} = \operatorname{pr}\pi_{i,k}$, then R_k, as well as R_i are isometries and by (2) and (3)

$$\|\varPsi_{ik}(X)\| = \|X\| \quad \text{(in fact } X \subset \tilde{X}\text{)}.$$

If $X_1, X_2 \in J_{ik}$, then $a_1 X_1 + a_2 X_2 \in J_{ik}$ $(a_i \in C^1)$. Also, since $\varPsi_{ik}(X_j)R_i = R_k X_j (j = 1, 2)$, we have

$$\left(a_1\varPsi_{ik}(X_1) + a_2\varPsi_{ik}(X_2)\right)R_i = R_k(a_1 X_1 + a_2 X_2),$$

which by the uniqueness property of conditions (a) and (b) of Theorem 1 shows that $\varPsi_{ik}$ is a linear mapping. Consequently $\varPsi_{ik}$ is one-to-one if $\operatorname{Ker}\varPsi_{ik} = 0$. It follows then that if $\operatorname{Ker}R_k = 0$, then $\varPsi_{ik}$ is one-to-one.

Suppose that we are given the operator functions φ_i: $\mathscr{S} \to L(H_i)$ $(i = 1, 2, 3)$ on the *-semi-group $\mathscr{S}$, and *-representations π_i: $\mathscr{S} \to L(K_i)$ $(i = 1, 2, 3)$ of $\mathscr{S}$ such that $\varphi_i = \varDelta_{R_i}\pi_i$ and $K_i = [\pi_i(\mathscr{S})R_i H_i]$. We denote by J_{ik} the space of operators which intertwine φ_i and φ_k, by J^{ik} the space of operators which intertwine π_i and π_k, and by $\varPsi_{ik}$: $J_{ik} \to J^{ik}$ the corresponding lifting map given by Theorem 1.

The uniqueness statement of Theorem 1 will be involved in the proof of the formula

$$(4) \qquad \Psi_{kj}(X)\Psi_{ik}(Y) = \Psi_{ij}(XY) \qquad (i,j,k = 1,2,3),$$

valid for $X \in J_{kj}$, $Y \in J_{ik}$.

To prove (4) we notice first that $\Psi_{kj}(X)R_k = R_j X$, $\Psi_{ik}(Y)R_i = R_k Y$, which implies that

$$\Psi_{kj}(X)\Psi_{ik}(Y)R_i = \Psi_{kj}(X)R_k Y = R_j XY.$$

Since $XY \in J_{ij}$ and $\Psi_{kj}(X)\Psi_{ik}(Y) \in J^{ij}$ by the uniqueness property of Theorem 1, we must have $\Psi_{ij}(XY) = \Psi_{kj}(X)\Psi_{ik}(Y)$.

The next property of lifting maps is the following one:

$$(5) \qquad \Psi_{ik}(X)^* = \Psi_{ki}(X^*) \qquad \text{for } X \in J_{ik}.$$

We appeal to the proof of Theorem 1. Since $\Psi_{ki}(X^*)\pi_k(a)R_k g = \pi_i(a)R_i X^* g$ for $g \in H_k$, $a \in \mathscr{S}$, it is enough to prove that $\tilde{X}^* = \Psi_{ik}(X)^*$ satisfies the equality

$$\tilde{X}^*\pi_k(a)R_k g = \pi_i(a)R_i X^* g.$$

To see this we take $a, b \in \mathscr{S}$, $g \in H_k$, $f \in H_i$ and infer by (a) and (b) of Theorem 1 that

$$\left(\tilde{X}^*\pi_k(a)R_k g, \pi_i(b)R_i f\right) = \left(\pi_k(a)R_k g, \tilde{X}\pi_i(b)R_i f\right) = \left(\pi_k(a)R_k g, \pi_i(b)\tilde{X}R_i f\right)$$

$$= \left(\pi_k(a)R_k g, \pi_k(b)R_k Xf\right) = \left(R_k^*\pi_k(b^*a)R_k g, Xf\right) = \left(X^*\varphi_k(b^*a)g, f\right)$$

$$= \left(\varphi_i(b^*a)X^* g, f\right) = \left(R_i^*\pi_i(b^*a)R_i X^* g, f\right) = \left(\pi_i(a)R_i X^* g, \pi_i(b)R_i f\right).$$

Since the elements $\pi_i(b)R_i f$ span K_i, the desired equality follows.

It follows from all the above properties of lifting mappings that Ψ_{ii} is linear, multiplicative and involution preserving mapping. It is an $*$-isomorphism if $\operatorname{Ker}R_i = 0$. The next proposition includes some simple properties of lifting mapping which easily follow from all that we have proved already.

PROPOSITION 3. *Let the assumptions of Theorem 1 be satisfied and suppose that* $X \in J_{12}$. *Then the following conditions hold true:*

(a) *If X is an isometry, then $\Psi_{12}(X)$ is also an isometry.*

(b) *If the range of X is dense, then the range of $\Psi_{12}(X)$ is dense.*

(c) *If X is a similarity (quasi-similarity), then $\Psi_{12}(X)$ is a similarity (quasi-similarity resp.).*

(d) *If X is unitary, then $\Psi_{12}(X)$ is unitary.*

Proof. It is plain that (d) follows from (a) and (b).

To prove (a) we first notice that since $\Psi_{ii}(I_{H_i})R_i = R_i$, we have $\Psi_{ii}(I_{H_i})\pi_i(a)R_i f = \pi_i(a)R_i f$, which by the minimality of K_i proves that $\Psi_{ii}(I_{H_i}) = I_{K_i}$. Now, if $X^* X = I_{H_1}$, then by (4) and (5)

$$\Psi_{12}(X)^* \Psi_{12}(X) = \Psi_{21}(X^*)\Psi_{12}(X)$$
$$= \Psi_{11}(X^* X) = \Psi_{11}(I_{H_1}) = I_{K_1},$$

which completes the proof of (a).

To prove (b) we notice that $\Psi_{12}(X)R_2 = R_2 X$ which implies that $\Psi_{12}(X)\pi_1(a)R_1 = \pi_2(a)R_2 X$. Hence, if $g \perp \mathscr{R}(\Psi_{12}(X))$, then, since $\overline{\mathscr{R}(X)} = H_2$, we have $g \perp \pi_2(a)R_2 H_2$ and by the minimality of K_2 we get $g = 0$.

To prove (c) suppose that X is a similarity and let $Y = X^{-1}$. Then by (4) $\Psi_{12}(X)\Psi_{21}(Y) = \Psi_{22}(XY) = I_{K_2}$ and $\Psi_{21}(Y)\Psi_{12}(X) = \Psi_{11}(YX) = I_{K_1}$.

If X is a quasi-similarity, so is X^*. It follows from (b) that the range of $\Psi_{12}(X)$ is dense and, since $\overline{\mathscr{R}(X^*)} = H_2$, the range of $\Psi_{21}(X^*)$, equal to the range of $\Psi_{12}(X)^*$, is also dense. Consequently $\Psi_{12}(X)$ is one-to-one, which completes the proof.

Notice that if $\varphi_1 = \varphi_2$ ($H = H_1 = H_2$), then $I_H \in J_{12}$ and the operator $U = \Psi_{12}(I_H)$ is unitary by (d). Also $U\pi_1(a) = \pi_2(a)U$ for $a \in \mathscr{S}$ and by (2) $UR_1 = R_2$. This is just the uniqueness property of minimal dilations stated in Theorem 2 of Section 4.

The following proposition results immediately from Corollary 1 and (c) of Proposition 3:

PROPOSITION 4. *Let* $\varphi_i: \mathscr{S} \to L(H_i)$ ($i = 1, 2$) *have the canonical expression* $\varphi_i = \Delta_{R_i}\pi_i$. *If there is a quasi-similarity which intertwines* φ_1 *and* φ_2 *or* φ_2 *and* φ_1, *then* π_1 *and* π_2 *are unitarily equivalent.*

We next prove a theorem concerning the range of a lifting mapping.

THEOREM 2. *Let be satisfied the assumptions of Theorem 1. Then*

$$\Psi_{ik}(X)R_i R_i^* = R_k R_k^* \Psi_{ik}(X)$$

for $X \in J_{ik}$ *and, moreover, the range of* Ψ_{ik} *is the totality of all* $Y \in J^{ik}$ *such that* $YR_i R_i^* = R_k R_k^* Y$.

Proof. Since $\Psi_{ik}(X)^* = \Psi_{ki}(X^*)$, by the equalities

$$\Psi_{ik}(X)R_i = R_k X, \qquad \Psi_{ki}(X^*)R_k = R_i X^*,$$

we have

$$\Psi_{ik}(X)R_i R_i^* = R_k X R_i^* = R_k(R_i X^*)^*$$
$$= R_k(\Psi_{ik}(X)^* R_k)^* = R_k R_k^* \Psi_{ik}(X).$$

To prove the second part of the assertion we take $Y \in J^{ik}$ such that $YR_i R_i^* = R_k R_k^* Y$. We have to show that $Y = \Psi_{ik}(X)$ for some $X \in J_{ik}$.

Let $R_j^* = W_j^* |R_j^*|$ be the polar decomposition of R_j^* $(j = 1, 2)$, where $\operatorname{Ker} W_j = \operatorname{Ker} |R_j^*|$. Then $|R_j^*| = W_j R_j^*$ and $R_j = |R_j^*| W_j$. We define $X = W_k^* Y W_i$. Then $R_k X = |R_k^*| Y W_i = R_k W_k^* Y W_i$. Since $Y R_i R_i^* = R_k R_k^* Y$, we have $Y |R_i^*| = |R_k^*| Y$. It follows that $R_k X = Y |R_i^*| W_i = Y R_i$. Now, if $a \in \mathscr{S}$, then $X \varphi_i(a) = W_k^* Y W_i R_i^* \pi_i(a) R_i = W_k^* Y |R_i^*| \pi_i(a) R_i$ $= W_k^* |R_k^*| Y \pi_i(a) R_i = W_k^* |R_k^*| \pi_k(a) Y R_i = R_k^* \pi_k(a) R_k X = \varphi_k(a) X$ because $Y \in J^{ik}$ and $W_k^* |R_k^*| = R_k^*$. Hence $X \in J_{ik}$ and the above relations together with the uniqueness property related to (a) and (b) of Theorem 1 show that $Y = \Psi_{ik}(X)$.

Notes. Proposition 1 is a routine theorem, which appears in different forms: for representations of C^*-algebras, for unitary representations of groups etc. The basic Lemma 1 is modelled after Theorem 1.3.1 and Lemma 1.4.1 of Arveson [1] and after Proposition 2, which is due to Lebow [11]. All the other lifting results are due to Arveson [1], who proved the corresponding results for linear positive definite functions on C^*-algebras: but, as shown, his arguments work well for operator functions on $*$-semi-groups. Proposition 3 and Proposition 4 appear as easy consequences of the methods involved.

9. Dilations of operator functions on star algebras

The present section deals with positive definite linear operator valued functions on $*$-algebras. We apply here the general theory developed in previous section. To begin with, we introduce some definitions and notation.

Let B be a complex algebra. An algebra involution or simply an involution on B is a mapping $u \to u^*$ of B into B such that the following conditions hold true $(u, v \in B; a, \beta \in C^1)$:

(a) $\qquad\qquad (u + v)^* = u^* + v^*,$
(b) $\qquad\qquad (au)^* = \bar{a} u^*,$
(c) $\qquad\qquad (u^*)^* = u,$
(d) $\qquad\qquad (uv)^* = v^* u^*.$

An algebra B with involution will be called a *star algebra* or simply a *$*$-algebra*. If B has a multiplicative unit e, then $e = e^*$. Notice that by (c) and (d) a $*$-algebra is a $*$-semi-group with respect to multiplication. The element u in the $*$-algebra B with a unit e is called *unitary* if u^{-1} exist and equals $u^* - u^{-1} = u^*$. B is called a *U^*-algebra* if it is a linear span of its unitary elements. Recall also that a linear functional p: $B \to C^1$ on a $*$-algebra B is called *positive* if $p(u^* u) \geqslant 0$ for $u \in B$.

A *-algebra *B* which is a Banach algebra with respect to some norm will be called a *Banach *-algebra*. We do not require here that the involution be continuous. If the involution in a Banach *-algebra *B* is continuous, then there is in B an equivalent norm $\|\cdot\|$ such that $\|u\| = \|u^*\|$ for $u \in B$, that is the involution is an isometric mapping. If, moreover, *B* has a unit *e*, then automatically $\|e\| = 1$ ($B \neq 0$).

A Banach *-algebra *B* will be called a *C *-algebra* if the norm satisfies

$$\|u^*u\| = \|u\|^2$$

for every $u \in B$. Notice that a Banach *-algebra can have at most one norm with respect to which it is a C^*-algebra.

The theorem of Gelfand and Naimark states that every C^*-algebra is isometrically *-isomorphic to a C^*-algebra of Hilbert space operators, the *-operation being the taking of adjoints. If such an algebra is commutative, then it is isometrically *-isomorphic to the sup norm algebra $C(X)$ of all continuous complex valued functions on a locally compact Hausdorff space *X* with natural involution of taking complex conjugate functions.

The net $\{e_a\}$ in the Banach algebra *B* is called an *approximate identity* (approximate unit) if

$$\lim \|u - e_a u\| = 0 = \lim \|u - u e_a\|$$

for every $u \in B$, and $\sup \|e_a\| < +\infty$.

Let B_1 and B_2 be two *-algebras. The linear mapping $\tau \colon B_1 \to B_2$ is called a *-homomorphism* if $\tau(uv) = \tau(u)\tau(v)$ and $\tau(u^*) = \tau(u)^*$ for $u, v \in B_1$.

Let *H* be a complex Hilbert space. The operator valued function $\varphi \colon B \to L(H)$ on the *-albegra will be called *positive definite* (weakly positive definite) if it is positive definite (weakly positive definite) on *B* treated as *a**-semi-group with the given involution, the multiplication in *B* being the inner semi-group operation. φ is positive definite (weakly positive definite) if, for every $f(\cdot) \in \mathscr{F}(B, H)$ $(\lambda_1, \ldots, \lambda_n \in C^1, u_1, \ldots, u_n \in B, f \in H)$,

$$\sum_{u,v}(\varphi(u^*v)f(v), f(u)) \geqslant 0 \qquad (\sum_{i,k}(\varphi(u_i^*u_k)f, f)\lambda_k \bar{\lambda}_i \geqslant 0 \text{ resp.}).$$

We then write $\varphi \geqslant 0$. Notice that a linear map $\varphi \colon B \to L(H)$ is weakly positive definite if and only if it is positive in the sense that $\varphi(u^*u)$ is a positive operator for every $u \in B$.

Let *K* be a complex Hilbert space. By a *-representation of *B* on *K* we understand a linear mapping $\pi \colon B \to L(K)$ such that $\pi(uv) = \pi(u)\pi(v)$ and $\pi(u^*) = \pi(u)^*$ for $u, v \in B$. Hence, a linear map π is a *-representation of the algebra *B* if it is a *-representation of *B* treated as a *-semi-

group. In other words, π is a *-homomorphism of B into the *-algebra $L(K)$, the involution in $L(K)$ being the taking of adjoints.

Notice that:

(e) Every *-representation on a Hilbert space K of a Banach *-algebra is continuous.

We endow here $L(K)$ with operator norm topology.

If $\varphi = \Delta_R\pi$, where π is a *-representation of the *-algebra B, then φ is positive definite. We are now in a position to formulate and prove the basic existence dilation theorems for positive definite linear functions on some star algebras. We shall apply the general theorems of Section 6.

The first theorem concerns functions on U^*-algebras.

THEOREM 1. *Let* $\varphi(\cdot)$: $B \to L(H)$ *be a positive definite and linear map defined on the* U^*-*algebra* B *with unit* e. *Then there is a Hilbert space* K, *an operator* R: $H \to K$ *and a* *-*representation* π: $B \to L(K)$ *such that* $\varphi(u) = \Delta_R\pi(u)$ *for* $u \in B$.

Proof. We shall apply Theorem 2. Sec. 6. First we check the boundedness condition. Let $v_1, \ldots, v_n \in B, f_1, \ldots, f_n \in H$ and $v \in B$. Since B is a U^*-algebra, there are unitary elements $u_1, \ldots, u_m$ such that $v = \sum_{i/1}^{m} \lambda_i u_i$ with suitable $\lambda_i \in C^1$. We define

$$\Delta_v = \sum_{i,k} \left(\varphi(v_i^* v^* v v_k) f_k, f_i \right).$$

Since φ is linear, $\varphi \geqslant 0$, we have

$$\Delta_v = \sum_{p,q/1}^{m} a_{pq} \lambda_p \bar{\lambda}_q \geqslant 0,$$

where

$$a_{pq} = \sum_{i,k/1}^{n} \left(\varphi(v_i^* u_p^* u_q v_k) f_k, f_i \right).$$

If we allow λ_i to vary over C^1 keeping u_i fixed, then by possitive definiteness of φ we infer that the matrix $\{a_{pq}\}$ is a positive definite one. Let A be the corresponding operator. Then

$$\|A\|_{C^m} \leqslant \text{Trace}(A).$$

But

$$\text{Trace}(A) = \sum_{p/1}^{m} \sum_{i,k/1}^{n} \left(\varphi(v_i^* u_p^* u_p v_k) f_k, f_i \right) = m \sum_{i,k/1}^{n} \left(\varphi(v_i^* v_k) f_k, f_i \right)$$

because $u_p^* u_p = e$. Since

$$\Delta_v \leqslant \|A\|_{C^m} \sum_{p/1}^{m} \|\lambda_p\|^2,$$

we get

$$A_v \leqslant m \Big(\sum_{p/1}^{m} \|\lambda_p\|^2 \Big) \Big(\sum_{i,k/1}^{n} \big(\varphi(v_i^* v_k) f_k, f_i \big) \Big),$$

which proves that the boundedness condition of Theorem 1. Sec 6 holds true with

$$\varphi(v) = m \Big(\sum_{p/1}^{m} \|\lambda_p\|^2 \Big).$$

It follows that $\varphi = \Delta_R \pi$, where π is $*$-representation of B treated as a $*$-semi-group. When choosing a minimal representation π, we conclude from the linearity of φ and from Proposition 2.6 (a) that π is linear. This completes the proof.

The clasical results relating linear positive functionals on $*$-algebras to $*$-representations of those algebras may be expressed in terms of dilations via the above theorem. In fact, in Theorem 1 the assumptions about the $*$-algebra guarantee that positive definite operator functions satisfy the boundendess condition required in the general dilation theorems of Section 6. If, in general, B is a $*$-algebra and φ is a linear positive functional on B, then φ can be viewed as a positive definite operator function $\varphi \colon B \to L(H)$ with one dimensional space H, just as positive definite scalar functions on groups where interpreted in Section 7. The boundedness condition needed in general dilation theorems reduces then to the following one:

(1)   There is a finite positive function $\varphi(\cdot)$ on B such that

$$\varphi(u^* a^* a u) \leqslant \varrho(a) \varphi(u^* u)$$

for $a, u \in B$.

Now, if the positive linear functional φ satisfies (1) and B has a unit e, then there is a $*$-representation π of B with a cyclic vector f (i.e., $\pi(u)f$, $u \in B$ span the representation space) such that $\varphi(u) = \big(\pi(u)f, f \big)$ for all $u \in B$. The above statement can easily be deduced from Theorem 1 Section 6 simply by repeating suitable parts of its proof. The direct classical proof reads as follows:

Let $\langle u, v \rangle = \varphi(v^* u)$ be the semi-inner product induced by φ and $K_0 = B/J$ the unitary quotient space with $J = \{u \colon \langle u, u \rangle = 0\}$ and for $u \in B$ denote by $\dot{u}$ the coset of u. We define the linear map π_0 of B into the algebra of linear operators on K_0 by the formula $\pi_0(u)\dot{v} = $ the coset of the product uv. Then by (1) $\|\pi_0(u)\dot{v}\|^2 = \varphi(v^* u^* u v) \leqslant \varrho(u)\varphi(v^* v)$ $= \varrho(u)\|\dot{v}\|^2$. Hence $\pi_0(u)$ extends by continuity to $\pi(u)$ on the completion K of K_0, and plainly π is a $*$-representation of B. By definition $\varphi(u)$ $= \varphi(ue) = \big(\pi(u)\dot{e}, \dot{e} \big)$, which proves the claim. If we define $f = Ru_0 = \dot{e}$

$(u_0 = 1)$, then $\varphi(u) = R^*\pi(u)R$. In this way we proved a dilation theorem for φ. Now, if B is a U^*-algebra, then ϱ can be computed as in the proof of Theorem 1, which together with all we said above proves independently of Theorem 1, in an elementary fashion that

(2)   If φ is a linear positive functional on the U^*-algebra B, then there is a cyclic $*$-representation π with a cyclic vector f such that

$$\varphi(u) = (\pi(u)f, f) \quad \text{for all } u \in B.$$

We can deduce from Theorem 1 a suitable theorem for Banach $*$-algebras with unit. The crucial fact here is that:

(f) Every Banach $*$-algebra with a unit is a U^*-algebra.

It is now plain that the following theorem is true:

THEOREM 2. *Let B be a Banach $*$-algebra with unit and $\varphi\colon B \to L(H)$ a positive definite linear mapping. Then $\varphi = \Delta_R\pi$, where π is a $*$-representation of B on a suitable Hilbert space.*

It follows from the above theorem and from condition 1. (e) that if $\varphi \geqslant 0$, where B satisfies assumptions of Theorem 2, then φ is a bounded map from B into $L(H)$. In fact, the reason of this property of φ lies in the continuity of positive functionals on Banach $*$-algebras with unit. But something more is true, namely the following deep theorem is true:

(g) Let $p\colon B \to C^1$ be a linear positive functional on a Banach $*$-algebra possessing an approximate identity. Then p is bounded.

Now, if the linear map $\varphi\colon B \to L(H)$ is positive, then for every $f \in H$ the linear functional $p_f(u) = (\varphi(u)f, f)$ is positive and by (g) continuous. Consequently, by polarization formula every linear functional $u \to (\varphi(u)f, g)$ is bounded for $f, g \in H$. Using the uniform boundedness principle, we infer therefore that φ is bounded itself. We have just proved that:

(h) If $\varphi\colon B \to L(H)$ is a positive linear mapping on the Banach $*$-algebra having an approximate identity, then φ is bounded.

Notice also that the following condition holds true:

(i) If p is a linear positive functional on a Banach $*$-algebra B, then $|p(v^*hv)| \leqslant r(h)p(v^*v)$ for every $h, v \in B$, where $h = h^*$ and $r(h)$ stands for the spectral radius of h.

We are now ready to prove the following

THEOREM 3. *Suppose that B is a Banach $*$-algebra with continuous involution. Assume that B has an approximate identity. Suppose that $\varphi\colon B \to L(H)$ is a linear positive definite mapping. Then $\varphi = \Delta_R\pi$, where π is a $*$-representation of B onto a suitable Hilbert space.*

Proof. It follows from (h) that φ is continuous. Let $\{e_a\}$ be an approximate identity of B. Since the involution in B is continuous, $\{e_a^*\}$ is also an approximate identity.

Suppose now that $f(\cdot) \in \mathscr{F}(B, H)$. Then the functional $p_f(u)$ $= \sum_{v,z} \big(\varphi(v^*uz)f(z), f(v)\big)$ is positive and by (i)

$$|p_f(v^*hv)| \leqslant r(h)p(v^*v) \quad (v, h \in B),$$

if $h = h^*$. It follows that for $h = a^*a$ we have

$$\sum_{x,y} \big(\varphi(x^*e_a^*a^*ae_a y)f(y), f(x)\big) = p_f(e_a^*a^*ae_a) \leqslant r(a^*a)p_f(e_a^*e_a)$$

$$= r(a^*a)\sum_{x,y}\big(\varphi(x^*e_a^*e_a y)f(y), f(x)\big)$$

which by the limit passage proves that

$$\sum_{x,y}\big(\varphi(x^*a^*ay)f(y), f(x)\big) \leqslant r(a^*a)\sum_{x,y}\big(\varphi(x^*y)f(y), f(x)\big).$$

It follows that φ satisfies the boundedness condition (1) of Section 6 with $\varrho(a) = r(a^*a)$. On the other hand, since φ is continuous, $\lim\varphi(xe_a) = \varphi(x)$ for every $x \in B$ and $\sup_a \|\varphi(e_a^*e_a)\| < +\infty$. We are now in a position to apply Theorem 2 Section 6 and conclude that $\varphi = \Delta_R\pi$, where π is a *-representation of the *-semi-group B. Cutting down π to the minimal dilation and using the linearity of φ and Proposition 2 Section 6, we complete the proof.

Having in view that the involution in a C^*-algebra is continuous and every C^*-algebra has an approximate identity we arrive automatically at the following:

THEOREM 4. Let φ: $B \to L(H)$ be a positive definite linear mapping on the C^*-algebra B. Then $\varphi = \Delta_R\pi$, where π is a *-representation of B on a suitable Hilbert space.

For linear operator maps of C^*-algebras positive definiteness is equivalent to complete positivity, a property first defined by Stinespring [27]. The concept of complete positivity is quite natural in dilation theory over C^*-algebras and in fact is widely used by several authors, replacing the property we called positive definiteness.

Recall that an element u of a C^*-algebra B is called positive if $u = v^*v$ with some $v \in B$. Since B is isometrically *-isomorphic with a C^*-algebra of operators on a suitable Hilbert space K_B, u is positive, if it corresponds to a positive operator A_u in that algebra. Let us consider the algebra $B^{(n)}$ of $n \times n$ matrices over B. We can think about $B^{(n)}$ as the algebra of $n \times n$ operator matrices, that is an C^*-algebra of operators on $\underbrace{K_B \oplus \cdots \oplus K_B}_{n}$. Let $a = (a_{ij}) \in B^{(n)}$. It is now plain what it means that a is positive. Suppose that φ: $B \to L(H)$ is a linear map and define φ_n

as a map from $B^{(n)}$ into $L(\underbrace{H \oplus \ldots \oplus H}_{n})$ which maps the matrix (a_{ij}) onto the matrix $(\varphi(a_{ij}))$, applying φ element by element. We say that φ is *completely positive* if φ_n is a positive map for every $n = 1, 2, 3, \ldots$ If φ is completely positive, then it is positive definite. Indeed, for a fixed n, the matrix $(a_i^* a_k)_{i,k=1,\ldots,n}$ is a positive element of $B^{(n)}$. This can be seen simply by interpreting a_j as operators over K_B. Hence φ_n is a positive map and consequently the matrix $(\varphi(a_i^* a_k))$ defines on $H \oplus \ldots \oplus H$ a positive operator, which proves our claim.

Suppose now that φ is positive definite. Fix n and consider the positive element $(b_{ik}) \in B^{(n)}$. There is an $n \times n$ matrix $(a_{pq}) \in B^{(n)}$ such that for $i, k = 1, \ldots, n$, $b_{ik} = \sum_{j/1}^{n} a_{ji}^* a_{jk}$, and consequently, by the linearity of φ, $\varphi(b_{ik}) = \sum_{j/1}^{n} \varphi(a_{ji}^* a_{ik})$. Let $f_1, \ldots, f_n \in H$. Then

$$\Delta \overset{\mathrm{df}}{=} \sum_{i,k} (\varphi(b_{ik})f_k, f_i) = \sum_{i,k} \sum_{j} (\varphi(a_{ji}^* a_{jk})f_k, f_i).$$

It follows from Proposition 1 of Section 5 that there is a Hilbert space L and a map $h\colon B \times H \to L$ such that

$$(\varphi(u^* v)f, g) = (h(v, f), h(u, g))_L \quad \text{for } u, v \in B;$$

$f, g \in H$. We conclude that

$$\Delta = \sum_{i,k} \sum_{j} (h(a_{jk}, f_k), h(a_{ji}, f_i))_L$$

$$= \sum_{j} \left(\sum_{k} h(a_{jk}, f_k), \sum_{i} h(a_{ji}, f_i) \right)_L$$

$$= \sum_{j} \left\| \sum_{p} h(a_{jp}, f_p) \right\|_L^2 \geqslant 0. \quad \blacksquare$$

Consequently Theorem 4 can be stated as follows: every completely positive linear operator map of a C^*-algebra is dilatable to a $*$-representation. This is the original version of Stinespring's result [27].

In what follows we need some factorization construction concerning $*$-algebras. Namely suppose, that B is a $*$-algebra. We introduces the following condition:

(3)   For every $u \in B$ the number $\gamma(u) = \sup\{\|\pi(u)\|\colon \pi$ is a $*$-representation of $B\}$ is finite.

Suppose that (3) holds true. Then $\gamma(u)$ is a submultiplicative semi-norm because

$$\gamma(uv) = \sup_{\pi \in Z} \|\pi(u\,v)\| \leqslant \sup_{\pi \in Z} \|\pi(u)\|\,\|\pi(v)\| \leqslant \gamma(u)\gamma(v),$$

where $Z = \{\pi \colon \pi$ a $*$-representation of $B\}$. Moreover, since $\|\pi(u^*)\| = \|\pi(u)^*\| = \|\pi(u)\|$ for $\pi \in Z$ and $u \in B$, we have $\gamma(u^*) = \gamma(u)$. On the other hand, $\gamma(u^*u) = \sup_Z \|\pi(u^*u)\| = \sup_Z \|\pi(u)\|^2 = \gamma(u)^2$.

The set $J = \{u \in B \colon \gamma(a) = 0\} = \bigcap_{\pi \in Z} \operatorname{Ker}\pi$ is an ideal in B. J is symmetric, that is $u \in J$ if and only if $u^* \in J$. Let $\dot{u}$ be the coset of u modulo J and $\dot{u} = k(u)$. Then $\gamma(\dot{u}) = \gamma(u)$ is a norm in the quotient $*$-algebra B/J with involution $(\dot{u})^* = (\dot{u}^*)$. Let $\hat{B}$ be the completion of B/J with respect to this norm. It is plain that $\hat{B}$ with this norm is a C^*-algebra. $\hat{B}$ is called the C^*-algebra associated to B. It is clear that

(4)   If B is commutative, then $\hat{B}$ is commutative.

Notice that (3) holds true if B is a Banach $*$-algebra. Indeed, suppose for a moment that B has a unit e. Let π be a $*$-representation of B. Without loss of generality we can assume that π is unital. Let $h = u^*u$, where $u \in B$ and suppose that $|\lambda| > r(h) = $ spectral radius of h. Then $h_\lambda^{-1} = (e - -h/\lambda)^{-1}$ exists and equals $e + h/\lambda + h^2/\lambda^2 + \dots$; since π is unital, $\pi(h_\lambda)\pi(h_\lambda^{-1}) = \pi(h_\lambda h_\lambda^{-1}) = \pi(e)$, which proves that $\pi(h_\lambda)^{-1}$ exists; consequently λ is not in the spectrum of $\pi(h)$, which proves that $\|\pi(h)\| = r(\pi(h)) \leqslant r(h)$. Hence

$$\|\pi(u)\|^2 = \|\pi(h)\| \leqslant r(u^*u).$$

If B has no unit, then taking its unitization $B' = B + C^1$ and norm $\|(u, \lambda)\| = \|u\| + |\lambda|$ and extending π to π' over B' by the formula $\pi'(u + \lambda) = \lambda I + +\pi(u)$, we prove that $\|\pi(u)\| \leqslant r(u^*u)^{1/2}$ for $u \in B$. Also, if B is a U^*-algebra, then (3) holds true. Indeed if π is a unital $*$-representation of such B and

$$a = \sum_{i/1}^{n} a_i u_i \in B, \text{ with unitary } u_1, \dots, u_n, \text{ then}$$

$$\|\pi(a)\|^2 = \|\pi(a^*a)\| = \left\| \sum_{i,k} \bar{a}_i a_k \pi(u_i^* u_k) \right\| \leqslant \sum_{i,k} |\bar{a}_i a_k|$$

because $\pi(u_i^* u_k)$ are unitary operators. If is plain that the constant $\sum_{i,k} |\bar{a}_i a_k|$ does not depend on π but merely on a, which proves our claim.

We shall prove now that:

(5)   If $\varphi \colon B \to L(H)$ is a positive linear map of the U^*-algebra, then φ is γ-continuous.

Proof. Let f be a unit vector in H. Then $p_f(u) = (\varphi(u)f, f)$ is a positive functional on B. It follows from (2) that for all $u \in B$ $(\varphi(u)f, f)$ $= (\pi(u)\eta, \eta)$, where π is a $*$-representation on some space K, $\eta \in K$. π is a unital representation, which implies that $\|\eta\|^2 = (\varphi(e)f, f) \leqslant \|\varphi(e)\|$ and consequently $|p_f(u)| \leqslant \pi(u)\| \|\eta\|^2 \leqslant \|\varphi(e)\|\gamma(u)$. Hence $\|\varphi(u)\| \leqslant 2\|\varphi(e)\|$ $\gamma(u)$. ∎

Suppose now that B is a Banach $*$-algebra with continuous involution, having an approximate identity. Let $\varphi\colon B \to L(H)$ be a linear positive map. Then by (h) φ is continuous and since by the results of Section 5 if f is a unit vector

$$| (\varphi(e_a u)f, f) |^2 \leqslant (\varphi(e_a^* e_a)f, f) (\varphi(u^* u)f, f)$$

we have

$$| (\varphi(u)f, f) |^2 \leqslant \varphi(\varphi(u^* u)f, f),$$

where $\varphi = \|\varphi\| \cdot \sup \|e_a^* e_a\|$. Let $B' = B + C^1$ and define $\varphi'(u + \lambda) = \varphi(u) + \lambda \varrho I_H$. Since $p_f(u) = (\varphi(u)f, f)$ is a positive functional, we have $p_f((u e_a)^*)$ $= \overline{p_f(u e_a)}$, which proves that $p_f(u^*) = \overline{p_f(u)}$ for all $u \in B$. It follows that

$$(\varphi'((u + \lambda)^*(u + \lambda))f, f)$$

$$\leqslant p_f(u^* u) - 2|\lambda| |p_f(u)| + \varrho |\lambda|^2 \geqslant \frac{1}{\varrho} (|p_f(u)| - \varrho |\lambda|)^2 \geqslant 0$$

that is φ' is a positive map. Using (5) for $B' = B + C^1$ with norm $\|(u, \lambda)\|$ $= \|u\| + |\lambda|$ we infer that

$$\|\varphi'(v)\| \leqslant \varepsilon \gamma'(v)$$

for $v \in B'$ and some constant ε. Let π be a $*$-representation of B on K. Then $\pi'(u + \lambda) = \pi(u) + \lambda I_K$ is a $*$-representation of B'. We fix $u \in B$. For $\delta > 0$ there is a π' such that

$$\|\varphi(u)\| \leqslant \|\varphi'(u)\| \leqslant \varepsilon (\|\pi'(u)\| + \delta) = \varepsilon (\|\pi(u)\| + \delta),$$

where $\pi = \pi'|_B$. It follows that $\|\varphi(u)\| \leqslant \varepsilon\gamma(u)$.

We have just proved the following condition:

(6) Every operator valued linear positive map of a Banach $*$-algebra with continuous involution having an approximate identity is γ-continuous.

As a byproduct of the proof of (6) we get the following classical corollary:

(7) If $\varphi(u)$ is a positive linear functional on a Banach $*$-algebra B with continuous involution, having an approximate unit, then $\varphi(u) = (\pi(u)\eta, \eta)$, where π is a $*$-representation of B with a cyclic vector η.

Our next purpose is to prove the following technical proposition (recall, that k is the canonical map of B in $\hat{B}$):

PROPOSITION 1. *Suppose that B is a $*$-algebra which satisfies (3). Then to every linear γ bounded mapping φ: $B \to L(H)$ there corresponds a linear map $\hat{\varphi}$: $\hat{B} \to L(H)$ such that $\varphi = \hat{\varphi} \circ k$. Moreover:*

(8) φ *is positive if and only if $\hat{\varphi}$ is positive.*

(9) φ *is positive definite if and only if $\hat{\varphi}$ is positive definite.*

(10) φ *is a $*$-representation if and only if $\hat{\varphi}$ is a $*$-representation.*

(11) $\varphi = \varDelta_R \pi$, *where π is a linear γ-bounded map of B into $L(K)$ if and only if $\hat{\varphi} = \varDelta_R \hat{\pi}$*

Proof. It follows from our assumptions that $\|\varphi(u)\| \leqslant K\gamma(u)$ for some finite K and all $u \in B$. Consequently φ vanishes on J, which implies that the map $\hat{\varphi}$: $B/J \to L(H)$ determined by the formula $\hat{\varphi}(\dot{u}) = \varphi(u)$ is well-defined. Since $\|\hat{\varphi}(\dot{u})\| \leqslant K\gamma(\dot{u})$ $\hat{\varphi}$ extends uniquely all over $\hat{B}$ by continuity. It is plain that $\hat{\varphi} \circ k = \varphi$. It follows also that for $u_1, \ldots, u_n \in B, f_1, \ldots, f_n \in H$,

$$\sum_{i,k} \left(\hat{\varphi}(\dot{u}_i^* \dot{u}_k) f_k, f_i \right) = \sum_{i,k} \left(\varphi(u_i^* u_k) f_k, f_i \right),$$

which shows that if $\hat{\varphi}$ is positive (positive definite), then φ is positive (positive definite). If $\varphi \geqslant 0$, then since B/J is γ dense in $\hat{B}$, the above equality proves that $\hat{\varphi} \geqslant 0$. By a similar token the positivity of φ implies that of $\hat{\varphi}$. Hence (6), (7) hold true.

Condition (8) is trivial. To prove (9) we notice simply observe that

$$\hat{\varphi}(\dot{u}) = \varphi(u) = R^* \pi(u) R = R^* \hat{\pi}(\dot{u}) R$$

for $u \in B$ if $\varphi = \varDelta_R \pi$.

The importance of the above proposition lies in the fact that the study of suitable operator valued map φ of B can be reduced within certain limits to a study of the map $\hat{\varphi}$ defined on a C^*-algebra $\hat{B}$, which, as a rule, is much richer than the original algebra B. Having in view all that was said above, we arrive at the following

PROPOSITION 2. *Suppose that B is a $*$-algebra and φ: $B \to L(H)$ is a linear positive map. If B is a U^*-algebra or if B is a Banach $*$-algebra with continuous involution having an approximate unit, then φ is γ-continuous and consequently $\hat{\varphi}$ makes sense and is a linear positive map of $\hat{B}$ into $L(H)$.*

A nice application of Proposition 2 for commutative B is at hand. To begin with, we shall prove the following proposition, which characterizes dilatable linear operator valued maps of commutative C^*-algebras.

PROPOSITION 3. *Let* A *be a commutative* C^*-*algebra. Suppose that* $\varphi\colon A \to L(H)$ *is a linear positive map, that is* φ *is linear and* $\varphi(u^*u)$ *is a positive operator in* H *for every* $u \in A$. *Then* φ *is positive definite and consequently* $\varphi = \varDelta_R\pi$, *where* π *is a* $*$-*representation of* A.

Proof. We can think about A as an supnorm algebra of continuous complex functions vanishing at infinity over a locally compact Hausdorff space X with natural involution $u^* = \bar{u}(u \in A)$.

Suppose $f_1, \ldots, f_n \in H$ and $u_1, \ldots, u_n \in A$. Since φ is a positive linear map, $p_f(u) = (\varphi(u)f, f)$ is a positive linear functional over A. Hence, there is a regular Borel measure m on X such that

$$\sum_j (\varphi(u)f_j, f_j) = \int_X u\, dm.$$

Since $\varphi(u^*u)$ is a positive operator, we have for every u

$$|(\varphi(\bar{u}u)f_i, f_k)| \leqslant \|\varphi(\bar{u}u)^{1/2}f_i\| \, \|\varphi(\bar{u}u)^{1/2}f_k\|$$
$$= \sqrt{(\varphi(\bar{u}u)f_i, f_i)} \cdot \sqrt{(\varphi(\bar{u}u)f_k, f_k)}.$$

It follows by the Radon–Nikodym theorem that there are functions s_{ik} such that for all $u \in A$

$$(\varphi(u)f_i, f_k) = \int_X u s_{ik}\, dm \qquad (i, k = 1, \ldots, n).$$

Notice now that for $\lambda_1, \ldots, \lambda_n \in C^1$

$$\int_X u \sum_{i,k} s_{ik}\lambda_i\bar{\lambda}_k\, dm = \left(\varphi(u)\sum_i \lambda_i f_i, \sum_k \lambda_k f_k\right) \geqslant 0$$

if $u \geqslant 0$. Consequently, $\sum_{i,k} s_{ik}\lambda_i\bar{\lambda}_k \geqslant 0$ almost everywhere m (a.e. m). If $u_1, \ldots, u_n \in A$, then the matrix $\{\bar{u}_k u_i\}$ is positive definite at every $x \in X$, which by Schur's lemma proves that $\sum_{i,k} \bar{u}_k u_i s_{ik} \geqslant 0$ a.e. m. It follows that

$$\sum_{i,k} (\varphi(\bar{u}_k u_i)f_i, f_k) = \int_X \left(\sum_{i,k} \bar{u}_k u_i s_{ik}\right) dm \geqslant 0.$$

We have proved that $\varphi \geqslant 0$; the rest of the assertion follows from Theorem 4.

Theorem 4 of Section 6 can be deduced from the above proposition. In fact, the proofs of both of them are essentially the same. What is common to both results is the fact that every linear positive φ on $C(X)$ can be written as a so called *semi-spectral integral* $\varphi(u) = \int_X u\, dF$. Such integrals will be discussed later on in detail.

The trick with Schur's lemma, which appears above as well in Theorem 4 of Section 6 will be used in the proof of the proposition below:

PROPOSITION 4. *Let the operator function* K: $X \times X \to L(H)$ *be weakly positive definite and assume that the values* $K(x, y)$ *commute. Then* K *is positive definite.*

Proof. Since K is weakly positive definite, we have $K(x, y)$ $= K(y, x)^*$ for all x, y, which implies that $\{K(x, y), K(u, v)^*; x, y, u, v \in X\}$ is a commutative family of normal operators.

We fix $x_1, \ldots, x_n \in X$ and consider the C^*-algebra generated by I_H and by the operators $K(x_i, x_k)$ $(i, k = 1, \ldots, n)$. This algebra is commutative and by a suitable form of the spectral theorem (see for instance [6]) there is a spectral measure E on Borel sets of the spectrum S of this algebra such that

$$\big(K(x_i, x_k)f, g\big) = \int_S \varphi_{ik}(s) d(E_s f, g)$$

for $f, g \in H$ and suitable continuous functions φ_{ik}. Let $f_1, \ldots, f_n \in H$ and define the measure $\mu = \sum_k (Ef_k, f_k)$. Let h_{ik} be the Radon–Nikodym derivative of the measure (Ef_k, f_i) with respect to μ. Since K is weakly positive definite we have for a Borel set σ and $\lambda_1, \ldots, \lambda_n \in C^1$

$$\sum_{i,k} \big(K(x_i, x_k)E(\sigma)f_j, E(\sigma)f_j\big)\lambda_k \bar{\lambda}_i = \sum_{i,k} \big(K(x_i, x_k)E(\sigma)f_j, f_j\big)\lambda_k \bar{\lambda}_i$$

$$= \sum_{i,k} \Big(\int_\sigma \varphi_{ik}(s) h_{jj} d\mu\Big)\lambda_k \bar{\lambda}_i \geqslant 0.$$

It follows that for any $(\lambda_1, \ldots, \lambda_n) \in C^n$

$$h_{jj}(s) \sum_{i,k} \varphi_{ik}(s)\lambda_k \bar{\lambda}_i \geqslant 0$$

for $s \in S - \sigma_j$, where $\mu(\sigma_j) = 0$. Consequently

$$\Big(\sum_{j=1}^n h_{jj}\Big)\Big(\sum_{i,k=1}^n \varphi_{ik}\lambda_k \bar{\lambda}_i\Big) \geqslant 0$$

for arbitrary $\lambda_1, \ldots, \lambda_n \in C^1$ and

$$s \in \bigcap_{j=1}^n (S - \sigma_j) = S - \Big(\bigcup_{j=1}^n \sigma_j\Big).$$

Obviously $\mu(\bigcup \sigma_j) = 0$. On the other hand, the set $\sigma_0 = \big\{s: \sum_{j=1}^n h_{jj}(s) = 0\big\}$ is of measure μ zero. Indeed $\mu(\sigma_0) = \sum_j \big(E(\sigma_0)f_j, f_j\big) = \sum_j \int h_{jj} d\mu$

$= \int \left(\sum h_{jj} \right) d\mu = 0$. Hence $\sum h_{jj} > 0$ a.e. which by the previous inequality shows that

$$\sum_{i,k} \varphi_{ik}(s) \lambda_k \bar{\lambda}_i \geqslant 0$$

for all $(\lambda_1, \ldots, \lambda_n) \in C^n$ and for s off a set of zero μ measure.

Since $\left\| E(\sigma) \sum_j \lambda_j f_j \right\|^2 = \int_\sigma \left(\sum_{i,k} h_{ik} \lambda_k \bar{\lambda}_i \right) d\mu \geqslant 0$ for each σ, we have

$$\sum_{i,k} h_{ik}(s) \lambda_k \bar{\lambda}_i \geqslant 0$$

for all $(\lambda_1, \ldots, \lambda_n) \in C^n$ and for s off a set of zero μ measure. By Schur's lemma we can now conclude that

$$\sum_{i,k} h_{ik}(s) \varphi_{ik}(s) \geqslant 0$$

for μ almost all s, which by integrating gives us

$$\sum_{i,k} \left(K(x_i, x_k) f_k, f_i \right) = \sum_{i,k} \int \varphi_{ik} h_{ik} d\mu \geqslant 0,$$

completing the proof.

Having in view Theorem 4, we conclude by the above proposition that the following holds true:

PROPOSITION 5. *Let* $\varphi: B \to L(H)$ *be a positive linear operator valued map of the* C^*-*algebra* B. *If* $\varphi(B)$ *is a commutative family of operators, then* $\varphi = \Delta_R \pi$, *where* π *is a* *-*representation of* B.

We shall now prove the following theorem:

THEOREM 5. *Let* B *be a commutative* *-*algebra and* $\varphi: B \to L(H)$ *a positive linear map. Suppose that*

(12)   *B is a* U^*-*algebra*

or

(13)   *B is a Banach* *-*algebra with continuous involution, having an approximate unit.*

Then φ *is positive definite and consequently* $\varphi = \Delta_R \pi$, *where* π *is a* *-representation of* B.

Proof. It follows from Proposition 2 that $\hat{\varphi}$ makes sense and by Proposition 1 $\hat{\varphi}$ is a positive map. Hence by Proposition 3 $\hat{\varphi}$ is positive definite because $\hat{B}$ is commutative by (4). Using (11) of Proposition 1 we complete the proof.

Bounded positive definite operator functions on *-semi-groups can be studied by using dilation properties of functions on star algebras.

For, with every ∗-semi-group we can associate a discrete semi-group algebra in a way modelled after the manner in which the group algebras are related to groups. To begin with, let us consider a ∗-semi-group $\mathscr{S}$ with a unit e (for simplicity). We define $l_1(\mathscr{S})$ as the collection of all complex valued functions ξ on $\mathscr{S}$ with countable supports and such that $\sum_s |\xi(s)| < +\infty$. $l_1(\mathscr{S})$ is a Banach space with the norm

$$\|\xi\| = \sum_{s \in \mathscr{S}} |\xi(s)|$$

with pointwise defined linear operations. $l_1(\mathscr{S})$ is a Banach algebra with multiplication defined as "convolution", namely

$$(\xi \cdot \eta)(s) = (\xi * \eta)(s) = \sum_{uv=s} \xi(u)\eta(v).$$

The mapping $\xi \to \xi^*$, where $\xi^*(s) = \overline{\xi(s^*)}$ defines in $l_1(\mathscr{S})$ an isometric involution.

Notice that the Dirac δ-function at e denoted by δ_e is a unit in $l_1(\mathscr{S})$. $l_1(\mathscr{S})$ is commutative if $\mathscr{S}$ is commutative.

Suppose now that the operator valued map φ: $\mathscr{S} \to L(H)$ is bounded, i.e., $\sup_s \|\varphi(s)\| < +\infty$. Then the map $\tilde{\varphi}$: $l_1(\mathscr{S}) \to L(H)$ defined by formula $\tilde{\varphi}(\xi) = \sum \xi(s)\varphi(s)$ makes sense. It is now easy to check that:

(14) φ is positive definite (weakly positive definite) if and only if $\tilde{\varphi}$ is positive definite (weakly positive definite).

Moreover, if $\tilde{\varphi} = \Delta_R \psi$, where ψ is a ∗-representation of $l_1(\mathscr{S})$, then $\varphi = \Delta_R \pi$, where $\pi(s) = \psi(\delta_s)$ is a ∗-representation of $\mathscr{S}$. δ_s stands here for the Dirac δ-function at the point s.

Using now Theorem 2 we can prove Arveson's Theorem 3.6 as follows: Suppose that φ is bounded and positive definite. Then $\tilde{\varphi}$ is positive definite and by Theorem 2 $\tilde{\varphi} = \Delta_R \psi$, where ψ: $l_1(\mathscr{S}) \to L(K)$ is a ∗-representation of $l_1(\mathscr{S})$. It follows that φ itself has an R-dilation which is a ∗-representation of $\mathscr{S}$ and the proof of Theorem 3.6 is complete.

Arveson's original of Theorem 3.6 uses the γ-continuity properties of $l_1(\mathscr{S})$ and runs as follows: if $\varphi \gg 0$ and φ is bounded, then $\tilde{\varphi}$ makes sense by Proposition 2 and $\overset{\circ}{\tilde{\varphi}} \gg 0$ by Proposition 1, because $l_1(\mathscr{S})$ is a ∗-Banach algebra with a unit and continuous involution. Now we apply Stinespring's Theorem 4 and prove the claim.

The next example is the following one:

PROPOSITION 6. *Let $\mathscr{S}$ be a commutative ∗-semi-group with a unit. If φ: $\mathscr{S} \to L(H)$ is a bounded weakly positive definite function, then $\varphi = \Delta_R \pi$, where π is a unital ∗-representation of $\mathscr{S}$.*

Proof. Since φ is bounded, $\tilde{\varphi}$ makes sense and is a positive map. But $l_1(\mathscr{S})$ is commutative. By Theorem 5 $\tilde{\varphi} = \varDelta_R \psi$, where ψ is a unital *-representation of $l_1(\mathscr{S})$, which completes the proof.

Since weakly positive definite operator functions on groups are bounded, we get as a product of Proposition 6 the following

COROLLARY 1. *Let G be an abelian group and* $\varphi\colon G \to L(H)$ *an operator function. Then φ is positive definite if and only if φ is weakly positive definite. In particular, every weakly positive definite operator function φ on G can be written as* $\varphi = \varDelta_R \pi$, *where π is a unitary representation of G* (by Theorem 1, Section 7).

Suppose that G is the additive group Z of integers and let $T_n \in L(H)$ $(n \in Z)$ be a sequence of operators. Then taking $\varphi(n) = T_n$ we infer from our corollary that T_n is a positive definite sequence if and only if for every $f \in H$ the scalar sequence $(T_n f, f)$ is a positive definite one. It follows then by Proposition 4 Section 7 that

(15) $T_n = \varDelta_R U^n$ $(n = 0, \pm 1, \pm 2, \ldots)$, where U is a unitary operator
        if and only if for every f the sequence $(T_n f, f)$ is positive definite.

Suppose now that $\mathscr{S}$ is a *-semi-group with a unit e and $a \in \mathscr{S}$. Call $a \in \mathscr{S}$ normal if $aa^* = a^*a$. Let φ be a bounded and weakly positive definite operator valued map on $\mathscr{S}$. Let $\mathscr{S}_a$ be the *-subsemi-group of $\mathscr{S}$ generated by e and a. Since a is normal $\mathscr{S}_a$ is commutative and by Proposition 6 the restriction φ_a of φ to $\mathscr{S}_a$ is dilatable to a *-representation of $\mathscr{S}_a$. If $\varphi(e) = I_H$, then we can apply Corollary 1 Section 6 and obtain

COROLLARY 2. *Let a be a normal element of the unital *-semi-group $\mathscr{S}$ and φ a bounded and weakly positive definite operator valued map of φ into $L(H)$ such that $\varphi(e) = I_H$. Then* $\varphi(a)^*\varphi(a) \leqslant \varphi(a^*a)$.

A nice application of the above corollary in operator theory is at hand.

PROPOSITION 7. *Let B be a C^*-algebra of operators in the Hilbert space K and suppose $I_K \in B$. Suppose that $\varphi\colon B \to L(H)$ is a unital positive linear map. Then*

(16) $\varphi(A)^*\varphi(A) \leqslant \varphi(A^*A)$

for every normal operator $A \in B$.

Proof. Since φ is linear and $\|A^*\| \, \|A\| = \|A^*A\|$ we can assume that A is a contraction. Let $\mathscr{S}$ be the multiplicative *-semi-group of contractions in B. Since φ is positive φ is weakly positive definite on $\mathscr{S}$ and by (h) $\varphi|_{\mathscr{S}}$ is bounded. Using Corollary 2, we complete the proof.

Notice that (16) holds true for every positive definite φ on B for any $A \in B$ (see Corollary 1 Section 6).

Notes. The theorems of this section were initiated by Theorem 4 of Stinespring [27] (for C^*-algebras with a unit), who also proved Proposition 3.

For references on *-algebras, cyclic representations etc. see Bonsall–Duncan [2] and Naimark [22]. Proofs of (e), (f), (g), (i) will be found in [2]. For U^*-algebras see Palmer [24], for C^*-algebras Dixmier [5].

The γ-continuity and related properties are classical material to be found in different terms in Naimark [22].

Theorem 1 belongs to Paschke [25]; Theorem 2 is a simple consequence — for its direct proof for algebras with continuous involution see Mlak–Szymański [18], where one finds also Theorem 3. Condition (5) and the part of Proposition 2 concerning U^*-algebras was proved by Paschke [25], to whom belongs the suitable part (ad (12)) of Theorem 5. Theorem 5 for Banach algebras with unit and continuous involution (which is a superfluous assumption) appears in Maserick [14]. In fact, dilation theorems for operator functions on Banach *-algebras with approximate identity can be reduced to those concerning algebras with unit, see Szafraniec [29]. The approximate unit arguments in dilation theory appear in group setting in Umegaki [39]. Proposition 5 belongs to Arveson [1] — Proposition 4 is only a slight extension of his arguments. Proposition 6 appears in Maserick [14] with an analytic proof via semi-spectral measures and semi-characters.

Corollary 2 is essentially due to Davis [4], to whom belongs also Proposition 7 which contains a result of Kadison [9], who proved (16) for $A = A^*$. Choi in [3] proved that (16) holds true for 2-positive definite linear mappings. More information on operator inequalities will be found in Størmer [28] and Choi [3], containing suitable references. For a summary and references on harmonic analysis on semi-groups and related l_1 algebras see Dunkl–Ramirez [7] and Lindashl–Maserick [12].

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