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GEOMETRIC ASPECT OF WEYL SUMS

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The idea of making visible the Weyl sums $\sum_{n\leqslant x} e(u_n)$ associated with a sequence (u_n) of real numbers is due to F. Dekking and M. Mendès-France (cf. [1]): they draw in the complex plane the polygon $\Gamma[u]$, the successive vertices of which are the complex numbers $z_0, z_1, \ldots, z_N = \sum_{n\leqslant N} e(u_n), \ldots$ Our aim is to show how one can make more precise the concluding remarks of Mendès-France's talk in the 1981 János Bolyai Colloquium in Budapest.

Using a program written by M. Pallard,(1) F. Dress obtained the curious curve $\Gamma[(n^{3/2})]$ (Fig. 1). The vicinity of the curve with the first bisector reminded me the $(1+i)$ factor appearing in the van der Corput treatment of Weyl sums. Van der Corput results, as they can be found in Titchmarsh's book [3], for example, are sufficient to prove the observed vicinity, but too big an error term hides the extreme regularity of the curve. In order to simplify the numerical aspect, we shall restrict ourselves to the curve $\Gamma[(\frac{4}{9}n^{3/2})]$ (Fig. 2).

Some numerical evidence leads one to the conviction that it is possible to locate the "attractors" with a $o(1)$-precision, despite the $\Omega(1)$-length of each side of the polygon! Indeed, if one replaces the polygon $\Gamma[u]$ by $\Gamma^*[u]$ obtained by linking the mid-points of two consecutive sides of $\Gamma[u]$ (Fig. 3 and 4), one observes such a good approximation. The mistery starts to fade

(1) I take this opportunity to express my thanks to all persons and institutions who made this lecture possible, specially to Michel Pallard who is responsible for the graphical part, Henryk Iwaniec for his successful organization of the Meeting, the Polish Academy of Sciences and the Franco-Polish cooperation for permitting my participation to the Meeting.

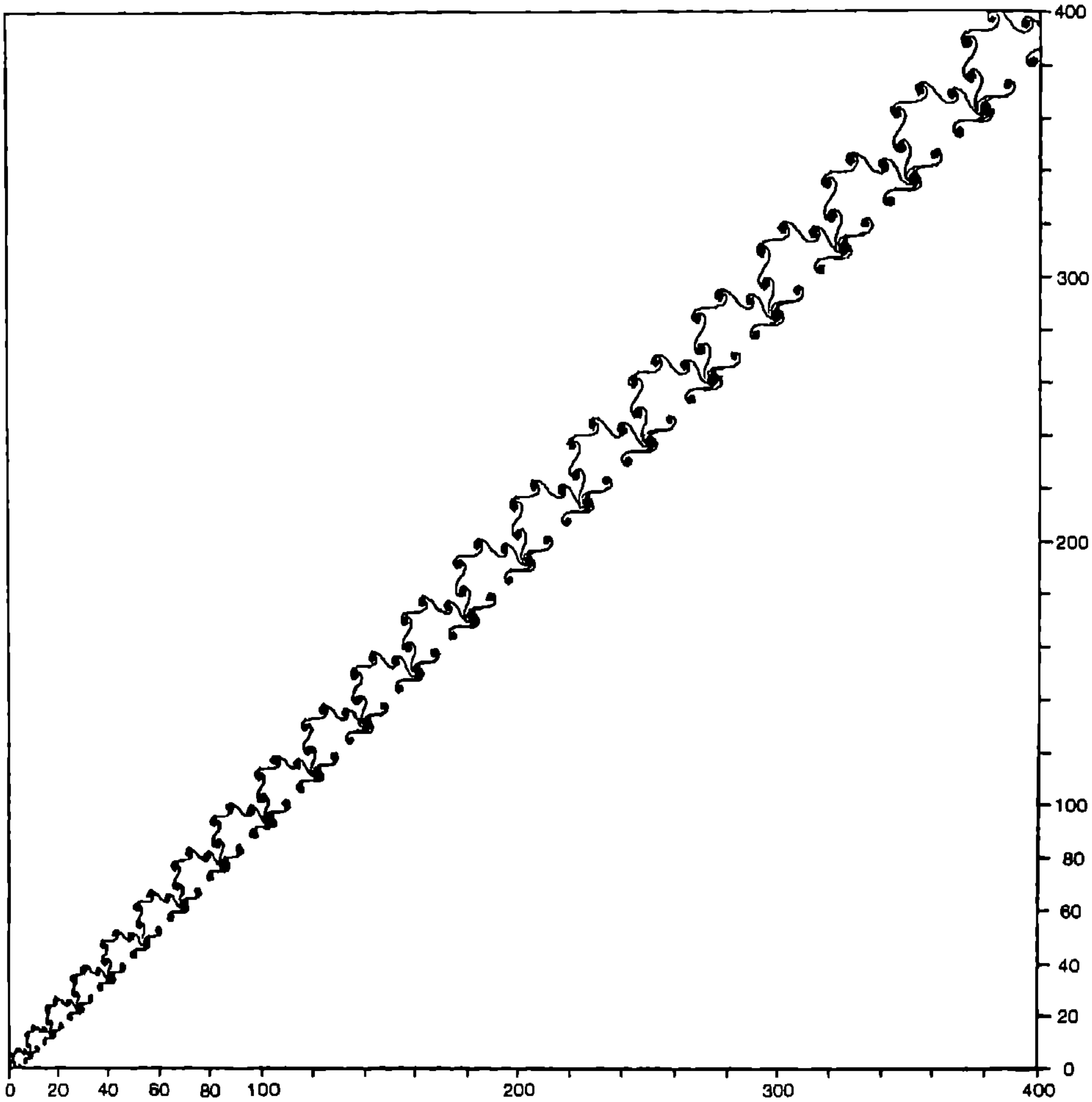

Fig. 1. $\Gamma[(n^{3/2})]$

when one notices that $\Gamma^*[u]$ is connected to

$$S(N) = \tfrac{1}{2} \sum_{n \leqslant N} \big(e(u_n) + e(u_{n-1})\big), \tag{1}$$

which is more adequate to Poisson summation formula.

Let us look at our special case

$$u_n = f(n) = \tfrac{4}{9} n^{3/2}. \tag{2}$$

The Poisson summation formula leads to

$$S(t) = \sum_{h \in \mathbf{Z}} \int_0^t e\big(f(\xi) - h\xi\big)\, d\xi \tag{3}$$

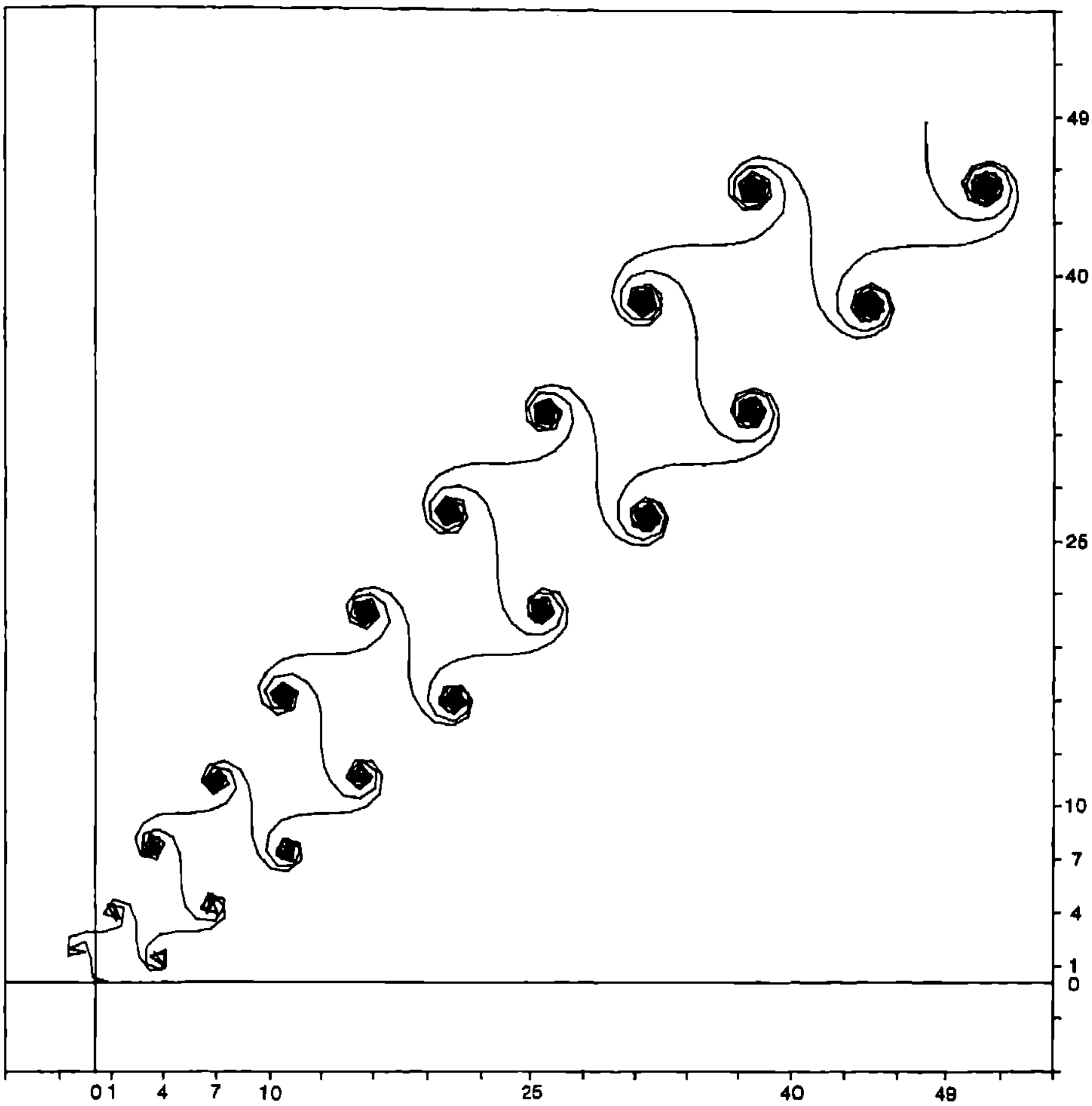

Fig. 2. $\Gamma[(\frac{4}{9}n^{3/2})]$, $n \leqslant 992$

for integral t, and this relation may be used to define $S(t)$ for non-integral t as well (in this way, one replaces $\Gamma^*[u]$ by a smoother curve $\Gamma^*[f]$ which coincides at integral arguments). For a positive integral H, we shall denote by t_H the real number such that

(4) $$f'(t_H) = H + 1/2$$

in our special case, one has

(4′) $$t_H = \tfrac{9}{4}(H+1/2)^2 \sim \tfrac{9}{4}H^2.$$

In the expansion (3) for $S(t_H)$, the contribution of the terms corresponding to $h \leqslant 0$ or $h > H$ may be seen to be $O(\log H)$, by integration by parts (cf. [3], ch. IV); on the other hand, the contribution of the terms corresponding to

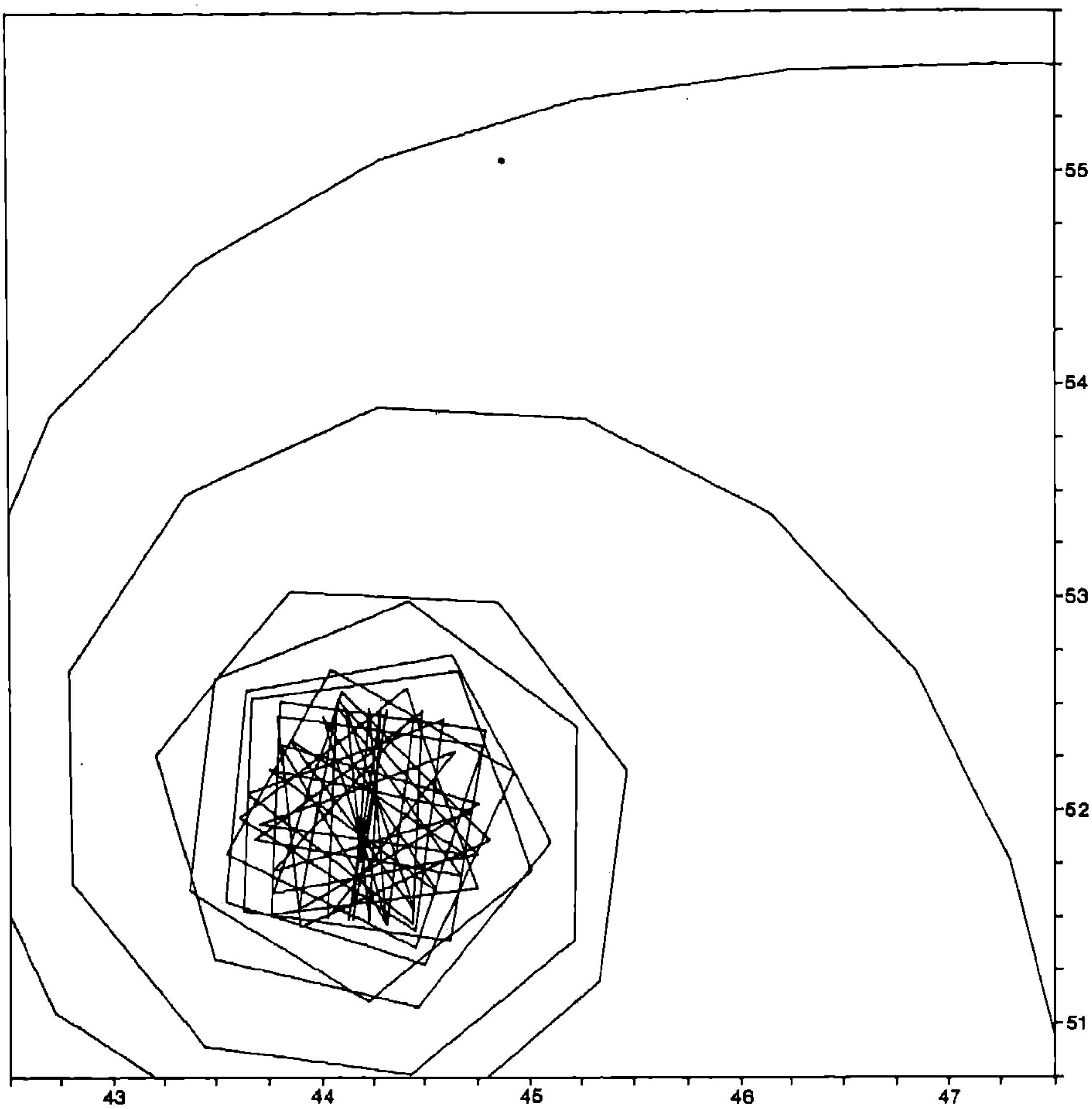

Fig. 3. $\Gamma[(\frac{4}{9}n^{3/2})]$, $n \sim 1040$

$0 < h \leqslant H$ may be estimated by the method of the stationary phase (cf. [2], ch. 2, § 9 for a description of the method). In this way, one obtains the relation

$$(5) \qquad S(t_H) = e^{-i\pi/4} \sum_{0<h\leqslant H} e\big(f(x_h) - hx_h\big)|f''(x_h)|^{-1/2} + O(\log H)$$

where x_h is defined by

$$(6) \qquad f'(x_h) = h$$

is our case, this formula reads

$$(5') \qquad S(t_H) = \tfrac{3}{2}(1+i) \sum_{0<h\leqslant H} h^{1/2} e(h^3/4) + O(\log H)$$

and one should notice that the error-term may even be improved upon by a

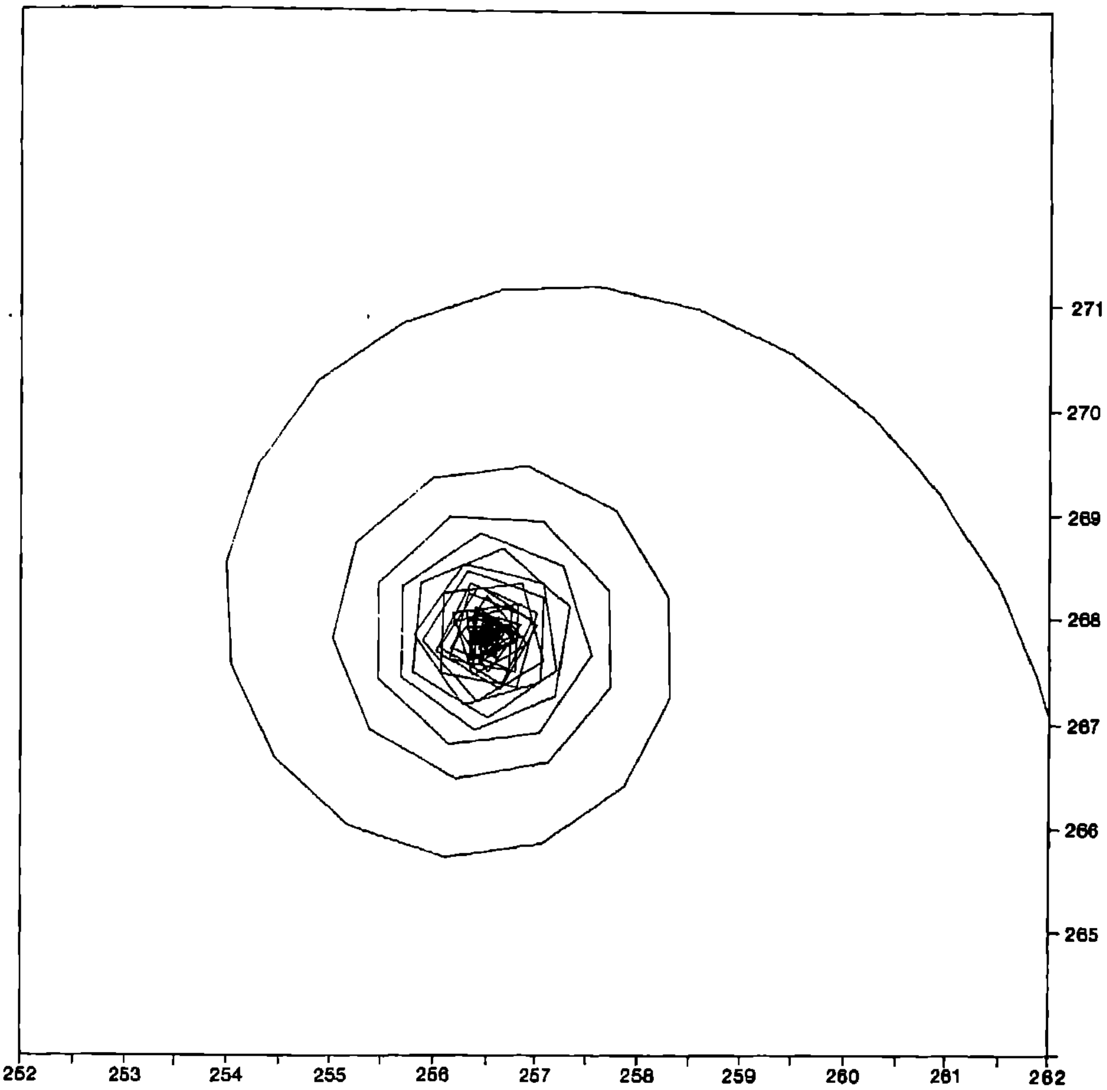

Fig. 4. $\Gamma^*[(\frac{4}{9}n^{3/2})]$, $n \sim 9653$

more careful treatment. As it is, relation (5) shows that if f is such that

$$f'(t) = o(t) \tag{7}$$

holds, then certain points $S(t)$ may be well approximated by trigonometrical sums of length $o(t)$. On the other hand, the fact that, in our special case, $f(x_h) - hx_h$ is a rational polynomial (cf. (5′)) is responsible for the "periodical" pattern of the points $S(t_H)$; the reader may prove that it is also the case (with period 9) for $f(x) = x^{3/2}$ (Dress curve) (it is not difficult to determine all functions $f(x) = \beta x^\alpha$ for which $f(x_h) - hx_h$ is a rational polynomial); the case $f(n) = \frac{1}{3}n^{3/2}$ leads to the nice curve of Fig. 5 and 6, the zero mean-value of $e(f(x_h) - hx_h)$ being responsible for the balance of the curve around the origin.

When t goes away from t_H, one may at first neglect the variations in all

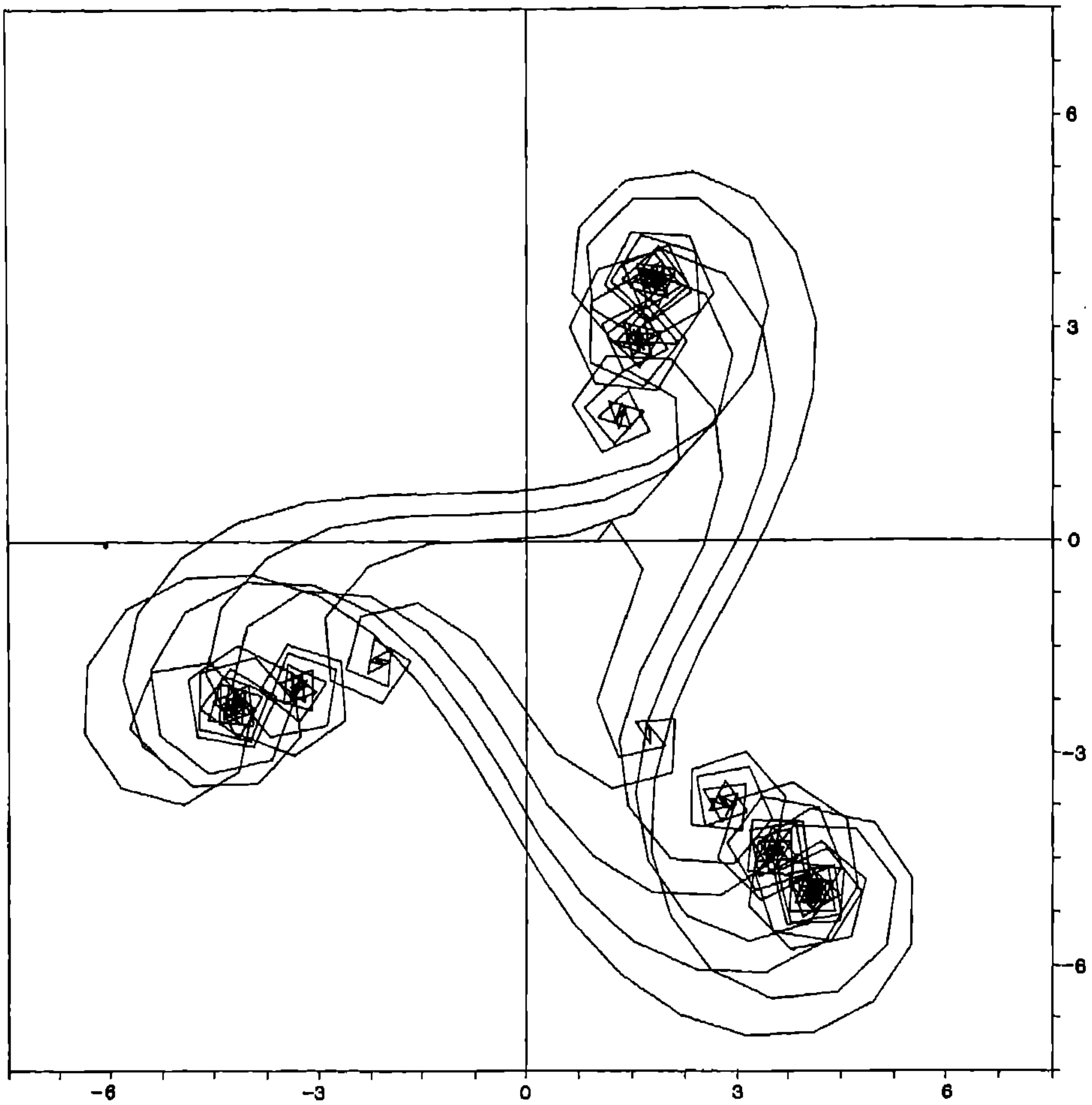

Fig. 5. $\Gamma^*[(\frac{1}{3}n^{3/2})]$, $n \leqslant 500$

the terms in (3) corresponding to $h \neq H$, that of the remaining term being responsible for the spiral aspect of the curve $\Gamma^*[f]$ around the attractor $S(t_H)$.

The limit for the curve $\Gamma^*[f]$ being composed of spirals connecting attractors in condition (7). However, a somewhat alike situation may be observed for some curves $\Gamma[\alpha n^2]$, but not all (cf. Fig. 3 and 4 of [1]), a similar situation arising when α has a good rational approximation $\alpha = \dfrac{a}{q} + \dfrac{\theta}{q^2}$, θ being small compared with 1. In this case, one may group q by q the terms in $\sum e(\alpha n^2)$. Let us choose an integer n_0 such that $\theta n_0/q$ is small modulo 1 (one may choose n_0 in connection with the rational approximations of α with

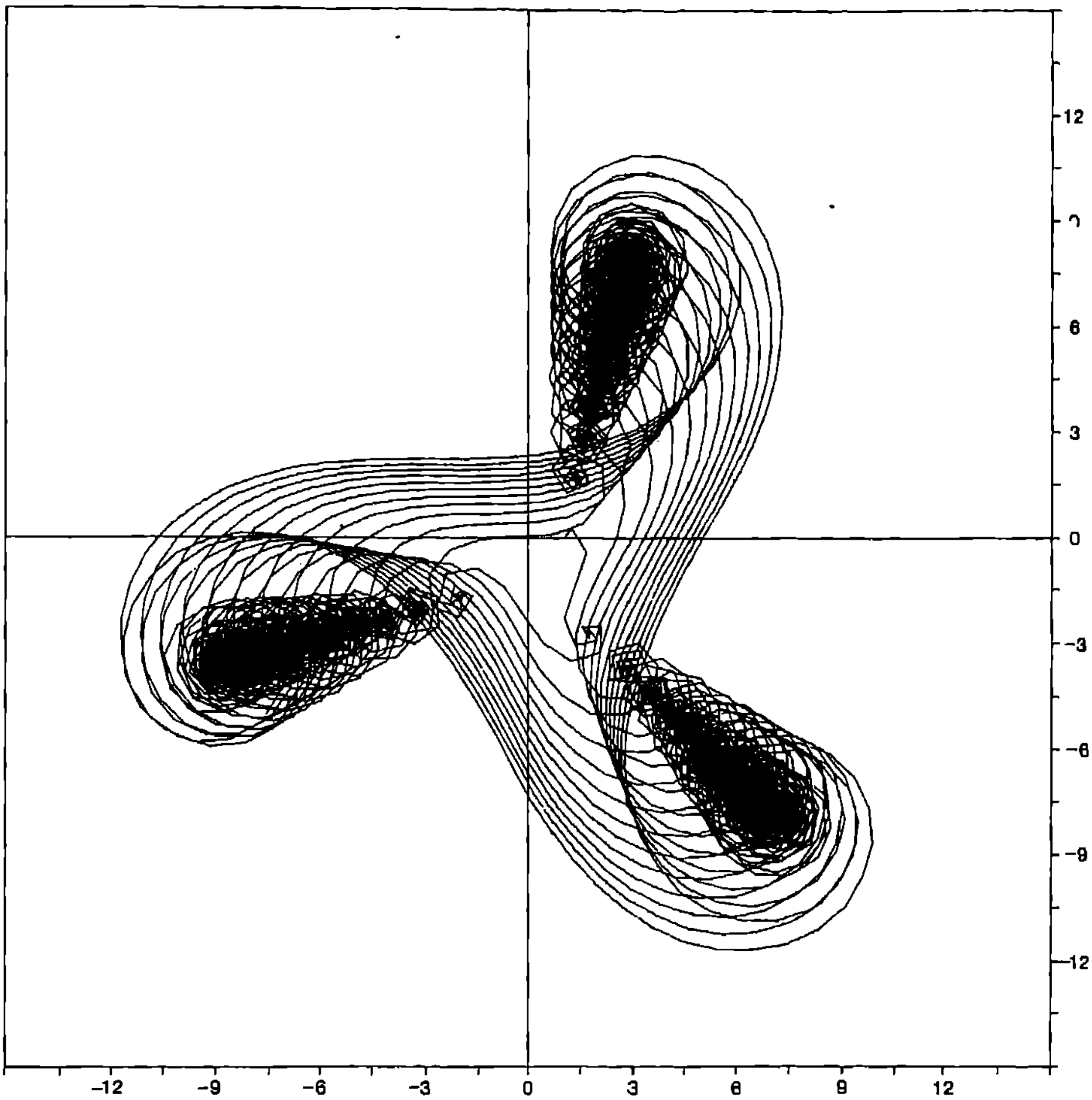

Fig. 6. $\Gamma^*[(\frac{1}{3}n^{3/2})]$, $n \leqslant 5000$

denominator larger than q), and define

$$T_{n_0}(k) = \sum_{n=n_0+kq+1}^{n_0+(k+1)q} e(\alpha n^2). \tag{8}$$

A standard computation shows that $T_{n_0}(k+1)$ is very close to $e(\theta(2k+1))\,T_{n_0}(k)$, as long as k is not too large, so that $T_{n_0}(k)$ is well approximated by $e(\theta k^2)\,T_{n_0}(0)$, and finally $T_{n_0}(0)+\ldots+T_{n_0}(k)$ is itself well approximated by $\int_0^k e(\theta t^2)\,dt.\ T_{n_0}(k)$, the Fresnel integral being responsible of the Cornu spiral, one may see in the curve.

Each good approximation gives raise to such spirals, whence the fractal

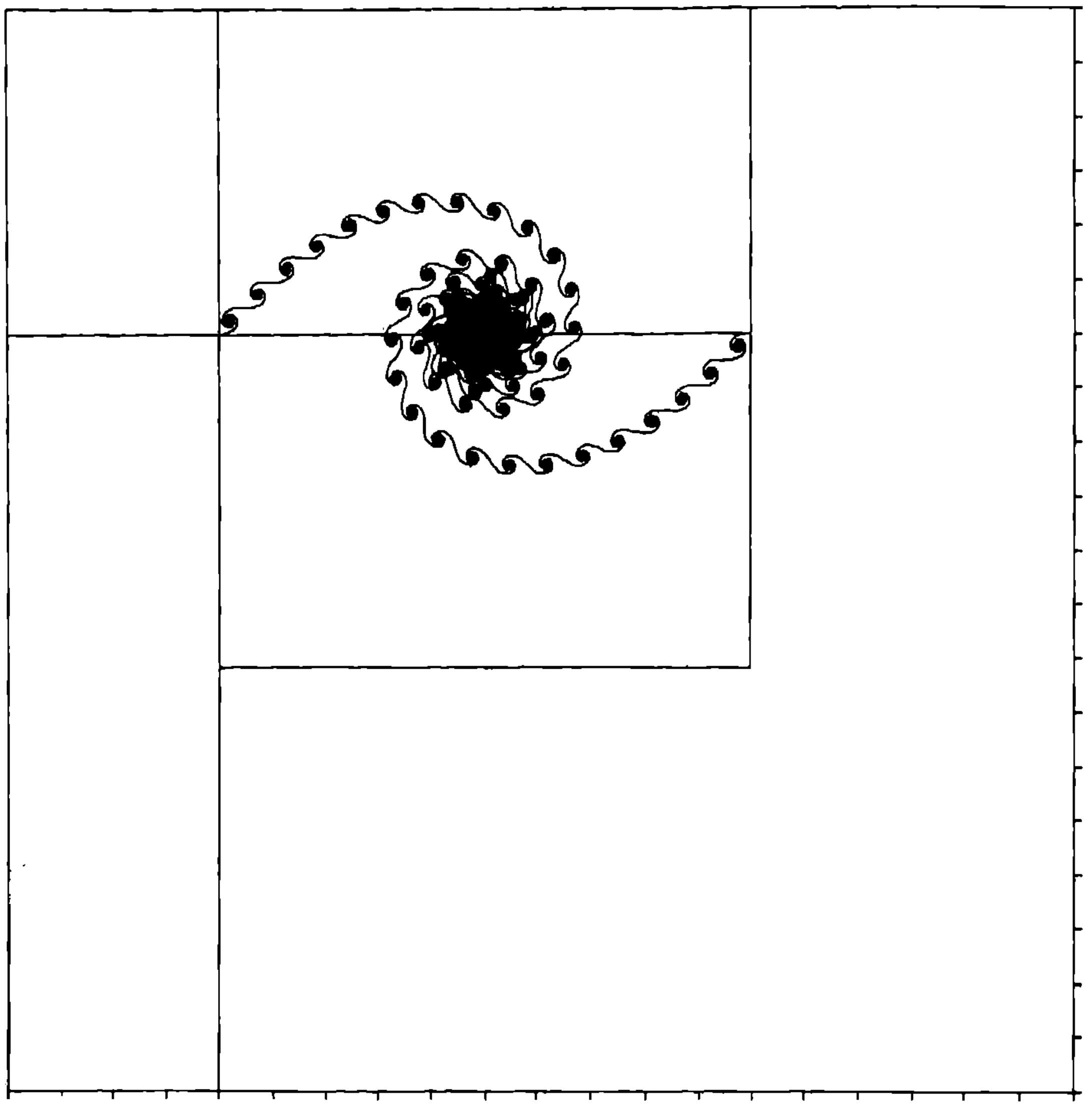

Fig. 7. $\Gamma[(\frac{100}{10001}n^2)]$, $n \leqslant 10000$

aspect of the curve. Fig. 7 shows the case $\alpha = \dfrac{100}{10001} = \dfrac{1}{100+\dfrac{1}{100}}$ with two rational approximations (each small spiral being a polygon with 100 sides).

References

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