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**Reciprocity theorems in the theory of representations  
of groups and algebras**

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## Introduction

The aim of this work is to obtain a uniform approach to duality phenomena in the theory of induced representations of groups and group algebras. The starting point to the investigation on different types of induced representations is a simple duality theorem of a more general character.

We consider here a topological algebra  $A$  with involution and approximate unit, which is an  $S$ -module ( $S$  being an arbitrary set), so that the action of  $S$  on  $A$  commutes with the left regular representation of  $A$ .

For a given  $S$ -module  $V$  we can now apply the following procedure: in the space of the inductive tensor product  $A \overline{\otimes} V$  we determine a subspace  $Y$ , which is a closed linear span of the set of elements of the form

$$sa \otimes v - a \otimes sv, \quad a \in A, s \in S, v \in V.$$

The quotient space  $A \overline{\otimes} V / Y$  with the natural action of the algebra on the first argument is a representation, which in this paper is referred to as the representation of  $A$  induced by the  $S$ -module  $V$  and denoted by  $(\lambda^{(S, V)}, A \overline{\otimes}_S V)$ .

For an arbitrary continuous representation  $(\pi, E)$  of the algebra  $A$ , in a complete bornological space we can construct a subspace  $\Phi(E) \subset E$  equipped with a topology finer than that inherited from  $E$  and such that  $\pi|_{\Phi(E)}$  is also the continuous representation. Theorem 2.4 of the present work, which forms the basis for further investigations, states a natural isomorphism of spaces of bilinear separately continuous invariant forms

$$(0.1) \quad B_A(A \overline{\otimes}_S V, E) = B_S(\Phi(E), V).$$

The algebra  $A$  is here assumed to be a bornological space and  $V$  as well as  $E$  are complete and bornological. The space  $\Phi(E)$  admits a natural structure of  $S$ -module inherited from  $A$ .

The relation of this theorem to Bruhat's theory of differentiable induced representations is investigated in Section 3. If  $G$  is a Lie group countable at infinity and the convolution algebra  $\mathcal{D}(G)$  of the Schwartz functions is taken as  $A$ , then the space  $\Phi(E)$  appears to be a well-known object and a recognized tool of the theory. Namely, if  $(\pi, E)$  is a repre-

sentation in a Fréchet space, associated with a continuous representation of  $G$ , i.e.

$$\pi(f) = \int_G f(g)\pi(g)dg, \quad f \in \mathcal{D}(G),$$

it can be proved that  $\Phi(E)$  is identical with  $E_\infty$ , the space of smooth vectors of the representation  $\pi$ .

If the differentiable representation of a subgroup  $\Gamma \subset G$  in the Fréchet space  $V$  is substituted for  $\mathcal{S}$ -module, the space  $A \overline{\otimes}_\Gamma V$  is isomorphic to the space of the differentiable induced representation  $\mathcal{D}^\sigma$ , which is the subject of Bruhat's theory [4].

The isomorphism appears to be identical with the theorem which Bruhat calls the Frobenius reciprocity theorem. Our generalization of Bruhat's theory is twofold. Firstly, instead of a Lie group, a metrizable Yamabe group is considered. The convolution algebra  $\mathcal{D}(G)$  of a Yamabe group has been investigated by several authors (Bruhat [5], Kac [13], K. Maurin [17]). What has been left to do is to give an interpretation to isomorphism (0.1). The space  $\Phi(E)$  is still composed of smooth vectors of the group representation. The space  $A \overline{\otimes}_\Gamma V$  is also isomorphic to a function space with the action of  $G$  by means of left translations. Making use of (0.1) we obtain the generalized Bruhat theorem

$$(0.2) \quad {}^!B_G(\mathcal{D}^\sigma, E) = B_\Gamma(V, E_\infty).$$

A non-trivial improvement of Bruhat's theory lies in the fact that our theorem can be equally well applied to non-differentiable inducing representations. As it has been shown in the case of a Lie group by J. Szmidt and the author [38] this version of the Bruhat theorem leads, for  $G = \Gamma$ , to theorem of N. Skovhus Poulsen concerning the structure of bilinear invariant forms on the space of smooth vectors [26]. The latter theorem is now generalized to the Yamabe groups and representations in Fréchet spaces.

In Section 4 an arbitrary unitary representation is considered, in which the representation  $\mathcal{D}^\sigma$  is continuously imbedded. We denote it by  $(U^\sigma, H^\sigma)$ . With the help of theorem (0.2) we obtain a description of the space of operators closeable on  $H^\sigma$ , whose domains contain  $H_\infty^\sigma$ , and intertwining for  $U^\sigma$  and  ${}^! \pi$ . The space appears to be isomorphic to the space of the so-called  $(\pi, \sigma)$ -automorphic forms (Theorem 4.10).

It is a well-connected theorem. It is associated with the Gelfand–Piatecki–Šapiro theorem in the theory of automorphic functions [10], which, in the papers of L. Maurin [22], has been considered as a theorem from the theory of induced representations and, in this respect, generalized. Next papers which followed this approach were given by Olšanskii [25] and then by K. and L. Maurin [18], [19]. Somewhat different approach can be found in the papers of the author [40] and A. Strasburger and the

author [35], [36]. The most advanced result is the General Duality Theorem of K. and L. Maurin, which however concerns only induced representations in the sense of Mackey. Our Theorem 4.10 is closely connected with the latter.

In Section 5, the Banach algebras are taken for an algebra  $A$ . In this case  $\Phi(E) = E$  occurs for any continuous representation in the Fréchet space. The duality (0.1) is in turn closely connected with theorems of M. A. Rieffel [27]. In particular, it provides appropriate approach to investigations of corepresenting group representations.

The last section is of a different character than the previous part of this work and it makes use of formula (0.1) only in a limited degree. This section is concerned with Hilbert algebras. The structure of the Hilbert space in  $A$  makes that it is more natural to use the topology of Hilbert tensor product for the construction of an induced representations. An investigation of these representations has been initiated by M. A. Rieffel [29]. He has pointed out that for  $A = L^2 \cap L^1(G)$ , and for a representation  $(\sigma, V)$  of a compact subgroup  $K \subset G$  the product  $A \otimes_K V$  is equivalent to the representation  $U^\sigma$  induced in the sense of Mackey. The continuous representation in the sense here considered is, in turn, a square-integrable representation of the group  $G$ .

The concept of a square-integrable representation of an arbitrary Hilbert algebra has been introduced also by Rieffel [28].

For the irreducible representation  $(\pi, H)$  of a Hilbert algebra  $A$  we find  $\Phi(H) = H$ . In this case (0.1) leads to the isomorphism  $L_A(A, H) = H'$ , which with the help of orthogonality relations may be found to be an isometry. In the case of reducible representation this isomorphism does not occur. However, in the space of intertwining operators we can separate a subspace equipped with a Hilbert structure determined by the canonical trace of the algebra  $A$ , in such a way that

$$(0.3) \quad L_A^2(A \otimes_S V, H) = \text{HS}_S(V, H),$$

where  $\otimes_S$  denotes the Hilbert space tensor product over  $S$ ,  $L_A^2(\cdot, \cdot)$  denotes the above-mentioned subspace of intertwining operators, and  $\text{HS}_S$  denotes the space of Hilbert-Schmidt operators intertwining for the action of the set  $S$  on  $H$  and  $V$ .

This isometry, applied to the theory of compact groups, takes Rieffel's form of the Frobenius theorem

$$(0.4) \quad \text{HS}_G(U^\sigma, U) = \text{HS}_K(\sigma, U)$$

from which foundations of the Peter-Weyl theory can be derived.

If we apply (0.3) to the group algebra  $L^1 \cap L^2(G)$  for the unimodular group  $G$  and an inducing unitary representation  $(\sigma, V)$  of the compact

subgroup  $K$ , we obtain for any irreducible square-integrable representation

$$(0.5) \quad L_G(U^\sigma, U) = \text{HS}_K(\sigma, U)$$

which is an isometry proved by K. and L. Maurin [19]; see also [41], [37].

The existence of an involution in the algebra  $A$  is assumed in Theorem 2.4 (see (0.1)) for formal reasons. In this way we only avoid introducing anti-representations of  $A$ , apart from representations.

It seems that the present work may be considered, even if it were not intended by the author as a support to the tendency observed in the literature, to construct a general theory of induced representations of topological algebras. A detailed survey of investigations in this field can be found in [30].

However, it should be noted that apart from the number of similarities (which motivate the terminology used in the present work) with the theory of induced representations of groups, there are also many arguments which contravene the approach to consider representations in the space  $A \widehat{\otimes}_S V$  as generalized induced representations (see especially [29], p. 159).

Nevertheless the argument that the Frobenius duality theorem, which is so characteristic, of the induced representations theory has here its good analogue cannot be neglected in this discussion.

I would like to express my gratitude to Professor Krzysztof Maurin whose views and ideas helped me to clarify this subject and whose constant encouragement made possible completing the work.

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## 1. Regular representations of algebras with approximate unit. A duality theorem

All algebras here are associative algebras over the field  $C$  of complex numbers, provided with a locally convex Hausdorff topology such that the algebra multiplication

$$A \times A \ni (a, b) \rightarrow ab \in A$$

is separately continuous. An algebra  $A$  is called *involution* if there is a linear continuous operator  $\checkmark : A \rightarrow A$ , such that

$$(a \checkmark) \checkmark = a \quad \text{and} \quad (ab) \checkmark = b \checkmark a \checkmark, \quad a, b \in A.$$

By an *approximate left (right) unit* in  $A$  we mean a generalized sequence  $\{a_i\}_{i \in I}$  in  $A$  such that  $a_i a \rightarrow a$  (respectively  $aa_i \rightarrow a$ ) for all  $a \in A$ . We say that an approximate unit  $\{a_i\}_{i \in I}$  is *equicontinuous* if the set of operators:

$$\lambda(a_i): A \ni a \rightarrow a_i a \in A$$

(respectively  $r(a_i): A \ni a \rightarrow aa_i \in A$ ) is equicontinuous.

Let  $E$  be a locally convex Hausdorff topological vector space. Denote by  $L(E)$  the algebra of continuous endomorphisms of  $E$ . A homomorphism  $A \ni a \rightarrow \pi(a) \in L(E)$  is called a *continuous representation* of  $A$  on  $E$  if the bilinear map

$$A \times E \ni (a, x) \rightarrow \pi(a)x \in E$$

is separately continuous.

Partly for completeness and partly to clarify the situation and introduce a notation convenient for us we include the definitions and standard generalities concerning the projective and inductive tensor products of topological vector spaces. (For details, see the fundamental work of Grothendieck [12].)

Let  $V$  and  $W$  be two locally convex Hausdorff topological vector spaces over  $C$ , and let  $V \otimes W$  denote the algebraic tensor product of the spaces.

1.1. LEMMA. *There exists a unique locally convex topology  $\mathcal{F}$  on  $V \otimes W$  with the following universal property.*

Let  $\beta$  be a bilinear map from  $V \times W$  into a locally convex space  $F$ , and  $\tilde{\beta}$  the unique extension of  $\beta$  to  $V \otimes W$ . In order that  $\tilde{\beta}$  be continuous in the topology  $\mathcal{T}$  it is necessary and sufficient that  $\beta$  be separately continuous.

The topology  $\mathcal{T}$  is called the *inductive topology* for  $V \otimes W$ . The completion of the space  $V \otimes W$  in the topology  $\mathcal{T}$  is called the *inductive tensor product* of the spaces  $V$  and  $W$  and is denoted by  $V \overline{\otimes} W$ .

By the very definition of the inductive topology it follows that the dual space  $(V \overline{\otimes} W)'$  may be identified with the space of bilinear separately continuous forms on  $V \times W$ . Equicontinuous sets of functionals correspond under this identification to separately equicontinuous sets of bilinear forms.

1.2. LEMMA ([12]). Suppose that  $E = \lim_{\rightarrow} E_{\mu}$  and  $F = \lim_{\rightarrow} F_{\nu}$ . Then

$$E \overline{\otimes} F = \lim_{\rightarrow} E_{\mu} \overline{\otimes} F_{\nu}.$$

1.3. LEMMA. The inductive tensor product of bornological (barrelled) spaces is bornological (resp. barrelled).

Let  $(E_1, E_2)$  and  $(F_1, F_2)$  be two pairs of locally convex Hausdorff vector spaces, and let  $A_i \in L(E_i, F_i)$ ,  $i = 1, 2$ . The mapping

$$E_1 \times E_2 \ni (x_1, x_2) \rightarrow A_1 x_1 \otimes A_2 x_2 \in F_1 \otimes F_2$$

extends to a continuous linear operator from  $E_1 \overline{\otimes} E_2$  into  $F_1 \overline{\otimes} F_2$ . It is denoted by  $A_1 \overline{\otimes} A_2$ .

The next result is certainly well-known, however, we have not found a good reference for it.

1.4. PROPOSITION. Let  $\{A_1^{\mu}\}_{\mu \in M}$ ,  $\{A_2^{\nu}\}_{\nu \in N}$  be two equicontinuous subsets of  $L(E_1, F_1)$  and  $L(E_2, F_2)$ , respectively. Then the operators  $\{A_1^{\mu} \overline{\otimes} A_2^{\nu}\}_{\mu, \nu}$  constitute an equicontinuous subset of  $L(E_1 \overline{\otimes} E_2, F_1 \overline{\otimes} F_2)$ .

Proof. Absolutely convex neighbourhoods of zero in a locally convex linear space  $X$  are polars of equicontinuous subsets of the dual  $X'$ . In order to prove the assertion it suffices to verify that transposed operators  ${}^t(A_1^{\mu} \overline{\otimes} A_2^{\nu})$  map equicontinuous subsets of  $(F_1 \overline{\otimes} F_2)'$  into equicontinuous subsets of  $(E_1 \overline{\otimes} E_2)'$ . A linear functional  $\tilde{\beta}$  on  $F_1 \overline{\otimes} F_2$ , when identified with a bilinear form  $\beta$  on  $F_1 \times F_2$ , is mapped under  ${}^t(A_1^{\mu} \overline{\otimes} A_2^{\nu})$  into the form  $(e_1, e_2) \rightarrow \beta(A_1^{\mu} e_1, A_2^{\nu} e_2)$ . Now it is obvious that a family of separately equicontinuous bilinear forms on  $F_1 \times F_2$  is mapped into a set of separately equicontinuous forms on  $E_1 \times E_2$  and the proof follows.

Let  $\|\cdot\|_{\alpha}^1, \|\cdot\|_{\beta}^2$  denote systems of seminorms, which define the topologies on  $E_1$  and  $E_2$ . For each  $x \in E_1 \otimes E_2$  we put

$$(1.1) \quad \|x\|_{\alpha, \beta} = \inf \sum_i \|e_1^i\|_{\alpha} \|e_2^i\|_{\beta}$$

where the inf is taken over all decompositions

$$x = \sum_i e_1^i \otimes e_2^i.$$

The system of seminorms  $\|\cdot\|_{\alpha,\beta}$  defines a topology  $\mathcal{P}$  on  $E_1 \otimes E_2$  which will be referred to as the *projective topology* for  $E_1 \otimes E_2$ .

The completion of the space  $E_1 \otimes E_2$  equipped with the projective topology is called the *projective tensor product* of  $E_1$  and  $E_2$  and is denoted by  $E_1 \hat{\otimes} E_2$ .

The projective topology has the following universal property.

1.5. LEMMA. *Let  $\beta$  be a bilinear map from  $E_1 \times E_2$  into a locally convex space  $F$ , and  $\tilde{\beta}$  the unique extension of  $\beta$  to  $E_1 \otimes E_2$ . Then  $\tilde{\beta}$  is continuous in the topology  $\mathcal{P}$  iff  $\beta$  is continuous on  $E_1 \times E_2$ .*

The inductive topology is in general finer than the projective topology. On the other hand

1.6. LEMMA. *Let  $E_1, E_2$  be metrizable locally convex spaces, and  $E_2$  in addition a barrelled space. Then each separately continuous bilinear form on  $E_1 \times E_2$  is continuous (Bourbaki, [2], III, § 4) and, consequently,*

$$\mathcal{T} = \mathcal{P}, \quad E_1 \overline{\otimes} E_2 = E_1 \hat{\otimes} E_2.$$

1.7. PROPOSITION. *Let  $E_1, E_2$  be bornological spaces,  $E_2$  complete, and  $F$  a barrelled locally convex space. Let  $\beta$  be a trilinear separately continuous map on  $E_1 \times E_2 \times F$ , valued in a complete locally convex space  $G$ . Then the formula*

$$(1.2) \quad \tilde{\beta}(e_1 \otimes e_2, f) := \beta(e_1, e_2, f)$$

*defines a separately continuous bilinear mapping on  $(E_1 \overline{\otimes} E_2) \times F$ .*

Proof. Plainly, formula (1.2) defines a bilinear form on  $(E_1 \otimes E_2) \times F$  which is separately continuous in the inductive topology on  $E_1 \otimes E_2$ . We have to prove that the mapping is extendable to the completion  $(E_1 \overline{\otimes} E_2) \times F$ . Since  $F$  is a barrelled space, the separate continuity of  $\beta$  implies an apparently stronger property: Given any neighbourhood of zero, say  $U$  in  $G$ , and any bounded set  $B$  in  $(E_1 \otimes E_2, \mathcal{T})$ , there is a neighbourhood of zero  $O$  in  $F$  such that  $\beta(B, O) \subset U$ . (Schaefer [31], III, 5.2.) This property of  $\beta$  is called  $\mathcal{B}$ -hypocontinuity. A fundamental theorem on  $\mathcal{B}$ -hypocontinuous mappings (see Bourbaki [2], III, § 4) implies that the form  $\beta$  extends in a unique manner to a separately continuous map on  $(E_1 \overline{\otimes} E_2)_0 \times F$ , where the subspace  $(E_1 \overline{\otimes} E_2)_0 \subset E_1 \overline{\otimes} E_2$  consists of closures in  $E_1 \overline{\otimes} E_2$  of all bounded subsets of  $(E_1 \otimes E_2, \mathcal{T})$ .

It remains to prove that  $(E_1 \overline{\otimes} E_2)_0 = E_1 \overline{\otimes} E_2$ . Bearing in mind Lemmas 1.2 and 1.6, we obtain  $E_1 \overline{\otimes} E_2 = \lim_{\rightarrow} E_1^\mu \otimes E_2^\nu$ , where  $E_1^\mu$  are normed and  $E_2^\nu$  are Banach spaces. Then  $E_1 \overline{\otimes} E_2 = \lim_{\rightarrow} E_1^\mu \hat{\otimes} E_2^\nu$ . Since  $(E_1^\mu \otimes E_2^\nu, \mathcal{P})$

is a normed space, its completion consists of closures in  $E_1^\mu \overline{\otimes} E_2^\nu$  of bounded subsets of  $E_1^\mu \otimes E_2^\nu$ . Consequently

$$E_1^\mu \overline{\otimes} E_2^\nu = (E_1^\mu \overline{\otimes} E_2^\nu)_0 \quad \text{for all } \mu, \nu.$$

This finishes the proof.

Given a continuous representation  $(\pi, E)$  of the algebra  $A$  and a locally convex topological space  $V$ , we define a representation of  $A$  in the space  $E \overline{\otimes} V$  as follows

$$(1.3) \quad \pi^V(a) := \pi(a) \overline{\otimes} I,$$

$I$  denoting the identity operator on  $V$ .

In general, the representation  $\pi^V$  is not continuous, but we are able to determine a broad class of algebras for which this construction leads to a continuous representation.

**1.8. PROPOSITION.** *Let  $A$  be a barrelled topological algebra and  $(\pi, E)$  a continuous representation of  $A$  in a bornological complete space  $E$ . If  $V$  is a bornological space, then the representation  $\pi^V$  is continuous.*

*Proof.* The trilinear mapping  $A \times E \times V \ni (a, e, v) \rightarrow (\pi(a)e) \otimes v$  is separately continuous. By Proposition 1.7 the proof follows.

We denote by  $\hat{\pi}$  the representation of  $A$  contragredient to  $\pi$ , i.e. the homomorphism  $A \ni a \rightarrow {}^t\pi(a^\vee) \in L(E')$ , where  ${}^t\pi(a)$  is the usual adjoint of  $\pi(a)$ .

Suppose that  $A$  is a barrelled algebra, and  $(\pi, E)$  a continuous representation of  $A$ . Then the contragredient representation  $(\hat{\pi}, E')$  is continuous in the strong topology on  $E'$ , i.e. the topology of uniform convergence on bounded sets. In fact, the mapping

$$A \ni a \rightarrow \hat{\pi}(a)e' \in E'$$

is continuous in the strong topology on  $E'$  iff the bilinear form

$$A \times E \ni (a, e) \rightarrow \langle \hat{\pi}(a)e', e \rangle = \langle e', \pi(a^\vee)e \rangle$$

is  $\mathcal{B}$ -hypocontinuous (here  $\mathcal{B}$  stands for the family of bounded subsets of  $E$ ). Since  $A$  is assumed to be barrelled, the separate continuity of the form implies its  $\mathcal{B}$ -hypocontinuity (Schaefer [31], III, 5.2).

We are now in a position to carry out a construction which occupies a central position in further investigations. Given a continuous representation of  $A$  in a quasicomplete space  $E$ , we introduce a dense and  $A$ -invariant subspace  $\Phi(E) \subset E$  which may be regarded as an analogue of the space of smooth vectors in the theory of Lie groups representations.

Well, now let  $\alpha_0$  denote the operator on  $A \otimes E$  into  $E$  defined on simple tensors  $a \otimes x$  by the formula

$$(1.4) \quad \alpha_0(a \otimes x) = \pi(a)x.$$

Since the representation  $(\pi, E)$  is assumed to be continuous, the operator  $\alpha_0$  extends to a continuous operator  $\alpha$  on  $A \overline{\otimes} E$ . We put  $\Phi(E) = \alpha(A \overline{\otimes} E)$  and place on  $\Phi(E)$  the topology inductive with respect to the operator  $\alpha$ .

Recall that a representation  $(\pi, E)$  is said to be *essential* if  $\pi(A)E$  is dense in  $E$ .

**1.9. PROPOSITION.** *Let  $(\pi, E)$  be a continuous essential representation of  $A$  in a quasicomplete space  $E$ . Then  $\Phi(E)$  is a continuously imbedded into  $E$ , dense and  $A$ -invariant linear subspace of  $E$ . The action of  $\pi(a)$  on  $\Phi(E)$  is continuous for all  $a \in A$ .*

*Proof.* The formula

$$(1.5) \quad \alpha(\lambda^E(a)y) = \pi(a)\alpha(y)$$

is obvious for all  $y \in A \otimes E$ , and by the continuity of  $\alpha$  and  $\lambda^E$  it is true for all  $y \in A \overline{\otimes} E$ . All assertions of the proposition follow now immediately from the formula.

The above conclusions enable us to define a representation of  $A$  in the space  $\Phi(E)$ . Let us denote it by the same symbol:

$$(\pi, \Phi(E)) := (\pi|_{\Phi(E)}, \Phi(E)).$$

Formula (1.5) and Proposition 1.7 imply

**1.10. PROPOSITION.** *Retain the assumptions of Proposition 1.9 and suppose that  $A$  is a barrelled space. Then the representation  $(\pi, \Phi(E))$  is continuous.*

Let  $(\pi_1, E_1)$  and  $(\pi_2, E_2)$  be two representations of  $A$ . A continuous operator  $T: E_1 \rightarrow E_2$  is said to be an *intertwining* operator for  $\pi_1$  and  $\pi_2$  if  $T\pi_1(a) = \pi_2(a)T$  for all  $a \in A$ . The set of intertwining operators forms a vector space which will be denoted by  $L_A(\pi_1, \pi_2)$  or, if no confusion will arise, simply by  $L_A(E_1, E_2)$ .

We denote by  $B_A(\pi_1, \pi_2)$  (or  $B_A(E_1, E_2)$ ) the space of bilinear separately continuous forms on  $E_1 \times E_2$  satisfying

$$\beta(\pi_1(a)x_1, x_2) = \beta(x_1, \pi_2(a)x_2) \quad \text{for all } a \in A.$$

**1.11. THEOREM.** *Let  $T \in L_A(E_1, E_2)$ . Then  $T|_{\Phi(E_1)}$  belongs to  $L_A(\Phi(E_1), \Phi(E_2))$ .*

**Proof.** The operator  $I \otimes T$  maps  $A \overline{\otimes} E_1$  into  $A \overline{\otimes} E_2$  continuously. The formula

$$(1.6) \quad a_2 \circ (I \otimes T) = T \circ a_1$$

is easily verified for elementary tensors. If we treat both sides of the formula as operators from  $A \overline{\otimes} E_1$  into  $E_2$ , they are evidently continuous. By uniqueness of the continuous extension, formula (1.6) is true for all  $y \in A \overline{\otimes} E_1$ . Thus we see that  $T(\Phi(E_1)) \subset \Phi(E_2)$ . To prove that  $T: \Phi(E_1) \rightarrow \Phi(E_2)$  is continuous it suffices to verify that  $T \circ a_1$  maps  $A \overline{\otimes} E_1$  into  $\Phi(E_2)$  continuously. Since the left-hand side of (1.6) is continuous, the proof is finished.

The following theorem will be referred to as a duality theorem for the regular representation  $\lambda^V$ .

**1.12. THEOREM.** *Let  $A$  be a bornological involutive algebra with an equicontinuous approximate unit, and  $(\pi, E)$  a continuous representation of  $A$  in a complete bornological space  $E$ . Let  $V$  be a locally convex bornological complete space. Then*

$$(1.7) \quad B_A(\lambda^V, \pi) = B(\Phi(E), V)$$

where  $=$  denotes a linear isomorphism defined by the relation

$$(1.8) \quad \tau(a^\vee \otimes v, x) = \eta(\pi(a)x, v), \quad a \in A, x \in E, v \in V,$$

(here  $\tau \in B_A(\lambda^V, \pi)$  and  $\eta \in B(\Phi(E), V)$ ).

**Proof.** First we prove that, given  $\tau$  the formula (1.8) defines the bilinear form  $\eta \in B(\Phi(E), V)$ . In view of Proposition 1.7 the trilinear mapping  $A \times E \times V \ni (a, x, v) \rightarrow \tau(a^\vee \otimes v, x)$  defines a bilinear separately continuous form  $\tilde{\eta}$  on  $(A \overline{\otimes} E) \times V$  by the formula  $\tilde{\eta}(a \otimes x, v) = \tau(a^\vee \otimes v, x)$ . (Recall: a complete bornological space is barrelled; [31].) So we have to prove that for fixed  $v \in V$  the functional  $\tilde{\eta}(\cdot, v)$  on  $A \overline{\otimes} E$  vanishes on the kernel of  $a$  and passes to the quotient to define a bilinear form on  $\Phi(E) \times V$ . Thus assume  $a(x) = 0$  and let  $x^\mu = \sum_i a_i^\mu \otimes x_i^\mu$  denote a generalized sequence convergent to  $x$  in  $A \overline{\otimes} E$ . Then, for all  $a \in A$ , we have

$$\begin{aligned} \tilde{\eta}(\lambda^E(a)x^\mu, v) &= \sum_i \tau((aa_i^\mu)^\vee \otimes v, x_i^\mu) = \sum_i \tau(\lambda^V((a_i^\mu)^\vee) a \otimes v, x_i^\mu) \\ &= \tau(a \otimes v, \sum_i \pi(a_i^\mu) x_i^\mu) = \tau(a \otimes v, a(x^\mu)) \xrightarrow{\mu} \tau(a \otimes v, a(x)) = 0. \end{aligned}$$

The kernel of  $a$  is  $A$ -invariant, so the result above tells us that the functional  $\tilde{\eta}(\cdot, v)$  vanishes at each point of  $\text{Ker } a$  which is of the form  $\lambda^E(a)x$  for some  $a \in A$ . In fact,

$$\tilde{\eta}(\lambda^E(a)x, v) = \lim_{\mu} \tilde{\eta}(\lambda^E(a)x^\mu, v) = 0.$$

Now, it is our aim to show that the set  $\lambda^E(A)\text{Ker } a$  is dense in  $\text{Ker } a$ . After Proposition 1.4 we know that the set of operators  $\lambda^E(a_i)$  is equicontinuous ( $\{a_i\}_{i \in I}$  denotes as before an equicontinuous approximate unit). On the other hand,  $\lambda^E(a_i)$  converges pointwise to the identity operator on the total subset  $A \otimes E \subset A \overline{\otimes} E$ . This implies the pointwise convergence on the whole  $A \overline{\otimes} E$  (Bourbaki [2], III, § 3). Accordingly, each element of  $\text{Ker } a$  is approximated by elements of the set  $\lambda^E(A)\text{Ker } a$ . Thus we obtain the claimed implication

$$a(x) = 0 \Rightarrow \tilde{\eta}(x, v) = 0.$$

Conclusion: the formula  $\eta(a(x), v) = \tilde{\eta}(x, v)$  defines the desired assignment  $\tau \rightarrow \eta$ .

Now we pass to determine the converse map. By Proposition 1.7 it is immediate that, having defined  $\eta$ , formula (1.8) determines a bilinear form  $\tau$  on  $(A \overline{\otimes} V) \times E$ . The invariance property is verified by the following computation:

$$\begin{aligned} \tau(a \otimes v, \pi(b)x) &= \eta(\pi(a^\vee)\pi(b)x, v) = \eta(\pi((b^\vee a)^\vee)x, v) \\ &= \tau((b^\vee a) \otimes v, x) = \tau(\lambda^V(b^\vee)(a \otimes v), x). \end{aligned}$$

Since under the assumptions of Theorem 1.12 the space  $A \overline{\otimes} V$  is bornological and  $E$  is barrelled, we have a natural linear bijection between  $B_A(A \overline{\otimes} V, E)$  and  $L_A(A \overline{\otimes} V, E')$  given by the formula

$$(1.9) \quad \langle T\varphi, v \rangle = \tau(\varphi, x).$$

The formula above defines evidently an operator on  $A \overline{\otimes} V$  valued in  $E'$ . In our case, the bilinear form  $\tau$  is  $\mathcal{B}$ -hypocontinuous by the theorem mentioned in the proof of Proposition 1.7. This is equivalent to the claimed continuity of  $T$  in the strong topology on  $E'$ . This yields

**1.13. THEOREM.** *Under assumptions of Theorem 1.12 the space  $L_A(A \overline{\otimes} V, E')$  is linearly isomorphic to  $(\Phi(E) \overline{\otimes} V)'$ . The assignment  $T \rightarrow \eta$  is defined by the formula*

$$(1.10) \quad \langle T(a^\vee \otimes v), x \rangle = \eta((\pi(a)x) \otimes v).$$

Assume now that the algebra  $A$  is provided with an antilinear involution:  $A \ni a \rightarrow a^* \in A$ , such that

$$(ab)^* = b^*a^*.$$

A representation  $(\pi, H)$  of  $A$  is called *unitary* if  $\pi(a^*) = \pi(a)^*$ ,  $\pi(a)^*$  denoting the usual Hilbert space adjoint of  $\pi(a)$ .

**1.14. THEOREM.** *Let  $(\pi, H)$  be a unitary representation of a bornological  $*$ -algebra  $A$  with an equicontinuous approximate unit. For each*

complete bornological space  $V$  the formula

$$(1.11) \quad (T(a \otimes v) | x) = \eta(v, \pi(a^*)x)$$

defines a linear isomorphism of  $L_A(A \overline{\otimes} V, H)$  onto the space of sesquilinear (antilinear in the first variable) forms on  $V \times \Phi(E)$ .

We omit the proof since it is virtually the same as the proof of Theorem 1.12.

Before we shall formulate and prove deeper results in the representation theory of involutive algebras we indicate some applications.

The example of the convolution algebra of the Schwartz functions on a Lie group  $G$  is of the special importance and will lead us in the sequel to Bruhat's theory of differentiable induced representations.

Let  $G$  be a Lie group with the left Haar measure  $dg$  and the modular function  $\delta_G$ .  $\mathcal{D}(G, V)$  will denote the space of infinitely differentiable functions compactly supported and valued in a locally convex space  $V$ .  $\mathcal{D}(K, V)$  denotes the subspace of  $\mathcal{D}(G, V)$  consisting of those functions with supports contained in the compact set  $K \subset G$ . By  $\mathcal{S}(G, V)$  we denote the space of all infinitely differentiable  $V$ -valued functions on  $G$ . All spaces are provided with their usual topologies.  $\mathcal{S}(G, V)$  is topologized by the system of seminorms

$$(1.12) \quad |f|_{K, X, \alpha} = \sup_{g \in K} \|Xf(g)\|_\alpha$$

where  $K$  runs over the family of all compact subsets of  $G$ ,  $X$  over the family of right invariant differential operators on  $G$ , and  $\|\cdot\|_\alpha$  over the set of seminorms determining the topology of  $E$ . It is due to Grothendieck [12] that  $\mathcal{S}(G, V)$  is isomorphic to  $\mathcal{S} \hat{\otimes} V$ , where  $\mathcal{S}$  stands for  $\mathcal{S}(G, C)$ . This natural isomorphism sends  $f \otimes v$  into  $g \rightarrow f(g)v$ .

The space  $\mathcal{D}(K, V)$  is equipped with the topology inherited from  $\mathcal{S}(G, V)$ . The Schwartz topology on  $\mathcal{D}(G, V)$  is the topology of the inductive limit of the spaces  $\mathcal{D}(K, V)$ , the family of compacts being ordered by inclusion. If  $V$  is a Fréchet space then  $\mathcal{D}(G, V) = \mathcal{D} \overline{\otimes} V$  (Grothendieck [12], p. 84). The space  $\mathcal{D} := \mathcal{D}(G, C)$  is a bornological complete convolution algebra with a linear involution defined by the formula

$$f^\vee(g) = \delta_G(g^{-1})f(g^{-1}).$$

An equicontinuous left approximate unit in  $\mathcal{D}$  may be defined as a sequence  $\{\varphi_n\}_1^\infty \subset \mathcal{D}$  with  $\varphi_n \geq 0$ ,  $\int_G \varphi_n(g) dg \leq 1$  and  $\text{spt} \varphi_n \searrow e$ , i.e. a regularization of the distribution  $\delta_e$ .

By a continuous representation of  $G$  in a topological vector space  $E$  we mean a homomorphism  $G \ni g \rightarrow \pi(g)$  of  $G$  into the group of topological automorphisms of  $E$  with additional properties:

(a) for every  $x \in E$ , the mapping  $G \ni g \rightarrow \pi(g)x \in E$  is continuous,  
 (b) for every compact  $K \subset G$ , the set of operators  $\{\pi(g): g \in K\}$  is equicontinuous.

When  $E$  is barrelled (b) follows from (a) by the Banach–Steinhaus theorem. A continuous representation of  $G$  on a complete space  $E$  lifts in the well-known way to a continuous representation of  $\mathcal{D}$  on  $E$ . Namely, one defines

$$(1.13) \quad \pi(f) = \int_G f(g)\pi(g)dg, \quad f \in \mathcal{D}.$$

An essential representation of  $\mathcal{D}$  on  $E$  corresponds to a continuous representation of  $G$  iff it satisfies the following requirement:

The set of operators  $\pi(f)$  for  $f$  with support contained in a fixed compact subset of  $G$  and such that  $\int_G f(g)dg = 1$ , is equicontinuous.

If we restrict our attention to this class of representations we have the natural isomorphism of spaces of intertwining operators

$$L_{\mathcal{D}}(\pi_1, \pi_2) = L_G(\pi_1, \pi_2).$$

In order to formulate the most of Theorems 1.12, 1.13, 1.14 we have to describe the space in a more convenient way.

We shall state that  $\Phi(E)$  coincides, at least for a Fréchet space  $E$ , with the space of smooth vectors of the representation  $(\pi, E)$  of  $G$ . For completeness we recall the definition.

A vector  $x \in E$  is said to be *differentiable* (or *smooth*) for  $\pi$  if the mapping  $\tilde{x}: g \rightarrow \tilde{x}(g) := \pi(g)x$  belongs to  $\mathcal{S}(G, E)$ . The linear space of smooth vectors for  $(\pi, E)$  is denoted by  $E_{\infty}$ . The map  $x \rightarrow \tilde{x}$  identifies  $E_{\infty}$  with a closed subspace of  $\mathcal{S}(G, E)$ , hence we may equip  $E_{\infty}$  with the relative topology induced by  $\mathcal{S}(G, E)$ .

1.15. PROPOSITION. *Let  $(\pi, E)$  be a representation of  $\mathcal{D}$  in a complete bornological space  $E$ , associated with a continuous representation  $G$ , and denoted by the same symbol. Then*

$$\Phi(E) \subset E_{\infty}.$$

Proof. First of all we derive a useful formula for the operator  $\alpha: \mathcal{D} \otimes E \rightarrow E$ . When expressed by means of the group representation  $g \rightarrow \pi(g)$  the operator is of the integral form:

$$(1.14) \quad \alpha(f \otimes x) = \int_G f(g)\pi(g)x dg = \int_G \pi(g)(f \otimes x)(g) dg.$$

Let us notice that the space  $\alpha(\mathcal{D} \otimes E)$  is just the Gårding subspace of  $E$ . Since

$$\pi(g')\alpha(f \otimes x) = \int_G f(g)\pi(g'g)x dg = \int_G f(g'^{-1}g)\pi(g)x dg,$$

the assertion  $a(f \otimes x) \in E_\infty$  follows by a simple computation. In the same way we verify that the mapping

$$\mathcal{D} \times E \ni (f, x) \rightarrow a(f \otimes x) \in E_\infty$$

is separately continuous. Since  $E_\infty$  is complete, the operator  $a$  extends to a continuous mapping  $a: \mathcal{D} \overline{\otimes} E \rightarrow E_\infty$ . Thus  $\Phi(E) \subset E_\infty$ . Since the topology of  $\Phi(E)$  is just the inductive topology relative to the map  $a$ , the proof is complete.

We cannot prove here that  $\Phi(E) = E_\infty$  without making additional assumptions on the topology of the space  $E$ , although we believe that this is so in a more general case.

**1.16. THEOREM.** *Let  $(\pi, E)$  be a representation of  $\mathcal{D}$  associated with a continuous representation of a Lie group  $G$  and assume  $E$  to be a strict inductive limit of Fréchet spaces. Then  $\Phi(E) = E_\infty$ .*

*Proof.* Write  $E = \lim_{\rightarrow} E_n$ . In virtue of Lemma 1.2 we have

$$\mathcal{D} \overline{\otimes} E = \lim_{\rightarrow} \mathcal{D} \overline{\otimes} E_n = \lim_{\rightarrow} \mathcal{D}(G, E_n).$$

Denote by  $j$  the natural continuous injection of  $\mathcal{D} \overline{\otimes} E$  into  $\mathcal{D}(G, E)$ . We can in turn define a continuous operator  $\tilde{a}: \mathcal{D}(G, E) \rightarrow E$ , which coincides with  $a$  on  $\mathcal{D} \overline{\otimes} E$ :

$$\tilde{a}(f) := \int_G \pi(g) f(g) dg.$$

By uniqueness of the continuous extension we obtain  $\tilde{a} \circ j = a$ . Let us choose a function  $\gamma \in \mathcal{D}$  such that  $\gamma \geq 0$  and  $\int_G \gamma dg = 1$ , and consider the following continuous injection:

$$E_\infty \ni x \rightarrow \beta(x) \in \mathcal{D}(G, E) \quad \text{where} \quad \beta(x)(g) := \gamma(g) \pi(g^{-1}) x.$$

Observe that  $\tilde{a} \circ \beta = \text{id}$ .

To complete the proof it remains to show that the injection  $j: \mathcal{D} \overline{\otimes} E \rightarrow \mathcal{D}(G, E)$  is surjective. For given  $\psi \in \mathcal{D}(G, E)$  we consider the bounded subset  $B \subset E$  of the values of  $\psi$ . Since we deal with the strict inductive limit, we can choose  $n$  such that  $B \subset E_n$  (see Schaefer [31], II.65.) The Fréchet topology of  $E_n$  coincides with that inherited from  $E$ , therefore the function  $\psi$  may be viewed as an element of  $\mathcal{D}(G, E_n) = \mathcal{D} \overline{\otimes} E_n$ , i.e.  $\psi = j(\psi_0)$  for some  $\psi_0 \in \mathcal{D} \overline{\otimes} E_n \subset \mathcal{D} \overline{\otimes} E$ . Finally we have

$$(1.15) \quad \text{id}_{E_\infty} = a \circ j^{-1} \circ \beta$$

and the proof follows.

Now we shall be concerned with the problem of topologies of  $\Phi(E)$  and  $E_\infty$ . Our first observation is based on (1.15).

1.17. PROPOSITION. *Let  $E$  be a Fréchet space. Then  $E_\infty$  is topologically isomorphic to  $\Phi(E)$ .*

In fact, this being the case  $j$  is a natural isomorphism and the right-hand side of (1.15) defines the continuous injection of  $E_\infty$  into  $\Phi(E)$ .

1.18. THEOREM. *Retain the assumptions of Theorem 1.16 and assume in addition that  $G$  is separable and  $E_\infty$  is a bornological space. Then  $E_\infty$  is isomorphic to  $\Phi(E)$ .*

Proof. After Theorem 1.16 and Proposition 1.15 it remains to verify that the assumptions of the Open Map Theorem are satisfied. We shall use the version of the theorem due to de Wilde [44]. It claims  $E_\infty$  to be a bornological complete space and  $\Phi(E)$  to be so called de Wilde's space. This class of topological vector spaces contains in particular countable inductive limits of Fréchet spaces as well as its quotient spaces. Since  $G$  is assumed to be separable we have

$$\mathcal{D} \overline{\otimes} E = \lim_{\rightarrow t, n} \mathcal{D}(K_t, E_n)$$

and at the same time

$$\Phi(E) = \mathcal{D}(G, E)/\text{Ker } \alpha,$$

so the assumptions of de Wilde's theorem are valid.

Now, we pass to the example  $E = \mathcal{D}(G, F)$  with  $F$  being a Fréchet space. In this case the assumption of separability of  $G$  may be removed. To this end we shall employ the Schwartz Kernel Theorem and the argument used in the course of the proof of Theorem 1.16, instead of the Open Map Theorem. The space  $\mathcal{D} \overline{\otimes} E$  is now isomorphic to  $\mathcal{D}(G \times G, F)$  and the operator  $\alpha$  is defined by the formula:

$$\alpha(f)(g) = \int_G f(g_1, g_1^{-1}g) dg_1.$$

The operator  $\beta$  which takes the form

$$\beta(x)(g, h) = \gamma(g)x(gh)$$

acts continuously into  $\mathcal{D}(G \times G, F)$ . Therefore, the natural injection of  $E_\infty$  into  $\Phi(E)$  may be viewed as a superposition of two continuous operators:  $\alpha \circ \beta$ . This yields

1.19. PROPOSITION. *Let  $G$  be a Lie group, and  $F$  a Fréchet space. Then the space  $\Phi(\mathcal{D}(G, F))$  corresponding to the regular representation  $\lambda^F$  is isomorphic to  $\mathcal{D}(G, F)$  itself.*

We shall conclude this section with simple applications which have just an illustrative character. After Proposition 1.19 the duality Theorem 1.12 implies the well-known result:

1.20. COROLLARY (Bruhat [4]). *Let  $F$  be a Fréchet space, and  $G$  a Lie group. Every bilinear separately continuous  $G$ -invariant form on*

$$\mathcal{D}(G) \times \mathcal{D}(G, F)$$

*defines an  $F$ -distribution  $T$  on  $G$  such that*

$$\tau(f, \psi) = T(f \check{*} \psi).$$

Let  $L^p(G)$  denote the space of smooth vectors for the left regular representation of  $G$  in the Banach space  $L^p(G, dg)$ . A nice description of the space  $L^p(G)$  is given in Skovhus Poulsen's paper [26]. Namely, we have:  $f \in L^p_\infty(G)$  iff for every right-invariant differential operator  $X$  on  $G$  the function  $Xf$  belongs to  $L^p(G)$ .

Our Theorem 1.13 together with Proposition 1.17 lead to the following characterization of the dual space  $(L^p_\infty(G))'$ .

1.21. COROLLARY. *Let  $T \in \mathcal{D}(G)'$ . Then  $T \in L^p(G)'$  iff for every  $f \in \mathcal{D}(G)$  the convolution  $f * T$  belongs to  $L^p(G)$ , where  $1/p' + 1/p = 1$ .*

This result generalizes a classical theorem due to Schwartz [32] concerning distributions on  $R^n$ .

## 2. Induced representations of algebras. The main duality

The first part of this section is devoted to the definition and some properties of the inductive tensor product over  $S$  of two abstract  $S$ -modules, which lead to the definition of the induced representation of an algebra. Hereafter we proceed to the investigation of duality phenomena in the theory of induced representations. The main results of this number are contained in the Theorems 2.4, 2.5, 2.6.

We begin with a general comment about our terminology and notation. Let  $S$  be a set. By  $S$ -module we mean a topological vector space  $V$  with a map  $S \ni s \rightarrow s(\cdot)$  valued in the space of linear endomorphisms of  $V$ . If  $V$  is both  $S$ - and  $T$ -module and operators  $s(\cdot)$  and  $t(\cdot)$  commute for all  $s \in S$  and  $t \in T$ , the space  $V$  is called the  $S$ - $T$ -bimodule. A topological algebra  $A$  will be usually viewed as an  $A$ -module with the action defined by the left regular representation.

If  $A$  is in addition an  $A$ - $S$ -bimodule, the action of  $S$  on  $A$  will be denoted by  $A \ni a \rightarrow as \in A$ . Let  $V$  and  $W$  be two  $S$ -modules. A bilinear map  $\beta: V \times W \rightarrow F$  is said to be  $S$ -balanced if

$$\beta(sv, w) = \beta(v, sw) \quad \text{for all } s \in S, v \in V, w \in W.$$

The space of all separately continuous  $S$ -balanced maps is denoted by  $B_S(V, W; F)$ . The action of  $S$  on  $V \otimes W$  may be defined in several

different ways. Now we define

$$(2.1) \quad s_l(v \otimes w) = (sv) \otimes w,$$

$$(2.2) \quad s_r(v \otimes w) = v \otimes (sw), \quad s \in S, v \in V, w \in W.$$

Let  $Y$  denote the closed linear span of the set

$$\{y \in V \overline{\otimes} W : y = s_l(v \otimes w) - s_r(v \otimes w), v \in V, w \in W, s \in S\}.$$

The quotient space  $V \overline{\otimes} W / Y$  is denoted by  $V \overline{\otimes}_S W$  and referred to as the inductive tensor product over  $S$  of  $V$  and  $W$ . The natural projection  $V \overline{\otimes} W \rightarrow V \overline{\otimes}_S W$  is denoted by  $p$ . Plainly, this construction is influenced by Rieffel's construction of projective tensor product over  $S$  of Banach  $S$ -modules carried out in [27]. As expected, the tensor product over  $S$  have an universal property.

**2.1. PROPOSITION.** *Let  $F$  be a complete locally convex vector space. Then the assignment  $L(V \overline{\otimes}_S W, F) \ni A \rightarrow A \circ d \in B_S(V, W; F)$  is a linear isomorphism (here  $d(v, w) = p(v \otimes w)$ ).*

We omit the standard proof based on the universal property of the inductive tensor product.

If the space  $V$  is an  $R$ - $S$ -bimodule the tensor product over  $S$  becomes in a natural way an  $R$ -module:

$$(2.3) \quad r(p(v \otimes w)) = p((rv) \otimes w).$$

Since the action of  $r_l$  commutes with  $s_l$  and  $s_r$  in  $V \overline{\otimes} W$  the subspace  $Y$  is  $r_l$ -invariant and the definition (2.3) is legitimate. Throughout what follows we shall deal entirely with the case of  $A$ -modules (strictly speaking, representations) on the tensor product  $A \overline{\otimes}_S V$ .

**DEFINITION.** Let  $A$  be a topological algebra, which is  $A$ - $S$ -bimodule and let  $V$  be an  $S$ -module. The representation of  $A$  on  $A \overline{\otimes}_S V$  defined by the formula (2.3) will be called the *representation of  $A$  induced by the  $S$ -module  $V$*  and denoted by  $(\lambda^{(S, V)}, A \overline{\otimes}_S V)$ .

The following statement is a straightforward corollary from Proposition 2.1.

**2.2. PROPOSITION.** *Let the algebra  $A$  be an  $A$ - $S$ -bimodule, and  $V$  be a  $S$ -module. Let  $(\pi, E)$  be a representation of  $A$  in a complete topological vector space  $E$ . Then the space  $L_A(\lambda^{(S, V)}, \pi)$  is isomorphic to the subspace of  $B_S(A, V; E)$  of forms verifying*

$$\beta(ab, v) = \pi(a)\beta(b, v), \quad \text{for all } a, b \in A, v \in V.$$

*The isomorphism assigns to  $T \in L_A(\lambda^{(S, V)}, \pi)$  the form  $T \circ d$ .*

Let us return to the case of involutive algebra  $A$  with an uniform approximate left unit, and suppose that the action of  $S$  on  $A$ , denoted by  $a \rightarrow as$ , commutes with the left regular representation.

In addition, we define the following action of  $S$  on  $A$ :

$$(2.4) \quad s \checkmark a := (a \checkmark s) \checkmark .$$

Then the formula holds:

$$(2.5) \quad \begin{aligned} \lambda(s \checkmark a)b &= (s \checkmark a)b = (a \checkmark s) \checkmark b = (b \checkmark (a \checkmark s)) \checkmark = ((b \checkmark a \checkmark) s) \checkmark \\ &= s \checkmark (ab) = s \checkmark \circ \lambda(a)b . \end{aligned}$$

Accordingly, the action of  $S$  on the tensor product  $A \overline{\otimes} E$  is defined as follows:

$$s \checkmark (a \otimes x) = (s \checkmark a) \otimes x .$$

Formula (2.5) yields

$$(2.6) \quad \lambda^E(s \checkmark a) = s \checkmark \circ \lambda^E(a) .$$

Given a continuous representation  $(\pi, E)$  of  $A$ , a structure of a  $S$ -module may be defined on the space  $\Phi(E)$ . We need the following

**2.3. LEMMA.** *Suppose that  $S$  acts on  $A$  in such a way that formula (2.6) holds. Then the kernel of the operator  $\alpha$  is  $s \checkmark$ -invariant.*

**Proof.** Let  $x \in \text{Ker } \alpha$ . Then for every  $a \in A$

$$\alpha(s \checkmark (\lambda^E(a)x)) = \alpha(\lambda^E(s \checkmark a)x) = \pi(s \checkmark a)\alpha(x) = 0 .$$

Thus the operator  $s \checkmark$  maps the dense in  $\text{Ker } \alpha$  subspace  $\lambda^E(A)\text{Ker } \alpha$  into itself. This completes the proof.

This lemma legitimates the definition of the  $S$ -module  $\Phi(E)$ :

$$(2.7) \quad s \checkmark \alpha(x) = \alpha(s \checkmark x), \quad x \in A \overline{\otimes} E, s \in S .$$

Thanks to formula (2.6) we obtain

$$(2.8) \quad s \checkmark \pi(a) = \pi(s \checkmark a) .$$

**EXAMPLES. 1.** Let  $A$  be a subalgebra of the convolution algebra  $L^1(G, dg)$  of integrable functions on a locally compact group  $G$ , provided with the involution  $f \checkmark(x) = \delta_G(x^{-1})f(x^{-1})$ . Suppose that  $A$  admits an approximate unit. The algebra  $A$  may be viewed as an  $A$ - $G$ -bimodule, the action of  $G$  being the right translations:

$$r(g)f(x) = f(xg) .$$

The operators which correspond to the action of  $G$  defined by (2.4) have the form:

$$A \ni f \rightarrow (r(g)f \checkmark) \checkmark = \delta_G(g^{-1})\lambda(g)f, \quad \text{where} \quad \lambda(g)f(x) = f(g^{-1}x) .$$

For given representation  $(\pi, E)$  of  $A$  we can define a representation of  $G$  in  $\Phi(E)$ , say  $\tilde{\pi}$ , by means of formula (2.7). If the representation  $\pi$  is

associated in a usual way to a continuous representation of  $G$  on  $E$ , then (2.8) implies

$$\tilde{\pi}(g) = \delta_G(g^{-1})\pi(g).$$

2. Let  $G$  be a Lie group, and  $A = \mathcal{D}$ ;  $U(G)$  denotes the universal enveloping algebra of the Lie group  $G$ , viewed as the space of right invariant differential operators on  $G$ . The action of  $X \in U(G)$  on  $\mathcal{D}$  satisfies condition (2.6) which takes the form

$$X(f * h) = (Xf) * h, \quad f, h \in \mathcal{D}.$$

The action of  $U(G)$  on  $\varphi(E)$  defined by formula (2.7) is just the usual representation of  $U(G)$  on  $E_\infty$  associated to the representation  $g \rightarrow \pi(g)$  by the differentiation.

Remark. We can also regard the elements of  $U(G)$  as left invariant operators on  $G$ . The algebra  $\mathcal{D}$  is then an  $\mathcal{D}$ - $U(G)$ -bimodule and one can define corresponding operators  $X$  on  $\mathcal{D}$  and  $\Phi(E)$  via (2.4) and (2.7). However the final formula is more complicated since one, in general, will have to differentiate the modular function  $\delta_G$ .

Now, we are in a position to formulate the main duality of the Frobenius–Bruhat type.

**2.4. THEOREM.** *Let  $A$  be a bornological involutive algebra with an equicontinuous approximate unit, and assume  $A$  to be an  $A$ - $S$ -bimodule. Let  $V$  be a bornological complete  $S$ -module and let  $(\pi, E)$  be a continuous representation of  $A$  in a complete bornological space. Then*

$$B_A(A \overline{\otimes}_S V, E) = B_S(\Phi(E), V),$$

where the isomorphism is defined by the formula

$$\tau(p(a \otimes v), x) = \beta(\pi(a^\vee)x, v)$$

for  $\tau \in B_A(A \overline{\otimes}_S V, E)$  and  $\beta \in B_S(\Phi(E), V)$ .

Proof. Thanks to Theorem 1.12 it remains to prove that every  $S$ -balanced bilinear form  $\beta$  defines such a form  $\tau_0$  on  $A \overline{\otimes} V$ , which passes to the quotient. We begin with the verifying that for fixed  $x \in E$  the functional  $(a, v) \rightarrow \tau_0(a \otimes v, x)$  is  $S$ -balanced. We have:

$$\begin{aligned} (2.9) \quad \tau_0(as \otimes v, x) &= \beta(\pi((as)^\vee)x, v) = \beta(\pi(s^\vee a^\vee)x, v) \\ &= \beta(s^\vee \pi(a^\vee)x, v) = \beta(\pi(a^\vee)x, sv) = \tau_0(a \otimes (sv), x). \end{aligned}$$

Now, by the universal property of the tensor product over  $S$  the following formula is legitimate:

$$\tau(p(a \otimes v), x) := \tau_0(a \otimes v, x).$$

On the other hand every form  $\tau \in B_A(A \overline{\otimes}_S V, E)$  defines by the latter formula a bilinear form on  $(A \overline{\otimes} V) \times E$  and in turn Theorem 1.12 leads

to  $\beta \in B(\Phi(E), V)$ . In order to verify that  $\beta$  is  $S$ -balanced it suffice to read (2.9) in the opposite direction.

In a similar way we can obtain counterparts of Theorems 1.13 and 1.14 in the theory of induced representations.

2.5. THEOREM. *Retain the assumptions of the latter theorem. Then the map*

$$L_A(A \overline{\otimes}_S V, E') \ni T \rightarrow \beta \in B_S(\Phi(E), V)$$

which assigns to  $T$  the bilinear form

$$\beta(\pi(a) x, v) := \langle T(p(a \otimes v)), x \rangle$$

is a linear isomorphism.

2.6. THEOREM. *Let  $(\pi, H)$  be a unitary continuous representation of a bornological  $*$ -algebra  $A$  with an equicontinuous approximate unit. Assume that a structure of  $A$ - $S$ -bimodule on  $A$  is defined. Then for every complete bornological  $S$ -module  $V$  we have an isomorphism of  $L_A(A \overline{\otimes}_S V, H)$  onto the space  $B_S(\Phi(E, V))$  of sesquilinear, separately continuous forms on  $\Phi(E) \times V$ , defined by:*

$$(T(p(a \otimes v)) | h) = \beta(\pi(a^*) h, v).$$

### 3. Specialization; differentiable induced representations of Yamabe groups

In this section we develop the theory of induced representations in spaces of "appropriately smooth" vector valued functions on locally compact groups which are not assumed to be Lie groups.

By a *Yamabe group* we mean a projective limit of Lie groups. To establish a notation, let  $\{K_i\}$  be a family of normal compact subgroups of the group  $G$  such that 1° the quotient group  $G_i := G/K_i$  is a Lie group for all  $i$  and 2°  $\bigcap_i K_i = e$  the neutral element of  $G$ . It is known (see [23]) that the topology of  $G$  is equivalent to the projective topology relative to the family of natural projections  $p_i: G \rightarrow G/K_i$ .

Recall that the existence of family  $\{K_i\}$  satisfying 1° and 2° is equivalent to the requirement:  $G/G_0$  is compact,  $G_0$  denoting the connected component of  $e$  in  $G$ .

To make the exposition more clear we restrict our attention to metrizable locally compact Yamabe groups. This assumption enables us to take a countable decreasing family  $\{K_n\}$ . Let  $E$  be a locally convex topological vector space. Then for  $n > m$  the space  $\mathcal{D}(G_m, E)$  is imbedded as a closed subspace into  $\mathcal{D}(G_n, E)$ . Let us define

$$\mathcal{D}(G, E) := \lim_{\rightarrow} \mathcal{D}(G_i, E), \quad \mathcal{D} := \mathcal{D}(G, C).$$

A function  $\psi$  on  $G$  is called *regular* if for every  $f \in \mathcal{D}$  the product  $f\psi$  belongs to  $\mathcal{D}(G, E)$ . This is equivalent to the statement: for every  $g \in G$  there exists a neighbourhood  $O(g)$  of  $g$  and  $f \in \mathcal{D}(G, E)$  such that  $f(x) = \psi(x)$  for  $x \in O(g)$ .

The space  $\mathcal{E}(G, E)$  of regular functions on  $G$  is provided with the projective topology associated to the family of mappings  $P_f: \psi \rightarrow \psi f \in \mathcal{D}(G, E)$ ,  $f$  ranging over  $\mathcal{D}$ . Regular functions with compact supports are just those from the space  $\mathcal{D}(G, E)$ .

3.1. LEMMA. *Let  $E$  be a Fréchet space. Then  $\mathcal{D}(G, E)$  is isomorphic to the tensor product  $\mathcal{D} \otimes E$ . In particular,  $\mathcal{D}(G, E)$  is a bornological complete (then barrelled) vector space.*

Proof.  $\mathcal{D}(G, E) = \lim_{\rightarrow} \mathcal{D}(G_n, E) = \lim_{\rightarrow} \mathcal{D}(G_n) \otimes E = (\lim_{\rightarrow} \mathcal{D}(G_n)) \otimes E = \mathcal{D} \otimes E$ .

The space  $\mathcal{D} = \lim_{\rightarrow} \mathcal{D}(G_n)$  is bornological as an inductive limit of bornological spaces. The proof follows from Lemma 1.3.

Since regularity is a local property the completeness of  $\mathcal{D}(G, E)$  implies

3.2. LEMMA. *If  $E$  is a Fréchet space then  $\mathcal{E}(G, E)$  is complete.*

3.3. LEMMA. *The space  $\mathcal{D}$  is a convolution subalgebra, dense in  $C_0(G)$  and invariant with respect to the usual involution. The algebra  $\mathcal{D}$  admits an equicontinuous approximate unit.*

The density of  $\mathcal{D}$  in  $C_0(G)$  was proved by Bruhat [5] and Maurin [17]. Both proofs use implicitly approximative units of  $\mathcal{D}$ . For completeness we recall this construction. Let  $d g_n$  denote the projection of the measure  $d g$  onto the group  $G_n$ ;

$$\int_G f(p_n(g)) d g = \int_{G_n} f(y) d g_n(y), \quad f \in \mathcal{D}(G_n).$$

We have seen in Section 1, that the algebra  $\mathcal{D}(G_n)$  admits an approximate unit  $\{\varphi_n^k\}_{k=1}^{\infty}$ . Then the sequence  $\psi_n := \varphi_n^n$  constitute an approximate unit in  $\mathcal{D}$ . Since  $\mathcal{D}$  is a barrelled space and operators  $\lambda(\psi_n)$  are pointwise bounded, the family  $\lambda(\psi_n)$  is equi-continuous by Banach–Steinhaus theorem.

Let  $(\pi, E)$  be a continuous representation of  $G$  on a Fréchet space  $E$ . The formula

$$\pi(f) = \int_G f(g) \pi(g) d g$$

defines a continuous representation of the algebra  $\mathcal{D}$  on  $E$ .

A vector  $x \in E$  is called *differentiable* (relative to  $\pi$ ) if the map  $\tilde{x}: G \rightarrow \pi(g)x \in E$  belongs to  $\mathcal{E}(G, E)$ .

We place on the space  $E_{\infty}$  of all differentiable vectors the topology induced by  $\mathcal{E}(G, E)$  via the identification  $x \leftrightarrow \tilde{x}$ .

It is easily seen, that for every  $x \in E$  there exists an  $i$ , such that  $\pi(k)x = x$  for  $k \in K_i$ . In fact, let  $\varphi \in \mathcal{D}$  be a function equal to 1 on some neighbourhood of  $e \in G$ . Then there exists an  $i_0$  such that  $\varphi$  is constant and nonzero on  $K_{i_0}$ . Since  $\varphi \tilde{x} \in \mathcal{D}(G, E)$  we can choose  $i > i_0$  such that in turn  $\varphi \tilde{x} \in \mathcal{D}(G_i, E)$ . Therefore, for  $k \in K_i$ , we have

$$\pi(k)x = \tilde{x}(k) = \varphi(k)x(k) = \varphi(e)x(e) = x,$$

as desired.

Let us write

$$E_\infty^i = \{x \in E_\infty : \pi(k)x = x, k \in K_i\}.$$

The subspace  $E_\infty^i$  is closed in  $E_\infty$  and is  $G$ -invariant.

3.4. PROPOSITION.  $E_\infty$  is a strict inductive limit of the spaces  $E_\infty^i$ .

Proof. We have shown above that  $E_\infty = \bigcup_{i=1}^{\infty} E_\infty^i$ . Obviously, the injection of  $\lim_{\rightarrow} E_\infty^i$  into  $E_\infty$  is continuous. For the purpose of proving the continuity of the converse injection we employ the mapping

$$\beta: E_\infty \rightarrow \mathcal{D}(G, E)$$

defined by the formula

$$(3.1) \quad \beta(x)(g) = \gamma(g)\pi(g^{-1})x = \gamma \check{\tilde{x}}(g^{-1}),$$

where  $\gamma$  is a non-negative function in  $\mathcal{D}$ , satisfying  $\int_G \gamma dg = 1$ . The mapping is continuous by the very definition of the topology of  $E_\infty$ . On the other hand we have a continuous map  $\tilde{\alpha}: \mathcal{D}(G, E) \rightarrow E$  defined by

$$(3.2) \quad \tilde{\alpha}(f) = \int_G \pi(g)f(g)dg.$$

Plainly,  $\tilde{\alpha}(\mathcal{D}(G_i, E))$  belongs to  $E_\infty^i$  and bearing in mind our results concerning representations of Lie groups (Section 1) we found that  $\tilde{\alpha}$  maps continuously  $\mathcal{D}(G_i, E) = \mathcal{D}(G_i) \otimes E$  into  $E_\infty^i$ . Accordingly  $\tilde{\alpha}$  is a continuous mapping from  $\mathcal{D}(G, E) = \lim_{\rightarrow} \mathcal{D}(G_i, E)$  into  $\lim_{\rightarrow} E_\infty^i$ . Since, by a simple computation, we verify  $\tilde{\alpha} \circ \beta = \text{id}$ , the proof follows.

Thanks to Lemma 3.1 we can identify the space  $\mathcal{D}(G, E)$  with  $\mathcal{D} \otimes \overline{E}$ , and consequently the operator  $\tilde{\alpha}$  with  $\alpha$  used in our construction of the space  $\Phi(E)$ . In the course of the latter proof we have observed that  $\alpha$  maps  $\mathcal{D} \otimes \overline{E}$  onto  $E_\infty$ . It means that  $\Phi(E) \supset E_\infty$ . Now, viewing in turn  $\alpha$  as a continuous operator onto  $\Phi(E)$  and using the identity  $\alpha \circ \beta = \text{id}$  we establish the topological isomorphism between these spaces. We have proved

3.5. THEOREM. Let  $E$  be a Fréchet space. Then  $E_\infty = \Phi(E) = \lim_{\rightarrow} E_\infty^i$ .

In virtue of the above theorem the restriction of  $\pi$  to  $\Phi(E)$  is a representation which admits a repeated construction  $\Phi(\Phi(E)) = \Phi(E_\infty)$ .

In fact, the space  $E_\infty$  is complete and the representation of  $A = \mathcal{D}$  on  $\Phi(E)$  is continuous.

The following result plays an important role in applications.

**3.6. THEOREM.** *Let  $G$  be a group countable at infinity. Retain other assumptions on  $G$  and  $(\pi, E)$ . Then the space  $\Phi(\Phi(E))$  is isomorphic to  $\Phi(E)$ .*

*Proof.* The spaces  $E_\infty^i$  are  $G$ - (as well as  $\mathcal{D}$ ) invariant subspaces of  $E_\infty$  and to prove the assertion it suffices to show, that  $\Phi(E_\infty^i) = E_\infty^i$ . By the very definition  $\Phi(E_\infty^i)$  is continuously imbedded into  $E_\infty^i$  and it remains to find a converse map. If  $G$  is countable at infinity the spaces  $\mathcal{E}(G_i, E)$  and consequently  $E_\infty^i$  are Fréchet spaces. Therefore

$$\Phi(E_\infty^i) = \alpha(\mathcal{D} \otimes E_\infty^i) = \alpha(\mathcal{D}(G, E_\infty^i)).$$

The proof will be complete if we shall be able to inject  $E_\infty^i$  continuously into  $\mathcal{D}(G, E_\infty^i)$ . To this end, remark that in the case of a Lie group  $G_i$  the space  $(E_\infty^i)_\infty$  is equal to  $E_\infty^i$ , because the regular representation of  $G_i$  in  $\mathcal{E}(G_i, E)$  is differentiable (i.e. all vectors in the carrier space are smooth). Therefore the mapping  $\beta$  defined by (3.1) with  $\gamma \in \mathcal{D}(G_i)$  may be viewed as a continuous map from  $E_\infty^i$  into  $\mathcal{D}(G_i, E_\infty^i)$ . The latter space is in turn continuously imbedded into  $\mathcal{D}(G, E_\infty^i)$ . Using the formula  $\alpha \circ \beta = \text{id}$  once again, we finish the proof.

Now we will detail the representations of  $\mathcal{D}$  (and  $G$ ) induced by a representation of a subgroup  $\Gamma \subset G$ . The carrier space of this representation, originally defined as quotient spaces of the tensor product  $\mathcal{D} \otimes V$ , may be identified with some function space, which in the special case of the Lie group  $G$  and differentiable inducing representations is just the space of Bruhats differentiable induced representations.

Let  $(\sigma, V)$  be a continuous representation of a closed subgroup  $\Gamma \subset G$  in a Fréchet space  $V$ . The algebra  $\mathcal{D}$  will be regarded as an  $\mathcal{D}$ - $\Gamma$ -bimodule with the action of  $\Gamma$  on  $\mathcal{D}$  by means of right translations:  $r(\gamma)f(g) := f(g\gamma)$ .

Recall some basic results concerning quasi-invariant measures on homogeneous spaces (for more details see Bourbaki [3], VIII, 2). We drop for the moment the supposition  $G$  to be a Yamabe group.

In general, positive  $G$ -invariant measures on  $G/\Gamma$  do not exist, nevertheless quasi-invariant measures are always present and any two of them are equivalent. They could be obtained in the following way. To any Borel function  $\varrho > 0$  on  $G$  bounded above and below on compact subsets and verifying for every  $\gamma \in \Gamma$

$$\varrho(g\gamma) = \delta_\Gamma(\gamma) / \delta_G(\gamma) \varrho(g)$$

there is associated a quasi-invariant measure  $\mu$  on  $G/\Gamma$  defined by

$$(3.3) \quad \int_G f(g) e(g) dg = \int_{G/\Gamma} d\mu(g\Gamma) \int_{\Gamma} f(g\gamma) d\gamma, \quad f \in C_0(G).$$

By  $d\gamma$  we denote the left Haar measure on  $\Gamma$  and  $\delta_H$  stands for the modular function on  $H$ . Let us return to our case of a metrizable Yamabe group  $G$ . Given  $f \in \mathcal{D}(G, V)$ , put

$$(3.4) \quad l_\sigma(f)(g) = \int_{\Gamma} e(\gamma)^{-1/2} \sigma(\gamma) f(g\gamma) d\gamma \quad (e(\gamma) = \delta_R/\delta_G(\gamma)).$$

**3.7. LEMMA.** *Ker  $l_\sigma$  is equal to the closed linear span of the set*

$$Z = \{\psi \in \mathcal{D}(G, E) : \psi(g) = \tau(\gamma^{-1})f(g) - (\delta_R \delta_G)^{1/2}(\gamma) \sigma(\gamma) f(g), \\ \text{for some } f \in \mathcal{D}(G, V), \gamma \in \Gamma\}.$$

**Proof.** The closed linear span of  $Z$  will be denoted by  $Y$ . First we verify the inclusion  $\text{Ker } l_\sigma \supset Y$ .

$$\begin{aligned} l_\sigma(\tau(\gamma_1^{-1})f)(g) &= \int_{\Gamma} e(\gamma)^{-1/2} \sigma(\gamma) f(g\gamma\gamma_1^{-1}) d\gamma \\ &= e^{-1/2}(\gamma_1) \delta_R(\gamma_1) \int_{\Gamma} e(\gamma)^{-1/2} \sigma(\gamma_1) \sigma(\gamma) f(g\gamma) d\gamma \\ &= l_\sigma(\sigma(\gamma_1)f)(g) (\delta_G \delta_R(\gamma_1))^{1/2}. \end{aligned}$$

In order to establish the converse inclusion it suffices to show that every continuous functional  $T$  on  $\mathcal{D}(G, V)$  which annihilates  $Y$ , vanishes on  $\text{Ker } l_\sigma$ . Suppose  $f \in \text{Ker } l_\sigma$ , and let

$$\begin{aligned} f_1(x) = \varphi * f(x) &= \int_G \varphi(g) f(g^{-1}x) dg = \int_G \varphi(g^{-1}) \delta_G(g^{-1}) f(gx) dg \\ &= \delta_G(x) \int_G \tilde{\varphi}(gx^{-1}) \delta_G(g^{-1}) f(g) dg, \end{aligned}$$

where  $\varphi \in \mathcal{D}$  and  $\tilde{\varphi}(g) = \varphi(g^{-1})$ .

Therefore, by the formula (3.3)

$$\begin{aligned} f_1(x) &= \delta_G(x) \int_{G/\Gamma} d\mu(g\Gamma) \int_{\Gamma} e^{-1}(g\gamma) \delta_G^{-1}(g\gamma) \tilde{\varphi}(g\gamma x^{-1}) f(g\gamma) d\gamma \\ &= \delta_G(x) \int_{G/\Gamma} d\mu(g\Gamma) e^{-1}(g) \delta_G^{-1}(g) \int_{\Gamma} \delta_G(\gamma^{-1}) \tilde{\varphi}(g(x\gamma^{-1})^{-1}) f(g\gamma) d\gamma. \end{aligned}$$

Since the integral is strongly convergent we can write

$$\begin{aligned} T(f_1) &= \int_{G/\Gamma} d\mu(g\Gamma) e^{-1}(g) \delta_G^{-1}(g) \int_{\Gamma} \delta_G(\gamma^{-1}) T_x(\delta_G(x) \tilde{\varphi}(g(x\gamma^{-1})^{-1}) f(g\gamma) d\gamma) \\ &= \int_{G/\Gamma} d\mu(g\Gamma) e^{-1}(g) \delta_G^{-1}(g) T_x(\delta_G(x) \tilde{\varphi}(gx^{-1}) \int_{\Gamma} e^{1/2}(\gamma) \sigma(\gamma) f(g\gamma) d\gamma) \\ &= \int_{G/\Gamma} d\mu(g\Gamma) e^{-1}(g) \delta_G^{-1}(g) T_x(\delta_G(x) \tilde{\varphi}(gx^{-1}) l_\sigma f) = 0, \end{aligned}$$

since by our assumption  $l_\sigma f = 0$ . Therefore

$$T(f) = \lim_n T(\varphi_n * f) = 0$$

( $\varphi_n$  denotes the approximate unit in  $\mathcal{D}$ ).

It is easily seen that the quotient space  $\mathcal{D}(G, V)/Y$  coincides with the carrier space of the induced representation  $\lambda^{(G, V)}$  corresponding to the action of  $\Gamma$  on  $V$  by means of the operators  $(\delta_\Gamma \delta_G) \sigma(\gamma^{-1}) \in L(V)$ . Thanks to Lemma 3.7 we may identify this space with the range of the operator  $l_\sigma$ . It would be more desirable to introduce an "internal" description of this space as well as of its topology induced by the identification with  $\mathcal{D}(G, V)/Y$ .

In the case of a Lie group countable at infinity it was made by Bruhat [4] (see also an excellent exposition in [39], p. 368).

Although our space  $\mathcal{D}(G, V)$  is defined as an inductive limit of function spaces on Lie groups we may not appeal to the theorem of Bruhat, because we have employed non differentiable inducing representations  $(\sigma, V)$  and on the other hand the subgroup  $\Gamma$  may not be treated as a subgroup of  $G_i = G/K_i$ . It will be more convenient to legitimate the Bruhat's procedure in this more general case.

Let us denote by  $\mathcal{D}^\sigma$  the subspace of  $\mathcal{E}(G, V)$  of functions satisfying

1°  $\text{spt} f$  is compact modulo  $\Gamma$  (i.e.  $(\text{spt} f)\Gamma$  is compact in  $G/\Gamma$ ),

2°  $r(\gamma)f(g) = \sigma(\gamma^{-1})\rho^{1/2}(\gamma)f(g)$ .

It is easily seen that  $l_\sigma(\mathcal{D}(G, V))$  belongs to  $\mathcal{D}^\sigma$ .

3.8. LEMMA. *Let  $G/\Gamma$  be countable at infinity. Then  $l_\sigma(\mathcal{D}(G, V)) = \mathcal{D}^\sigma$ .*

We have to prove that  $l_\sigma$  is surjective. Let us fix  $i$ , and select a covering  $\{\mathcal{O}^j\}$  of  $G_i/\Gamma_i$  (where  $\Gamma_i = \Gamma K_i$ ), such that in each of  $\mathcal{O}^j$  there exists a smooth local section  $x\Gamma \rightarrow s_j(x\Gamma)$  of the principal bundle  $G_i$  over  $G_i/\Gamma_i$ . Let us choose  $\psi \in \mathcal{D}(\Gamma_i)$  such that

$$\int_\Gamma \psi(\gamma) \rho^{-1/2}(\gamma) d\gamma = 1.$$

Given  $f \in \mathcal{D}^\sigma \cap \mathcal{E}(G_i, V)$  we define

$$f^\#(x) = \sum_j \Phi_j(x\Gamma) \psi(s_j(x\Gamma)^{-1}x) \sigma(x^{-1}s_j(x\Gamma))f(s_j(x\Gamma)),$$

where  $\{\Phi_j\}$  denotes a partition of unity subordinate to this covering.

Plainly  $f^\# \in \mathcal{D}(G, V)$  and by a direct computation we found  $l_\sigma(f^\#) = f$ . It remains to show that for every  $f \in \mathcal{D}^\sigma$  there exists  $i$ , such that  $f$  is  $K_i$ -invariant, that is  $f \in \mathcal{E}(G_i, V)$ .

We shall need a technical

3.9. LEMMA (Bourbaki [3], VII, § 2). *There exists a regular function  $\varphi_0$  on  $G_i$  such that  $\text{spt} \varphi_0 \cap g\Gamma_i$  is compact for every  $g \in G_i$ , and  $\text{spt} \varphi_0 \Gamma_i = G_i$ .*

We denote by  $\varphi$  the lift of  $\varphi_0$  to  $G$ . Return to the proof of Lemma 3.8. The function  $\varphi f$  regular on  $G$  and by 1° it has a compact support. Hence  $\varphi f$  belongs to  $\mathcal{D}(G_j, V)$  for appropriate  $j$ , and for  $m = \max(i, j)$  and  $g \in \text{spt}\varphi \cap \text{spt}f$  we have

$$f(gk_m) = f(g), \quad k_m \in K_m.$$

By 2° we have

$$\begin{aligned} f(g\gamma k_m) &= f(g\gamma k_m \gamma^{-1} \gamma) = \varrho^{1/2}(\gamma) \sigma(\gamma^{-1}) f(g\gamma k_m \gamma^{-1}) \\ &= \varrho^{1/2}(\gamma) \sigma(\gamma^{-1}) f(g) = f(g\gamma). \end{aligned}$$

Since the set  $(\text{spt}\varphi \cap \text{spt}f) \Gamma$  covers the support of  $f$  the proof follows.

The space  $\mathcal{D}(G_i, V)$  is mapped under the action of  $l_\sigma$  onto  $\mathcal{E}(G_i, V) \cap \mathcal{D}^\sigma$ . Let us provide the latter space with the topology of the inductive limit of spaces  $\mathcal{D}_K^\sigma(G_i)$ , where

$$\mathcal{D}_K^\sigma(G_i) := \{f \in \mathcal{D}^\sigma \cap \mathcal{E}(G_i, V) : \text{spt}f \subset K\Gamma\}$$

is equipped with the topology inherited from  $\mathcal{E}(G_i, V)$ . Plainly  $l_\sigma$  is a continuous operator from  $\mathcal{D}(G, V)$  onto  $\mathcal{D}^\sigma$ . Using the open map theorem we assert the isomorphism of these spaces. We have proved

**3.10. PROPOSITION.** *There is a topological isomorphism of  $\mathcal{D}(G, V)/Y$  onto  $\mathcal{D}^\sigma$  intertwining for the representation  $\lambda^{(L, V)}$  and the left regular representation of  $G$  on  $\mathcal{D}^\sigma$ .*

The latter representation will be denoted by  $(U^\sigma, \mathcal{D}^\sigma)$  and referred to as differentiable representation induced by the representation  $(\sigma, V)$ .

At last let us recall, that the action of  $\Gamma$  in the space  $\Phi(E)$  defined by formula (2.8) has the form  $\gamma \rightarrow \pi(\gamma) \delta_G(\gamma^{-1})$  (Section 2, example 1). The main results of this section are the immediate consequences of Theorems 2.4, 2.5 and 3.5, 3.10.

**3.11. THEOREM.** *Let  $G$  be a metrizable Yamabe group, let  $\Gamma \subset G$  be a closed subgroup of  $G$  such that  $G/\Gamma$  is countable at infinity. Let  $(\pi, E)$  be a continuous representation of  $G$  in a Fréchet space  $E$ . Let  $(\sigma, V)$  be a continuous representation of  $\Gamma$  in a Fréchet space  $V$ . Then we have the following isomorphisms*

$$L_G(\mathcal{D}^\sigma, E') = B_G(\mathcal{D}^\sigma, E) = B_\Gamma(E_\infty, V) = L_\Gamma(E_\infty, V'),$$

where  $B_\Gamma$  ( $L_\Gamma$ ) denotes the space of forms (resp. operators) balanced (intertwining) with respect to the action of  $\Gamma$  via  $\varrho^{1/2}\sigma(\gamma)$  on  $V$  and  $\pi(\gamma)$  on  $E$ .

The first and the latter isomorphisms are just natural connections between bilinear forms and operators, the central one is defined by the formula

$$\tau(l_\sigma(f \otimes v), x) = \beta(\pi(f^\vee)x, v).$$

Bearing in mind Theorem 3.6 we obtain an additional isomorphism:

$$3.12. \text{ THEOREM. } B_G(\mathcal{D}^\sigma, E) = B_G(\mathcal{D}^\sigma, E_\infty) = B_\Gamma(E_\infty, V).$$

$$3.13. \text{ COROLLARY. } B_G(V_\infty, E) = B_G(V_\infty, E_\infty).$$

This is so, because in case  $G = \Gamma$  the space  $V_\infty$  is isomorphic to  $\mathcal{D}^\sigma$  via identification of  $v \in V_\infty$  with the function  $g \rightarrow \pi(g^{-1})v$ .

A natural anti-linear involution on  $\mathcal{D}$  is defined by the formula

$$f^* = \bar{f}^\vee.$$

Then in turn Theorems 2.6 and 3.5, 3.6, 3.10 imply

3.14. THEOREM. *Retain all assumptions of 3.11 and assume that  $(\pi, H)$  is a unitary representation of the group  $G$  on a Hilbert space  $H$ . Then*

$$L_G(\mathcal{D}^\sigma, H) = L_G(\mathcal{D}^\sigma, H_\infty) = L_\Gamma(E_\infty, \bar{V}'),$$

where the latter isomorphism assigns to  $T \in L_G(\mathcal{D}^\sigma, H)$  the operator  $\eta$  defined by the formula

$$(T(l_\sigma(f \otimes v) | h)) = \langle \eta(f^*)h, v \rangle.$$

( $\bar{V}'$  denotes the space of antilinear, continuous functionals on  $V$  made a  $\Gamma$ -module by means of the operators  ${}^t\sigma(\gamma) \varrho^{-1/2}(\gamma)$ .)

As we have mentioned above these theorems generalize the reciprocity theorems obtained by F. Bruhat in case of a Lie group  $G$  and differentiable induced representation [4].

We refer the reader to the monograph [39] for various applications of theorems of this type, especially for their interpretation in terms of intertwining numbers and for relations to classical theorems of G. Frobenius, A. Weil and G. Mackey.

In the subsequent number we explain their connections with the General Duality Theorem of K. and L. Maurin [19]. In the course of the proof we obtain a notable consequence of Theorem 3.12, namely a generalization of Skovhus Poulsen's structure theorem of bilinear invariant forms on spaces of smooth vectors.

#### 4. Unitary induced representations of groups

We retain all assumptions and notation of the preceding section. In the present we consider unitary representations obtained by the completion of the spaces  $\mathcal{D}^\sigma$  in a prehilbert norm invariant under the action of the group. In the preparatory step the carrier space  $\mathcal{D}^\sigma$  is imbedded into a space  $\mathcal{C}^\sigma$  of continuous functions on  $G$ . Several connections between continuous intertwining operators on  $\mathcal{D}^\sigma$  and closed operators on  $\mathcal{C}^\sigma$  are investigated.

Let  $\varphi$  be a function on  $G$ , introduced in Lemma 3.9. The set

$$\Omega = \{g \in G: \varphi(g) > 0\}$$

is an open set which satisfies:

$$1^\circ p(\Omega) = G/\Gamma,$$

2 $^\circ$  for every compact  $K \subset G/\Gamma$ ,  $p^{-1}(K) \cap \Omega$  is precompact.

We define a family of continuous seminorms on  $\mathcal{D}^\sigma$  which is labelled by the set of seminorms describing the topology of  $V$ :

$$|||f|||_j = \sup_{g \in \Omega} \|f(g)\|_j,$$

where  $\|\cdot\|_j$  is a continuous seminorm on  $V$ .

We denote by  $\mathcal{E}^\sigma$  the Fréchet space obtained by the completion of the space  $\mathcal{D}^\sigma$  in the topology defined by the family  $\{|||\cdot|||\}_j$  of semi-norms.

4.1. PROPOSITION. *The induced representation  $U^\sigma$  extends to a continuous representation of  $G$  on  $\mathcal{E}^\sigma$ .*

Proof. First of all we verify the continuity of left translations in the topology of  $\mathcal{E}^\sigma$ . Let  $f \in \mathcal{D}^\sigma$ ,  $x \in G$ ,

$$|||U^\sigma(x)f|||_j = \sup_{g \in \Omega} \|f(x^{-1}g)\|_j = \sup_{g \in x\Omega} \|f(g)\|_j.$$

Plainly, the set  $\mathcal{O} = x\Omega \cap \text{spt} f$  is relatively compact. We put

$$\mathcal{O} = \{\gamma \in \Gamma: \gamma = s^{-1}y, s \in \Omega \cap \text{spt} f, y \in (x\Omega) \cap \text{spt} f\}.$$

Since  $\mathcal{O}$  is relatively compact we obtain the following estimation

$$\begin{aligned} |||U^\sigma(x)f|||_j &\leq \sup_{\substack{g \in \Omega \cap \text{spt} f \\ \gamma \in \mathcal{O}}} \|f(g\gamma)\|_j = \sup_{g, \gamma} \|\sigma(\gamma) \varrho^{1/2}(\gamma) f(g)\|_j \\ &\leq L \sup_{g \in \Omega} \|f(g)\|_j = L |||f|||_j \end{aligned}$$

for some  $L > 0$  and  $i$ . Here we have employed the equicontinuity of the set of operators  $\sigma(\gamma)$ ,  $\gamma$  ranging over the compact set  $\mathcal{O}$ .

Now remark, that all arguments used in the course of the proof remain valid if  $x$  ranges over a compact set. Accordingly, left translations extend to continuous operators  $U^\sigma$  on  $\mathcal{E}^\sigma$  and for every compact set  $K$  the operators  $\{U^\sigma(g): g \in K\}$  constitute an equicontinuous set. In order to prove that this representation is continuous it suffice to verify the continuity on a total subset of  $\mathcal{O}$ , i.e.  $\mathcal{D}^\sigma$  (see Bourbaki [3], VIII, 2.1). We have to show, that for every  $f \in \mathcal{D}^\sigma$  the set

$$\mathcal{U}_\varepsilon = \{y \in G: |||(U^\sigma(y) - U^\sigma(x))f|||_i < \varepsilon\}$$

is open. It is a consequence of the continuity of  $f$  and the compactness of the set  $A = \Omega \cap \text{spt} f$ . For every  $s \in A$  we can choose an open symmetric

neighbourhood  $V_s$  of the neutral element  $e \in G$  such that

$$\|U_{\mathbf{a}}^{\sigma}(x)f(s) - U_{\mathbf{a}}^{\sigma}(x)f(g)\|_t < \varepsilon/2$$

for  $g \in V_s^2$ .

Since the compact set  $A = \bigcup_s V_s s$  admits a finite covering  $\bigcup_{i=1}^n V_{s_i} s_i$  we obtain an open set  $(\bigcap_{i=1}^n V_{s_i})x$  contained in  $\mathcal{U}_x$ .

In the process of proving we can easily observe that the elements of the completion  $\mathcal{C}^{\sigma}$  are represented by continuous functions, thus the action of  $G$  on  $\mathcal{C}^{\sigma}$  continues to be the left regular representation.

**4.2. PROPOSITION.** *Let  $T$  be a sequentially closed operator intertwining for the representation  $(U^{\sigma}, \mathcal{C}^{\sigma})$  and a representation  $(\pi, F')$ ,  $F'$  being Fréchet space. Assume that  $D_T \supset \mathcal{D}^{\sigma}$ . Then the operator  $T|_{\mathcal{D}^{\sigma}}$  is a continuous operator intertwining for  $U^{\sigma}|_{\mathcal{D}^{\sigma}}$  and  $\pi|(F')_{\infty}$ .*

(A closed operator is called interwining if  $TU^{\sigma}(g) \supset \pi(g)T$ ,  $g \in G$ .)

*Proof.* Let us denote  $T_0 = T|_{\mathcal{D}^{\sigma}}$ . The space  $\mathcal{D}^{\sigma}$  is a bornological space and  $F'$  as a dual to a Fréchet space is a de Wilde's space (see [44]). Since the injection  $\mathcal{D}^{\sigma} \rightarrow \mathcal{C}^{\sigma}$  is continuous, the operator  $T_0$  is closed on  $\mathcal{D}^{\sigma}$ . Thus we observe that de Wilde's version of the Closed Graph Theorem is in force, and implies the continuity of  $T_0: \mathcal{D}^{\sigma} \rightarrow F'$ . Actually  $T$  maps  $\mathcal{D}^{\sigma}$  continuously into  $(F')_{\infty}$ . For the purpose of proving this it suffice to show that the operator  $T_0 \circ l_{\sigma}: \mathcal{D}(G, V) \rightarrow F'$ , acts into  $(F')_{\infty}$  continuously. By the very definition of the space  $\mathcal{D}(G, V)$  and Proposition 1.19, we assert  $(\mathcal{D}(G, V))_{\infty} = \mathcal{D}(G, V)$ . Owing to Theorem 1.11 we obtain

$$T \circ l_{\sigma} \in L_G(\mathcal{D}^{\sigma}, \Phi(F')).$$

Since  $\Phi(F')$  is continuously injected into  $(F')_{\infty}$  the proof follows.

After additional (and restrictive) assumptions on the homogeneous space  $G/\Gamma$  we shall be able to prove that to every  $G$ -invariant bilinear separately continuous form on  $\mathcal{D}^{\sigma} \times E$  there is associated a closeable on  $\mathcal{C}^{\sigma}$  operator ranged on  $\mathcal{D}^{\sigma}$  and valued in  $(E')_{\infty}$ .

**4.3. PROPOSITION.** *Retain the assumptions of Theorem 3.11 and suppose, in addition, that  $G/\Gamma$  is compact.*

*Then the operator  $T$  defined by the formula*

$$\langle T(f), x \rangle = \tau(f, x), \quad f \in \mathcal{D}^{\sigma}, x \in E, \tau \in B_G(\mathcal{D}^{\sigma}, E)$$

*acts continuously from  $\mathcal{D}^{\sigma}$  into  $(E')_{\infty}$  and is closeable on  $\mathcal{C}^{\sigma}$ .*

*Proof.* With the aid of our Theorem 3.11 we see that the above formula defines a continuous intertwining operator  $T: \mathcal{D}^{\sigma} \rightarrow E'$ . The duality Theorem 3.11 states that  $T$  determines a bilinear form  $\beta \in B_{\Gamma}(E_{\infty}, V)$ , so that

$$(4.1) \quad \langle T(l_{\sigma}(\varphi \otimes v)), x \rangle = \beta(\pi(\varphi^{\vee})x, v).$$

We need the following technical

4.4. LEMMA. *Formula (4.1) is equivalent to*

$$(4.2) \quad \langle T(\psi), x \rangle = \int_{G/\Gamma} \varrho^{-1}(g) \beta(\pi(g^{-1})x, \psi(g)) d\mu(g\Gamma), \quad \psi \in \mathcal{D}^\sigma,$$

where  $\varrho$  denotes a rho function on  $G$  and  $d\mu$  is the quasi-invariant measure on  $G/\Gamma$  associated to  $\varrho$ .

**Proof.** We turn formula (4.1) into (4.2) by means of (3.3)

$$\begin{aligned} \beta(\pi(\varphi^\vee)x, v) &= \beta\left(\int_G \delta_G(g^{-1}) \varphi(g^{-1}) \pi(g)x dg, v\right) \\ &= \int_G \varphi(g) \beta(\pi(g^{-1})x, v) dg \\ &= \int_{G/\Gamma} d\mu(g\Gamma) \int_\Gamma \varrho(g\gamma)^{-1} \varphi(g\gamma) \beta(\pi(g\gamma)^{-1}x, v) d\gamma \\ &= \int_{G/\Gamma} d\mu(g\Gamma) \varrho^{-1}(g) \beta(\pi(g^{-1})x, \int_\Gamma \varrho^{-1/2}(\gamma) \varphi(g\gamma) \sigma(\gamma) v d\gamma) \\ &= \int_{G/\Gamma} d\mu(g\Gamma) \varrho(g)^{-1} \beta(\pi(g^{-1})x, l_\sigma(\varphi \otimes v)). \end{aligned}$$

The formula extends to  $\mathcal{D}^\sigma$  by continuity.

Let us return to the proof of Proposition 4.3. Since  $G/\Gamma$  is compact, there exists a compact set  $\mathcal{O} \subset G$  such that  $\mathcal{O}\Gamma = G$ . For every  $\psi \in \mathcal{D}^\sigma$

$$\sup_{g \in G} |\varrho(g)^{-1} \beta(\pi(g^{-1})x, \psi(g))| = \sup_{g \in \mathcal{O}} |\varrho(g)^{-1} \beta(\pi(g^{-1})x, \psi(g))|$$

since the function at hand is right  $\Gamma$ -invariant.

Consequently, by Lemma 4.4, we conclude

$$|\tau(\psi, x)| \leq \mu(G/\Gamma) \sup_{g \in \mathcal{O}} |\varrho(g)^{-1} \beta(\pi(g^{-1})x, \psi(g))|.$$

The set  $\{\pi(g^{-1})x : g \in \mathcal{O}\}$  is compact in  $E_\infty$ , therefore for every  $x$  there exists a seminorm  $\|\cdot\|_x$  on  $V$  and a constant  $L > 0$  such that

$$|\beta(\pi(g^{-1})x, v)| \leq L \|v\|_x \quad \text{for all } g \in \mathcal{O}$$

(recall that  $\beta$  is hypocontinuous, cf. proof of 1.8).

Finally, we obtain the estimation

$$(4.3) \quad |\tau(\psi, x)| \leq L_1 \sup_{g \in \mathcal{O}} \varrho(g)^{-1} \sup_{g \in \mathcal{O}} \|\psi(g)\|_x.$$

Now, suppose  $\psi_n \rightarrow 0$  in  $\mathcal{E}^\sigma$  and  $T(\psi_n) \rightarrow y$  in  $E'$ . For fixed  $x \in E_\infty$  we obtain

$$\langle y, x \rangle = \lim_n \langle T(\psi_n), x \rangle = \lim_n \tau(\psi_n, x) = 0$$

in virtue of (4.3). Thus  $y$  vanishes on a total subset of  $E$  and the proof follows.

We collect the above results in the following

**4.5. THEOREM.** *Let  $G$  be a metrizable Yamabe group, and  $\Gamma$  a closed subgroup, such that  $G/\Gamma$  is compact. Let  $(\pi, E)$  and  $(\sigma, V)$  be continuous representations of  $G$  and  $\Gamma$  respectively in Fréchet spaces  $E$  and  $V$ . Then we have:*

1° *Every bilinear form  $\beta \in B_\Gamma(E_\infty, V)$  defines a sequentially closed operator  $T: \mathcal{C}^\sigma \rightarrow E'$  such that  $D_T \supset \mathcal{D}^\sigma$  and the restriction of  $T$  to  $\mathcal{D}^\sigma$  belongs to  $L_G(\mathcal{D}^\sigma, (E')_\infty)$ . The operator  $T$  is determined in a unique manner by the formula*

$$(4.4) \quad \langle T\psi, x \rangle = \int_{G/\Gamma} \varrho^{-1}(g) \beta(\pi(g^{-1})x, \psi(g)) d\mu(g\Gamma)$$

for all  $\psi \in \mathcal{D}^\sigma, x \in E_\infty$ .

2° *Suppose that  $T$  is a closeable operator on  $\mathcal{C}^\sigma$ , intertwining for  $U^\sigma$  and  $\hat{\pi}$  and satisfying  $D_T \supset \mathcal{D}^\sigma$ .*

*Then formula (4.4) defines a bilinear  $\Gamma$ -balanced form on  $E_\infty \times V$ .*

*(The space  $E_\infty$  is treated as before, as a  $G$ -module with the action of  $g \in G$  by means of  $\pi(g)$  and  $V$  as a  $\Gamma$ -module with the action of  $\gamma$  via  $\varrho^{1/2} \sigma(\gamma)$ .)*

The significance of this result becomes clearer if applied to a very special case  $G = \Gamma$ . It is easily seen that, this being the case,  $\mathcal{D}^\sigma$  is isomorphic to  $V_\infty$  (evaluation of a function at  $g = e$  realizes the isomorphism intertwining for  $U^\sigma$  and  $\sigma$ ). Likewise,  $\mathcal{C}^\sigma = V$  by the same argument.

**4.6. COROLLARY.** *Let  $G$  be as before, and let  $(\pi, E)$  and  $(\sigma, V)$  be two continuous representations of  $G$  on Fréchet spaces  $E$  and  $V$ .*

*Then every bilinear separately continuous invariant form  $\beta$  on  $V_\infty \times E_\infty$  extends in a natural way to  $V_\infty \times E$  and defines a closed operator  $T: V \rightarrow E'$  by the formula*

$$\langle Tv, x \rangle = \beta(v, x), \quad v \in V_\infty, x \in E.$$

*This operator has the following properties:*

1°  $D_T \supset V_\infty$ ,

2°  $T\sigma(g) \supset {}^t\pi(g)T, g \in G$ ,

3°  $T|V_\infty$  acts into  $(E')_\infty$  continuously.

**4.7. COROLLARY.** *A closeable operator  $T: V \rightarrow E'$  intertwining for  $\sigma$  and  $\pi$ , and such that  $D_T \supset V_\infty$  maps  $V_\infty$  into  $(E')_\infty$  continuously.*

**4.8. COROLLARY.** *A continuous operator  $T: V_\infty \rightarrow E'$  intertwining for  $\sigma$  and  ${}^t\pi$  is closeable on  $V$ .*

In the case of Banach space representations  $(\pi, E)$  and  $(\sigma, V)$  of a Lie group  $G$  these theorems are due to N. Skovhus Poulsen [26]. This

being the case a closed intertwining operator is uniquely determined by its restriction to  $V_\infty$ . In the excellent work of N. Skovhus Poulsen the reader may also find various applications of these theorems to the investigation of irreducibility and the Neumark equivalence of representations in spaces of smooth vectors.

We shall now proceed to investigate unitary representations of groups associated with induced representations. There is no general method of putting on  $\mathcal{D}^\sigma$  a prehilbert structure invariant with respect to the regular representation. In the case where the inducing representation  $(\sigma, V)$  is unitary, there exists a natural scalar product  $(\cdot|\cdot)$  on  $\mathcal{D}^\sigma$  relative to which  $U^\sigma$  acquires the structure of unitary representation. We shall refer to the case as the representation induced in the sense of Mackey. For completeness we recall this construction. Denote by  $(\cdot|\cdot)_V$  the scalar product on  $V$  invariant relative to  $\sigma$ . Then for  $f_1, f_2 \in \mathcal{D}^\sigma$  we put

$$(4.5) \quad (f_1|f_2) = \int_{G/\Gamma} e^{-1}(g) (f_1(g)|f_2(g))_V d\mu(g\Gamma).$$

The completion of  $\mathcal{D}^\sigma$  in the norm  $\|\cdot\| := (\cdot|\cdot)^{1/2}$  is denoted by  $H^\sigma$ . The regular representation of  $G$  on  $\mathcal{D}^\sigma$  extends to a unitary continuous representation of  $G$  on  $H^\sigma$ . It will be as usual denoted by  $(U^\sigma, H^\sigma)$ .

We shall, however, deal with a more general notion. By a *unitary induced representation* we shall mean a system  $(\mathcal{D}^\sigma, I, U, H)$  where  $(U, H)$  is a unitary representation of  $G$  and  $I \in L_G(\mathcal{D}^\sigma, H)$  is a continuous bijection onto a dense subset of  $H$ . Evidently, the system is uniquely determined by the inner product

$$(I(\cdot)|I(\cdot)) =: q(\cdot, \cdot) \quad \text{on } \mathcal{D}^\sigma.$$

Consequently, we shall denote the unitary induced representation defined by  $(\mathcal{D}^\sigma, I, U, H)$  also by  $({}^qU^\sigma, {}^qH^\sigma)$ .

For the purpose of proving an analogue of Theorem 4.5 in the theory of unitary induced representations we have to replace the original condition of compactness of  $G/\Gamma$  by certain growth condition for the function  $G \ni g \rightarrow \beta(\pi(g)x, \cdot) \in V'$ . It will be more convenient to formulate this condition with the aid of the operator  $\eta: E \rightarrow V'$ :

$$\langle \eta(x), v \rangle = \beta(x, v).$$

DEFINITION. Let  $(\pi, E)$  and  $(\sigma, V)$  be as before. By  $(\pi, \sigma, q)$ -*morphic form* we mean an operator  $\eta \in L_\Gamma(E_\infty, V')$  such that for every  $x \in E_\infty$  the functional

$$(4.6) \quad \mathcal{D}^\sigma \ni \psi \rightarrow F_x(\psi) = \int_{G/\Gamma} e^{-1}(g) \langle \eta(\pi(g^{-1})x), \psi(g) \rangle d\mu(g\Gamma)$$

is continuous in the  ${}^qH^\sigma$ -norm.

The space of  $(\pi, \sigma, q)$ -morphic forms is denoted by  $\mathcal{A}(\pi, \sigma, q)$ . Bearing in mind Lemma 4.4, we see that the functional  $F_x$  takes on elementary tensors  $f \otimes v$  the value  $\beta(\pi(f^\vee)x, v)$ . This remark, together with the duality Theorem 3.11, enables us to sight a connection between  $(\pi, \sigma, q)$ -morphic forms and operators intertwining for  ${}^aU^\sigma$  and  $\pi$ . In the case of the unitary representation induced in the sense of Mackey the  $(\pi, \sigma, q)$ -morphic forms are just  $(\pi, \sigma)$ -automorphic forms defined by K. and L. Maurin [18]. The desired continuity of the functional  $F_x$  is then equivalent to the following condition: the function

$$f: G \ni g \rightarrow \eta(\pi(g^{-1})x) \in V$$

is  $d\mu$ -square-integrable in the following sense:

$$\int_{G/\Gamma} (f(g) | f(g))_V e^{-1}(g) d\mu(g\Gamma) < \infty.$$

4.9. THEOREM. 1° Every  $(\pi, \sigma, q)$ -morphic form  $\eta$  defines a closed operator

$$T_\eta: {}^aH^\sigma \rightarrow E',$$

which has the following properties:

(a)  $D_T \supset {}^aH_\infty^\sigma$ ,

(b)  $T_\eta {}^aU^\sigma(g) \supset {}^t\pi(g)T_\eta$  and  $T_\eta \in L_G({}^aH^\sigma, (E')_\infty)$ .

2° Every closed operator  $T: {}^aH^\sigma \rightarrow E'$  verifying (a) and (b) defines a  $(\pi, \sigma, q)$ -morphic form  $\eta_T$  such that

$$(4.7) \quad \langle T\psi, x \rangle = \int_{G/\Gamma} e^{-1}(g) \langle \eta(\pi(g^{-1})x), \psi(g) \rangle d\mu(g\Gamma)$$

for all  $\psi \in \mathcal{D}^\sigma$ .

3°  $\eta_{T_\eta} = \eta$ .

Proof. Ad 1°. Denote by  $\tau_\eta(\psi, x)$  the right-hand side of formula (4.7). By use of Theorem 3.11 we state that  $\tau_\eta$  is a separately continuous form on  $\mathcal{D}^\sigma \times E$ , therefore it defines by restriction a separately continuous form on  $\mathcal{D}^\sigma \times E_\infty$ . The continuity property of a  $(\pi, \sigma, q)$ -morphic form implies that  $\tau_\eta$  is separately continuous relative to the  ${}^aH^\sigma$ -norm on  $\mathcal{D}^\sigma$ . Therefore  $\tau_\eta$  extends to a separately continuous form on  ${}^aH^\sigma \times E_\infty$  (see Bourbaki [2], III, § 4), which is easily seen to be  $G$ -invariant. Now assertion 1° follows by Corollary 4.6.

Ad 2°. By use of Corollary 4.7, the operator  $T$  defines a bilinear form on  ${}^aH_\infty^\sigma \times E$ , with the formula  $\tau(\psi, x) = \langle T\psi, x \rangle$ . In virtue of the duality Theorem 3.11 formula (4.7) determines an operator  $\eta_T: E_\infty \rightarrow V'$ . On the other hand, by Corollary 3.13 the form  $\tau$  can be continuously extended to  ${}^aH^\sigma \times E_\infty$ , as well. Thus  $\eta_T$  is a  $(\pi, \sigma, q)$ -morphic form. The assertion 3° is obvious.

Unfortunately, we are not able to derive a full duality between closed intertwining operators and  $(\pi, \sigma, q)$ -morphic forms in the general case of a Fréchet space representation  $(\pi, E)$ . In the event that  $(\pi, E)$  is a Banach space representation the mentioned theorem of Skovhus Poulsen holds: a closed intertwining operator with the domain containing  $H_\infty$  is uniquely determined by its restriction to  $H$ . Then we have

**4.10. THEOREM.** *The space  $C_G({}^qH^\sigma, E')$  of closed operators intertwining for  ${}^qU^\sigma$  and  ${}^t\pi$  and verifying (a) is isomorphic to the space of  $(\pi, \sigma, q)$ -morphic forms.*

This theorem when specified to the case of a Lie group  $G$  and  $(U^\sigma, H^\sigma)$  being induced in the sense of Mackey, is due to K. and L. Maurin [21]. The latter theorem follows, in turn, on the lines of a theorem of Strasburger and the author [36], which is concerned with the case of compact homogeneous space  $G/\Gamma$ .

**4.11. THEOREM.** *Let  $(\pi, H)$  be a unitary representation of the group  $G$  and let  $\Gamma, (\sigma, V), q$  be as before. Then the space  $C_G({}^qH^\sigma, H)$  is isomorphic to the space of  $(\pi, \sigma, q)$ -morphic forms. The isomorphism assigns to  $T \in C_G({}^qH^\sigma, H)$  the form  $\eta$  by means of the formula*

$$(4.8) \quad (T\psi|h) = \int_{G/\Gamma} \langle \eta(\pi(g^{-1})h), \psi(g) \rangle e^{-1}(g) d\mu(g\Gamma).$$

Plainly the case of an irreducible unitary representation  $(\pi, E)$  is worthy of notice. It is well known (cf. Neumark, *Normed Rings* (Russian), p. 298) that this being the case any closeable intertwining operators is continuous.

**4.12. THEOREM.** *Retain the assumptions of Theorem 4.11 and assume in addition  $(\pi, H)$  to be irreducible. Then there is a linear isomorphism*

$$L_G({}^qH^\sigma, H) = \mathcal{A}(\pi, \sigma, q)$$

*defined by formula (4.8).*

A similar result was obtained by K. and L. Maurin [18] in the case of an arbitrary locally compact group  $G$  and a unitary representation  $U$  induced in the sense of Mackey.

For the domain of an  $(\pi, \sigma)$ -morphic form a dense subspace  $\Phi \subset E$  is taken, which, roughly speaking, corresponds to our construction  $\Phi(E)$  associated with the convolution algebra  $O_0(G)$  of continuous compactly supported functions.

In the mentioned works [19] and [20], the reader may find an expository account of these developments as well as their further applications, especially to the determination of intertwining numbers for induced representations.

## 5. Induced representations of Banach algebras

In this section we investigate relations of our results to theorems of Rieffel [27] on induced representations of Banach algebras with approximate units. As we have mentioned above our approach to the theory of induced representations of algebras was strongly influenced by these developments. Nevertheless, as we shall prove, both methods lead to the same results just in the case of representations contragredient to Banach space representations.

In the sequel  $A$  will denote a Banach involutive algebra with bounded approximate unit. An approximate unit is said to be of norm one if  $\|a_i\| \leq 1$ . All modules and representations are assumed to be Fréchet space modules. Being so, we deal in this section with projective tensor products, which are isomorphic to inductive ones, in virtue of Lemma 1.5. In case of Banach space representation  $(\pi, E)$  we shall always assume  $\|\pi(a)\| \leq \|a\|$ .

From our point of view Banach algebras are distinguished by the following property

**5.1. LEMMA.** *Let  $(\pi, E)$  be a continuous representation of  $A$  in a Fréchet space  $E$ . Then the space  $\Phi(E)$  coincides with  $E$  and the natural injection is a topological isomorphism. If  $E$  is a Banach space and  $A$  admits an approximate unit of norm one then the isomorphism is isometric.*

**Proof.** Both spaces  $E$  and  $\Phi(E)$  are now Fréchet spaces. Denote by  $\{p_i\}$  and  $\{q_j\}$  families of seminorms describing the topologies in  $E$  and  $\Phi(E)$ , respectively. By the very definition of the topology in  $\Phi(E)$  the map

$$A \times E \ni (a, e) \rightarrow \pi(a)e \in \Phi(E)$$

is separately continuous, hence by Lemma 1.6 it is continuous. It means that for every  $q_i$  there exists a seminorm  $p_j$  and a constant  $c > 0$ , such that

$$q_i(\pi(a)x) \leq c\|a\|p_j(x).$$

It was proved in the course of the proof of Theorem 1.12 that for every bounded approximate unit  $\{a_i\}$  the set of operators  $\lambda^E(a_i)$  converges to the identity operator pointwise on  $A \hat{\otimes} E$ . Consequently,  $\pi(a_i)x = a(\lambda^E(a_i)y)$  converges to  $x = a(y)$  in  $\Phi(E)$  for all  $x \in \Phi(E)$ . Thus we obtain the estimation

$$q_i(x) = \lim_i q_i(\pi(a_i)x) \leq c\|a_i\|q_j(x) \leq c'q_j(x).$$

It means that on the total subspace  $\Phi(E) \subset E$  the topologies inherited from  $E$  and  $\Phi(E)$  are equivalent. Since both  $E$  and  $\Phi(E)$  are complete, we have  $E = \Phi(E)$ .

In the case of a Banach space  $E$ , a more precise estimations are needed. Let us denote by  $\|\cdot\|_E$ ,  $q(\cdot)$ ,  $\|\cdot\|^\wedge$  the norms in  $E$ ,  $\Phi(E)$ ,  $A \hat{\otimes} E$  respectively. Since  $q$  is the quotient norm in  $A \hat{\otimes} E / \text{Ker } a$  we have an obvious inequality

$$q(\pi(a)x) \leq \|a \hat{\otimes} x\|^\wedge.$$

Bearing in mind the definition of the norm  $\|\cdot\|^\wedge$  we obtain in turn

$$q(\pi(a)x) \leq \|a\| \|x\|_E,$$

and applying the bounded approximate unit  $\{a_t\}$  we assert

$$q(x) \leq \|x\|_E.$$

On the other hand, by the continuity of the representation  $\pi$  on  $E$  we have for

$$x = a \left( \sum_k a_k \otimes x_k \right) = \sum_k \pi(a_k) x_k$$

the following inequality

$$\|x\|_E \leq \sum_k \|a_k\| \|x_k\|_E.$$

Consequently,

$$(5.1) \quad \|x\|_E \leq \inf \sum_k \|a_k\| \|x_k\|_E,$$

where the inf is taken over all decompositions  $x = a \left( \sum_k a_k \otimes x_k \right)$ . It is easily seen that the right-hand side of (5.1) is just the norm of the quotient space  $A \hat{\otimes} E / \text{Ker } a$ . The proof is complete.

The following theorem results now immediately from our Theorem 2.4.

**5.2. THEOREM.** *Let  $(\pi, E)$  be a Fréchet space representation of a Banach involutive algebra  $A$  with a bounded approximate unit. Assume  $A$  to be an  $A$ - $S$ -bimodule. Then we have the following isomorphism*

$$L_A(A \hat{\otimes}_S V, E') = L_S(V, E')$$

for every  $S$ -module  $V$ . The isomorphism is defined by the formula

$$\langle T(p(a \otimes v)), x \rangle = \langle \eta(v), \pi(a^\sim)x \rangle.$$

In the special case of the trivial one dimensional module we obtain

**5.3. COROLLARY.** *Let  $(\pi, E')$  be a representation contragredient to a Fréchet space representation. Then*

$$L_A(A, E') = E'.$$

On the other hand, if  $(\lambda, A)$  is taken for  $(\pi, E)$  we obtain

5.4. COROLLARY. *For every bilinear continuous invariant form on  $A \times A$  there exists a functional  $\eta \in A'$  such that*

$$\tau(a, b) = \eta(a \check{v} b).$$

The above results permits one to investigate regular representations of locally compact groups in some function spaces associated with the space  $L^1(G, V, dg)$  by means of the induction process. Let  $G$  be a locally compact group and  $\Gamma$  a closed subgroup of  $G$ . We shall take for  $A$  the group algebra  $L^1(G)$  viewed as an  $A$ - $\Gamma$ -bimodule, with the action of  $\Gamma$  via right translations. Given a representation  $(\sigma, V)$  of  $\Gamma$  we produce the representation  $(\lambda^\sigma, A \hat{\otimes}_\Gamma V)$  in accordance with the general method described in Section 2. If  $V$  is assumed to be a Fréchet space we have  $L^1(G) \hat{\otimes} V = L^1(G) \bar{\otimes} V$ . Being so, the original space  $L^1(G) \bar{\otimes} V$  becomes isomorphic to the function space  $L^1(G, V)$  (Grothendieck [12], p. 59) viewed as a  $G$ -module with respect to left translations  $\lambda(g)$  and with the action  $L^1(G)$  by means of convolution operators

$$\lambda(\varphi)f = \int_G \varphi(g)\lambda(g)f dg, \quad \varphi \in L^1(G), f \in L^1(G, V).$$

It is our purpose to legitimate the method used in case of the regular representation in  $\mathcal{D}(G, V)$ , and to prove that the representation of  $G$  on  $A \hat{\otimes}_\Gamma V$  is equivalent to the regular representation on the space  $l_\sigma(L^1(G, V))$ . We recall

$$(5.2) \quad l_\sigma(f)(g) = \int_\Gamma (\delta_\Gamma / \delta_G)^{-1/2}(\gamma) \sigma(\gamma) f(g\gamma) d\gamma.$$

5.5. LEMMA. *Let  $(\sigma, V)$  be a representation of  $\Gamma$  on a Fréchet space  $V$ , such that the set of operators  $\{(\delta_\Gamma / \delta_G)^{-1/2}(\gamma) \sigma(\gamma) : \gamma \in \Gamma\} \subset L(V)$  is equicontinuous. Then for  $dg$  almost all  $g$  the integral (5.2) does exist and the function  $g \rightarrow l_\sigma f(g)$  is locally integrable on  $G$ .*

Proof. We apply a version of formula (3.3) (see e.g. Bourbaki [3], VII, § 2) which is concerned with functions on  $G$  valued in a Banach space. Let us fix a seminorm  $q$  on  $V$ , denote by  $i_q$  the natural map of  $V$  into the Banach space  $V_q$  obtained by the completion of  $V/\text{Ker } q$  in the norm  $q$ . For every  $f \in L^1(G, V)$ , the function  $i_q f$  belongs to  $L^1(G, V_q)$  and the mentioned theorem is in force. It states that for  $dg$  almost all  $g$  the function

$$h(g, \cdot) : \Gamma \ni \gamma \rightarrow i_q f(g\gamma) \varrho^{-1}(g\gamma) \in V_q$$

is  $d\gamma$  integrable.

Moreover, the function

$$G/\Gamma \ni g\Gamma \rightarrow \int_\Gamma h(g, \gamma) d\gamma \in V_q$$

is Bochner integrable with respect to the quasi-invariant measure  $d\mu$  related to the rho-function  $\varrho$  and the following formula holds

$$(5.3) \quad \int_G f(g) dg = \int_{G/\Gamma} d\mu(g\Gamma) \varrho(g)^{-1} \int_F \varrho^{-1}(\gamma) f(g\gamma) d\gamma.$$

Consequently the natural lift of this function to the group  $G$  is locally integrable. Since the topology of  $V$  is determined by a countable family of seminorms, the assertions remain true when  $f$  is viewed as  $V$ -valued function.

By our hypotheses, for every seminorm  $p$  there exists a seminorm  $q$  on  $V$  such that

$$p(\varrho^{1/2}(\gamma) \sigma(\gamma) v) \leq q(v).$$

The function

$$\Gamma \ni \gamma \rightarrow \varrho^{-1/2}(\gamma) \sigma(\gamma) f(g\gamma) \in V_p$$

is for  $dg$  almost all  $g$   $d\gamma$ -measurable and by the inequality

$$(5.4) \quad p(\varrho^{-1/2}(\gamma) \sigma(\gamma) f(g\gamma)) \leq \varrho^{-1}(\gamma) q(f(g\gamma)) \leq q(\varrho(g) h(g, \gamma))$$

it is  $d\gamma$ -integrable. Accordingly, formula (5.2) defines a measurable function on  $G$ . On the other hand formula (5.4) implies that the function  $p(l_\sigma f(g))$  satisfies the inequality

$$(5.5) \quad p(l_\sigma f(g)) \leq \int_F \varrho^{-1/2}(\gamma) p(\sigma(\gamma) f(g\gamma)) d\gamma \leq \int_F \varrho^{1/2}(\gamma) q(f(g\gamma)) d\gamma.$$

As we have mentioned above the right-hand side of (5.5) is a locally integrable function on  $G$ . This completes the proof of the lemma.

5.6. LEMMA. *Ker  $l_\sigma$  coincides with the closed linear span of the set*

$$W = \{f \in L^1(G, V) : f(g) = r(\gamma) \varphi(g) - (\delta_\Gamma \delta_G)^{1/2}(\gamma) \sigma(\gamma) \varphi(g), \\ \text{for some } \varphi \in L^1(G, V), \gamma \in \Gamma\}.$$

*Proof.* Let us define a family of seminorms on  $L^1(G, V)$  by the formula

$$A_{p,\psi}(f) = \int_G \psi(g) p(l_\sigma f(g)) dg, \quad \psi \in C_0(G).$$

Taking into account formulas (5.5) and (5.3) we obtain

$$|A_{p,\psi}(f)| \leq \int_G \psi(g) \left( \int_F \varrho^{-1}(\gamma) q(f(g\gamma)) d\gamma \right) dg \\ \leq \sup_g |\psi|^p(g) \int_G q(f)(g) dg, \quad \text{where } |\psi|^p(g) = \int_F |\psi|(g\gamma) d\gamma.$$

Therefore  $A_{p,\psi}$  is a continuous seminorm on  $L^1(G, V)$ . The function  $l_\sigma f$  equals zero for  $dg$  almost all  $g$  iff  $A_{p,\psi}(f) = 0$  for all  $p$  and  $\psi$ . Equivalently

$$\text{Ker } l_\sigma = \bigcap_{p,\psi} \text{Ker } A_{p,\psi}$$

and it becomes clear that  $\text{Ker } l_\sigma$  is closed. The assertion of the lemma follows now by direct computations which are quite similar to this ones, which we have performed while proving Lemma 3.7.

The above lemmas give rise to an identification of the space  $A \hat{\otimes}_R V = A^{(r,v)}$  and the image of the operator  $l_\sigma$ .

It is easily verified that every  $f \in l_\sigma L^1(G, V)$  satisfies the following conditions:

- 1°  $f$  is measurable;
- 2° for  $dg$  almost all  $g$

$$f(g\gamma) = (\delta_r/\delta_G)^{1/2}(\gamma) \sigma(\gamma^{-1}) f(g);$$

3° for every continuous seminorm  $q$  on  $V$  there exists a non-negative measurable function  $\psi^q$  such that

- (a)  $e^{-1}\psi^q$  is right  $\gamma$ -invariant,
- (b)  $\int_{G/\Gamma} d\mu(g\Gamma) e^{-1}(g) \psi^q(g) < \infty$ ,
- (c)  $q(f)(g) \leq \psi^q(g) dg$  almost everywhere.

Denote by  $L^\sigma$  the space of all functions verifying 1°, 2°, 3° and determine a locally convex topology in  $L^\sigma$  by means of the family of seminorms

$$\|f\|_q = \inf_{\psi^q} \|e^{-1}\psi^q\|_{L^1(G/\Gamma, \mu)}$$

where  $\psi^q$  ranges over the family of all functions verifying (a), (b), (c).

**5.7. PROPOSITION.** *Let  $G/\Gamma$  be paracompact. Then  $l_\sigma(L^1(G, V)) = L^\sigma$ .*

*Proof.* We have to prove that  $l_\sigma$  is surjective. To this end we follow on the lines of our proof of Lemma 3.8.

Well now, let  $\{\mathcal{O}_i\}$  denote a locally finite covering of the space  $G/\Gamma$  by open sets in each of which there exists a continuous section of the bundle  $G \rightarrow G/\Gamma$ . Let  $\Phi_i$  be a continuous partition of unit subordinate to this covering and let  $\psi$  be a continuous compactly supported function on  $\Gamma$  such that

$$\int_\Gamma e^{-1/2} \psi d\gamma = 1.$$

We define

$$f^\#(g) = \sum_i \Phi_i(g\Gamma) \psi(s_i(g\Gamma)^{-1}g) \sigma(g^{-1}s_i(g\Gamma)) f(s_i(g\Gamma)).$$

Plainly the series makes sense and the function  $f^\#$  is measurable. The formal computation leads to the formula  $l_\sigma(f^\#) = f$ . In order to legiti-

mate this calculus and complete the proof it remains to show that  $f^\# \in L^1(G, V)$ .

$$\begin{aligned} \int_G q(f^\#(g)) dg &= \int_{G/\Gamma} d\mu(g\Gamma) e^{-1}(g) \int_\Gamma q\left(\sum_i \Phi_i(g\Gamma)\right) \psi(s_i(g\Gamma)^{-1}g\gamma) \times \\ &\quad \times \sigma(\gamma^{-1}g^{-1}s_i(g\Gamma)) f(s_i(g\Gamma)) d\gamma \\ &\leq \int_{G/\Gamma} d\mu(g\Gamma) e^{-1}(g) \sum_i \Phi_i(g\Gamma) \int_\Gamma \psi(s_i(g\Gamma)^{-1}g\gamma) (\delta_\Gamma/\delta_G)^{-1}(\gamma) \times \\ &\quad \times q(\sigma(\gamma^{-1}g^{-1}s_i(g\Gamma))) f(s_i(g\Gamma)) d\gamma \\ &\leq \int_{G/\Gamma} d\mu(g\Gamma) e^{-1}(g) \sum_i \Phi_i(g\Gamma) \int_\Gamma \psi(s_i(g\Gamma)^{-1}g\gamma) (\delta_\Gamma/\delta_G)^{1/2} \times \\ &\quad \times (s_i(g\Gamma)^{-1}g\gamma) (\delta_\Gamma/\delta_G)(\gamma^{-1}) d\gamma q'(f(s_i(g\Gamma))), \end{aligned}$$

where we have employed equicontinuity property of operators  $\sigma(\gamma)$ .

Finally, using 3° we obtain

$$\begin{aligned} \int_G q(f^\#(g)) dg &\leq \int_{G/\Gamma} d\mu(g\Gamma) e^{-1}(g) \sum_i \Phi_i(g\Gamma) (\delta_\Gamma/\delta_G)(s_i(g\Gamma)x) \psi^\alpha(s_i(g\Gamma)) \\ &= \int_{G/\Gamma} d\mu(g\Gamma) e^{-1}(g) \sum_i \Phi_i(g\Gamma) \psi^\alpha(g) = \int_{G/\Gamma} d\mu(g\Gamma) e^{-1}(g) \psi^\alpha(g). \end{aligned}$$

The algebraic bijection between  $A \hat{\otimes}_\Gamma V$  and  $L^\sigma$  is then established: Moreover, the latter inequality states, that the inverse mapping  $L^\sigma \rightarrow A \hat{\otimes}_\Gamma V$  is continuous. Since both spaces are Fréchet spaces the proof follows by Banach theorem.

Finally the reciprocity Theorem 5.2 takes in this case the form:

$$L_{L(G)}(L^\sigma, E') = L_\Gamma(V, E')$$

for any representation of  $\Gamma$  satisfying hypotheses of Lemma 5.5 and any continuous representation of  $G$  connected with a continuous representation of  $L^1(G)$ .

The space  $L^\sigma$  was introduced by Rieffel in the case of Banach space isometric representation  $(\sigma, V)$ .

The objective of the Rieffel treatment is the Frobenius reciprocity for representations of a Banach algebra  $A$  induced (in the sense of the present work) by isometric representations of such subalgebras of  $A$  which contain an approximate unit for  $A$ .

The theorem of Rieffel asserts then the isometry

$$(5.6) \quad L_A(A \hat{\otimes}_B V, E) = L_B(V, E)$$

for any isometric representation  $(\sigma, V)$  of  $B$  and isometric representation  $(\pi, E)$  of  $A$ .

Our Theorem 5.2 concerns much larger class of induced representations, however as the representation  $(\pi, E)$  merely contragredient representations are  $(\pi, E')$  are admitted. If this assumption on  $(\pi, E)$  is removed the isomorphism fails to be true, as a theorem of Wendel shows  $(L_A(A, A) \neq A$  in case of  $A = L^1(G)$  for nondiscrete group  $G$ ).

The theorem (5.6), in contrast to our duality Theorem 5.2, is not directly applicable to investigations of group representations, because there is no correspondence between (nonopen) subgroups of  $G$  and subalgebras of the group algebra  $L^1(G)$ . Nevertheless, Rieffel investigated independently representations of the algebra  $L^1(G)$  induced by a representation of a subgroup  $\Gamma \subset G$ , and found a sufficient condition, for  $(\pi, E)$  the duality,

$$L_G(L^\sigma, E) = L_\Gamma(V, E)$$

to be valid for all  $(\sigma, V)$  and all  $\Gamma \subset G$ . This condition is formulated as follows:

$$L_G(L(G), E) = E;$$

if it is satisfied, the module  $E$  is called *corepresenting*.

Corollary 5.3 applied to the algebra  $L^1(G)$  implies that all representations contragredient to essential Fréchet space representations are corepresenting.

At the same time Rieffel has conjectured that the isomorphism

$$L_A(A, W) = W$$

holds for any  $A$ -module  $W$  which is the dual of a Banach space. Now, Corollary 5.3 gives an affirmative answer to the conjecture, as far as contragredient representations of  $A$  on  $E'$  are concerned.

## 6. The Frobenius reciprocity in the theory of square-integrable representations of Hilbert algebras

An associative  $*$ -algebra  $A$  provided with a prehilbert structure  $(\cdot | \cdot)$  is called a *Hilbert algebra* if the following axioms are satisfied

$$(6.1) \quad (x | y) = (y^* | x^*);$$

$$(6.2) \quad (xy | z) = (y | x^*z), \quad x, y, z \in A;$$

$$(6.3) \quad \text{the left regular representation is continuous and essential.}$$

Throughout what follows  $A$  will denote a separable Hilbert algebra. The completion of  $A$  in the norm  $(\cdot | \cdot)^{1/2}$  is denoted by  $\mathcal{H}(A)$ . The isometric antihomomorphism  $*$  extends uniquely to the antilinear isometry on  $\mathcal{H}(A)$

denoted by  $J$ . Accordingly, the operators of the left (right) regular representation extend to operators on  $\mathcal{H}(A)$ , which are denoted by  $U_x$  (respectively  $V_x$ ). The following identities result less or more immediately from the axioms:

- 6.1. LEMMA ([6]). 1°  $U_x^* = U_{x^*}$ ,  $V_x^* = V_{x^*}$ ,  $x \in A$ ;  
2°  $JU_xJ = V_{x^*}$ ,  $JV_xJ = U_{x^*}$ .

As proved by Godement [11] the axiom (6.3) is equivalent to the following one

(6.3')  $A$  admits a bounded approximate unit.

The algebra multiplication  $(x, y) \rightarrow xy$  does not in general extend to a bilinear map on  $\mathcal{H}(A)$ . The objective of the preliminary study in the theory of Hilbert algebras is a maximal subspace of  $\mathcal{H}(A)$  which acquires the continuous multiplication extending the original multiplication from  $A$ . More precisely, a vector  $a \in \mathcal{H}(A)$  is said to be bounded if there exists a bounded operator  $U_a \in L(\mathcal{H}(A))$ , such that  $U_ax = V_xa$ ,  $x \in A$ . The linear space of bounded elements is called the fulfillment of  $A$  and denoted by  $\bar{A}$ . Plainly  $A \subset \bar{A}$ . In order to see that  $\bar{A}$  admits a structure of \*-algebra we need the following

6.2. LEMMA ([6]). An element  $a \in \mathcal{H}(A)$  is bounded iff there exists an operator say  $V_a \in L(\mathcal{H}(A))$ , such that  $V_ax = U_xa$ ,  $x \in A$ . Moreover

$$(6.4) \quad V_a^* = V_{Ja} = JU_aJ$$

and

$$(6.5) \quad U_{Ja} = U_a^*.$$

By the above lemma the space  $\bar{A}$  is invariant under the action of the operator  $J$ . Consequently, we can define on  $A$  the structure of Hilbert algebra putting

$$ab := U_ab = V_ba \quad \text{and} \quad a^* := Ja.$$

One proves that  $\bar{A} = \bar{\bar{A}}$ . A Hilbert algebra satisfying  $\bar{A} = A$  is called a full algebra.

Let  $\mathcal{U}(A)$  (resp.  $\mathcal{V}(A)$ ) denote the von Neumann algebra generated by the family  $U_a$ ,  $a \in A$ , of operators (resp.  $V_a$ ,  $a \in A$ ). If  $\sigma \in L(\mathcal{H}(A))$ ,  $\sigma'$  will denote the commutator of  $\sigma$ . The following theorem plays a fundamental role in the theory of Hilbert algebras.

6.3. COMMUTATION THEOREM ([11]).  $\mathcal{U}(A)' = \mathcal{V}(A)$ ,  $\mathcal{V}(A)' = \mathcal{U}(A)$ .

6.4. LEMMA. Suppose  $T \in \mathcal{U}(A)$  and  $a \in A$ . Then  $Ta \in \bar{A}$  and  $U_{Ta} = TU_a$ . Respectively for  $T \in \mathcal{V}(A)$  we have  $V_{Ta} = TV_a$ .

In particular, any projective operator  $P$  commuting with the left regular representation of  $A$  satisfies

$$(6.6) \quad (P\mathcal{H}(A)) \cap \bar{A} = P\bar{A} \subset \bar{A}.$$

Thus every closed subspace of  $\mathcal{H}(A)$  which is invariant under the action of the left (right) regular representation contains a dense subset of bounded elements.

**EXAMPLES.** 1. Let  $G$  be a unimodular locally compact group, and let  $A = L^1 \cap L^2(G)$  be viewed as a convolution algebra with the involution  $x^*(g) = x(g^{-1})$ . It is easily verified that  $A$  is a Hilbert algebra relative to the scalar product induced by  $L^2(G)$ .

More generally, let  $\mu$  be a central ( $\mu(x*y) = \mu(y*x)$ ) positive definite measure on  $G$ . Then the quotient space  $C_0(G)/N$ , where  $N = \{x: \mu(x**x) = 0\}$  can be turned into a Hilbert algebra with the following rules

$$([x] | [y]) := \mu(x**y) \quad \text{and} \quad [x][y] := [x*y], \quad [x]^* := [x^*].$$

The previous example corresponds to the case  $\mu = \delta_e$ .

2. One may expect that Hilbert algebras which satisfy  $\mathcal{H}(A) = A$  are of special interest. This is the case if in the first example we assume in addition  $G$  to be a compact group. Since the Haar measure is then finite, we have  $L^2 \subset L^1$ , and therefore, the convolution is well defined for all  $x, y \in \mathcal{H}(A)$ .

3. Consider now the space  $\text{HS}(H_1, H_2)$  of all Hilbert-Schmidt operators from  $H_1$  to  $H_2$ . We recall:  $T \in L(H_1, H_2)$  is said to be a *Hilbert-Schmidt operator* if there exists an orthonormal basis  $\{h_i\}$  in  $H_1$  such that

$$\sum_i \|Te_i\|_2^2 = \sum_i (T^*Te_i | e_i) < \infty.$$

One proves that for every  $T, S \in \text{HS}(H_1, H_2)$  the series

$$(6.7) \quad (T | S) := \sum_i (Te_i | Se_i)$$

is convergent and defines a sesquilinear form on  $\text{HS}(H_1, H_2)$  which is independent of the choice of an orthonormal basis.

The space  $\text{HS}(H_1) := \text{HS}(H_1, H_1)$  provided with the structure of \*-algebra induced by  $L(H_1)$  and with the scalar product (6.7) is a Hilbert algebra.

The algebras described in examples 2 and 3 are the special cases of  $H^*$ -algebras. A \*-algebra, which is a Hilbert space is called  $H^*$ -algebra if these two structures are connected by the axioms (6.1), (6.2) and  $x^*x = 0$  implies  $x = 0$ . If a Hilbert algebra is complete it is a  $H^*$ -algebra since the latter condition is automatically satisfied. In fact, taking an approximate unit for  $A$  we state

$$0 = (x^*x | x_n) = (x | x_n) \rightarrow \|x\|,$$

what was to be proved.

It was W. Ambrose [1] who proved the fundamental structure theorem for  $H^*$ -algebras. Before we formulate this result we recall some basic terminology. An element  $e \in A$  is called *idempotent* if  $e^2 = e$ , it is *self-adjoint* if  $e^* = e$ . An idempotent  $e$  is said to be *minimal* provided that every decomposition of  $e$  into a sum of mutually orthogonal idempotents is a trivial one. Because the norm of an idempotent is no less than 1, every idempotent admits a decomposition into a finite sum of minimal ones.

6.5. THEOREM ([1]). *Let  $A$  be a  $H^*$ -algebra. Then*

1°  $A = \bigoplus_a \bar{A}_a$ , where  $A_a$  are minimal two-sided ideals in  $A$ ,

2° there is an orthonormal system of self-adjoint idempotents  $\{a_{\alpha\beta}\}$  such that

$$A_a = \bigoplus_{\beta} A_a a_{\alpha\beta}.$$

The constituents  $A_a a_{\alpha\beta}$  of the decomposition are easily seen to be irreducible with respect to the left regular representation. When applied to the group algebra  $L^2(G)$  of a compact group  $G$  the theorem leads to the Peter-Weyl theorem.

Now, we define what we will mean in this section by a square-integrable representation of a Hilbert algebra.

Let  $(\pi, H)$  be a unitary representation of a Hilbert algebra  $A$  in a Hilbert space  $H$ . A vector  $h \in H$  is called *square-integrable* if the operator  $S_h: A \ni a \rightarrow \pi(a)h \in H$  is continuous. By a *square-integrable representation* of  $A$  we shall mean an essential representation  $(\pi, H)$ , such that every vector in the carrier space  $H$  is square-integrable. Fundamental theorems concerning the structure of square-integrable representation have been derived by M.A. Rieffel in [28]. We shall set down these results in a form convenient to our applications.

First of all we remark that the set of square-integrable vectors corresponding to the left regular representation of  $A$  in  $\mathcal{K}(A)$  is identical with the fulfillment of  $A$ . On the other hand, a continuous operator  $T: H_1 \rightarrow H_2$  intertwining for the representations  $\pi_1$  and  $\pi_2$  maps square-integrable vectors of the representation  $\pi_1$  into square-integrable vectors associated with the representation  $\pi_2$ .

6.6. PROPOSITION (see [14]). *Let  $(\pi, H)$  be a cyclic square-integrable representation of the algebra  $A$ . Then  $(\pi, H)$  is unitarily equivalent to a sub-representation of the left regular representation of  $A$  on  $\bar{A}$ .*

Proof. Let us denote by  $h_0$  the cyclic vector in  $H$ . The mapping  $S_{h_0}: A \ni a \rightarrow \pi(a)h_0$  is a continuous operator intertwining for  $U$  and  $\pi$ . The Hilbert space adjoint operator  $S_{h_0}^*$  is a continuous operator from  $H$  in  $\mathcal{K}(A)$  intertwining for  $\pi$  and  $U$ . It is injective. In fact,

$$(S_{h_0}^* h' | a) = (h' | S_{h_0} a) = (h' | \pi(a)h_0).$$

Since  $h_0$  is cyclic  $S_{h_0}^* h' = 0$  implies  $h' = 0$ . The polar decomposition  $S_h^* = IT$  consists of injective operators. Thus  $I$  is just the desired isometry intertwining for  $\pi$  and  $U$ . Now, by the remarks preceding the proposition the proof follows.

As proved by Rieffel a representation  $\pi$  of  $A$  admits a square-integrable cyclic vector iff it is equivalent to a subrepresentation of the left regular representation of  $A$  on  $\mathcal{H}(A)$ .

Now we direct our attention in more detailed study of unitary irreducible representations of  $A$ .

Let  $(\pi, H)$  denote an irreducible representation and suppose  $h \in H$  to be square-integrable. The operator  $S_h S_h^*: H \rightarrow H$  commutes with operators of the irreducible representation  $\pi$ , hence it is proportional to the identity operator. There exists a positive constant  $c_h$ , such that

$$(6.8) \quad (S_h^* h' | S_h^* h'') = c_h (h' | h'')_H.$$

At the point we make the following observation. The operator  $S_h: \mathcal{H}(A) \rightarrow H$ , being proportional to the adjoint operator of the isometry  $S_h^* c_h^{-1/2}$  appears to be surjective and partially isometric.

Therefore there exists  $h_1 \in H$  such that  $S_{h_1}(B_{\mathcal{H}(A)}) = B_H$  ( $B_X$  denoting the unit ball in the Hilbert space  $X$ ).

Putting  $h' = h'' = h_1$  in (6.8) we obtain

$$\begin{aligned} c_h (h_1 | h_1) &= \|S_h^* h_1\|^2 = \sup_{a \in B_A} |(a | S_h^* h_1)|^2 = \sup_{a \in B_A} |(S_h a | h_1)|^2 \\ &= \sup_{a \in B_A} |(\pi(a)h | h_1)|^2 = \sup_{a \in B_A} |(h | \pi(a^*)h_1)|^2 \\ &= \sup_{a \in B_A} |(h | \pi(a)h_1)|^2 = \sup_{h' \in B_H} |(h | h')|^2 = \|h\|^2. \end{aligned}$$

Formula (6.8) takes the form

$$(6.9) \quad (S_h^* h' | S_h^* h'') = \|h_1\|^{-2} (h | h) (h' | h'').$$

The constant  $n_\pi = \|h_1\|^2$  is called the *formal dimension* of the representation  $\pi$  for reasons which will be explained in the due course. The formula is valid for all  $h', h'' \in H$  and  $h$  ranging over the set of square-integrable elements of  $H$  which constitute at least a dense subset of  $H$ . Using the polarization formula we can turn (6.9) into:

$$(6.10) \quad (S_x^* h' | S_y^* h'') = n_\pi^{-1} (y | x) (h' | h'').$$

Thus we have at once

$$\|S_h\| = \|S_h^*\| = n_\pi^{-1/2} \|h\|$$

and consequently

$$\|\pi(a)h\| \leq n_\pi^{-1/2} \|h\| \|a\|,$$

for  $a, h$  ranging over dense subsets of  $\mathcal{H}(A)$  and  $H$ , respectively. Then the formula extends in a unique manner to the whole  $H$ , what means that all vectors in  $H$  are square-integrable.

6.7. THEOREM. *Let  $(\pi, H)$  be an irreducible unitary representation of  $A$  which have a square-integrable vector. Then  $(\pi, H)$  is square-integrable and the orthogonality relations (6.10) are valid for all  $x, y, h', h' \in H$ .*

Now we return to our main subject, namely the Frobenius duality.

6.8. THEOREM. *Retain the assumptions of Theorem 6.7. Then the mapping*

$$H \ni h \rightarrow n_\pi^{-1/2} S_h \in L_A(U, \pi)$$

*is an isometric isomorphism.*

**Proof.** After orthogonality relations (6.10) it suffice to prove that this mapping is surjective. To this end it suffice to verify  $\Phi(H) = H$  and then apply Theorem 1.14. Recall that  $\Phi(H) = \alpha(A \bar{\otimes} H)$  and  $\alpha(a \otimes h) = S_h a$ . Since both  $A$  and  $H$  are normed spaces and  $H$  is complete we have the isomorphism  $A \bar{\otimes} H = A \hat{\otimes} H = \mathcal{H}(A) \hat{\otimes} H$ . This implies  $\Phi(H) = \alpha(\mathcal{H}(A) \otimes H)$ , and in particular  $\Phi(H) \supset S_h(\mathcal{H}(A)) = H$ . Now, in virtue of Theorem 1.14 every intertwining operator  $T \in L_A(U, \pi)$  determines a functional  $\eta = (h | \cdot) \in H'$ , such that for every  $x \in H$

$$(T(a) | x) = \eta(\pi(a^*)x) = (h | \pi(a^*)x) = (\pi(a)h | x) = (S_h a | x),$$

and  $T = S_h$  as desired.

Each unitary irreducible square-integrable representation of  $A$  is equivalent to a subrepresentation of the left regular representation of  $A$  (Proposition 6.6). For simplicity we may assume  $H \subset \bar{A}$ . Denote by  $P_H$  the orthogonal projector on  $H$ . Because  $P_H$  is the intertwining operator for  $U$  and  $\pi$ , it follows by Theorem 6.8 that  $P_H = S_h$  for some  $h \in H$ . It is easy to verify that fundamental properties of a projector:  $P_H P_H = P_H$  and  $P_H^* = P_H$  are equivalent to  $h^2 = h$  and  $h^* = h$ .

It means that each irreducible component of the regular representation is determined by a self-adjoint idempotent. Plainly, this idempotent satisfies  $S_h(B_{\mathcal{H}(A)}) = B_H$ , hence the norm of the idempotent connected with the representation  $\pi$  is equal to  $n_\pi^{1/2}$ .

Before presenting more advanced results along this line we give fundamental examples, which motivate the terminology introduced by M. A. Rieffel.

Let  $A$  be the group algebra  $L^1 \cap L^2(G)$  of a separable locally compact group  $G$ . Let  $(\pi, H)$  be a representation of  $A$  defined by a unitary representation of  $G$ . A vector  $h \in H$  is square-integrable iff the mapping

$$A \ni f \rightarrow \int_G \pi(g)f(g)h dg$$

is continuous in the  $L^2$ -norm. In particular, for every  $h' \in H$  the functional

$$A \ni f \rightarrow \int_G \bar{f}(g) (\pi(g)h | h') dg$$

is continuous in the norm  $L^2$ . By the Fréchet–Riesz theorem, it follows that the matrix element

$$G \ni g \rightarrow (\pi(g)h | h') =: \omega_{h,h'}(g)$$

is a square-integrable function on  $G$ .

The operator  $S_h^*$  assigns to  $h' \in H$  the matrix element  $\omega_{h,h'} \in L^2(G)$ . The orthogonality relations are now turned into the form

$$\int_G \bar{\omega}_{h_1,h'}(g) \omega_{h_2,h''}(g) dg = n_\pi^{-1}(h_2 | h_1) (h' | h'').$$

When compared with the orthogonality relations of matrix elements of irreducible representations of a compact group the formula shows that in this special case the formal dimension is equal to the “true” dimension of a representation.

If the supposition of irreducibility in Theorem 6.8 is dropped the duality fails to be true. At any rate the set of operators constitutes a linear subspace of  $L_A(U, \pi)$  which admit a natural Hilbert structure inherited from  $H$ . The problem that interests us is to describe this structure without referring to the correspondence  $h \rightarrow S_h$ . The solution of this problem rests on a closer study of the structure of unitary algebras from the point of view of the theory of traces.

Let  $\mathcal{U}$  be a von Neumann algebra, and  $\mathcal{U}^+$  the cone of positive elements of  $\mathcal{U}$ . A function on  $\mathcal{U}^+$  with values  $\geq 0$  finite or infinite is called a *trace* if the following conditions are satisfied:

- 1°  $\varphi(S+T) = \varphi(S) + \varphi(T)$ ;
- 2°  $\varphi(aS) = a\varphi(S)$ ,  $a > 0$ ,  $S, T \in \mathcal{U}^+$  (convention:  $0 \cdot \infty = 0$ );
- 3°  $\varphi(SS^*) = \varphi(S^*S)$ ,  $S \in \mathcal{U}$ .

A trace is said to be *normal* if  $\varphi(\sup T_i) = \sup \varphi(T_i)$  for every bounded increasing net  $\{T_i\}$  in  $\mathcal{U}^+$ . A trace  $\varphi$  is called *faithful* if  $\varphi(T) = 0$  implies  $T = 0$ ; it is called *semi-finite* if every non-zero  $T \in \mathcal{U}^+$  majorizes a non-zero  $S \in \mathcal{U}^+$ , with  $\varphi(S) < \infty$ .

For a given faithful, semi-finite, normal trace  $\varphi$  there exists a two-sided ideal  $\mathfrak{m}^{1/2}$  in  $\mathcal{U}$ , such that

$$(\mathfrak{m}^{1/2})^2 \cap \mathcal{U}^+ = \{T \in \mathcal{U}^+ : \varphi(T) < \infty\}.$$

The trace  $\varphi$  extends to a linear functional on  $\mathfrak{m} := (\mathfrak{m}^{1/2})^2$ , such that  $\varphi(ST) = \varphi(TS)$ . The ideal  $\mathfrak{m}^{1/2}$  has the prehilbert structure defined by

$$(6.11) \quad [S | T] := \varphi(S^*T).$$

6.9. LEMMA ([6]). Let  $(A, (\cdot|\cdot))$  be a Hilbert algebra and  $\mathcal{V}(A)$  the von Neuman algebra generated in  $L(\mathcal{K}(A))$  by the operators  $V_x$  of right multiplication. Then there exists a unique trace on  $\mathcal{V}(\mathcal{K}(A))$ , which is faithful, normal and semi-finite and such that

$$1^\circ \mathfrak{m}^{1/2} = \{V_a : a \in \bar{A}\},$$

2° for every  $T, S \in \mathfrak{m}^{1/2}$  such that  $T = V_a, S = V_b$  we have

$$\varphi(T^*S) = (a|b).$$

Conversely, for every faithful, normal and semi-finite trace on  $\mathcal{U}$  the ideal  $\mathfrak{m}^{1/2}$  provided with the scalar product (6.11) is a Hilbert algebra.

Let us give a simple illustration. An orthogonal projector  $P$  onto the space of an irreducible subrepresentation of  $(U, A)$  has the form  $P = V_h$ , for some symmetric idempotent  $h \in \bar{A}$ . Thus

$$\varphi(P^*P) = \varphi(P) = \|h\|^2 = n_\pi.$$

As expected the trace of the projector is equal to the formal dimension of the representation.

Let  $(\pi, H)$  be a unitary representation of a Hilbert algebra  $A$  with the canonical trace  $\varphi$ . If  $T \in L_A(U, \pi)$  then  $T^*T \in \mathcal{U}(A)' = \mathcal{V}(A)$  by the commutation theorem.

DEFINITION.  $L_A^2(U, \pi) = \{T \in L_A(U, \pi) : T^*T \in \mathfrak{m}\}$ .

The elements of the space  $L_A^2(U, \pi)$  will be referred to as the Hilbert-Schmidt operators, relatively to the trace  $\varphi$ .

There is an inner product on  $L_A^2(U, \pi)$  defined by the formula

$$[T|S] = \varphi(T^*S).$$

The formula makes sense since  $T^*S \in \mathfrak{m}$  by the polarization formula.

6.10. THEOREM. Let  $(\pi, H)$  be a unitary square-integrable representation of a Hilbert algebra  $A$ . Then the mapping

$$H \ni h \rightarrow S_h \in L_A(U, \pi)$$

is a linear isometry of  $H$  onto  $L_A^2(U, \pi)$ .

Proof. Let us first suppose that  $(\pi, H)$  is a cyclic subrepresentation of  $U$  with  $H \subset \bar{A}$ . This being the case the operator  $S_h$  coincides with  $V_h$ , thus in virtue of Lemma 6.9  $S_h \in L_A^2(U, \pi)$ .

Conversely, if  $T \in L_A^2(U, \pi)$  then by the lemma we have  $T = V_h$  for some  $h \in A$ . Denoting by  $P$  the orthogonal projection onto  $H$ , we obtain  $PV_h = V_h$ , and since  $P \in (\mathcal{U}(A))' = \mathcal{V}(A)$ , we have by Lemma 6.4

$$V_{Ph} = PV_h = V_h.$$

Since the regular representation is faithful, we conclude  $h \in H$ , as desired. The isometry of the mapping at hand follows immediately by Lemma 6.9.

Turning to the general case, let  $H = \bigoplus H_i$  be a decomposition of the separable space  $H$  into an orthogonal sum of subspaces invariant and cyclic relative to  $\pi$ , and let  $P_i$  denote the orthogonal projector onto  $H_i$ .

Given  $T \in L_A^2(U, \pi)$ , we have

$$T = \sum_i P_i T =: \sum_i T_i, \quad \text{with} \quad T_i \in L_A^2(U, \pi).$$

In virtue of the former result, together with Proposition 6.8, we have  $T_i = S_{h_i}$  with  $h_i \in H_i$ , and  $|||T_i||| = \|h_i\|$  ( $|||\cdot|||$  denoting the  $L_A^2$ -norm).

Since  $\text{Ker } T_i^* = \text{Coker } T_i$ , we see that  $T_i \perp T_j$  in  $L_A^2$  for  $i \neq j$ , and consequently

$$|||T|||^2 = \sum \|T_i\|^2 = \sum \|h_i\|^2 = h^2, \quad \text{where} \quad h = \sum_i h_i.$$

This completes the proof of the theorem.

In a large treatise [29], M. A. Rieffel has shown that unitary representations of locally compact groups induced from a compact subgroup can be defined in terms of certain Hilbert module tensor products. This construction is quite similar to that given in Section 2. The difference is that by using Hilbert space tensor product instead of the inductive tensor product one obtains unitary representations of groups.

We give a brief summary of notation and basic results which will be needed in the sequel. Let  $A$  be a Hilbert algebra and assume  $\mathcal{H}(A)$  to be an  $A$ - $S$ -bimodule. Let  $V$  be a Hilbert space  $S$ -module. We denote by  $A \otimes V$  the Hilbert space tensor product of  $A$  and  $V$ , i.e. the completion of the algebraic tensor product  $A \otimes V$  with respect to the unique inner product, whose value on elementary tensors is given by

$$(6.12) \quad \langle a \otimes v, a' \otimes v' \rangle = (a | a')(v | v')_V.$$

By  $A \otimes_S V$  we denote the quotient space  $A \otimes_S V / Y$ , where  $Y$  is the closed subspace spanned by elements of the form  $as \otimes v - a \otimes sv$ ,  $a \in A$ ,  $v \in V$ ,  $s \in S$ . The space  $A \otimes_S V$  becomes a unitary representation of  $A$ , when the action  $a \in A$  is defined by  $[a' \otimes v] \rightarrow [(aa') \otimes v]$ .

In this section,  $(U^{(\mathcal{V}, S)}, H^{(\mathcal{V}, S)})$  will denote the representation of  $A$  on  $A \otimes_S V$ . If  $S$  is an empty set we write simply  $(U^{\mathcal{V}}, H^{\mathcal{V}})$ . In order to determine a (pre)Hilbert structure on the space of operators intertwining for  $U^{(\mathcal{V}, S)}$  and a square-integrable representation  $U$ , we are looking for a natural trace on the commutator  $(U^{\mathcal{V}})'$ . Although we deal entirely with bounded operators we find it convenient to employ methods and terminology of non-commutative integration of I. Segal. Let  $\varphi$  denote as before

a normal faithful semi-finite trace on  $\mathcal{U}$  and  $\mathfrak{m} \subset \mathcal{U}$  the ideal of traceable operators in  $\mathcal{U}$ . One can define a norm on  $\mathfrak{m}$  by the formula

$$\|T\|_1 = \sup_{\substack{S \in \mathcal{U} \\ \|S\|=1}} |\varphi(ST)|.$$

The norm  $\|\cdot\|_1$  has the following properties:

$$(6.13) \quad \begin{aligned} \|ST\|_1 &\leq \|S\| \|T\|_1, & \|TS\|_1 &\leq \|T\|_1 \|S\|, & T \in \mathfrak{m}, S \in \mathcal{U}; \\ \|T\|_1 &= \|T^*\|_1 = \varphi(|T|) & \text{where } |T| &= (T^*T)^{1/2}. \end{aligned}$$

The completion of  $\mathfrak{m}$  in this norm is denoted by  $L^1(\mathcal{U}, \varphi)$  and referred to as the space of integrable operators (relative to  $\varphi$ ). The theory of Segal provides motivation for the terminology. To each  $T \in L^1(\mathcal{U}, \varphi)$  there is associated in a unique manner a closed, densely defined operator on  $H$ . The domain of  $T$  has additional properties called strong density relative to  $\mathcal{U}$ .

A linear subspace  $E \subset H$  is called *strongly dense* in  $H$  if

1°  $UE \subset E$  for every unitary operator  $U \in \mathcal{U}'$ ,

2° there exists a sequence of increasing subspaces  $E_n \subset E$  such that each of  $E_n$  has the property 1°, and

3° for every  $n$ , and a partially isometric operator  $S \in \mathcal{U}$ , from  $SE_n \subset E_n$  it follows  $S = \text{id}$  on  $E$ ,

4°  $E_n^\perp \searrow 0$ .

The functional  $\varphi$  extends uniquely to a symmetric positive and continuous functional on  $L^1(\mathcal{U}, \varphi)$  called the integral. The formulas (6.13) are valid for all  $T \in L^1(\mathcal{U}, \varphi)$  and  $S \in \mathcal{U}$ .

Now we formulate an analogue to the Fubini theorem, developed by Segal [33] and Stinespring [34]. As in the measure-theoretic situation the theorem asserts that the iterated integral equals the double integral.

Let  $(H_i, \mathcal{U}_i, \varphi_i)$  be two von Neumann algebras with the traces  $\varphi_i$  and let  $\mathcal{U}_1 \otimes \mathcal{U}_2$  denote the von Neumann algebras in  $L(H_1 \otimes H_2)$  generated by elementary operators  $T_1 \otimes T_2$ ,  $T_i \in \mathcal{U}_i$ . Then the formula

$$\varphi(T_1 \otimes T_2) = \varphi_1(T_1)\varphi_2(T_2)$$

defines a trace on  $\mathcal{U}_1 \otimes \mathcal{U}_2$  and consequently the space  $L^1(\mathcal{U}_1 \otimes \mathcal{U}_2, \varphi)$ . As proved by Stinespring, each  $T \in L^1(\mathcal{U}_1 \otimes \mathcal{U}_2, \varphi)$  determines a function  $q_T: H_1 \otimes H_2 \rightarrow \mathbb{C}$ , defined on a strongly dense set by means of the formula:

$$q_T(v) := \langle Tv | v \rangle.$$

6.11. THEOREM ([34]). *Let  $T \in L^1(\mathcal{U}_1 \otimes \mathcal{U}_2, \varphi)$ . Then*

(1) *There exists a strongly dense subset  $E_2 \subset H_2$  such that for every  $y \in H_2$  there exists unique operator  ${}^yT \in L^1(\mathcal{U}_1, \varphi_1)$  such that*

$$({}^yTx | x) = q_T(x \otimes y).$$

(2) *There exists an operator  $\xi(T) \in L^1(\mathcal{U}_2, \varphi)$  such that*

$$(\xi(T)y | y) = \varphi_1({}^vT).$$

(3)  $\varphi(T) = \varphi_2(\xi(T))$ .

**EXAMPLE 1.** Let  $L(H) = \mathcal{U}_2$  and let us define a trace on  $\mathcal{U}_2$  putting  $\varphi_2(P_E) = \dim E$ , where  $P_E$  denotes the orthogonal projection on the subspace  $E \subset H$ . It is easy to verify that the only strongly dense (relative to  $\varphi_2$ ) subspace of  $H$  is  $H$  itself. The space  $L^1(\mathcal{U}_2, \varphi_2)$  consists of operators which are traceable in the usual sense; the ideal  $\mathfrak{m}^{1/2}$  consists of Hilbert-Schmidt operators. The integral corresponding to this trace will be denoted by  $\text{tr}T$ .

**EXAMPLE 2.** Let  $\mathcal{U}_1$  be the von Neumann algebra  $\mathcal{V}(A)$  associated with the Hilbert algebra  $A$ . It was proved by Segal that for every  $a \in A$  there exists a closed (strongly) dense defined operator  $V_a$  such that  $V_a x = xa$ . Moreover,  $V_a^* V_a$  is well defined on a strongly dense domain and belongs to  $L^1(\mathcal{V}(A), \varphi_1)$  and

$$\varphi_1(V_a^* V_a) = (a | a).$$

In what follows  $(\mathcal{U}, \varphi)$  will denote the tensor product of measures described in Examples 2 and 1. To each operator  $T \in L_A(A \otimes_S V)$  we assign the unique operator  $\tilde{T} \in L_A(A \otimes V)$  defined by the formula

$$[\tilde{T}(a \otimes v)] = T([a \otimes v])$$

and the condition  $\tilde{T}(x) \in Y^\perp (= [0]^\perp)$ . Since the spaces  $Y$  and  $Y^\perp$  are  $U_a$  invariant, the assignment

$$L_A(H^{(V, S)}) \ni T \rightarrow \tilde{T} \in L_A(H^V)$$

is well defined and injective.

6.12. **LEMMA.** *If  $T \in L_A(H^{(V, S)})$  then  $\tilde{T} \in \mathcal{V}(A) \otimes L(V)$ .*

**Proof.** Since  $T$  commutes with operators  $U_a \otimes \text{id}$ , we have

$$\begin{aligned} \tilde{T} \in (\mathcal{U}(A) \otimes C \text{id})' &= \mathcal{U}(A)' \otimes L(V) \quad (\text{see Dixmier [6], p. 26}) \\ &= \mathcal{V}(A) \otimes L(V) \quad \text{by Commutation Theorem.} \end{aligned}$$

**DEFINITION.** Let  $(\pi, H)$  be a representation of  $A$  on a Hilbert space  $H$ . We denote by  $L_A^2(U^{(V, S)}, \pi)$  the subspace of  $L_A(U^{(V, S)}, \pi)$  of operators which satisfy

$$T^* T \in L^1(\mathcal{U}, \varphi).$$

We provide  $L_A^2(U^{(V, S)}, \pi)$  with the norm  $\|T\|_2 = \varphi(T^* T)^{1/2}$ .

6.13. **THEOREM.** *Let  $(A, (\cdot | \cdot))$  be a Hilbert algebra and let  $\mathcal{K}(A)$  be an  $A$ - $S$ -bimodule. Suppose that the carrier space of a unitary square-integrable representation  $(\pi, H)$  is a  $S$ -module such that*

$$(6.14) \quad \pi(as)h = \pi(a)sh, \quad a \in A, s \in S, h \in H.$$

Let  $(U^{(V, S)}, H^{(V, S)})$  be the representation of  $A$  induced by an  $S$ -module  $V$ . Then we have an isometry

$$L_A^2(H^{(V, S)}, H) = \text{HS}_S(V, H)$$

determined by the formula

$$(6.15) \quad T([a \otimes v]) = \pi(a)\eta(v)$$

for  $T \in L_A^2(H^{(V, S)}, H)$ ,  $\eta \in \text{HS}_S(V, H)$ .

Proof. First of all we show that the assignment  $T \rightarrow \eta$  is correct. It is clear that for fixed  $v \in V$  and  $T \in L_A^2(H^{(V, S)}, H)$  the formula

$$A \ni a \rightarrow \tilde{T}(a \otimes v) \in A \otimes V$$

defines a continuous operator, say  ${}^vQ$ . Now we apply Theorem 6.11. Due to Theorem 6.11 (1) for every  $v \in V$  there exists a unique integrable operator  ${}^vT$  on  $\mathcal{H}(A)$  such that

$$({}^vT a | a) = (\tilde{T}(a \otimes v) | \tilde{T}(a \otimes v)) = ({}^vQ(a) | {}^vQ(a)) = ({}^vQ^* {}^vQ(a) | a).$$

Thus  ${}^vQ^* {}^vQ = {}^vT$  is integrable, that is,  ${}^vQ \in L^2(U, \pi)$ . Thanks to Theorem 6.10 there exists  $\eta(v) \in H$  such that formula (6.15) holds. Moreover, with the aid of Theorem 6.11 (2) we state that there exists a traceable operator  $\xi(T) \in L(V)$  such that for every  $v \in V$

$$(\xi(T)v | v) = \varphi({}^vQ^* {}^vQ) = (\eta(v) | \eta(v))_H.$$

The latter equality follows by Theorem 6.10. Therefore the operator  $\eta^* \eta = \xi(T)$  has a finite trace, that is,  $\eta \in \text{HS}(V, H)$ . From Theorem 6.11 (3) it follows in turn that

$$\|T\|_2^2 = \varphi(T^* T) = \|\eta^* \eta\|_1 = \|\|\eta\|\|^2$$

where  $\|\|\cdot\|\|$  denotes the Hilbert-Schmidt norm in  $\text{HS}(V, H)$ . In order to prove that  $\eta$  is an intertwining operator for the action of  $S$  on  $H$  and  $V$  we observe that for every  $a \in A$ ,  $v \in V$ ,  $s \in S$  we have

$$(6.16) \quad \begin{aligned} 0 &= \tilde{T}(as \otimes v - a \otimes sv) = \pi(as)\eta(v) - \pi(a)\eta(sv) \\ &= \pi(a)s\eta(v) - \pi(a)\eta(sv). \end{aligned}$$

In particular,

$$(\pi(a)h | s\eta(v) - \eta(sv)) = 0$$

for all  $a \in A$ ,  $h \in H$ , and the assertion is proved.

It remains to verify that the mapping  $T \rightarrow \eta$  defined by (6.15) is surjective. Given  $\eta \in \text{HS}_S(V, H)$ , we define  $T_0$  on  $A \otimes V$  by the formula  $T_0(a \otimes v) = \pi(a)\eta(v)$ . In order to prove that this operator is bounded on

$A \otimes V$ , let us choose orthonormal bases in  $\mathcal{A}(A)$ :  $\{a_i\}$  and in  $V$ :  $\{v_j\}$ . The basis  $\{a_i\}$  can be chosen in such a way that  $a_i \in \bar{A}$ .

$$(6.17) \quad \sum_{i,j} |(x|T(a_i \otimes v_j))_H|^2 = \sum_{i,j} |(x|\pi(a_i)\eta(v_j))_H|^2 = \sum_{i,j} |(S_x a_i^* | \eta(v_j))_H|^2 \\ = \sum_{i,j} |(a_i^* | S_x^* \circ \eta(v_j))|^2 = \sum_j \|S_x^* \circ \eta(v_j)\|_A^2.$$

Here we have made use of the fact that  $\{a_i^*\}$  is also an orthonormal basis in  $\mathcal{A}(A)$  and the Parseval formula is in force. The final series is convergent since the operator  $S_x^* \circ \eta$  is a Hilbert-Schmidt operator. Now, it is easy to observe that the series (6.17) is equal to the norm of the functional

$$A \otimes V \ni y \rightarrow (x|T_0 y).$$

Thus we have proved that the formula

$$H \times (A \otimes V) \ni (x, a \otimes v) \rightarrow (x|T_0(a \otimes v))$$

defines a bilinear form, which is separately continuous with respect to the topology induced in  $A \otimes V$  by  $A \otimes V$ . This form extends to a continuous form on  $H \times (A \otimes V)$  and defines the operator  $T_0 \in L(A \otimes V, H)$ .

We define  $T$  by the formula

$$T([a \otimes v]) = T_0(a \otimes v).$$

To verify that this formula is correct it suffice to observe that reading (6.16) in the opposite direction we find that  $T_0$  vanishes on  $Y$ . We have yet to show that the operator  $T_0^* T_0$  is integrable. Since it is a bounded positive definite operator, it suffice to verify that  $\varphi(T_0^* T_0) < \infty$ . Write  $X = T_0^* T_0$  and define  ${}^v X := {}^v Q {}^* v Q$ .

Then  ${}^v X \in L^1(\mathcal{A}(A), \varphi_1)$  by Theorem 6.10 and moreover  $\varphi_1({}^v X) = (\eta^* \eta v | v)_V$ . The operator  $\eta^* \eta = \xi(X)$  is traceable and the following formula makes sense

$$\|T_0\|^2 = \varphi(X) = \text{tr } \xi(X) = \|\eta\|^2 < \infty,$$

since by Theorem 6.11 the double integral at hand is equal to the iterated integral  $\varphi(X)$ . This completes the proof.

It is worth indicating that in the case of an irreducible representation  $\pi$  the formula (6.17) together with the orthogonality relations (6.11) lead to the isometry of the map  $T \rightarrow \eta$  relative to the usual operator norm on  $L_A(H^{(V, S)}, H)$ .

6.14. THEOREM. *Retain assumptions of the preceding theorem and assume in addition that  $(\pi, H)$  is an irreducible representation with the formal dimension  $n_\pi$ . Then the mapping  $T \rightarrow n_\pi^{-1/2} \eta$  determines the isometry from  $L_A(H^{(V, S)}, H)$  into  $\text{HS}_S(V, H)$ .*

As we have mentioned above the construction of the representation  $U^{(V, S)}$  is especially applicable to the investigation of unitary representations of groups induced from compact subgroups. It was proved by M. A. Rieffel that taking  $L^1 \cap L^2(G)$  for  $A$  and a unitary representation  $(\sigma, V)$  of a compact  $K \subset G$  for  $(V, S)$  one obtains the representation  $U^{(V, S)}$  which is unitarily equivalent to the representation  $U^\sigma$  induced by  $\sigma$  in the sense of Mackey. The action of  $K$  on  $H$  by means of the operators  $\pi(k)$  satisfies condition (6.14). We obtain

6.15. COROLLARY. *Let  $(\pi, H)$  be a unitary square-integrable representation of a unimodular group  $G$ , and let  $(\sigma, V)$  be a unitary representation of a compact subgroup  $K \subset G$ . Then there exists an isometry*

$$L_G^2(U^\sigma, \pi) = \text{HS}_K(\sigma, \pi).$$

With the aid of Theorem 6.14 we obtain in turn

6.16. COROLLARY (K. and L. Maurin [19]). *Let  $G, K, (\sigma, V)$  be as before and let  $(\pi, H)$  be a unitary irreducible square-integrable representation of  $G$ . Then there exists an isometry*

$$L_G(U^\sigma, \pi) = \text{HS}_K(\sigma, \pi).$$

Plainly, the latter result (as well as Theorem 6.14) may be obtained without referring to the theory of non-commutative integration. The simple proof was given by J. Szmidi [37].

In the case of a compact group  $G$  we obtain the result mentioned in the introduction.

6.17. COROLLARY (M. A. Rieffel [29]). *Let  $(\sigma, V)$  be a unitary representation of a subgroup  $K$  of a compact group  $G$  and let  $(\pi, H)$  be a unitary representation of  $G$ . Then we have the following isometry*

$$\text{HS}_G(U^\sigma, \pi) = \text{HS}_K(\sigma, \pi).$$

The proof follows by the observation that in the case of a compact group  $G$  the space  $L^1(\mathcal{V}(A), \varphi)$  consists of operators which are traceable in the usual sense.

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